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A New Approach to Front Propagation Problems: Theory and Applications

Guy Barles & Panagiotis E. Souganidis

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Abstract

In this paper we present a new definition for the global-in-time propagation (motion) of fronts (hypersurfaces, boundaries) with a prescribed normal velocity, past the first time they develop singularities. We show that if this propagation satisfies a geometric maximum principle (inclusion-avoidance-type property), then the normal velocity must depend only on the position of the front, its normal direction and principal curvatures. This new approach, which is more geometric and, as it turns out, equivalent to the level-set method, is then used to develop a very general and simple method to rigorously validate the appearance of moving interfaces at the asymptotic limit of general evolving systems like interacting particles and reaction-diffusion equations. We finally present a number of new asymptotic results. Among them are the asymptotics of (i) reaction-diffusion equations with rapidly oscillating coefficients, (ii) fully nonlinear nonlocal (integral differential) equations, and (iii) stochastic Ising models with long-range anisotropic interactions and general spin-flip dynamics.

1. Introduction

In this article we present: (i) a new definition for the generalized (weak) propagation (motion) of fronts (hypersurfaces, boundaries of sets) in $\mathbb{R}^N$ with prescribed normal velocity, which satisfies some geometric conditions, past the first time that singularities develop, (ii) a simple and general method to establish the appearance of such fronts in the asymptotic (singular) limit of evolving systems, like reaction-diffusion equations and particle systems (stochastic Ising models), and (iii) a number of completely new and, in our opinion, striking examples including the asymptotics of reaction-diffusion equations with oscillating coefficients, non-local equations, and stochastic Ising models with long-range anisotropic interactions and general spin-flip dynamics.

The new definition is based on a few general geometric assumptions, namely, locality and regularity, monotonicity, i.e., an avoidance-inclusion property of a geometric maximum principle type, and local existence for smooth data. We show that (i) the locality and monotonicity yield that the normal velocity must depend only on the position in space and time, the normal direction and the principal curvatures of the front, with the dependence on the curvatures being nondecreasing, and (ii) the generalized evolution is equivalent, under the so-called no-interior condition, to the one obtained by the level-set method.

This approach is motivated by and is related to the abstract approach to image analysis by ALVAREZ, GUICHARD, LIONS & MOREL [AGLM]. It should be noted at this point, however, that our goal here is not to give yet another definition for the weak front propagation—there are already too many—but rather to develop a powerful new method to study the appearance of moving interfaces.

This new general method relies on an abstract approach, which, roughly speaking, looks like the classical formulation for the convergence of numerical schemes introduced by us in [BS] to prove the convergence of stable, monotone and consistent schemes. Our goal here is to show that the rigorous justification of the appearance of interfaces is reduced to checking some consistency-type properties, i.e., to proving a similar result but in the case where everything is smooth and for small time intervals, in other words, to justifying the formal asymptotics under the appropriate regularity assumptions. As a consequence we not only simplify and unify a number of results already obtained in this context (see below for references) but we also obtain new and, in our opinion, striking results of the type already described above.

Interfaces in $\mathbb{R}^N$ moving with normal velocity

$$V = v(Dn, n, x, t), \tag{1.1}$$

where n and Dn are the exterior normal vectors to the surface and its gradient, arise in addition to the situations already talked about, in geometry, in image processing, in turbulent flame propagation and combustion, in the phenomenological theory of phase transitions in continuum mechanics, etc.

The most typical example of interface dynamics appearing in the aforementioned areas is the general anisotropic motion

$$V = -\mathrm{tr}\,[A(n, x, t) Dn] + c(n, x, t), \tag{1.2}$$

where $A(n, x, t)$ is a matrix and $c(n, x, t)$ a scalar, a special case of which is the motion by mean curvature

$$V = -\mathrm{tr}\, Dn = \kappa_1 + \cdots + \kappa_{N-1},$$

$\kappa_1, \ldots, \kappa_{N-1}$ being the principal curvatures of the interface, and the anisotropic first-order motion

$$V = c(n, x, t).$$

The main characteristics of interface dynamics as in (1.1) are (i) the development of singularities in finite time, independently of the smoothness of the initial surface, and (ii) the fact that they satisfy a monotonicity, i.e., a geometric maximum principle-type avoidance and inclusion property; loosely speaking, if two fronts moving by (1.1) are separated at some time, then they remain separated. A great deal of work has been done during the last few years to find (i) a way to extend and interpret the evolution past singularities so that this maximum principle-type property is satisfied, and (ii) to use this weak interpretation to justify the appearance of such interfaces in the asymptotic analysis of general systems.

The outcome of this effort has been the development of a weak (generalized) notion of evolving fronts. The generalized evolution $\{\Gamma_t\}_{t \geqq 0}$ with normal velocity (1.1) starting with a given surface $\Gamma_0 \subset \mathbb{R}^N$ is defined for all $t \geqq 0$, although it may become extinct in finite time. Moreover, it agrees with the classical differential-geometric flow, as long as the latter exists. The generalized motion may, on the other hand, develop singularities, change topological type, and exhibit various other pathologies.

The main mathematical tool to study the generalized motion has been the level-set approach, which was introduced by OSHER & SETHIAN [OS] for numerical calculations—see also BARLES [B] for a first-order model for flame propagation. The level-set approach, which is based on the idea of representing the evolving front as the level set, for definiteness, the zero level set, of an auxiliary function satisfying an appropriately defined nonlinear partial differential equation, has been developed by EVANS & SPRUCK [ES] for the mean curvature motion and by CHEN, GIGA & GOTO [CGG] for (1.2) and later extended by BARLES, SONER & SOUGANIDIS [BSS], who introduced some concepts and tools which are used extensively in this paper, and by ISHII & SOUGANIDIS [IS], GOTO [G] and others. A related approach using the properties of the (signed) distance to the front was introduced by SONER [Son] and further developed in [BSS]. For a general review of these theories, their relationship as well as other related facts, we refer to SOUGANIDIS [Sou1,2].

In spite of the peculiarities described earlier, the generalized motion $\{\Gamma_t\}_{t \geqq 0}$ has been proved to be the right way to extend the classical motion past singularities. Some of the most definitive results in this direction were obtained by EVANS, SONER & SOUGANIDIS [ESS] (see also [BSS]) who proved that the generalized motion by mean curvature governs the asymptotic behavior of solutions to semilinear reaction-diffusion equations with bistable nonlinearities. We again refer to

[Sou1,2] for a general overview as well as asymptotic results for reaction-diffusion equations of a different type.

Another striking application of the generalized front propagation is the fact that it governs the macroscopic behavior, for large times and in the context of grain coarsening, of a number of stochastic interacting particle systems like the stochastic Ising model with long-range interactions and general spin-flip dynamics or nearest neighbor interaction and fast exchange dynamics. Such systems are standard Gibbsian models used in statistical mechanics to describe phase transitions. As was shown by KATSOULAKIS & SOUGANIDIS in [KS1,2,3], it turns out that the generalized front propagation not only describes the limiting behavior of such systems but also provides a theoretical justification, from the microscopic point of view, of several phenomenological sharp-interface models in phase transitions and of the Monte-Carlo numerical methods widely used in the physics literature to calculate moving fronts. It should also be noted that the generalized front propagation also describes the behavior of general threshold dynamics (cellular autonomon-type models) as was shown by ISHII, PIRES & SOUGANIDIS [IPS]. Once more we refer to [Sou1,2] for a general overview.

This paper is organized as follows: Section 2 is devoted to the development of the new approach to the generalized front propagation. In Section 3 we describe the general method to study the appearance of interfaces. In Section 4 we revisit the results of [ESS] and [BSS] about the asymptotics of reaction-diffusion equations and describe how the abstract method applies to them. Since these are relatively simple cases, we present all the details here. Section 5 is devoted to some new results regarding asymptotics of reaction-diffusion equations. In Section 6 we study the asymptotics of reaction-diffusion equations with oscillating coefficients. Section 7 is devoted to the asymptotics of general nonlocal, fully nonlinear equations. Finally in Section 8 we present asymptotic results about particle systems.

2. The New Definition

2.1. The General Framework

The aim of this section is to develop a new approach for defining the (weak) geometric motion of hypersurfaces $(\Gamma_t)_{t\in(a,b)}$ in $\mathbb{R}^N$ with a prescribed normal velocity past the first time that singularities develop. This new approach applies to motions which satisfy certain geometric assumptions, namely, monotonicity, locality and regularity.

Throughout this discussion we consider hypersurfaces $(\Gamma_t)_{t\in(a,b)}$ in $\mathbb{R}^N$, which are boundaries of open subsets $(\Omega_t)_{t\in(a,b)}$ of $\mathbb{R}^N$, and introduce the signed-distance function $d(x,t)$ from x to Γ_t defined by

$$d(x,t) = \begin{cases} d(x,\Gamma_t) & \text{if } x \in \Omega_t, \\ -d(x,\Gamma_t) & \text{otherwise,} \end{cases}$$

where $d(x,\Gamma_t)$ denotes the usual nonnegative distance from $x \in \mathbb{R}^N$ to Γ_t. If Γ_t is a smooth hypersurface, then d is a smooth function in a neighborhood of Γ_t, and for $x \in \Gamma_t$, $n(x,t) = -Dd(x,t)$ is the unit normal to Γ_t pointing away from Ω_t.

We begin considering smooth motions of smooth hypersurfaces $(\Gamma_t)_{t\in(a,b)}$ with a general prescribed normal velocity v. To this end we recall that the normal velocity $V(x,t)$ of such smoothly evolving hypersurfaces is defined at $x \in \Gamma_t$ by

$$V(x,t) = n(x,t) \cdot \dot{X}(t),$$

where $X : (a,b) \to \mathbb{R}^N$ is a C^∞-curve such that $X(s) \in \Gamma_s$ for all $s \in (a,b)$ and $X(t) = x$, and $p \cdot q$ denotes the usual inner product of the vectors $p, q \in \mathbb{R}^N$. It is, by the way, easy to check that such curves X exist and that the definition is independent of the particular choice of the curve.

The collection of smooth hypersurfaces $(\Gamma_t)_{t\in(a,b)}$ (or of smooth open subsets $(\Omega_t)_{t\in(a,b)}$) is said to propagate with the prescribed normal velocity v if and only if, for all $x \in \Gamma_t$ and $t \in (a,b)$,

$$V(x,t) = v(x,t,\Gamma_t).$$

In the same way, we say that the normal velocity is respectively larger or smaller than v if "$=$" is replaced in the above equation by "$\geqq$" or "$\leqq$".

A priori one may consider normal velocities v depending on (x,t) and on the different characteristics of Γ_t. Here, however, we focus our attention on motions which satisfy a number of assumptions, stated below.

To this end, let D^∞ be the set of C^∞-jets, and, if d is the signed distance to $(\Gamma_t)_{t\in(a,b)}$, define $[d] : \mathbb{R}^N \times (a,b) \to D^\infty$ by

$$[d](x,t) = (Dd(x,t), D^2 d(x,t), \cdots, D^k d(x,t), \cdots).$$

Our first assumption is

(A1) Locality and Regularity. *The normal velocity v at $x \in \Gamma_t$ depends only on x, t and the fundamental forms of Γ_t at x, i.e., there exists a function $v : \mathbb{R}^N \times (a,b) \times D^\infty \to \mathbb{R}$, such that the prescribed normal velocity is given by*

$$V(x,t) = v(x,t,[d](x,t)).$$

Moreover, v is a continuous function of x, t and Γ_t, in the sense that, if, as $n \to \infty$, $(x_n,t_n) \to (x,t)$ and $D^k d_n(x_n,t_n) \to D^k d(x,t)$ for all $k \in \mathbb{N}$, then

$$v(x_n,t_n,[d_n](x_n,t_n)) \to v(x,t,[d](x,t)).$$

Note that if, as $n \to \infty$, $D^k d_n \to D^k d$, uniformly in a neighborhood of Γ_t for all $k \in \mathbb{N}$, then $(x_n,t_n) \to (x,t)$ yields that $D^k_n(x_n,t_n) \to D^k d(x,t)$ for all $k \in \mathbb{N}$.

The second assumption is

(A2) Motonicity. *If $(\Omega_t)_{t\in(a,b)}$ and $(\tilde{\Omega}_t)_{t\in(a,b)}$ are two collections of C^∞-open subsets of $\mathbb{R}^N$ such that $\Gamma_t = \partial\Omega_t$ and $\tilde{\Gamma}_t = \partial\tilde{\Omega}_t$ move smoothly with a normal velocity smaller and larger, respectively, than v and if $\Omega_t \subset \tilde{\Omega}_t$ for some $t \in (a,b)$, then*

$$\Omega_s \subset \tilde{\Omega}_s \quad \textit{for any } s \in [t,b).$$

To state the definition of (weak) geometric front propagation, it is necessary to extend the domain of v to functions which are not necessarily distance functions. A natural way to understand this extension is to remark that, with the above notations, the set $\Omega_t = \{x : d(x,t) > 0\}$ can also be written as $\Omega_t = \{x : \phi(x,t) > 0\}$ for other suitable functions ϕ such that $|D\phi(x,t)| \neq 0$ on $\partial\Omega_t$. Then for all $x \in \partial\Omega_t$, one has

$$Dd(x,t) = \widehat{D\phi}(x,t),\ D^2 d(x,t) = D\big(\widehat{D\phi}(x,t)\big), \cdots,$$
$$D^k d(x,t) = D^{k-1}\big(\widehat{D\phi}(x,t)\big), \cdots,$$

where, for $p \in \mathbb{R}^N \backslash \{0\}$,

$$\hat{p} = |p|^{-1} p.$$

We set

$$[\phi](x,t) = (\widehat{D\phi}(x,t), D\big(\widehat{D\phi}(x,t)\big), \cdots, D^{k-1}\big(\widehat{D\phi}(x,t)\big), \cdots),$$

and consider the extension $\bar{v}$ of v given by

$$\bar{v}(x,t,[\phi](x,t)) = |D\phi(x,t)| v(x,t,[\phi](x,t)).$$

Notice that since v is defined on $\mathbb{R}^N \times (a,b) \times D^\infty$, the right-hand side of this last equality is well-defined at each point where $|D\phi(x,t)| \neq 0$. Moreover, if we fix the function ϕ, then $\bar{v}$ can also be seen as a continuous function of (x,t), which is defined on the open subset $\{(x,t) : |D\phi(x,t)| \neq 0\}$.

In order to simplify the presentation below, we assume throughout the paper that, for each fixed smooth function ϕ, $\bar{v}$ is locally bounded in $\mathbb{R}^N \times (a,b)$ in the following sense.

(A3) Local Boundedness. *For any compact subset K of $\mathbb{R}^N \times (a,b)$, there exists a constant $C(K)$ such that, for all C^∞-functions ϕ and for all $(x,t) \in \{(y,s) : |D\phi(y,s)| \neq 0\} \cap K$,*

$$|\bar{v}(x,t,[\phi](x,t))| \leqq C(K).$$

This type of assumption is not satisfied when the normal velocity grows superlinearly on the curvature tensor, as, for example, is the case of motion by Gaussian curvature. We indicate in Remark 2.3 below how to remove this assumption at the expense of a slightly more complicated definition.

In what follows it is also necessary to consider the upper- and lower-semicontinuous envelopes of the locally bounded function $\bar{v}$ considered as a function of x and t, which we denote by $\bar{v}^*$ and $\bar{v}_*$ respectively. Recall that for a locally bounded function $f : A \to \mathbb{R}$, where A is a subset of some $\mathbb{R}^k$, the upper- and lower-semicontinuous envelopes f^* and f_* of f are given by

$$f^*(y) = \limsup_{z \to y} f(z), \quad f_*(y) = \liminf_{z \to y} f(z).$$

We have

Definition 2.1. A family $(\Omega_t)_{t\in(a,b)}$ of open subsets of $\mathbb{R}^N$ is respectively called a *generalized super-flow* or *sub-flow* with normal velocity v if and only if for all $x_0 \in \mathbb{R}^N$, $t \in (a,b)$, $r > 0$, $\alpha > 0$ and for all smooth functions $\phi : \mathbb{R}^N \to \mathbb{R}$ such that $\{x \in \mathbb{R}^N : \phi(x) \geqq 0\} \subset \Omega_t \cap B_r(x_0)$, or $\{x \in \mathbb{R}^N : \phi(x) \leqq 0\} \subset \bar{\Omega}_t^c \cap B_r(x_0))$, with $|D\phi| \neq 0$ on $\{x \in \mathbb{R}^N : \phi(x) = 0\}$, there exists $h_0 > 0$ depending only on α and the C^∞-function ϕ through the properties of $[\phi]$ in $\bar{B}_r(x_0)$ such that, for all $h \in (0, h_0)$,

$$\{x \in \mathbb{R}^N : \phi(x) + h[\bar{v}_*(x,t,[\phi](x)) - \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega_{t+h}$$

or

$$\{x \in \mathbb{R}^N : \phi(x) + h[\bar{v}^*(x,t,[\phi](x)) + \alpha] < 0\} \cap \bar{B}_r(x_0) \subset \bar{\Omega}^c_{t+h}).$$

A family $(\Omega_t)_{t\in(a,b)}$ of open subsets of $\mathbb{R}^N$ is called a *generalized flow with normal velocity* v if it is both a sub- and super-flow.

The following remark is essential to better understand the definitions of sub- and super-flow.

Remark 2.1. For smooth classical flows, if we change the orientation, i.e., if we replace d by $\tilde{d} = -d$ which, in other words, can be expressed by saying that we consider the motion of the family $(\bar{\Omega}_t^c)_{t\in(a,b)}$ instead of $(\Omega_t)_{t\in(a,b)}$, then the prescribed normal velocity has to be changed into $-v(x,t,\big[-\tilde{d}\big])$. This elementary fact is used in Definition 2.1. Indeed, we easily see that $(\Omega_t)_{t\in(a,b)}$ is a generalized sub-flow with normal velocity $v(x,t,[d])$ if and only if $(\bar{\Omega}_t^c)_{t\in(a,b)}$ is a generalized super-flow with normal velocity $-v(x,t,\big[-\tilde{d}\big])$.

From now we always assume that the assumptions (A1)–(A3) are satisfied. We prove that (i) if $(\Omega_t)_{t\in(a,b)}$ is a collection of open subsets which depends smoothly on t, then $(\Omega_t)_{t\in(a,b)}$ is respectively a generalized super-flow or sub-flow with normal velocity v if and only if the collection $(\Gamma_t)_{t\in(a,b)}$, where $\Gamma_t = \partial\Omega_t$ for $t \in (a,b)$, propagates with normal velocity larger or smaller than v, (ii) the monotonicity assumption (A2) yields that there exists a continuous function G such that $\bar{v}(x,t,[\phi])$ is necessarily of the form

$$\bar{v}(x,t,[\phi](x,t)) = -|D\phi(x,t)|G(D(\widehat{D\phi})(x,t), \widehat{D\phi}(x,t), x, t), \tag{2.1}$$

and (iii) G is nonincreasing, i.e., degenerate elliptic, in $D^2\phi$.

To formulate these results and to emphasize the dependence of G on $D^2\phi$ and $D\phi$ we introduce the function $F : \mathcal{S}^N \times \mathbb{R}^N\backslash\{0\} \times \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$, $\mathcal{S}^N$ being the space of $N \times N$ symmetric matrices, given by

$$F(X,p,x,t) = |p|G\big(|p|^{-1}\left(X - X\hat{p}\otimes\hat{p}\right), \hat{p}, x, t\big). \tag{2.2}$$

The monotonicity of G is expressed by saying that for all $X, Y \in \mathcal{S}^N$ and $(p,x,t) \in \mathbb{R}^N\backslash\{0\} \times \mathbb{R}^N \times (0,\infty)$, F satisfies the ellipticity condition

$$F(X,p,x,t) \leqq F(Y,p,x,t) \quad \text{if } X \geqq Y. \tag{2.3}$$

Notice that (2.1)–(2.2) and the local boundedness assumption (A3) yield that F is itself locally bounded. As we did for $\bar{v}$, we extend the locally bounded function F

to $p = 0$ by taking upper or lower semi-continuous envelopes. Finally we remark that in the rest of the paper we use any of the symbols v, $\bar{v}$ and F to denote the prescribed normal velocity.

The first result is

Theorem 2.1. *Assume that* (A1)–(A3) *hold and that* $(\Omega_t)_{t\in(a,b)}$ *is a collection of smooth, open subsets of* $\mathbb{R}^N$ *which depend smoothly on* t. *The collection* $(\Omega_t)_{t\in(a,b)}$ *is a generalized super-flow or sub-flow with normal velocity* v *if and only if respectively the collection* $(\Gamma_t)_{t\in(a,b)}$, *where* $\Gamma_t = \partial\Omega_t$ *for all* $t \in (a, b)$, *propagates with a normal velocity larger or smaller than* v.

Proof. 1. We prove the result only in the super-flow case. The other case follows by either using similar arguments or by considering the collection $(\bar{\Omega}_t^c)_{t\in(a,b)}$.

2. Assume that $(\Omega_t)_{t\in(a,b)}$ is a generalized super-flow with normal velocity v. For $t \in (a, b)$, denote by $\phi : \mathbb{R}^N \to \mathbb{R}$ a C^∞-function which is equal to the signed-distance to Γ_t in a neighborhood of Γ_t and such that $\Omega_t = \{y : \phi(y) > 0\}$. It is clear that, without loss of generality, ϕ may be taken to be bounded.

3. The following lemma plays an important role not only in this proof but also in the proof of Theorem 2.4 below. We state it here and present its proof after we complete the proof of Theorem 2.1.

Lemma 2.1. *Let* $\phi, \chi : \mathbb{R}^N \to \mathbb{R}$ *be* C^∞*-functions such that* $\phi(x) = 0$, $D\phi(x) \neq 0$ *for some* $x \in \mathbb{R}^N$ *and* $\chi(0) = 0$. *Moreover, assume that* χ *is radially symmetric,* $\chi > 0$ *in* $\mathbb{R}^N \backslash \{0\}$ *with* $D\chi(y) \cdot y \geqq 2\chi(y)$ *in a neighborhood* $\mathcal{O}$ *of* 0 *and* $(\chi(y - x))^{-1}\phi(y) \to 0$, *as* $|y| \to +\infty$. *For each* $k \in \mathbb{R}$, *define the function* $\phi_k : \mathbb{R}^N \to \mathbb{R}$ *by* $\phi_k(y) = \phi(y) + k\chi(y - x)$. *Then, for* $|k|$ *sufficiently large, the set* $A_k = \{y \in \mathbb{R}^N : \phi_k(y) = 0\}$ *is bounded and* $D\phi_k(y) \neq 0$ *on* A_k.

