Arch. Rational Mech. Anal. 141 (1998) 1–61.

Slow Dynamics for the Cahn-Hilliard Equation in Higher Space Dimensions: The Motion of Bubbles

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Contents

Abstract

It is known that the Van der Waals-Cahn-Hilliard (W-C-H) dynamics can be approximated by a Quasi-static Stefan problem with surface tension. It turns out that the Stefan problem has a manifold of equilibria equal in dimension to that of the domain Ω: any sphere of fixed radius with interface contained in the domain is an equilibrium (indistinguishable from the point of view of the perimeter functional). We resolve this degeneracy by showing that at the W-C-H level this manifold is replaced by a quasi-invariant stable manifold, on which the typical solution preserves its "bubble" like shape until it reaches the boundary. Moreover, we show that the "bubble" moves superslowly. We also obtain an equation that determines those special spheres that correspond to equilibria at the W-C-H level. Our work establishes the phenomenon of superslow motion in higher space dimensions in the class of single interface solutions.

1. Introduction

A. General Remarks

The Cahn-Hilliard equation

$$(1.1)\qquad \begin{aligned} u_t &= -\Delta(\varepsilon^2\Delta u - F'(u)), & x\in\Omega,\\ \frac{\partial u}{\partial n} &= \frac{\partial}{\partial n}(\varepsilon^2\Delta u - F'(u)) = 0, & x\in\partial\Omega \end{aligned}$$

is a known model describing the phase separation and subsequent coarsening of binary alloys [C1, C2, C3, C-H, F1, F2, F3, E, E-Z, Gu1, Gu2, Gu3, B-E1, B-E2, Ca, Ca-F, P-F, Pe]. Here $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain which represents the container, u is the concentration of one of the species, and $F(u)$ is the bulk free energy per unit volume. F is assumed smooth with two equal nondegenerate minima, at $u=\pm1$. A typical example is $F(u)=\frac{1}{4}(u^2-1)^2$. The constants $u=\pm1$ are stable solutions of (1.1), and model the situation resulting when two different stable phases having two different concentrations (conventionally assumed to be $u=\pm1$) coexist at thermodynamical equilibrium. The term containing the singular parameter ε in (1.1) models the effect of interfacial energy on the separation phenomenon, where ε is a measure of the relative importance of surface energy to bulk free energy.

As was observed by Fife [F4], the Cahn-Hilliard equation can be realized as the gradient flow for the free-energy functional

$$(1.2)\qquad J_\varepsilon(u) = \int_\Omega \left[\frac{\varepsilon^2}{2}|\nabla u|^2 + F(u)\right] dx$$

on the Hilbert space H_0^{-1} (the closed subspace of H^{-1} of functions with zero average):

$$(1.3)\qquad \frac{du}{dt} = -\mathrm{grad}_{H_0^{-1}} J_\varepsilon(u).$$

Consequently $J_\varepsilon(\cdot)$ is nonincreasing along solutions and therefore one can expect that, for $\varepsilon\ll1$, solutions of (1.1) stay mostly near $u=-1$ or $u=+1$, the minima of $F(u)$. There is ample numerical evidence [E-Fr, McK, Ey1, Ey2] supported by some theoretical work [G1] that the typical initial condition for $0<\varepsilon\ll1$ evolves into a layered function in space. Because of this, as soon as this initial stage is completed, we can think of Ω as split into subdomains on which $u_\varepsilon(\cdot,t)$ takes approximately the constant values -1 and 1, with boundaries ε-localized about an interface $\Gamma_\varepsilon(t)$. In agreement with the physical situation the Cahn-Hilliard equation (1.1) preserves the mass of each component:

$$(1.4)\qquad \int_\Omega u\,dx = \text{ constant along the evolution.}$$

This puts a constraint on the relative size of the regions $u\simeq-1$ and $u\simeq1$ corresponding to the two phases, and therefore also on the dynamics of the thin zone about $\Gamma_\varepsilon(t)$ separating these regions. The initial separation can happen in two ways, depending on whether the initial datum is a perturbation of a constant state in the *spinodal* region s [B-F1, E-Fr, G1], or in the *metastable* region m [B-F2] (see Fig. 1).

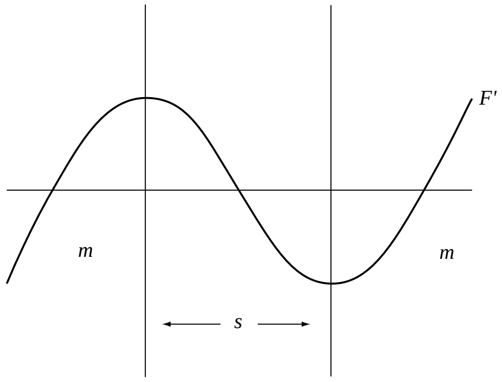

Fig. 1.

This paper deals with fully layered solutions and is focused on a special stage in the evolution of $\Gamma_\varepsilon(t)$. Formal analysis performed by PEGO [Pe], recently supported by rigorous work of STOTH [St] and ALIKAKOS, BATES & CHEN [A-B-Ch] in general, establishes the following geometric evolution law of the fronts Γ_ε in the limit $\varepsilon \to 0$:

$$v = b\left[\frac{\partial\mu}{\partial n}\right]_{\Gamma(t)} \tag{1.5}$$

where

$$\begin{aligned} \Delta\mu &= 0, & x &\in \Omega/\Gamma(t), \\ \frac{\partial\mu}{\partial n} &= 0, & x &\in \partial\Omega, \\ \mu &= \varepsilon a K, & x &\in \Gamma(t). \end{aligned}$$

Problem (1.5) will be referred to as the *Mullins-Sekerka problem*. Here $\Gamma(t) = \lim \Gamma_\varepsilon(t)$ is the interface at time t; a, b are constants; K is the mean curvature of $\Gamma(t)$ at x; $\left[\frac{\partial\mu}{\partial n}\right]$ is the jump of $\frac{\partial\mu}{\partial n}$ across $\Gamma(t)$; and v is the normal component of the velocity of $\Gamma(t)$. We note that by switching to the slow time scale $\tau = \varepsilon t$, ε can be scaled out from (1.5) confirming its geometric character. It is easy to see that formally (1.5) is a perimeter-shortening volume-preserving law; these facts reflect the monotonicity of $J_\varepsilon(u(t))$ and the conservation (1.4). Rigorous results on the well-posedness of (1.5) are due to X. CHEN and his collaborators [Ch1, Ch-H-Y]. There is also related work of CONSTANTIN & PUGH [C-Pu] and ESCHER & SIMONETT [E-Si] on the one-phase Hele-Shaw problem. Numerics suggest that certain initial curves may pinch off and subsequently decompose into many components [B, Ch2].

It is easy to see that a sphere, or more generally a (disconnected) surface, consisting of a finite number of equal non-overlapping spheres contained in Ω, is an equilibrium for (1.5). This paper is concerned with the degeneracy of (1.5) — the existence of a continuum of spherical equilibria — and it carries out a resolution via (1.1). First considered is the manifold of equilibria of (1.5) constructed as follows: A radius $\rho > 0$ is fixed such that the set $\Omega_\rho = \{\xi \in \Omega / d(\xi, \partial\Omega) - \rho > 0\}$ is nonempty. Each point ξ in this set can be identified with the center of a sphere having radius ρ. The set of single spheres with radius ρ and center in $\Omega_{\rho+\delta}$, $0 < \delta \ll 1$ a fixed number, is denoted by M_ρ. The following related questions are now addressed (see Fig. 2):

(1) Is there an invariant manifold M_ρ^ε for (1.1), $0 < \varepsilon \ll 1$, which corresponds to M_ρ?
(2) Which equilibria on M_ρ correspond to equilibria at the $\varepsilon > 0$ level?
(3) What is the stability of M_ρ^ε?

More generally one could consider the manifold of equilibria obtained by moving N spheres of the same radius around Ω_ρ, with the condition that they stay distance δ apart. Unlike the single sphere case, the corresponding invariant manifold at the ε-level is highly unstable, and for this reason it is not considered in this paper.

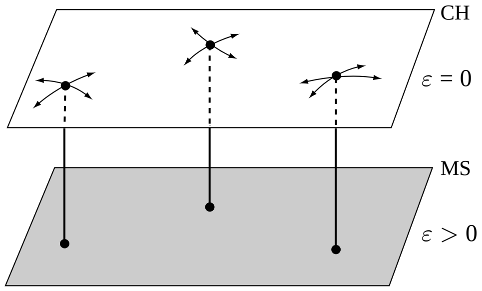

Fig. 2.

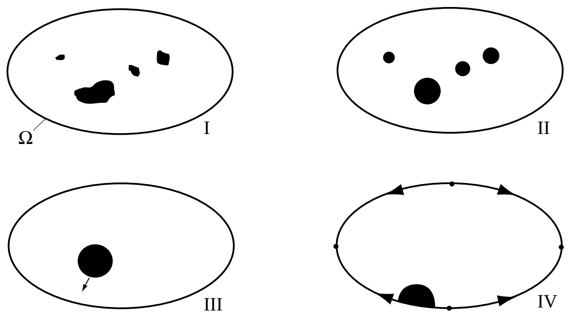

Fig. 3.

We remark that here lies an important difference between one and more space dimensions: In one dimension the analog of the multisphere manifold is stable (see BATES & XUN [B-X1, B-X2]. To put our results in perspective, we feel it is useful to give an idea of the whole evolution of a typical layered initial condition for (1.1), from the beginning of time to the end. The picture put forward has been only partially justified and so it should be taken, on the whole, as a speculation. In Fig. 3, four easily distinguishable stages in the evolution $0 < \varepsilon \ll 1$ are depicted. Stage I persists for a time of order $1/\varepsilon$; see ALIKAKOS, BATES & CHEN [A-B-Ch]. The transition from I to II is speculative. So is the transition from II to III which is also suggested in part by the instability of the multisphere manifold mentioned above. Stage III persists for a time of the order $e^{c/\varepsilon}$. The "*bubble*" retains its shape until it reaches the boundary $\partial\Omega_\rho$, and it does so with a speed that is exponentially small. The motion is directed, roughly, towards the point on the boundary $\partial\Omega$ that is closest to the bubble. Stage III is studied in the present paper. The transitions from III to IV should be abrupt, similar to the disappearance of layers for (1.1) in one space dimension (MCKINNEY [McK]). On stage IV there is related rigorous work of ALIKAKOS, CHEN & FUSCO [A-Ch-F]. It is established in that work (for a related equation) that the "*droplet*" shape, when sufficiently small, persists and crawls on the boundary $\partial\Omega$ towards points where the curvature attains a local maximum. More precisely, the center $s(t) \in \partial\Omega$ of the "droplet" moves, approximately, according to

$$\frac{ds(t)}{dt} = c\varepsilon \frac{\partial K}{\partial s}(s(t)) \tag{1.6}$$

where c is a constant independent of ε and K is the mean curvature of $\partial\Omega$ with the sign convention $K > 0$ for a sphere. A corresponding result has been established for the evolution law (1.5) [A-B-Ch-F]. For the equilibrium theory we refer to MODICA [M1–2], to KOHN & STERNBERG [K-Ste], STERNBERG [Ste] and to CHEN & KOWALCZYK [Ch-K].

Why is the "bubble" drawn to the boundary? This can be understood at various levels with the free energy offering the most direct explanation: Recall that the evolution happens so that $J_\varepsilon(u(t))$ is monotone in t, and that for small ε, J_ε registers the perimeter of the interface lying inside Ω. Therefore spheres are the favored intermediate states, while interfaces intersecting the boundary are the favored asymptotic states (Fig. 4).

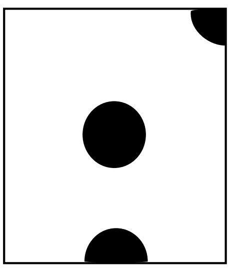

Fig. 4.

For a deeper comprehension of this phenomenon, understanding of the "connections" leading to the final state is required. This approach has been carried out to completion only in the one-dimensional case; see CARR & PEGO [C-Pe] and FUSCO & HALE [F-Ha] for the second-order equation and ALIKAKOS, BATES & FUSCO [A-B-F], and BATES & XUN [B-X1–2] for the Cahn-Hilliard equation.

The phenomenon of exponentially slow motion, in one space dimension, was first pointed out in NEU [N]. BRONSARD & KOHN [Bro-K1–2] have introduced an energy approach for justifying slow motion of layers in one dimension that later was implemented for the Cahn-Hilliard equation by BRONSARD & HILHORST [Bro-H]. GRANT [G2] later succeeded in refining and extending the results in [Bro-H] to a vector analogue of the Cahn-Hilliard equation known as Cahn-Moral system. The Bronsard-Kohn idea is elegant and relatively simple. However, it does not establish persistence of shape until the layers come together. For a refinement which allows that conclusion, and which can be applied also to higher space dimensions, see ALIKAKOS, BRONSARD & FUSCO [A-Bro-F].

1.1. B. Description of the Contents of the Paper and of Its Structure

As said before, the present work utilizes the Cahn-Hilliard equation for resolving a certain type of degeneracy of the limiting geometric problem (1.5). Specifically, it shows that when the singular parameter $\varepsilon > 0$ is sufficiently small, the Cahn-Hilliard equation (1.1) admits solutions which exhibit an (almost) spherical interface, which persists and either remains in equilibrium inside Ω, or migrates towards the boundary $\partial\Omega$ at a very slow speed. These solutions as functions of x are very close to step functions, with a steep transition from -1 to 1 across a spherical interface, and represent the physical "bubble" of a homogeneous phase that slowly moves inside another homogeneous phase with a different concentration. We estimate the time T needed for reaching the boundary to be transcendentally large in ε:

$$T > \text{Constant } e^{cd^{\xi}/\varepsilon} \tag{1.7}$$

where $c > 0$ is a constant independent of ε, and d^{ξ} is the distance of the bubble from the boundary $\partial\Omega : d^{\xi} = d(\xi, \partial\Omega) - \rho$. We show that, when $0 < \varepsilon \ll 1$, the dynamics of the center $\xi \in \Omega$ is determined to a very high degree of accuracy by an ordinary differential equation (cf. Theorem 7.2)

$$\dot{\xi} = c^{\xi}. \tag{1.8}$$

Here $\xi \to c^{\xi}$ is a vector field (defined on $\Omega_{\rho+\delta}$) which is transcendentally small in ε

$$|c^{\xi}| = O\left(e^{-cd^{\xi}/\varepsilon}\right) \tag{1.9}$$

and can, in principle, be computed and used to detect those special points in $\Omega_{\rho+\delta}$ which can be the centers of spherical interfaces corresponding to stationary solutions of (1.1). We establish the following estimate for the function $\xi(t)$ determining the location of the center of the bubble (cf. Theorem 7.2):

$$\dot{\xi}(t) = c^{\xi(t)} + O(e^{-c/\varepsilon}|c^{\xi(t)}|). \tag{1.10}$$

From this it follows that ξ is the center of a bubble remaining at equilibrium inside Ω if and only if

$$c^{\xi} = 0 \tag{1.11}$$

(see Theorem 5.1 for a precise statement). A careful analysis of (1.11) is outside the scope of the present work and will appear in a forthcoming paper, but the expression for c^{ξ} derived in the proof of Theorem 5.1 indicates that the bubble moves towards the closest points on the boundary $\partial\Omega$, as if it were attracted to its mirror image with respect to the boundary. This leads to the conjecture (backed up by formal calculations based on the form of c^{ξ}) that the bubble can remain in equilibrium inside Ω only if its center ξ is inside the convex hull of the points of $\partial\Omega$ having distance from ξ equal to $d(\xi, \partial\Omega)$ (cf. Fig. 5). Recently M. Ward [Wa] has done the complete asymptotics for a related equation. His work provides more evidence in favor of the speculation above.

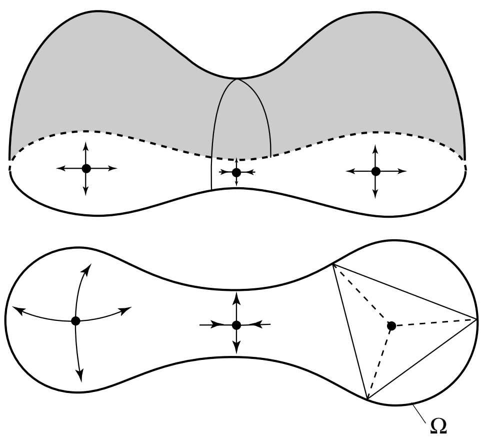

Fig. 5.

From the general theory of semilinear equations [H] and from the gradient nature of equation (1.1) it follows that (1.1) generates a semiflow in a suitable Sobolev space X^{α} and that this semiflow admits a global attractor $\mathscr{A}^{\varepsilon}$ [Ha1]. This is a compact connected invariant set which attracts bounded sets. $\mathscr{A}^{\varepsilon}$ is expected to depend on ε in a singular way as $\varepsilon \to 0$ (see [A-B-F]) and therefore in general one cannot hope to derive a limit problem, a system of differential equations which could capture the global limit behavior for $\varepsilon \ll 1$. In such a situation, when studying the limit as $\varepsilon \to 0$, it is preferable instead of focusing on the global attractor, to concentrate on certain "slices" of it. These are invariant subsets of the attractor of complexity bounded uniformly in ε and therefore with meaningful limits as $\varepsilon \to 0$. Regions II, III, and IV in Fig. 3 correspond to different energy slices of the attractor. We surmise that the set of step functions M_ρ mentioned above is the limit, as $\varepsilon \to 0$, of an invariant manifold M_ρ^ε of (1.1) which is made up of functions with approximately spherical interface. Strictly speaking, a proof of the existence of M_ρ^ε is not presented here although a related manifold $\tilde{M}_\rho^\varepsilon$ is constructed which contains the equilibria of (1.1) with almost spherical interface contained in Ω and which is approximately invariant in the sense that solutions starting near it stay near it (cf. Theorem 7.2). The construction of $\tilde{M}_\rho^\varepsilon$ is basic in the derivation of the results presented here. $\tilde{M}_\rho^\varepsilon$ would be, clearly, a very good approximation to the invariant manifold M_ρ^ε.

In this paper only the two-dimensional case is considered. However all arguments and techniques apply to the general case $N > 2$ with the following modifications. First the logarithmic Green function has to be replaced by the appropriate higher-dimensional Green function. This point is straightforward. The second point involves the spectrum, specifically estimate (6.6) in Theorem 6.1, the higher-dimensional analog of which has not yet been established. Instead the seemingly suboptimal result

$$\lambda_{N+1}^{\xi} \geqq C'\varepsilon^2$$

is available, which, however, is sufficient for the purposes of this paper. In fact anything algebraic would do (see the Introduction in [A-F1]). In Section 2 we construct bounded radial solutions to

$$\Delta(\Delta u - F'(u)) = 0 \tag{1.12}$$

on the whole space, rescaled versions of which are used for constructing good approximations for solutions to (1.1) with almost spherical interface. These radial bounded solutions of (1.12) (Fig. 6) are used as the "*building blocks*" of the quasi-invariant manifold $\tilde{M}_\rho^\varepsilon$. Equation (1.12) is solved by a perturbation argument

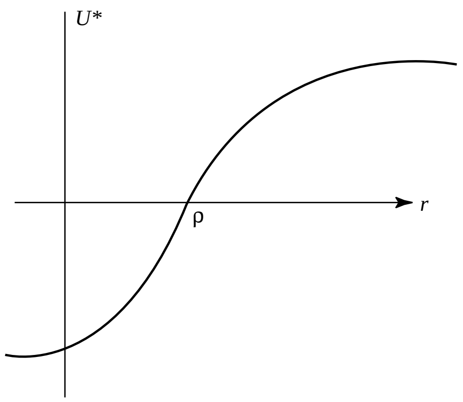

Fig. 6.

which is of some independent interest but on the other hand is not essential in understanding the rest of the paper. The reader mostly interested in the dynamics of bubbles can skip the proofs of Propositions 2.1 and 2.4, keeping in mind the statements and the definition of the bubble u^ξ in Section 3. In Section 4 various results concerning the operator

$$-\varepsilon^2\Delta + F''(u^\xi) \tag{1.13}$$

in $L^2(\mathbb{R}^2)$ are collected. In particular, we prove an estimate (cf. Theorem 4.2) for the generalized inverse of (1.13), which is then systematically employed in the proof of Theorem 5.1. Knowledge of the proof of Theorem 4.2 is not essential for the comprehension of the results in Sections 5 and 7. The reader mostly interested in the main results of the paper needs to be only familiar with the statements of Theorems 4.1 and 4.2. In Section 5, building on the results of Sections 3 and 4 we construct the quasi-invariant manifold $\tilde{M}^\varepsilon_\rho$, which has three important properties:

(a) The elements of $\tilde{M}^\varepsilon_\rho$ are step-like functions of x changing abruptly from -1 to 1 across a spherical cell of radius ρ and thickness ε.
(b) $\tilde{M}^\varepsilon_\rho$ contains the equilibria of (1.1) with an almost spherical interface of radius ρ.
(c) $\tilde{M}^\varepsilon_\rho$ satisfies a quasi-invariance condition (5.2) and solutions starting near $\tilde{M}^\varepsilon_\rho$ remain near $\tilde{M}^\varepsilon_\rho$ for a very long time (Theorem 7.2).

