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Asymptotic Behavior of Minimizers for the Ginzburg-Landau Functional with Weight. Part I

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0. Introduction

Let G be a bounded, simply-connected smooth domain of $\mathbb{R}^2$, $g:\ \partial G \to S^1$ a smooth boundary function of degree d and $p(x)$ a smooth positive function on $\bar{G}$. For each $\varepsilon > 0$ let u_ε be a minimizer for the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G p|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2 \tag{0.1}$$

of Ginzburg-Landau type on the set

$$H_g^1 = \{u \in H^1(G,\circ);\ u = g \text{ on } \partial G\}\,.$$

In this paper we are interested in studying the asymptotic behavior of u_ε as $\varepsilon \to 0$. In their fundamental works BETHUEL, BREZIS & HÉLEIN have studied the case $p \equiv 1$ for boundary data satisfying $d = 0$ [BBH1] and $d \neq 0$ [BBH2]. In this latter work only the case of G starshaped was treated. Later, STRUWE [S] proposed a method which works for an arbitrary domain (recently DEL PINO & FELMER [DPF1,2] gave a very simple argument for reducing the general case to the starshaped case). Z.C. HAN & Y.Y. LI [HL] have generalized the results of [BBH1,2] to higher dimensions (when the Dirichlet energy is replaced by the n-energy). The method of STRUWE is found to be very useful for the case of a nonconstant p. In the case $d = 0$ the analysis of [BBH1] carries over without any difficulty to the case p nonconstant and one can prove the $C^{1,\alpha}$ convergence $u_\varepsilon \to u_0$ as $\varepsilon \to 0$, with $u_0 \in C^\infty(\bar{G}, S^1)$ a solution of $-\operatorname{div}(p\nabla u_0) = p|\nabla u_0|^2 u_0$ in G. Hence, from now on we assume that $d \neq 0$ and, without loss of generality, that $d > 0$.

Let us now give some motivation for studying the functional (0.1). The first motivation is a physical one. The functional (0.1) is related to the Ginzburg-Landau energy in superconductivity. It should be noted that here, in contrast with the physical problem, we ignore the magnetic vector potential A. Moreover, there are no boundary data in the physical problem. So from the physical point of view we can consider our problem only as a "model problem". In the usual model $p \equiv 1$. The presence of a nonconstant weight function is motivated by the problem of pinning of vortices, that is, of forcing the location of the vortices to some restricted sites. One possible mechanism for modeling pinning was introduced by DU & GUNZBURGER [DG]. They consider a thin (three-dimensional) film whose thickness in the z-direction varies as a function of the two-dimensional variable x and so it equals $\delta p(x)$, with small δ for some positive smooth function $p(x)$. By an averaging process in the z-direction, they are led to a functional of the type (0.1) (in fact, in the resulting functional $p(x)$ multiplies also the term $(1-|u|^2)^2$, but this does not affect our analysis, which can be modified easily to cover this case too, see the remark at the end of Section 2 in [ASh2]). As we shall see below, the zeros of u_ε are indeed located near the minima of p for small ε. Another model for pinning is obtained by introducing impurities into the material. This leads to the functional

$$F_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (a(x)^2 - |u|^2)^2; \tag{0.2}$$

see RUBINSTEIN [R]. Here we assume that $a(x)$, describing the maximal density of the superconducting electrons at x, is a smooth function satisfying $0 < a(x) \leqq 1$ in G. As it turns out (see Appendix B) the study of the minimizers of F_ε is essentially the same as that of the minimizers of E_ε. The asymptotic behavior of minimizers of the functional F_ε, in the special case where the number of minima of $a(x)$ in G is greater than or equal to d, was studied independently by LASSOUED [L].

Another motivation for introducing the weight function p is purely mathematical. One can view the results of [BBH2] as a way of realizing an S^1-valued harmonic map with isolated singularities as a limit of minimizers of a relaxed problem, i.e., as a limit when $\varepsilon \to 0$ of the minimizers u_ε of E_ε with $p \equiv 1$. Now one may do the same by introducing an arbitrary Riemannian metric on the domain. This leads to the functional

$$H_\varepsilon(u) = \frac{1}{2}\int_G \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2, \tag{0.3}$$

with $A(x) = (a_{ij}(x))_{i,j}$ a positive definite matrix with smooth entries. This remark, due to S. AGMON, was communicated to us by H. BREZIS. The functional E_ε is thus a special case of H_ε corresponding to a metric which is conformal to the Euclidean metric. Looking over the proofs of the present paper we see that they can be generalized with simple adaptations to the more general functional H_ε. The modification becomes easier if we use the classical result (see [W]) which states that any smooth metric $\sum a_{i,j}(x)dx_i dx_j$ can be transformed by a suitable change of variables, at least locally, to a metric of the form $p(x)(dx_1^2 + dx_2^2)$. Hence all the results we shall establish for E_ε are valid, with suitable modifications of the statements, for H_ε. In particular, the zeros of the minimizers for H_ε are located, for small enough ε, near the minima of $\det(a_{i,j}(x))$. The results of [ASh2] also have appropriate analogues for the minimizers of H_ε (see [ASh2] for details).

Now we describe the main results of this paper. We denote $p_0 \equiv \min\{p(x);\ x \in \bar{G}\}$ and $\Lambda = p^{-1}(p_0) = \Lambda_i \cup \Lambda_b$ where $\Lambda_i = \Lambda \cap G$ and $\Lambda_b = \Lambda \cap \partial G$. We set $K = \operatorname{card}\Lambda$, $K_i = \operatorname{card}\Lambda_i$ and $K_b = \operatorname{card}\Lambda_b$. Our first result, which is proved in Section 1, is a general convergence result. It was proved by Beaulieu & Hadiji [BH1,2] for the cases $K_i \geqq d$ and $K = 1$.

Theorem 1. *$u_{\varepsilon_n} \to u_*$ in $C^{1,\alpha}_{\mathrm{loc}}(\bar{G} \setminus \{b_1, \dots, b_N\})$ for a subsequence $\varepsilon_n \to 0$ for every $\alpha < 1$, where the N distinct points $\{b_1, \dots, b_N\}$ lie in Λ. The limit $u_* \in C^\infty(\bar{G} \setminus \{b_1, \dots, b_N\}, S^1)$ is a solution of*

$$-\operatorname{div}(p\nabla u_*) = p|\nabla u_*|^2 u_* \quad \text{in } \bar{G} \setminus \{b_1, \dots, b_N\}, \quad u_* = g \text{ on } \partial G.$$

Finally, around each b_j, u_* is of degree $D_j > 0$ and $\sum_{j=1}^N D_j = d$.

In this statement, for $b_j \in \Lambda_b$ the degree of u_* around b_j is defined as the degree on $\partial(B(b_j, R) \cap G)$ (for small $R > 0$) of the map which equals g on $B(b_j, R) \cap \partial G$ and u_* on $\partial B(b_j, R) \cap G$.

Recall that when $K_i \geqq d$, Beaulieu & Hadiji [BH1,2] showed that $N = d$, $D_j = 1$ for all j, and that the configuration $\{b_1, \dots, b_d\}$ is minimizing for a certain renormalized energy defined on Λ^d. The more interesting case is $K < d$ since then singularities of degree > 1 must occur, and in some cases there are singularities on the boundary.

We would thus like to understand the "selection process" for the points $\{b_1, \dots, b_N\}$ and the associated degrees $\{D_1, \dots, D_N\} \in \mathbb{Z}^N$. As a first step in this direction we establish in Section 2 a precise description of u_{ε_n} for n large. Namely, u_{ε_n} has its zeros located in d discs of radius $\sim \varepsilon_n$ called "bad discs". Outside these discs $|u_{\varepsilon_n}|$ is close to 1. Using a result of Bauman, Carlson & Phillips [BCP] we can show that for n large each bad disc contains exactly one zero. There are thus exactly D_k zeros approaching each b_k (as $n \to \infty$). In case D_k is bigger than 1 (this must be the case for at least one k if $K < d$), we expect to observe an "interaction energy" between zeros approaching the same limit b_k. Similarly, for $b_k \in \partial G$ we expect an interaction energy between zeros and the boundary to appear. In order to get estimates for the energy of u_{ε_n} it is essential to study the mutual distances between bad discs approaching the same b_k and, for $b_k \in \partial G$, the distances between zeros and the boundary. It turns out that these distances depend in a crucial way on the behavior of p around its minima points Λ. We will study this problem in detail in the second part of this work [ASh2] under some particular assumptions on the behavior of p near its minima. For general p in Section 2 we prove:

Theorem 2. *Let $\{u_{\varepsilon_n}\}$ be a sequence of minimizers as in Theorem 1. Then, for n large enough, u_{ε_n} has exactly d zeros $z_1^{(n)}, \dots, z_d^{(n)}$. The degree of u_{ε_n} around each $z_i^{(n)}$ is equal to 1. Moreover for all $\beta \in (0, 1)$, $\operatorname{dist}(z_i^{(n)}, \partial G) \geqq \varepsilon_n^\beta$ for every i and $|z_i^{(n)} - z_j^{(n)}| \geqq \varepsilon_n^\beta$ for every $i \neq j$.*

The results of this paper were announced in [ASh1].

1. Proof of Theorem 1

Our proof of Theorem 1 consists of two main ingredients. First we use the method of Struwe [S] in order to locate the "bad discs", i.e., a finite collection of discs of radius $O(\varepsilon)$, or $O(\varepsilon^\alpha)$, which cover the set $\{x;\ |u_\varepsilon(x)| < \frac{1}{2}\}$. Then we use some modifications of the results of Brezis, Merle & Rivière [BMR] in order to bound from below the energy of u_ε near the singularities by the energy of a "reference map" of the form $\prod \big(\frac{x-x_i^\varepsilon}{|x-x_i^\varepsilon|}\big)^{d_i}$. This lower bound is used to get an upper bound for the energy of u_ε away from the singularities, which in turn implies the desired convergence. This method follows essentially the strategy of [BBH2] and [S]. There is however a significant difference coming from the fact here that we do not have any *explicit* lower bound for the energy (in comparison with $E_\varepsilon(u_\varepsilon) \geqq \pi d|\log\varepsilon| - C$ in the case $p \equiv 1$ for example). Indeed, a posteriori we know that the asymptotic behavior of $E_\varepsilon(u_\varepsilon)$ may be different for different p's. What makes the method work after all is the "good behavior" of the reference map away from the singularities. This idea will become clear in the sequel.

We start by quoting three lemmas. The first is a simple upper bound for the energy, which is proved in [BH2]. The next two lemmas require obvious modifications of those of [S]. Hence, all three proofs are omitted.

Lemma 1.1. *For every $\delta > 0$ there exists a constant $C(\delta)$ such that*

$$E_\varepsilon(u_\varepsilon) \leqq (p_0 + \delta)\pi d|\log\varepsilon| + C(\delta) \quad \forall\varepsilon > 0.$$

Lemma 1.2. *There exists an integer J_0 such that $J_\varepsilon \leqq J_0$ for any collection of mutually disjoint discs $\{B(x_j^\varepsilon, \frac{\varepsilon}{5});\ 1 \leqq j \leqq J_\varepsilon,\ x_j^\varepsilon \in G\}$ with $|u_\varepsilon(x_j^\varepsilon)| < \frac{1}{2}$, for all j.*

Lemma 1.3. *For every $\alpha \in (0,1)$ there exists a constant $C(\alpha)$ such that*

$$\frac{1}{\varepsilon^2}\int_{B(x,\varepsilon^\alpha)\cap G} (1 - |u_\varepsilon|^2)^2 \leqq C(\alpha) \quad \forall x \in G,\ \forall\varepsilon > 0.$$

For each $\varepsilon > 0$ we can find by a simple recursive process a finite collection of disjoint discs $\{B(x_j, \frac{\varepsilon}{5}),\ 1 \leqq j \leqq J_\varepsilon\}$ such that $\{x \in G;\ |u_\varepsilon(x)| < \frac{1}{2}\} \subset \bigcup_{j=1}^{J_\varepsilon} B(x_j, \varepsilon)$ (actually $x_j = x_j^\varepsilon$ but we omit ε for simplicity). By Lemma 1.2 we know that $J_\varepsilon \leqq J_0$ uniformly in ε. Starting with the collection $\{B(x_j,\varepsilon)\}_{j=1}^{J_\varepsilon}$ we can obtain after a finite number of iterations (no more than J_0), each consisting of multiplying all the radii by 9 and deleting some discs (see [BBH2, p. 50]), a new collection of discs $\{B(x_j,\lambda\varepsilon)\}_{j=1}^{N_s}$ (again $x_i = x_i^\varepsilon$ and $N_s = N_s^\varepsilon$ but we have $N_s \leqq J_\varepsilon \leqq J_0$, also $\lambda = \lambda_\varepsilon \leqq 9^{J_0}$) satisfying

$$\{x \in G;\ |u_\varepsilon(x)| < \tfrac{1}{2}\} \subset \bigcup_{j=1}^{N_s} B(x_j, \lambda\varepsilon), \tag{1.1}$$

$$|x_i - x_j| \geqq 8\lambda\varepsilon \quad \forall\, i \neq j. \tag{1.2}$$

Similarly, starting with the collection $\{B(x_j, \varepsilon^{1/2})\}_{j=1}^{N_s}$ we obtain after $k \leqq N_s$ iterations, each consisting of multiplying all the radii by 9 and deleting some discs, a new collection of discs $\{B(y_i, \nu\varepsilon^{1/2})\}_{i=1}^{N_b}$ with $N_b \leqq N_s$, $\{y_1, \dots, y_{N_b}\} \subset \{x_1, \dots, x_{N_s}\}$, and $\bigcup_{j=1}^{N_s} B(x_j, \lambda\varepsilon) \subset \bigcup_{i=1}^{N_b} B(y_i, \nu\varepsilon^{1/2})$ satisfying

$$|y_i - y_j| \geqq 8\nu\varepsilon^{1/2} \quad \forall\, i \neq j. \tag{1.3}$$

By a possible further modification we can also assume that

$$\text{for all } i \text{ either } y_i \in \partial G \text{ or } \operatorname{dist}(y_i, \partial G) \geqq 2\nu\varepsilon^{1/2}, \tag{1.4}$$

$$\text{for all } j \text{ either } x_j \in \partial G \text{ or } \operatorname{dist}(x_j, \partial G) \geqq 2\lambda\varepsilon. \tag{1.5}$$