4. Since ϕ is bounded, we can apply Lemma 2.1 for any $x \in \Gamma_t$ with a C^∞-function χ with quadratic growth at infinity and such that $D^l\chi(0) = 0$ for any $l \in \mathbb{N}$. Since $\Omega_t = \{y : \phi(y) > 0\}$, it follows, for $k < 0$ and $|k|$ large enough, $\eta > 0$ small enough, and for some $r > 0$, that

$$\{y : \phi_k(y) - \eta \geqq 0\} \subset \Omega_t \cap B_r(x), \quad D(\phi_k(y) - \eta) \neq 0 \quad \text{on } \{y : \phi_k(y) - \eta = 0\}.$$

5. That the family $(\Omega_t)_{t\in(a,b)}$ is a generalized super-flow with normal velocity v, yields, for $\alpha > 0$ and for sufficiently small h, depending only on x, t, r, α and the properties of $[\phi_k - \eta] = [\phi_k]$ in $\bar{B}_r(x)$ and not on η, that

$$\{y : \phi_k(y) - \eta + h[\bar{v}_*(y, t, [\phi_k - \eta](y)) - \alpha] > 0\} \cap \bar{B}_r(x) \subset \Omega_{t+h}.$$

Letting $\eta \to 0$, we get

$$\{y : \phi_k(y) + h[v_*(y, t, [\phi_k](y)) - \alpha] > 0\} \cap \bar{B}_r(x) \subset \Omega_{t+h}.$$

6. Let $X : (t, b) \to \mathbb{R}^N$ be a C^∞-curve such that $X(t) = x$ and $X(s) \in \Gamma_s$ for all $t \in [s, b)$. Since $X(t + h) \in \Gamma_{t+h} \cap B_r(x)$ for sufficiently small $h > 0$, it follows that $X(t + h) \notin \Omega_{t+h}$ and therefore

(2.4) $$\phi_k(X(t+h)) + h[v_*(X(t+h), t, [\phi_k](X(t+h))) - \alpha] \leqq 0.$$

7. Since ϕ is equal to the signed-distance function to Γ_t in a neighborhood of Γ_t, we have $D\phi(x) = -n(x,t)$. The smoothness of ϕ_k and X yields

$$\begin{aligned}\phi_k(X(t+h)) &= \phi_k(X(t)) + \int_t^{t+h} \frac{d}{ds}\big[\phi_k(X(s))\big]ds \\ &= \phi_k(x) + \int_t^{t+h} D\phi_k(X(s)) \cdot \dot{X}(s)ds \\ &= \phi_k(x) - hn(x,t) \cdot \dot{X}(t) + o(h).\end{aligned}$$

8. Substituting this last expression in (2.4) and recalling that $\phi_k(x) = 0$, we get

$$h\left[v_*(X(t+h), t, [\phi_k](X(t+h))) - n(x,t) \cdot \dot{X}(t) - \alpha\right] + o(h) \leqq 0.$$

9. Dividing by h, letting $h \to 0$, using the regularity of v and the fact that every derivative of χ vanishes at 0, we obtain, for any $\alpha > 0$,

$$v_*(x, t, [\phi]) - n(x,t) \cdot \dot{X}(t) - \alpha \leqq 0.$$

Letting $\alpha \to 0$, yields the desired inequality.

10. Assume that the collection $(\Gamma_t)_{t\in(a,b)}$ propagates with a normal velocity larger than v and let $x_0 \in \mathbb{R}^N$, $t \in (a,b)$, $r > 0$, $\alpha > 0$ and a smooth function $\phi : \mathbb{R}^N \to \mathbb{R}$ be such that $\{x \in \mathbb{R}^N : \phi(x) \geqq 0\} \subset \Omega_t \cap B_r(x_0)$, with $|D\phi| \neq 0$ on $\{x \in \mathbb{R}^N : \phi(x) = 0\}$. We need to prove the existence of $h_0 > 0$, depending only on x_0, t, α, r and the C^∞-function ϕ, through the properties of $[\phi]$ in $\bar{B}_r(x_0)$, such that, for all $h \in (0, h_0)$,

$$\{x \in \mathbb{R}^N : \phi(x) + h[\bar{v}_*(x, t, [\phi](x)) - \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega_{t+h}.$$

11. This claim is a consequence of (A2) and

Lemma 2.2. *For any smooth function $\phi : \mathbb{R}^N \to \mathbb{R}$ such that $\{x \in \mathbb{R}^N : \phi(x) \geqq 0\} \subset B_r(x_0)$, with $|D\phi| \neq 0$ on $\{x \in \mathbb{R}^N : \phi(x) = 0\}$, there exists $h_0 > 0$, depending only on x_0, t, α, r and the C^∞-function ϕ through the properties of $[\phi]$ in $\bar{B}_r(x_0)$, and a family of smooth open subsets $(\Omega^1_{t+s})_{s\in[0,h_0)}$ such that*

(i) $\Omega^1_t = \{x \in \mathbb{R}^N : \phi(x) = 0\}$,

(ii) $(\partial\Omega^1_{t+s})_{s\in[0,h_0)}$ *evolves smoothly with a normal velocity smaller than v,*

(iii) $\{x \in \mathbb{R}^N : \phi(x) + h[\bar{v}_*(x, t, [\phi](x)) - \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega^1_{t+h}$,

(iv) $\Omega^1_{t+s} \subset B_r(x_0)$ *for $s \in [0, h_0)$.*

12. We continue with the proof of Theorem 2.1 and prove Lemma 2.2 next. It follows from (A2) and the properties (i) and (ii) of the collection $(\Omega^1_{t+s})_{s\in[0,h_0)}$ given by Lemma 2.2 that, for $h \in [0, h_0)$,

$$\Omega^1_{t+h} \subset \Omega_{t+h}.$$

Properties (iii) and (iv) of $(\Omega^1_{t+s})_{s\in[0,h_0)}$ also yield

$$\{x \in \mathbb{R}^N : \phi(x) + h[\bar{v}_*(x,t,[\phi](x)) - \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega^1_{t+h}.$$

The result now follows from these last two inclusions. □

The following remark, we hope, clarifies the meaning of Lemma 2.2.

Remark 2.2. Lemma 2.2 gives a justification to the basic idea beyond the definition of generalized sub- and super-flows. Indeed we have in mind that the boundary of the sets $\{x \in \mathbb{R}^N : \phi(x) + h[\bar{v}_*(x,t,[\phi](x)) - \alpha] > 0\}$ should evolve (in some weak sense) with a normal velocity which is smaller than v. This idea together with (A2) leads to the inclusion required in the definition. On the other hand, since these sets are not smooth, it is not possible to justify this idea directly. The few lines of proof (Step 12) show how Lemma 2.2 allows us to do it.

We continue with the

Proof of Lemma 2.2. 1. Since the map $(x,t) \mapsto \bar{v}_*(x,t,[\phi](x))$ is bounded and continuous in a neighborhood of $\{x \in \mathbb{R}^N : \phi(x) = 0\}$, there exists a smooth function $\psi : \mathbb{R}^N \to \mathbb{R}$ such that $\psi \geqq \bar{v}_*$ in $\bar{B}_r(x_0)$ and $|\psi - \bar{v}_*| \leqq \frac{1}{2}\alpha$ in a neighborhood $\mathscr{V}$ of the set $\{x \in \mathbb{R}^N : \phi(x) = 0\}$.

2. Define

$$\Omega^1_{t+s} = \{x \in \mathbb{R}^N : \phi(x) + s[\psi(x) - \alpha] > 0\} \cap B_r(x_0).$$

The assumptions on the function ψ easily yield that for $h_1 > 0$ small enough, the family $(\Omega^1_{t+s})_{s\in[0,h_1)}$ satisfies the properties (i), (iii) and (iv) of Lemma 2.2. It remains to prove that (ii) holds for sufficiently small h_0.

3. Consider a C^∞–curve $X : [t, t+h_1) \to \mathbb{R}^N$ such that for any $h \in [t, t+h_1)$, $X(t+h) \in \partial\Omega^1_{t+h}$, or, in other words,

$$\phi(X(t+h)) + h[\psi(X(t+h)) - \alpha] = 0.$$

4. Differentiating this equality with respect to h, we obtain

$$[D\phi(X(t+h)) + hD\psi(X(t+h))] \cdot \dot{X}(t+h) + \psi(X(t+h)) - \alpha = 0.$$

5. Set

$$p_h = D\phi(X(t+h)) + hD\psi(X(t+h)).$$

For h small, it follows that $X(t+h) \in \mathscr{V}$, $|p_h| \neq 0$ and $n(X(t+h), t+h) = -\hat{p}_h$ is the unit normal vector to $\partial\Omega^1_{t+h}$ pointing away to Ω^1_{t+h}. The last equality and the property of ψ in $\mathscr{V}$ yield

$$-n(X(t+h), t+h) \cdot \dot{X}(t+h) + |p_h|^{-1}\bar{v}_*(X(t+h), t+h, [\phi + h(\psi - \alpha)]) \geqq 0.$$

6. The conclusion follows from the fact that, if d_h is the sign-distance to $\partial\Omega^1_{t+h}$, then

$$|p_h|^{-1}\bar{v}_*(X(t+h), t+h, [\phi + h(\psi - \alpha)]) = v(X(t+h), t+h, [d_h]).\square$$

We continue with the

Proof of Lemma 2.1. 1. Assume that there exist sequences $(k_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ such that $|k_i| \to +\infty$ as $i \to +\infty$, $y_i \in A_{k_i}$ and $D\phi_{k_i}(y_i) = 0$. It is easy to see that $y_i \to x$ as $i \to +\infty$, and, therefore, $y_i - x \in \mathcal{O}$ for i sufficiently large.

2. The facts that $\phi_{k_i}(y_i) = 0$ and $D\phi_{k_i}(y_i) = 0$ yield

$$\phi(y_i) + k_i\chi(y_i - x) = 0, \tag{2.5}$$

$$D\phi(y_i) + k_i D\chi(y_i - x) = 0. \tag{2.6}$$

Taking the inner product of the expression in the left-hand side of (2.6) with $y_i - x$ and subtracting (2.5) we get

$$k_i[D\chi(y_i - x)\cdot(y_i - x) - \chi(y_i - x)] - [\phi(y_i) - D\phi(y_i)\cdot(y_i - x)] = 0.$$

3. Taylor's expansion of ϕ at x and the fact that $\phi(x) = 0$ give

$$\phi(y_i) - D\phi(y_i)\cdot(y_i - x) = O(|y_i - x|^2).$$

4. Since $y_i - x \in \mathcal{O}$, the assumptions on χ yield

$$D\chi(y_i-x)\cdot(y_i-x)-\chi(y_i-x) \geqq \tfrac{1}{2}D\chi(y_i-x)\cdot(y_i-x) = \tfrac{1}{2}|D\chi(y_i-x)||y_i-x|,$$

the last equality being a consequence of the radial symmetry of χ.

5. Combining the above results yields

$$k_i|D\chi(y_i - x)| = O(|y_i - x|),$$

and, in view of (2.6),

$$|D\phi(y_i)| = O(|y_i - x|).$$

6. This leads to a contradiction for i large enough, since $|D\phi(y_i)| \to |D\phi(x)| \neq 0$ and $O(|y_i - x|) \to 0$ as $i \to \infty$. $\square$

The next result is the

Theorem 2.2. *Assume that* (A1)–(A3) *hold. Then there exists a continuous function G such that, for all $(x, t) \in \mathbb{R}^N \times [0, \infty)$ and for all smooth functions ϕ,*

$$\bar{v}(x, t, [\phi]) = -|D\phi(x)|G(D(\widehat{D\phi})(x), \widehat{D\phi}(x), x, t) \quad \textit{on } \{x \in \mathbb{R}^N : |D\phi(x)| \neq 0\}.$$

Proof. 1. Following the proof of the analogous result of [AGLM], we want to show that if ϕ and ψ are C^∞-functions such that, for some $x \in \mathbb{R}^N$, $D\phi(x) \neq 0$, $D\psi(x) \neq 0$ and

$$\widehat{D\phi}(x) = \widehat{D\psi}(x), \quad D\left(\widehat{D\phi}(x)\right) = D\left(\widehat{D\psi}(x)\right),$$

then

$$\bar{v}(x, t, [\phi](x)) = \bar{v}(x, t, [\psi](x)).$$

2. Changing if necessary ϕ into $\chi_1(\phi)$ and ψ into $\chi_2(\psi)$, where $\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}$ are C^∞-functions such that $\chi_1', \chi_2' > 0$ in $\mathbb{R}$, we may assume, without loss of generality, that

$$\phi(x) = \psi(x), \quad D\phi(x) = D\psi(x), \quad D^2\phi(x) = D^2\psi(x).$$

Notice that these changes on ϕ and ψ preserve $v(x,t,[\phi])$ and $v(x,t,[\psi])$. Moreover, in the same way, we may assume that ϕ and ψ are bounded.

3. We argue by contradiction assuming that

$$\bar{v}(x,t,[\phi](x)) > \bar{v}(x,t,[\psi](x)),$$

which, in turn, implies the existence of $\beta \in \mathbb{R}$ such that

$$\bar{v}(x,t,[\phi](x)) + \beta > 0 > \bar{v}(x,t,[\psi](x)) + \beta.$$

4. We introduce the functions

$$\bar{\phi}(y) = \phi(y) - \varepsilon|y-x|^2 - K\chi(y-x),$$
$$\bar{\psi}(y) = \psi(y) + K\chi(y-x),$$

where χ is as in Lemma 2.1 with at least a quadratic growth at infinity and such that $D^k\chi(0) = 0$ for all $k \in \mathbb{N}$.

The assumptions on ϕ and ψ easily yield that, for K large enough,

$$\bar{\phi} \leqq \psi \leqq \bar{\psi} \quad \text{in } \mathbb{R}^N.$$

5. Lemma 2.1 yields, that, for some $r > 0$,

$$\emptyset \neq \{y : \bar{\phi}(y) > 0\} \subset \{y : \psi(y) > 0\} \cap B_r(x)$$
$$\text{with } |D\bar{\phi}(y)| \neq 0 \text{ on } \{y : \bar{\phi}(y) = 0\},$$

$$\emptyset \neq \{y : \bar{\psi}(y) < 0\} \subset \{y : \psi(y) < 0\} \cap B_r(x)$$
$$\text{with } |D\bar{\psi}(y)| \neq 0 \text{ on } \{y : \bar{\psi}(y) = 0\}.$$

6. Applying Lemma 2.2 to $\bar{\phi}$ and $-\bar{\psi}$, we find two families of smooth open subsets $(\Omega_h^1)_{h\in[0,h_0)}$ and $(\Omega_h^2)_{h\in[0,h_0)}$ of $\mathbb{R}^N$ evolving smoothly with normal velocities smaller than $v(x,t,[d])$ and $-v(x,t,[-d])$ respectively, and such that

$$\Omega_0^1 = \{y : \bar{\phi}(y) > 0\}, \quad \Omega_0^2 = \{y : -\bar{\psi}(y) > 0\}.$$

Recall that, in view of Remark 2.1, the family $([\bar{\Omega}_h^2]^c)_{h\in[0,h_0)}$ moves smoothly with a normal velocity larger than $v(x,t,[d])$.

7. Step 5 yields

$$\Omega_0^1 \subset [\bar{\Omega}_0^2]^c.$$

Hence, in view of Assumption (A2), for all $h \in [0,h_0)$ we have

$$\Omega_h^1 \subset \left[\bar{\Omega}_h^2\right]^c,$$

and, therefore,

$$\Omega_h^1 \cap \Omega_h^2 = \emptyset. \tag{2.7}$$

Lemma 2.2 also implies, for all $h \in [0, h_0)$, that

$$\{y : \bar{\phi}(y) + h[\bar{v}_*(y, t, \left[\bar{\phi}\right](y)) - \alpha] > 0\} \subset \Omega_h^1,$$

$$\{y : \bar{\psi}(y) + h[\bar{v}^*(y, t, \left[\bar{\psi}\right](y)) + \alpha] < 0\} \subset \Omega_h^2.$$

8. Consider the point $x_h = x + h\beta\widehat{D\phi}(x)$. Since $D\phi(x) = D\psi(x)$, it follows that

$$\bar{\phi}(x_h) = h\beta + o(h), \quad \bar{\psi}(x_h) = h\beta + o(h).$$

Examining the quantities $v(x_h, t, \left[\bar{\phi}\right](x_h))$ and $v(x_h, t, \left[\bar{\psi}\right](x_h))$, we deduce from (A1) that

$$v(x_h, t, \left[\bar{\phi}\right](x_h)) = v(x, t, [\phi](x)) + o_h(1) + o_\varepsilon(1),$$

$$v(x_h, t, \left[\bar{\psi}\right](x_h)) = v(x, t, [\psi](x)) + o_h(1).$$

9. If we choose $\alpha > 0$ sufficiently small, a simple computation gives, for sufficiently small h and ε, that

$$\bar{\phi}(x_h) + h[\bar{v}_*(x_h, t, \left[\bar{\phi}\right](x_h)) - \alpha]$$
$$= h[\bar{v}_*(x, t, [\phi](x)) + \beta - \alpha + o_h(1) + o_\varepsilon(1)] > 0,$$

$$\bar{\psi}(x_h) + h[\bar{v}^*(x_h, t, \left[\bar{\psi}\right](x_h)) + \alpha] = h[\bar{v}^*(x, t, [\psi](x)) + \beta + \alpha + o_h(1)] < 0.$$

Hence $x_h \in \Omega_h^1$ and $x_h \in \Omega_h^2$ which contradicts (2.7).

10. The above arguments show that, if $D\phi(x) \neq 0$, then $v(x, t, [\phi](x))$ depends only on $(D(\widehat{D\phi})(x), \widehat{D\phi}(x), x, t)$ and, therefore, there exists a function G such that

$$v(x, t, [\phi]) = -|D\phi(x)| G(D(\widehat{D\phi})(x), \widehat{D\phi}(x), x, t).$$

11. The continuity of G just follows from the regularity assumption on v. □

Finally, we have

Theorem 2.3. *Assume that* (A1)–(A3) *hold. The function F defined by* (2.1), (2.2) *satisfies the ellipticity condition* (2.3).

Proof. 1. Since the proof is based on exactly the same ideas as the proof of Theorem 2.2, we only present a sketch here.

2. We argue by contradiction, assuming that there exists $(x, t) \in \mathbb{R}^N \times (0, \infty)$, $p \in \mathbb{R}^N \setminus \{0\}$ and $A, B \in \mathscr{S}^N$ such that

$$A \leqq B, \quad F(A, p, x, t) < F(B, p, x, t)\ .$$

It follows that there must exist $\beta \in \mathbb{R}$ such that

$$F(A, p, x, t) - \beta < 0 < F(B, p, x, t) - \beta.$$

Finally, notice that, in view of (2.2), we may assume without loss of generality that $|p| = 1$.

3. Consider the functions

$$\bar{\phi}(y) = p \cdot (y-x) + \tfrac{1}{2} B(y-x) \cdot (y-x) - K\chi(y-x),$$
$$\bar{\psi}(y) = p \cdot (y-x) + \tfrac{1}{2} A(y-x) \cdot (y-x) + K\chi(y-x),$$

where χ is as in Lemma 2.1 with cubic growth at infinity and such that $D^k\chi(0) = 0$ for all $k \in \mathbb{N}$. Since $A \leqq B$, we clearly have

$$\bar{\phi} \leqq \psi \leqq \bar{\psi} \quad \text{in } \mathbb{R}^N.$$

4. Arguing as in the proof of Theorem 2.2, we use Lemma 2.2. It follows that there exist two families of smooth open subsets $(\Omega_h^1)_{h\in[0,h_0)}$ and $(\Omega_h^2)_{h\in[0,h_0)}$ of $\mathbb{R}^N$ evolving smoothly with normal velocity smaller than $v(x, t, [d])$ and $-v(x, t, [-d])$ respectively and such that

$$\Omega_0^1 = \{y : \bar{\phi}(y) > 0\}, \quad \Omega_0^2 = \{y : -\bar{\psi}(y) > 0\}.$$

5. Lemma 2.2 and the arguments in the proof of Theorem 2.2 yield that, for $h \in [0, h_0)$,

$$\Omega_h^1 \cap \Omega_h^2 = \emptyset,$$

$$\{y : \bar{\phi}(y) + h[\bar{v}_*(y, t, \left[\bar{\phi}\right](y)) - \alpha] > 0\} \subset \Omega_h^1,$$
$$\{y : \bar{\psi}(y) + h[\bar{v}^*(y, t, \left[\bar{\psi}\right](y)) + \alpha] < 0\} \subset \Omega_h^2.$$

6. Let $x_h = x + h\beta p$. Since $|p| = 1$, we have

$$\bar{\phi}(x_h) = h\beta + o(h), \quad \bar{\psi}(x_h) = h\beta + o(h).$$

7. The continuity of F at (A, p, x, t) and (B, p, x, t) yields

$$\begin{aligned}&\bar{v}_*(x_h, t, \left[\bar{\phi}\right](x_h))\\ &\quad = -F(D^2\bar{\phi}(x_h, t), D\bar{\phi}(x_h, t), x_h, t) = -F(A, p, x, t) + o_h(1),\end{aligned}$$

$$\begin{aligned}&\bar{v}^*(x_h, t, \left[\bar{\psi}\right](x_h))\\ &\quad = -F(D^2\bar{\psi}(x_h, t), D\bar{\psi}(x_h, t), x_h, t) = -F(B, p, x, t) + o_h(1).\end{aligned}$$

8. For α and h sufficiently small we have

$$\bar{\phi}(x_h) + h[\bar{v}_*(x_h, t, \left[\bar{\phi}\right](x_h)) - \alpha] = h[-F(A, p, x, t) + o_h(1) + \beta - \alpha] > 0,$$
$$\bar{\psi}(x_h) + h[\bar{v}^*(x_h, t, \left[\bar{\psi}\right](x_h)) + \alpha] = h[-F(B, p, x, t) + o_h(1) + \beta + \alpha] < 0.$$

9. Hence $x_h \in \Omega_h^1 \cap \Omega_h^2$, which contradicts the first assertion in Step 5 above. □

We continue with a brief discussion, formulated as a remark, about how Assumption (A3) can be removed.

Remark 2.3. If Assumption (A3) does not hold, we argue by replacing in Definition 2.1 $\bar{v}_*$ by $\inf(\bar{v}_*, R)$ and $\bar{v}^*$ by $\sup(\bar{v}^*, R)$ and by requiring that the inclusions have to hold for all $h \in (0, h_0)$, where h_0 may now also depend on R, and that the assertion has to be true for all $R > 0$. It is worth remarking that if ϕ is as in Definition 2.1, then for R large enough, $\bar{v}_* = \inf(\bar{v}_*, R)$ and $\bar{v}^* = \sup(\bar{v}^*, R)$ in a neighborhood of $\{x : \phi(x) = 0\}$. The truncations are only effective away from this set, to rule out difficulties at the points where $\bar{v}$ is unbounded. Finally, from the technical point of view, truncating by R allows us, more or less, to come back to the case where (A3) holds.

It turns out that it is occasionally more convenient to restate Definition 2.1 using F in place of v.

Definition 2.2. A family $(\Omega_t)_{t\in(a,b)}$ of open subsets of $\mathbb{R}^N$ is called respectively a *generalized super-flow or sub-flow* with normal velocity $-F(D^2d, Dd, x, t)$ if and only if for all $x_0 \in \mathbb{R}^N$, $t \in (a, b)$, $r > 0$, $\alpha > 0$ and for all smooth functions $\phi : \mathbb{R}^N \to \mathbb{R}$ such that $\{x \in \mathbb{R}^N : \phi(x) \geqq 0\} \subset \Omega_t \cap B_r(x_0)$, or $\{x \in \mathbb{R}^N : \phi(x) \leqq 0\} \subset \bar{\Omega}_t^c \cap B_r(x_0),$) with $|D\phi| \neq 0$ on $\{x \in \mathbb{R}^N : \phi(x) = 0\}$, there exists $h_0 > 0$ depending only on α and ϕ through its C^4-norm in $\bar{B}_r(x_0)$ such that, for all $h \in (0, h_0)$,

$$\{x \in \mathbb{R}^N : \phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega_{t+h}$$

or

$$\{x \in \mathbb{R}^N : \phi(x) - h[F_*(D^2\phi(x), D\phi(x), x, t) - \alpha] < 0\} \cap \bar{B}_r(x_0) \subset \bar{\Omega}^c_{t+h}).$$

A family $(\Omega_t)_{t\in(a,b)}$ of open subsets of $\mathbb{R}^N$ is called a *generalized flow with normal velocity* $-F(D^2d, Dd, x, t)$ if it is both a sub- and super-flow.

The next remark emphasizes an observation, which plays an important role in this paper, stemming from our discussion above.

Remark 2.4. In view of Theorems 2.1, 2.2 and 2.3, any motion with prescribed normal velocity which satisfies (A1)–(A3), locality, regularity and monotonicity being the most important assumptions, reduces to a generalized evolution as in Definition 2.2, i.e., the normal velocity must depend only on (x, t), n and Dn and must satisfy (2.3).