The basic geometric ideas behind the construction of $\tilde{M}^\varepsilon_\rho$ can be summarized as follows: We seek to determine v^ξ and $c(\xi)$ satisfying the two conditions

$$\mathscr{L}(u^\xi + v^\xi) = c(\xi)\cdot u^\xi_\xi \ \left(= c_i(\xi)\frac{\partial u^\xi}{\partial \xi_i}\right), \quad v^\xi \perp u^\xi_\xi \tag{1.14}$$

where $\mathscr{L}$ is the Cahn-Hilliard vector field,

$$\mathscr{L}(u) = \Delta\left(-\varepsilon^2\Delta u + F'(u)\right), \tag{1.15}$$

u^ξ a “bubble” function (see Section 3) and $\perp$ denotes orthogonality in H_0^{-1}. The intuition behind (1.14) is based on the fact that the manifold of bubbles $M^\varepsilon_\rho = \{u^\xi / \xi \in \Omega_{\rho+\delta}\}$ is already an excellent approximation of the true invariant manifold. Indeed, if the manifold of bubbles were truly invariant, then the vector field $\mathscr{L}$ would be tangent to it at every point:

$$\mathscr{L}(u^\xi) \in T_{u^\xi} M^\varepsilon_\rho. \tag{1.16}$$

The tangent space $T_{u^\xi} M^\varepsilon_\rho$ is spanned by $\partial u^\xi/\partial\xi_i$, $i = 1, \ldots, N$ and therefore the condition (1.16) can be stated equivalently in the form

$$\mathscr{L}(u^\xi) = c_i(\xi)\frac{\partial u^\xi}{\partial \xi_i}, \tag{1.17}$$

for appropriate c’s. Although (1.17) is false in the sense that there are generally no c_i’s for which it holds, it can be amended by adding a small correction v^ξ as an extra unknown. This correction gives rise to (1.14), which can be considered as a global Liapunov-Schmidt reduction. In fact, once (1.14) is solved, the bifurcation equation (1.11) can be utilized to determine the equilibria with almost spherical interface. A linear version of the "v-equation" (1.14) was already introduced in ALIKAKOS, BATES & FUSCO [A-B-F] and in FUSCO & HALE [F-Ha]. Both of these works however concern the one-dimensional case where equilibria can be easily determined by phase-plane analysis.

In Section 7, Theorem 7.2 is established. It contains the results on the dynamics of bubbles (cf. (1.9), (1.10)). The proof uses semigroup theory and is based on the spectral estimates for the linearized Cahn-Hilliard operator

$$\begin{aligned} &\Delta(-\varepsilon^2\Delta\psi + F''(u^\xi)\psi) = -\lambda\psi, \quad x \in \Omega, \\ &\frac{\partial\psi}{\partial n} = \frac{\partial\Delta\psi}{\partial n} = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.18}$$

that were derived in [A-F1] and are stated in Theorem 6.1, and on the "slow-channel" ideas of CARR & PEGO [C-Pe]. In Fig. 7 we show the quasi-invariant manifold $\tilde{M}_\rho^\varepsilon$ together with the "slow channel" around it. According to Theorem 7.2, if the initial condition is chosen in the inner neighborhood, then the solution can leave the outer neighborhood, described by $c(\xi)$ where c as in (1.14), only from the top or from the bottom. This implies that in physical space either the bubble stays in Ω forever or it persists until it reaches $\partial\Omega$.

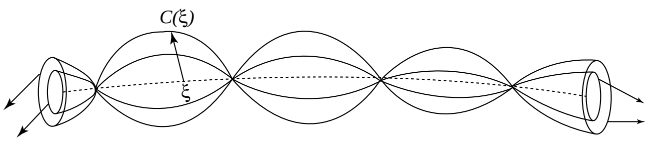

Fig. 7.

Some of the results in the present paper appeared without proof in the Barcelona EQUADIFF proceedings [A-F2].

2. Radial Equilibria on the Whole Space

The equilibria of (1.5) contained entirely in $\Omega \subset \mathbb{R}^2$ are circles. This suggests the existence of bounded radial stationary solutions to the Cahn-Hilliard equation considered in the whole of $\mathbb{R}^2$. A function $u \in C^2(\mathbb{R}^2)$ is such a solution if, and only if, it is radial and satisfies

$$\varepsilon^2\Delta u - F'(u) = \sigma, \quad x \in \mathbb{R}^2, \tag{2.1}$$

for some constant σ.

The following proposition concerns the existence of radial solutions of the rescaled version of (2.1)

$$\Delta u - F'(u) = \sigma, \quad x \in \mathbb{R}^2. \tag{2.2}$$

Proposition 2.1. *There exists a number* $\bar{\rho} > 0$ *and smooth functions* $\sigma : (\bar{\rho}, \infty) \to \mathbb{R}$, $U^* : [0, \infty) \times (\bar{\rho}, \infty) \to \mathbb{R}$, *such that for each* $\rho \in (\bar{\rho}, \infty)$, $\sigma(\rho)$ *and* $u(x, \rho) = U^*(|x|, \rho)$ *satisfy equation* (2.2). *Moreover,* $U^*(r, \rho)$ *is increasing in* r *and*

(i) $\sigma(\rho) = \mathscr{X}\rho^{-1} + O(\rho^{-2})$,
(ii) $U^*(\rho, \rho) = O(\rho^{-1})$,
(iii) $1 + U^*(0, \rho) = O(\rho^{-1})$
(iv) $\lim_{r\to\infty} U^*(r, \rho) = \alpha(\rho)$,

where $\mathscr{X} > 0$ *is a constant and* $\alpha(\rho)$ *is the root near 1 of the equation* $F'(u) + \sigma(\rho) = 0$.

(v) $\alpha(\rho) - U^*(r, \rho) = O(e^{-\nu(\rho)(r-\rho)}), \quad r > \rho, \quad \nu(\rho) = (F''(\alpha(\rho)))^{1/2}$,

and similar exponential estimates hold for the derivatives of U^* *with respect to* r.

Proof. If σ, U^* as in the proposition exist, then the function $U^\rho(s) = U^*(s - \rho, \rho)$ satisfies

$$\ddot{U}^\rho + \frac{1}{\rho + s}\dot{U}^\rho - F'(U^\rho) = \sigma(\rho), \quad -\rho < s < \infty.$$

Therefore we can expect that, as $\rho \to \infty$, U^ρ tends to U, the unique bounded solution of

$$\ddot{U} - F'(U) = 0, \quad \lim_{s\to\pm\infty} U(s) = \pm 1, \quad U(0) = 0. \tag{2.3}$$

On the other hand, away from the interface, we can expect U^ρ to be close to one of the roots of F'. The proof is a perturbation argument based on this observation.

a. *An equivalent problem.* Fix $\rho > 0$ and define

$$\begin{aligned} u(r) &= -1 + w(r), \quad 0 \leqq r \leqq \tfrac{1}{2}\rho, \\ u(\rho + s) &= U(s) + v(s), \quad -\tfrac{1}{2}\rho \leqq s. \end{aligned} \tag{2.4}$$

Set $a = \frac{1}{2}\rho$ and consider the problem

$$\begin{cases} r^{-1}(r\dot{w})^{\cdot} - \bar{\nu}^2 w = 1_{(0,a]}[F'(-1 + w) - \bar{\nu}^2 w + \sigma] + \alpha a^{-1}\delta_a, \\ \qquad\qquad r \in (0, \infty) \\ \ddot{v} - F''(U)v = 1_{(-a,\infty)}[F'(U + v) - F'(U) - F''(U)v \\ \qquad -(\rho + s)^{-1}(\dot{U} + \dot{v}) + \sigma)] + \beta\delta_{-a}, \quad s \in \mathbb{R}. \\ \dot{w}(0) = 0, \\ -1 + w(a) = U(-a) + v(-a), \quad \dot{w}(a^-) = \dot{U}(-a) + \dot{v}(-a^+), \\ \int_{-\infty}^{\infty} v\dot{U} = 0, \end{cases} \tag{2.5}$$

where $1_{[a,b]}$ is the characteristic function of the interval [a,b], $\bar{\nu}^2 = F''(1)$, and δ_a is the Dirac function at a. If $w : [0, \infty) \to \mathbb{R}$, $v : \mathbb{R} \to \mathbb{R}$, σ, α, $\beta \in$

$\mathbb{R}$ is a solution of (2.5) with w, v bounded, then the function u defined by (2.4) corresponds to a bounded radial solution of (2.2). On the other hand, if u corresponds to a bounded radial solution of (2.2), then the functions w, v defined by (2.4) satisfy equations (2.5) in the intervals $[0, a]$, $[-a, \infty)$, and it can be shown that they can be extended uniquely to $[0, \infty)$ and $\mathbb{R}$ as solutions of (2.5) for suitable values of $\rho, \sigma, \alpha, \beta$.

Remark. We owe an explanation to the reader about the role of the distributions δ_a and δ_{-a}. As we have mentioned, we split the problem on $(0,\infty)$ into two half-interval problems and a matching condition. The distributions introduced allow us to recast each of the half-interval problems into whole-space problems. The idea is general and can be utilized for taking a Dirichlet problem on a bounded domain and then modifying the equation so as to produce a problem on the whole space whose solution, when restricted to the bounded domain, satisfies the original Dirichlet problem. The advantage of the whole-space point of view is that the relevant operations now have known inverses expressible in terms of Green's functions on the whole space. The same idea is used again in the proof of Theorem 5.1.

b. *The map* $(w, v) \to (\hat{w}, \hat{v})$. For studying the solvability of the problem (2.5) we need the following lemmata. The proofs are quite standard and are omitted.

Lemma 2.2. *The problem*

$$\begin{aligned} r^{-1}(rK_r)_r - \bar{\nu}^2 K &= \tau^{-1}\delta_\tau, \quad (r, \tau) \in (0, \infty) \times (0, \infty), \\ K_r(0, \tau) = 0, &\quad \lim_{r \to \infty} rK_r(r, \tau) = 0 \end{aligned} \tag{2.6}$$

has a unique solution $K\colon [0, \infty)^2 \to \mathbb{R}$. *The function* K *satisfies*

$$K(r, \tau) < 0, \quad \int_0^\infty K(r, \tau)\tau d\tau = -\frac{1}{\bar{\nu}^2}, \tag{2.7}$$

$$\left| K(r, \tau)\tau + \frac{1}{2\bar{\nu}} e^{-\bar{\nu}|r-\tau|} \right| \leqq \frac{C}{1 + \tau} e^{-c|r-\tau|}, \quad r > 0, \ \tau > 0, \tag{2.8}$$

for some constants $C, c > 0$. *Similar estimates hold for derivatives of* K.

Lemma 2.3. *The problem*

$$\begin{aligned} g_{ss} - F''(U)g &= \delta_\tau - \frac{\dot{U}(\tau)\dot{U}}{\|\dot{U}\|^2}, \quad (s, \tau) \in \mathbb{R}^2, \\ \int_{-\infty}^\infty & g(s, \tau)\dot{U}(\tau)\, d\tau = 0 \end{aligned} \tag{2.9}$$

has a unique solution $g : \mathbb{R}^2 \to \mathbb{R}$. *The function* g *satisfies*

$$\left| g(s, \tau) + \frac{1}{2\bar{\nu}} e^{-\bar{\nu}|s-\tau|} \right| \leqq C\ e^{-c|\tau|} e^{-c|s|}, \quad (s, \tau) \in \mathbb{R}^2, \tag{2.10}$$

for some constants $C, c > 0$. *Similar estimates hold for derivatives of* g.

Given $w \in C^0[0,a],\ v \in C^1[-a,\infty), \alpha, \beta, \sigma \in \mathbb{R}$ set

$$
\begin{aligned}
q(w) &= F'(-1+w) - \bar{\nu}^2 w \\
p(v) &= F'(U+v) - F'(U) - F''(U)v - (\rho+s)^{-1}\dot{v},
\end{aligned}
\tag{2.11}
$$

and define

$$
\begin{aligned}
\hat{w}(r) &= \int_0^a K(r,\tau)[q(w)](\tau)\tau\,d\tau + \sigma\int_0^a K(r,\tau)\tau\,d\tau + \alpha K(r,a), \\
\hat{v}(s) &= -\int_{-a}^{\infty} g(s,\tau)\frac{\dot{U}(\tau)}{\rho+\tau}d\tau + \int_{-a}^{\infty} g(s,\tau)[p(v)](\tau)\,d\tau \\
&\quad + \sigma\int_{-a}^{\infty} g(s,\tau)\,d\tau + \beta g(s,-a).
\end{aligned}
\tag{2.12}
$$

By Lemma 2.2, the function $\hat{w}$ satisfies $\dot{\hat{w}}(0) = 0$ and is a solution of $(2.5)_1$ when w in the right-hand side is considered a known function. By Lemma 2.2, the function $\hat{v}$ is orthogonal to $\dot{U}$ and, provided the right-hand side of $(2.5)_2$ is also orthogonal to $\dot{U}$, is a solution of $(2.5)_2$ when v in the right-hand side is a known function. We now show that, if $a = \frac{1}{2}\rho$ is sufficiently large, then the numbers α, β, σ can be chosen so that this orthogonality condition is satisfied together with the two matching conditions at a in (2.5). We obtain the following linear system

(2.13)

$$
\begin{bmatrix}
\int_{-a}^{\infty}\dot{U} & 0 & \dot{U}(-a) \\
\left(-\int_{-a}^{\infty} g(-a,\tau)\,d\tau + \int_0^a K(a,\tau)\tau d\tau\right) & K(a,a) & -g(-a,-a) \\
\left(-\int_{-a}^{\infty} g_1(-a,\tau)\,d\tau + \int_0^a K_1(a,\tau)\tau d\tau\right) & K_1(a^-,a) & -g_1(-a^+,-a)
\end{bmatrix}
\begin{bmatrix} \sigma \\ \alpha \\ \beta \end{bmatrix}
=
\begin{bmatrix} l \\ m \\ n \end{bmatrix}
$$

where the subscript 1 denotes differentiation with respect to the first variable and l, m, n are given by the expressions

$$
\begin{aligned}
l &= \int_{-a}^{\infty} \frac{\dot{U}^2(\tau)}{\rho+\tau}d\tau - \int_{-a}^{\infty}[p(v)]\dot{U}(\tau)\,d\tau, \\
m &= 1 + U(-a) - \int_{-a}^{\infty} g(-a,\tau)\frac{\dot{U}(\tau)}{\rho+\tau}d\tau - \int_0^a K(a,\tau)[q(w)](\tau)\tau\,d\tau \\
&\quad + \int_{-a}^{\infty} g(-a,\tau)[p(v)](\tau)\,d\tau, \\
n &= \dot{U}(-a) - \int_{-a}^{\infty} g_1(-a,\tau)\frac{\dot{U}(\tau)}{\rho+\tau}d\tau - \int_0^a K_1(a,\tau)[q(w)](\tau)\tau\,d\tau \\
&\quad + \int_{-a}^{\infty} g_1(-a,\tau)[p(v)](\tau)\,d\tau.
\end{aligned}
\tag{2.14}
$$

Using the estimate

$$
|\dot{U}(s)| \leqq Ce^{-\bar{\nu}s},
\tag{2.15}
$$

we see that

$$\int_{-a}^{\infty} \dot{U} = 2 + O(e^{-\bar{\nu}/2\rho}), \quad |\dot{U}(-a)| = O(e^{-\bar{\nu}/2\rho}). \tag{2.16}$$

On the other hand, Lemmata 2.2, 2.3 imply that the other two elements in the first column of the matrix (A_{ij}) of system (2.13) are bounded uniformly in ρ and also imply the estimates

$$\begin{aligned} K(a,a) &= -\frac{1}{\bar{\nu}\rho} + O(\rho^{-2}), \quad g(-a,-a) = -\frac{1}{2\bar{\nu}} + O(e^{-\bar{\nu}/2\rho}), \\ K_1(a^-,a) &= -\frac{1}{\rho} + O(\rho^{-2}), \quad g_1(-a^+,a) = \frac{1}{2} + O(e^{-\bar{\nu}/2\rho}). \end{aligned} \tag{2.17}$$

The determinant d of A_{ij} is then given by

$$d = \frac{2}{\bar{\nu}\rho} + O(\rho^{-2}), \tag{2.18}$$

and we conclude that system (2.13) is solvable if ρ is sufficiently large. Therefore, under the condition $\rho \gg 1$, σ, α, and β are uniquely determined by (w, v). In turn, the functions defined by (2.12) are uniquely determined by (w, v). It follows that if we again denote by $\hat{w}, \hat{v}$ their restrictions to the intervals $[0, a]$ and $[-a, -\infty)$ we have a map $(w, v) \xrightarrow{T} (\hat{w}, \hat{v})$ from $C^0[0, a] \times C^1[-a, \infty)$ into itself and from the above discussion we see that fixed points of this map correspond to radial solutions of (2.2).

c. $\rho \gg 1$ *implies that* T *is a contraction*. Using (2.8), (2.10) and the corresponding estimates for the derivatives of K and g, one obtains

$$A_{21} = O(\rho^{-1}), \quad A_{31} = 1 + O(\rho^{-1}).$$

From this and previous estimates for the other coefficients of (A_{ij}) we conclude that

$$\sigma = l/2 + \cdots, \quad \alpha = B_0\rho l + \cdots, \quad \beta = B_1 l + B_2 m + B_3 n + \cdots, \tag{2.19}$$

where B_i, $i = 1, 2, 3, 4$ are bounded functions of $\rho \gg 1$ and dots denote linear combinations of l, m, n with coefficient of order $O(e^{-c\rho})$ for some $c > 0$.

We now show that T is a contraction on the closed subset X of $C^0[0, a] \times C^1[-a, \infty)$ defined by

$$X = \{(w, v) |\ \|(w, v)\|_X = \max\{\|w\|_0, \|v\|_1\} \leqq \eta\}, \tag{2.20}$$

for a suitable choice of the number $\eta > 0$.

Definition (2.11) and $(w, v) \in X$ imply that

$$\begin{aligned} \|q(w)\|_0 &\leqq C\eta\|w\|_0, \quad \|q(w) - q(\bar{w})\|_0 \leqq C\eta\|w - \bar{w}\|_0, \\ \|p(v)\|_0 &\leqq C(v + \rho^{-1})\|v\|_1, \quad \|p(v) - p(\bar{v})\|_0 \leqq C(\eta + \rho^{-1})\|v - \bar{v}\|_1 \end{aligned} \tag{2.21}$$

where $C > 0$ is a constant that does not always need to be the same.