We call $\{B(y_i, \nu\varepsilon^{1/2}\}_{i=1}^{N_b}$ the "big bad discs" (BBD's) and $\{B(x_j, \lambda\varepsilon)\}_{j=1}^{N_s}$ the "small bad discs" (SBD's). We recall that the x_j's and the y_i's all depend on ε and so do λ and ν but these last two are uniformly bounded independently of ε. Relabeling the SBD's we may assume we have the following situation: Each BBD $B(y_i, \nu\varepsilon^{1/2})$ contains $n_i \geqq 1$ SBD's $\{B(x_{i,j}, \lambda\varepsilon)\}_{j=1}^{n_i}$ which are actually contained in $B(y_i, \frac{1}{4}\nu\varepsilon^{1/2})$. By our construction all the degrees $d_i = d_i^\varepsilon = \deg(u_\varepsilon, \partial B(y_i, \nu\varepsilon^{1/2}))$, $d_{i,j} = d_{i,j}^\varepsilon = \deg(u_\varepsilon, \partial B(x_{i,j}, \lambda\varepsilon))$ are well defined. Moreover, as in [BBH2] we easily see that

$$|d_i|, |d_{i,j}| \leqq \kappa \quad \forall\, \varepsilon,\ \forall\, i, j, \tag{1.6}$$

for some constant κ. Passing to a subsequence $\{\varepsilon_n\}$ we may assume that all the quantities $\lambda, \nu, N_b, N_s, \{n_i\}, \{d_i\}, \{d_{i,j}\}$ are independent of n. Passing to a further subsequence, still denoted by $\{\varepsilon_n\}$, we may assume that $y_i = y_i^{\varepsilon_n} \to l_i \in \bar{G}$, $i = 1, \dots, N_b$. Let $b_1, \dots, b_N$ be the distinct points among the $\{l_i\}_{i=1}^{N_b}$ and set

$$I_k = \left\{ i \in \{1, \dots, N_b\};\ y_i^{\varepsilon_n} \to b_k \right\}, \qquad k = 1, \dots, N.$$

Denoting $D_k = \sum_{i \in I_k} d_i$, $k = 1, \dots, N$, we clearly have $\sum_{k=1}^N D_k = d$.

Next, before continuing with the rigorous, and rather technical proof, let us give some explanation of the (simple) basic idea behind it. For any subdomain D of G we use the notation $E_{\varepsilon_n}(u|D) = \frac{1}{2}\int_D p|\nabla u|^2 + \frac{1}{4\varepsilon_n^2}\int_D (1 - |u|^2)^2$. As in [BBH2, S], the main step toward proving a converegence result like Theorem 1, is to get an upper bound for the energy "away from the singularities". More precisely, we claim that for all $\eta > 0$,

$$E_{\varepsilon_n}(u_{\varepsilon_n}|\Omega_\eta) \leqq C(\eta) \quad \forall n \geqq n(\eta), \tag{i}$$

where $\Omega_\eta \equiv G \setminus \bigcup_{i=1}^N B(b_i, \eta)$. Once (i) is established, the convergence of $\{u_{\varepsilon_n}\}$ in $C^{1,\alpha}_{\text{loc}}(\bar{G} \setminus \{b_1, \dots, b_N\})$ can be deduced as in [BBH2]. Assume for simplicity that all the points $b_1, \dots, b_N$ lie in the interior of G (of course, the case where some of the points lie on ∂G must also be treated by the rigorous proof). Next, for each n we define a "reference map" u_0^n on $G \setminus \bigcup_{i,j} B(x_{i,j}, \lambda\varepsilon_n)$ by

$$u_0^n(z) = \prod_{i,j} \left(\frac{z - x_{i,j}}{|z - x_{i,j}|} \right)^{d_{i,j}}.$$

Using a modification of a result of BREZIS, MERLE & RIVIÈRE [BMR], we establish the lower bound

$$\text{(ii)} \quad \begin{aligned} &\sum_{i=1}^{N} \frac{1}{2} \int_{B(b_i,\eta)\setminus\bigcup_j B(x_{i,j},\lambda\varepsilon_n)} p|\nabla u_{\varepsilon_n}|^2 \\ &\geqq \sum_{i=1}^{N} \frac{1}{2} \int_{B(b_i,\eta)\setminus\bigcup_j B(x_{i,j},\lambda\varepsilon_n)} p|\nabla u_0^n|^2 - C\,. \end{aligned}$$

In fact, we cannot get (ii) directly, since a priori we *do not* have the global estimate

$$\frac{1}{\varepsilon_n^2} \int_G (1 - |u_{\varepsilon_n}|^2)^2 \leqq C.$$

So in order to establish (ii) we have to proceed in two steps, using the BBD's. This point is treated carefully in the detailed proof, but we ignore it for the moment. Now we complete each u_0^n to a competing map $\tilde{u}_n$ for the infimum of E_{ε_n} over H_g^1. So we first set

$$\tilde{u}_n = u_0^n \text{ in } \bigcup_{i=1}^{N} \left(B(b_i, \eta) \setminus \bigcup_j B(x_{i,j}, \lambda\varepsilon_n) \right).$$

Then on each "hole" $B(x_{i,j}, \lambda\varepsilon_n)$, using polar coordinates around the point $x_{i,j}$, we define

$$\tilde{u}_n(x_{i,j} + re^{i\theta}) = \frac{r}{\lambda\varepsilon_n} u_0^n(x_{i,j} + \lambda\varepsilon_n e^{i\theta}).$$

It is easy to see that

$$\text{(iii)} \qquad E_{\varepsilon_n}(\tilde{u}_n | B(x_{i,j}, \lambda\varepsilon_n)) \leqq C \quad \text{for all } i \text{ and } j.$$

Finally, since the tangential energies of $\{u_0^n\}$ on each $\partial B(b_i, \eta)$ are uniformly bounded, it is clear that we can find for each n a map $w_n : \Omega_\eta \to S^1$ which equals g on ∂G, and u_0^n on each $\partial B(b_i, \eta)$, and which satisfies

$$\text{(iv)} \qquad \tfrac{1}{2} \int_{\Omega_\eta} p|\nabla w_n|^2 \leqq C(\eta).$$

So we set $\tilde{u}_n = w_n$ on Ω_η. Since u_{ε_n} is a minimizer, the inequality $E_{\varepsilon_n}(u_{\varepsilon_n}) \leqq E_{\varepsilon_n}(\tilde{u}_n)$ yields

$$E_{\varepsilon_n}\left(u_{\varepsilon_n} | \bigcup_{i=1}^{N} B(b_i, \eta)\right) + E_{\varepsilon_n}(u_{\varepsilon_n} | \Omega_\eta) \leqq E_{\varepsilon_n}\left(\tilde{u}_n | \bigcup_{i=1}^{N} B(b_i, \eta)\right) + E_{\varepsilon_n}(w_n | \Omega_\eta).$$

But then (i) follows immediately from (ii), (iii) and (iv). This concludes the sketch of the proof.

Going back to the rigorous proof, we need two lemmas which generalize a result of Brezis, Merle & Rivière [BMR, Th. 4]. The proof of Lemma 1.4 requires an obvious modification of the proof in [BMR] and is omitted. The proof of Lemma 1.5 is given in Appendix A.

Lemma 1.4. *On a domain* $\Omega = B(0, R) \setminus \bigcup_{j=1}^{m} \overline{B(a_j, R_0)}$ *with*

$$R_0 \leqq \tfrac{1}{4}R, \tag{1.7}$$

$$a_j \in B(0, \tfrac{1}{4}R) \quad \forall j, \tag{1.8}$$

$$|a_j - a_k| \geqq 4R_0 \quad \forall j \neq k, \tag{1.9}$$

a smooth positive function $q(x)$ *is given. Assume that* $u \in H^1(\Omega, \mathbb{C}) \cap C(\partial\Omega, \mathbb{C})$ *satisfies*

$$0 < a \leqq |u| \leqq 1 \quad \text{in } \Omega, \tag{1.10}$$

$$\frac{1}{R_0^2}\int_\Omega (1 - |u|^2)^2 \leqq K, \tag{1.11}$$

$$\deg(u, \partial B(a_j, R_0)) = d_j\ , \ \ j = 1, \dots, m. \tag{1.12}$$

Then,

$$\int_\Omega q|\nabla u|^2 \geqq \int_\Omega \left\{q|\nabla u_0|^2 + \tfrac{1}{2}q_0 a^2 |\nabla\psi|^2 + q|\nabla\rho|^2\right\} - C(a, K, m, q_0, ||q||_\infty, R||\nabla q||_\infty, \max_j |d_j|),$$

where $u_0(z) = \prod_{j=1}^{m}\left(\frac{z-a_j}{|z-a_j|}\right)^{d_j}$, $\rho = |u|$, $q_0 = \inf_\Omega q(z)$ *and* ψ *is defined by* $u = |u|u_0 e^{i\psi}$.

We denote by $\mathbb{R}^2_+$ the upper half space $\{\operatorname{Im} z > 0\}$.

Lemma 1.5. *Let the domain* $\Omega_+ = \mathbb{R}^2_+ \cap B(0, R) \setminus \bigcup_{j=1}^{m} \overline{B(a_j, R_0)}$ *be given with* R, R_0 *and* $\{a_j\}_{j=1}^m \subset \Omega_+$ *satisfying* (1.7)–(1.9) *and also*

$$\begin{aligned} &\operatorname{Im} a_j \geqq 2R_0, \ \ j = 1, \dots, L, \\ &\operatorname{Im} a_j = 0, \ \ j = L+1, \dots, m \ (0 \leqq L \leqq m). \end{aligned} \tag{1.13}$$

Let q *be a smooth positive function on* $\bar{\Omega}_+$ *and let* $\omega \in C^\infty((-R, R); S^1)$ *be given.* $(-R, R)$ *is identified with* $B(0, R) \cap \{\operatorname{Im} z = 0\}$. *Consider a map* $u \in H^1(\Omega_+; \mathbb{C}) \cap C(\partial\Omega_+, \mathbb{C})$ *which satisfies*

$$u = \omega \text{ on } (-R, R). \tag{1.14}$$

Assume also that (1.10)–(1.12) *are satisfied* (Ω *is replaced by* Ω_+ *in* (1.10), (1.11) *and* $\partial B(a_j, R_0)$ *by* $\partial(B(a_j, R_0) \cap \mathbb{R}^2_+)$ *for* $j \geqq L+1$ *in* (1.12)). *Then,*

$$\int_{\Omega_+} q|\nabla u|^2 \geqq \int_{\Omega_+} \left\{q|\nabla u_0|^2 + \tfrac{1}{2}q_0 a^2|\nabla\psi|^2 + q|\nabla\rho|^2\right\} - C,$$

with $C = C(a, K, m, \omega, q_0, ||q||_\infty, R||\nabla q||_\infty, \max_j |d_j|)$, $u_0(z) = \prod_{j=1}^m \left(\frac{z-a_j}{|z-a_j|}\right)^{d_j} \left(\frac{z-\bar{a}_j}{|z-\bar{a}_j|}\right)^{d_j}$, $\rho = |u|$, $q_0 = \inf_{\Omega_+} q$, ψ *is defined by* $u = |u|u_0 e^{i\psi}$, *and* $\bar{a}_j$ *is the complex conjugate of* a_j.