It is also worth remarking that the quantities

$$\phi(x) - hF^*(D^2\phi(x), D\phi(x), x, t), \quad \phi(x) - hF_*(D^2\phi(x), D\phi(x), x, t),$$

which appear in the definition, can be seen as the Euler approximation for solving the partial differential equation

$$u_t + F(D^2u, Du, x, t) = 0 \quad \text{in } \mathbb{R}^N \times (a, b),$$

with the initial datum

$$u = \phi \quad \text{on } \mathbb{R}^N \times \{a\}.$$

Such equations, with F of the special form we are considering here, arise in the so-called level-set approach to define weak motions of hypersurfaces with normal velocity $-F$. We briefly present the level-set approach in the next subsection and show the connections with our approach. Here we just want to point out that our definition mixes ideas coming from the level-set approach and also from a more geometric point of view, as it incorporates properties like (A2), inspired by what is known as avoidance and inclusion properties in the literature. We refer to ILMANEN [Il] where properties of this type were remarked for the motion by mean curvature without, however, being put forward as possible definition for such motions. The notion of generalized flow we introduce here is closely related to the notion of barriers introduced by DE GIORGI [DG], who proposed the use of some geometric maximum principle-type ideas to define the propagation of manifolds in $\mathbb{R}^N$ (see BELLETTINI & PAOLINI [BP] which expands on these ideas).

We conclude this part of Section 2 with the remark that with the appropriate modifications the notion of the generalized flow introduced earlier can also be used to study propagating fronts in bounded domains with appropriate boundary conditions. Although we will use this fact later in the paper, we do not expand more on it, since this is more or less a straightforward adaptation of the above.

2.2. The Generalized Level-Set Evolution

We begin with a brief description of the classical derivation of the level set approach. To this end let $(\Gamma_t)_{t\in(a,b)}$ be a collection of smooth hypersurfaces moving with normal velocity $-F$ and let $(D_t)_{t\in(a,b)}$ be a collection of smooth open subsets of $\mathbb{R}^N$ such that $\Gamma_t = \partial D_t$. Assume that $u : \mathbb{R}^N \times (a, b) \to \mathbb{R}$ is a C^∞-function such that

$$D_t = \{x \in \mathbb{R}^N : u(x,t) > 0\},\ \Gamma_t = \{x \in \mathbb{R}^N : u(x,t) = 0\},$$
$$|Du| \neq 0 \quad \text{on } \textstyle\bigcup_{t\in(a,b)} \Gamma_t \times \{t\}.$$

A straightforward computation (see for example [ES]) yields, under the additional assumption that all the smooth level sets of u move with the same normal velocity, that u must satisfy the partial differential equation

$$u_t + F(D^2u, Du, x, t) = 0 \quad \text{in } \mathbb{R}^N \times (a,b), \tag{2.8}$$

where $F : \mathcal{S}^N \times (\mathbb{R}^N \backslash \{0\}) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is related to the normal velocity by (2.2).

To justify and extend this approach to the case of non-smooth motions, one has to use the notion of viscosity solutions for fully nonlinear elliptic and parabolic partial differential equations. This theory provides the existence and uniqueness of viscosity solutions of (2.8) under rather general assumptions. We refer to [ES, CGG, BSS, IS and G] for such results and to the "User's Guide" by CRANDALL, ISHII & LIONS [CIL] for a general overview of the theory of viscosity solutions.

The level-set approach can then be described in the following way. Let $\mathcal{E}$ be the collection of triplets (Γ, D^+, D^-) of mutual disjoint subsets of $\mathbb{R}^N$ such that Γ is closed and $D^\pm$ is open and $\mathbb{R}^N = \Gamma \cup D^+ \cup D^-$. For any $(\Gamma_0, D_0^+, D_0^-) \in \mathcal{E}$, first choose $u_0 \in BUC(\mathbb{R}^N)$, the space of bounded uniformly continuous functions defined on $\mathbb{R}^N$, so that

$$D_0^+ = \{x \in \mathbb{R}^N : u_0(x) > 0\},\ \ D_0^- = \{x \in \mathbb{R}^N : u_0(x) < 0\},$$
$$\Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\},$$

and then consider the initial-value problem

$$\text{(2.9)} \qquad \begin{aligned} &\text{(i)} \quad u_t + F(D^2u, Du, x, t) = 0 \quad &&\text{in } \mathbb{R}^N \times [0, \infty), \\ &\text{(ii)} \quad u = u_0 &&\text{on } \mathbb{R}^N \times \{0\}, \end{aligned}$$

where F is defined by (2.2). In view of the existence and uniqueness theory of [CGG, ES, BSS, IS, G], etc., under general assumptions on F, there exists, for every $u_0 \in UC(\mathbb{R}^N)$, a unique viscosity solution u of (2.9) in $UC(\mathbb{R}^N \times [0, T))$ for all $T > 0$.

Finally set

$$\Gamma_t = \{x \in \mathbb{R}^N : u(x, t) = 0\}\ ,$$
$$D_t^+ = \{x \in \mathbb{R}^N : u(x, t) > 0\}, \quad D_t^- = \{x \in \mathbb{R}^N : u(x, t) < 0\}.$$

Since F is geometric, i.e., it satisfies, for all $(X, p, x, t) \in \mathscr{S}^N \times (\mathbb{R}^N \backslash \{0\}) \times \mathbb{R}^N \times (0, \infty)$, $\mu \in \mathbb{R}$ and $\lambda > 0$,

$$\text{(2.10)} \qquad F(\lambda X + \mu p \otimes p, \lambda p, x, t) = \lambda F(X, p, x, t),$$

the collection $\{(\Gamma_t, D_t^+, D_t^-)\}_{t \geqq 0} \subset \mathscr{E}$ is uniquely determined, independently of the choice of u_0, by the initial triplet (Γ_0, D_0^+, D_0^-). We recall that the main consequence of the property (2.10) is that the partial differential equation in (2.9) is invariant under the changes $u \to \varphi(u)$ for all nondecreasing functions $\varphi : \mathbb{R} \to \mathbb{R}$.

Next, for each $t > 0$ we define the mapping $E_t : \mathscr{E} \to \mathscr{E}$ by

$$E_t(\Gamma_0, D_0^+, D_0^-) = (\Gamma_t, D_t^+, D_t^-),$$

and notice that $\{E_t\}_{t \geqq 0}$ satisfies the properties $E_0 = \mathrm{id}_{\mathscr{E}}$ and $E_{t+s} = E_t \circ E_s$ for all $t, s \geqq 0$ (see, for example, [ES, CGG, IS, G]).

Definition 2.3. (i) The collection $\{E_t\}_{t \geqq 0}$ is called the *generalized level-set evolution with normal velocity* $-F$.

(ii) Given $(\Gamma_0, D_0^+, D_0^-) \in \mathscr{E}$, the collection $\{\Gamma_t\}_{t \geqq 0}$ of closed sets is called the *generalized level-set front propagation of* Γ_0 *with normal velocity* $-F$.

Notice that the level set propagation is determined not only by Γ_0 but also by the choice of D_0^+ and D_0^-, which corresponds to fixing an orientation for the normal to Γ_0. In particular, the evolution differs, in general, if D_0^+ and D_0^- are interchanged.

The properties of the generalized level evolution have been the object of extended study. One of the most intriguing issues is whether the so-called *no-interior condition* holds, i.e., whether the set Γ_t does not develop an interior. We say that the no-interior condition holds if and only if

(2.11) $\{x,t) : u(x,t) = 0\} = \partial\{(x,t) : u(x,t) > 0\} = \partial\{(x,t) : u(x,t) < 0\}.$

It turns out that there are general geometric conditions on Γ_0 yielding (2.11) (see [BSS]) for such conditions as well as examples where (2.11) fails. [BSS] also considered the issue of the existence and uniqueness of discontinuous solutions to (2.9) with $u_0 = \mathbb{1}_{D_0^+} - \mathbb{1}_{D_0^-}$, where, if A is a subset of some $\mathbb{R}^k$, $\mathbb{1}_A$ denotes the characteristic function of A, i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \in A^c$.

Notice that in view of (2.10) one can expect that any such solution only takes the values $+1$ and -1. In fact, this is true if and only if (2.11) holds (see [BSS]). Since this plays some role in our analysis below, we state the relevant result of [BSS] in

Proposition 2.1. (i) *The initial-value problem* (2.9) *with* $u_0 = \mathbb{1}_{D_0^+} - \mathbb{1}_{D_0^-}$ *has a unique discontinuous solution if and only if the no interior condition* (2.11) *holds.*

(ii) *If* (2.11) *fails and* $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ *is a discontinuous solution of* (2.9)*, then, for all* $t > 0$,

$$D_t^{\pm} \subset \{x \in \mathbb{R}^N : u(x,t) = \pm 1\} \subset D_t^{\pm} \cup \Gamma_t,$$

where $(\Gamma_t, D_t^+, D_t^-) = E_t(\Gamma_0, D_0^+, D_0^-)$.

2.3. *The Relationship Between the Generalized Flow and Level-Set Evolution*

The next theorem is the main result of this section. As before, to simplify the presentation we assume that (A3) holds, i.e., that v and, hence, F are locally bounded. This assumption can be removed using Remark 2.2 and the results of [IS], which treats unbounded F's by changing the class of allowed test functions.

Theorem 2.4. *Assume that* (A1)–(A3) *hold. A family* $\{\Omega_t\}_{t\in[0,T]}$ *of open subsets of* $\mathbb{R}^N$ *is a generalized flow or a super-flow or sub-flow with normal velocity* v *if and only if the function* $\chi = \mathbb{1}_{\Omega_t} - \mathbb{1}_{\bar{\Omega}_t^c}$ *is respectively a viscosity solution or a super-solution or a sub-solution of* (2.9)(i) *in* $\mathbb{R}^N \times (0,\infty)$.

Before we present the proof of Theorem 2.4 we state in the next proposition the relationship between the generalized flow and the level-set evolution. Since its proof is immediate from Proposition 2.1 and Theorem 2.4, we omit it.

Proposition 2.2. (i) *Let* $\{\Omega_t\}_{t\in[0,T]}$ *be a generalized flow with normal velocity* v*, let* $(\Gamma_t, D_t^+, D_t^-)_{t\in[0,T]}$ *be the generalized level-set evolution of* (Γ_0, D_0^+, D_0^-)*, where* $D_0^+ = \Omega_0$ *and* $D_0^- = \bar{\Omega}_0^c$*, and assume that the no-interior condition* (2.11) *holds. Then, for all* $t > 0$,

$$\Omega_t = D_t^+.$$

(ii) *If the no-interior condition* (2.11) *fails, then, for all* $t > 0$,

$$D_t^+ \subset \Omega_t \subset D_t^+ \cup \Gamma_t.$$

We now continue with the

Proof of Theorem 2.4. 1. The fact that, if $\chi = 1\!\!1_{\Omega_t} - 1\!\!1_{\bar{\Omega}_t^c}$ is a solution or a sub-solution or super-solution of (2.9)(i), then $\{\Omega_t\}_{t\in[0,T]}$ is respectively a generalized flow or a sub-flow or a super-flow with normal velocity v, is an immediate consequence of the definition and the comparison properties of (2.9). We leave the details to the reader.

2. Next we show that if $\{\Omega_t\}_{t\in[0,T]}$ is a generalized super-flow, then χ is a super-solution of (2.9)(i). The case of the generalized sub-flow is studied similarly.

3. Let $(x,t) \in \mathbb{R}^N \times (0,T)$ be a strict local minimum point of $\chi_* - \varphi$ where $\varphi \in C^4(\mathbb{R}^N \times [0,T])$. Changing if necessary φ to $\varphi - \varphi(x,t)$ we may assume that $\varphi(x,t) = 0$. Moreover, since χ is bounded, we may also assume that φ is bounded. We then need to show the inequality

$$\varphi_t(x,t) + F^*(D^2\varphi(x,t), D\varphi(x,t), x, t) \geqq 0. \tag{2.12}$$

This inequality is obvious if (x,t) is in the interior of either $\{\chi_* = 1\}$ or $\{\chi_* = -1\}$. Indeed, in both cases, χ_* is constant in a neighborhood of (x,t). Hence

$$\varphi_t(x,t) = 0, \quad D\varphi(x,t) = 0, \quad D^2\varphi(x,t) \geqq 0,$$

and (2.12) follows, since, in view of (2.10), we have

$$F^*(0,0,x,t) \geqq 0.$$

4. Assume that $(x,t) \in \partial(\{\chi_* = 1\} \cup \{\chi_* = -1\})$. The lower semicontinuity of χ_* yields

$$\chi_*(x,t) = -1.$$

Since (x,t) is a strict local minimum point of $\chi_* - \varphi$, there exists some $r > 0$ such that, if $0 < |y-x| + |t-s| < 2r$, then

$$\chi_*(x,t) - \varphi(x,t) = -1 < \chi_*(y,s) - \varphi(y,s).$$

Hence, if $0 < |y-x| + |t-s| < 2r$, then

$$-1 + \varphi(y,s) < \chi_*(y,s).$$

It follows that $\chi_*(y,s) = 1$ if $\varphi(y,s) > 0$ and $(y,s) \neq (x,t)$, since χ_* takes only the values -1 and 1. For any $h \in (0,r)$ this implies that

$$\{y : \varphi(y,t-h) \geqq 0\} \cap B_r(x) \subset \Omega_{t-h}.$$

5. Next consider the case $|D\varphi(x,t)| \neq 0$ and introduce, for $k > 0$, the functions $\varphi_k : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ defined by

$$\varphi_k(y,s) = \varphi(y,s) - k|x-y|^4.$$

All the previous arguments hold true for φ_k. Moreover, in view of Lemma 2.1 and the smoothness of φ, there exist $\bar{k} > 0$ and $\bar{h} > 0$ such that, for all $h \in (0,\bar{h})$,

$$
(2.13)\qquad \begin{aligned} &\text{(i)} \quad \{y : \varphi_{\bar{k}}(y, t-h) \geqq 0\} \subset \Omega_{t-h} \cap B_r(x), \\ &\text{(ii)} \quad |D\varphi_{\bar{k}}(y, t-h)| \neq 0 \qquad \text{on } \{y : \varphi_{\bar{k}}(y, t-h) = 0\} \end{aligned}
$$

6. Since the family $\{\Omega_t\}_{t\in[0,T]}$ is a generalized super-flow with normal velocity v and the C^4-norms of the functions $\varphi_{\bar{k}}(\cdot, t-h)$ are uniformly bounded in $\bar{B}_r(x)$ for h small, it follows that for $\alpha > 0$ sufficiently small, we have

$$
\begin{aligned} \{y : \varphi_{\bar{k}}(y, t-h) - h[F^*(D^2\varphi_{\bar{k}}(y, t-h), \\ D\varphi_{\bar{k}}(y, t-h), y, t-h) + \alpha] > 0\} \cap \bar{B}_r(x) \subset \Omega_t. \end{aligned}
$$

In particular, since $x \notin \Omega_t$,

$$
\varphi_{\bar{k}}(x, t-h) - h[F^*(D^2\varphi_{\bar{k}}(x, t-h), D\varphi_{\bar{k}}(x, t-h), x, t-h) + \alpha] \leqq 0.
$$

Next recall that $\varphi(x, t) = 0$ and, therefore, $\varphi_{\bar{k}}(x, t) = 0$. Dividing the last inequality by h, letting $h \to 0$ and using the lower semicontinuity of $-F^*$, we obtain

$$
-\varphi_{\bar{k},t}(x, t) - [F^*(D^2\varphi_{\bar{k}}(x, t), D\varphi_{\bar{k}}(x, t), x, t) + \alpha] \leqq 0.
$$

But

$$
\varphi_{\bar{k},t}(x, t) = \varphi_t(x, t), \quad D\varphi_{\bar{k}}(x, t) = D\varphi(x, t), \quad D^2\varphi_{\bar{k}}(x, t) = D^2\varphi(x, t).
$$

Letting $\alpha \to 0$ yields (2.12).

7. If $|D\varphi(x, t)| = 0$, we may assume without any loss of generality that $D^2\varphi(x, t) = 0$ (see, for example, BARLES & GEORGELIN [BG]). Since $F^*(0, 0, x, t) \geqq 0$, to conclude we only need to show that

$$
\varphi_t(x, t) \geqq 0.
$$

Using once again the facts that (x, t) is a local minimum point of $\chi_* - \varphi$ and that $\varphi(x, t) = 0$, $D\varphi(x, t) = 0$ and $D^2\varphi(x, t) = 0$, we have, if $|y - x| < r$,

$$
-1 + \varphi(x, t) - h\phi_t(x, t) + O(|y - x|^3) + o(h) \leqq \chi_*(y, t-h).
$$

8. If there exists a sequence $(y_n, t-h_n) \to (x, t)$ such that $\chi_*(y_n, t-h_n) = -1$ and $|x - y_n|^3 = o(h_n)$, then the proof is complete.

9. If not, for all $C > 0$, there exists $h_0 > 0$ such that, for all $h \in (0, h_0)$ and $|y - x|^2 \leqq Ch$,

$$
\chi_*(y, t-h) = 1.
$$

10. Consider the function $\phi(y) = Ch - |y - x|^2$. It is clear that, for h small enough,

$$
\{y : \phi(y) \geqq 0\} \subset \Omega_{t-h} \cap B_r(x), \quad |D\phi(y)| \neq 0 \quad \text{on } \{y : \phi(y) = 0\}.
$$

That the family $\{\Omega_t\}_{t\in[0,T]}$ is a generalized super-flow with normal velocity v yields that there exists h_0, depending only on α and on ϕ through its C^4-norm in $\bar{B}_r(x)$ (which is independent of h if, say, $h \leqq 1$), such that, for $h < h_0$,

$$
\{y : \phi(y) - h[F^*(D^2\phi(y), D\phi(y), y, t) + \alpha] > 0\} \cap \bar{B}_r(x) \subset \Omega_t.
$$

But $x \notin \Omega_t$. Hence

$$\phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] \leqq 0,$$

i.e.,

$$h\left[C - F^*(-2I, 0, x, t) - \alpha\right] \leqq 0,$$

which is a contradiction if C is large enough, since F is locally bounded. □

3. The Abstract Method

In most of the asymptotic problems we have in mind, we are given a family $(u_\varepsilon)_{\varepsilon>0}$ of functions bounded in $\mathbb{R}^N \times (0, T)$, uniformly in ε, typically the solutions of a reaction-diffusion equation with a small parameter ε or the total magnetization of a stochastic system with interaction range ε^{-1}, etc. The goal is to prove that there exists a generalized flow $(\Omega_t)_{t\geqq 0}$ with normal velocity v determined by the problem such that, as $\varepsilon \to 0$,

$$u_\varepsilon(x, t) \to b \quad \text{if} \quad (x, t) \in \Omega = \bigcup_{t\in(0,T)} \Omega_t \times \{t\},$$

$$u_\varepsilon(x, t) \to a \quad \text{if } (x, t) \in \bar{\Omega}^c,$$

where $a, b \in \mathbb{R}$ are equilibrium states of this system.

In this section we present an abstract formulation of a new general method for proving such results. In the next sections, we will show how this method applies to several concrete examples.

We assume that for all $\varepsilon > 0$, $t \geqq 0$ and $h > 0$, the family $(u_\varepsilon)_{\varepsilon>0}$ satisfies the following properties, where $B(\mathbb{R}^N)$ denotes the set of real-valued bounded functions on $\mathbb{R}^N$:

(H1) Causality. *There exists a family of maps* $S^\varepsilon_{t,t+h} : B(\mathbb{R}^N) \to B(\mathbb{R}^N)$, *such that*

$$u_\varepsilon(\cdot, t+h) = S^\varepsilon_{t,t+h} u_\varepsilon(\cdot, t) \quad \text{in } \mathbb{R}^N.$$

(H2) Monotonicity. *For all functions* $u, v \in B(\mathbb{R}^N)$,

$$\text{if } u \leqq v \text{ in } \mathbb{R}^N, \text{ then } S^\varepsilon_{t,t+h} u \leqq S^\varepsilon_{t,t+h} v.$$

(H3) Existence of equilibria. *There exist* $a_\varepsilon, b_\varepsilon, a, b \in \mathbb{R}$ *such that*

$$a < b, \quad S^\varepsilon_{t,t+h} a_\varepsilon = a_\varepsilon, \quad S^\varepsilon_{t,t+h} b_\varepsilon = b_\varepsilon, \quad a_\varepsilon \leqq u_\varepsilon \leqq b_\varepsilon \quad \text{in } \mathbb{R}^N \times \{0\},$$

and, as $\varepsilon \to 0$,

$$a_\varepsilon \to a, \qquad b_\varepsilon \to b.$$

(H4) Consistency. *There exists a locally bounded function* $F : \mathscr{S}^N \times \mathbb{R}^N \backslash \{0\} \times \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ *such that*

(i) *For all* $(x_0, t) \in \mathbb{R}^N \times [0, T)$, $r > 0$, $\alpha > 0$ *and for all smooth functions* $\phi : \mathbb{R}^N \to \mathbb{R}$ *such that* $\{x : \phi(x) \geqq 0\} \subset B_r(x_0)$ *and* $|D\phi(x)| \neq 0$ *on* $\{x : \phi(x) = 0\}$, *there exists* $\bar{\delta} > 0$ *and* $h_0 > 0$, h_0 *depending only on* α *and* ϕ *through its* C^4*-norm in* $\bar{B}_r(x_0)$*, such that, if* $\delta \leqq \bar{\delta}$, $x \in B_r(x_0)$ *and* $\phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] > 0$ *for* $h \leqq h_0$*, then*

$$\liminf_* (S^\varepsilon_{t,t+h}[(b_\varepsilon - \delta)\mathbb{1}_{\{\phi \geqq 0\}} + a_\varepsilon \mathbb{1}_{\{\phi < 0\}}])(x) = b.$$

(ii) *For all* $(x_0, t) \in \mathbb{R}^N \times [0, T)$, $r > 0$, $\alpha > 0$ *and for all smooth functions* $\phi : \mathbb{R}^N \to \mathbb{R}$ *such that* $\{x : \phi(x) \leqq 0\} \subset B_r(x_0)$ *and* $|D\phi(x)| \neq 0$ *on* $\{x : \phi(x) = 0\}$*, there exists* $\bar{\delta}' > 0$ *and* $h'_0 > 0$, h'_0 *depending only on* α *and* ϕ *through its* C^4*-norm in* $\bar{B}_r(x_0)$ *such that, if* $\delta' \leqq \bar{\delta}'$, $x \in B_r(x_0)$ *and* $\phi(x) - h[F_*(D^2\phi(x), D\phi(x), x, t) - \alpha] < 0$ *for* $h \leqq h'_0$*, then*

$$\limsup{}^* (S^\varepsilon_{t,t+h}[b\mathbb{1}_{\{\phi > 0\}} + (a_\varepsilon + \delta')\mathbb{1}_{\{\phi \leqq 0\}}])(x) = a.$$

We recall that if $f^\varepsilon : A \to \mathbb{R}$ is a family of uniformly bounded functions, then

$$\liminf_* f^\varepsilon(y) = \liminf_{\substack{z \to y \\ \varepsilon \to 0}} f^\varepsilon(z), \qquad \limsup{}^* f^\varepsilon(y) = \limsup_{\substack{z \to y \\ \varepsilon \to 0}} f^\varepsilon(z).$$

The result is

Theorem 3.1. *Assume that* (H1)–(H4) *hold for all* $\varepsilon > 0$, $t \geqq 0$ *and* $h > 0$*. For* $t > 0$*, set*

$$\Omega^1_t = \{x \in \mathbb{R}^N \colon \liminf_* u_\varepsilon(x,t) = b\}, \quad \Omega^2_t = \{x \in \mathbb{R}^N \colon \limsup{}^* u_\varepsilon(x,t) = a\},$$

and define Ω^1_0 *and* Ω^2_0 *by*

$$\Omega^1_0 = \bigcap_{t>0} \Big(\bigcup_{0<h\leqq t} \Omega^1_t \Big) \quad \textit{and} \quad \Omega^2_0 = \bigcap_{t>0} \Big(\bigcup_{0<h\leqq t} \Omega^2_t \Big).$$

If Ω^1_0 *or* Ω^2_0 *respectively is not empty, then* Ω^1_t *or* Ω^2_t *is not empty for sufficiently small* $t \geqq 0$*, and the family* $(\Omega^1_t)_{t \geqq 0}$ *or* $\big((\bar{\Omega}^2_t)^c\big)_{t \geqq 0}$ *is a generalized super-flow or sub-flow with normal velocity* $-F$*. Moreover,* $\overline{F}$ *is degenerate elliptic, i.e., it satisfies* (2.3).