We use the superscript 0 to denote the value of $l, m, n, \sigma, \ldots$ corresponding to $w = v = 0$ and the superscript 1 to denote the remaining part. Making use of (2.9)(ii) and of its consequence

$$\int_{-\infty}^{\infty} g_1(s,\tau)\dot{U}(\tau)\,d(\tau) = 0$$

and observing that the quantity $1 + U(-a)$ is of order $O(e^{-c\rho})$, we obtain

$$(2.22)\quad \begin{aligned} l^0 &= \frac{1}{\rho}\int_{-\infty}^{\infty} \dot{U}^2 + O(\rho^{-2}),\\ m^0 &= O(e^{-c\rho}) + \frac{1}{\rho}\int_{-a}^{\infty} g(-a,\tau)\dot{U}(\tau)\,d\tau + O(\rho^{-2}) = O(\rho^{-2}),\\ n^0 &= O(e^{-c\rho}) + \frac{1}{\rho}\int_{-a}^{\infty} g_1(-a,\tau)\dot{U}(\tau)\,d\tau + O(\rho^{-2}) = O(\rho^{-2}). \end{aligned}$$

On the other hand (2.14), (2.21) imply that

$$\begin{aligned} |l^1| &\leqq C(\eta + \rho^{-1})\|v\|_1,\\ |l^1 - \bar{l}^1| &\leqq C(\eta + \rho^{-1})\|v - \bar{v}\|_1\\ |m^1| &\leqq C(\eta + \rho^{-1})\|(w, v)\|_X\\ |m^1 - \bar{m}^{-1}| &\leqq C(\eta + \bar{\rho})^{-1}\|(w, v) - (\bar{w}, \bar{v})\|_X \end{aligned}$$

and similar estimates for n^1. Therefore, from (2.19) we conclude that

$$(2.23)\quad \begin{aligned} \sigma &= \frac{1}{2\rho}\int_{-\infty}^{\infty} \dot{U}^2 + O(\rho^{-2} + (\eta + \rho^{-1})\|(w, v)\|_X),\\ |\alpha| &\leqq C(1 + \rho(\eta + \rho^{-1})\|(w, v)\|_X),\\ |\beta| &\leqq C\left(\frac{1}{\rho} + (\eta + \rho^{-1})\|(w, v)\|_X\right). \end{aligned}$$

Using (2.12), (2.21), (2.23) and Lemmata 2.2, 2.3 we finally get

$$\begin{aligned} \|(\hat{w}, \hat{v})\|_X &\leqq C(\rho^{-1} + (\eta + \rho^{-1})\|(w, v)\|_X),\\ \|(\hat{w}, \hat{v}) - (\hat{\bar{w}}, \hat{\bar{v}})\|_X &\leqq C(\eta + \rho^{-1})\|(w, v) - (\bar{w}, \bar{v})\|_X. \end{aligned}$$

Under the assumption that $\eta = 2C\rho^{-1}$, these estimates show that if ρ is larger than some $\bar{\rho} > 0$, then T is a map from X into itself and is a contraction.

For each ρ larger than some $\bar{\rho} > 0$, the fixed point (w^*, v^*) of T defines via (2.4) a function U^* which is a bounded radial solution of (2.2). From (2.23) with $\eta = 2C\rho^{-1}$ and $(w, v) = (w^*, v^*)$, and from

$$(2.24)\quad \|(w^*, v^*)\|_X \leqq \text{Const. } \rho^{-1}$$

the estimate (i) follows with

$$(2.25)\quad \mathscr{X} = \int_{-\infty}^{\infty} \dot{U}^2 \Big/ \int_{-\infty}^{\infty} \dot{U}.$$

Estimates (ii), (iii) follow from (2.4), (2.24).

Once the existence of a radial bounded solution U^* of (2.2), a perturbation of U, is established, standard phase-plane analysis applied to the equation

$$\ddot{U}^* + \frac{\dot{U}^*}{r} - F'(U^*) = \sigma$$

implies that U^* is increasing in r and also implies the statements (iv) and (v). The proof of Proposition 2.1 is complete. □

Proposition 2.4. *There is a number $C > 0$, independent of ρ, such that the functions σ, U^* constructed in Proposition 2.1 satisfy the following estimates:*

(i) $$\sigma'(\rho) = \mathscr{K}\rho^{-2} + O(\rho^{-3}),$$

(ii) $$\begin{aligned} &U^*(r,\rho) = U(r-\rho) + V(r-\rho,\rho) + O(\rho^{-2}), \quad r-\rho \in [-C\rho, \infty),\\ &U^*_\rho(r,\rho) = -\dot{U}(r-\rho) + V_\rho(r-\rho,\rho) + O(\rho^{-3}), \quad r-\rho \in [-C\rho, \infty)\end{aligned}$$

where

(2.26) $$V(r,\rho) = \mathscr{K}\rho^{-1}\int_{-\infty}^{\infty} g(r,s)\,ds, \quad \mathscr{K} = \int_{-\infty}^{\infty}\dot{U}^2 \setminus \int_{-\infty}^{\infty}\dot{U}.$$

Moreover

(iii) $$\int_{-\infty}^{\infty} F'''(U)\dot{U}^2 V = 0.$$

Proof. We drop the superscript $*$ and write (w,v) instead of (w^*,v^*).

1. We begin by noting that the contraction map T constructed in the proof of Proposition 2.1 depends C^∞-smoothly on ρ. Indeed, given (w,v) in X, σ,α,β are chosen so that the orthogonality condition (necessary for the solvability of the second equation in (2.5)) and the matching conditions are satisfied, and therefore the pair $(\hat{w},\hat{v})$ produced by (2.12) depends smoothly on ρ. By a well-known result (see for example Th. 3.2, Chapter 0 in [Ha2]), the fixed point depends smoothly on ρ, and so does σ since it is a smooth function of (w,v).

2. Next we establish (ii) 1. If we set $v = V + R$, where V is the function defined by (2.26), and R is a remainder, equation $(2.5)_2$ can be rewritten in the form

$$\begin{aligned}\ddot{R} - F''(U)R = &\left[\rho^{-1}(\mathscr{K} - \dot{U}) - \ddot{V} + F''(U)V\right]\\ &+ 1_{[-\rho/2,\infty)}\left[F'(U+v) - F'(U) - F''(U)v + \frac{s\dot{U}}{\rho(\rho+s)} - \frac{\dot{v}}{\rho+s}\right]\\ &- 1_{(-\infty,-\rho/2]}\rho^{-1}(\mathscr{K} - \dot{U}) + \beta\delta_{-\rho/2} + O(\rho^{-2})\end{aligned} \tag{2.27}$$

where we have used the expression (i) for σ in Proposition 2.1. Now observe that the function $\rho^{-1}(\mathscr{K} - \dot{U})$ is orthogonal to $\dot{U}$ and therefore V is the unique solution orthogonal to $\dot{U}$ to the equation

$$\ddot{V} - F''(U)V = \rho^{-1}(\mathscr{K} - \dot{U}).$$

Moreover, from (2.24) the expression $1_{[-\rho/2,\infty)}\,[\cdots]$ on the right-hand side of (2.27) is $O\left(\rho^{-2}\right)$. Then (2.27) implies that

$$R = \rho^{-1} \int_{-\infty}^{-\rho/2} g(s, \sigma)(\mathscr{X} - \dot{U})\, d\sigma + \beta g(s, -\rho/2) + O(\rho^{-2}).$$

From this expression for R, from Lemma 2.3 and the estimate $(2.23)_3$, it follows that for $-C\rho < s < \infty$,

$$\begin{aligned} |R(s)| &\leqq \text{ Constant } \left(\int_{-\infty}^{-\rho/2} e^{-\bar{\nu}(s-\sigma)} d\sigma + e^{-\bar{\nu}(s+\rho/2)} \right) + O(\rho^{-2}) \\ &\leqq \text{ Constant } e^{-(\bar{\nu}/2 - C)\rho} + O(\rho^{-2}), \end{aligned}$$

which, provided C is chosen $< \bar{\nu}/2$, implies that

$$R(s) = O(\rho^{-2}).$$

3. To verify the second equation of (ii), we start by differentiating (2.11), (2.12) (together with the derivative of $(2.12)_2$ with respect to r), (2.13), (2.14) with respect to ρ. This gives a linear system for the unknowns $\sigma',\ \alpha',\ \beta' \in \mathbb{R},\ (w', v') \in C^0(0, \rho/2] \times C^1[-\rho/2, \infty)$ (here $'$ denote differentiation with respect to ρ).

From (2.11) and (2.24) we see that

$$\begin{aligned} \left\| \frac{d}{d\rho}[q(w)] \right\|_0 &\leqq \text{ Const. } \rho^{-1} \|w^1\|_0, \\ \left\| \frac{d}{d\rho}[p(v)] \right\|_0 &\leqq \text{ Const. } \left(\rho^{-3} + \rho^{-1}\|v^1\|_1\right). \end{aligned} \tag{2.28}$$

These estimates and (2.14) imply that

$$\begin{aligned} \left| l' + \frac{1}{\rho^2} \int_{-\infty}^{\infty} \dot{U}^2 \right| &\leqq \text{ Const. } \left(\rho^{-3} + \rho^{-1}\|v'\|_1\right), \\ |m'| + |n'| &\leqq \text{ Const. } \left(\rho^{-2} + \rho^{-1}\left(\|w'\|_0 + \|v'\|_1\right)\right). \end{aligned} \tag{2.29}$$

Differentiating (2.14) with respect to ρ yields

$$A \begin{bmatrix} \sigma' \\ \alpha' \\ \beta' \end{bmatrix} = -A' \begin{bmatrix} \sigma \\ \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} l' \\ m' \\ n' \end{bmatrix}. \tag{2.30}$$

One can show that

$$A' = \begin{bmatrix} O(e^{-c\rho}) & 0 & O(e^{-c\rho}) \\ O(1) & O(\rho^{-2}) & O(e^{-c\rho}) \\ O(1) & O(\rho^{-2}) & O(e^{-c\rho}) \end{bmatrix}. \tag{2.31}$$

From (2.30) and the estimates (2.23), (2.24), (2.29) and (2.31), it follows that

$$\begin{aligned} |\sigma' + \mathscr{X}\rho^{-2}| &\leqq \text{ Const. } \rho^{-1}\|v'\|_1, \\ |\alpha'| &\leqq \text{ Const. } \rho^{-1}(1 + \|v'\|_1), \\ |\beta'| &\leqq \text{ Const. } \rho^{-1}(1 + \|w'\|_0 + \|v'\|_1). \end{aligned} \tag{2.32}$$

Next we consider the expressions of w', v', and $\dot{v}'$ that one obtains by differentiating equations (2.12) with respect to ρ and by differentiating $(2.12)_2$ with respect to r and ρ. Only the expression for v' is written, the expressions of w' and $\dot{v}'$ being similar.

$$\begin{aligned} v'(s) = &-\rho^{-1} g\left(s, -\frac{1}{2}\rho\right)\dot{U}\left(-\frac{1}{2}\rho\right) + \int_{-\rho/2}^{\infty} g(s,\tau)\frac{\dot{U}(\tau)}{(\rho+\tau)^2}d\tau \\ &+\frac{1}{2}g\left(s,-\frac{1}{2}\rho\right)p\left(v\left(-\frac{1}{2}\rho\right)\right) + \int_{-\rho/2}^{\infty} g(\tau,\tau)\left(\frac{d}{d\rho}p(v)\right)(\tau)\,d\tau \\ &+\sigma'\int_{-\rho/2}^{\infty} g(s,\tau)\,d\tau + \left(\frac{1}{2}\sigma+\beta'\right)g\left(s,-\frac{1}{2}\rho\right) - \frac{1}{2}\beta g_2\left(s,-\frac{1}{2}\rho\right) \\ = &\; I_1+I_2+I_3+I_4+I_5+I_6+I_7 \end{aligned} \tag{2.33}$$

where g_2 is the derivative of g with respect to the second variable. Equation (2.15) implies that $I_1 = O(e^{-c\rho})$. If $s > -c\rho$ with $c < \frac{1}{2}\overline{\nu}$, then Lemma 2.3 implies that

$$\left|g\left(s,-\tfrac{1}{2}\rho\right)\right| < \text{Const.}\, e^{-\text{Const.}\ \rho}. \tag{2.34}$$

This, (2.24), (2.23), and (2.32) imply that

$$\begin{aligned} I_2 &= O(\rho^{-3}), \\ I_4 &< \text{Const.}\ (\rho^{-3}+\rho^{-1}\|v'\|_1), \\ I_6 &< \text{Const.}\ e^{-\text{Const.}\ \rho}\left(\rho^{-1}+\|w'\|_0+\|v'\|_1\right), \\ I_7 &= O(e^{-c\rho}). \end{aligned} \tag{2.35}$$

On the other hand, also using $(2.32)_1$, we have for $s > -C\rho$ that

$$\left|I_5 + \mathscr{K}\rho^{-2}\int_{-\infty}^{\infty} g(s,\tau)\right| \leqq \text{Const.}\ \left(\rho^{-3}+\rho^{-1}\|v'\|_1\right). \tag{2.36}$$

From equation (2.33), the above estimates for $I_i,\ i = 1,\dots,7$, and a similar discussion for the terms appearing in the expressions of w' and $\dot{v}'$, we obtain that

$$\|v'\|_1 = O(\rho^{-2}), \tag{2.37}$$

which in turn implies via $(2.32)_1$ and (2.36) that

$$\sigma' = -\mathscr{K}\rho^{-2} + O(\rho^{-3}), \tag{2.38}$$

$$v'(s) = \mathscr{K}\rho^{-2}\int_{-\infty}^{\infty} g(s,\tau)d\tau = V_\rho(s,\rho). \tag{2.39}$$

4. Finally we verify (iii). We begin by deriving

$$\dddot{U} - F''(U)\ddot{U} = F'''(U)\dot{U}^2.$$

Then by multiplying this by V and integrating over $(-\infty,\infty)$ we find

$$\int_{-\infty}^{\infty} F'''(U)\dot{U}^2 V = \int_{-\infty}^{\infty} (\ddddot{U} - F''(U)\ddot{U})V$$
$$= \int_{-\infty}^{\infty} \ddot{U}\ddot{V} - F''(U)\ddot{U}V = \int_{-\infty}^{\infty} (\ddot{V} - F''(U)V)\ddot{U}$$
$$= \int_{-\infty}^{\infty} \rho^{-1}(\dot{\mathscr{X}} - \dot{U})\ddot{U} = 0.$$

The proof of Proposition 2.4 is complete. □

3. The "Bubble" $u^\xi(x)$

By means of Propositions 2.1 and 2.4 we can associate with each $\xi \in \Omega_{\rho+\delta} = \{\xi | d(\xi, \partial\Omega) - \rho > \delta\}$ a function $u^\xi : \Omega \to \mathbb{R}$ with the following properties:

(a) It is an almost stationary solution of the Cahn-Hilliard equation in the sense that it fails to satisfy the equation, or the boundary conditions, by terms which are of the order $O(e^{-c/\varepsilon})$;

(b) It jumps from near -1 to near 1 in a thin layer of size of order ε around the circle of radius ρ and center ξ.

For $\varepsilon \ll 1$ we set

$$u^\xi(x) = U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right), \quad x \in \Omega, \tag{3.1}$$

where the number a^ξ is chosen to be zero at some fixed $\xi_0 \in \Omega_{\rho+\delta}$ and is determined for generic $\xi \in \Omega_{\rho+\delta}$ by imposing that the "mass" of u^ξ is constant on $\Omega_{\rho+\delta}$

$$\int_\Omega u^\xi = \int_\Omega u^{\xi_0}, \quad \forall \xi \in \Omega_{\rho+\delta}. \tag{3.2}$$

We choose ξ_0 to be a point of maximal distance from $\partial\Omega$.

Lemma 3.1. *The number a^ξ is uniquely determined by the condition* (3.2) *and the assumption $a^{\xi_0} = 0$. Moreover*

$$0 \leqq a^\xi < C\ e^{-(\nu_\varepsilon/\varepsilon)\, d^\xi}, \tag{3.3}$$

where $\nu_\varepsilon = \nu\left(\frac{\rho - a^\xi}{\varepsilon}\right)$ (see notation in Proposition 2.1); $d^\xi = d(\xi, \partial\Omega) - \rho$. Similar estimates hold for derivatives of a^ξ with respect to ξ_i, $i = 1, 2$.

Proof. We can write

$$\int_\Omega u^\xi = \int_\Omega U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx + \frac{a^\xi}{\varepsilon}\int_0^1 \int_\Omega U^*_\rho\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho - s a^\xi}{\varepsilon}\right) dx\, ds.$$

On the other hand based on Propositions 2.1 and 2.4, we have

$$
\begin{aligned}
\int_\Omega u^{\xi_0} - \int_\Omega U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx \\
&= \int_\Omega U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx - \int_\Omega U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx \\
&= \int_\Omega U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx - \int_{\Omega_\xi} U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx \\
&= \int_{\Omega\setminus\Omega_\xi} U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx - \int_{\Omega_\xi\setminus\Omega} U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) dx \\
&= \int_{\Omega\setminus\Omega_\xi} \left[U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) - \alpha\left(\frac{\rho}{\varepsilon}\right)\right] dx \\
&\quad - \int_{\Omega_\xi\setminus\Omega} \left[U^*\left(\frac{|x-\xi_0|}{\varepsilon}, \frac{\rho}{\varepsilon}\right) - \alpha\left(\frac{\rho}{\varepsilon}\right)\right] dx \\
&= O\left(e^{-(\nu^\varepsilon/\varepsilon)d^\xi}\right) < 0
\end{aligned}
$$

where $\Omega_\xi = \{y | y = x - \xi + \xi_0,\ x \in \Omega\}$. From Proposition 2.4 we also obtain

$$
\frac{1}{\varepsilon}\int_\Omega U_\rho^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho + sa^\xi}{\varepsilon}\right) dx < -C < 0.
$$

Therefore we see that, if $\varepsilon > 0$ is sufficiently small, then (3.1) can be uniquely solved for a^ξ and also the estimate for a^ξ follows. The proof of Lemma 3.1 is complete. □

From the definition of u^ξ it follows that u^ξ satisfies the differential equation (1.1) with $u_t = 0$ and the second boundary condition. Furthermore, from Proposition 2.1 it can be concluded that

$$
\frac{\partial u^\xi}{\partial n} = O\left(\varepsilon^{-1} e^{-(\nu_\varepsilon/\varepsilon)\, d^\xi}\right) . \tag{3.4}
$$

4. The Operator $-\varepsilon^2\Delta + F''(u^\xi)$ on the Whole Space

In this section we collect some results on the operator defined, for $\varepsilon \ll 1$, by

$$
A\phi = -\varepsilon^2\Delta\phi + F''(u^\xi)\phi, \tag{4.1}
$$

on $L^2(\mathbb{R}^2)$ that are used systematically below. Here u^ξ is the function defined as in (3.1) with $x \in \mathbb{R}^2$.

A. The Spectrum

Theorem 4.1. *A is self-adjoint and there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,*

(i) *A has a unique negative eigenvalue μ_0 and*

$$-C\varepsilon^2 < \mu_0 < -C'\varepsilon^2$$

with $C, C' > 0$ *independent of* ε. *The eigenvalue* μ_0 *is simple and the corresponding eigenfunction* V_0 *can be chosen to be positive. Also,* V_0 *is a function of* $|x-\xi|$ *and*

$$|V_0(\rho+s) - \alpha\varepsilon^{-1/2}\dot{U}(s/\varepsilon)| < C\varepsilon^{1/2}, \quad -\rho < s < \infty$$

where α *is a constant of normalization.*

(ii) $0 = \mu_1 = \mu_2$ *is a double eigenvalue of A and the corresponding eigenspace is the span of* $V_i = \dfrac{\partial u^\xi}{\partial x_i} \Big/ \left\| \dfrac{\partial u^\xi}{\partial x_i} \right\|_{L_2}, \quad i = 1, 2.$

(iii) *If* $\mu > 0$ *is in the spectrum of A, then*

$$\mu > C\varepsilon^2.$$

(iv) *If* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *is Lipschitz continuous, then*

$$\int_{\mathbb{R}^2} \varphi V_0 = \varepsilon^{1/2} \frac{\sqrt{2}}{\sqrt{\pi\rho \int_{-\infty}^{\infty} \dot{U}^2}} \int_\Gamma \varphi + O\left(\varepsilon^{3/2}\right),$$

$$\int_{\mathbb{R}^2} \varphi V_i = \varepsilon^{1/2} \frac{2}{\sqrt{\pi\rho \int_{-\infty}^{\infty} \dot{U}^2}} \int_\Gamma \varphi \cos\alpha_i + O(\varepsilon^{3/2}), \quad i = 1, 2,$$

where $\Gamma = \{x / |x-\xi| = \rho\}$ *and* $\alpha_i(x)$ *is the angle between the vector* $x - \xi$ *and the* x_i *axis.*

The proof of this theorem is a consequence of [A-F1] and [St], in particular Lemma 2.5 in [A-F1]. We omit the details and refer the reader to [A-F1].