Relabeling if necessary, we may assume that $b_1, \dots, b_M \in G$ while $b_{M+1}, \dots, b_N \in \partial G$ (note that $0 \leqq M \leqq N$, so we may have all the b_i's in G or all of them on ∂G). Since G is a smooth simply-connected domain, we may choose a point $x_\infty \in G \setminus \{b_1, \dots, b_N\}$ and get by the Riemann mapping theorem a smooth (up to the boundary) conformal map $h : G \setminus \{x_\infty\} \to \mathbb{R}^2_+$ sending x_∞ to ∞. We then define a "reflection" r on G by $r(z) = h^{-1}(\overline{h(z)})$. We mention in passing that all our results can be generalized easily to multiply connected domains by using local conformal coordinates. We fix a positive η_0 satisfying

$$\eta_0 < \tfrac{1}{2} \min\big\{\min\{|b_i - b_j|;\ i \neq j\},\ \min\{\operatorname{dist}(b_i, \partial G);\ i = 1, \dots, M\}\big\},$$

and such that for all k the domain $G \cap B(b_k, \eta_0)$ is a connected domain with $\partial G \cap B(b_k, \eta_0)$ and $G \cap \partial B(b_k, \eta_0)$ both homeomorphic to a segment. Our main tool in proving Theorem 1 is

Proposition 1.1. *For every* $\eta \in (0, \eta_0)$

$$E_{\varepsilon_n}\Big(u_{\varepsilon_n}\big|G \setminus \bigcup_{k=1}^N \overline{B(b_k, \eta)}\Big) \leqq \pi |\log \eta| \left\{\sum_{k=1}^M p(b_k) D_k^2 + 2 \sum_{k=M+1}^N p(b_k) D_k^2\right\} + C(G, g, p) \quad \textit{for } n \geqq n_0(\eta).$$

Almost all the rest of Section 1 is devoted to the proof of Proposition 1.1. It is based on a comparison of the energy of u_{ε_n} around each b_k with that of a suitable "reference map". We start with b_k, $k = 1, \dots, M$. For every $i \in I_k$ we define on $\Omega_i^{(n)} = B(y_i, \nu\varepsilon_n^{1/2}) \setminus \bigcup_{j=1}^{n_i} \overline{B(x_{i,j}, \lambda\varepsilon_n)}$ the map $u_{0,i}(z) = u_{0,i}^{(n)}(z) = \prod_{j=1}^{n_i} \left(\frac{z-x_{i,j}}{|z-x_{i,j}|}\right)^{d_{i,j}}$. Applying Lemma 1.4 (note that Lemma 1.3 ensures that (1.11) is satisfied) we get

$$\int_{\Omega_i^{(n)}} p|\nabla u_{\varepsilon_n}|^2 \geqq \int_{\Omega_i^{(n)}} p|\nabla u_{0,i}|^2 - C(p, N, \kappa). \tag{1.15}$$

On $A_k^{(n)} = B(b_k, \eta) \setminus \bigcup_{i \in I_k} \overline{B(y_i, 2\nu\varepsilon_n^{1/2})}$ we define the map $v_{0,k}(z) = v_{0,k}^{(n)}(z) = \prod_{i \in I_k} \left(\frac{z-y_i}{|z-y_i|}\right)^{d_i}$. By Lemma 1.1 we have

$$\frac{1}{\varepsilon_n} \int_{A_k^{(n)}} (1 - |u_{\varepsilon_n}|^2)^2 \leqq \varepsilon_n E_{\varepsilon_n}(u_{\varepsilon_n}) \leqq C\varepsilon_n |\log \varepsilon_n| \leqq C,$$

hence we may apply Lemma 1.4 to infer that

$$\int_{A_k^{(n)}} p|\nabla u_{\varepsilon_n}|^2 \geqq \int_{A_k^{(n)}} p|\nabla v_{0,k}|^2 - C(p, N, \kappa). \tag{1.16}$$

Next we deal with the case $M+1 \leqq k \leqq N$, i.e. when $b_k \in \partial G$. For $i \in I_k$ we have either $y_i \in G$, in which case (1.15) continues to hold, or $y_i \in \partial G$. In the latter case, for n large, with $c_k = |h'(b_k)|$, we have

$$\begin{aligned}\tilde{\Omega}_i^{(n)} &= B_+\big(h(y_i), \tfrac{1}{2}\nu c_k \varepsilon_n^{1/2}\big) \setminus \textstyle\bigcup_{j=1}^{n_i} \overline{B(h(x_{i,j}), 2\lambda c_k \varepsilon_n)} \\ &\subset h(\Omega_i^{(n)}) \subset B_+(h(y_i), 2\nu c_k \varepsilon_n^{1/2}) \setminus \textstyle\bigcup_{j=1}^{n_i} \overline{B(h(x_{i,j}), \tfrac{1}{2}\lambda c_k \varepsilon_n)},\end{aligned} \tag{1.17}$$

where $\Omega_i^{(n)} = G \cap B(y_i, \nu\varepsilon_n^{1/2}) \setminus \bigcup_{j=1}^{n_i} \overline{B(x_{i,j}, \lambda\varepsilon_n)}$. Here and in the sequel we use the notation $B_+ \doteq B \cap \mathbb{R}_+^2$ for any disc B. Setting

$$\tilde{v}_{0,i}(z) = \tilde{v}_{0,i}^{(n)}(z) = \prod_{j=1}^{n_i} \left(\frac{z - h(x_{i,j})}{|z - h(x_{i,j})|} \right)^{d_{i,j}} \left(\frac{z - \overline{h(x_{i,j})}}{|z - \overline{h(x_{i,j})}|} \right)^{d_{i,j}} \quad \text{on } \tilde{\Omega}_i^{(n)},$$

we have by Lemma 1.5, applied to $\tilde{u}_{\varepsilon_n}(x) = u_{\varepsilon_n}(h^{-1}(x))$ with $q(x) = p(h^{-1}(x))$, that

$$\int_{\tilde{\Omega}_i^{(n)}} q|\nabla \tilde{u}_{\varepsilon_n}|^2 \geqq \int_{\tilde{\Omega}_i^{(n)}} q|\nabla \tilde{v}_{0,i}|^2 - C(p, N, G, g, \kappa). \tag{1.18}$$

Looking for a "cleaner form" for the reference map on the original domain, we define

$$\tilde{u}_{0,i}(z) = \prod_{j=1}^{n_i} \left(\frac{h^{-1}(z) - x_{i,j}}{|h^{-1}(z) - x_{i,j}|} \right)^{d_{i,j}} \left(\frac{h^{-1}(z) - r(x_{i,j})}{|h^{-1}(z) - r(x_{i,j})|} \right)^{d_{i,j}}.$$

Clearly $\tilde{u}_{0,i}(z) = \frac{F_i^{(n)}(z)}{|F_i^{(n)}(z)|}\, \tilde{v}_{0,i}(z)$ with

$$F_i^{(n)}(z) = \prod_{j=1}^{n_i} \left(\frac{h^{-1}(z) - x_{i,j}}{z - h(x_{i,j})} \right)^{d_{i,j}} \left(\frac{h^{-1}(z) - r(x_{i,j})}{z - \overline{h(x_{i,j})}} \right)^{d_{i,j}}.$$

Using the fact that h is a one-to-one smooth conformal map on $\bar{G}$ with a nonvanishing derivative we see easily that the possible singularities at $h(x_{i,j}), \overline{h(x_{i,j})}$, $j = 1, \ldots, n_i$, are removable. It follows that $F_i^{(n)}$ is a smooth conformal map on $\overline{\mathbb{R}_+^2}$ with no zeros. Using the Taylor expansion of h^{-1} we see that

$$||\nabla F_i^{(n)}||_{L^\infty(h(G \cap B(b_k, \eta_0)))} \leqq C(G, N, \kappa). \tag{1.19}$$

We claim that

$$\left| \int_{\tilde{\Omega}_i^{(n)}} q(|\nabla \tilde{v}_{0,i}|^2 - |\nabla \tilde{u}_{0,i}|^2) \right| \leqq C(N, \kappa, p, G, g). \tag{1.20}$$

The estimate (1.20) follows immediately from

Lemma 1.6. *Let* $\Omega_+, R, R_0, \{a_j\}_{j=1}^m, \{d_j\}_{j=1}^m, u_0$ *and* q *be as in Lemma 1.5. Let* $\psi \in C^1(\overline{\Omega_+})$ *be given, and set* $u = u_0 e^{i\psi}$*. Then,*

$$\left|\int_{\Omega_+} q(|\nabla u|^2 - |\nabla u_0|^2)\right| \leqq C(m, \max_j |d_j|, ||q||_\infty, R||\nabla q||_\infty, R||\nabla\psi||_\infty).$$

Proof. We may write $u_0 = e^{i\phi_0}$ locally in Ω_+; hence $u = e^{i(\phi_0+\psi)}$. We then have locally

$$q|\nabla u|^2 = q\big\{|\nabla u_0|^2 + 2(\nabla\phi_0, \nabla\psi) + |\nabla\psi|^2\big\}.$$

The proof of Lemma 1.5 (in particular the estimate for X_3) shows (see Appendix A) that $\left|\int_{\Omega_+} q(\nabla\phi_0, \nabla\psi)\right| \leqq C(m, \max_j |d_j|, ||q||_\infty, , R||\nabla q||_\infty, R||\nabla\psi||_\infty)$, which clearly implies the result. □

From the particular form of $\tilde{u}_{0,i}$ it follows immediately that

$$\int_{h(\Omega_i^{(n)})\setminus\tilde{\Omega}_i^{(n)}} q|\nabla\tilde{u}_{0,i}|^2 \leqq C(p, N, G, g, \kappa). \tag{1.21}$$

Returning to the original domain, we obtain from (1.18), (1.20) and (1.21) that for

$$u_{0,i}(z) = u_{0,i}^{(n)}(z) = \prod_{j=1}^{n_i} \left(\frac{z - x_{i,j}}{|z - x_{i,j}|}\right)^{d_{i,j}} \left(\frac{z - r(x_{i,j})}{|z - r(x_{i,j})|}\right)^{d_{i,j}}$$

we have the inequality

$$\int_{\Omega_i^{(n)}} p|\nabla u_{\varepsilon_n}|^2 \geqq \int_{\Omega_i^{(n)}} p|\nabla u_{0,i}|^2 - C(p, N, G, g, \kappa). \tag{1.22}$$

Next, there exists a constant $\bar{c} = \bar{c}(G)$ such that

$$B_+(h(b_k), \eta/\bar{c}) \subset h(G \cap B(b_k, \eta)) \subset B_+(h(b_k), \bar{c}\eta) \quad \text{for all } k\ .$$

Using a similar argument, we conclude that

$$\int_{A_k^{(n)}} p|\nabla u_{\varepsilon_n}|^2 \geqq \int_{A_k^{(n)}} p|\nabla v_{0,k}|^2 - C(p, N, G, g, \kappa), \tag{1.23}$$

with

$$A_k^{(n)} = G \cap B(b_k, \eta) \setminus \bigcup_{i\in I_k} \overline{B(y_i, 2\nu\varepsilon_n^{1/2})},$$

$$v_{0,k}(z) = \prod_{i\in I_k} \left(\frac{z - y_i}{|z - y_i|}\right)^{d_i} \left(\frac{z - r(y_i)}{|z - r(y_i)|}\right)^{d_i}.$$

Next we are looking for a lower bound for the energy of u_{ε_n} on $D_\eta = G \cap \big(\bigcup_{k=1}^N B(b_k, \eta)\big)$ using a map w_n that will be constructed in the sequel. We first define $w_n = u_{0,i}^{(n)}$ on $\Omega_i^{(n)}$ for all $i = 1, \ldots, N_b$. Then $w_n = v_{0,k}^{(n)}$ on $A_k^{(n)}$, $k = 1, \ldots, N$. We need to extend the definition of w_n to the domains $D_i^{(n)} = G \cap B(y_i, 2\nu\varepsilon_n^{1/2}) \setminus \overline{B(y_i, \nu\varepsilon_n^{1/2})}$, $i = 1, \ldots, N_b$. Assume first that $y_i \in G$; hence $D_i^{(n)}$ is an annulus. From our definition of w_n outside $D_i^{(n)}$ the following estimate for the tangential energy follows:

$$\int_{\partial D_i^{(n)}} \left|\frac{\partial w_n}{\partial \tau}\right|^2 d\tau \leqq \frac{C(N,\kappa)}{\varepsilon_n^{1/2}} \quad \forall i,\ \forall n.$$

Hence we may extend w_n as an S^1-valued map inside $D_i^{(n)}$ in such a way that

$$\int_{D_i^{(n)}} |\nabla w_n|^2 \leqq C(N,\kappa). \tag{1.24}$$

Actually, the same conclusion still holds in case $y_i \in \partial G$ since from our construction $|\partial w_n/\partial \tau| \leqq C(N,\kappa,G)$ on $\partial D_i^{(n)} \cap \partial G$.