An immediate consequence of Theorem 3.1 is

Corollary 3.1. *Assume that the hypotheses of Theorem 3.1 hold and, in addition, that* $\Omega^1_0 = (\bar{\Omega}^2_0)^c$*. Let* $(\Gamma_t, \Omega^+_t, \Omega^-_t)_{t \geqq 0}$ *be the generalized level set evolution of* $(\partial\Omega^1_0, \Omega^1_0, \Omega^2_0)$ *with normal velocity* $-F$*.*

(i) *Then, for all* $t \geqq 0$,

$$\Omega^+_t \subset \Omega^1_t \subset \Omega^+_t \cup \Gamma_t, \quad \Omega^-_t \subset \Omega^2_t \subset \Omega^-_t \cup \Gamma_t.$$

(ii) *If the no-interior condition* (2.11) *holds, then, for all* $t \geqq 0$,

$$\Omega^1_t = \Omega^+_t, \quad \Omega^2_t = \Omega^-_t.$$

Before we present any proofs, we discuss the meaning of (H1)–(H4) as well as what is involved in checking them. In most of the examples, (H1) is satisfied by definition as a consequence of a semi-group-type property, while (H2) follows, in general, from a maximum principle-type property and (H3) follows from the structure of the problem. Checking (H4) is in general the only difficult step.

Verifying (H4) consists in proving a result similar to the one we want to derive for u_ε but only for smooth data (as smooth as we want), for compact smooth fronts and for small time (as small as we want). It is clear enough that these properties are a priori far easier to obtain—although they are not completely trivial—than those for the general case for u_ε. This reduction to an easier case is the main contribution of this new method.

The additional assumption that Ω_0^1 or Ω_0^2 is not empty, which is used to initialize the moving front, is checked, in general, exactly in the same way as (H4), i.e., by studying the small-time behavior of solutions which, locally in time, generate a smooth front. Finally, the reason for giving different definitions for Ω_0^1 and Ω_0^2 is to take into account the boundary layer which may occur at $t = 0$.

The intuitive idea here is that the map S^ε defines some approximation to a certain flow on sets and that (H4) amounts to finding the generator of this flow. It is in this context that this approach resembles, as mentioned in the Introduction, the formulation introduced in [BS] to study convergence of numerical schemes.

We now present the

Proof of Corollary 3.1. Theorem 3.1 yields that the families $(\Omega_t^1)_{t\geqq 0}$ and $(\Omega_t^2)_{t\geqq 0}$ are generalized super- and sub-flows with normal velocity $-F$. The conclusion now follows from Proposition 2.1, in view of the assumption on Ω_0^1 and Ω_0^2. □

Now we turn to the

Proof of Theorem 3.1. 1. That F satisfies (2.3) follows, once the rest of the theorem is proved, as in Theorem 2.3. The key point here is that the fact that (H2) holds yields for the evolving fronts a monotonicity property like (A2).

2. We only prove the result for $(\Omega_t^1)_{t\geqq 0}$, the one for $(\Omega_t^2)_{t\geqq 0}$ following similarly.

3. Let $(x_0, t) \in \mathbb{R}^N \times (0, T)$, $r > 0$ and a smooth function $\phi : \mathbb{R}^N \to \mathbb{R}$ be such that $\{x : \phi(x) \geqq 0\} \subset \Omega_t^1 \cap B_r(x_0)$ and $|D\phi(x)| \neq 0$ on $\{x : \phi(x) = 0\}$. We need to show that, for all sufficiently small $\alpha > 0$, there exists $h_0 > 0$ depending only on α, r and the C^4-norm of ϕ in $\bar{B}_r(x_0)$ such that, for all $0 < h < h_0$ and for α small enough,

$$\{x : \phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega_{t+h}.$$

Since

$$a_\varepsilon = S^\varepsilon_{0,t} a_\varepsilon \leqq S^\varepsilon_{0,t} u_\varepsilon(\cdot, 0) = u_\varepsilon(\cdot, t) \leqq S^\varepsilon_{0,t} b_\varepsilon = b_\varepsilon \quad \text{in } \mathbb{R}^N,$$

assumption (H2) yields

$$a_\varepsilon \leqq u_\varepsilon \leqq b_\varepsilon \quad \text{in } \mathbb{R}^N \times [0, T],$$

and, hence,

$$\limsup^* u_\varepsilon \leqq b \quad \text{in } \mathbb{R}^N \times [0, T].$$

4. Classical arguments from the theory of viscosity solutions yield that, as $\varepsilon \to 0$, $u_\varepsilon \to b$, locally uniformly, in $\bigcup_{t>0} \Omega_t^1 \times \{t\}$. Therefore, for ε small enough and δ as in (H4), we have

$$u_\varepsilon(\cdot, t) \geqq b_\varepsilon - \delta \quad \text{on } \{x : \phi(x) \geqq 0\} \subset \Omega_t^1.$$

Hence

$$u_\varepsilon(\cdot, t) \geqq (b_\varepsilon - \delta)\mathbb{1}_{\{\phi \geqq 0\}} + a_\varepsilon \mathbb{1}_{\{\phi < 0\}} \quad \text{in } \mathbb{R}^N,$$

and, by (H2),

$$S_{t,t+h}^\varepsilon u_\varepsilon(\cdot, t) \geqq S_{t,t+h}^\varepsilon[(b_\varepsilon - \delta)\mathbb{1}_{\{\phi \geqq 0\}} + a_\varepsilon \mathbb{1}_{\{\phi < 0\}}] \quad \text{in } \mathbb{R}^N.$$

5. Choose $\alpha > 0$ and apply (H4). There exist $\bar{\delta} > 0$ and $h_0 > 0$ such that, if $\delta \leqq \bar{\delta}$, $0 < h \leqq h_0$, $x \in B_r(x_0)$ and

$$\phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] > 0,$$

then

$$\begin{aligned} \liminf_* u_\varepsilon(x, t+h) &= \liminf_* S_{t,t+h}^\varepsilon u_\varepsilon(\cdot, t)(x) \\ &\geqq \liminf_* (S_{t,t+h}^\varepsilon[(b_\varepsilon - \delta)\mathbb{1}_{\{\phi \geqq 0\}} + a_\varepsilon \mathbb{1}_{\{\phi < 0\}}])(x) = b. \end{aligned}$$

This yields that, for any $0 < h \leqq h_0$, where h_0 depends only on α and ϕ through its C^4-norm on $\bar{B}_r(x_0)$,

$$\{\phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] > 0\} \cap \bar{B}_r(x_0) \subset \Omega_{t+h}^1.$$

The proof is now complete. □

In view of the discussion at the end of Subsection 2.2, it is clear that in the definition of the generalized sub- and super-flows, we may replace "if $x \in B_r(x_0)$ and $\phi(x) - h[F^*(D^2\phi(x), D\phi(x), x, t) + \alpha] > 0$ for some $\alpha > 0$" and "if $x \in B_r(x_0)$ and $\phi(x) - h[F_*(D^2\phi(x), D\phi(x), x, t) - \alpha] < 0$ for some $\alpha > 0$" in (H4), respectively, by "if $x \in B_r(x_0)$ and $\phi_\alpha^+(x, t) > 0$" and "if $x \in B_r(x_0)$ and $\phi_\alpha^-(x, t) < 0$" where ϕ_α^+ and ϕ_α^- are solutions of

$$\begin{aligned} \phi_{\alpha,t}^\pm + F(D^2\phi_\alpha^\pm, D\phi_\alpha^\pm, x, t) \pm \alpha|D\phi_\alpha^\pm| = 0 \quad &\text{in } \mathbb{R}^N \times (0, T), \\ \phi_\alpha^\pm = \phi \quad &\text{on } \mathbb{R}^N \times \{0\}. \end{aligned}$$

Since we may assume without loss of generality that ϕ is uniformly continuous in $\mathbb{R}^N$, these equations have a unique solution under general assumptions on F (see, for example, [CGG, BSS, IS, G], etc.).

4. The Asymptotics of Semilinear Reaction-Diffusion Equations

One of the most striking applications of the theory of the generalized front propagation is the rigorous study of the asymptotics of solutions to semilinear reaction-diffusion equations, which has been done in [ESS] and [BSS]. A canonical example of such a problem is the study of the asymptotics of the reaction-diffusion equation

$$u_t - \Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{4.1}$$

where $f = W'$, W being a double-wells potential. A special case of (4.1) is the so-called Allen-Cahn equation which corresponds to the choice of

$$f(u) = 2u(u^2 - 1) \qquad (u \in \mathbb{R}). \tag{4.2}$$

The Allen-Cahn equation was introduced in [AC] to model the motion of the sharp interface—the antiphase boundary—between regions of different phases of a material. The conjecture of [AC], which was proved rigorously and for all times in [ESS] (see also [BSS]), was that, if the wells of W have equal depth, then the asymptotics of

$$u_t^\varepsilon - \Delta u^\varepsilon + \varepsilon^{-2} f(u^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{4.3}$$

which is obtained from (4.1) by scaling (x, t) to $(\varepsilon^{-1}x, \varepsilon^{-2}t)$, are controlled by the (generalized) mean curvature flow. Here we revisit this result by presenting another proof based on Theorem 3.1. Of course the main issue is to verify (H4).

Checking (H4) can be done in a number of different ways. The first possibility is to use already existing results on the small-time behavior of the reaction-diffusion equation for smooth data. In this case the asymptotics of (4.3) can be treated in a straightforward way using the results of Chen [C]. As we already mentioned, the fact that the method we propose transforms results on the small-time behavior of the solutions of reaction-diffusion equations for smooth data into complete results is one of its main interests.

The second possibility to check the consistency requirement is to follow ideas introduced in [ESS] (see also [BSS]) to build suitable sub- and super-solutions to (4.3) using the travelling-wave solution of (4.1) and the distance function to the moving front. Our method simplifies this approach since we have to build such sub- and super-solutions only for small time using the smooth distance function.

Here we want to describe a third possibility, which, although similar in spirit to the second one, avoids the use of the distance function and allows us to treat the more complicated problems we present later in this paper. This approach closely follows the formal asymptotic analysis of Keller, Rubinstein & Sternberg [KRS] to study (4.3). As a matter of fact, the power of the method introduced here is that we can make all these formal asymptotics rigorous.

In order to emphasize the main new ideas, we concentrate on (4.3) although our arguments work for second-order operators more general than the Laplacian Δ and for nonlinearities which also depend on (x, t). We refer to [BSS] for such results.

As far as the reaction term $f : \mathbb{R} \to \mathbb{R}$ goes, throughout the paper, we assume that

$$\begin{aligned} &f \in C^2(\mathbb{R}) \text{ has exactly three zeroes } m_- < m_0 < m_+,\\ &f(s) > 0 \text{ in } (m_-, m_0), \quad f(s) < 0 \text{ in } (m_0, m_+),\\ &f'(m_\pm) > 0,\ f''(m_-) < 0, \quad f''(m_+) > 0. \end{aligned} \tag{4.4}$$

We also assume that for each $e \in S^{N-1}$, (4.1) admits travelling-wave solutions connecting m_- and m_+, i.e., solutions of the form

$$u(x,t) = q(x \cdot e - ct),$$

where $q : \mathbb{R} \to \mathbb{R}$ is such that $q(\pm\infty) = m_\pm$. Indeed, we assume that there exists a unique pair (c, q) such that

$$\begin{aligned} &c\dot{q} + \ddot{q} = f(q) \text{ on } \mathbb{R}, \quad \dot{q} > 0 \text{ on } \mathbb{R}, \quad q(0) = m_0,\\ &q(s) \to \pm m_\pm \text{ exponentially fast as } s \to \pm\infty,\\ &\sup_{s\in\mathbb{R}}[(1+|s|)\dot{q}(s) + (|s| + s^2)|\ddot{q}|] < \infty. \end{aligned} \tag{4.5}$$

The existence and the properties of such pairs (c, q) are studied, for example, in ARONSON & WEINBERGER [AW], where we refer for details.

In the case where the wells of the potential W have the same depth, i.e.,

$$W(m_+) - W(m_-) = 0, \tag{4.6}$$

it follows that $c = 0$ in (4.5) and q satisfies

$$\ddot{q} = f(q) \quad \text{in } \mathbb{R}. \tag{4.7}$$

(Recall that $W : \mathbb{R} \to \mathbb{R}$ is such that $f = W'$.)

Linearizing around q the equation satisfied by the travelling wave leads to the unbounded, self-adjoint operator $\mathscr{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\mathscr{A}p = c\dot{p} + \ddot{p} - f'(q)p.$$

Moreover a straightforward computation shows that $\dot{q} \in \ker \mathscr{A} = \ker \mathscr{A}^*$. In the sequel we assume that

$$\ker \mathscr{A}^* = \dot{q}\mathbb{R}. \tag{4.8}$$

In our analysis below we will need to solve, for appropriate functions $\chi : \mathbb{R} \times \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$, the equation

$$\mathscr{A}p = \chi \quad \text{on } \mathbb{R}. \tag{4.9}$$

We assume that for any compact subset of K of $\mathbb{R}^N \times [0,\infty)$, and any smooth $\chi : \mathbb{R} \times K \to \mathbb{R}$, such that, for all $(s, x, t) \in \mathbb{R} \times K$ and for some $B > 0$,

$$\int_{-\infty}^{\infty} \chi(s,x,t)\dot{q}(s)ds = 0, \quad \|\chi(s,\cdot,\cdot)\|_{C^2(K)} \leqq B\,[\dot{q}(s) + |s\ddot{q}(s)|],$$

there exists a solution $p \in C^2(\mathbb{R} \times K)$of (4.9) such that

$$\begin{aligned} &p(s) \to 0 \text{ exponentially fast as } |s| \to \infty,\\ &\sup_{s\in\mathbb{R}} \|p(s,\cdot,\cdot)\|_{C^2(K)} < \infty, \end{aligned} \tag{4.10}$$

$$\sup_{(s,x,t)\in\mathbb{R}\times K} [\dot{p}(s,x,t), |D_x\dot{p}(s,x,t)|,\ (1+|s|)|\ddot{p}|(s,x,t)] < \infty.$$

This last assumption is, of course, rather technical but also essential to our analysis. Given (4.8) it is immediate that at least the integral condition on χ is a consequence of Fredholm's alternative applied to $\mathscr{A}$.

In the special case of the Allen-Cahn equation, when f satisfies (4.2), it follows that

$$q(s) = \tanh(s)$$

and that (4.8) and (4.10) are satisfied. As a matter of fact, a straightforward calculation yields that in this case the solution p of (4.9) has the form

$$p(s, x, t) = q(s) \int_0^s (\dot{q}(\tau))^{-1} \Big(\int_{-\infty}^{\tau} \chi(\eta, x, t) \dot{q}(\eta) d\eta \Big) d\tau$$

and satisfies the conditions in (4.10).

To state the main result of this section we recall that, for all $u_0 \in UC(\mathbb{R}^N)$, the initial-value problem

$$\begin{aligned} u_t - \operatorname{tr}\big[(I - \widehat{Du} \otimes \widehat{Du}) D^2 u\big] &= 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \\ u &= u_0 \quad \text{on } \mathbb{R}^N \times \{0\}, \end{aligned} \tag{4.11}$$

admits a unique viscosity solution $u \in UC(\mathbb{R}^N \times [0, T])$ for all $T > 0$.

Theorem 4.1. *Assume that* (4.4)–(4.6), (4.8) *and* (4.10) *hold and let* u_ε *be the solution of* (4.3) *associated with the initial datum* $u_\varepsilon = g$ *on* $\mathbb{R}^N \times \{0\}$, *where* $g : \mathbb{R}^N \to [m_-, m_+]$ *is such that* $\Gamma_0 = \{x : g(x) = m_0\}$ *is a non-empty subset of* $\mathbb{R}^N$. *Then, as* $\varepsilon \to 0$,

$$u^\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} \end{Bmatrix},$$

where u *is the unique viscosity solution of* (4.11) *with* $u_0 = d_0$ *the signed distance to* Γ_0 *which is positive in the set* $\{g > m_0\}$ *and negative in the set* $\{g < m_0\}$.

If, in addition, the no-interior condition (2.11) *holds, then, as* $\varepsilon \to 0$,

$$u^\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} = \Big(\overline{\{u > 0\}}\Big)^c \end{Bmatrix}.$$

Proof. 1. In view of Theorem 3.1, we only need to check assumptions (H1)–(H4).

2. Let T^ε be the semigroup associated with (4.3). It is immediate that (H1)–(H3) are satisfied with $a = a_\varepsilon = m_-$ and $b = b_\varepsilon = m_+$.

3. Below we check only "half" of (H4), i.e., the part about the $\liminf_*$. Since the other "half", i.e., the $\limsup^*$ part, can be checked similarly, we leave the details to the reader. The proof is then complete if we prove Theorem 4.2 below.

Theorem 4.2. *For all* $x_0 \in \mathbb{R}^N$, $r > 0$, $\alpha > 0$ *and all smooth functions* $\phi : \mathbb{R}^N \to \mathbb{R}$ *such that* $\{x : \phi(x) = 0\} \subset B_r(x_0)$ *and* $|D\phi(x)| \neq 0$ *on* $\{x : \phi(x) = 0\}$, *and for all* $0 < \delta < m_+ - m_0$, *there exists* $\bar{h} > 0$ *depending only on* ϕ *through its* C^4*-norm in* $\bar{B}_r(x_0)$ *such that, for all* $h \in (0, \bar{h}]$,

$$\liminf_* \, (T_\varepsilon(h)[(m_+ - \delta)\mathbb{1}_{\{\phi \geqq 0\}} + m_- \mathbb{1}_{\{\phi<0\}}])(x) = m_+,$$

if $\phi(x) - h[F^*(D^2\phi(x), D\phi(x)) + \alpha] > 0$ *and* $x \in \bar{B}_r(x_0)$, *where*

$$F^*(X,p) = \begin{cases} -\mathrm{tr}\,\left[(I - \hat{p}\otimes\hat{p})X\right] & \text{if } |p| \neq 0, \\ -\mathrm{tr}\,(X) + \lambda_{\max} & \text{if } p = 0, \end{cases}$$

$\lambda_{\max}$ *being the largest eigenvalue of* X.

It is worth mentioning that Theorem 4.2 yields a bit more information than the consistency requirement (H4). Indeed, the assertion is valid for any $0 < \delta < m_+ - m_0$, while (H4) only requires this result to be valid for some $0 < \delta < m_+ - m_0$. This stronger formulation is in fact necessary to initialize the front.

Proving Theorem 4.2 (and of any such results in the other examples we present later) consists of two main steps. The first step, roughly speaking, initializes the front, while the second one is about its propagation.

The proof we provide below for the second step in particular is clearly far from being the simplest one we could give. We take, however, this opportunity to describe a new general argument which allows us to treat more complicated examples. We refer the reader to the end of this section where we make more comments about this point.

The two steps of the proof of Theorem 4.2 are described in the following two lemmas.

Lemma 4.1. (Initialization of the Front). *Under the assumptions of Theorem* 4.2, *for any* $\beta > 0$, *there exist a constant* $\tau > 0$ *such that, if* $t_\varepsilon = \tau\varepsilon^2|\ln\varepsilon|$, *then, for all sufficiently small* ε,

$$T^\varepsilon(t_\varepsilon)[(m_+ - \delta)\mathbb{1}_{\{\phi\geqq 0\}} + m_-\mathbb{1}_{\{\phi<0\}}] \geqq (m_+ - \beta\varepsilon)\mathbb{1}_{\{\phi\geqq\beta\}} + m_-\mathbb{1}_{\{\phi<\beta\}}.$$

This result, which is due to CHEN [C], describes the "very small-time" behavior of the solutions of (4.3), which is essentially controlled by the reaction term f. Roughly speaking, Lemma 4.1 reduces the proof of Theorem 4.2 to the case where $\delta = \beta\varepsilon$ for some sufficiently small $\beta > 0$.

The second step is

Lemma 4.2. (Propagation of the Front). *For all sufficiently small* $\alpha > 0$ *there exists* $\bar{h} > 0$, *depending only on* ϕ *through its* C^4*-norm in* $\bar{B}_r(x_0)$, *such that, if* $\beta \leqq \bar{\beta}(\alpha,\phi)$ *and* $\varepsilon \leqq \bar{\varepsilon}(\alpha,\beta,\phi)$, *then there exists a subsolution* $w^{\varepsilon,\beta}$ *of* (4.3) *in* $\mathbb{R}^N \times (0,\bar{h})$ *such that*

$$w^{\varepsilon,\beta}(\cdot,0) \leqq (m_+ - \beta\varepsilon)\mathbb{1}_{\{\phi\geqq\beta\}} + m_-\mathbb{1}_{\{\phi<\beta\}} \quad \text{in } \mathbb{R}^N.$$

Moreover, if for $(x,t) \in B_r(x_0)\times(0,\bar{h})$, $\phi(x) - t[F^*(D^2\phi(x), D\phi(x)) + \alpha] > 2\beta$, *then*

$$\liminf_* \, w^{\varepsilon,\beta}(x,t) = m_+.$$

Assuming for the moment Lemmas 4.1 and 4.2 we may proceed with the

Proof of Theorem 4.2. 1. Let $u^\varepsilon : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ be defined by

$$u^\varepsilon(x,t) = T_\varepsilon(t)[(m_+ - \delta)1\!\!1_{\{\phi \geqq 0\}} + m_- 1\!\!1_{\{\phi<0\}}](x).$$

2. Lemmas 4.1 and 4.2 yield that

$$w^{\varepsilon,\beta}(\cdot,0) \leqq u^\varepsilon(\cdot,t_\varepsilon) \quad \text{in } \mathbb{R}^N,$$

and, by the maximum principle,

$$w^{\varepsilon,\beta} \leqq u^\varepsilon(\cdot,\cdot + t_\varepsilon) \quad \text{in } \mathbb{R}^N \times (0,\bar{h}).$$

It follows that, if $t \in (0,\bar{h})$, $x \in \bar{B}_r(x_0)$ and $\phi(x) - t[F^*(D^2\phi(x), D\phi(x)) + \alpha] > 2\beta$, then

$$m_+ \leqq \liminf{}_* \; u^\varepsilon(x,t).$$

3. Since β is arbitrary and does not depend on $\bar{h}$, the result follows. □

We continue with the sketch of the proof of Lemma 4.1 borrowed from [C], to which we refer for all the details.