B. Mapping Properties of the Inverse

Theorem 4.2. *The problem*

$$A\phi = \psi, \quad \phi, \psi \in L^2(\mathbb{R}^2),$$
$$\int_{\mathbb{R}^2} \phi V_i = \int_{\mathbb{R}^2} \psi V_i = 0, \quad i = 0, 1, 2, \tag{4.2}$$

has a unique solution. The solution ϕ *can be represented in the form*

$$\phi(x) = \int_{\mathbb{R}^2} g(x, y)\psi(y)\, dy \tag{4.3}$$

through a function $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *which satisfies*

$$\int_{\mathbb{R}^2} g(x, y) V_i(y)\, dy = 0, \quad i = 0, 1, 2. \tag{4.4}$$

Moreover,

(i) *there exist* $C, C', c, \beta > 0$ *such that, for* $0 < \varepsilon < \varepsilon_0$,

(4.5) $|g(x,y) - \overline{g}(x,y)| \leqq C\varepsilon^{-\beta} e^{-(c/\varepsilon)(d(x)+d(y))}, \quad |x-\xi|, |y-\xi| > \rho + C'\varepsilon.$

where $d(x) = ||x-\xi| - \rho|$ *and* $\overline{g}$ *is the fundamental solution of*

$$-\varepsilon^2 \Delta_x \overline{g} + \nu_\varepsilon^2 \overline{g} = \delta_y. \tag{4.6}$$

(ii) *If* $\psi \in C^0(\overline{\Omega})$, *then*

$$\left| \int_\Omega \int_\Omega g(x,y)\psi(y)\, dy\, dx \right| \leqq C\|\psi\|_0. \tag{4.7}$$

(iii) *If* $\psi \in C^0(\partial\Omega)$, *then*

$$\left| \int_\Omega \int_{\partial\Omega} g(x,y)\psi(y)\, dy\, dx \right| \leqq C\varepsilon^{-1}\|\psi\|_0, \tag{4.8}$$

where $\|\cdot\|_0$ *stands for the* C^0 *norms.*

Proof. The first part of the theorem is standard. Only the estimates (i), (ii), (iii) are proved. We start from the observation that $\gamma = g - \overline{g}$ satisfies the equation

$$\varepsilon^2 \Delta_x \gamma - F''(u^\xi)\gamma = (F''(u^\xi) - \nu_\varepsilon^2)\overline{g} + \sum_{i=0}^{2} V_i(y) V_i, \tag{4.9}$$

and derive an estimate for the L^2 norm of γ. A standard computation shows that the right-hand side h of (4.9) is orthogonal to V_1 and V_2. Therefore, from Theorem 4.1, we conclude that

$$\gamma = \sum_{i=1}^{2} \langle \gamma, V_i \rangle V_i + A^{-1} h \tag{4.10}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^2)$ and A^{-1} is the inverse of A restricted to the subspace orthogonal to V_1 and V_2. We need the following classical result:

Lemma 4.3. *The function* $\overline{g}$ *is given by*

$$\overline{g}(x,y) = \frac{1}{2\pi\varepsilon^2} K_0\left(\frac{\nu_\varepsilon}{\varepsilon}|x-y|\right) \tag{4.11}$$

where $\frac{1}{2\pi}K_0$ *is the fundamental solution of*

$$-\Delta\phi + \phi = \delta, \tag{4.12}$$

and satisfies the estimates

(4.13) $K_0(r) = -\left(\ln\frac{r}{2} + C\right) + O(r), \quad r \ll 1$ (C *is the Euler constant*),

(4.14) $K_0(r) = \sqrt{\frac{\pi}{2r}} e^{-r}(1 + O(r^{-1})), \quad r \gg 1.$

To simplify the notation we take $\xi = 0$, assuming $0 \in \Omega_\rho$. This can always be achieved by a translation.

a. Estimating $\langle \gamma, V_i \rangle$. From (4.4) and the symmetry of $g, \overline{g}$ we obtain

$$\langle \gamma, V_i \rangle = -\int_{\mathbb{R}^2} \overline{g}(y, x) V_i(x)\, dx, \quad i = 1, 2. \tag{4.15}$$

From Theorem 4.1, and also Lemma 2.2 in [A-F1] (with $c > 0$ a constant $< \nu_\varepsilon$) we obtain

$$|V_i(x)| \leqq C\varepsilon^{-1/2} e^{-(c/\varepsilon)||x|-\rho|}. \tag{4.16}$$

It follows that

$$\begin{aligned} |\langle \gamma, V_i \rangle| &\leqq C\varepsilon^{-5/2} \int_{\mathbb{R}^2} K_0\left(\frac{\nu_\varepsilon}{\varepsilon}|x-y|\right) e^{-(c/\varepsilon)||x|-\rho|} dx \\ &= \frac{C\varepsilon^{-5/2}}{2\pi|y|} \int_{S_y}\int_{\mathbb{R}^2} K_0\left(\frac{\nu_\varepsilon}{\varepsilon}|x-y'|\right) e^{-(c/\varepsilon)||x|-\rho|} dx\, dy' \\ &= C\varepsilon^{-5/2} \int_{\mathbb{R}^2} e^{-(c/\varepsilon)||x|-\rho|} \left(\frac{1}{|y|}\int_{S_y} K_0\left(\frac{\nu_\varepsilon}{\varepsilon}|x-y'|\right) dy'\right) dx, \end{aligned} \tag{4.17}$$

where $S_y = \{y' \,|\, |y'| = |y|\}$. Equations (4.13), (4.14) imply that for each $0 < \sigma \leqq \frac{1}{2}$ there is a constant $C_\sigma > 0$ such that

$$K_0(r) < C_\sigma r^{-\sigma} e^{-r}, \tag{4.18}$$

and therefore

$$\begin{aligned} &\frac{1}{|y|}\int_{S_y} K_0\left(\frac{\nu_\varepsilon}{\varepsilon}|x-y'|\right) dy' < \frac{C\varepsilon^{1/2}}{|y|}\int_{S_y} \frac{e^{-(\nu_\varepsilon/\varepsilon)}|x-y'|}{|x-y'|^{1/2}} dy' \\ &\quad < C\varepsilon^{1/2} e^{-(c/\varepsilon)||x|-|y||} \frac{1}{|y|}\int_{S_y} \frac{dy'}{|x-y'|^{1/2}} \leqq C\varepsilon^{1/2} \frac{e^{-(c/\varepsilon)||x|-|y||}}{|x|^{1/2}}, \end{aligned} \tag{4.19}$$

where we have made use of

$$\frac{1}{|y|}\int_{S_y} \frac{dy'}{|x-y'|^{1/2}} = \frac{1}{|x|^{1/2}}\int_{|z|=1} \frac{dz}{\left|\frac{x}{|x|} - \frac{|y|}{|x|}z\right|^{1/2}} < \frac{C}{|x|^{1/2}}.$$

The estimates (4.17), (4.19) imply that

$$\begin{aligned} |\langle \gamma, V_i \rangle| &< C\varepsilon^{-2} \int_0^\infty e^{-(c/\varepsilon)||x|-\rho|} e^{-(c/\varepsilon)||x|-|y||} |x|^{1/2} d|x| \\ &< C\varepsilon^{-1} e^{-(c/\varepsilon)||y|-\rho|}. \end{aligned} \tag{4.20}$$

b. Estimating $h = h_1 + h_2 = \left(F''(u^0) - \nu_\varepsilon^2\right)\overline{g} + \sum_{i=0}^{2} V_i(y) V_i$. From (4.16) and the normalization $\|V_i\| = 1$, it can be seen that

$$\|h_2\| < C\varepsilon^{-1/2} e^{-(c/\varepsilon)||y|-\rho|}. \tag{4.21}$$

To estimate $\|h_1\|$ we observe Proposition 2.1 and the definition of ν_ε, which imply that

$$(4.22)\qquad \begin{aligned} &|F''(u^0(x)) - \nu_\varepsilon^2| < Ce^{-(c/\varepsilon)||x|-\rho|}, \quad |x| > \rho,\\ &|F''(u^0(x)) - \nu_\varepsilon^2| < C\left(\varepsilon + e^{-(c/\varepsilon)||x|-\rho|}\right), \quad |x| < \rho. \end{aligned}$$

This and (4.18) with $\sigma < \frac{1}{2}$ imply that

$$(4.23)\qquad \begin{aligned} \|h_1\|^2 \leqq{}& C\varepsilon^{2\sigma-4}\int_{\mathbb{R}^2} e^{-(2c/\varepsilon)||x|-\rho|}\frac{e^{-(2\nu_\varepsilon/\varepsilon)}|x-y|}{|x-y|^{2\sigma}}\,dx\\ &+C\varepsilon^{2\sigma-2}\int_{|x|<\rho}\frac{e^{-(2\nu_\varepsilon/\varepsilon)}|x-y|}{|x-y|^{2\sigma}}dx = C\varepsilon^{2\sigma}\left(\varepsilon^{-4}I_1+\varepsilon^{-2}I_2\right), \end{aligned}$$

$$(4.24)\qquad \begin{aligned} I_1 &\leqq \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-(2c/\varepsilon)||x|-\rho|}e^{-(2\nu_\varepsilon/\varepsilon)||y|-|x||}\left(\frac{1}{|y|}\int_{S_y}\frac{dy'}{|x-y'|^{2\sigma}}\right)dx\\ &\leqq C\int_0^\infty e^{-(2c/\varepsilon)||x|-\rho|}e^{-(2\nu_\varepsilon/\varepsilon)||y|-|x||}|x|^{1-2\sigma}d|x|\\ &\leqq \text{Const.}\, e^{-(\text{Const.}/\varepsilon)||y|-\rho|}. \end{aligned}$$

Similarly we obtain

$$(4.25)\qquad \begin{aligned} &I_2 \leqq C\varepsilon^{2-2\sigma} &&\text{if } |y| < \rho,\\ &I_2 \leqq Ce^{-(c/\varepsilon)(|y|-\rho)} &&\text{if } |y| > \rho, \end{aligned}$$

and therefore from (4.21), (4.23), and (4.24),

$$(4.26)\qquad \begin{aligned} &\|h\| = C\varepsilon^{-2+\sigma} &&\text{if } |y| < \rho,\\ &\|h\| = C\varepsilon^{-2+\sigma}e^{-(c/\varepsilon)(|y|-\rho)} &&\text{if } |y| > \rho. \end{aligned}$$

From Theorem 4.1 it can be inferred that

$$(4.27)\qquad \|L^{-1}\phi\| \leqq C\varepsilon^{-2}\|\phi\| \quad \text{for } \phi \text{ such that } \langle\phi, V_i\rangle = 0, \quad i = 1, 2.$$

This and (4.10), in view of (4.20) and (4.26), yield the following estimate for the L^2 norm of γ:

$$(4.28)\qquad \begin{aligned} &\|\gamma\| < C\varepsilon^{-4+\sigma} &&\text{if } |y| < \rho,\\ &\|\gamma\| < C\varepsilon^{-4+\sigma}e^{-(c/\varepsilon)(|y|-\rho)} &&\text{if } |y| > \rho. \end{aligned}$$

c. Local L^2 Estimates of γ. Let $z : \mathbb{R}^2 \to \mathbb{R}$, $0 \leqq z \leqq 1$, be a C^∞ function. Multiplying (4.9) by γz^2 and integrating over $\mathbb{R}^2$ yields [1]

[1] We remark that it is not necessary for z to have compact support for the integration by parts below to be valid. It is enough that at infinity $g, \overline{g}$ and their derivatives decay exponentially. See the remark after the proof.

$$\tag{4.29} \begin{aligned} &\varepsilon^2 \int_{\mathbb{R}^2} |\nabla(\gamma z)|^2 + \int_{\mathbb{R}^2} F''(u^0)(\gamma z)^2 \\ &= \int_{\mathbb{R}^2} \left(\nu_\varepsilon^2 - F''(u^0)\right) \bar{g}\gamma z^2 + \sum_{i=0}^{2} V_i(y) \int_{\mathbb{R}^2} \gamma z^2 V_i + \varepsilon^2 \int_{\mathbb{R}^2} \gamma^2 |\nabla z|^2. \end{aligned}$$

It follows that

$$\tag{4.30} \begin{aligned} &\varepsilon^2 \int \mathbb{R}^2 |\nabla(\gamma z)|^2 + \int_{\mathbb{R}^2} F''(u^0)(\gamma z)^2 \leqq \varepsilon^2 \int_{\mathbb{R}^2} \gamma^2 |\nabla z|^2 \\ &+ \left\{ \left[\int_{\mathbb{R}^2} \left(\left(\nu_\varepsilon^2 - F''(u^0)\right) \bar{g} z \right)^2 \right]^{1/2} \right. \\ &\left. + \sum_{i=0}^{2} |V_i(y)| \left[\int_{\mathbb{R}^2} (z V_i)^2 \right]^{1/2} \right\} \left(\int_{\mathbb{R}^2} (\gamma z)^2 \right)^{1/2}. \end{aligned}$$

If we assume that $\operatorname{supp} z$ is contained in the complement of the circle $B^0 = \{x \mid |x| \leqq \rho\}$, then from (4.22) and (4.18) it can be concluded that

$$\tag{4.31} \begin{aligned} &\left| \left(\nu_\varepsilon^2 - F''(u^0)\right) \bar{g} z \right| \\ &\leqq C \varepsilon^{-2+\sigma} e^{-(c/\varepsilon)(|x|-\rho)} e \, \frac{e^{-(\nu_\varepsilon/\varepsilon)|y-x|}}{|y-x|^\sigma}, \quad x \in \operatorname{supp} z. \end{aligned}$$

Therefore, proceeding as in (4.24), we obtain

$$\tag{4.32} \begin{aligned} &\int_{\mathbb{R}^2} \left(\left(\nu_\varepsilon^2 - F''(u^0)\right) \bar{g} z \right)^2 \\ &\leqq C \varepsilon^{-4+2\sigma} \int_{|x|-\rho > d_0} e^{-(2c/\varepsilon)(|x|-\rho)} \frac{e^{-(2\nu_\varepsilon/\varepsilon)||y|-|x||}}{|x|^{2\sigma}} \, dx \\ &\leqq C \varepsilon^{-4+2\sigma} \int_{s=\rho+d_0}^{\infty} e^{-(2c/\varepsilon)(s-\rho)} e^{-(2\nu_\varepsilon/\varepsilon)||y|-s|} s^{1-2\sigma} ds, \end{aligned}$$

where $d_0 = d(\operatorname{supp} z, B^0)$.

Now we observe that $\rho < s < |y|$ implies that

$$s - \rho + |\,|y| - s\,| = |y| - \rho > \tfrac{1}{2}\left(|y| - \rho\right) + \tfrac{1}{2}(s - \rho),$$

while $\rho < |y| < s$ implies that

$$s - \rho + |\,|y| - s\,| > s - \rho > \tfrac{1}{2}\left(|y| - \rho\right) + \tfrac{1}{2}(s - \rho).$$

Therefore, if we let c' be a positive constant satisfying $c' < \frac{1}{4}\min(c, \nu_\varepsilon)$, then the estimate (4.32) yields

$$\tag{4.33} \begin{aligned} &\int_{\mathbb{R}^2} \left(\left(\nu_\varepsilon^2 - F''(u^0)\right) \bar{g} z \right)^2 \\ &\leqq C \varepsilon^{-4+2\sigma} e^{-(2c'/\varepsilon)(|y|-\rho)} \int_{s=\rho+d_0}^{\infty} e^{-(2c'/\varepsilon)(s-\rho)} s^{1-2\sigma} ds, \\ &\leqq C \varepsilon^2 e^{-(2c'/\varepsilon)(|y|-\rho)} e^{-(\mathrm{Const.}/\varepsilon)\, d_0} \end{aligned}$$

which implies that

$$\left[\int_{\mathbb{R}^2}\left(\left(\nu_\varepsilon^2 - F''(u^0)\right)\bar{g}z\right)^2\right]^{1/2} \leqq C\varepsilon^{-1}e^{-(c/\varepsilon)(|y|-\rho)-(c/\varepsilon)\,d_0}. \tag{4.34}$$

From (4.16) we also obtain

$$\sum_{i=0}^{2}|V_i(y)|\left[\int_{\mathbb{R}^2}(zV_i)^2\right]^{1/2} \leqq C\varepsilon^{-1}e^{-(c/\varepsilon)(|y|-\rho)}e^{-(c/\varepsilon)\,d_0}. \tag{4.35}$$

Now we consider a sequence of domains defined by

$$\Omega_i = \{x | d\left(x, B^0\right) \geqq \varepsilon\eta i\}, \quad i = 0, 1, \ldots, \tag{4.36}$$

where $\eta > 0$ is a number to be chosen later. Let z_i, $i = 1, 2, \ldots$, be a corresponding sequence of C^∞ cut-off functions such that

$$\text{supp } z_i = \Omega_{i-1},\;\; z_i(x) = 1 \text{ for } x \in \Omega_i,\;\; i = 1, 2, \ldots,\;\; 0 \leqq z_i(x) \leqq 1. \tag{4.37}$$

We can also assume that

$$|\bigtriangledown_x z_i| < \frac{2}{\eta\varepsilon}, \quad i = 1, 2, \ldots. \tag{4.38}$$

Let

$$a_i^2 = \int_{\Omega_i}\gamma^2. \tag{4.39}$$

Then, from the definition of Ω_i and z_i, it follows that

$$\begin{aligned} a_i^2 &\leqq \int_{\mathbb{R}^2}\gamma^2 z_i^2 \leqq a_{i-1}^2, \\ \varepsilon^2\int_{\mathbb{R}^2}\gamma^2|\bigtriangledown_x z_i|^2 &\leqq \frac{4}{\eta^2}\int_{\Omega_{i-1}\setminus\Omega_i}\gamma^2 \leqq \frac{4}{\eta^2}(a_{i-1}^2 - a_i^2). \end{aligned} \tag{4.40}$$

This, (4.30) and the estimates (4.34) and (4.35) imply that

$$Ka_i^2 \leqq \alpha e^{-c\eta i}a_{i-1} + \frac{4}{\eta^2}(a_{i-1}^2 - a_i^2), \tag{4.41}$$

where $K > 0$ is a fixed number $K < \min_{\Omega_1} F''(u^0(x))$ and

$$\alpha = C\varepsilon^{-1}e^{-(c/\varepsilon)\left(|y|-\rho\right)}. \tag{4.42}$$

From (4.41) it can be concluded that

$$\left(K + \frac{4}{\eta^2}\right)a_i^2 \leqq \alpha e^{-c\eta i}a_{i-1} + \frac{4}{\eta^2}a_{i-1}^2 \leqq \left(\frac{\eta}{4}\alpha e^{-c\eta i} + \frac{2}{\eta}a_{i-1}\right)^2,$$

which implies that

$$a_i \leqq \left(1 + \frac{K\eta^2}{4}\right)^{-1/2} \left(a_{i-1} + \frac{\eta^2}{8}\alpha e^{-c\eta i}\right), \tag{4.43}$$

and therefore

$$a_i \leqq p^i a_0 + p\frac{\eta^2}{8}\alpha e^{-c\eta i}\sum_{j=0}^{i-1}(pe^{c\eta})^j, \tag{4.44}$$

where $p = \left(1 + K\eta^2/4\right)^{-1/2}$. By choosing η large and c small we can obtain

$$pe^{c\eta} < 1.$$

Then

$$a_i \leqq e^{-c\eta i}(a_0 + C\alpha) \tag{4.45}$$

and accordingly, by recalling the expression (4.42) of α, and the estimate (4.28), we obtain

$$\left(\int_{|x|-\rho>r_i}\gamma^2\right)^{1/2} \leqq C\varepsilon^{-4+\sigma}e^{-(c/\varepsilon)(r_i+d(y))}, \tag{4.46}$$

where $r_i = d(\Omega_i,\ B^0)$.

d. Pointwise Estimates. From (4.16), (4.18), and (4.22), we infer that for $|x|, |y| > \rho$ the right-hand side h of (4.9) satisfies

$$|h(x,y)| \leqq C\left(\varepsilon^{-3/2}\frac{e^{-(c/\varepsilon)d(x)}e^{-(c/\varepsilon)|x-y|}}{|x-y|^{1/2}} + \varepsilon^{-1}e^{-(c/\varepsilon)(d(x)+d(y))}\right)$$

for some constants C, c. This and

$$d(x) + |x-y| \geqq d(x) + |d(x) - d(y)| \geqq \tfrac{1}{2}(d(x) + d(y))$$

imply that

$$|h(x,y)| \leqq C\varepsilon^{-3/2}\frac{e^{-(c/\varepsilon)(d(x)+d(y))}}{|x-y|^{1/2}} \tag{4.47}$$

for some constants $C, c > 0$ and $|x|,\ |y| > \rho$. Define j by

$$r_j \leqq d(x) \leqq r_j + \varepsilon\eta, \tag{4.48}$$

and let

$$X(x) = \bar{C}\varepsilon^{-3/2}e^{-(\bar{c}/\varepsilon)(r_j+d(y))}, \quad \delta(x) = \tfrac{4}{3}X|x-y|^{3/2}. \tag{4.49}$$

Then

$$\Delta\delta = X\frac{1}{|x-y|^{1/2}}. \tag{4.50}$$

From this, the equation

$$\varepsilon^2\Delta\gamma = F''(u^0)\gamma + h$$

and Kato's inequality

$$\Delta|\gamma| \geqq \text{sign }\gamma\Delta\gamma,$$

it follows that

$$(4.51) \qquad \varepsilon^2 \Delta(|\gamma| + \delta) \geqq F''(u^0)|\gamma| + X \frac{1}{|x-y|^{1/2}} + (\text{sign }\gamma)h.$$

By taking $\bar{C}$ large and $\bar{c}$ small in (4.49) we can make the sum of the last two terms in (4.51) nonnegative. On the other hand, $|x| > \rho + C'\varepsilon$ implies that $F''(u^0) > 0$. Consequently, from (4.51) we conclude that $|\gamma| + \delta$ is subharmonic for $|x| > \rho + C'\varepsilon$ and therefore that

$$\begin{aligned} |\gamma|(x) + \delta(x) &\leqq \frac{4}{\pi\varepsilon^2\eta^2} \int_{|z-x|<\varepsilon\eta/2} (|\gamma| + \delta)\, dz \\ &\leqq \frac{4}{\sqrt{\pi}\varepsilon\eta} \left[\left(\int_{|z-x|<\varepsilon\eta/2} |\gamma|^2 dz \right)^{1/2} + \left(\int_{|z-x|<\varepsilon\eta/2} \delta^2 dz \right)^{1/2} \right]. \end{aligned}$$

From this, the local L^2 estimate of γ, and the L^2 estimate for δ that one obtains from (4.49) by using the fact that

$$|x-y|^{3/2} \leqq \text{Const.} \left(1 + r_j^{3/2} + (d(y))^{3/2} \right),$$

statement (i) follows with $\beta = -5 + \sigma$.