Next we need to complete the definition of w_n in the "holes" $\{B(x_{i,j}, \lambda\varepsilon_n) \cap G\}$. Consider first the case $x_{i,j} \in G$. Using polar coordinates around $x_{i,j}$ we define

$$w_n(x_{i,j} + re^{i\theta}) = \left(\frac{r}{\lambda\varepsilon_n}\right) u_{0,i}(x_{i,j} + \lambda\varepsilon_n e^{i\theta}).$$

Using the estimate $\int_{\partial B(x_{i,j},\lambda\varepsilon_n)} \left|\frac{\partial u_{0,i}}{\partial \tau}\right|^2 d\tau \leqq C(N,\kappa)/\varepsilon_n$, we easily find that

$$E_{\varepsilon_n}(w_n | B(x_{i,j}, \lambda\varepsilon_n)) \leqq C(N,\kappa,p). \tag{1.25}$$

In the case $x_{i,j} \in \partial G$, we argue similarly. Using the fact that for ε_n small enough, $G \cap B(x_{i,j}, \lambda\varepsilon_n)$ is a star-shaped domain which is close to a half disc, we choose a point $x'_{i,j} \in G \cap B(x_{i,j}, \lambda\varepsilon_n)$ such that $G \cap B(x_{i,j}, \lambda\varepsilon_n)$ is star-shaped with respect to $x'_{i,j}$ and $\operatorname{dist}(x'_{i,j}, \partial(G \cap B(x_{i,j}, \lambda\varepsilon_n))) \geqq \frac{1}{4}\lambda\varepsilon_n$. Note that the rescaled domains $\left\{\frac{1}{\varepsilon_n}\big(G \cap B(x_{i,j}, \lambda\varepsilon_n)\big)\right\}_{n=1}^{\infty}$ have Lipschitz boundary with a uniform Lipschitz character. For any $z \in G \cap B(x_{i,j}, \lambda\varepsilon_n)$ (different from $x'_{i,j}$) there is a unique $\tilde{z} \in \partial(G \cap B(x_{i,j}, \lambda\varepsilon_n))$ such that z lies on the segment $[x'_{i,j}, \tilde{z}]$. We then define

$$w_n(z) = \frac{|z - x'_{i,j}|}{|\tilde{z} - x'_{i,j}|}\, u_{0,i}(\tilde{z}).$$

Again, a standard calculation leads to

$$E_{\varepsilon_n}(w_n | G \cap B(x_{i,j}, \lambda\varepsilon_n)) \leqq C(N,\kappa,p,G,g). \tag{1.26}$$

Summing up (1.15), (1.16), (1.22)–(1.26) we are led to

$$\frac{1}{2}\int_{D_\eta} p|\nabla u_{\varepsilon_n}|^2 \geqq E_{\varepsilon_n}(w_n | D_\eta) - C(N,\kappa,p,G,g). \tag{1.27}$$

Now in case $M < N$, i.e., when at least one b_k lies on the boundary, we further modify the map w_n to a new map w'_n which satisfies $w'_n = g$ on $\partial D_\eta \cap \partial G$. For this purpose we consider for $k = M+1, \ldots, N$ a smooth map $f_k^{(n)} : G \cap B(b_k, \eta_0) \to S^1$ satisfying $f_k^{(n)} = g/w_n$ on $\partial G \cap B(b_k, \eta_0)$. Since $||\nabla w_n||_{L^\infty(\partial G \cap B(b_k,\eta_0))} \leqq C$, we can clearly find such a map with $||\nabla f_k^{(n)}||_\infty \leqq C(N,\kappa,g,G)$. Next we define w'_n by

$$w'_n = \begin{cases} w_n & \text{on } B(b_k, \eta), \quad k = 1, \dots, M, \\ w_n f_k^{(n)} & \text{on } G \cap B(b_k, \eta), \quad k = M+1, \dots, N. \end{cases}$$

Note that the energies of w_n and w'_n on the "holes" $\{B(x_{i,j}, \lambda\varepsilon_n) \cap G\}$ are uniformly bounded. Combining it with an argument similar to that which led to (1.20) yields

$$\left| \int_{D_\eta} p\, (|\nabla w'_n|^2 - |\nabla w_n|^2) \right| \leqq C(N, \kappa, p, G, g). \tag{1.28}$$

If a modification of w_n has been made as above, we rename w'_n as w_n, and from (1.27) and (1.28) we conclude that in all cases

$$\tfrac{1}{2} \int_{D_\eta} p |\nabla u_{\varepsilon_n}|^2 \geqq E_{\varepsilon_n}(w_n | D_\eta) - C(N, \kappa, p, G, g). \tag{1.29}$$

For any $\eta > 0$ let $\Omega_\eta = G \setminus \bigcup_{k=1}^N \overline{B(b_k, \eta)}$. We continue with

Lemma 1.7. *For any $\eta \in (0, \eta_0)$ there exists a map $v = v_\eta^{(n)} \in C^1(\Omega_\eta; S^1)$ such that $v = w_n$ on $\partial(G \cap B(b_k, \eta))$ for $k = 1, \dots, N$ and such that for all $n \geqq n(\eta)$:*

$$\left| \frac{1}{2} \int_{\Omega_\eta} p |\nabla v|^2 - \pi \left\{ \sum_{k=1}^M p(b_k) D_k^2 + 2 \sum_{k=M+1}^N p(b_k) D_k^2 \right\} \log \frac{1}{\eta} \right| \leqq C(N, \kappa, p, G, g). \tag{1.30}$$

Proof. On $\Omega_{2\eta}$ we define

$$v(z) = e^{i\psi_0} \prod_{k=1}^M \left(\frac{z - b_k}{|z - b_k|} \right)^{D_K} \prod_{k=M+1}^N \left(\frac{h(z) - h(b_k)}{|h(z) - h(b_k)|} \right)^{2D_K},$$

where ψ_0 is a smooth function on G chosen so that $v = g$ on ∂G (we use the elementary fact that for $b_k \in \partial G$ the restriction of $\left(\frac{h(z)-h(b_k)}{|h(z)-h(b_k)|}\right)^{2D_k}$ to ∂G is a smooth S^1-valued map of degree D_k with a removable singularity at b_k). We easily find that

$$\left| \frac{1}{2} \int_{\Omega_{2\eta}} p |\nabla v|^2 - \pi \left\{ \sum_{k=1}^M p(b_k) D_k^2 + 2 \sum_{k=M+1}^N p(b_k) D_k^2 \right\} \log \frac{1}{\eta} \right| \leqq C(N, \kappa, p, G, g). \tag{1.31}$$

Finally, by an elementary direct construction we can extend the definition of v to each of the domains $A_{k,\eta} = G \cap B(b_k, 2\eta) \setminus \overline{B(b_k, \eta)}$ so that both $v = w_n$ on $\partial(G \cap B(b_k, \eta))$ and $\int_{A_{k,\eta}} |\nabla v|^2 \leqq C$ are satisfied. Here we use the fact that for n large we have by the definition of w_n that $|\nabla w_n| \leqq C/\eta$ on $\partial(G \cap B(b_k, \eta))$. □

Proof of Proposition 1.1. Fix any $\eta \in (0, \eta_0)$. We define a map $\tilde{u}_n \in H_g^1(G; \mathbb{C})$ by

$$\tilde{u}_n = \begin{cases} w_n & \text{on } D_\eta, \\ v_\eta^{(n)} & \text{on } \Omega_\eta. \end{cases}$$

Recall that $D_\eta = G \cap \bigcup_{k=1}^N B(b_k, \eta)$ and $\Omega_\eta = G \setminus \overline{D_\eta}$. Since u_{ε_n} is a minimizer for E_{ε_n}, we have

$$E_{\varepsilon_n}(u_{\varepsilon_n}|G) \leqq E_{\varepsilon_n}(\tilde{u}_n|G). \tag{1.32}$$

Hence

$$E_{\varepsilon_n}(u_{\varepsilon_n}|\Omega_\eta) + E_{\varepsilon_n}(u_{\varepsilon_n}|D_\eta) \leqq E_{\varepsilon_n}(v_\eta^{(n)}|\Omega_\eta) + E_{\varepsilon_n}(w_n|D_\eta). \tag{1.33}$$

Using (1.29) we conclude that

$$\frac{1}{4\varepsilon_n^2}\int_{D_\eta}(1-|u_{\varepsilon_n}|^2)^2 + E_{\varepsilon_n}(u_{\varepsilon_n}|\Omega_\eta) \leqq E_{\varepsilon_n}(v_\eta^{(n)}|\Omega_\eta) + C(N,\kappa,p,G,g). \tag{1.34}$$

Combining this with (1.30) we find

$$\begin{aligned} &\frac{1}{4\varepsilon_n^2}\int_{D_\eta}(1-|u_{\varepsilon_n}|^2)^2 + E_{\varepsilon_n}(u_{\varepsilon_n}|\Omega_\eta) \\ &\leqq C(N,\kappa,p,G,g) + \pi\left\{\sum_{k=1}^M p(b_k)D_k^2 + 2\sum_{k=M+1}^N p(b_k)D_k^2\right\}\log\frac{1}{\eta}, \end{aligned} \tag{1.35}$$

and the result follows. □

We mention an important corollary to our previous results:

Lemma 1.8. *For some constant C independent of n,*

$$\frac{1}{4\varepsilon_n^2}\int_G (1-|u_{\varepsilon_n}|^2)^2 \leqq C. \tag{1.36}$$

Proof. Fix any $\eta < \eta_0$ and apply (1.35). □

Proof of Theorem 1. Once Proposition 1.1 has been proved, we can apply the arguments of Theorem VI.1 in [BBH2] to obtain the convergence of $\{u_{\varepsilon_n}\}$ in $C^{1,\alpha}_{\text{loc}}(\overline{G} \setminus \{b_1,\ldots,b_N\})$ to a map $u_* \in C^\infty(\overline{G}\setminus\{b_1,\ldots,b_N\})$, satisfying

$$-\operatorname{div}(p\nabla u_*) = p|\nabla u_*|^2 u_* \quad \text{in } G\setminus\{b_1,\ldots,b_N\}.$$

Indeed, the proofs of [BBH1] for the case $d = 0$, which form the basis of the proof of [BBH2, Th. VI.1], can be easily generalized to the case p nonconstant. In order to complete the proof of Theorem 1 we need to prove two more properties:

$$D_k > 0 \quad \forall k, \tag{1.37}$$

$$b_k \in \Lambda \quad \forall k. \tag{1.38}$$

We shall use

Lemma 1.9. *For every k and $\eta < \eta_0$,*

$$\frac{1}{2}\int_{G\cap B(b_k,\eta)} p|\nabla u_{\varepsilon_n}|^2 \geqq \pi D_k \min\{p(x);\ \ x\in G\cap B(b_k,\eta)\}\log\left(\frac{\eta}{\varepsilon_n}\right) - C(N,\kappa,p,G,g).$$

for $n \geqq n(\eta)$.

Proof. In case $b_k \in G$, we can apply the result of [HS] (or the method of [BBH2, Th. V.2]) to u_{ε_n} on the perforated domain $B(b_k,\eta)\setminus\bigcup\overline{B(x_{i,j},\lambda\varepsilon_n)}$ to conclude the result. If $b_k \in \partial G$, then we can essentially reduce this case to the previous one by smoothly extending g and p to $G'\setminus G$ where $G'\supset G$ without reducing the minimum value of p. □

Proof of (1.37) **and** (1.38)**.** Fix any $\eta < \eta_0$. By Lemma 1.9 for n large enough we get

$$\frac{1}{2}\int_G p|\nabla u_{\varepsilon_n}|^2 \geqq \pi\log\frac{\eta}{\varepsilon_n}\sum_{k=1}^N |D_k|\min\{p(x);\ x\in G\cap B(b_k,\eta)\} - C.$$

Combining this with Lemma 1.1 and letting $n\to\infty$ we obtain

$$\sum_{k=1}^N |D_k|\min\{p(x);\ \ x\in G\cap B(b_k,\eta)\} \leqq p_0(d+\delta) = p_0\left(\sum_{k=1}^N D_k + \delta\right) \quad \forall\delta > 0.$$

Since δ is arbitrary, we conclude that necessarily $|D_k| = D_k$ for all k, i.e., $D_k \geqq 0$ and $\min\{p(x);\ \ G\cap B(b_k,\eta)\} = p_0$ for all k. Since η is arbitrary, we conclude that $p(b_k) = p_0$ for all k. Finally, the argument of [BBH2, p. 61] can be applied (now that we know that (1.36) holds) to infer that $D_k \neq 0$ for all k; hence both (1.37) and (1.38) are established. □

2. Proof of Theorem 2

The proof of Theorem 2 requires a simplification of the configuration of the bad discs associated with each minimizer. We begin by modifying further our BBD's and SBD's. Starting with the collection $\{B(y_i,\varepsilon_n^{\alpha_0})\}_{i=1}^{N_b}$ with $\alpha_0 = \frac{1}{2}$ we can obtain after a finite number of iterations (no more than N_b), each consisting of replacing α_k by $\alpha_{k+1} = \frac{1}{2}\alpha_k$ and deleting some discs, a new collection, which may be written after relabeling as $\{B(y_i,\varepsilon_n^{\alpha})\}_{i=1}^{N_b'}$, satisfying

$$|y_i - y_j| \geqq 2\varepsilon_n^{\alpha/2} \quad \forall\, i \neq j, \tag{2.1}$$

$$\text{for all } i \text{ either } \ y_i\in\partial G \ \text{ or } \ \operatorname{dist}(y_i,\partial G) \geqq 2\varepsilon_n^{\alpha/2}\,. \tag{2.2}$$

Here $N'_b \leqq N_b$ and $\alpha = 2^{-m}$ for some $m \leqq N_b + 1$. Of course, the regrouping of the SBD's is also affected by the above modification. By passing to a further subsequence we may assume that α, N'_b and the regrouping of the SBD's are all independent of n. In any case, we are now going to modify the SBD's too. Starting from the original collection $\{B(x_{i,j}, \lambda\varepsilon_n)\}$ we carry out a finite number of iterations (no more than N_s), each consisting of multiplying all radii by a constant, and deleting some discs, until a new collection, which after relabeling may be written as $\{B(x_{i,j}, \lambda'\varepsilon_n)\}$ for $i = 1, \ldots, N'_b$ $j = 1, \ldots, n_i$, is obtained which satisfies

$$\frac{|x_{i,j} - x_{i,j'}|}{\varepsilon_n} \to \infty, \quad \text{for all } i \text{ and all } j \neq j', \tag{2.3}$$

(2.4)

$$\text{for } i = 1, \ldots, N'_b,\ j = 1, \ldots, n_i \ \text{ either } x_{i,j} \in \partial G \text{ or } \frac{\operatorname{dist}(x_{i,j}, \partial G)}{\varepsilon_n} \to \infty,$$

$$x_{i,j} \in G \cap B\big(y_i, \tfrac{1}{4}\varepsilon_n^\alpha\big), \quad i = 1, \ldots, N'_b,\ j = 1, \ldots, n_i. \tag{2.5}$$

We may perform this construction so that $y_i = x_{i,1}$ if $y_i \in G$. As in Section 1, we set

$$d_i = \deg\big(u_{\varepsilon_n}, \partial(G \cap B(y_i, \varepsilon_n^\alpha))\big), \quad d_{i,j} = \deg\big(u_{\varepsilon_n}, \partial(G \cap B(x_{i,j}, \lambda'\varepsilon_n))\big).$$

By passing to a further subsequence we may assume that all the quantities α, N'_b, $\{n_i\}$,$\{d_i\}$, and $\{d_{i,j}\}$ are independent of n. Recall that the y_i's and the $x_{i,j}$'s do depend on n but we do not indicate this for the sake of simplicity. As in Section 1 we set $I_k = \{i\ ;\ y_i \to b_k\}$. In the sequel we shall denote by C different constants which do not depend on n. Next we have

Lemma 2.1. $d_i = 1$ *for all* i.