Sketch of Proof of Lemma 4.1. 1. Standard arguments from the theory of ordinary differential equations and (4.4) yield the existence of a unique solution $\chi \in C^2(\mathbb{R} \times [0,\infty))$ of

(4.12) $$\dot{\chi} + f(\chi) = 0 \quad \text{in } [0,\infty) \text{ with } \chi(0) = \xi \in \mathbb{R},$$

satisfying, in addition,

(4.13) $$\chi_\xi(\xi,s) > 0 \quad \text{in } \mathbb{R} \times [0,+\infty),$$

(4.14) $$\begin{gathered}\text{for every } 0 < \delta < m_+ - m_0, \text{ there exists } a > 0 \text{ such that}\\ \chi(\xi,s) \geqq m_+ - \beta\varepsilon\\ \text{for } s \geqq a|\ln\varepsilon| \text{ and } \xi \geqq 2^{-1}(m_+ + m_0 - \delta),\end{gathered}$$

(4.15) $$\begin{gathered}\text{for every } a > 0, \text{ there exists } M(a) \in \mathbb{R}\\ \text{such that, for } \varepsilon \text{ small enough,}\\ (\chi_\xi(\xi,s))^{-1}|\chi_{\xi\xi}(\xi,s)| \leqq \varepsilon^{-1}M(a) \quad \text{for } 0 < s \leqq a|\ln\varepsilon|.\end{gathered}$$

2. Let ψ be a smooth function such that

$$m_- \leqq \psi \leqq m_+ - \delta \;\text{ in } \mathbb{R}^N, \quad \psi = m_- \;\text{ in } \{\phi < 0\}, \quad \psi = m_+ - \delta \;\text{ on } \{\phi \geqq \beta\}.$$

It is now clear that

$$\psi \leqq (m_+ - \delta)1\!\!1_{\{\phi\geqq 0\}} + m_- 1\!\!1_{\{\phi<0\}} \qquad \text{in } \mathbb{R}^N.$$

3. Define $\bar{w} : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ by

$$\bar{w}(x,t) = \chi(\psi(x) - \varepsilon^{-1}Kt, \varepsilon^{-2}t).$$

It follows that, for $K > 0$ large enough, $\bar{w}$ is a subsolution of (4.3) in $\mathbb{R}^N \times (0, a\varepsilon^2|\ln\varepsilon|)$, where a is as in (4.14). Indeed, a simple calculation yields

$$\bar{w}_t - \Delta\bar{w} + \varepsilon^{-2} f(\bar{w}) = -\chi_\xi \varepsilon^{-1} K + \varepsilon^{-2}\dot{\chi} - \chi_\xi \Delta\psi - \chi_{\xi\xi}|D\psi|^2 + \varepsilon^{-2} f(\chi)$$
$$= -\chi_\xi\big[\varepsilon^{-1}K - \Delta\psi - (\chi_\xi)^{-1}\chi_{\xi\xi}|D\psi|^2\big].$$

Given (4.15) and the fact that, by definition, the functions ψ, $\Delta\psi$ and $D\psi$ have compact supports, it is now clear that, for K large enough, the quantity inside the brackets above is positive.

4. Since the definitions of $\bar{w}$ and ψ yield

$$\bar{w} \leqq u^\varepsilon \quad \text{in } \mathbb{R}^N \times \{0\},$$

Step 3 and the maximum principle imply that

$$\bar{w} \leqq u^\varepsilon \quad \text{in } \mathbb{R}^N \times (0, a\varepsilon^2|\ln\varepsilon|).$$

5. Evaluating this last inequality for $t = a\varepsilon^2|\ln\varepsilon|$ and for x such that $\phi(x) > \beta$ yields

$$\chi(m_+ - \delta - Ka\varepsilon|\ln\varepsilon|, a|\ln\varepsilon|) \leqq u^\varepsilon(x, a\varepsilon^2|\ln\varepsilon|).$$

But since for ε small enough

$$m_+ - \delta + Ka\varepsilon|\ln\varepsilon| \geqq 2^{-1}(m_+ + m_- - \delta),$$

it follows from (4.14) that

$$m_+ - \beta\varepsilon \leqq u^\varepsilon(x, a\varepsilon^2|\ln\varepsilon|).$$

This last inequality, together with the fact that $m_- \leqq u^\varepsilon$ in $\mathbb{R}^N \times (0, \infty)$, finally gives

$$(m_+ - \beta\varepsilon)\mathbb{1}_{\{\phi\geqq\beta\}} + m_-\mathbb{1}_{\{\phi<\beta\}} \leqq u^\varepsilon(\cdot, a\varepsilon^2|\ln\varepsilon|) \quad \text{in } \mathbb{R}^N.$$

6. The conclusion now follows for $\tau = a$. □

It is worth pointing out that the proof of Lemma 4.1 relies entirely on the properties of the ordinary differential equation (4.12). This is related to the aforementioned fact that, for very small time, the reaction effects dominate the diffusion ones.

We now turn to the

Proof of Lemma 4.2. 1. Since $|D\phi(x)| \neq 0$ on the compact set $\{\phi = 0\}$, it follows that $|D\phi(x)| \neq 0$ on the set $\{|\phi| \leqq \gamma\}$ for some $\gamma > 0$ small enough.

2. Choose β small compared to γ and consider the function

$$\varphi(x,t) = \phi(x) - t[F^*(D^2\phi(x), D\phi(x)) + \alpha] - 2\beta. \tag{4.16}$$

Notice that one of the main points for arguing in $\{|\phi| < \gamma\}$ is that the functions ϕ and $F^*(D^2\phi, D\phi)$ are smooth in this domain. Moreover, by choosing h sufficiently small, we have

$$|D\varphi(x,t)| \neq 0, \quad \varphi_t + F^*(D^2\varphi, D\varphi) \leqq -\tfrac{1}{2}\alpha \text{ in } \{|\phi| < \gamma\} \times (0, h).$$

3. Next we construct a subsolution v of (4.3) in $\{|\phi| < \gamma\} \times (0, h)$ of the form

$$v(x,t) = Q\big(\varepsilon^{-1}\varphi(x,t), x, t\big) + \varepsilon\big[P\big(\varepsilon^{-1}\varphi(x,t), x, t\big) - 2\beta\big], \tag{4.17}$$

where Q and P are smooth functions to be chosen below. Since the idea is to mimic the formal asymptotic expansion of u^ε in ε (cf. [KRS]), subsolutions of this form are rather natural.

Substituting v in (4.3), performing the needed algebraic manipulations, using

$$f(v) = f(Q) + \varepsilon f'(Q)(P - 2\beta) + O(\varepsilon^2)(P - 2\beta)^2,$$

and dropping the arguments of Q and P for the sake of notational clarity, we obtain

$$v_t - \Delta v + \varepsilon^{-2} f(v) = \varepsilon^{-2}\mathrm{I}_\varepsilon + \varepsilon^{-1}\mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon,$$

where

$$\mathrm{I}_\varepsilon = -\ddot{Q}|D\varphi|^2 + f(Q),$$

$$\mathrm{II}_\varepsilon = \dot{Q}(\varphi_t - \Delta\varphi) - 2(D_x\dot{Q}\cdot D\varphi) - \ddot{P}|D\varphi|^2 + f'(Q)(P - 2\beta),$$

$$\mathrm{III}_\varepsilon = Q_t - \Delta Q + O(1)(P - 2\beta)^2 + \varepsilon(P_t - \Delta P) + \dot{P}(\varphi_t - \Delta\varphi) - 2(D_x\dot{P}\cdot D\varphi).$$

4. The first goal is to choose Q so that $I_\varepsilon = 0$. Indeed if q is the travelling-wave solution of (4.1), which in the case at hand satisfies (4.7), we set

$$Q(s,x,t) = q(|D\varphi(x,t)|^{-1}s)$$

for $s \in \mathbb{R}$ and $(x,t) \in \{|\phi| < \gamma\} \times (0,\eta)\}$. Notice that, in view of (4.5), Q is a smooth function of s, x and t. It is now immediate that $\mathrm{I}_\varepsilon = 0$ with this choice Q.

5. We rewrite II_ε as $\mathrm{II}_\varepsilon = \bar{\mathrm{II}}_\varepsilon - 2\beta f'(q)$ and continue with the analysis of

$$\bar{\mathrm{II}}_\varepsilon = \dot{Q}(\varphi_t - \Delta\varphi) - 2(D_x\dot{Q}\cdot D\varphi) - \ddot{P}|D\varphi|^2 + f'(Q)P.$$

Our aim is to choose P so that, for ε sufficiently small,

$$\varepsilon^{-1}\mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon \leqq 0.$$

This is exactly the place where (4.10) comes into play.

6. Using the form of Q in Step 4 above and choosing P such that

$$P(s,x,t) = p(|D\varphi(x,t)|^{-1}s, x, t),$$

we can rewrite the equality $\bar{\mathrm{II}}_\varepsilon = 0$ as

$$\ddot{p} - f'(q)p = \chi(s,x,t), \tag{4.18}$$

where

$$\begin{aligned}\chi(s,x,t) = {} & |D\varphi|^{-1}\dot{q}(\lambda)[\varphi_t - \Delta\varphi] \\ & -2|D\varphi|^{-3}(\dot{q} + \lambda\ddot{q})(\lambda)(D^2\varphi D\varphi\cdot D\varphi).\end{aligned} \tag{4.19}$$

In this expression, φ, $D\varphi$ and $D^2\varphi$ are evaluated at (x,t) and q is evaluated at $\lambda = |D\varphi(x,t)|^{-1}s$.

7. In view of (4.10), there exists p satisfying (4.18), provided that

$$\int_{-\infty}^{\infty} \chi(s,x,t)\dot{q}(s)ds = 0.$$

It is straightforward to check, using the elementary fact that

$$2\int_{-\infty}^{\infty} \lambda\ddot{q}(\lambda)\dot{q}(\lambda)d\lambda = -\int_{-\infty}^{\infty} |\dot{q}(\lambda)|^2 d\lambda,$$

that this last condition leads to

$$\varphi_t - \Delta\varphi + (D^2\varphi\widehat{D\varphi}\cdot\widehat{D\varphi}) = 0,$$

which is the formal justification of the connection with the mean curvature equation.

Here, however, φ does not exactly satisfy the mean curvature equation. To overcome this difficulty, we solve (4.18) replacing the χ of (4.19) by $\tilde{\chi}$ given by

$$\tilde{\chi}(s,x,t) = \chi(s,x,t) - |D\varphi|^{-1}\dot{q}(\lambda)[\varphi_t - \Delta\varphi + (D^2\varphi D\varphi\cdot D\varphi)].$$

Indeed $\tilde{\chi}$ satisfies the right orthogonality condition required in (4.10). Of course we have a remainder which helps us to control the order-1 terms in the expansion.

8. Making this choice of p and using the definition of φ we find that

$$\mathrm{II}_\varepsilon = \dot{Q}[\varphi_t - \Delta\varphi + D^2\varphi\widehat{D\varphi}\cdot\widehat{D\varphi}] + 2f'(Q)\beta \leqq -2^{-1}\alpha\dot{Q} + 2f'(Q)\beta.$$

9. To conclude we need Lemma 4.3 which we now state, postponing its proof.

Lemma 4.3. *If β is small compared to α, then there exists $\nu(\alpha,\beta) < 0$ such that, for all $s \in \mathbb{R}$ and $(x,t) \in \{|\phi| < \gamma\} \times [0,\eta]$,*

$$-\tfrac{1}{2}\alpha\dot{Q}(s) + 2f'(Q(s))\beta \leqq \nu(\alpha,\beta) < 0.$$

10. We continue with the analysis of III_ε. Given (4.4)–(4.6), (4.8) and (4.10), tedious but straightforward computations show that all terms of III_ε are bounded, independently of ε. This allows us to conclude the proof of the assertion of Step 3, since Lemma 4.3 yields that, in $\{|\phi| < \gamma\} \times [0,\eta]$,

$$v_t - \Delta v + \varepsilon^{-2} f(v) \leqq \varepsilon^{-1}\nu(\alpha,\beta) + O(1),$$

with the right-hand side negative for ε small enough.

11. Next we need to extend the subsolution v to the whole domain $\mathbb{R}^N \times [0,\eta]$. We do so in a series of lemmas below. The first is about extending v to $\{\phi \leqq \gamma\} \times [0,\eta]$.

Lemma 4.4. *If η and β are so small that*

$$\eta\Big[\max_{|\phi(x)|\leqq\gamma} |F^*(D^2\phi(x), D\phi(x))| + \alpha\Big] + 2\beta \leqq \tfrac{1}{4}\gamma, \tag{4.20}$$

then, for ε sufficiently small, the function $\bar{v}$ defined on $\{\phi \leqq \gamma\} \times [0,\eta]$ by

$$\bar{v}(x,t) = \begin{cases} \sup(v(x,t), m_-) & \text{if } \phi(x) > -\gamma, \\ m_- & \text{if } \phi(x) \leqq -\gamma, \end{cases}$$

is a viscosity subsolution of (4.3) *in $\{\phi < \gamma\} \times (0,\eta)$.*

Proof. 1. Since the constant m_- is a solution of (4.3) in $\mathbb{R}^N \times (0, +\infty)$, the result is clear in $\{-\gamma < \phi < \gamma\} \times (0, \eta)$, $\bar{v}$ being the supremum of two subsolutions.

2. Now we examine the function v in the domain $\{x : \phi(x) \leqq -\frac{1}{2}\gamma\}$. In view of (4.20), it follows that in this domain

$$\varphi(x, t) \leqq -\tfrac{1}{4}\gamma.$$

The asymptotic behavior of q and p near $-\infty$ then yields, for some constant $c > 0$,

$$v(x, t) \leqq m_- + \exp\big(-(4\varepsilon)^{-1}c\gamma\big) + \varepsilon(o_\varepsilon(1) - 2\beta).$$

It is therefore clear that, for ε sufficiently small,

$$v(x, t) < m_- \quad \text{if } \phi(x) \leqq -\tfrac{1}{2}\gamma.$$

Hence,

$$\bar{v}(x, t) = m_- \quad \text{if } \phi(x) \leqq -\tfrac{1}{2}\gamma,$$

and the result follows. □

12. To complete the construction, we introduce a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi' \leqq 0$ in $\mathbb{R}$, $\psi = 1$ in $(-\infty, \frac{1}{2}\gamma)$, $0 < \psi < 1$ in $(\frac{1}{2}\gamma, \frac{3}{4}\gamma)$, $\psi = 0$ in $(\frac{3}{4}\gamma, +\infty)$, and, finally, $\psi'' \leqq 0$ in a neighborhood of $\frac{1}{2}\gamma$.

13. We need

Lemma 4.5. *Assume that* (4.20) *holds, set* $C = \max |\Delta\phi| + 1$ *on* $\{|\phi| < \gamma\}$ *and denote by* χ *the function defined, for* $(x, t) \in \mathbb{R}^N \times [0, \infty)$, *by* $\chi(x, t) = \phi(x) - Ct$. *The function* $w : \mathbb{R}^N \times [0, \eta] \to \mathbb{R}$ *defined by*

$$w(x, t) = \begin{cases} \psi(\chi(x, t))\bar{v}(x, t) + (1 - \psi(\chi(x, t)))(m_+ - \beta\varepsilon) & \text{if } \phi(x) < \gamma, \\ m_+ - \beta\varepsilon & \text{otherwise} \end{cases}$$

is a viscosity subsolution of (4.3) *in* $\mathbb{R}^N \times [0, \eta]$, *if* ε *and* η *are sufficiently small. Moreover,*

$$w(\cdot, 0) \leqq (m_+ - \beta\varepsilon)\mathbb{1}_{\{\phi \geqq \beta\}} + m_- \mathbb{1}_{\{\phi < \beta\}} \quad \text{in } \mathbb{R}^N.$$

Proof. 1. Choose η small enough in order to have $C\eta < \frac{1}{8}\gamma$. If $\phi(x) > \frac{7}{8}\gamma$, then $w(x, t) = m_+ - \beta\varepsilon$. Since $f(m_+ - \beta\varepsilon) < 0$ for ε small enough, it is clear that w is a subsolution of (4.3) for $\phi(x) > \frac{7}{8}\gamma$ as well as in $\{\phi(x) - Ct < \frac{1}{2}\gamma\}$, where $w = \bar{v}$.

2. It is enough to check the subsolution property in the set where $\{\frac{1}{2}\gamma < \phi < \gamma\}$. We have

$$\begin{aligned} w_t - \Delta w + \varepsilon^{-2} f(w) = {} & \psi(\bar{v}_t - \Delta\bar{v}) + [\psi'(-C - \Delta\phi) \\ & - \psi''|D\phi|^2](\bar{v} - (m_+ - \beta\varepsilon)) \\ & + \varepsilon^{-2} f(\psi\bar{v} + (1 - \psi)(m_+ - \beta\varepsilon)), \end{aligned} \tag{4.21}$$

where once again we have dropped the arguments of the function ψ for the sake of clarity.

Using (4.20) and the asymptotic behavior of q and p at $+\infty$, we obtain, for some constant $\tilde{c} > 0$, that

$$\bar{v}(x,t) = m_+ - \exp\Big(-(4\varepsilon)^{-1}\tilde{c}\gamma\Big) + \varepsilon(o_\varepsilon(1) - 2\beta) = m_+ - 2\beta\varepsilon + \varepsilon o(1),$$

and, hence, for ε small enough,

$$\bar{v}(x,t) - (m_+ - \beta\varepsilon) = -\beta\varepsilon + \varepsilon o(1) \leqq 0.$$

Since $\psi' \leqq 0$ in $\mathbb{R}$ and $-\Delta\phi \leqq C$, we also have

$$\psi'(-C - \Delta\phi)(\bar{v} - (m_+ - \beta\varepsilon)) \leqq 0.$$

But f is convex in a neighborhood of m_+. Therefore, if ε is sufficiently small,

$$f(w) \leqq \psi f(\bar{v}) + (1 - \psi) f(m_+ - \beta\varepsilon).$$

Substituting all this information in (4.21) yields

$$w_t - \Delta w + \varepsilon^{-2} f(w) \leqq -\psi''|D\phi|^2(\bar{v} - (m_+ - \beta\varepsilon)) + (1 - \psi)\varepsilon^{-2} f(m_+ - \beta\varepsilon).$$

3. Since $\psi''(s) \leqq 0$ if $s \leqq \frac{1}{2}\gamma + \nu$ for some $\nu > 0$, the right-hand side of this inequality is negative for $\phi(x) - Ct \leqq \nu + \frac{1}{2}\gamma$.

If $s > \nu + \frac{1}{2}\gamma$, then $1 - \psi(s) \geqq c(\nu) > 0$ and, hence,

$$w_t - \Delta w + \varepsilon^{-2} f(w) \leqq O(\varepsilon) + c(\nu)\varepsilon^{-2} f(m_+ - \beta\varepsilon).$$

The right-hand side of this last inequality is negative for ε small enough since, $f(m_+) = 0$ and $f'(m_+) > 0$.

4. It remains to examine $w(\cdot, 0)$. To this end, we first consider

$$v(x,0) = Q\big(\varepsilon^{-1}(\phi(x) - 2\beta), x, 0\big) + \varepsilon\big[P\big(\varepsilon^{-1}(\phi(x) - 2\beta), x, 0\big) - 2\beta\big].$$

Since $P(s,x,t) \to 0$ when $|s| \to +\infty$ uniformly with respect to (x,t) in $\{|\phi| < \gamma\} \times [0, \eta]$, there exists $\bar{c} > 0$ such that

$$|P(s,x,t)| \leqq \beta \quad \text{if } |s| \geqq \bar{c}.$$

In particular, if $\phi(x) \leqq 2\beta - \bar{c}\varepsilon$, we have

$$v(x,0) \leqq Q\big(\varepsilon^{-1}(\phi(x) - 2\beta), x, 0\big) - \varepsilon\beta.$$

From now on, we assume that ε is such that $2\bar{c}\varepsilon \leqq \beta$.

5. If $\phi(x) \leqq \frac{3}{2}\beta \leqq 2(\beta - \bar{c}\varepsilon)$, it follows that for ε sufficiently small,

$$v(x,0) \leqq Q\big(-2\varepsilon^{-1}\beta, x, 0\big) - \varepsilon\beta < m_-.$$

Therefore $\bar{v}(x,0) = m_-$ and if, in addition, $\frac{3}{2}\beta < \frac{1}{2}\gamma$, then

$$w(x,0) = \bar{v}(x,0) = m_-,$$

which yields the result for $\phi(x) \leqq \frac{3}{2}\beta$.

6. If $\phi(x) \geqq 2\beta + \bar{c}\varepsilon$, a similar argument yields

$$v(x,0) \leqq Q\big(\varepsilon^{-1}(\phi(x) - 2\beta), x, 0\big) - \varepsilon\beta \leqq m_+ - \varepsilon\beta,$$

which implies that

$$w(x,0) \leqq m_+ - \varepsilon\beta \quad \text{if } \phi(x) \geqq 2\beta + \bar{c}\varepsilon.$$

7. If $\phi(x) \leqq 2\beta + \bar{c}\varepsilon$, we use the fact that there exists $\nu(\bar{c}) > 0$ such that, for $s \leqq \nu(\bar{c})$ and $|\phi(x)| < \gamma$,

$$Q(s,x,0) \leqq m_+ - \nu(\bar{c}).$$

It follows, for ε sufficiently small, that

$$v(x,0) \leqq m_+ - \nu(\bar{c}) + \varepsilon(\|P\|_\infty + \beta) \leqq m_+ - \beta\varepsilon.$$

Therefore

$$w(x,0) \leqq m_+ - \beta\varepsilon \quad \text{if } \phi(x) \leqq 2\beta + \bar{c}\varepsilon,$$

and the proof is complete. □

We conclude this section with the

Proof of Lemma 4.3. Since

$$-\tfrac{1}{2}\alpha\dot{Q}(s,x,t) - 2f'(Q(s,x,t))\beta = -\tfrac{1}{2}|D\varphi(x,t)|^{-1}\alpha\dot{q}(\lambda) - 2f'(q(\lambda))\beta,$$

where $\lambda = s|D\varphi(x,t)|^{-1}$, it is enough to show that for $\lambda \in \mathbb{R}$ and $(x,t) \in \{|\phi| < \gamma\} \times [0,\eta]$,

$$-(\tfrac{1}{2}|D\varphi(x,t)|)^{-1}\alpha\dot{q}(\lambda) - 2f'(q(\lambda))\beta \leqq \nu(\alpha,\beta) < 0.$$

2. We first recall that $f'(m_\pm) > 0$. Since $q(\lambda) \to m_\pm$, when $\lambda \to \pm\infty$, and $\dot{q} \geqq 0$ on $\mathbb{R}$, it follows that there exists a constant $C > 0$ such that, for $|\lambda| > C$,

$$f'(q(\lambda)) \geqq k = \tfrac{1}{2}\min(f'(m_+), f'(m_-)),$$

$$-(2|D\varphi(x,t)|)^{-1}\alpha\dot{q}(\lambda) - 2f'(q(\lambda))\beta \leqq -2\beta k < 0.$$

3. If $|\lambda| \leqq C$, then there exists $K = K(C) > 0$ such that $\dot{q}(\lambda) \geqq K$. Using the fact that f' is bounded on $[m_-, m_+]$, we obtain

$$-(2|D\varphi(x,t)|)^{-1}\alpha\dot{q}(\lambda) - 2f'(q(\lambda))\beta \leqq -(2|D\varphi(x,t)|)^{-1}\alpha K(C) + 2\|f'\|_\infty\beta.$$

The right-hand side of this inequality is negative if β is small compared to α in $\{|\phi| < \gamma\} \times [0,\eta]$. □

As mentioned earlier, the proof of Theorem 4.2 we presented above is not only intuitive, since it closely follows the formal asymptotics, but also "wrong", in the sense that it is obviously rather complicated and relies on very special properties of the travelling wave.