Before completing the proof, for later reference, we note

Remark. From part (i) we can deduce information on the decay properties of $g(x, y)$ out of $\bar{g}(x, y)$. However, this is under a restriction on x, y while below in a number of places we need information on decay properties of $g(\cdot, y)$, uniformly for y in some arbitrary compact set. Such information can be deduced directly as follows. Equation (4.9) can be written, by using obvious notation, in the form

$$P\gamma = h.$$

Fix y in some ball of radius R. h is in $L^2(\mathbb{R}^2)$ and decays like $e^{-(c/\varepsilon)|x|}$ for $|x| \geqq R > |y|$. Utilizing the behavior of the potential F'' at infinity and Lemma 1.7 in [Ag] we obtain the existence of a $\mathscr{X} \in C_0^\infty(\mathbb{R}^2)$, $\mathscr{X} \geqq 0$, such that $P_{\mathscr{X}} = P + \mathscr{X}$ satisfies

$$\int_{\mathbb{R}^2} (P_{\mathscr{X}}\varphi)\varphi \geqq \nu^2 \int_{\mathbb{R}^2} |\varphi|^2 \qquad \forall\ \varphi \in C_0^\infty(\mathbb{R}^2),$$

where ν^2 is a positive constant.

As in [Ag, p. 26] set

$$v(x) = (1 - z(x))\gamma(x)$$

where $z \in C_0^\infty(\mathbb{R}^2)$, $z = 1$ for $|x| \leqq R + \frac{1}{2}$, $z = 0$ for $|x| \geqq R + 1$. Also notice that v is in $H^1_{\text{loc}}(\mathbb{R}^2)$, $v(x) = \gamma(x)$ for $|x| \geqq R + 1$ and

$$P_{\mathscr{X}} v = h + q = \tilde{h}$$

where $q(x) = -2\nabla z \cdot \nabla\gamma + (\mathscr{X}(1-z) - \Delta z)\gamma$. Notice that q has compact support and that

$$\|q\|_{L^2(\mathbb{R}^2)} \leqq \frac{C}{\varepsilon}. \tag{4.52}$$

Applying Theorem 4.1 in [Ag] we obtain that for every Lipschitz f with $|\nabla f(x)|^2 < \nu^2/\varepsilon^2$ the following estimate holds:

$$\int_{\mathbb{R}^2} \left(\frac{\nu^2}{\varepsilon^2} - |\nabla f|^2 \right) e^{2f} v^2 \leqq \int_{\mathbb{R}^2} \left(\frac{\nu^2}{\varepsilon^2} - |\nabla f|^2 \right)^{-1} e^{2f} \tilde{h}^2. \tag{4.53}$$

Taking into account that $\tilde{h}$ decays like $e^{-(c/\varepsilon)|x|}$ for $|x| \geqq R+1$, by choosing $f \approx \frac{c}{2\varepsilon}|x|$ we can easily deduce from (4.52), (4.53) that

$$\int_{|x| \geqq R+1} e^{(c/\varepsilon)|x|} v^2(x) < \frac{C}{\varepsilon}.$$

From this we can obtain an exponential estimate for $\int_{B(x;r)} v^2$, where $B(x;r)$ is the ball with center x and radius r, and then by using the subharmonicity of γ (hence of v) for $|x|$ large we can upgrade this to a pointwise estimate. Analogous estimates can also be obtained for the derivatives of v, and hence γ, for $|x| \geqq R+1$. $\square$

To prove (ii) a $\psi \in C^0(\bar{\Omega})$ is taken and extended by zero over $\mathbb{R}^2/\Omega$. Then the extended ψ is decomposed to

$$\psi = \psi_1 + \sum_{i=0}^{2} a_i V_i, \quad \psi_1 \perp V_0, V_1, V_2, \tag{4.54}$$

and ψ_1 is further decomposed into its radial part $\tilde{\psi}_1$ and the rest

$$\psi_1 = \tilde{\psi}_1 + \widehat{\psi}_1, \quad \tilde{\psi}_1 = \frac{1}{2\pi} \int_0^{2\pi} \psi_1(r, \Theta)\, d\Theta. \tag{4.55}$$

Note that

$$\begin{aligned} \int_\Omega g(x,y)\psi(y)\, dy &= \int_{\mathbb{R}^2} g(x,y)\tilde{\psi}_1(y)\, dy + \int_{\mathbb{R}^2} g(x,y)\widehat{\psi}_1(y)\, dy \\ &= \tilde{\phi}_1(x) + \widehat{\phi}_1(x), \end{aligned} \tag{4.56}$$

which is a consequence of the definition of the projections $\sim$ and $\widehat{}$ and of the symmetry properties of $g(x,y)$. The function $\tilde{\phi}_1$ is the solution of the problem

$$\begin{gathered} \varepsilon^2 \tilde{\phi}_1'' + \frac{\varepsilon^2}{r}\tilde{\phi}_1' - F''(u^0)\tilde{\phi}_1 = \tilde{\psi}_1, \quad r \in (0,\infty), \\ \int_0^\infty \tilde{\phi}_1 V_0 r\, dr = \int_0^\infty \tilde{\psi}_1 V_0 r\, dr = 0. \end{gathered} \tag{4.57}$$

After changing variables to $s = (r-\rho)/\varepsilon$, $\Phi(s) = \tilde{\phi}_1(\varepsilon s+\rho)$, and $\Psi(s) = \tilde{\psi}_1(\varepsilon s+\rho)$, equation (4.57) becomes

$$\begin{aligned}
&\Phi_{ss} + \frac{\varepsilon}{\varepsilon s + \rho}\Phi_s - F''\left(U^*\left(s + \frac{\rho}{\varepsilon},\ \frac{\rho - a^\xi}{\varepsilon}\right)\right)\Phi = \Psi, \quad -\frac{\rho}{\varepsilon} < s < \infty,\\
&\int_{-(\rho/\varepsilon)}^{\infty} (\varepsilon s + \rho)\Phi(s)V_0(\varepsilon s + \rho)\, ds = \int_{-(\rho/\varepsilon)}^{\infty} (\varepsilon s + \rho)\Psi(s)V_0(\varepsilon s + \rho)\, ds.
\end{aligned} \tag{4.58}$$

Based on the estimate (ii) in Proposition 2.4 and the estimate (i) in Theorem 4.1, problem (4.58) can be viewed as a perturbation of the problem

$$\begin{aligned}
&\Phi_{ss} - F''(U)\Phi = \Psi\ , \quad -\infty < s < \infty,\\
&\int_{-\infty}^{\infty} \Phi\dot{U}\, ds = \int_{-\infty}^{\infty} \Psi\dot{U}\, ds.
\end{aligned}$$

Lemma 2.3 implies that this problem has a unique solution, which satisfies the estimate

$$\|\Phi\|_0 \leqq \text{ Const. } \|\Psi\|_0.$$

Based on this, we can show that this estimate also holds for the perturbed problem (4.58) too and therefore for problem (4.57),

$$\|\tilde{\phi}_1\|_0 \leqq \text{ Const. } \|\tilde{\psi}_1\|_0.$$

From this and the inequality

$$\|\tilde{\psi}_1\|_0 \leqq \text{Const. } \|\psi\|_0,$$

which is a consequence of the definition of the projection $\sim$ and of the inequalities (4.16) and

$$|a_i| \leqq \|\psi\|_0 \int_\Omega |V_i| \leqq \text{Const. } \varepsilon^{1/2}\|\psi\|_0, \tag{4.59}$$

it follows that

$$\|\tilde{\phi}_1\|_0 \leqq \text{Const } \|\psi\|_0. \tag{4.60}$$

Next $\widehat{\phi}_1$ is estimated. From (4.54) and (4.55) it follows that outside a big ball $\widehat{B} \supset \Omega$,

$$\tilde{\psi}_1 = -a_0 V_0, \quad \widehat{\psi}_1 = -\sum_{i=1}^{2} a_i V_i.$$

Also,

$$\int_{\mathbb{R}^2} \widehat{\phi}_1(x)\, dx = 0. \tag{4.61}$$

Consequently

$$\begin{aligned}
\int_\Omega \widehat{\phi}_1(x)\, dx &= -\int_{\mathbb{R}^2/\Omega} \widehat{\phi}_1(x)\, dx = -\int_{\mathbb{R}^2/\Omega}\int_{\widehat{B}} g(x,y)\widehat{\psi}_1(y)\, dy\, dx\\
&\int_{\mathbb{R}^2/\Omega}\int_{\mathbb{R}^2/\widehat{B}} g(x,y)\left(\sum_{i=1}^{2} a_i V_i(y)\right) dy\, dx\\
&= -\int_{\mathbb{R}^2/\Omega}\int_{\widehat{B}} g(x,y)\widehat{\psi}_1(y)\, dy\, dx + O\left(e^{-c/\varepsilon}\right)\|\psi\|_{L^2(\Omega)},
\end{aligned} \tag{4.62}$$

where the estimate (4.16) for V_i, the estimate $|a_i| \leqq \text{Const.}\ \|\psi\|_{L^2(\Omega)}$, and part (i) above have been used. Now

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2/\Omega} \int_{\widehat{B}} g(x,y)\widehat{\psi}_1(y)\,dy\,dx \right| \\
(4.63)\qquad &\leqq \left| \int_{\mathbb{R}^2/\Omega} \int_{\bar{B}} g(x,y)\widehat{\psi}_1(y)dy\,dx \right| + \left| \int_{\mathbb{R}^2/\Omega} \int_{\widehat{B}/\bar{B}} g(x,y)\widehat{\psi}_1(y)\,dy\,dx \right|,
\end{aligned}
$$

where $\bar{B}$ is a ball of radius $\bar{\rho} > \rho$ contained in Ω. Using the definition of $\widehat{\psi}_1$ and the decay properties of g established in (i), we see that the first integral I_1 on the right of (4.63) can be estimated by

$$
(4.64)\qquad I_1 \leqq \text{Const.}\, e^{-(c/\varepsilon)}\|\psi\|_0, \quad c < \operatorname{dist}(\mathbb{R}^2/\Omega, \bar{B})\nu_\varepsilon.
$$

To estimate the second integral I_2, part (i) and the last Remark are used; they imply that

$$
\begin{aligned}
&I_2 \leqq \left| \int_{\mathbb{R}^2/\Omega} \int_{\widehat{B}/\bar{B}} \bar{g}(x,y)\widehat{\psi}_1(y)\,dy\,dx \right| + \text{Const.}\, e^{-(c/\varepsilon)}\|\psi\|_0 \\
(4.65)\qquad &\leqq C\,\|\psi\|_0 \left(e^{-(c/\varepsilon)} + \int_{\mathbb{R}^2} \int_{\widehat{B}} \bar{g}(x,y)\,dy\,dx \right) \leqq C'\|\psi\|_0
\end{aligned}
$$

where the fact that $\bar{g}$ is a positive function and

$$
(4.66)\qquad \int_{\mathbb{R}^2} \bar{g}(x,y)\,dx = 1
$$

has been employed. Putting together these estimates, we obtain (ii). The proof of (iii) is similar and is omitted. The proof of Theorem 4.2 is complete. □

5. The Quasi-Invariant Manifold $\tilde{M}^\varepsilon_\rho$ and Equilibria

This section is devoted to the construction of a manifold $\tilde{M}^\varepsilon_\rho$ of “bubbles” of the form $\xi \to u^\xi + v^\xi$ (v^ξ is a very small perturbation) which is an approximate invariant manifold for equation (1.1). The construction of $\tilde{M}^\varepsilon_\rho$ is made in such a way that stationary solutions to (1.1) with approximately circular interface are in $\tilde{M}^\varepsilon_\rho$ and can be detected by the vanishing of a vector field $\xi \to c^\xi$ that, as we show in the next section, describes to a very high degree of accuracy the dynamics of the center ξ of a bubble.

Theorem 5.1. *Assume that* $\rho > 0$ *is such that* $\Omega_\rho = \{\xi \in \Omega :\ d(\xi, \partial\Omega) > \rho\}$ *is non-empty and let* $\delta > 0$ *be a fixed small number. Then there is an* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$ *there exist* C^1 *functions*

$$
(5.1)\qquad \xi \to v^\xi \in C^4(\bar{\Omega}), \quad \xi \to c^\xi = \left(c^\xi_1, c^\xi_2\right) \in \mathbb{R}^2
$$

defined in $\Omega_{\rho+\delta}$ *and such that* $\int_\Omega v^\xi = 0$, *for which*

(i) $\|v^\xi\|_0 \leqq C\varepsilon^{-2}e^{-(\nu_\varepsilon/\varepsilon)\,d^\xi}$,

(ii) $|c^\xi| \leqq C\varepsilon^{-4}e^{-2(\nu_\varepsilon/\varepsilon)\,d^\xi}$.

(iii) *Similar estimates with* C *replaced by* $C\varepsilon^{-k}$, *with* k *the order of differentiation, hold for the derivatives of* v^ξ, c^ξ *with respect to* x, ξ.

(iv) *The function* $\tilde{u}^\xi = u^\xi + v^\xi$, *where* u^ξ *is defined in Section 3 satisfies the boundary conditions in* (1.1) *and*

$$\mathscr{L}(\tilde{u}^\xi) = c_1^\xi u_{,1}^\xi + c_2^\xi u_{,2}^\xi, \tag{5.2}$$

where $\mathscr{L}(\phi) = \Delta(-\varepsilon^2\Delta\phi + F'(\phi))$ *and* $u_{,i}^\xi$ *is the derivative of* u^ξ *with respect to* ξ_i, $i = 1, 2$.

(v) *Let* $\tilde{M}_\rho^\varepsilon \subset C^0(\overline{\Omega})$ *be the two-dimensional manifold*

$$\tilde{M}_\rho^\varepsilon = \{u = \tilde{u}^\xi,\ \xi \in \Omega_{\rho+\delta}\}$$

and let $\tilde{\mathscr{N}} \subset C^0(\overline{\Omega})$ *be the open neighborhood of* $\tilde{M}_\rho^\varepsilon$ *defined by*

$$\tilde{\mathscr{N}} = \left\{u \mid \exists\, \xi \in \Omega_{\rho+\delta},\ w \in C^0(\overline{\Omega}),\ \|w\|_0 < C\varepsilon^\eta,\ u = \tilde{u}^\xi + w\right\},$$

where $\|\cdot\|_0$ *stands for the* C^0 *norm. Then there is* $\eta > 0$ *such that* $u \in \tilde{\mathscr{N}}$ *is an equilibrium of* (1.1) *if and only if*

$$u = \tilde{u}^\xi, \quad c^\xi = 0 \tag{5.3}$$

for some $\xi \in \Omega_{\rho+\delta}$.

Remark. From the estimate (ii) it follows that on the manifold $\tilde{M}_\rho^\varepsilon$ the Cahn-Hilliard vector field is exponentially small in ε:

$$\|\mathscr{L}(\tilde{u}^\xi)\|_0 \leqq C\varepsilon^{-3}e^{-2(\nu_\varepsilon/\varepsilon)d^\xi}. \tag{5.4}$$

Moreover, the manifold has the property that $\mathscr{L}(\tilde{u}^\xi)$ is almost tangent to it at $\tilde{u}^\xi$ (if it were exactly tangent, then the manifold would be invariant). Indeed, Theorems 4.1, 5.1 imply that

$$\left|\langle u_{,1}^\xi, u_{,2}^\xi\rangle_{L^2}\right| \leqq Ce^{-(\nu_\varepsilon/\varepsilon)\,d^\xi}, \quad \|u_{,i}^\xi\|_{L^2} \geqq C\varepsilon^{-\frac{1}{2}}.$$

$$\begin{aligned}\min_{c_1,c_2}\frac{\|\mathscr{L}(\tilde{u}^\xi) - c_1\tilde{u}_{,1}^\xi - c_2\tilde{u}_{,2}^\xi\|_0}{\|\mathscr{L}(\tilde{u}^\xi)\|_0} &\leqq \frac{\|\mathscr{L}(\tilde{u}^\xi) - c_1^\xi\tilde{u}_{,1}^\xi - c_2^\xi\tilde{u}_{,2}^\xi\|_0}{\|\mathscr{L}(\tilde{u}^\xi)\|_0}\\ &= \frac{\|c_1^\xi v_{,1}^\xi + c_2^\xi v_{,2}^\xi\|_0}{\|c_1^\xi u_{,1}^\xi + c_2^\xi u_{,2}^\xi\|_0} \leqq |\Omega|^{\frac{1}{2}}\frac{\|c_1^\xi v_{,1}^\xi + c_2^\xi v_{,2}^\xi\|_0}{\|c_1^\xi u_{,1}^\xi + c_2^\xi u_{,2}^\xi\|_{L^2}}\\ &\leqq C\frac{|c_1^\xi|\|v_{,1}^\xi\|_0 + |c_2^\xi|\|v_{,2}^\xi\|_0}{\sqrt{(c_1^\xi)^2 + (c_2^\xi)^2}} \leqq Ce^{-(\nu_\varepsilon/\varepsilon)\,d_\xi}.\end{aligned}$$

Remark. Theorem 5.1 gives a criterion for deciding which circular curves $\Gamma \subset \Omega$ can be "continued" into true equilibria of (1.1) for positive ε, in the sense that there is a family u^ε of equilibria of (1.1) such that

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = \begin{cases} -1 & \text{inside } \Gamma, \\ +1 & \text{outside } \Gamma. \end{cases} \tag{5.5}$$

Proof of Theorem 5.1. Consider the problem

$$\begin{aligned} &\Delta\mu = c_1 u^\xi_{,1} + c_2 u^\xi_{,2}, \quad x \in \Omega, \\ &Av = \begin{cases} \mu + \sigma + F^\xi(v) + \beta\delta_{\partial\Omega}, & x \in \overline{\Omega}, \\ 0, & x \in \mathbb{R}^2 \setminus \overline{\Omega}, \end{cases} \\ &\int_\Omega v = 0, \ \int_{\mathbb{R}^2} vV_i = 0, \quad i = 1, 2, \\ &\frac{\partial\mu}{\partial n} = 0, \ \frac{\partial v}{\partial n} = -\frac{\partial u^\xi}{\partial n}, \quad x \in \partial\Omega, \end{aligned} \tag{5.6}$$

where c_1, c_2, β, μ, and v are to be considered unknowns with β a continuous function defined on $\partial\Omega$; $\delta_{\partial\Omega}$ is a unitary distribution on $\partial\Omega$; A is the second-order operator defined in (4.1); σ is the Lagrange multiplier in (2.1) and F^ξ is defined by

$$F^\xi(v) = -F'(u^\xi + v) + F'(u^\xi) + F''(u^\xi)v. \tag{5.7}$$

From (5.6), (5.7) and the definition (3.1) of u^ξ it follows that, given a solution of problem (5.6), the function $\tilde{u}^\xi = u^\xi + v$ satisfies the boundary conditions in (1.1) and equation (5.2). Therefore, aside from the "only if" part of (v), in order to establish Theorem 5.1 it is sufficient to show that (5.6) has a solution satisfying the estimates (i) and (ii).