Proof. Fix any $\eta < \eta_0$. For each k, by the same proof as in Lemma 1.9 we have

$$\frac{1}{2}\int_{G\cap B(b_k,\eta)\setminus\bigcup_{i\in I_k} B(y_i,\varepsilon_n^{\alpha/2})} p|\nabla u_{\varepsilon_n}|^2 \geqq \pi p_0 D_k \log\frac{\eta}{\varepsilon_n^{\alpha/2}} - C. \tag{2.6}$$

The same argument yields

$$\frac{1}{2}\int_{G\cap B(y_i,\varepsilon_n^{\alpha/2})\setminus B(y_i,\varepsilon_n^\alpha)} p|\nabla u_{\varepsilon_n}|^2 \geqq \pi p_0 d_i^2 \log\frac{\varepsilon_n^{\alpha/2}}{\varepsilon_n^\alpha} - C \quad \forall\, i \in I_k. \tag{2.7}$$

Finally, the same argument applied to each $B(y_i, \varepsilon_n^\alpha) \setminus \bigcup_{j=1}^{n_i} \overline{B(x_{i,j}, \lambda'\varepsilon_n)}$ gives

$$\frac{1}{2}\int_{G\cap B(y_i,\varepsilon_n^\alpha)} p|\nabla u_{\varepsilon_n}|^2 \geqq \pi p_0 |d_i| \log\frac{\varepsilon_n^\alpha}{\lambda'\varepsilon_n} - C \quad \forall\, i. \tag{2.8}$$

Summing up (2.6)–(2.8) we conclude that

$$
\begin{aligned}
E_{\varepsilon_n}(u_{\varepsilon_n}|G) &\geqq \sum_{k=1}^{N} \frac{1}{2} \int_{G\cap B(b_k,\eta)} p|\nabla u_{\varepsilon_n}|^2 \\
&\geqq \pi p_0 d \log \frac{1}{\varepsilon_n} + \frac{\pi p_0 \alpha}{2} \sum_{i=1}^{N_b'} (d_i^2 - d_i) \log \frac{1}{\varepsilon_n} - C(\eta).
\end{aligned} \tag{2.9}
$$

Combining (2.9) with Lemma 1.1 we get that $\sum_{i=1}^{N_b'}(d_i^2 - d_i) = 0$; hence,

$$
d_i = 0 \text{ or } 1 \quad \forall i. \tag{2.10}
$$

We only need to exclude the possibility that $d_i = 0$. Looking for a contradiction, assume that $d_i = 0$ for some i. We first claim that for this i we have

$$
E_{\varepsilon_n}(u_{\varepsilon_n}|G \cap B(y_i, \varepsilon_n^{\alpha})) \leqq C. \tag{2.11}
$$

Indeed, by Lemma 1.1 and the Fubini Theorem we may find (as in [S]) for all n some $r_n \in (\varepsilon_n^{\alpha}, \varepsilon_n^{\alpha/2})$ such that

$$
\int_{G\cap\partial B(y_i,r_n)} \left\{ \frac{1}{2} p|\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2)^2 \right\} \leqq \frac{C}{r_n}. \tag{2.12}
$$

Using (2.12) we now construct a map v_n on $G \cap B(y_i, r_n)$ which equals u_{ε_n} on $\partial(G \cap B(y_i, r_n))$ such that $E_{\varepsilon_n}(v_n|G \cap B(y_i, r_n)) \leqq C$. Since u_{ε_n} is a minimizer, this would certainly imply (2.11). We use an argument which is due to F. H. LIN [Lin]. Look first at the case $y_i \in G$. Since by assumption, $d_i = 0$ and $|u_{\varepsilon_n}| \geqq \frac{1}{2}$ on $\partial B(y_i, r_n)$, we may write on $\partial B(y_i, r_n)$ that $u_{\varepsilon_n} = \rho_n e^{i\phi_n}$, with $\rho_n = |u_{\varepsilon_n}|$ and $\min_{\partial B(y_i,r_n)} \phi_n \in [0, 2\pi)$. Since by (2.12) $\int_{\partial B(y_i,r_n)} \left|\frac{\partial \phi_n}{\partial \tau}\right|^2 \leqq C/r_n$, it follows that $\max_{\partial B(y_i,r_n)} \phi_n \leqq C$. Using polar coordinates around y_i we first define $v_n = \tilde{\rho}_n e^{i\tilde{\phi}_n}$ on $B(y_i, r_n) \setminus B(y_i, r_n - \varepsilon_n)$ where

$$
\tilde{\rho}_n(r, \theta) = \left(\frac{r_n - r}{\varepsilon_n}\right) + \left(1 - \frac{r_n - r}{\varepsilon_n}\right) \rho_n(r_n, \theta), \qquad \tilde{\phi}_n(r, \theta) = \phi_n(r_n, \theta).
$$

Then, we extend the definition of v_n to $B(y_i, r_n - \varepsilon_n)$ by

$$
\tilde{\rho}_n(r, \theta) = 1, \quad \tilde{\phi}_n(r, \theta) = \frac{r}{r_n - \varepsilon_n} \phi_n(r_n, \theta).
$$

A direct calculation shows that v_n satisfies $E_{\varepsilon_n}(v_n|G \cap B(y_i, r_n)) \leqq C$; hence (2.11) holds in case $y_i \in G$.

In case $y_i \in \partial G$, we use a similar argument. This time the domain $G \cap B(y_i, r_n)$ is close to a half disc. The only difference with respect to the above construction is that for the extension along rays we use a point $y_i' \in G \cap B(y_i, r_n)$ satisfying $\text{dist}(y_i', \partial(G \cap B(y_i, r_n))) \in \left[\frac{1}{20} r_n \frac{1}{10} r_n\right]$ as a center instead of y_i. We leave the details to the reader. As a result of the previous analysis we have established (2.11) in all cases.

Note that by our construction, each $G \cap B(y_i, \varepsilon_n^\alpha)$ contains a point z_i with $|u_{\varepsilon_n}(z_i)| \leqq \frac{1}{2}$. A simple modification of the argument of [S] shows that there exists a constant $\gamma_0 > 0$ such that if

$$\int_{\partial B(x,r)\cap G} \left\{ \frac{1}{2} p |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2)^2 \right\} \leqq \frac{\gamma_0}{r} \tag{2.13}$$

for some $x \in G$ and $r \in \left[\varepsilon_n^\alpha, \varepsilon_n^{\alpha/2}\right]$, then $|u_{\varepsilon_n}(x)| \geqq \frac{2}{3}$. We claim that there exists some $r = r(n) \in (\varepsilon_n^\alpha, \varepsilon_n^{\alpha/2})$ for which (2.13) is satisfied for $x = z_i$. Indeed, if not, we would have

$$\begin{aligned} E_{\varepsilon_n}(u_{\varepsilon_n} | G \cap B(y_i, \varepsilon_n^\alpha)) &\geqq E_{\varepsilon_n}\left(u_{\varepsilon_n} | G \cap B\left(z_i, \tfrac{1}{2}\varepsilon_n^\alpha\right)\right) \\ &\geqq \gamma_0 \int_{\varepsilon_n^\alpha}^{\varepsilon_n^{\alpha/2}} \frac{ds}{s} \to \infty \text{ as } \varepsilon_n \to 0, \end{aligned}$$

a contradiction to (2.11). So we must have $|u_{\varepsilon_n}(z_i)| \geqq \frac{2}{3}$, a contradiction. Hence $d_i = 1$ is established in all cases. □

Lemma 2.2. *$y_i \in G$ for all $i = 1, \ldots, N_b'$.*

Proof. Looking for a contradiction, assume that $y_i \in \partial G$ for some i. As in the proof of (2.12) we see that there exists some $r_n \in (\varepsilon_n^\alpha, \varepsilon_n^{\alpha/2})$ with

$$\int_{G\cap\partial B(y_i,r_n)} \left\{ \frac{1}{2} p |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2)^2 \right\} \leqq \frac{C}{r_n}. \tag{2.14}$$

We choose $z_i \in G \cap B(y_i, r_n)$ satisfying $\operatorname{dist}(z_i, \partial(G \cap B(y_i, r_n))) \in \left[\frac{1}{20} r_n, \frac{1}{10} r_n\right]$. For n large enough, $G \cap B(y_i, r_n)$ is star-shaped with respect to z_i. Using polar coordinates around z_i we may write $u_{\varepsilon_n} = \rho_n e^{i(\theta+\psi_n)}$ on $\partial(G \cap B(y_i, r_n))$ with $\rho_n = |u_{\varepsilon_n}|$ and ψ_n a smooth function, which by (2.14) has an extension $\tilde{\psi}_n$ in $G \cap B(y_i, r_n)$ satisfying

$$\int_{G\cap B(y_i,r_n)} |\nabla \tilde{\psi}_n|^2 \leqq C \quad \text{uniformly in } n.$$

Using this and (2.14) we readily construct a map v_n on $G \cap B(y_i, r_n)$ satisfying $v_n = u_{\varepsilon_n}$ on $\partial(G \cap B(y_i, r_n))$ and

$$E_{\varepsilon_n}(v_n | G \cap B(y_i, r_n)) \leqq \pi p_0 \log(r_n/\varepsilon_n) + C. \tag{2.15}$$

Indeed, first we define $v_n = e^{i(\theta+\tilde{\psi}_n)}$ on

$$A_n = \{x \in G \cap B(y_i, r_n);\ |x - z_i| \geqq \varepsilon_n, \quad \operatorname{dist}(x, \partial(G \cap B(y_i, r_n))) \geqq \varepsilon_n\}.$$

On $B_n = \{x \in G \cap B(y_i, r_n);\ \operatorname{dist}(x, \partial(G \cap B(y_i, r_n))) \leqq \varepsilon_n\}$ we can easily extend v_n to a map which coincides with u_{ε_n} and v_n, respectively on the two components of the boundary of B_n, and which satisfies $E_{\varepsilon_n}(v_n | B_n) \leqq C$. So far we have

$$E_{\varepsilon_n}(v_n|G \cap B(y_i, r_n) \setminus B(z_i, \varepsilon_n)) \leqq p_0\pi \log \frac{r_n}{\varepsilon_n} + C. \tag{2.16}$$

Finally on $B(z_i, \varepsilon_n)$ we define

$$v_n(z) = \left|\frac{z - z_i}{\varepsilon_n}\right| v_n\left(z_i + \varepsilon_n \frac{z - z_i}{|z - z_i|}\right). \tag{2.17}$$

By a direct computation, $E_{\varepsilon_n}(v_n|B(z_i, \varepsilon_n)) \leqq C$. We combine this inequality with (2.16) to prove (2.15).