The fact is that we can actually provide a much easier proof, which allows for far more flexibility and less reliance on very particular properties of q. Such a proof is based on using as a building block towards constructing subsolution and super-solutions of (4.3) the function

$$v(x,t) = q(\varepsilon^{-1}(d(x,t) - 2\beta))$$

instead of the one given by (4.17), where d is the signed-distance function to the set

$$\Gamma_t = \{x : \phi(x) - t[F^*(D^2\phi(x), D\phi(x)) + \alpha] = 0\},$$

which is normalized to have the same signs as $\phi - [F^*(D^2\phi, D\phi) + \alpha]$ in $\mathbb{R}^N$. Since everything is smooth, classical arguments yield the existence of $\bar{\gamma}, \bar{h} > 0$ such that

$$d \text{ is smooth in the set } Q_{\bar{\gamma},\bar{h}} = \{(x,t) : |d(x,t)| < \bar{\gamma}\ ,\ 0 \leqq t \leqq \bar{h}\},$$

$$d_t + F^*(D^2 d, Dd) = d_t - \Delta d \leqq -\frac{\alpha}{4|D\varphi|} \quad \text{in } Q_{\bar{\gamma},\bar{h}}, \tag{4.22}$$

where φ is defined by (4.16) (recall that we may assume that $|D\varphi| \neq 0$ in $Q_{\bar{\gamma},\bar{h}}$) and, in addition,

$$|Dd| = 1, \quad D^2 d Dd = 0 \quad \text{in } Q_{\bar{\gamma},\bar{h}}. \tag{4.23}$$

Essentially replacing φ by $d - 2\beta$ in the proof of Lemma 4.2 leads to far easier arguments, since now we can take $P = 0$, and to far easier computations, since (4.23) hold.

The P-term in the proof of Lemma 4.2 is needed to control the remainders which arise when, by using $|D\varphi|^{-1}\varphi$ instead of d in the Q-term, we create additional terms. These terms, which, in view of (4.23), do not appear when d is used, can be handled when φ is used by using some special identities about the travelling wave.

In more complicated problems, like those discussed in Sections 6 and 7 below, the P-term is important to balance additional terms arising in the asymptotic analysis of the solutions. Using d instead of $|D\varphi|^{-1}\varphi$ simplifies and, in some cases, makes the proofs possible. Indeed, (4.23) eliminates a number of perturbation terms, which at least, we were unable to handle in the more complicated cases. We refer the reader to Sections 6 and 7 where this remark is used in an essential way.

5. Other Results Related to the Reaction-Diffusion Equations

In this section we present two new results about the asymptotics of solutions to semilinear reaction-diffusion equations. The first is about the initial-value problem

$$\text{(5.1)} \quad \begin{aligned} &\text{(i) } u_{\varepsilon,t} - \operatorname{tr}(A(x, \widehat{Du_\varepsilon})D^2 u_\varepsilon) + \varepsilon^{-2} f(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty),\\ &\text{(ii) } u_\varepsilon = g \quad \text{on } \mathbb{R}^N \times \{0\}. \end{aligned}$$

Here the matrix $A = ((a_{ij})) \in C^2(\mathbb{R}^N \times \mathbb{R}^N, \mathcal{S}^N)$ is such that for all $i, j, k \in \{1, \ldots, N\}$,

$$a_{ij},\ a_{ij,x_k},\ a_{ij,p_k} \text{ are uniformly continuous on } \mathbb{R}^N \times \mathbb{R}^N, \tag{5.2}$$

for each $R > 0$ there exists $C_R > 0$ such that for all $p \in \mathbb{R}^N$,

$$A(\cdot, p) \in W^{2,\infty}(\mathbb{R}^N, \mathcal{S}^N), \qquad \sup_{|p| \leqq R} \|A(\cdot, p)\|_{W^{2,\infty}} \leqq C_R, \tag{5.3}$$

and, finally, there exists $\nu > 0$ such that for all $(x, p, q) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{0\} \times \mathbb{R}^N$,

$$A(x, \hat{p}) q \cdot q \geqq \nu |q|^2. \tag{5.4}$$

Note that because of the special dependence of the matrix A on Du_ε (recall that $\hat{p} = |p|^{-1} p$), the initial-value problem (5.1) must be interpreted in the viscosity sense.

To state the result about the asymptotics of (5.1) we need to recall that, for every $u_0 \in UC(\mathbb{R}^N)$, the initial-value problem

$$\begin{aligned} u_t + F(D^2 u, Du, x) = 0 \quad &\text{in } \mathbb{R}^N \times (0, \infty), \\ u = u_0 \quad &\text{on } \mathbb{R}^N \times \{0\}, \end{aligned} \tag{5.5}$$

with

$$\begin{aligned} F(X, p, x) = &-\operatorname{tr}\{A(x, \hat{p}) X \big[I - (A(x, \hat{p}) p \cdot p)^{-1} (Ap \otimes p)\big]\} \\ &+ (2A(x, \hat{p}) p \cdot p)^{-1} \operatorname{tr}\big\{A(x, \hat{p}) p \otimes [D_x A(x, \hat{p}) p \cdot p \\ &+ (X - X\hat{p} \otimes \hat{p}) D_p A(x, \hat{p}) p \cdot p]\big\}, \end{aligned}$$

has a unique viscosity solution $UC(\mathbb{R}^N \times [0, T])$ for all $T > 0$. The proof of this fact is a tedious, but nevertheless straightforward, adaptation of the usual uniqueness proofs for geometric equations—see, for example, [CIL, CGG, IS, G], etc. We leave the details to the reader.

The result is

Theorem 5.1. *Let* (4.4)–(4.6), (4.8), (4.10), (5.2)–(5.4) *and let u_ε be the solution of* (5.1) *with $g : \mathbb{R}^N \to [m_-, m_+]$ such that the set $\Gamma_0 = \{x : g(x) = m_0\}$ is a nonempty subset of $\mathbb{R}^N$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} \end{Bmatrix},$$

where u is the unique viscosity solution of (5.5) *with $u_0 = d_0$, the signed distance to Γ_0 such that $d_0 > 0$ in $\{g > m_0\}$ and $d_0 < 0$ in $\{g < m_0\}$.*

If, in addition, the no-interior condition (2.11) *holds, then, as $\varepsilon \to 0$,*

$$u_\varepsilon(x, t) \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \overline{\{u > 0\}}^c \end{Bmatrix}.$$

Proof. 1. The proof here follows closely the one given in Section 4 except for a minor additional argument that we give below.

2. In view of our assumptions on A and, in particular, on the way it depends on p, the term $F(D^2\phi(x), D\phi(x), x)$ is not of class C^2. This difficulty is resolved by a standard regularization argument. We refer to the next section where such a strategy is presented for a more complicated situation.

3. To prove the existence of subsolutions $w^{\varepsilon,\beta}$, we set, as in Section 4,

$$v(x,t) = Q\big(\varepsilon^{-1}\varphi(x,t), x, t\big) + \varepsilon\big(P\big(\varepsilon^{-1}\varphi(x,t), x, t\big) - 2\beta\big)$$

where

$$\begin{aligned} Q(s,x,t) &= q\big(\big[A(x, \widehat{D\varphi})D\varphi \cdot D\varphi\big]^{-1/2}s\big),\\ P(s,x,t) &= p\big(\big[A(x, \widehat{D\varphi})D\varphi \cdot D\varphi\big]^{-1/2}s\big),\\ \varphi(x,t) &= \phi(x) - t\big[F_\alpha(D^2\phi(x), D\phi(x), x) + \alpha\big] - 2\beta, \end{aligned}$$

where F_α is a suitable regularization of F, or alternatively, following the discussion at the end of Section 4 we set

$$v(x,t) = q(\varepsilon^{-1}(d(x,t) - 2\beta))$$

where d is the signed-distance from the set $\{\varphi = 0\}$, where φ is given by (4.16).

4. All the computations of Section 4 extend easily to this more complicated case. $\square$

The second result is about the asymptotic behavior of the usual reaction-diffusion equation (4.3), which is now set in a bounded domain with Neumann boundary conditions, i.e., the initial-boundary-value problem

$$(5.6)\qquad \begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + b(x)\cdot Du_\varepsilon + \varepsilon^{-2}f(u_\varepsilon) = 0 & \text{in } \Omega\times(0,\infty),\\ Du_\varepsilon \cdot n = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u_\varepsilon = g & \text{on } \Omega\times\{0\}, \end{cases}$$

where

(5.7) $\quad b : \mathbb{R}^N \to \mathbb{R}^N$ is a bounded Lipschitz continuous vector field,

f is as in Theorem 4.1 and $\Omega \subset \mathbb{R}^N$ is a bounded subset of $\mathbb{R}^N$. The asymptotics of (5.6) were studied by KATSOULAKIS, KOSSIORIS & REITICH [KKR] and CHEN [C] under the assumption that the resulting interface is smooth and by [KKR] globally in time but for convex domains Ω.

The front evolution associated with the asymptotics of (5.6) is motion by mean curvature transported by b with Neumann boundary conditions. The corresponding geometric partial differential equation, which was studied by GIGA & SATO [GS], is

$$(5.8)\qquad \begin{aligned} u_t - \operatorname{tr}[(I - \widehat{Du} \otimes \widehat{Du}) D^2 u] + b(x) \cdot Du = 0 &\quad \text{in } \Omega \times (0, \infty), \\ Du \cdot n = 0 &\quad \text{on } \partial\Omega \times (0, \infty), \\ u = u_0 &\quad \text{on } \Omega \times \{0\}. \end{aligned}$$

Our result is

Theorem 5.2. *Let* (4.4), (4.5), (4.6), (4.8), (4.10) *and* (5.7) *hold and let* u_ε *be the solution of* (5.8) *with* $g : \Omega \to [m_-, m_+]$ *such that the set* $\Gamma_0 = \{x : g(x) = m_0\}$ *is a nonempty subset of* Ω. *Then, as* $\varepsilon \to 0$,

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \textit{ locally uniformly in } \begin{Bmatrix} \{> 0\}, \\ \{u < 0\{u < 0\} \end{Bmatrix},$$

where u *is the unique viscosity solution of* (5.7) *with* $u_0 = d_0$ *the signed distance to* Γ_0 *which is positive on the set* $\{g > m_0\}$ *and negative on the set* $\{g < m_0\}$.

If, in addition, the no-interior condition (2.11) *holds, then, as* $\varepsilon \to 0$,

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_0 \end{Bmatrix} \textit{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \overline{\{u > 0\}^c} \end{Bmatrix}.$$

The proof of Theorem 5.2 is based on Theorem 3.1 and the short-time analysis performed in [C] and [KKR]. We leave the details to the reader.

6. Reaction-Diffusion Equations with Oscillatory Coefficients

In this section we study the asymptotic behavior, as $\varepsilon \to 0$, of reaction-diffusion equations of the form

$$(6.1)\qquad u_{\varepsilon,t} - \varepsilon \mathscr{L}^\varepsilon \left(x, \frac{x}{\varepsilon}\right) u_\varepsilon + \varepsilon^{-1} f(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

$$(6.2)\qquad u_{\varepsilon,t} - \mathscr{L}^\varepsilon \left(x, \frac{x}{\varepsilon}\right) u_\varepsilon + \varepsilon^{-1} f(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

with initial datum

$$(6.3)\qquad u_\varepsilon = g \quad \text{in } \mathbb{R}^N,$$

where, as usual, $f = W'$, W being a double-well potential, and

$$(6.4)\qquad \mathscr{L}^\varepsilon \left(x, \frac{x}{\varepsilon}\right) v = \operatorname{div}\left[A\left(\frac{x}{\varepsilon}\right) Dv\right] + \frac{1}{\varepsilon}\left(b\left(\frac{x}{\varepsilon}\right) + \varepsilon B\left(\frac{x}{\varepsilon}\right)\right) \cdot Dv.$$

The matrix $A = ((a_{ij})) : \mathbb{R}^N \to \mathscr{S}^N$ and the transport coefficients $b : \mathbb{R}^N \to \mathbb{R}^N$ and $B : \mathbb{R}^N \to \mathbb{R}^N$ are assumed to satisfy, for some compact subset Π of $\mathbb{R}^N$,

(6.5) $\qquad A \in C^2(\mathbb{R}^N, \mathscr{S}^N)$ is positive-definite and periodic in Π,

(6.6) $\qquad b$ and B are Lipschitz continuous and periodic in Π.

Asymptotic problems like (6.1) and (6.2) arise in the study of the behavior for large x and t of the solution of the reaction-diffusion equation

$$u_t - \operatorname{div}(A(x)Du) - (b(x) + \varepsilon B(x)) \cdot Du + f(u) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{6.7}$$

To study the asymptotics of (6.1) and (6.2), as in the previous sections, we assume that f satisfies (4.4) and that (6.7) admits travelling-wave solutions. The main difficulty as well as novelty here are that these travelling waves depend nontrivially on x and the direction e. More precisely, we assume that

$$\begin{aligned} &\text{for each } e \in S^{N-1}, \text{ there exists a unique pair } (c(e), q(\cdot, \cdot, e)) \\ &\text{where } c(e) \in \mathbb{R} \text{ and } q : \mathbb{R} \times \mathbb{R}^N \times S^{N-1} \to \mathbb{R} \text{ are such that} \\ &\quad c(e)\dot{q} + (D_y + e\partial_s)^T (A(y)(D_y + e\partial_s))q \\ &\qquad + b \cdot (D_y + e\partial_s)q = f(q) \quad \text{in } \mathbb{R} \times \Pi, \\ &\quad q(0, y, e) = m_0, \quad \dot{q}(\cdot, y, e) > 0 \quad \text{on } \mathbb{R} \times \Pi, \\ &\quad y \to q(s, y, e) \text{ periodic in } \Pi \text{ for each } s \in \mathbb{R}, \\ &\text{and, as } |s| \to \pm\infty, q(s, y, e) \to m_\pm, \text{ exponentially fast} \\ &\text{with rate depending on } e \text{ and uniformly on } S^{N-1} \times \Pi. \end{aligned} \tag{6.8}$$

The existence, uniqueness and properties of such pairs (c, q) have been studied, under some additional assumptions on A, b and f, by XIN [X1,2,3,4], to which we refer for the details. In what follows we take the point of view that (6.8) is satisfied and proceed with the study of the asymptotics of (6.1) and (6.2). To this end we also assume that

$$\begin{aligned} &q \in C^2(\mathbb{R} \times \mathbb{R}^N \times S^{N-1}), \quad c \in C(S^{N-1}), \\ &c \quad \text{depends continuously on } f, \end{aligned} \tag{6.9}$$

$$\begin{aligned} &\sup_{\substack{(s,y,e)\in \\ \mathbb{R}\times\mathbb{R}^N\times S^{N-1}}} \big[|s|\dot{q}(s, y, e) + (|s| + s^2)|\ddot{q}(s, y, e)|\big] < \infty, \\ &\sup_{\substack{(s,y,e)\in \\ \mathbb{R}\times\mathbb{R}^N\times S^{N-1}}} \big[|D_e q(s, y, e)| \\ &\qquad + (1 + |s|)[|D_y q(s, y, e)| + |D_e \dot{q}(s, y, e)|]\big] < \infty, \\ &\sup_{\substack{(s,y,e)\in \\ \mathbb{R}\times\mathbb{R}^N\times S^{N-1}}} \big[|D_e^2 q(s, y, e)| + |D_{y,e}^2 q(s, y, e)|\big] < \infty. \end{aligned} \tag{6.10}$$

To state the result about the asymptotics of (6.1) we recall that, for all $u_0 \in UC(\mathbb{R}^N)$, the initial-value problem

$$\begin{aligned} u_t + F(Du) = 0 \quad &\text{in } \mathbb{R}^N \times (0, \infty), \\ u = u_0 \quad &\text{on } \mathbb{R}^N \times \{0\}, \end{aligned} \tag{6.11}$$

where $F : \mathbb{R}^N \to \mathbb{R}$ is the continuous function given by

$$F(p) = \begin{cases} c(\hat{p})|p|, & p \neq 0, \\ 0, & p = 0, \end{cases} \tag{6.12}$$

has a unique viscosity solution $u \in UC(\mathbb{R}^N \times [0, T])$ for all $T > 0$.

We have

Theorem 6.1. *Let* (4.4), (6.5), (6.6), (6.8)–(6.10) *hold and let* u_ε *be the solution of* (6.1), (6.3) *with* $g : \mathbb{R}^N \to [m_-, m_+]$ *such that the set* $\Gamma_0 = \{x : g(x) = m_0\}$ *is a nonempty subset of* $\mathbb{R}^N$. *Then, as* $\varepsilon \to 0$,

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{> 0\} \\ \{u < 0\} \end{Bmatrix},$$

where u *is the unique viscosity solution of* (6.11), *with* $u_0 = d_0$ *the signed distance to* Γ_0 *which is positive in the set* $\{g > m_0\}$ *and negative in the set* $\{g < m_0\}$.

If, in addition, the no-interior condition (2.11) *holds, then, as* $\varepsilon \to 0$,

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \overline{\{u > 0\}}^c \end{Bmatrix}.$$

Proof. 1. The proof follows exactly by using the same strategy as the one we adopted for the proof of Theorem 4.1.

2. If T^ε is the semigroup associated with (6.1), that (H1)–(H3) are satisfied. Again we only have to check (H4) and the initialization of the front. Changing $f(u)$ to $f(m_+ + m_- - u)$ shows that we only have to prove "half" of these properties.

3. The main step in the proof is

Theorem 6.2. *For all* $x_0 \in \mathbb{R}^N$, $r > 0$, $\alpha > 0$ *and for any smooth function* $\phi : \mathbb{R}^N \to \mathbb{R}$ *such that* $\{x : \phi(x) = 0\} \subset B_r(x_0)$ *and* $|D\phi(x)| \neq 0$ *on* $\{x : \phi(x) = 0\}$ *and for any* $0 < \nu < m_+ - m_0$, *there exists* $\bar{h} > 0$ *depending only on* ϕ *through its* C^4*-norm in* $\bar{B}_r(x_0)$ *such that*

$$\liminf{}_* \left(T_\varepsilon(h)[(m_+ - \nu)\mathbb{1}_{\{\phi \geqq 0\}} + m_- \mathbb{1}_{\{\phi < 0\}}]\right)(x) = m_+$$

if $\phi(x) - h[F(D\phi(x)) + \alpha] > 0$ *and* $x \in B_r(x_0)$ *for* $0 < h \leqq \bar{h}$, *with* F *given by* (6.12).

4. As for Theorem 6.2, its proof consists of two steps given by the following two lemmas.

Lemma 6.1. (Initialization of the front). *Under the assumptions of Theorem 6.1 and for any* $\beta > 0$, *there exists a constant* $\tau > 0$ *such that, if* $t_\varepsilon = \tau\varepsilon$, *then for* ε *sufficiently small,*

$$T^\varepsilon(t_\varepsilon)[(m_+ - \nu)\mathbb{1}_{\{\phi \geqq 0\}} + m_- \mathbb{1}_{\{\phi < 0\}}] \geqq (m_+ - \beta)\mathbb{1}_{\{\phi \geqq \beta\}} + m_- \mathbb{1}_{\{\phi < \beta\}}.$$

Lemma 6.2. (Propagation of the front). *If* α *is small enough, there exists* $\bar{h} > 0$ *depending only on* ϕ *through its* C^4*-norm in* $\bar{B}_r(x_0)$ *such that for* $0 < \beta \leqq \bar{\beta}(\alpha, \phi)$ *and* $\varepsilon \leqq \bar{\varepsilon}(\alpha, \beta, \phi)$, *there exists a subsolution* $w^{\varepsilon,\beta}$ *of* (6.1) *in* $\mathbb{R}^N \times (0, \bar{h})$ *satisfying*

$$w^{\varepsilon,\beta}(\cdot, 0) \leqq (m_+ - \beta)\mathbb{1}_{\{\phi \geqq \beta\}} + m_- \mathbb{1}_{\{\phi < \beta\}} \quad \text{in } \mathbb{R}^N,$$

and, for any $(x, t) \in B_r \times (0, \bar{h})$ *such that* $\phi(x) - t[F(D\phi(x)) + \delta] > 2\beta$,

$$\liminf{}_* [w^{\varepsilon,\beta}(x, t)] \geqq m_+ - o_\beta(1).$$

5. By using these two lemmas, the proof of Theorem 6.2 is again a straightforward adaptation of the proof of Theorem 4.2. □

The proof of Lemma 6.1 follows along the lines of the proof of the analogous result in Section 4. We therefore leave it up to the reader to fill in the details. We continue with the

Proof of Lemma 6.2. 1. To simplify the presentation, throughout the proof we assume that $A = I$, the identity matrix. The general case follows by similar but a bit more tedious arguments.

2. Since $|D\phi(x)| \neq 0$ on the compact set $\{\phi = 0\}$, we may choose $\gamma > 0$ such that $|D\phi(x)| \neq 0$ on $\{|\phi| \leqq \gamma\} \subset \bar{B}_r(x_0)$.

3. As in Section 4, we first build a subsolution $w^{\varepsilon,\beta}$ in $\Omega_{\gamma,\bar{h}} = \{|\phi| \leqq \gamma\} \times (0, \bar{h})$, where $\bar{h}$ is to be chosen below.

4. The first additional difficulty we encounter here is that, in view of (6.8) and (6.9), F is not a smooth function of Du. To overcome this difficulty, we consider a C^2-approximation F_α of F such that

$$F(D\phi(x)) - \tfrac{1}{4}\alpha \leqq F_\alpha(D\phi(x)) \leqq F(D\phi(x)) \quad \text{on } \{|\phi| \leqq \gamma\}.$$

5. Choose β small compared to γ and introduce the function $\varphi : \Omega_{\gamma,\bar{h}} \to \mathbb{R}$ defined by

$$\varphi(x,t) = \phi(x) - t[F_\alpha(D\phi(x)) + \alpha] - 2\beta.$$

Choosing $\bar{h}$ small enough, we may assume that

$$|D\varphi(x,t)| \neq 0 \text{ in } \Omega_{\gamma,\bar{h}}, \quad \varphi_t + F(D\varphi) \leqq -\tfrac{1}{2}\alpha \text{ in } \Omega_{\gamma,\bar{h}}.$$

6. Next we build a subsolution of (6.1) of the form

$$v(x,t) = Q\left(\varepsilon^{-1}\varphi(x,t), \frac{x}{\varepsilon}, x, t\right), \tag{6.13}$$

where $Q(s, y, x, t)$ is a smooth function to be chosen below.

7. Substituting v into (6.1) yields

$$v_t - \varepsilon\Delta v - \left(b\Big(\frac{x}{\varepsilon}\Big) + \varepsilon B\Big(\frac{x}{\varepsilon}\Big)\right) \cdot Dv + \varepsilon^{-1} f(v) = 0 = \varepsilon^{-1}\mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon,$$

with

$$\begin{aligned}\mathrm{I}_\varepsilon &= \dot{Q}(\varphi_t - \varepsilon\Delta\varphi - b\cdot D\varphi) + b\cdot D_y Q\\ &\quad -\ddot{Q}|D\varphi|^2 - 2D_y\dot{Q}\cdot D\varphi - \Delta_y Q + f(Q),\\ \mathrm{II}_\varepsilon &= Q_t - \varepsilon\Delta_x Q - b\cdot D_x Q - B\cdot D_y Q - 2D_x\dot{Q}\cdot D\varphi - 2\Delta_{x,y}Q,\end{aligned}$$

where $\Delta_{x,y}Q = \sum_{i=1}^N Q_{x_i y_i}$ and y stands for the argument x/ε. Once again for the sake of notational simplicity we suppress the explicit dependence on the arguments.

8. Next we observe that since f satisfies (4.4), there exists $\bar{\delta} > 0$ such that, for all $\delta \in [-\bar{\delta}, \bar{\delta}]$, the function $f^\delta = f + \delta$ also satisfies (4.4). Moreover, for

each $\delta \in [-\bar\delta, \bar\delta]$, there exist pairs (c^δ, q^δ) satisfying (6.8)–(6.10) with constants depending on δ. Finally, it follows that $c^\delta(e) \to c(e)$ as $\delta \to 0$ for all $e \in S^{N-1}$.