The strategy of the proof is to construct a map $v \to \hat{v}$ by assuming that v is a given function in the right-hand side of (5.6) and by showing that (5.6) can be solved, yielding a new value $\hat{v}$ for v. The map $v \to \hat{v}$ is then shown to be a contraction for $\varepsilon > 0$ sufficiently small.

From (3.2) it is inferred that

$$\int_\Omega u^\xi_{,i} = 0, \quad i = 1, 2. \tag{5.8}$$

Therefore $(5.6)_1$ with Neumann boundary conditions is solvable and

$$\mu = c_1 \int_\Omega h(x, y) u^\xi_{,1}(y)\, dy + c_2 \int_\Omega h(x, y) u^\xi_{,2}(y)\, dy + \overline{\mu}, \tag{5.9}$$

where $\overline{\mu}$ is a constant and h is the Green's function for the problem

$$\Delta\phi = \psi, \quad x \in \Omega,$$

$$\int_\Omega \phi = \int_\Omega \psi = 0,$$
$$\frac{\partial \phi}{\partial n} = 0, \quad x \in \partial\Omega.$$

We remark that equation (5.9) defines μ as a function $\mu(c, \overline{\mu})$ of the constants c_1, c_2, and $\overline{\mu}$. If we assume that the right-hand side of equation $(5.6)_2$ satisfies the orthogonality conditions

$$\int_\Omega (\mu + \sigma + F^\xi(v))V_i + \int_{\partial\Omega} \beta V_i = 0, \quad i = 1, 2, \tag{5.10}$$

then equation $(5.6)_2$ is solvable and the solution $\hat{v}$ can be represented in the form

$$\begin{aligned} \hat{v}(x) = &\int_\Omega g(x,y)(\mu + \sigma + F^\xi(v))\, dy + \int_{\partial\Omega} g(x,y)\, \beta\, dS_y \\ &+ \mu_0^{-1} \left(\int_\Omega (\mu + \sigma + F^\xi(v))V_0\, dy + \int_{\partial\Omega} \beta V_0\, dS_y \right) V_0(x), \end{aligned} \tag{5.11}$$

where μ_0 is the principal eigenvalue of A and g is Green's function for A (see (4.3)). The function $\hat{v}$ defined by (5.11) depends on β, v and on c_1, c_2, and $\overline{\mu}$ through μ. The function $\hat{v}$ automatically satisfies the condition of being orthogonal to V_i, $i = 1, 2$. We now show how β is determined by the requirement that $\hat{v} + u^\xi$ satisfy Neumann boundary conditions. In order to do that, we need to study the operator from $C^0(\partial\Omega)$ into itself defined by

$$\begin{aligned} \beta(x) \to &\frac{\partial}{\partial n_x} \int_{\partial\Omega} g(x,y)\beta\, dS_y + \mu_0^{-1} \left(\int_{\partial\Omega} \beta V_0\, dS_y \right) \frac{\partial V_0}{\partial n}(x) \\ &= \lim_{s\to 0^-} \frac{d}{ds} \int_{\partial\Omega} g(x+sn, y)\beta\, dS_y + \mu_0^{-1} \left(\int_{\partial\Omega} \beta\, V_0\, dS_y \right) \frac{\partial V_0}{\partial n}(x), \end{aligned} \tag{5.12}$$

where n is the outward normal to $\partial\Omega$ at x.

Lemma 5.2. *Assume that* $\beta \in C^0(\partial\Omega)$. *Then there is* $\varepsilon_0 > 0$ *such that, for* $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} &\frac{\partial}{\partial n_x} \int_{\partial\Omega} g(x,y)\beta\, dS_y + \mu_0^{-1} \left(\int_{\partial\Omega} \beta V_0\, dS_y \right) \frac{\partial V_0}{\partial n}(x) \\ &\qquad = \frac{1}{2}\varepsilon^{-2}\beta(x) + (K\beta)(x), \end{aligned} \tag{5.13}$$

where K *is an integral operator such that*

$$\|K\beta\|_0 \leqq C\varepsilon^{-1}\|\beta\|_0. \tag{5.14}$$

Therefore, for $\varepsilon < C^{-1}$, *the equation*

$$\tfrac{1}{2}\varepsilon^{-2}\beta + K\beta = \gamma \tag{5.15}$$

has a unique solution and

$$\|\beta\|_0 \leqq \text{Const.}\, \varepsilon^2 \|\gamma\|_0. \tag{5.16}$$

Proof. By Theorem 4.2(i), for $x \in \partial\Omega$ we have the estimate

$$\left| \frac{\partial}{\partial n_x} \int_{\partial\Omega} (g(x,y) - \overline{g}(x,y))\, \beta(y)\, dS_y \right| \leqq Ce^{-(c/\varepsilon)} \|\beta\|_0.$$

Accordingly, it suffices to prove the lemma with g replaced by $\overline{g}$. The analysis of the first term in (5.13) is then reduced to standard potential theory estimates (cf. e.g. [Fo]) and is omitted. The analysis of the second term in (5.13) which is part of the operator K is done by using the estimate implied in Theorem 4.1:

$$|V_0(x)| \leqq Ce^{-(c/\varepsilon)}, \quad x \in \partial\Omega,$$

and similar estimates for the partial derivatives of V_0 together with the estimate for μ_0 given in Theorem 4.1(i). □

From Lemma 5.2 it is seen that the equation

$$\frac{\partial \hat{v}}{\partial n} = -\frac{\partial u^{\xi}}{\partial n}$$

uniquely determines β (and therefore $\hat{v}$) as a function $\beta = \beta(c, v)$ of $c = (c_0, c_1, c_2)$, and v, where c_0 stands for the sum $c_0 = \overline{\mu} + \sigma$. We now show that equations (5.10) with $\beta = \beta(c, v)$, together with the condition

$$\int_{\Omega} \hat{v}(c, v) = 0 \tag{5.17}$$

make up a linear system in $c = (c_0, c_1, c_2)$, which ultimately determines c as a function of v and therefore gives meaning to the map $v \to \hat{v}$. In what follows we carry out the analysis of this linear system for establishing solvability. We also obtain estimates that will be needed for computing the norm of the map. We begin by studying the coefficients $B = (B_0, B_1, B_2)$ of c_0, c_1, c_2 in $\beta(c, v)$. The estimates of B, and other estimates needed later, are collected in the following lemma:

Lemma 5.3. *The solution $\beta(c, v)$ of the equation $\frac{\partial \hat{u}}{\partial n} = -\frac{\partial u^{\xi}}{\partial n}$ can be represented in the form*

$$\beta(c, v) = B^T c + \beta^0 + \beta^1(v), \tag{5.18}$$

where $\beta^1(0) = 0$ and the functions $\beta^0 : \partial\Omega \to \mathbb{R}$, $\beta^1 : \partial\Omega \times C^0(\overline{\Omega}) \to \mathbb{R}$, $B : \partial\Omega \to \mathbb{R}^3$ satisfy, for $0 < \varepsilon \ll 1$,

$$\begin{aligned} B_0(x) &= -\varepsilon \frac{k}{\pi \nu_\varepsilon} + O(\varepsilon^2), \\ B_i(x) &= -2\varepsilon \frac{k}{\pi \nu_\varepsilon} \int_{\Gamma} h(x, \cdot) \cos\alpha_i + O(\varepsilon^2), \quad i = 1, 2, \\ \beta^0(x) &= -2\varepsilon^2 \frac{\partial u^{\xi}}{\partial n}(x) + O\left(\varepsilon^3 \left\| \frac{\partial u^{\xi}}{\partial n} \right\|_0 \right), \\ \beta^1(v)(x) &= \varepsilon \frac{k}{\pi \nu_\varepsilon} F^{\xi}(v)(x) + O\left(\varepsilon^2 \|F^{\xi}(v)\|_0\right), \end{aligned} \tag{5.19}$$

where $\alpha_i(x)$ *is the angle of the vector* $x - \xi$ *with the* x_i *axis and*

$$\| \ \|_0 = \max_{\partial\Omega} |\ |, \quad k = \int_{\langle z,n\rangle>0} K_0'(|z|)\frac{\langle z,n\rangle}{|z|}dz$$

with K_0 *the function defined in Lemma 4.3. Moreover, from* (5.19) *it follows that*

$$\|B\|_0 \leqq C\varepsilon, \quad \|\beta^0\|_0 \leqq C\varepsilon e^{-(\nu_\varepsilon/\varepsilon)d^\xi}, \quad \|\beta^1(v)\|_0 \leqq C\varepsilon\|v\|_0^2. \tag{5.20}$$

Proof. Let the known terms in equation (5.17), that is, the terms which do not contain β, be denoted by $\gamma(c, v)$; then

$$\gamma(c, v) = D^T c + \gamma^0 + \gamma^1(v),$$

where $D = (D_0, D_1, D_2)$ is given by

$$D_0(x) = -\frac{\partial}{\partial n_x}\int_\Omega g(x,\cdot) - \mu_0^{-1}\left(\int_\Omega V_0\right)\frac{\partial V_0}{\partial n}(x) =: D_0^1(x) + D_0^2(x),$$

$$\begin{aligned}D_i(x) &= -\frac{\partial}{\partial n_x}\int_\Omega g(x,y)\left(\int_\Omega h(y,z)u^\xi_{,i}(z)dz\right)dy\\ &\quad -\mu_0^{-1}\left[\int_\Omega V_0(y)\left(\int_\Omega h(y,z)u^\xi_{,i}(z)dz\right)dy\right]\frac{\partial V_0}{\partial n}(x) =: D_i^1(x) + D_i^2(x),\end{aligned} \tag{5.21}$$

$$\gamma^0(x) = -\frac{\partial u^\xi}{\partial n}(x),$$

$$\gamma^1(v)(x) = -\frac{\partial}{\partial n_x}\int_\Omega g(x,\cdot)F^\xi(v) - \mu_0^{-1}\left(\int_\Omega V_0 F^\xi(v)\right)\frac{\partial V_0}{\partial n}(x).$$

To estimate D_0^1 we write

$$D_0^1(x) = -\frac{\partial}{\partial n_x}\int_{\bar B} g(x,\cdot) - \frac{\partial}{\partial n_x}\int_{\Omega/\bar B} g(x,\cdot),$$

where $\bar B$ is a ball of radius $\bar\rho > \rho$ contained in Ω. Using the decay properties of g (see the Remark in the proof of Theorem 4.2) we see that the first integral is of order $O(e^{-c/\varepsilon})$. Similarly, from Theorem 4.2(i), we see that replacing g by $\bar g$ in the second integral changes the integral by a term which is again of order $O(e^{-(c/\varepsilon)})$. Therefore we can write

$$D_0^1(x) = -\frac{\partial}{\partial n_x}\int_{\Omega/\bar B}\bar g(x,\cdot) + O\left(e^{-c/\varepsilon}\right). \tag{5.22}$$

Therefore by (4.11),

$$\begin{aligned}\frac{\partial}{\partial n_x}\int_{\Omega/\bar B}\bar g(x,\cdot) &= \frac{\nu_\varepsilon}{2\pi\varepsilon^3}\int_{\Omega/\bar B} K_0'\left(\frac{\nu^\varepsilon}{\varepsilon}|x-y|\right)\frac{\langle x-y,\ n\rangle}{|x-y|}dy\\ &= \frac{1}{2\pi\nu_\varepsilon\varepsilon}\int_{E_x} K_0'(|z|)\frac{\langle z,n\rangle}{|z|}dz,\end{aligned} \tag{5.23}$$

where $E_x = \{z\,|\,z = \varepsilon^{-1}\nu_\varepsilon(x-y),\ y \in \Omega/\bar B\}$.

From (5.22), (5.23), the smoothness of Ω and the exponential decay of K_0' at infinity it is understood that

$$D_0^1(x) = -\frac{k}{2\pi\nu^\varepsilon\varepsilon} + \eta_0(x), \tag{5.24}$$

where

$$k = \lim_{\varepsilon\to 0}\int_{E_x} K_0'(|z|)\frac{\langle z, n\rangle}{|z|}dz = \int_{\langle z,n\rangle>0} K_0'(|z|)\frac{\langle z, n\rangle}{|z|}dz, \tag{5.25}$$

$$\|\eta_0\|_0 \leqq C. \tag{5.26}$$

From Theorem 4.1(i) (see also Lemma 2.2 in [A-F1]) it follows that

$$\|D_0^2\|_0 < Ce^{-c/\varepsilon}. \tag{5.27}$$

To estimate $D_i^1,\ i = 1, 2$, we start by observing that, as $\varepsilon \to 0,\ u^\xi_{,i}$ approaches a distribution supported on Γ with density $-2\cos\alpha_i(x),\ \alpha_i(x)$ being the angle of the vector $x - \xi$ with the x_i axis. This is derived from the definition of u^ξ, from Proposition 2.1 and from the fact that $\int_{-\infty}^{\infty}\dot U = 2$. Setting

$$\varphi(x) = \int_\Omega h(x,\cdot)u^\xi_{,i},$$

we deduce by this observation that

$$\lim_{\varepsilon\to 0}\varphi(x) = -2\int_\Gamma h(x,\cdot)\cos\alpha_i, \tag{5.28}$$

where $\Gamma = \{y | y - \xi| = \rho\}$. By the same arguments used for estimating D_0^1 and D_0^2 we see that

$$\begin{aligned} D_i^1(x) &= -\frac{\nu_\varepsilon}{2\pi\varepsilon^3}\int_{\Omega/\bar B} K_0'\left(\frac{\nu_\varepsilon}{\varepsilon}|x-y|\right)\frac{\langle x-y, n\rangle}{|x-y|}\varphi(y)\,dy + O(e^{-c/\varepsilon}) \\ &= -\frac{k}{\pi\nu_\varepsilon\varepsilon}\int_\Gamma h(x,\cdot)\cos\alpha_i + \eta_i(x), \end{aligned} \tag{5.29}$$

$$||\eta_i||_0 < C, \tag{5.30}$$

$$\left|D_i^2(x)\right| < Ce^{-c/\varepsilon}, \tag{5.31}$$

$$\gamma^1(v)(x) = \frac{k}{2\pi\nu_\varepsilon\varepsilon}F^\xi(v)(x) + O(1). \tag{5.32}$$

From (3.4) it follows that

$$\|\gamma^0\|_0 \leqq C\varepsilon^{-1}e^{-(\nu^\varepsilon/\varepsilon)\,d^\xi}. \tag{5.33}$$

We also have

$$\|\gamma^1(v)\|_0 \leqq C\varepsilon^{-1}\|v\|_0^2. \tag{5.34}$$

From the definition of γ, the above estimates and Lemma 5.2 the lemma follows. □

We now begin the discussion of the linear system determining $c(v)$.

Lemma 5.4. *The linear system of the three equations* (5.17) *and* (5.10) (*with* $\beta = \beta(c,v)$) *in the three unknowns* $c = (c_0, c_1, c_2)$ *can be written in the form*

$$Hc = p \tag{5.35}$$

where $H = (H_{ij})$, $i, j = 0, 1, 2$, *is a* 3×3 *matrix and* p *is a* 3*-vector. Furthermore,*

$$\begin{aligned}
H_{00} &= -\varepsilon^{-1} \frac{8\pi\rho^3}{\int_{-\infty}^{\infty} \dot{U}^2} + O(1), \\
H_{0i} &= -\varepsilon^{-1} \frac{8\rho^2}{\int_{-\infty}^{\infty} \dot{U}^2} \int_\Gamma \int_\Gamma h(x,y) \cos\alpha_i(y)\, dy\, dx + O(1), \quad i = 1,2, \\
H_{i0} &= O\left(\varepsilon^{1/2} e^{-(\nu^\varepsilon/\varepsilon) d^\xi}\right), \quad i = 1,2, \\
H_{ij} &= \varepsilon^{1/2} \frac{2\sqrt{2}}{\sqrt{\pi\rho \int_{-\infty}^{\infty} \dot{U}^2}} \int_\Gamma \int_\Gamma h(x,y) \cos\alpha_j(y) \cos\alpha_i(x)\, dy\, dx \\
&\quad + O(\varepsilon^{3/2}), \quad i, j = 1,2,
\end{aligned} \tag{5.36}$$

$$\begin{aligned}
p_0 &= O\left(\varepsilon^{-1} e^{-(\nu_\varepsilon/\varepsilon) d^\xi} + \varepsilon^{-1} \|v\|_0^2\right), \\
p_i &= O\left(\varepsilon^{1/2} e^{-(2\nu_\varepsilon/\varepsilon) d^\xi} + \varepsilon^{1/2} \|v\|_0^2\right), \quad i = 1,2.
\end{aligned} \tag{5.37}$$

Proof. By inserting (5.9) and (5.18) into (5.11) and by writing equations (5.17) and (5.10) explicitly we obtain the following expressions for H, p:

$$\begin{aligned}
H_{00} &= \mu_0^{-1} \left[\left(\int_\Omega V_0 \right)^2 + \left(\int_{\partial\Omega} B_0 V_0 \right) \int_\Omega V_0 \right] + \int_\Omega \int_\Omega g(x,y)\, dx\, dy \\
&\quad + \int_\Omega \int_{\partial\Omega} g(x,y) B_0(y)\, dx\, dy, \\
H_{o,i} &= \mu_0^{-1} \left[\int_\Omega \int_\Omega h(x,y) u^\xi_{,i}(y) V_0(x)\, dy\, dx + \int_{\partial\Omega} B_i V_0 \right] \int_\Omega V_0 \\
&\quad + \int_\Omega \int_\Omega \int_\Omega g(x,y) h(y,z) u^\xi_{,i}(z)\, dz\, dy\, dx \\
&\quad + \textstyle\int_\Omega \int_{\partial\Omega} g(x,y) B_i(y)\, dy\, dx, \quad i = 1,2, \\
H_{io} &= \int_\Omega V_i + \int_{\partial\Omega} B_0 V_i, \quad i = 1,2, \\
H_{ij} &= \int_\Omega \int_\Omega h(x,y) u^\xi_j(y) V_i(x)\, dx\, dy + \int_{\partial\Omega} B_j V_i, \quad i,j = 1,2,
\end{aligned} \tag{5.38}$$

$$
\begin{aligned}
p_0 &= p_0^0 + p_0(v),\\
p_0^0 &= -\mu_0^{-1}\left(\int_{\partial\Omega}\beta^0 V_0\right)\int_\Omega V_0 - \int_\Omega\int_{\partial\Omega} g(x,y)\beta^0(y)\,dy\,dx\\
p_0(v) &= -\mu_0^{-1}\left[dst\int_\Omega F^\xi(v)V_0 + \int_{\partial\Omega}\beta^1(v)V_0\right]\int_\Omega V_0\\
&\quad - \int_\Omega\int_\Omega g(x,y)F^\xi(v)(y)\,dy\,dx - \int_\Omega\int_{\partial\Omega} g(x,y)\beta^1(v)(y)\,dy\,dx,
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
p_i &= p_i^0 + p_i(v),\\
p_i^0 &= -\int_{\partial\Omega}\beta^0 V_i, \quad i = 1,2.\\
p_i(v) &= -\int_\Omega F^\xi(v)V_i - \int_{\partial\Omega}\beta^1(v)V_i.
\end{aligned}
\tag{5.40}
$$

The main contribution to H_{00} is the first term in $(5.38)_1$. This term can be estimated by using Theorem 4.1(i),(iv). The second term is of order $O(e^{-c/\varepsilon})$ because V_0 is exponentially small in ε on $\partial\Omega$ (cf. (4.16) and Lemma 2.2 in [A-F1]). The third and fourth terms are handled by Theorem 4.2(ii) and (iii) and Lemma 5.3 and are of order $O(1)$. The estimates in (5.36) are obtained in a similar way from (5.38), by keeping in mind that

$$
u^\xi_{,i} = \frac{\partial u^\xi}{\partial x_i} + \frac{1}{\varepsilon}U^*_\rho a_i^\xi, \tag{5.41}
$$

with $a_i^\xi = O(e^{-c/\varepsilon})$ by Lemma 3.1 and by using Theorem 4.2(ii), (iii) with

$$
\varphi(x) = \int_\Omega h(x,\cdot)u_i^\xi, \quad \psi(x) = B_i(x).
$$

The estimates (5.37) are obtained from (5.39) and (5.40) by similar arguments utilizing Lemma 2.2 in [A-F1], Lemma 5.3, and Theorem 4.2 (ii), (iii). □

Lemma 5.5. *There exists a number $\varepsilon_0 > 0$ such that for any ε, $0 < \varepsilon < \varepsilon_0$, equation* (5.35) *has a unique solution $c(v)$ and*

$$
\begin{aligned}
|c_0(v)| &\leqq C\left(e^{-(\nu_\varepsilon/\varepsilon)\,d^\xi} + \|v\|_0^2\right),\\
|c_i(v)| &\leqq C\left(e^{-(2\nu_\varepsilon/\varepsilon)\,d^\xi} + \|v\|_0^2\right), \quad i = 1,2.
\end{aligned}
\tag{5.42}
$$

Proof. For ε small, the matrix H is nonsingular because Lemma 5.4 implies that $|H_{00}| > C\varepsilon^{-1}$, for some $C > 0$, $H_{i0} = O(e^{-c/\varepsilon})$, $i = 1,2$, $H_{ij} = O(\varepsilon^{1/2})$, $i,j = 1,2$ and, furthermore, this 2×2 submatrix is a negative-definite matrix because Δ with Neumann boundary conditions defines a negative operator on $L_0^2(\Omega)$, the subspace of $L^2(\Omega)$ of the functions with zero average. From this and from the estimates for p_0, p_i in Lemma 5.4 solvability follows together with the estimates (5.42). □

Substituting the function $c(v)$ given by Lemma 5.5 into the expression for $\beta(c, \underline{v})$ in Lemma 5.3 and that in turn into (5.11) yields a map $v \to \widehat{v}$ from $C^0(\bar{\Omega})$ into itself. We denote this map by T and show that T is a contraction on a suitable closed subset of $C^0(\bar{\Omega})$.