On the other hand, we now show that

$$E_{\varepsilon_n}(u_{\varepsilon_n}|G \cap B(y_i, r_n)) \geqq 2p_0\pi \log \frac{r_n}{\varepsilon_n} - C. \tag{2.18}$$

From the results of Section 1 it follows that

$$\int_{G\cap B(y_i,r_n)} \tfrac{1}{2} p|\nabla u_{\varepsilon_n}|^2 \geqq \tfrac{1}{2} p_0 \int_{\tilde{\Omega}_+} |\nabla \tilde{v}_{0,i}|^2 - C, \tag{2.19}$$

with

$$\tilde{v}_{0,i}(z) = \prod_{j=1}^{n_i} \left(\frac{z - h(x_{i,j})}{|z - h(x_{i,j})|}\right)^{d_{i,j}} \left(\frac{z - \overline{h(x_{i,j})}}{|z - \overline{h(x_{i,j})}|}\right)^{d_{i,j}},$$

$$\tilde{\Omega}_+ = h\Big(G \cap B(y_i, r_n) \setminus \bigcup_{j=1}^{n_i} \overline{B(x_{i,j}, \lambda'\varepsilon_n)}\Big) \subset \mathbb{R}^2_+.$$

Let us set $\tilde{\Omega} = \tilde{\Omega}_+ \cup \tilde{\Omega}_- \cup (\partial\tilde{\Omega}_+ \cap \{\mathrm{Im}\ z = 0\})$ where $\tilde{\Omega}_-$ is the reflection of $\tilde{\Omega}_+$ in the real axis. It is clear by symmetry considerations that

$$\int_{\tilde{\Omega}_+} |\nabla \tilde{v}_{0,i}|^2 = \tfrac{1}{2} \int_{\tilde{\Omega}} |\nabla \tilde{v}_{0,i}|^2. \tag{2.20}$$

Now $\tilde{\Omega}$ is a perforated domain with exterior boundary close to a circle, and with holes close to discs. Moreover the degree of $\tilde{v}_{0,i}$ on the exterior boundary equals $2d_i = 2$ (by Lemma 2.1). We may enlarge the holes a little bit, so that they become discs, and apply the lower bound of [BBH2, Ch. II] or [HS] to infer that

$$\frac{1}{2} \int_{\tilde{\Omega}} |\nabla \tilde{v}_{0,i}|^2 \geqq 4\pi \log\left(\frac{r_n}{\lambda'\varepsilon_n}\right) - C. \tag{2.21}$$

Now (2.18) clearly follows from (2.19)–(2.21). Combining (2.15) with (2.18) we obtain (since u_{ε_n} is a minimizer) that

$$2\pi \log\left(\frac{r_n}{\varepsilon_n}\right) \leqq \pi \log\left(\frac{r_n}{\varepsilon_n}\right) + C.$$

This leads to a contradiction for n large enough. □

Lemma 2.3. *$n_i = 1$ for all $i = 1, \ldots, N'_b$.*

Proof. Fix any $i \in \{1, \dots, N_b'\}$ and $\delta > 0$. By Lemma 2.2 we already know that $y_i \in G$. By the same argument as in the proof of (2.12) it follows that there exists some $r_n = r_n(\delta) \in (\varepsilon_n^\alpha, \varepsilon_n^{\alpha/2})$ with

$$\int_{\partial B(y_i,r_n)} \left\{ \frac{1}{2} p |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2)^2 \right\} \leqq \frac{p_0(\pi + \delta)}{r_n}. \tag{2.22}$$

Next we use the Pohožaev identity for u_{ε_n}, namely, we multiply the Euler-Lagrange equation

$$-\operatorname{div}(p \nabla u_{\varepsilon_n}) = \frac{1}{\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2) u_{\varepsilon_n}$$

by $(x - y_i) \cdot \nabla u_{\varepsilon_n}$ and integrate over $B(y_i, r_n)$. A direct calculation gives

$$\begin{aligned} &\int_{B(y_i,r_n)} \left\{ \frac{1}{2}(x - y_i, \nabla p)|\nabla u_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2)^2 \right\} \\ &= r_n \int_{\partial B(y_i,r_n)} \left\{ \frac{p}{2}|(u_{\varepsilon_n})_\tau|^2 - \frac{p}{2}|(u_{\varepsilon_n})_r|^2 + \frac{1}{4\varepsilon_n^2}(1 - |u_{\varepsilon_n}|^2)^2 \right\}. \end{aligned} \tag{2.23}$$

Note that

$$\begin{aligned} \left| \int_{B(y_i,r_n)} (x - y_i, \nabla p)|\nabla u_{\varepsilon_n}|^2 \right| &\leqq \varepsilon_n^{\alpha/2} ||\nabla p||_\infty \int_G |\nabla u_{\varepsilon_n}|^2 \\ &\leqq C \varepsilon_n^{\alpha/2} |\log \varepsilon_n| \to 0. \end{aligned} \tag{2.24}$$

Hence from (2.22)–(2.24) we conclude that

$$\limsup_{n\to\infty} \frac{1}{2\varepsilon_n^2} \int_{B(y_i,r_n)} (1 - |u_{\varepsilon_n}|^2)^2 \leqq p_0(\pi + \delta). \tag{2.25}$$

On the other hand, around each $x_{i,j}$ we may perform a "blow-up", that is to define $\tilde{u}_n(x) = u_{\varepsilon_n}(x_{i,j} + \varepsilon_n x)$ on $B(0, R)$ for each fixed $R > 0$ (it is well defined for n large enough). Note that $\tilde{u}_n$ satisfies the equation $-\operatorname{div}(\tilde{p}_n \nabla \tilde{u}_n) = (1 - |\tilde{u}_n|^2)\tilde{u}_n$ with $\tilde{p}_n(x) = p(x_{i,j} + \varepsilon_n x)$. Using standard elliptic estimates we deduce the convergence $\tilde{u}_n \to \tilde{u}$ in $C^m_{\text{loc}}(B(0, R))$ for every $m \geqq 0$. Taking a sequence $R_l \nearrow \infty$ and passing to a diagonal subsequence we may assume that $\tilde{u}_n \to \tilde{u}$ in $C^m_{\text{loc}}(\mathbb{R}^2)$ for every $m \geqq 0$. It is easy to see that $\tilde{u}$ is a solution of $-\Delta\tilde{u} = \frac{1}{p_0}(1-|\tilde{u}|^2)\tilde{u}$ on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2}(1-|\tilde{u}|^2)^2 < \infty$ (by (1.36)). Moreover, since by our construction $B(x_{i,j}, \lambda'\varepsilon_n)$ contains a point x with $|u_{\varepsilon_n}(x)| \leqq \frac{1}{2}$, it follows that $\tilde{u}$ is not identically constant. Such solutions were studied in [BMR] (see also [Sh]) and it is shown there, in particular, that

$$\int_{\mathbb{R}^2} (1 - |\tilde{u}|^2)^2 \geqq 2\pi p_0. \tag{2.26}$$

From (2.26) and (2.3), (2.5) it follows that

$$\liminf_{n\to\infty} \frac{1}{2\varepsilon_n^2} \int_{B(y_i,r_n)} (1 - |u_{\varepsilon_n}|^2)^2 \geqq \pi p_0 n_i. \tag{2.27}$$

Choosing $\delta < \pi$ and combining (2.27) with (2.25) we see that necessarily $n_i = 1$. □

From our previous analysis it follows that the configuration of the modified bad discs is quite simple. Since each BBD $B(y_i, \varepsilon_n^\alpha)$ contains exactly one SBD $B(x_{i,1}, \lambda'\varepsilon_n)$, and our construction ensures that $x_{i,1} = y_i$, we do not need the BBD's anymore. We now have most of the ingredients needed for the proof of Theorem 2.

Proof of Theorem 2. That there are exactly d bad discs each of degree 1 readily follows from Lemmas 2.1–2.3. In order to see that for every $\beta \in (0,1)$ $\operatorname{dist}(y_i, \partial G) \geqq \varepsilon_n^\beta$ for all i, and $|y_i - y_j| \geqq \varepsilon_n^\beta$ for all $i \neq j$, we may continue the modification procedure of the BBD's from the beginning of this section. More precisely, for any given $\beta \in (0,1)$ we arrive after a finite number of iterations of the form $\alpha_{k+1} = \frac{1}{2}\alpha_k$ at an $\alpha < \beta$ of the form $\alpha = (\frac{1}{2})^m$ for which both (2.1) and (2.2) hold. Then, for the resulting configuration of the bad discs we may apply the same arguments as above.

Finally, it is left to show that each bad disc $B(y_i, \lambda'\varepsilon_n)$ contains exactly one zero of u_{ε_n} (for n large enough). By Lemma 2.1 it follows that $B(y_i, \lambda'\varepsilon_n)$ contains at least one zero; let us denote it by $z^{(n)}$. Next we apply a blow-up argument as in the proof of Lemma 2.3. Namely, for each fixed $R > 0$ we define $\tilde{u}_n(x) = u_{\varepsilon_n}(z^{(n)} + \varepsilon_n x)$ on $B(0, R)$. Using standard elliptic estimates we deduce the convergence (of a diagonal subsequence) $\tilde{u}_n \to \tilde{u}$ in $C^m_{\text{loc}}(\mathbb{R}^2)$ for every $m \geqq 0$. $\tilde{u}$ is clearly a solution of $-\Delta\tilde{u} = \frac{1}{p_0}(1 - |\tilde{u}|^2)\tilde{u}$ on $\mathbb{R}^2$ satisfying $\tilde{u}(0) = 0$ and $\int_{\mathbb{R}^2}(1 - |\tilde{u}|^2)^2 < \infty$. Moreover, it is easy to see that $\tilde{u}$ is actually a *local minimizer*, that is, for every $R > 0$ $\tilde{u}$ is a minimizer, for its boundary data on $\partial B(0, R)$ of the energy

$$\int_{B(0,R)} \left\{\tfrac{1}{2}p_0|\nabla u|^2 + \tfrac{1}{4}(1 - |u|^2)^2\right\}.$$

Looking for a contradiction, assume that a subsequence, still denoted by $\{u_{\varepsilon_n}\}$, satisfies $u_{\varepsilon_n}(z^{(n)}) = u_{\varepsilon_n}(z_1^{(n)}) = 0$ for some $z_1^{(n)} \neq z^{(n)}$ in $B(y_i, \lambda'\varepsilon_n)$. For the rescaled sequence $\{\tilde{u}_n\}$ we get $\tilde{u}_n(\tilde{z}^{(n)}) = \tilde{u}_n(\tilde{z}_1^{(n)}) = 0$ for some $\tilde{z}_1^{(n)} \neq \tilde{z}^{(n)}$. Since the origin is the only zero of $\tilde{u}$ and $\tilde{u}_n \to \tilde{u}$ in $C^m_{\text{loc}}(\mathbb{R}^2)$, it follows that $\tilde{z}^{(n)}, \tilde{z}_1^{(n)} \to 0$. Passing to a further subsequence we may assume that

$$\frac{\tilde{z}^{(n)} - \tilde{z}_1^{(n)}}{|\tilde{z}^{(n)} - \tilde{z}_1^{(n)}|} \to v \quad \text{for some } v \text{ of norm 1 in } \mathbb{R}^2.$$

Passing to the limit as $n \to \infty$ we obtain that $\nabla\tilde{u}(0) \cdot v = 0$. In particular, it follows that $\det \nabla\tilde{u}(0) = 0$. This contradicts Corollary 2.4 of [BCP], which states that $|\det \nabla\tilde{u}(0)| > 0$. (The same argument as in [Sh, Th. 2] shows that for R large enough $\tilde{u} = \rho(\theta)e^{i\phi(\theta)}$ on $\partial B(0, R)$ with $\rho \geqq \frac{1}{2}$ and $\frac{d\phi}{d\theta} \geqq \frac{1}{2}$; hence, the result of [BCP] is applicable.) □

We close this section with two more estimates which will be useful in the second part of our study [ASh2]. Recall that $y_i = y_i^{\varepsilon_n} \to b_k$ for every $i \in I_k$, and that we are assuming that $b_1, \dots, b_M \in G$ and $b_{M+1}, \dots, N \in \partial G$. Next, on $\Omega_n \doteq G \setminus \bigcup_{i=1}^{d} \overline{B(y_i, \lambda' \varepsilon_n)}$ we may define

$$u_0(z) = u_0^{(n)}(z) = \prod_{k=1}^{N} \prod_{i \in I_k} \frac{z - y_i}{|z - y_i|} \prod_{k=M+1}^{N} \prod_{i \in I_k} \frac{z - r(y_i)}{|z - r(y_i)|}.$$

Then we may write

$$u_{\varepsilon_n}(z) = \rho_n(z) u_0(z) e^{i\psi_n(z)} \quad \text{on } \Omega_n, \tag{2.28}$$

with $\rho_n(z) = |u_{\varepsilon_n}(z)|$ and $\psi_n(z)$ a smooth function on Ω_n. Next we prove

Proposition 2.1. *The estimate*

$$\int_{\Omega_n} |\nabla \psi_n|^2 + |\nabla \rho_n|^2 \leqq C \tag{2.29}$$

holds uniformly in n.