9. For a suitable $\delta \in [-\bar\delta, \bar\delta]$, we choose Q of the form

$$Q(s, y, x, t) = q^\delta\big(|D\varphi(x, t)|^{-1} s, y, \widehat{D\varphi}(x, t)\big).$$

Denoting by $e(x, t)$ the vector $\widehat{D\varphi}(x, t)$, we may rewrite I_ε as

$$\mathrm{I}_\varepsilon = |D\varphi|^{-1}\dot q^\delta\,[\varphi_t - \varepsilon\Delta\varphi] - \dot q^\delta(b\cdot e) - b\cdot D_y q^\delta - \ddot q^\delta - 2D_y q^\delta\cdot e - \Delta_y q^\delta + f(q^\delta).$$

Substituting the equation satisfied by q^δ into the last expression, we obtain

$$\mathrm{I}_\varepsilon = |D\varphi|^{-1}\dot q^\delta\big[\varphi_t - \varepsilon\Delta\varphi + c^\delta(e)|D\varphi|\big] + \delta.$$

10. It remains to estimate the quantity inside the bracket in this last expression. The properties of c^δ, F_α and φ now yield the estimate

$$\begin{aligned}&|D\varphi|^{-1}\dot q^\delta\left[-\varepsilon\Delta\varphi - F(D\varphi) - \tfrac{\alpha}{2} + |D\varphi|c^\delta(e)\right] + \delta\\ &\qquad\leqq \dot q^\delta\big[c^\delta(e) - c(e) - (2|D\varphi|)^{-1}(\alpha + 2\varepsilon\Delta\varphi)\big] + \delta.\end{aligned}$$

It is then clear that, for $|\delta|$ small enough compared to $\alpha > 0$, the quantity inside the bracket is negative for ε small enough, since, for fixed α, the function $\Delta\varphi$ is bounded—note that the bound may depend on α. Since $\dot q^\delta > 0$, choosing $\delta < 0$ small, we conclude that $\varepsilon^{-1}\mathrm{I}_\varepsilon \leqq \varepsilon^{-1}\delta$.

Finally, using the assumptions on q^δ and carrying out some tedious computations, we can show that there is some constant $\tilde K_\delta$. Such that II_ε can be estimated by

$$v_t - \varepsilon\Delta v - \left(b\left(\frac{x}{\varepsilon}\right) + \varepsilon B\left(\frac{x}{\varepsilon}\right)\right)\cdot Dv + \varepsilon^{-1} f(v) \leqq \varepsilon^{-1}\delta + \tilde K_\delta \quad \text{in } \Omega_{\gamma,\bar h}.$$

Hence, for any fixed δ, v is a subsolution of (6.1), provided that ε is sufficiently small.

11. Next, if β is small enough compared to α, we claim that we can choose δ so that

$$v(\cdot, 0) \leqq (m_+ - \beta)\mathbb{1}_{\{\phi\geqq\beta\}} + m_-\mathbb{1}_{\{\phi<\beta\}} \quad \text{on } \{|\phi| \leqq \gamma\}.$$

Indeed, we first remark that the assumption that $f'(m_\pm)$ yields $m_-^\delta \leqq m_- + d\delta$ and $m_+^\delta \leqq m_+ + d\delta$ for some constant $d > 0$. And, if β is small enough compared to α, we can choose $-\bar\delta \leqq \delta < 0$ satisfying the requirements of Step 10 such that $d\delta = -\beta$. It is then clear that

$$v(x, 0) \leqq m_+^\delta \leqq m_+ + d\delta \leqq m_+ - \beta \;\text{ on } \{|\phi| \leqq \gamma\}$$

and the above inequality obviously holds on $\{\phi > \beta\}$.

12. On the set $\{\phi \leqq \beta\}$, we have

$$\begin{aligned}v(x, 0) &= q^\delta\big(\varepsilon^{-1}(\phi(x) - 2\beta), \varepsilon^{-1}x, D\varphi(x, t)\big)\\ &\leqq q^\delta\left(-\varepsilon^{-1}\beta, \varepsilon^{-1}x, D\varphi(x, t)\right) \leqq m_- - \beta - o_\varepsilon(1),\end{aligned}$$

where the $o_\varepsilon(1)$ is uniform in x in view of (6.10). Hence, for ε small enough, the initial data satisfy the desired inequality and the proof of this step is complete.

13. To conclude, we need to extend the function v to be a subsolution defined on $\mathbb{R}^N \times (0, \bar{h})$. This can be done exactly as in the previous section by first extending $\sup(v, m_-)$ to $\{\phi \leqq \gamma\} \times (0, \bar{h})$ and then by proving a result analogous to Lemma 4.2 where $m_+ - \beta\varepsilon$ is replaced by $m_+ - \beta$. Since this extension is a straightforward adaptation of the argument of Section 4, we leave it to the reader. □

Before we continue it is worth remarking that the proof of Lemma 6.2 can be considerably simplified, if, as in the discussion at the end of Section 4, instead of the v in (6.13), we use

$$v(x,t) = q\left(\varepsilon^{-1}(d_\alpha(x,t) - 2\beta), \frac{x}{\varepsilon}, Dd_\alpha(x,t)\right)$$

where d_α is the signed distance to the set

$$\Gamma_{\alpha,t} = \{x \in \mathbb{R}^N : \phi(x) - t[F_\alpha(D\phi(x)) + \alpha] = 0\},$$

which is normalized to have the same sign as $\phi - t[F_\alpha(D\phi) + \alpha]$ in $\mathbb{R}^N$.

We now discuss the asymptotics of (6.2). To this end, we assume that

$$\text{if (4.6) holds and if } (c, q) \text{ is as in (6.8),} \quad \text{then } c(e) = 0 \text{ for all } e \in S^{N-1}, \tag{6.14}$$

and note that the fact that $c(e) = 0$ can be easily verified if $b = 0$, but is an assumption in general.

Next we need assumptions similar to (4.8) and (4.10). To this end, observe that, for each $e \in S^{N-1}$, linearizing around $q(\cdot, \cdot, e)$ the equation satisfied by the travelling wave leads to the unbounded operator $\mathscr{A}(e) : L^2(\mathbb{R} \times \Pi) \to L^2(\mathbb{R} \times \Pi)$ given by

$$\mathscr{A}(e)p = (D_y + e\partial_s)^T[A(y)(D_y + e\partial_s)]p + b \cdot (D_y + e\partial_s)p - f'(q)p,$$

which, unless $b = 0$, is not self-adjoint.

As before, for $\chi : \mathbb{R} \times \mathbb{R}^N \times S^{N-1} \times \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ we need to find solutions $p : \mathbb{R} \times \mathbb{R}^N \times S^{N-1} \times R^N \times [0, \infty) \to \mathbb{R}$ of the equation

$$\mathscr{A}(e)p = \chi(s, y, e, x, t) \quad \text{in } \mathbb{R} \times \mathbb{R}^N \tag{6.15}$$

such that $p \in C^2(\mathbb{R} \times \mathbb{R}^N \times S^{N-1} \times R^N \times [0, \infty))$ and, for all compact subsets K of $\mathbb{R}^N \times [0, \infty)$,

$$\begin{aligned} &p \to 0 \text{ as } |s| \to \infty, \text{ exponentially fast and uniformly in } \Pi \times S^{N-1} \times K,\\ &\sup_{\substack{(s,y,e,x,t) \in \\ \mathbb{R} \times \mathbb{R}^N \times S^{N-1} \times K}} [(|p| + |D_x p| + |D_e p| + |D_e^2 p| + |D_{y,e}^2 p|)(s, y, e, x, t) \\ &\qquad + (1 + |s|)[(\dot{p} + |s||\ddot{p}| + |D_x \dot{p}| + |D_e \dot{p}|)(s, y, e, x, t)]] < \infty. \end{aligned} \tag{6.16}$$

We next assume that

there exists a solution $X : \mathbb{R} \times \mathbb{R}^N \times S^{N-1} \to \mathbb{R}$ of

$$\mathscr{A}^*(e)X = (D_y + e\partial_s)^T[A(y)(D_y + e\partial_s)]X \\ -(D_y + e\partial_s)^T(bX) - f'(q)X = 0 \ \text{ in } \mathbb{R} \times \mathbb{R}^N \tag{6.17}$$

such that

$$X_s > 0 \text{ on } \Pi,\ \ y \mapsto X(s, y, e) \text{ is periodic in } \Pi \text{ for each } s \in \mathbb{R}, \\ \text{as } |s| \to \infty,\ X(s, y, e) \to 0 \text{ exponentially fast} \\ \text{and uniformly in } \Pi \times S^{N-1}, \tag{6.18}$$

and that

$$\ker \mathscr{A}^*(e) = X\mathbb{R}. \tag{6.19}$$

Note that if $\mathscr{A}$ is self-adjoint, i.e., if $b = 0$, we may choose $X = \dot{q}$, in which case (6.19) becomes

$$\ker \mathscr{A}^*(e) = \dot{q}\mathbb{R}.$$

Our next assumption is that

$$\text{for any compact subset } K \text{ of } \mathbb{R}^N \times [0, \infty) \text{ and for any smooth} \\ \chi : \mathbb{R} \times \mathbb{R}^N \times S^{N-1} \times K \to \mathbb{R} \text{ such that for all } (s, y) \in \mathbb{R} \times \Pi \\ \text{and all } (x, t) \in K \text{ and for some } C > 0, \\ \int\int_{-\infty}^{\infty} \chi(s, y, e, x, t)X(s, y, e)\,ds\,dy = 0, \\ \|\chi(s, y, e, \cdot, \cdot)\|_{C^2(K)} \leqq C\left[\dot{q} + |s\ddot{q}| + |q_e| + |\dot{q}_e| + |q_{ye}|\right](s, y, e), \\ \text{there exists a solution } p \text{ of (6.15) satisfying (6.16).} \tag{6.20}$$

Next for each $e \in S^{N-1}$ define the scalar

$$\mu(e) = \Big(\int_{\mathbb{R}}\int_{\Pi} \dot{q}(s, y, e)X(s, y, e)\,ds\,dy\Big)^{-1}, \tag{6.21}$$

the symmetric matrix

$$\bar{A}(e) = \int_{\mathbb{R}}\int_{\Pi} X(s, y, e)\big[\dot{q}(s, y, e)A(y) \\ + A(y)e \otimes D_e\dot{q}(s, y, e) + D_e\dot{q}(s, y, e) \otimes A(y)e +\ 2A(y)D^2_{y,e}q(s, y, e) \\ +\tfrac{1}{2}(D_eq(s, y, e) \otimes (b + \tilde{A}) + (b + \tilde{A}) \otimes D_eq(s, y, e))\big]\,ds\,dy, \tag{6.22}$$

where $\tilde{A}(y)$ is the vector whose i^{th} component is $\sum_{j=1}^N D_{y_i}a_{ij}(y)$, the vector

$$\bar{B}_1(e) = \int_{\mathbb{R}}\int_{\Pi}[X(s, y, e)\dot{q}(s, y, e)B(y)]\,ds\,dy, \tag{6.23}$$

the scalar

$$\bar{B}_2(e) = \int_{\mathbb{R}} \int_{\Pi} [X(s, y, e)B(y) \cdot D_y q(s, y, e)] ds dy, \tag{6.24}$$

and, finally, for each $(X, p) \in \mathscr{S}^N \times \mathbb{R}^N \backslash \{0\}$, the function

$$F(X, p) = -\mu(\hat{p}) \left\{ \mathrm{tr}[\bar{A}(\hat{p}) X (I - \hat{p} \otimes \hat{p})] + B_1(\hat{p}) \cdot p + \bar{B}_2(\hat{p})|p| \right\}. \tag{6.25}$$

Since F is bounded, as usual, we may extend the definition of F at $p = 0$ by considering lower and upper semi-continuous envelopes. Note also that it is easy to check that F is geometric, i.e., that it satisfies (2.10), but it is not clear a priori, or at least it is not clear to us, that it is elliptic, i.e., that it satisfies (2.3).

Consider next the initial-value problem

$$\begin{aligned} u_t + F(D^2 u, Du) = 0 & \text{ in } \mathbb{R}^N \times (0, \infty), \\ u = u_0 \qquad & \text{ on } \mathbb{R}^N \times \{0\}. \end{aligned} \tag{6.26}$$

If F is elliptic, it turns out (cf. [CGG, BSS, IS, G], etc.) that, for every $u_0 \in UC(\mathbb{R}^N)$, there exists a unique viscosity solution $u \in UC(\mathbb{R}^N \times [0, T])$ for all $T > 0$.

The result about the asymptotic behavior of (6.2) is:

Theorem 6.3. *Assume that* (4.4), (4.6), (6.5), (6.8)–(6.10), (6.14), *and* (6.17)–(6.20) *hold. Then*

(i) *The function F defined by* (6.25) *is degenerate-elliptic, i.e., it satisfies* (2.3).

(ii) *Let u_ε be the solution of* (6.2), (6.3) *with $g : \mathbb{R}^N \to [m_-, m_+]$ being a function such that the set $\Gamma_0 = \{x \in \mathbb{R}^N : g(x) = m_0\}$ is a nonempty subset of $\mathbb{R}^N$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} \end{Bmatrix},$$

where u is the unique viscosity solution of (6.26), *with $u_0 = d_0$ the signed distance to Γ_0, which is positive in the set $\{g > m_0\}$ and negative in the set $\{g < m_0\}$.*

If the no-interior condition (2.11) *holds, then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} = \left(\overline{\{u > 0\}}\right)^c \end{Bmatrix}.$$

Theorem 6.3 is proved as Theorem 4.1, provided we show that the assumptions of Theorem 3.1 are satisfied, as usual (H4) being the most important one. Instead of reproducing all the details here, we choose to show only the formal expansion argument, which explains the result. As we hope we have made clear so far, the actual proof is nothing else than a justification of these asymptotics.

An important point is, however, that here we need to argue as discussed at the end of Section 4, i.e., it is essential to consider expansions using the signed-distance function. Contrary to the situation in Section 4, here we need to go up to order ε in the expansion, i.e., to have a P-term, in order to control the additional terms which exist due to the oscillating coefficients.

To this end we write

$$u_\varepsilon(x,t) = Q\left(\varepsilon^{-1}d(x,t), \frac{x}{\varepsilon}, x, t\right) + \varepsilon P\left(\varepsilon^{-1}d(x,t), \frac{x}{\varepsilon}, x, t\right) + O(\varepsilon^2),$$

where d is the signed distance to a smooth front, and P and Q are to be chosen and substituted in (6.2). The goal, of course, is to identify an equation satisfied by d which, in turn, leads to an expression for the normal velocity of the front.

It is a simple calculation to see that

$$\begin{aligned} u_{\varepsilon,t} - \operatorname{div}\left(A\left(\tfrac{x}{\varepsilon}\right)Du_\varepsilon\right) - \varepsilon^{-1}\left(b\left(\tfrac{x}{\varepsilon}\right) + \varepsilon B\left(\tfrac{x}{\varepsilon}\right)\right)\cdot Du_\varepsilon + \varepsilon^{-2}f(u_\varepsilon) \\ = \varepsilon^{-2}\mathrm{I}_\varepsilon + \varepsilon^{-1}\mathrm{II}_\varepsilon + O(1), \end{aligned}$$

where

$$\begin{aligned} \mathrm{I}_\varepsilon &= -[(D_y + Dd(x,t)\partial_s)^T(A(y)(D_y + Dd(x,t)\partial_s)) \\ &\quad + b(y)\cdot(D_y + Dd(x,t)\partial_s)]Q + f(Q), \\ \mathrm{II}_\varepsilon &= -[(D_y + Dd(x,t)\partial_s)^T[A(y)(D_y + Dd(x,t)\partial_s)] \\ &\quad + b\cdot(D_y + Dd(x,t)\partial_s)]P + f'(Q)P \\ &\quad + \dot{Q}[d_t - \operatorname{tr}(A(y)D^2d)] - 2A(y)Dd\cdot D_x\dot{Q} - 2\operatorname{tr}\big(A(y)D^2_{x,y}Q\big) \\ &\quad - (b+\tilde{A})\cdot D_xQ - B\cdot(D_yQ + \dot{Q}Dd(x,t)). \end{aligned}$$

To simplify these expressions as usual we do not exhibit the arguments of the functions unless it is necessary to avoid confusion and we denote by y the x/ε argument.

Choosing

$$Q(s,y,x,t) = q(s,y,Dd(x,t)),$$

where q is as in (6.8) for $e = Dd(x,t)$, we immediately see that $\mathrm{I}_\varepsilon = 0$.

Moreover, if

$$P(s,y,x,t) = p(s,y,Dd(x,t),x,t),$$

a simple calculation shows that $\mathrm{II}_\varepsilon = 0$ is equivalent to p satisfying (6.15) with $e = Dd(x,t)$ and

$$\begin{aligned} \chi(s,y,Dd(x,t),x,t) &= -\dot{q}[d_t - \operatorname{tr} A(y)D^2d] + 2A(y)Dd\cdot D^2dD_e\dot{q} \\ &\quad + 2\operatorname{tr}(A(y)D^2dD^2_{y,e}q) + (b(y)+\tilde{A}(y))\cdot D^2dD_eq \\ &\quad + B(y)\cdot(\dot{q}Dd(x,t) + D_yq). \end{aligned}$$

In view of (6.20) such a p exists provided that

$$\iint \chi(s,y,Dd(x,t),x,t)X(s,y,Dd(x,t))dsdy = 0,$$

with X as in (6.17), (6.18) for $e = Dd(x,t)$.

By the periodicity of A, the compatibility condition leads to

$$d_t = \left(\iint \dot q(s,y,Dd)X(s,y,Dd)dsdy\right)^{-1}$$
$$\iint X(s,y,Dd)\big\{\mathrm{tr}\big[\dot q(s,y,Dd)A(y)D^2d + (A(y)Dd\otimes D^2dD_e\dot q(s,y,Dd)$$
$$+ D^2dD_e\dot q(s,y,Dd)\otimes A(y)Dd) + 2A(y)D^2dD^2_{y,e}q(s,y,Dd)$$
$$+\tfrac{1}{2}\big((b(y)+\tilde A(y))\otimes D_eq(s,y,Dd)$$
$$+ D_eq(s,y,Dd)\otimes(b(y)+\tilde A(y))\big)D^2d\big]$$
$$+ B(y)\cdot\big(D_yq(s,y,Dd)+\dot q(s,y,Dd)Dd\big)\big\}dsdy$$

which, of course, justifies the claim at least formally.

7. Asymptotics of Nonlocal Equations

In this section we study the asymptotics of two very general nonlocal fully nonlinear integral-differential equations which arise in the theory of stochastic Ising models with ferromagnetic long-range interactions and general spin-flip dynamics, which is briefly described in the next section.

The two equations we consider are

(7.1) $$u_t + \Phi(\beta(J*u))[u - \tanh(\beta(J*u))] = 0 \quad \text{in } \mathbb{R}^N\times(0,\infty),$$

(7.2) $$u_t + (u - J*u) + f(u) = 0 \quad \text{in } \mathbb{R}^N\times(0,\infty).$$

Here β is a positive constant, Φ is a continuous positive function, $f = W'$, W being a double-well potential, and $J \in C^1(\mathbb{R}^N,\mathbb{R})$ is assumed to be nonnegative, even, and to have compact support, i.e.,

(7.3) $$J(r) = J(-r) \geqq 0, \quad \text{and} \quad J(r) = 0 \quad \text{if } |r| > R \text{ for some } R > 0.$$

The assumption that J has compact support is made only to simplify the arguments and can be easily removed by specifying appropriate growth and integrability conditions on J at infinity. We leave this task to the reader. That J is nonnegative is, however, very important both from the physical and analytical point of view.

Before we make precise assumptions about the rest of the terms in (7.1) and (7.2), we remark that since we are interested in the asymptotic behavior of the solutions for large (x,t), it is appropriate to introduce the scaling $(x,t)\mapsto(\varepsilon^{-1}x,\varepsilon^{-2}t)$. To this end, we define

$$u_\varepsilon(x,t) = u(\varepsilon^{-1}x,\varepsilon^{-2}t),$$

and observe that, if u satisfies (7.1), then u_ε satisfies the equation

(7.4) $$u_{\varepsilon,t} - \varepsilon^{-2}\Phi(\beta(J^\varepsilon*u_\varepsilon))[u_\varepsilon - \tanh\big(\beta(J^\varepsilon*u_\varepsilon)\big)] = 0 \quad \text{in } \mathbb{R}^N\times(0,\infty),$$

and, if u is a solution of (7.2), then u_ε satisfies

(7.5) $$u_{\varepsilon,t} + \varepsilon^{-2}[u_\varepsilon - J^\varepsilon * u_\varepsilon] + \varepsilon^{-2} f(u_\varepsilon) = 0 \ \text{ in } \mathbb{R}^N \times (0,\infty),$$

where, for $x \in \mathbb{R}^N$,

$$J^\varepsilon(x) = \varepsilon^{-N} J(\varepsilon^{-1}x).$$

We are going to consider the equations (7.4) and (7.5) together with the initial data

(7.6) $$u_\varepsilon(x,0) = g(x) \ \text{ in } \mathbb{R}^N.$$

This section is divided into three parts. The first two are about the asymptotics of (7.4) and (7.5), while in the third part we discuss the meaning of the results.

7.1. The Asymptotics of (7.4)

In addition to (7.3) here we assume that

(7.7) $$\beta \bar{J} = \beta \int J(x)dx > 1.$$

It turns out that (7.1) and, therefore (7.4), admits three steady-state solutions $-m_\beta$, 0 and m_β, with $m_\beta > 0$, i.e., solutions of the algebraic equation

$$s = \tanh\big(\beta \bar{J} s\big),$$

which are, by the way, independent of Φ.

The precise assumptions on $\Phi : \mathbb{R} \to \mathbb{R}$ are

(7.8) $$\begin{gathered} \Phi > 0, \quad \Phi \in C(\mathbb{R}), \ \text{ and for all } m \in [-m_\beta, m_\beta] \\ \text{and } r \in [-\beta \bar{J} m_\beta, \beta \bar{J} m_\beta], \\ r \mapsto \Phi(r)(m - \tanh r) \text{ is nonincreasing in } r. \end{gathered}$$

An immediate consequence of (7.8) is that (7.1) and (7.4) satisfy a comparison principle. Since this is more or less straightforward, we leave it up to the reader to fill in the details.

We need to assume that (7.1) admits, for all directions $e \in S^{N-1}$, travelling-(standing-) wave solutions connecting $-m_\beta$ and m_β, i.e., solutions of the form

$$u(x,t) = q(x \cdot e, e),$$

where $q : \mathbb{R} \times S^{N-1} \to \mathbb{R}$ is such that $q(\pm\infty, e) = \pm m_\beta$.

We assume that

(7.9) $$\begin{gathered} \text{for all } e \in S^{N-1}, \text{ there exists } q : \mathbb{R} \times S^{N-1} \to \mathbb{R} \text{ such that} \\ q(\xi, e) = \tanh\left[\beta \textstyle\int_{\mathbb{R}^N} J(y) q(\xi + e \cdot y, e) dy\right], \\ q(0,e) = 0 \quad \text{and} \quad \dot{q}(\cdot, e) > 0 \ \text{ on } \mathbb{R}, \\ \text{and, as } \xi \to \pm\infty,\ q(\xi, e) \to \pm m_\beta \text{ exponentially fast with the} \\ \text{rate depending on } e \text{ and } q \in C^2(\mathbb{R} \times S^{N-1}) \cap W^{2,\infty}(\mathbb{R} \times S^{N-1}). \end{gathered}$$

The existence of such q's, which is by no means trivial, is discussed in BATES, FIFE, REN & WANG [BFRW], KATSOULAKIS & SOUGANIDIS [KS3], and ORLANDI & TRIOLO [OT], to which we refer for details. In the isotropic case, i.e., when $J(r) = J(|r|)$, the standing-wave solutions are independent of the direction e. For a detailed study of such solutions we refer to the paper of DE MASI, GOBRON & PRESUTTI [DGP].