Lemma 5.6. *There exist numbers K, α, $\varepsilon_0 > 0$, such that for any ε, $0 < \varepsilon < \varepsilon_0$, the map $T : B_\delta \to B_\delta$, with $B_\delta = \{v \in C^0(\bar{\Omega}) | \|v\|_0 \leqq \delta = K\varepsilon^\alpha\}$, is a contraction. The fixed point v^ξ satisfies the estimate*

$$\|v^\xi\|_0 \leqq C\varepsilon^{-2} e^{-(\nu_\varepsilon/\varepsilon)\, d^\xi}. \tag{5.43}$$

Proof. By standard elliptic regularity theory we obtain from (4.9) the gross estimate

$$\|g(\cdot, y)\|_{W^{1,2}(\Omega)} \leqq \frac{C}{\varepsilon}$$

uniformly for $y \in \Omega$. From this and the symmetry of $g(x, y)$ we deduce that

$$\int_{\partial\Omega} |g(x, y)|\, dS_y \leqq \frac{C}{\varepsilon}, \quad \int_\Omega |g(x, y)|\, dy \leqq \frac{C}{\varepsilon} ,$$

and by utilizing the second of these estimates we obtain for $v, w \in B_\delta$ that

$$\begin{aligned} \left| \int_\Omega g(x, \cdot) F^\xi(v) \right| &\leqq C\varepsilon^{-1}\delta^2, \\ \left| \int_\Omega g(x, \cdot) \left(F^\xi(v) - F^\xi(w)\right) \right| &\leqq C\varepsilon^{-1}\delta \|v - w\|_0. \end{aligned} \tag{5.44}$$

Now, from Lemma 5.3, for $v, w \in B_\delta$ we have

$$\|\beta^1(v)\|_0 \leqq C\varepsilon\delta^2, \quad \|\beta^1(v) - \beta^1(w)\|_0 \leqq C\varepsilon\delta \|v - w\|_0. \tag{5.45}$$

From (5.42) it follows that

$$\begin{aligned} |c_0(v)| &\leqq C(e^{-(\nu_\varepsilon/\varepsilon)\, d^\xi} + \delta^2), \\ |c_i(v)| &\leqq C(e^{-(2\nu_\varepsilon/\varepsilon)\, d^\xi} + \delta^2), \quad i = 1, 2. \end{aligned} \tag{5.46}$$

From Lemma 5.4 and in particular from (5.35) and the expressions for $p_0(v), p_i(v)$, it follows that

$$|c(v) - c(w)| \leqq C\delta \|v - w\|_0. \tag{5.47}$$

Using these estimates, Theorem 4.2, Lemma 5.3 and the estimates on g above, we obtain

$$
(5.48)\qquad
\begin{aligned}
\left|\int_{\Omega} g(x,\cdot)(\mu(v)+\sigma)\right| &\leqq C\varepsilon^{-1}\left(e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}}+\delta^2\right),\\
\left|\int_{\Omega} g(x,\cdot)(\mu(v)-\mu(w))\right| &\leqq C\varepsilon^{-1}\delta\|v-w\|_0,\\
\left|\int_{\partial\Omega} g(x,\cdot)\beta(v)\right| &\leqq C\varepsilon^{-1}\left(e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}}+\delta^2\right),\\
\left|\int_{\partial\Omega} g(x,\cdot)\,(\beta(v)-\beta(w))\right| &\leqq C\varepsilon^{-1}\delta\|v-w\|_0,
\end{aligned}
$$

where $\mu(v)$ is the function obtained by inserting $c_1(v)$, $c_2(v)$, $\bar{\mu} = c_0(v) - \sigma$ into (5.9), and $\beta(v)$ is obtained by inserting $c(v)$ into (5.18). Taking into account Theorem 4.1 we also obtain

$$
(5.49)\qquad
\begin{aligned}
\left\|\mu_0^{-1}\left[\int_{\Omega}(\mu(v)+\sigma)V_0\right]V_0\right\|_0 &\leqq C\varepsilon^{-2}\left(e^{-(\nu_\varepsilon d^{\xi}/\varepsilon)}+\delta^2\right),\\
\left\|\mu_0^{-1}\left[\int_{\Omega}(\mu(v)-\mu(w))V_0\right]V_0\right\|_0 &\leqq C\varepsilon^{-2}\delta\|v-w\|_0,\\
\left\|\mu_0^{-1}\left[\int_{\partial\Omega}\beta(v)V_0\right]V_0\right\|_0 &\leqq C\varepsilon^{-2}e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}}\left(e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}}+\delta^2\right),\\
\left\|\mu_0^{-1}\left[\int_{\partial\Omega}(\beta(v)-\beta(w))V_0\right]V_0\right\|_0 &\leqq C\varepsilon^{-2}e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}}\delta\|v-w\|_0.
\end{aligned}
$$

$$
(5.50)\qquad
\begin{aligned}
\left\|\mu_0^{-1}\left[\int_{\Omega}F^{\xi}(v)V_0\right]V_0\right\|_0 &\leqq C\varepsilon^{-2}\delta^2,\\
\left\|\mu_0^{-1}\left[\int_{\Omega}\left(F^{\xi}(v)-F^{\xi}(w)\right)V_0\right]V_0\right\|_0 &\leqq C\varepsilon^{-2}\delta\|v-w\|_0.
\end{aligned}
$$

Therefore, from (5.11) we obtain

$$
(5.51)\qquad
\begin{aligned}
\|T(v)\|_0 &\leqq C\varepsilon^{-2}(e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}}+\delta^2),\\
\|T(v)-T(w)\|_0 &\leqq C\varepsilon^{-2}\delta\|v-w\|_0,
\end{aligned}
$$

which show that T is a contraction on B_δ for $\delta = K\varepsilon^{\alpha}$ with $\alpha > 2$, $K > 0$ and $0 < \varepsilon \ll 1$. To derive the estimate (5.43) we observe that if we set

$$
(5.52)\qquad \delta = 4C\varepsilon^{-2}e^{-(\nu_\varepsilon/\varepsilon)\,d^{\xi}},
$$

then for $\varepsilon > 0$ sufficiently small, the estimates (5.51) imply

$$
(5.53)\qquad \|T(v)\|_0 < \delta, \quad \|T(v)-T(w)\|_0 < \eta\|v-w\|_0
$$

for some $\eta < 1$ and therefore the fixed point v^{ξ} satisfies $\|v^{\xi}\|_0 < \delta$, that is, (5.43). □

We are now in the position to completing the proof of Theorem 5.1: Part (ii) follows by inserting (i) in (5.42)(ii). Part (iii) is straightforward but requires some computations. We restrict ourselves to a few words. First, for obtaining estimates for the derivatives of v with respect to x_i, one can use the equation (5.2) to obtain estimates on Δv, and then by interpolation, estimates on v_{x_i}. For higher derivatives, one needs to differentiate (5.2), and this requires smoothness of F. For obtaining estimates on derivatives of v and c with respect to ξ, one has to argue differently. First, we note the smooth dependence of c on v. Next, by differentiating (5.2) with respect to ξ_i, we obtain an equation involving the linearized operator A. It is not difficult to see that the orthogonality condition $\int_{\mathbb{R}^2} vV_i\,dx = 0$ is almost preserved under differentiation with respect to ξ (since $\int_{\mathbb{R}^2} v_\xi V_i\,dx = -\int_{\mathbb{R}^2} vV_{i\xi}dx = 0$ and v is exponentially small) and therefore we can invert and obtain the desired estimates on v_ξ. Similarly, we can treat higher derivatives in ξ. Part (iv) is a straightforward consequence of the above constructions. The "if" part of (v) is obvious. To prove the "only if" part of (v) one need to show that, given $u_e \in \tilde{\mathcal{N}}$, a stationary solution of (1.1), there exist $\xi \in \Omega_\rho,\ \mu = \bar{\mu},\ \beta\colon \partial\Omega \to \mathbb{R}$ and a continuous function $v\colon \mathbb{R}^2 \to \mathbb{R}$ that satisfy (5.6) with $c_1 = c_2 = 0$ and $v\,|_\Omega = u - u^\xi$. To do this, we define $v^\xi = u_e - u^\xi$ for any $\xi \in \Omega_\rho$ and extend the function $v^\xi = u_e - u^\xi$ to a continuous bounded function defined in the whole of $\mathbb{R}^2$ by imposing that

$$Av^\xi = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{\Omega}.$$

Clearly,

$$\int_\Omega v^\xi = 0,$$

$$\frac{\partial v^\xi}{\partial n} = -\frac{\partial u^\xi}{\partial n},\ x \in \partial\Omega.$$

Then we choose ξ by imposing the conditions

$$\int_{\mathbb{R}^2} u^\xi V_i = 0, \quad i = 1, 2.$$

Once ξ is fixed, β is determined by the jump of the normal derivative of v^ξ across $\partial\Omega$ and $\bar{\mu} = -\sigma_e$ where

$$\varepsilon^2\Delta u_e - F'(u_e) = \sigma_e. \quad \square$$

6. The Linearized Cahn-Hilliard Operator

We begin by stating a theorem concerning the linearized Cahn-Hilliard operator

$$L^\xi = \Delta(-\varepsilon^2\Delta + F''(\tilde{u}^\xi)), \tag{6.1}$$

where $\tilde{u}^\xi = u^\xi + v^\xi$, with v^ξ the function constructed in Theorem 5.1. We consider L^ξ with the boundary conditions as an operator on H_0^{-1}, the subspace of the

Sobolev space H^{-1} consisting of functions with zero average. We let $\langle \cdot, \cdot \rangle$ be the standard inner product in $L^2(\Omega)$ and $(\cdot, \cdot)$ the inner product in H^{-1}. In the subspace H_0^{-1} we have

(6.2) $$(\phi, \psi) = \langle (-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\psi \rangle,$$

where $-\Delta$ is the self-adjoint positive operator defined in $L_0^2(\Omega) = \{\phi \in L^2(\Omega) | \int_\Omega \phi = 0\}$ by the negative Laplacian with Neumann boundary conditions. In this section $\| \cdot \|$ denotes the H^{-1} norm

(6.3) $$\|\phi\| = \sqrt{(\phi, \phi)}.$$

Theorem 6.1.

(i) *The operator L^ξ can be extended to a self-adjoint operator on H_0^{-1}, the subspace of the Sobolev space H^{-1} consisting of functions with zero average. $-L^\xi$ is bounded below.*

(ii) *Let $\lambda_1^\xi \leqq \lambda_2^\xi \leqq \lambda_3^\xi \leqq \cdots$ be the eigenvalues of*

(6.4) $$\begin{aligned} &\Delta(-\varepsilon^2 \Delta\psi + F''(\tilde{u}^\xi)\psi) = -\lambda\psi, \quad x \in \Omega, \\ &\frac{\partial\psi}{\partial n} = \frac{\partial\Delta\psi}{\partial n} = 0, \quad x \in \partial\Omega, \end{aligned}$$

and let $\delta > 0$ be fixed. Then there is $\varepsilon_0 > 0$ and constants $c, C, C' > 0$ independent of ε such that, for $0 < \varepsilon < \varepsilon_0$ and $\xi \in \Omega$ with $d^\xi > \delta$, the following estimates hold:

(6.5) $$-Ce^{-c/\varepsilon} \leqq \lambda_1^\xi \leqq \lambda_2^\xi \leqq Ce^{-c/\varepsilon},$$

(6.6) $$\lambda_3^\xi \geqq C'\varepsilon.$$

(iii) *In the two-dimensional subspace U^ξ corresponding to the small eigenvalues $\lambda_1^\xi, \lambda_2^\xi$ there is an orthonormal basis (in H^{-1}) ψ_1^ξ, ψ_2^ξ such that*

(6.7) $$\psi_i^\xi = \sum_{j=1}^{2} a_{ij}^\xi \frac{\tilde{u}_{,j}^\xi}{\|\tilde{u}_{,j}^\xi\|} + O(e^{-c/\varepsilon}), \quad i = 1, 2$$

where the matrix (a_{ij}^ξ) is nonsingular and a smooth function of ξ and $\tilde{u}_{,j}^\xi$ is the derivative of $\tilde{u}^\xi$ with respect to ξ_j. Moreover ψ_i^ξ is a smooth function of ξ and

(6.8) $$\|\psi_{i,j}^\xi\| = O(\varepsilon^{-1}), \quad i, j = 1, 2,$$

where $\psi_{i,j}^\xi$ is the derivative of ψ_i^ξ with respect to ξ_j.

Proof. Statement (i) is standard and is omitted. Statement (ii) is proved in [A-F1] with u^ξ instead of $\tilde{u}^\xi$. Since the difference between $\tilde{u}^\xi$ and u^ξ is of order $O(e^{-c/\varepsilon})$, it is not difficult to see that replacing u^ξ with $\tilde{u}^\xi$ changes the eigenvalues by a quantity of order $O(e^{-c/\varepsilon})$. To prove (iii) we start from the equations

$$L^{\xi}\tilde{u}^{\xi}_{,i} = \sum_{j=1}^{2} c^{\xi}_{j,i} u^{\xi}_{,j} + \sum_{j=1}^{2} c^{\xi}_{j} u^{\xi}_{,ij}, \quad i = 1, 2, \tag{6.9}$$

which are obtained by differentiating equation (5.2) with respect to ξ_i, $i = 1, 2$. In equation (6.9), $c^{\xi}_{j,i}$ stands for the derivative of c^{ξ}_j with respect to ξ_i and $u^{\xi}_{,ji}$ for the second derivative of u^{ξ} with respect to ξ_i, ξ_j. From equation (6.9) and Theorem 5.1 it follows that

$$\|L^{\xi}\tilde{u}^{\xi}_{,i}\| = O(e^{-c/\varepsilon}), \quad i = 1, 2. \tag{6.10}$$

Furthermore, since Theorem 5.1(i),(iii), and (5.41) imply that

$$\tilde{u}^{\xi}_{,i} = \frac{\partial u^{\xi}}{\partial x_i} + O(e^{-c/\varepsilon}), \tag{6.11}$$

$$\left(\frac{\partial u^{\xi}}{\partial x_i}, \frac{\partial u^{\xi}}{\partial x_j}\right) = -\int_{\Omega}\int_{\Omega} h(x, y)\frac{\partial u^{\xi}}{\partial x_i}(x)\frac{\partial u^{\xi}}{\partial x_j}(y)\, dx\, dy, \tag{6.12}$$

we obtain that

$$\left(\tilde{u}^{\xi}_{,i}, \tilde{u}^{\xi}_{,j}\right) = -\varepsilon^{-1/2}H_{ij} + O(\varepsilon), \tag{6.13}$$

which together with (5.36)(iv) and the fact that (H_{ij}) is a negative-definite matrix, imply that the matrix $\left(\tilde{u}^{\xi}_{,i}, \tilde{u}^{\xi}_{,j}\right)$ approaches a nonsingular limit when $\varepsilon \to 0$. This, the estimate (6.10) and the fact that (6.5), (6.6) imply the existence of a gap of order ε between λ^{ε}_2 and λ^{ε}_3 allow for the application of a basic perturbation result (Lemma A.1 in [A-F1]). It follows that the distance between U^{ξ} and span $\{\tilde{u}^{\xi}_{,1}, \tilde{u}^{\xi}_{,2}\}$ is of order $O(e^{-c/\varepsilon})$. By using this fact and the fact that $-\varepsilon^{-1/2}(H_{ij})$ is a nonsingular matrix which depends smoothly on ξ it is possible to construct the coefficients a^{ξ}_{ij} as claimed. Equation (6.8) follows from (6.7), Proposition 2.4 and Theorem 5.1. □

7. The Dynamics of Bubbles

In this section we discuss the dynamics of approximately circular interfaces and show that, if the initial condition is sufficiently close to the quasi-invariant invariant manifold $\tilde{M}^{\varepsilon}_{\rho}$ constructed in §5, then the interface keeps its almost circular shape and drifts very slowly across Ω. We show that the dynamics of its center is determined to a very high degree of accuracy by the system of ordinary differential equations

$$\dot{\xi}_1 = c^{\xi}_1, \quad \dot{\xi}_2 = c^{\xi}_2 \tag{7.1}$$

where $c^{\xi} = \left(c^{\xi}_1, c^{\xi}_1\right)$ is the vector field determined in Theorem 5.1. To show this we use a technique similar to the one developed in [A-B-F] for the one-dimensional case. We justify equations (7.1) by proving the existence of a set which is similar to the slow channel considered in [C-Pe] for the one-dimensional Allen-Cahn equation and in [B-X1, B-X2] for the one-dimensional Cahn-Hilliard equation.

A. The Equations for the New Variables (ξ, v)

The following proposition deals with the possibility of constructing a tubular neighborhood of $\tilde{M}_\rho^\varepsilon$ in H_0^{-1}.

Proposition 7.1. *Let* $\tilde{u}^\xi, \tilde{M}_\rho^\varepsilon, \Omega_\rho$ *be as in Theorem 5.1; then, for* $\eta > 1$, *the condition*

$$\inf_{\xi \in \tilde{\Omega}_{\rho+2\delta}} \|u - \tilde{u}^\xi\| < \varepsilon^\eta, \tag{7.2}$$

implies the existence of unique $\xi \in \Omega_{\rho+\delta},\ v \in H_0^{-1}$ *such that*

$$\begin{aligned} u &= \tilde{u}^\xi + v \\ (v, \psi_i^\xi) &= 0, \quad i = 1, 2, \end{aligned} \tag{7.3}$$

where ψ_i^ξ *as in Theorem 6.1. Moreover, the map* $u \to (\xi, v)$ *defined by* (7.3) *is a smooth map together with its inverse.*

The proof is a quite standard argument based on the implicit function theorem and on Theorem 6.1(iii), and is omitted (see [A-Bro-F]).