Proof. From the results of Section 1 (using the full statements of Lemmas 1.4 and 1.5) it follows that

$$\int_{\Omega_n} \tfrac{1}{2} p |\nabla u_{\varepsilon_n}|^2 \geqq \int_{\Omega_n} \tfrac{1}{2} p (|\nabla u_0|^2 + |\nabla \rho_n|^2) + \frac{p_0}{16} \int_{\Omega_n} |\nabla \psi_n|^2 - C. \tag{2.30}$$

Next we can define $w_n = u_0 e^{i\phi_n}$ on Ω_n, with ϕ_n uniquely determined (modulo 2π) by the requirement that $w_n = g$ on ∂G. The arguments of Section 1 show that

$$\left| \int_{\Omega_n} p |\nabla w_n|^2 - \int_{\Omega_n} p |\nabla u_0|^2 \right| \leqq C. \tag{2.31}$$

Finally, we can extend the definition of w_n to each of the "holes" $\{B(y_i, \lambda' \varepsilon_n)\}_{i=1}^{d}$ (as in the proof of (1.25)) so that

$$E_{\varepsilon_n}(w_n | B(y_i, \lambda' \varepsilon_n)) \leqq C \quad \forall i. \tag{2.32}$$

Using (2.31), (2.32) and the minimality of u_{ε_n} we find that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \leqq \int_{\Omega_n} \tfrac{1}{2} p |\nabla u_0|^2 + C\ . \tag{2.33}$$

The conclusion of the lemma follows by combining (2.33) with (2.30). □

Combining (2.30), (2.33) and the estimate $E_{\varepsilon_n}(u_{\varepsilon_n} | B(y_i, \lambda' \varepsilon_n)) \leqq C$ (which follows from $||\nabla u_{\varepsilon_n}||_\infty \leqq \frac{C}{\varepsilon_n}$) we immediately get an important corollary:

Proposition 2.2.

$$\left| E_{\varepsilon_n}(u_{\varepsilon_n}) - \int_{\Omega_n} \tfrac{1}{2} p |\nabla u_0|^2 \right| \leqq C \quad \textit{uniformly in } n. \tag{2.34}$$

From Proposition 2.2 it follows that in order to approximate the energy of u_{ε_n}, up to an $O(1)$ error, it is enough to calculate the energy of the reference map $u_0 = u_0^{(n)}$. This, in turn, requires the knowledge of the mutual distances between the y_i's and their distances from the boundary. These distances depend in a crucial manner on the particular weight function p. In [ASh2] we study this problem under some concrete assumptions on p.

Appendix A. Proof of Lemma 1.5

Recall that $u = \rho u_0 e^{i\psi}$ with ψ globally defined on Ω_+, and $\rho = |u|$. Moreover, since $u_0 \equiv 1$ on $(-R, R)$, we have $e^{i\psi} = \omega$ on $(-R, R)$. In the sequel we denote by C different universal constants. For $j = 1, \ldots, L$ we use Lemma 3 of [BMR] to extend ψ in each $B_+(a_j, R_0)$ ($= B(a_j, R_0)$ in this case) to a function $\bar{\psi}$ such that

$$\int_{B_+(a_j,R_0)} |\nabla\bar{\psi}|^2 \leqq C \int_{B(a_j,2R_0)\setminus B(a_j,R_0)} |\nabla\psi|^2 \leqq C \int_{\Omega_+} |\nabla\psi|^2 = C||\nabla\psi||_2^2,$$

with C a universal constant. For $j = L+1, \ldots, m$ we may find by a standard construction an extension $\bar{\psi}$ satisfying $e^{i\bar{\psi}} = \omega$ on $(a_j - R_0, a_j + R_0)$ and

$$\int_{B_+(a_j,R_0)} |\nabla\bar{\psi}|^2 \leqq C\left(\int_{B_+(a_j,2R_0)\setminus B(a_j,R_0)} |\nabla\psi|^2 + R_0^2||\nabla\omega||_\infty^2\right).$$

Hence,

$$\int_{B_+(0,R)} |\nabla\bar{\psi}|^2 \leqq C\big(||\nabla\psi||_2 + \sqrt{m}R_0||\nabla\omega||_\infty\big)^2. \tag{A.1}$$

We may write $u_0 = e^{i\phi_0}$ *locally* in Ω_+, with $|\nabla u_0| = |\nabla\phi_0|$ and

$$\nabla\phi_0(z) = \sum_j d_j\left(\frac{V_j(z)}{|z-a_j|} + \frac{\bar{V}_j(z)}{|z-\bar{a}_j|}\right)$$

where

$$V_j(z) = \left(-\frac{\operatorname{Im}(z-a_j)}{|z-a_j|}, \frac{\operatorname{Re}(z-a_j)}{|z-a_j|}\right), \quad \bar{V}_j(z) = \left(-\frac{\operatorname{Im}(z-\bar{a}_j)}{|z-\bar{a}_j|}, \frac{\operatorname{Re}(z-\bar{a}_j)}{|z-\bar{a}_j|}\right).$$

Next we have $|\nabla u|^2 = |\nabla\rho|^2 + \rho^2(|\nabla\phi_0|^2 + 2\nabla\phi_0\cdot\nabla\psi + |\nabla\psi|^2)$; hence

$$\int_{\Omega_+} q|\nabla u|^2 \geqq \int_{\Omega_+} q(|\nabla\rho|^2 + a^2|\nabla\psi|^2) + \int_{\Omega_+} q|\nabla u_0|^2 - X, \tag{A.2}$$

with

$$\begin{aligned} X &= \int_{\Omega_+} q(1-\rho^2)|\nabla u_0|^2 + \int_{\Omega_+} 2q(1-\rho^2)\nabla\phi_0\nabla\psi - \int_{\Omega_+} 2q\nabla\phi_0\nabla\psi \\ &= X_1 + X_2 + X_3. \end{aligned}$$

We estimate X_1 and X_2 exactly as in [BMR]. Setting $D \doteq \max_j |d_j|$ we get

$$|X_1| \leqq ||q||_\infty (\pi K)^{1/2} D^2 (2m)^2. \tag{A.3}$$

For X_2 we find

$$|X_2| \leqq 4||q||_\infty K^{1/2} m D ||\nabla\psi||_2. \tag{A.4}$$

since $|\nabla\phi_0| \leqq 2mD/R_0$. Only the estimate of X_3 requires some modifications of the argument in [BMR]. We have

$$\int_{\Omega_+} q\nabla\phi_0 \cdot \nabla\psi = \sum_j d_j \int_{\Omega_+} q\left(\frac{V_j \cdot \nabla\psi}{|z-a_j|} + \frac{\bar{V}_j \cdot \nabla\psi}{|z-\bar{a}_j|}\right). \tag{A.5}$$

Now,

$$\int_{\Omega_+} q\frac{V_j \cdot \nabla\psi}{|z-a_j|} = \int_{B_+(0,R)\setminus B(a_j,R_0)} q\frac{V_j \cdot \nabla\bar{\psi}}{|z-a_j|} - \sum_{k\neq j}\int_{B_+(a_k,R_0)} q\frac{V_j \cdot \nabla\bar{\psi}}{|z-a_j|}.$$

By the Cauchy-Schwarz inequality and (A.1),

$$\begin{aligned}\left|\sum_{k\neq j}\int_{B_+(a_k,R_0)} q\frac{V_j \cdot \nabla\bar{\psi}}{|z-a_j|}\right| &\leqq \frac{||q||_\infty}{R_0}\sum_{k\neq j}\int_{B_+(a_k,R_0)} |\nabla\bar{\psi}| \\ &\leqq C||q||_\infty\sqrt{m}(||\nabla\psi||_2 + \sqrt{m}R_0||\nabla\omega||_\infty).\end{aligned} \tag{A.6}$$

Denoting $\rho_j = R - |a_j|$ we write

$$\begin{aligned}&\int_{B_+(0,R)\setminus B(a_j,R_0)} q\frac{V_j \cdot \nabla\bar{\psi}}{|z-a_j|} \\ &\quad = \int_{B_+(0,R)\setminus B(a_j,\rho_j)} q\frac{V_j \cdot \nabla\psi}{|z-a_j|} + \int_{B_+(a_j,\rho_j)\setminus B(a_j,R_0)} q\frac{V_j \cdot \nabla\bar{\psi}}{|z-a_j|}.\end{aligned}$$

Note first that

$$\left|\int_{B_+(0,R)\setminus B(a_j,\rho_j)} q\frac{V_j \cdot \nabla\psi}{|z-a_j|}\right| \leqq \frac{||q||_\infty}{\rho_j}\int_{\Omega_+} |\nabla\psi| \leqq C||q||_\infty ||\nabla\psi||_2.$$

If $1 \leqq j \leqq L$, we find for every $r \in (R_0, \rho_j)$ (using $\int_{S_r(a_j)} V_j \cdot \nabla\bar{\psi} = \int_{S_r(a_j)} \frac{\partial\bar{\psi}}{\partial\tau} = 0$) that

$$\left|\int_{S_r(a_j)} q\frac{V_j \cdot \nabla\bar{\psi}}{|z-a_j|}\right| = \left|\frac{1}{r}\int_{S_r(a_j)} (q - q(a_j))V_j \cdot \nabla\bar{\psi}\right| \leqq ||\nabla q||_\infty \int_{S_r(a_j)} |\nabla\bar{\psi}|.$$

If $L+1 \leqq j \leqq m$, we have (denoting $S_r^+(a_j) = S_r(a_j) \cap \mathbb{R}^2_+$)

$$\int_{S_r^+(a_j)} q\frac{V_j\cdot\nabla\bar\psi}{|z-a_j|} = \frac{q(a_j)}{r}\int_{S_r^+(a_j)} V_j\cdot\nabla\bar\psi + \frac{1}{r}\int_{S_r^+(a_j)}(q-q(a_j))V_j\cdot\nabla\bar\psi$$
$$= \frac{q(a_j)}{r}(\bar\psi(a_j-r)-\bar\psi(a_j+r))$$
$$+\frac{1}{r}\int_{S_r^+(a_j)}(q-q(a_j))V_j\cdot\nabla\bar\psi.$$

Hence

$$\left|\int_{S_r^+(a_j)} q\frac{V_j\cdot\nabla\bar\psi}{|z-a_j|}\right| \leqq 2||q||_\infty||\nabla\omega||_\infty + ||\nabla q||_\infty\int_{S_r^+(a_j)}|\nabla\bar\psi|.$$

It follows that for every $j=1,\dots,m$,

$$\left|\int_{B_+(0,R)\setminus B(a_j,R_0)} q\frac{V_j\cdot\nabla\bar\psi}{|z-a_j|}\right| \leqq C\|q\|_\infty\|\nabla\psi\|_2 + 2R\|q\|_\infty\|\nabla\omega\|_\infty$$
$$+CR||\nabla q||_\infty(||\nabla\psi||_2+\sqrt{m}R_0||\nabla\omega||_\infty). \tag{A.7}$$

Combining (A.6) and (A.7) we are led to

$$\left|\int_{\Omega_+} q\frac{V_j\cdot\nabla\psi}{|z-a_j|}\right| \leqq C\sqrt{m}||q||_\infty||\nabla\psi||_2 + CR||\nabla q||_\infty||\nabla\psi||_2$$
$$+2R||q||_\infty||\nabla\omega||_\infty + C\sqrt{m}R_0||\nabla\omega||_\infty(\sqrt{m}||q||_\infty$$
$$+R||\nabla q||_\infty). \tag{A.8}$$

For $j=L+1,\dots,m$ the estimate (A.7) clearly continues to hold if we replace V_j by $\bar V_j$ (since $\bar a_j = a_j$), i.e.,

(A.9)
$$\left|\int_{B_+(0,R)\setminus B(a_j,R_0)} q\frac{\bar V_j\cdot\nabla\bar\psi}{|z-\bar a_j|}\right| \leqq C||q||_\infty||\nabla\psi||_2 + CR||q||_\infty||\nabla\omega||_\infty$$
$$+CR||\nabla q||_\infty(||\nabla\psi||_2+\sqrt{m}R_0||\nabla\omega||_\infty).$$

We will show in the sequel that (A.9) continues to hold also for $1\leqq j\leqq L$. First notice that, since Im $a_j \geqq R_0$,

$$\left|\int_{B_+(a_j,R_0)} q\frac{\bar V_j\cdot\nabla\bar\psi}{|z-\bar a_j|}\right| \leqq \frac{||q||_\infty}{R_0}\int_{B_+(a_j,R_0)}|\nabla\bar\psi| \tag{A.10}$$
$$\leqq C||q||_\infty(||\nabla\psi||_2+R_0||\nabla\omega||_\infty).$$

As above (note that by (1.8) $\{a_1,\dots,a_m\}\cap B_+(0,R)\setminus B(\bar a_j,\rho_j)=\emptyset$)

$$\left|\int_{B_+(0,R)\setminus B(\bar a_j,\rho_j)} q\frac{\bar V_j\cdot\nabla\psi}{|z-\bar a_j|}\right| \leqq C||q||_\infty||\nabla\psi||_2. \tag{A.11}$$

Finally, note that for each $r\in(R_0,\rho_j)$ we have either $S_r(\bar a_j)\cap\mathbb{R}^2_+=\emptyset$ or $S_r(\bar a_j)\cap\{\text{Im } z=0\}=\{b-\alpha,b+\alpha\}$ for some $b\in\left[-\frac14 R,\frac14 R\right]$ and $\alpha\in(0,r)$. In the latter case we have

$$\left| \int_{S_r^+(\bar{a}_j)} q \frac{\bar{V}_j \cdot \nabla \bar{\psi}}{|z - \bar{a}_j|} \right| = \left| \frac{q(a_j)}{r} (\bar{\psi}(b-\alpha) - \bar{\psi}(b+\alpha)) + \frac{1}{r} \int_{S_r^+(\bar{a}_j)} (q - q(a_j)) \bar{V}_j \cdot \nabla \bar{\psi} \right|$$

$$\leqq 2||q||_\infty ||\nabla \omega||_\infty + C||\nabla q||_\infty \int_{S_r^+(\bar{a}_j)} |\nabla \bar{\psi}|,$$

and this inequality is valid for every $r \in (R_0, \rho_j)$. Integrating this inequality and using (A.11), (A.1) we are led to

$$\text{(A.12)} \quad \begin{aligned} \left| \int_{B_+(0,R)} q \frac{\bar{V}_j \cdot \nabla \bar{\psi}}{|z - \bar{a}_j|} \right| &= \left| \int_{B_+(0,R) \setminus B(\bar{a}_j, R_0)} q \frac{\bar{V}_j \cdot \nabla \bar{\psi}}{|z - \bar{a}_j|} \right| \\ &\leqq C||q||_\infty ||\nabla \psi||_2 + 2R||q||_\infty ||\nabla \omega||_\infty \\ &\quad + CR||\nabla q||_\infty (||\nabla \psi||_2 + \sqrt{m} R_0 ||\nabla \omega||_\infty). \end{aligned}$$

Combining (A.10) and (A.12) we see that (A.9) holds also for $j = 1, \dots, L$.