For notational simplicity below we rewrite the equation satisfied by q as

$$q(\xi, e) = \tanh\left[\beta(J * q)(\xi, e)\right],$$

where, for all $(\xi, e) \in \mathbb{R}^N \times S^{N-1}$,

$$(J * q)(\xi, e) = \int J(y')q(\xi + y' \cdot e, e)dy'. \tag{7.10}$$

Finally we need some assumptions which play the role of (4.8) and (4.10). The linearization, the equation satisfied by the travelling wave around $q(\cdot, e)$ for each $e \in S^{N-1}$, leads to the unbounded, self-adjoint operator

$$\mathscr{A}(e)p = (1 - q^2(\xi, e))^{-1}p - \beta(J * p).$$

Notice that we write $(1-q^2)^{-1}p - \beta(J * p)$ instead of $p - \beta(1-q^2)J * p$ exactly in order for $\mathscr{A}$ to be self-adjoint (the operator $p \mapsto J * p$ is self-adjoint since J is even). It is now a straightforward exercise to check that $\dot{q}(\cdot, \cdot e) \in \ker \mathscr{A}(e)$ for all $e \in S^{N-1}$.

We assume that, for each $e \in S^{N-1}$,

$$\ker \mathscr{A}(e) = \dot{q}(\cdot, \cdot, e)\mathbb{R}. \tag{7.11}$$

In our analysis below we need to solve, for appropriate functions

$$\chi : \mathbb{R} \times S^{N-1} \times \mathbb{R}^N \times [0, \infty) \to \mathbb{R},$$

the equation

$$\mathscr{A}(e)p = \chi(\cdot, e, x, t) \quad \text{on } \mathbb{R}. \tag{7.12}$$

We assume that

for all compact subsets K of $\mathbb{R}^N \times [0, \infty)$ and for all smooth
function $\chi : \mathbb{R} \times S^{N-1} \times K \to \mathbb{R}$, such that
for all $(\xi, e, x, t) \in \mathbb{R} \times S^{N-1} \times K$ and for some $B > 0$,

$$\int_{-\infty}^{\infty} \chi(\xi, e, x, t)\dot{q}(\xi, e)d\xi = 0,$$

(7.13)

$$\|\chi(\xi, e, \cdot, \cdot)\|_{C^2(K)} \leqq B\left[[(|y|^2 J) * \dot{q}](\xi, e) + [(|y|J) * |q_e|](\xi, e)|\right],$$

there exists a solution $p \in C^2(\mathbb{R} \times S^{N-1} \times K)$ of (7.12) such that
$\|p\|_{W^{2,\infty}(\mathbb{R}\times S^{N-1}\times K)} < \infty$ and $p(\xi, e, x, t) \to 0$
exponentially fast as $|\xi| \to \infty$.

To state our result about the asymptotics of (7.4) we need to introduce the scalar $\mu : S^{N-1} \to \mathbb{R}$ and the matrix $A(e) : S^{N-1} \to \mathscr{S}^N$, given by

$$\mu(e) = \beta \left[\int \frac{(\dot{q}(\xi, e))^2}{\Phi(\beta(J * q)(\xi, e))(1 - q^2(\xi, e))} \, d\xi \right]^{-1}, \tag{7.14}$$

$$\begin{aligned} \tilde{A}(e) = \tfrac{1}{2} \iint J(r)\dot{q}(\xi, e)\big[\dot{q}(\xi + r \cdot e, e)(r \otimes r) \\ + D_e q(\xi + r \cdot e, e) \otimes r + r \otimes D_e q(\xi + r \cdot e, e)\big] dr d\xi. \end{aligned} \tag{7.15}$$

Notice that if J is radially symmetric, then q is independent of e and, hence, $\tilde{A}(e) = \theta I$ for the obvious choice of the constant θ.

Next define the function $F : \mathscr{S}^N \times \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ given by

$$F(X, p) = -\mu(\hat{p}) \operatorname{tr} \left[\tilde{A}(\hat{p}) X (\mathrm{I} - \hat{p} \otimes \hat{p}) \right]. \tag{7.16}$$

It is straightforward to check that F is geometric, i.e., that it satisfies (2.10) but it is not clear a priori (at least to us) whether it is elliptic, i.e., whether it satisfies (2.3). Finally as usual we extend F to $p = 0$ by considering upper- and lower-semicontinuous envelopes.

Consider next the initial-value problem

$$\begin{aligned} u_t + F(D^2 u, Du) = 0 \quad & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = u_0 \quad & \text{on } \mathbb{R}^N \times \{0\}, \end{aligned} \tag{7.17}$$

and recall (cf. [BSS, CGG, IS, G], etc.) that, under the above assumptions and if F is degenerate-elliptic, it admits, for all $u_0 \in UC(\mathbb{R}^N)$, a unique viscosity solution $u \in UC(\mathbb{R}^N \times [0, T])$ for all $T > 0$.

Our result about the asymptotics of (7.4) is

Theorem 7.1. *Assume that* (7.3), (7.8), (7.9) *and* (7.13) *hold. Then:*

(i) *The function F defined by* (7.16) *is degenerate-elliptic.*

(ii) *Let u_ε be the solution of* (7.4)–(7.6) *with $g : \mathbb{R}^N \to [-m_\beta, m_\beta]$ being a function such that the set $\Gamma_0 = \{x : g(x) = 0\}$ is a nonempty subset of $\mathbb{R}^N$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_\beta \\ -m_\beta \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} \end{Bmatrix},$$

where u is the unique viscosity solution of (7.16) *with $u_0 = d_0$ the signed distance to Γ_0, which is positive in $\{g > 0\}$ and negative in $\{g < 0\}$.*

If, in addition, the nonempty condition (2.11) *holds, then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_\beta \\ -m_\beta \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \overline{\{u > 0\}^c} \end{Bmatrix}.$$

Theorem 7.1 is proved by using the general abstract method of Theorem 3.1 along the lines described earlier in previous sections. As before, the main issue is to check (H4). This follows by a combination of the results of [DGP], who studied the initialization of the front for the isotropic case although their results can be adapted for our situation, and the results of [KS3], who studied Theorem 7.1 under the assumption that the propagating front is smooth.

7.2. The Asymptotics of (7.5)

We begin with an assumption about the existence, for each $e \in S^{N-1}$, of travelling-wave solutions of (7.2) connecting m_- and m_+, i.e., solutions of the form

$$u(x,t) = q(x \cdot e, e),$$

where $q : \mathbb{R} \times S^{N-1} \to \mathbb{R}$ is such that $q(\pm\infty, e) = m_\pm$. The existence of such solutions when f satisfies (4.4) and (4.6) is studied in [BFRW].

With the same notations as in (7.9) we assume that

$$\begin{gathered}\text{for each } e \in S^{N-1}\text{, there exists a unique solution } q : \mathbb{R} \times S^{N-1} \to \mathbb{R} \text{ of} \\ J * q - q = f(q) \quad \text{in } \mathbb{R} \\ \text{such that } q(0,e) = m_0, \ \ \dot{q}(\cdot, e) > 0 \text{ on } \mathbb{R}, \\ q \in C^2(\mathbb{R} \times S^{N-1}) \cap W^{2,\infty}(\mathbb{R} \times S^{N-1}) \text{ and, as } \xi \to \pm\infty, \\ q(\xi, e) \to m_\pm \text{ exponentially fast and with a rate depending on } e.\end{gathered} \tag{7.18}$$

Linearizing the equation of the standing wave around $q(\cdot, e)$, for each $e \in S^{N-1}$ leads to the unbounded self-adjoint operator

$$\mathscr{A}(e)p = J * p - p - f'(q)p.$$

It is immediate that $\dot{q}(\cdot, e) \in \ker \mathscr{A}(e)$. Below we assume that, for all $e \in S^{N-1}$,

$$\ker \mathscr{A}(e) = \dot{q}(\cdot, e)\mathbb{R}. \tag{7.19}$$

As before, we need to find solutions to the equation

$$\mathscr{A}(e)p = J * p - p - f'(q)p = \chi(\xi, e, x, t) \quad \text{in } \mathbb{R}, \tag{7.20}$$

for appropriate functions $\chi : \mathbb{R} \times S^{N-1} \times \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$. We assume that

$$\begin{aligned}
&\text{for all compact subsets } K \text{ of } \mathbb{R}^N \times [0,\infty) \text{ and for all}\\
&\text{smooth function } \chi : \mathbb{R}\times S^{N-1}\times K \to \mathbb{R}, \text{ such that, for all}\\
&(\xi,e,x,t) \in \mathbb{R}\times S^{N-1}\times K \text{ and for some } C>0,\\
&\int_{-\infty}^{\infty} \chi(\xi,e,x,t)\dot{q}(\xi,e)d\xi = 0,\\
&\|\chi(\xi,e,\cdot,\cdot)\|_{C^2(K\times S^{N-1})}\\
&\qquad \leqq C[((|y|^2J)*\dot{q})(\xi,e) + ((|y|J)*|q_e|)(\xi,e)],\\
&\text{there exists a solution } p\in C^2(\mathbb{R}\times S^{N-1}\times K) \text{ of (7.20) such that}\\
&\|p\|_{W^{2,\infty}(\mathbb{R}\times S^{N-1}\times K)} < \infty,\\
&\text{and, as } |\xi|\to\infty,\ p(\xi,e,x,t)\to 0\\
&\text{exponentially fast and uniformly in } S^{N-1}\times K.
\end{aligned} \tag{7.21}$$

Finally, we define the function $F : S^N \times \mathbb{R}^N\backslash\{0\} \to \mathbb{R}$ by

$$F(X,p) = -\mu(\hat{p})\,\mathrm{tr}\left[\tilde{A}(\hat{p})X(I-\hat{p}\otimes\hat{p})\right], \tag{7.22}$$

where, for each $p \in \mathbb{R}^N\backslash\{0\}$,

$$\mu(\hat{p}) = \Big(\int_{\mathbb{R}} \dot{q}^2(\xi,\hat{p})d\xi\Big)^{-1}, \tag{7.23}$$

$$\begin{aligned}\tilde{A}\big(X,\hat{p}\big) = \tfrac{1}{2}\iint J(y)\dot{q}(\xi,\hat{p})[\dot{q}(\xi+\hat{p}\cdot y,\hat{p})(y\otimes y)\\ + D_eq(\xi+y\cdot\hat{p},\hat{p})\otimes y + y\otimes D_eq(\xi+y\cdot\hat{p},\hat{p})]dyd\xi.\end{aligned} \tag{7.24}$$

Since F is bounded, as usual, we extend it to $p=0$, by using semicontinuous envelopes. Again it is immediate that the nonlinearity F is geometric, i.e., that it satisfies (2.10), but it is not clear a priori that F is also degenerate-elliptic, i.e., that it satisfies (2.3).

Consider next the initial-value problem

$$\begin{aligned} u_t + F(D^2u, Du) = 0 &\quad \text{in } \mathbb{R}^N\times(0,\infty),\\ u = u_0 &\quad \text{on } \mathbb{R}^N\times\{0\},\end{aligned} \tag{7.25}$$

and recall that if the above assumptions on q and J hold and if F is degenerate-elliptic, then (7.25) admits, for all $u_0 \in UC(\mathbb{R}^N)$, a unique viscosity solution $u \in UC(\mathbb{R}^N\times[0,T])$ for all $T>0$.

We have

Theorem 7.2. *Assume that f satisfies* (4.4) *and* (4.6) *and that there exists a q such that* (7.18) *and* (7.21) *hold. Then*

(i) *The function F defined by* (7.22) *is degenerate-elliptic.*

(ii) *Let u_ε be the solution of* (7.5), (7.6) *with $g : \mathbb{R}^N \to [m_-, m_+]$ such that the set $\Gamma_0 = \{x : g(x) = m_0\}$ is a nonempty subset of $\mathbb{R}^N$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \{u < 0\} \end{Bmatrix},$$

where u is the unique viscosity solution of (7.25) *with $u_0 = d_0$ the signed distance to Γ_0 which is positive in the set $\{g > m_0\}$ and negative in the set $\{g < m_0\}$.*

If, in addition, the no-interior condition (2.11) *holds, then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to \begin{Bmatrix} m_+ \\ m_- \end{Bmatrix} \text{ locally uniformly in } \begin{Bmatrix} \{u > 0\} \\ \overline{\{u > 0\}}^c \end{Bmatrix}.$$

Theorem 7.2 follows from the general abstract method as soon as (H4) is checked. This is done by proving two lemmas, one about initializing the front and the other about its propagation as long as it remains smooth. The first lemma is proved along the lines of the corresponding lemma in Section 4. The proof of the second lemma, which is, as always, more complicated, follows along the lines of the discussion at the end of Section 4, as soon as the correct asymptotics are identified. Since we have repeated this argument several times so far, we only show the formal asymptotics. It is worth mentioning, however, that here we can derive the necessary information to present a proof for the second lemma along the lines of the formal asymptotics. Since this takes a number of pages, even at the formal level, we choose not to present them here.

To this end we look for an expansion of u_ε of the form

$$u_\varepsilon(x,t) = Q(\varepsilon^{-1}d(x,t), x, t) + \varepsilon P(\varepsilon^{-1}d(x,t), x, t) + O(\varepsilon^2), \tag{7.26}$$

where d is the signed-distance to the front and Q and P are to be chosen.

Substituting in (7.5) and rearranging terms we find

$$u_{\varepsilon,t} + \varepsilon^{-2}(u_\varepsilon - J^\varepsilon * u_\varepsilon) + \varepsilon^{-2} f(u_\varepsilon) = \varepsilon^{-2}\mathrm{I}_\varepsilon + \varepsilon^{-1}\mathrm{II}_\varepsilon + O(1),$$

with

$$\mathrm{I}_\varepsilon = Q - \int J(y) Q(\xi + Dd\cdot y, x, t)dy + f(Q),$$

$$\mathrm{II}_\varepsilon = \dot{Q} d_t - \int J(y)\Big[\tfrac{1}{2}\dot{Q}(\xi + Dd\cdot y, x, t) D^2 dy\cdot y + D_x Q(\xi + Dd\cdot y, x, t)\cdot y\Big]dy$$
$$+ P - \int J(y) P(\xi + Dd\cdot y, x, t)dy + f'(Q)P,$$

where d and its derivatives are evaluated at (x,t) and ξ stands for the $\varepsilon^{-1}d$ argument.

It is now clear that, if Q is chosen so that

$$Q(\xi, x, t) = q(\xi, Dd(x,t)),$$

with q as in (7.17), then $\mathrm{I}_\varepsilon = 0$.

With this Q, if we also choose

$$P(\xi, x, t) = p(\xi, Dd(x,t), x, t),$$

the equation $\mathrm{II}_\varepsilon = 0$ takes the form

$$J * p - p - f'(q)p = \chi(\xi, Dd(x,t), x, t),$$

where, for $e = Dd(x,t)$,

$$\begin{aligned}\chi(\xi, e, x, t) = -\dot{q}(\xi, e, x, t)d_t \\ &+ \tfrac{1}{2}\int J(y)\dot{q}(\xi + e\cdot y, e)D^2 dy \cdot y dy \\ &+ \int J(y)D^2 d D_e q(\xi + e\cdot y, e)\cdot y dy.\end{aligned}$$

In view of (7.21) such a p exists provided that

$$\int \chi(\xi, e, x, t)\dot{q}(\xi, e)d\xi = 0,$$

which leads to

$$d_t + F(D^2 d, Dd) = 0,$$

with F as given by (7.22).

We leave the details to the reader.

7.3. A Few Comments

One may simplify (7.4) and (7.5) by substituting $J_2(\Delta m - m)$ for the convolution term $J * m$ (see, for example, PENROSE [P]), where $\bar{J}_2 = \int J(|r|)|r|^2 dr$, or even additionally linearize the hyperbolic tangent, thus obtaining a Ginzburg-Landau equation like (4.1). It is known (see, for example, [J, ESS, and BSS]) that, in the isotropic case, both simplified models have the same qualitative asymptotic behavior as (4.3) with different scalar coefficients for the curvature. In the anisotropic case, however, this picture is no longer true. The second-order approximations described earlier still give, in the limit $\varepsilon \to 0$, isotropic motion by mean curvature with a constant transport coefficient, while (7.4) and (7.5), according to our analysis, yield the anisotropic equations (7.17) and (7.25) respectively, with the μ and $\tilde{A}$ given by the Green-Kubo formulae (7.14) and (7.15) and (7.23) and (7.24) respectively. It appears that anisotropy is a higher-order effect which cannot be accounted for only with second-order approximating equations. This phenomenon is also pointed out by CAGINALP & FIFE [CF], where depending on the type of anisotropy expected, they "correct" (4.1) by suitably adding higher-order derivatives.

8. Stochastic Ising Models

In this section we use the results of Section 7 to obtain some new results regarding hydrodynamic (macroscopic) limits of ferromagnetic stochastic Ising models with long-range interactions and general spin-flip dynamics, which in the sequel we will call IPS (interacting particle systems) for short. Stochastic Ising models are the canonical Gibbsian models used in statistical physics to describe phase transitions. Below we briefly recall the basic facts about IPS. For a more detailed discussion we refer, for example, to De Masi, Orlandi, Presutti & Triolo [DOPT], Spohn [Sp], etc.

Ising models are interacting-particle (spin) systems on the lattice $\mathbf{Z}^N$. A spin configuration σ is an element of the state (configuration) space $\Sigma = \{-1,1\}^{\mathbf{Z}^N}$. We write $\sigma = \{\sigma(x) : x \in \mathbf{Z}^N\}$ and call $\sigma(x)$ the spin at x.

The energy H of the system, evaluated at σ, is given by

$$H(\sigma) = \sum_{x \neq y} J_\gamma(x,y)\sigma(x)\sigma(y) + h \sum \sigma(x),$$

where h is attributed to an external magnetization field and J_γ, $\gamma^{-1} > 0$ being the interaction range, is the Kač potential defined by

$$J_\gamma(x,y) = \gamma^N J(\gamma(x-y)) \qquad (x, y \in \mathbf{Z}^N). \tag{8.1}$$

Here $J : \mathbb{R}^N \to \mathbb{R}$ is assumed to be nonnegative, even, and to have compact support, i.e., to satisfy (7.3). We refer to Section 7 for a discussion of the meaning of these assumptions.

The dynamics of the model consist of a sequence of flips. If σ is the configuration before a flip at x, then after the flip at x the configuration is

$$\sigma^x(y) = \begin{cases} -\sigma(x) & \text{if } y = x, \\ \sigma(y) & \text{if } y \neq x. \end{cases}$$

We assume that a flip occurs at x, when the configuration is σ, with a rate $c_\gamma(x,\sigma)$, given by

$$c_\gamma(x,\sigma) = \Psi(-\beta h_\gamma(x)\sigma(x)), \tag{8.2}$$

where $\beta > 0$ is identified with the inverse temperature,

$$h_\gamma(x) = h + \sum_{y \neq x} J_\gamma(x,y)\sigma(y), \tag{8.3}$$

and $\Psi = \Psi(r) > 0$ satisfies the detailed balance law

$$\Psi(r) = \Psi(-r)e^{-r} \qquad (r \in \mathbb{R}). \tag{8.4}$$

It follows easily from the above that

$$h_\gamma = \Delta_x H = H(\sigma^x) - H(\sigma),$$

i.e., the change in the energy due to a flip at x.

The underlying process is a jump process in $L^\infty(\Sigma;\mathbb{R})$ with generator given by

$$L_\gamma f(\sigma) = \sum_{x\in\mathbf{Z}^N} c_\gamma(x,\sigma)[f(\sigma^{(x)}) - f(\sigma)].$$

Such processes leave the Gibbs measures, associated with the Hamiltonian H and the inverse temperature β, invariant. Typical choices of Ψ's are $\Psi(r) = (1+e^r)^{-1}$ (Glauber dynamics), $\Psi(r) = e^{-r/2}$ (Arrhenius dynamics) or $\Psi(r) = e^{-r^+}$ (Metropolis dynamics).

A very basic question in the theory of stochastic Ising models with Kač potentials is the behavior of the system as the interaction range tends to infinity, i.e., $\gamma \to 0$. The passage in the limit $\gamma \to 0$, which in the physics literature is identified with grain coarsening, of quantities like the thermodynamical pressure, total magnetization, etc., is known as the Lebowitz-Penrose limit (see, for example, [DP; K1,2,3], etc.).

Along these lines we study the asymptotics, as $\gamma \to 0$, of the averaged magnetization

(8.5) $$m_\gamma(x,t) = \mathbb{E}_{\mu^\gamma}\sigma_t(x) \qquad ((x,t)\in \mathbf{Z}^N \times [0,\infty))$$

of the system, where $\mathbb{E}_{\mu^\gamma}$ denotes the expectation of the IPS starting from a measure μ^γ. The relevant mesoscopic mean-field equation is

(8.6) $$m_t + \Phi(\beta(J*m))[m - \tanh\beta(J*m)] = 0 \quad \text{in } \mathbb{R}^N\times[0,\infty),$$

where

(8.7) $$\Phi(r) = \Psi(-2r)(1+e^{-2r}).$$

Indeed the following theorem is proved in KATSOULAKIS & SOUGANIDIS [KS3], where we refer for a discussion about its history, relevance, etc.

Theorem 8.1. *Assume that the IPS defined earlier has as initial measure a product measure μ^γ such that*

$$\mathbb{E}_{\mu^\gamma}(\sigma(x)) = m_0(\gamma x) \qquad (x\in\mathbf{Z}^N),$$

where m_0 is Lipschitz continuous and that (8.4) *holds. Then, for each $n\in\mathbf{Z}^+$,*

$$\lim_{\gamma\to 0}\sup_{\underline{x}\in\mathbf{Z}^N_n}\left|\mathbb{E}_{\mu^\gamma}\left(\prod_{i=1}^n \sigma_t(x_i)\right) - \prod_{i=1}^n m(\gamma x_i,t)\right| = 0,$$

where m is the unique solution of (8.6) *with initial datum m_0.*

In the above statement for each n,

$$\mathbf{Z}^N_n = \{\underline{x} = (x_1,\dots,x_n)\in\mathbf{Z}^N : x_1 \ne \cdots \ne x_n\}.$$

To state our result for the IPS, if u is the solution of (2.10), for $t>0$, we define the sets

$$P^\gamma_t = \{x\in\mathbf{Z}^N : u(\gamma\varepsilon(\gamma)x,t) > 0\},\quad N^\gamma_t = \{x\in\mathbf{Z}^N : u(\gamma\varepsilon(\gamma)x,t) < 0\},$$

$$M^n_{\gamma,t} = \{\underline{x}\in\mathbf{Z}^N_n : x_i \in P^\gamma_t \cup N^\gamma_t\}.$$

The result is

Theorem 8.2. *Assume that $\beta \bar{J} > 1$. Under the assumptions of Theorem 7.1 on the initial measure, there exists a $\rho^* > 0$ such that for any $\varepsilon(\gamma)$ with $\gamma^{-\rho^*}\varepsilon(\gamma) \to +\infty$ as $\gamma \to 0$, and for all $t > 0$,*

$$\lim_{\gamma\to 0} \sup_{\underline{x}\in M^n_{\gamma,t}} \left| E_{\mu^\gamma} \prod_{i=1}^{n} \sigma_{t\varepsilon(\gamma)^{-2}}(x_i) - m^n_\beta \prod_{i\in N^\gamma_t} (-1) \right| = 0,$$

with the limit local uniform in t.

Theorem 8.2 follows from Theorem 7.1 the same way as the analogous theorem in [KS2]; we therefore do not present its proof here.

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Laboratoire de Mathématiques et Physique Théorique
Faculté des Sciences et Techniques
Université de Tours
Parc de Grandmont
37200 Tours, France

and

Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin

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