Now for $k = 1, \dots, m$ different from j we have $|z - \bar{a}_j| \geqq |z - a_j| \geqq R_0$ for $z \in B_+(a_k, R_0)$ and so we get as above that

$$\text{(A.13)} \quad \left| \sum_{k \neq j} \int_{B_+(a_k, R_0)} q \frac{\bar{V}_j \cdot \nabla \bar{\psi}}{|z - \bar{a}_j|} \right| \leqq C||q||_\infty \sqrt{m} (||\nabla \psi||_2 + \sqrt{m} R_0 ||\nabla \omega||_\infty).$$

Combining (A.9) and (A.13) we are led to

$$\text{(A.14)} \quad \begin{aligned} &\left| \int_{\Omega_+} q \frac{\bar{V}_j \cdot \nabla \psi}{|z - \bar{a}_j|} \right| \\ &\leqq C\sqrt{m} ||q||_\infty ||\nabla \psi||_2 + CR||\nabla q||_\infty ||\nabla \psi||_2 + CR||q||_\infty ||\nabla \omega||_\infty \\ &\quad + C\sqrt{m} R_0 ||\nabla \omega||_\infty (\sqrt{m} ||q||_\infty + R||\nabla q||_\infty). \end{aligned}$$

By (A.5), (A.8) and (A.14) we finally conclude that

$$\text{(A.15)} \quad \begin{aligned} |X_3| \leqq CmD\big\{ &\sqrt{m} ||q||_\infty ||\nabla \psi||_2 + R||\nabla q||_\infty ||\nabla \psi||_2 \\ &+ R||q||_\infty ||\nabla \omega||_\infty + \sqrt{m} R_0 ||\nabla \omega||_\infty (\sqrt{m} ||q||_\infty + R||\nabla q||_\infty) \big\}. \end{aligned}$$

Combining (A.15) with (A.3) and (A.4) we get

$$\text{(A.16)} \quad \begin{aligned} |X| \leqq\ & C_1\big(m, D, K, ||q||_\infty, R||\nabla q||_\infty\big) ||\nabla \psi||_2 \\ &+ C_2(m, D, K, ||q||_\infty, R||\nabla q||_\infty, R||\nabla \omega||_\infty). \end{aligned}$$

By the Cauchy-Schwarz inequality we conclude that for any $\varepsilon > 0$,

$$\text{(A.17)} \quad |X| \leqq \frac{\varepsilon^2}{2} \int_{\Omega_+} |\nabla \psi|^2 + \frac{C_1^2}{2\varepsilon^2} + C_2.$$

Choosing $\varepsilon = a\sqrt{q_0}$ in (A.17) and returning to (A.2) we get the desired conclusion. □

Appendix B. Minimizers of $F_\varepsilon(x)$

In this appendix we show how to modify our methods in order to study the asymptotic behavior of minimizers of the functional

$$F_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (a(x)^2 - |u|^2)^2, \quad \varepsilon > 0.$$

See [R] and the Introduction for motivation. We assume that $a(x)$ is a smooth function on $\bar{G}$ which satisfies $0 < a(x) \leqq 1$ for all $x \in \bar{G}$ and that a smooth boundary function $g : \partial G \to \mathbb{C}$ of degree $d > 0$ is given with $|g(x)| = a(x)$ for all $x \in \partial G$. For all $\varepsilon > 0$ we denote by u_ε a minimizer for F_ε over $H^1_g(G, \mathbb{C})$. In analogy with our previous notations we define $a_0 \equiv \min\{a(x);\ x \in \bar{G}\}$ and $\Lambda = a^{-1}(a_0)$. We first sketch the proof of the following analogue of Theorem 1.

Proposition B.1. *$u_{\varepsilon_n} \to u_*$ in $C^{1,\alpha}_{\mathrm{loc}}(\bar{G} \setminus \{b_1, \dots, b_N\})$ for a subsequence $\varepsilon_n \to 0$ for every $\alpha < 1$, where the N distinct points $\{b_1, \dots, b_N\}$ lie in Λ. The limit u_* can be written as $u_* = av_*$ where $v_* \in C^\infty(\bar{G} \setminus \{b_1, \dots, b_N\}, S^1)$ is a solution of*

$$-\operatorname{div}(a^2 \nabla v_*) = a^2 |\nabla v_*|^2 v_* \quad \text{in } \bar{G} \setminus \{b_1, \dots, b_N\}, \quad v_* = g/a \text{ on } \partial G.$$

Around each b_j, u_* is of degree $D_j > 0$ and $\sum_{j=1}^N D_j = d$.

For the proof we first notice that the two basic estimates

$$\text{(B.1)} \qquad ||u_\varepsilon||_{L^\infty(G)} \leqq 1, \quad ||\nabla u_\varepsilon||_{L^\infty(G)} \leqq \frac{C}{\varepsilon}$$

are proved in a way similar to that for the case $a(x) \equiv 1$ (i.e., the case treated in [BBH1, BBH2]). Also, the analogue of Lemma 1.1, namely,

$$\text{(B.2)} \qquad F_\varepsilon(u_\varepsilon) \leqq \pi d(a_0 + \delta)^2 |\log \varepsilon| + C(\delta) \quad \forall \varepsilon > 0,$$

can be easily verified. For the proof of (B.2) and for our further analysis it is useful to introduce $v_\varepsilon = u_\varepsilon / a$. A simple calculation shows that $F_\varepsilon(u_\varepsilon) = G_\varepsilon(v_\varepsilon)$ where (B.3)

$$G_\varepsilon(v) = \int_G \left\{ \frac{1}{2}\left(a^2|\nabla v|^2 + |v|^2 |\nabla a|^2 + \frac{1}{2}(\nabla a^2, \nabla |v|^2) \right) + \frac{a^4}{4\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

From (B.3) we are led to conjecture that the asymptotic behavior of $\{v_\varepsilon\}$ is essentially the same as that of the minimizers $\{w_\varepsilon\}$ over $H^1_{g/a}(G, \mathbb{C})$ for the functional

$$\text{(B.4)} \qquad \tilde{G}_\varepsilon(v) = \int_G \left\{ \frac{1}{2} a^2 |\nabla v|^2 + \frac{a^4}{4\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

We shall see later that this is indeed the case. The main obstruction for showing it a priori comes from the fact that we do not know in advance whether $\left| \int_G (\nabla a^2, \nabla |v_\varepsilon|^2) \right|$ remain bounded as $\varepsilon \to 0$.

In any case, using (B.1), (B.2) we can apply the methods of Section 1 to locate the BBD's $\{B(y_i, \nu\varepsilon^\beta)\}_{i=1}^{N_b}$, for some $\beta \in (0, 1)$ and the SBD's $\{B(x_{i,j}, \lambda\varepsilon)\}_{j=1}^{n_i}$, $i =$

$1, \dots, N_b$ which cover the set $S_\varepsilon = \{x \in G\,;\ |v_\varepsilon(x)| < \frac{1}{2}\}$. Passing to a subsequence $\{u_{\varepsilon_n}\}$ we denote again by $b_1, \dots, b_N$ the distinct limit points of the y_i's, the first M points lying in G and the rest on ∂G. As is the case for the functional E_ε, the main point is to establish an upper bound for the energy away from the singularities, as in Proposition 1.1. More precisely, we want to show that for any small η there exists $n(\eta)$ such that for $n \geqq n(\eta)$ we have

(B.5)

$$F_{\varepsilon_n}(u_{\varepsilon_n}|\Omega_\eta) \leqq \pi|\log\eta| \left\{ \sum_{k=1}^{M} a^2(b_k) D_k^2 + 2 \sum_{k=M+1}^{N} a^2(b_k) D_k^2 \right\} + C(G, g, p),$$

where $\Omega_\eta \doteq G \setminus \bigcup_{k=1}^N \overline{B(b_k, \eta)}$. Once the estimate (B.5) is established we can easily modify the methods of [BBH1] to deduce the convergence of $\{u_{\varepsilon_n}\}$ in $C^{1,\alpha}_{\mathrm{loc}}(\bar{G} \setminus \{b_1, \dots, b_N\})$. In order to prove the estimate (B.5) we bound from below the Dirichlet energy of v_{ε_n} on perforated domains (like $\Omega_i^{(n)}$ and $A_k^{(n)}$ in the proof of Proposition 1.1) by the energy of reference maps as in Section 1, using Lemmas 1.4 and 1.5. For a typical perforated domain Ω we obtain by using these lemmas that

$$\int_\Omega a^2 |\nabla v_{\varepsilon_n}|^2 \geqq \int_\Omega a^2 \big(|\nabla u_0|^2 + \big|\nabla |v_{\varepsilon_n}|\big|^2\big) - C. \tag{B.6}$$

Here u_0 is a reference map as in Section 1. Actually from (B.6) we may infer that

$$G_{\varepsilon_n}(v_{\varepsilon_n}|\Omega) \geqq G_{\varepsilon_n}(u_0|\Omega) - C. \tag{B.7}$$

Indeed estimates analogous to (B.7) enabled us to prove Proposition 1.1. Here, at first glance, it is not clear that (B.6) implies (B.7) since it might be the case that

$$\int_\Omega (\nabla a^2, \nabla |v_{\varepsilon_n}|^2) \to -\infty \quad \text{as} \quad \varepsilon_n \to 0.$$

However, we can make use of the additional term on the right-hand side of (B.6), namely, $\int_\Omega a^2 |\nabla|v_{\varepsilon_n}||^2$. Using the Cauchy-Schwarz inequality we get

$$\left| \int_\Omega (\nabla a^2, \nabla |v_{\varepsilon_n}|^2) \right| \leqq C \int_\Omega \big|\nabla |v_{\varepsilon_n}|\big| \leqq \frac{a_0^2}{2} \int_\Omega \big|\nabla |v_{\varepsilon_n}|\big|^2 + \frac{C^2}{2a_0^2} |\Omega|. \tag{B.8}$$

Using (B.8) we see easily that (B.6) does imply (B.7). As explained above, the estimate (B.7) enables us to prove (B.5), which in turn is the main step towards proving Proposition B.1. The other parts of Proposition B.1, namely, that $D_k > 0$ for all k and $\{b_1, \dots, b_N\} \subset \Lambda$ are proved by using the arguments of Section 1. This completes the sketch of the proof of Proposition B.1.

Using the above arguments we can also obtain the estimate

$$\int_G \big|\nabla |v_{\varepsilon_n}|\big|^2 \leqq C, \quad \text{uniformly in } n,$$

which implies of course that

$$\int_G \left|(\nabla a^2, \nabla |v_{\varepsilon_n}|^2)\right| \leqq C, \quad \text{uniformly in } n.$$

It follows easily that

$$F_{\varepsilon_n}(u_{\varepsilon_n}) = \tilde{G}_{\varepsilon_n}(w_{\varepsilon_n}) + O(1) \tag{B.9}$$

(recall that $\{w_\varepsilon\}$ is a minimizer over $H^1_{g/a}(G, \mathbb{C})$ for the functional $\tilde{G}_\varepsilon$ given in (B.4)). In particular, we immediately get an analogue of Theorem 1 of [ASh2]. Indeed, assume in the sequel that

$$\Lambda = \{a_1, \ldots, a_K\} \subset G \text{ with } K = K_i < d, \tag{B.10}$$

and that there exist K positive-definite quadratic forms $Q_1, \ldots, Q_K$ such that

$$\begin{aligned} a(x) = a_0 + Q_j(x - a_j) + o(|x - a_j|^2) \\ \text{in a neighborhood of } a_j, \quad 1 \leqq j \leqq K. \end{aligned} \tag{B.11}$$

By (B.9) and Theorem 1 in [ASh2] (and its proof) it follows that under these assumptions

$$F_{\varepsilon_n}(u_{\varepsilon_n}) = \pi a_0^2\{d|\log\varepsilon_n| + \tfrac{1}{2}(F(d,k) - d)\log(|\log\varepsilon_n|)\} + O(1) \text{ as } \varepsilon_n \to 0,$$

$\{b_1, \ldots, b_N\} = \{a_1, \ldots, a_K\}$ and the configuration of degrees $(D_1, \ldots, D_K)$ associated with $(a_1, \ldots, a_K)$ attains the minimum for $F(d, K)$, where

$$F(d,k) = \min\left\{\sum_{j=1}^k d_j^2\,; \quad (d_1, \ldots, d_k) \in (\mathbb{Z}^+)^k,\ \sum_{j=1}^k d_j = d\right\}.$$

Analogues to Theorems 2 and 3 of [ASh2] can be obtained too.

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