Let $u(t)$ be a solution of (1.1) with initial condition $u(0)$ satisfying (7.2). Then there is a time interval where the change of variables $u \to (\xi, v)$ introduced in Propositon 7.1 is well defined and (1.1) is equivalent to

$$\frac{dv}{dt} + \dot{\xi}_1 \tilde{u}^\xi_{,1} + \dot{\xi}_2 \tilde{u}^\xi_{,2} = \mathscr{L}(\tilde{u}^\xi + v), \tag{7.4}$$

which is obtained by setting $u = \tilde{u}^\xi + v$ in (1.1). On the other hand, differentiating $(7.3)_2$ yields

$$\left(\frac{dv}{dt}, \psi_i^\xi\right) + \dot{\xi}_1 \left(v, \psi^\xi_{i,1}\right) + \dot{\xi}_2 \left(v, \psi^\xi_{i,2}\right) = 0, \quad i = 1, 2. \tag{7.5}$$

From Theorem 6.1(iii) the matrix $(\psi_j^\xi, \tilde{u}^\xi_{,i})$ is nonsingular and approaches a constant as $\varepsilon \to 0$. On the other hand, as long as $u(t)$ remains in the tubular neighborhood of $\tilde{M}_\rho^\varepsilon$ defined by Proposition 7.1, we have from the estimate (6.8) in Theorem 6.1 that

$$|(v, \psi^\xi_{i,j})| < \varepsilon^{\eta - 1}. \tag{7.6}$$

Therefore, if we multiply equation (7.4) by ψ_i^ξ and eliminate the term containing $\frac{dv}{dt}$ by means of (7.5), we obtain a linear system for $\dot{\xi}_1, \dot{\xi}_2$ which is solvable for $\varepsilon > 0$ sufficiently small, and we can write

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \left(Z^0 + Z^1(v)\right)^{-1} \begin{bmatrix} \left(\mathscr{L}\left(\tilde{u}^\xi + v\right),\ \psi_1^\xi\right) \\ \left(\mathscr{L}\left(\tilde{u}^\xi + v\right),\ \psi_2^\xi\right) \end{bmatrix} =: \vartheta^\xi(v), \tag{7.7}$$

where $Z_0 = (Z^0_{ij})$ and $Z^1(v) = (Z^1_{ij}(v))$ are 2×2 matrices defined by

$$Z^0_{ij} = \left(\psi_i^\xi, \tilde{u}^\xi_{,j}\right), \quad Z^1_{ij}(v) = -\left(v, \psi^\xi_{i,j}\right). \tag{7.8}$$

Let V^ξ be the orthogonal complement of U^ξ in H_0^{-1} (see Theorem 6.1(iii)). If $\phi \in H_0^{-1}$, we have $\phi = \phi^= + \phi^\perp$ with $\phi^= \in U^\xi$, $\phi^\perp \in V^\xi$. From (7.5), and the fact that ψ_1^ξ, ψ_2^ξ are orthonormal, we obtain

$$\left(\frac{dv}{dt}\right)^= = -\sum_i \sum_j \dot{\xi}_j \left(v, \psi_{i,j}^\xi\right) \psi_i^\xi =: K^{\xi,\dot{\xi}} v. \tag{7.9}$$

Moreover, we can write

$$\mathscr{L}(\tilde{u}^\xi + v) = \mathscr{L}(\tilde{u}^\xi) + L^\xi v + N^\xi(v), \tag{7.10}$$

where $N^\xi(v)$ is the nonlinear part of $\mathscr{L}(\tilde{u}^\xi + v)$. Therefore projecting equation (7.4) on the subspace V^ξ yields

$$\frac{dv}{dt} = L^\xi v + K^{\xi,\vartheta^\xi(v)} v + h^\xi(v), \tag{7.11}$$

where $\vartheta^\xi(v)$ stands for the right-hand side of (7.7) and

$$h^\xi(v) = \left(\mathscr{L}(\tilde{u}^\xi) + N^\xi(v) - \vartheta_1^\xi(v)\tilde{u}_{,1}^\xi - \vartheta_2^\xi(v)\tilde{u}_{,2}^\xi\right)^\perp. \tag{7.12}$$

Equations (7.7) and (7.11) make a system which in the tubular neighborhood defined by Proposition 7.1 is equivalent to (1.1), and which is basic for the rigorous justification of (7.1), stated in Theorem 7.2 below.

B. The Main Result

In the next theorem c^ξ, $\tilde{u}^\xi$, $\tilde{M}_\rho^\varepsilon$, Ω_ρ, and V^ξ are defined as before (cf. Theorem 6.1 and Proposition 7.1); $X^\alpha, 0 \leqq \alpha < 1$, is the fractional-power space of order α associated with the sectorial operator $-L^\xi$ on $H_0^{-1} = X^0$ (cf. [H]). The operator L^ξ depends on ξ, but the spaces corresponding to different choices of ξ coincide. We assume, as we can, that α is so large that the imbedding $X^\alpha \hookrightarrow C^1(\overline{\Omega})$ holds. The norm in X^α is denoted by $\|\cdot\|_\alpha$.

Theorem 7.2. *Let $\eta > 0$ be fixed and let $\mathscr{N}_\eta \subset X^\alpha$, with α so large that $X^\alpha \hookrightarrow C^1(\overline{\Omega}),\ W^{2,2}(\Omega)$, be the set defined for small $\varepsilon > 0$ by*

$$\mathscr{N}_\eta = \{u = \tilde{u}^\xi + v \mid \xi \in \Omega_{\rho+2\delta},\ v \in V^\xi \cap X^\alpha, \|v\|_\alpha \leqq \eta |c^\xi|\}. \tag{7.13}$$

Let $u : [0,\infty) \to X^\alpha$ be a solution to the Cahn-Hilliard equation with initial condition $u_0 \in \mathscr{N}_1$. Then there is $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$u(t) = \tilde{u}^{\xi(t)} + v(t) \in \mathscr{N}_2 \Rightarrow \begin{cases} \|v(t)\|_\alpha < 2\left|c^{\xi(t)}\right|, \\ \dot{\xi}(t) = c^{\xi(t)} + O\left(\left|c^{\xi(t)}\right| e^{-c/\varepsilon}\right), \end{cases} \tag{7.14}$$

where $c > 0$ is a constant independent of ε.

Remark. We note that the estimate (ii) in Theorem 5.1 and Proposition 7.1 imply that the set $\mathscr{N}_\eta$ is well defined for $\varepsilon > 0$ small. $\mathscr{N}_\eta$ is something like a tubular neighborhood of $\tilde{M}^\varepsilon_\rho$ but not exactly so because $\mathcal{N}_\eta$ shrinks to zero whenever c^ξ vanishes (Fig. 7).

Remark. Before proving Theorem 7.2 we note the implication of (7.14) that $u(t)$ can leave $\mathcal{N}_2$ only if $\xi(t) \in \partial\Omega_{\rho+2\delta}$ and that as long as $\xi(t)$ is in $\Omega_{\rho+2\delta}$, the evolution of ξ, the center of the bubble, is determined to an extremely high degree of accuracy by (7.1). These observations imply that

(a) The interface keeps its almost circular shape at least until $\xi(t)$ reaches the boundary of $\partial\Omega_{\rho+2\delta}$. In fact, all functions in $\mathcal{N}_2$ are very small perturbations of functions which have an exact circular interface.

(b) The motion of the bubble is extremely slow. Typical speeds are exponentially small in ε.

Proof of Theorem 7.2. The main point in the proof is to show that solutions starting near $\tilde{M}^\varepsilon_\rho$ remain near $\tilde{M}^\varepsilon_\rho$ for a very long time. To show this we use a kind of variation-of-constants argument applied to equation (7.11). To implement the argument, we need estimates for the nonlinear functions $\vartheta^\xi(v), h^\xi(v)$ and a careful analysis of the homogenous equation

$$\frac{dv}{dt} = L^\xi v + K^{\xi,\dot{\xi}} v \tag{7.15}$$

where $K^{\xi,\dot{\xi}}$ is defined in (7.9) and ξ is considered as a known function of t.

Lemma 7.3. *Let γ be a number satisfying*

$$\gamma < \tfrac{1}{2}\nu^\varepsilon \min_{\xi\in\Omega_{\rho+\delta}} d^\xi.$$

Then there is $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the condition

$$u \in \{u = \tilde{u}^\xi + v \mid \xi \in \Omega_{\rho+2\delta},\ v \in V^\xi \cap X^\alpha,\ \|v\|_\alpha < e^{-\gamma/\varepsilon}\} \tag{7.16}$$

implies that

$$\vartheta^\xi(v) = \left(I + O(e^{-c/\varepsilon})\right) c^\xi + O\left(e^{-c/\varepsilon}\|v\|_\alpha\right), \tag{7.17}$$

$$\|h^\xi(0)\|_\alpha \leqq e^{-c/\varepsilon} \left|c^\xi\right|, \tag{7.18}$$

$$\|h^\xi(v) - h^\xi(0)\| \leqq e^{-c/\varepsilon} \|v\|_\alpha, \tag{7.19}$$

where I is the identity matrix in $\mathbb{R}^2$ and c is a positive constant independent of ε.

Proof. Theorem 5.1(i),(iv), together with the definition (7.8) of Z^0, imply that

$$c^\xi = \left(Z^0 + O(e^{-c/\varepsilon})\right)^{-1} \begin{bmatrix} \left(\mathcal{L}(\tilde{u}^\xi), \psi^\xi_1\right) \\ \left(\mathcal{L}(\tilde{u}^\xi), \psi^\xi_2\right) \end{bmatrix}. \tag{7.20}$$

Therefore, if we insert equation (7.10) into the expression (7.7) of $\vartheta^\xi(v)$ and note that

$$\left(L^{\xi} v, \psi_i^{\xi}\right) = 0, \quad i = 1, 2. \tag{7.21}$$

(because $v \in V^{\xi}$ and V^{ξ} is invariant under L^{ξ}), we obtain

$$\vartheta^{\xi}(v) = \left(Z^0 + Z^1(v)\right)^{-1} \left((Z^0 + O(e^{-c/\varepsilon}))c^{\xi} + \begin{bmatrix} (N^{\xi}(v), \psi_1^{\xi}) \\ (N^{\xi}(v), \psi_2^{\xi}) \end{bmatrix} \right). \tag{7.22}$$

On the other hand, the definition of $Z^1(v)$, estimate (6.8), and the condition (7.16) imply, for $\varepsilon > 0$ small and some $c > 0$, that

$$Z^1(v) = O(\varepsilon^{-1}\|v\|) = O(e^{-c/\varepsilon}). \tag{7.23}$$

Moreover, taking into account that $N^{\xi}(v) = \Delta\left(-F'(\tilde{u}^{\xi} + v) + F'(\tilde{u}^{\xi}) + F''(\tilde{u}^{\xi})v\right)$, and using Theorem 6.1(iii) for estimating the L^1 norm of ψ_i^{ξ}, we find that

$$\begin{aligned} (N^{\xi}(v), \psi_i^{\xi}) &= \langle (-\Delta)^{-1/2}\Delta\tilde{F}^{\xi}(v), (-\Delta)^{-1/2}\psi_i^{\xi}\rangle = -\langle \tilde{F}^{\xi}(v), \psi_i^{\xi}\rangle \\ &= O(\varepsilon^{1/2}\|v\|_0^2) = O(e^{-c/\varepsilon}\|v\|_{\alpha}), \end{aligned} \tag{7.24}$$

where $\tilde{F}^{\xi}(v) = -F'(\tilde{u}^{\xi} + v) + F'(\tilde{u}^{\xi}) + F''(\tilde{u}^{\xi})v$ and $\| \ \|_0$ denotes the sup norm. In the last equality in (7.24) we have used the imbedding $X^{\alpha} \hookrightarrow C^0(\bar{\Omega})$ and the fact that on the subspace $X^{\alpha} \cap V^{\xi}$,

$$\|v\|_0 \leqq C\varepsilon^{-p}\|v\|_{\alpha},$$

for some number p. The estimate (7.17) is obtained from (7.22)–(7.24). From (7.12) it follows that

$$h^{\xi}(v) = \sum_{i=1}^{2} \left(c_i^{\xi} - \vartheta_i^{\xi}(v)\right)(\tilde{u}_{,i}^{\xi})^{\perp} + \sum_{i=1}^{2} c_i^{\xi}(u_{,i}^{\xi} - \tilde{u}_{,i}^{\xi})^{\perp} + (N^{\xi}(v))^{\perp}. \tag{7.25}$$

From Theorem 6.1, and its proof (cf. equation (6.10)), we have

$$\|(\tilde{u}_{,i}^{\xi})^{\perp}\| = O(e^{-c/\varepsilon}), \quad \|L^{\xi}(\tilde{u}_{,i}^{\xi})^{\perp}\| = O(e^{-c/\varepsilon}). \tag{7.26}$$

On the other hand, Theorem 5.1(i),(iii) implies that

$$\|u_{,i}^{\xi} - \tilde{u}_{,i}^{\xi}\| = O(e^{-c/\varepsilon}), \quad \|L^{\xi}(u_{,i}^{\xi} - \tilde{u}_{,i}^{\xi})\| = O(e^{-c/\varepsilon}). \tag{7.27}$$

The estimate (7.18) follows from (7.25), by also using (7.22), which implies that

$$\left|c^{\xi} - \vartheta^{\xi}(0)\right| \leqq \text{Const.}\left|c^{\xi}\right|, \tag{7.28}$$

by means of the first estimate (7.27), and by observing that $N^{\xi}(0) = 0$.

To prove the last estimate, we note that for $\varepsilon > 0$ small,

$$\begin{aligned} \|(N^{\xi}(v))^{\perp}\|^2 &\leqq \|N^{\xi}(v)\|^2 = \langle(-\Delta)^{-1/2}\Delta\tilde{F}^{\xi}(v), (-\Delta)^{-1/2}\Delta\tilde{F}^{\xi}(v)\rangle \\ &= \langle\tilde{F}^{\xi}(v), \Delta\tilde{F}^{\xi}(v)\rangle \leqq \text{Const.}\ \|v\|_{C^1(\bar{\Omega})}^2 \\ &\leqq e^{-c/\varepsilon}\|v\|_{\alpha}, \end{aligned} \tag{7.29}$$

where we have performed an integration by parts, and also employed the imbedding $X^\alpha \hookrightarrow C^1(\overline{\Omega})$ and the condition (7.16). From the above estimate, and (7.22), (7.23) (since $c^\xi = O(e^{-c/\varepsilon})$ it follows that

$$\left|\vartheta^\xi(v) - \vartheta^\xi(0)\right| = O(e^{-c/\varepsilon}\|v\|_\alpha). \tag{7.30}$$

We note that this estimate is not a direct consequence of (7.17), because the coefficient of c^ξ in (7.17) is actually a function of v.

The estimate (7.19) follows from this, the estimate (7.29) and (7.26), (7.27). □

Remark. The estimates (7.17), (7.18), and (7.19) hold in $\mathcal{N}_2^\cdot$. In fact, the definition in Lemma 7.4 of γ implies that $\mathcal{N}_2^\cdot$ is contained in the set defined by (7.16).

We now begin the study of the homogeneous equation (7.15). We follow an approach very similar to that of [A-B-F]. We assume that $\xi : [a,b] \to \Omega_\rho$ is a given smooth function. Equation (7.15) can be considered as a linear nonautonomous differential equation. Since the principal part of the operator is independent of t, the theory for nonautonomous parabolic equations in [H] applies and implies the existence of a bounded linear operator $S(t,\tau,\xi) : X^\alpha \to X^\alpha$; $S(t,\tau,\xi) = I$, such that $S(t,\tau,\xi)v$ is a smooth function of t,τ. As we shall see, the special structure of the linear operator in equation (7.15), in particular the presence of the operator $K^{\xi,\dot\xi}$, implies that (7.15) preserves the fibration $\xi \to V^\xi$. Since L^ξ is a sectorial operator, and the spectrum of the restriction of L^ξ to V^ξ is bounded below by a positive constant, we can expect exponential decay in t for $S(t,\tau,\xi)v$ when $v \in V^{\xi(\tau)}$. To prove this, and related facts, we start by analyzing the operator $K^{\xi,\dot\xi}$.

Lemma 7.4. *Let $\xi : [a,b] \to \Omega_\rho$ be a smooth function. Then*

(i) *The problem*

$$\frac{d\phi}{dt}(t) = K^{\xi(t),\dot\xi(t)}\phi(t), \quad \phi(\tau) = \overline{\phi} \in H_0^{-1} \tag{7.31}$$

has a unique solution $\phi(t) = \Phi(t,\tau,\xi)\overline{\phi} \in H_0^{-1}$ which is a smooth function of $(t,\tau,\xi,\overline{\phi})$.

(ii)
$$\Phi(t,\tau,\xi) : V^{\xi(\tau)} \to V^{\xi(t)},$$
$$\|\Phi(t,\tau,\xi)\phi\| = \|\phi\| \ \forall\phi \in V^{\xi(\tau)}.$$

Furthermore, for $t \geqq \tau$ and some number b independent of ε,

(iii)
$$\|\Phi(t,\tau,\xi)\phi\| \leqq (1 + C\varepsilon^{-1}\int_\tau^t |\dot\xi|)\|\phi\| \ , \phi \in H_0^{-1}.$$

(iv) $\|\Phi(t,\tau,\xi)\phi - \phi\|_\alpha$
$$\leqq C\varepsilon^{-b}\left[\int_\tau^t \left|\dot\xi(s)\right| \left(1 + C\varepsilon^{-1}\int_\tau^s \left|\dot\xi\right|\right) ds\right] \|\phi\|, \quad \phi \in H_0^{-1},$$

and the same is true with the $C^k(\overline{\Omega})$ norm replacing the X^α norm.

The adjoint of $\Phi(t,\tau,\xi)$ *can be identified with the solution operator* $\Phi^*(\tau,t,\xi)$ *of the equation*

$$\frac{d\phi}{dt} = \sum_i \sum_j \dot{\xi}_j (\phi, \psi_i^\xi)\psi_{i,j}^\xi, \tag{7.32}$$

and

(v) $$\|\Phi^*(t,\tau,\xi)\psi_{i,j}^{\xi(\tau)}\|_{C^0(\bar{\Omega})} \leqq C\varepsilon^{-2}\left(1+\varepsilon^{-2}\left[\int_\tau^t |\dot{\xi}|\right]\right)$$

Proof. Statement (i) follows from the definition (7.9), which implies that $K^{\xi(t),\dot{\xi}(t)}$ is a bounded linear operator on H_0^{-1} and a smooth function of t and $\xi : [a,b] \to \Omega_\rho$. To prove (ii), we observe that (7.31) and (7.9) imply that

$$\frac{d}{dt}(\phi,\psi_i^\xi) = \left(K^{\xi,\dot{\xi}}\phi,\psi_i^\xi\right) + \sum_j \dot{\xi}_j\left(\phi,\psi_{i,j}^\xi\right) = 0 \quad \forall\,\phi(t). \tag{7.33}$$

Therefore,

$$(\phi,\psi_i^\xi) = \text{Const.}, \tag{7.34}$$

and in particular $(\phi,\psi_i^\xi) = 0$ if it is zero at $t=\tau$. This shows that $\Phi(t,\tau,\xi)$ maps $V^{\xi(t)}$ into $V^{\xi(t)}$.

A solution $\phi(t)$ of (7.31) satisfies

$$\frac{d}{dt}(\phi,\phi) = 2\left(K^{\xi,\dot{\xi}}\phi,\phi\right). \tag{7.35}$$

From this, if $\phi \in V^\xi$, it follows that $\frac{d}{dt}(\phi,\phi) = 0$ and therefore $\|\phi\| = \text{Const.}$ In order to prove (iii), we decompose ϕ as

$$\phi = \sum_i \alpha_i\psi_i^\xi + \sigma, \quad \sigma \in V^\xi. \tag{7.36}$$

We have seen in (7.34) that for a generic solution $\phi(t)$ of (7.31), $\alpha_i = (\phi,\psi_i^\xi)$ is a constant quantity; therefore differentiating (7.36) and also using (7.9) yields

$$K^{\xi,\dot{\xi}}\sigma - \sum_i\sum_h\sum_j \bar{\alpha}_i\dot{\xi}_j\left(\psi_i^\xi,\,\psi_{h,j}^\xi\right)\psi_h = \sum_i\sum_j \bar{\alpha}_i\dot{\xi}_j\psi_{i,j}^\xi + \dot{\sigma}, \tag{7.37}$$

where $\bar{\alpha}_i$ is the value of α_i for $t=\tau$.

From this, and the orthonormality of the basis $\psi_1^\xi,\ \psi_2^\xi$, which implies that

$$\sum_j \dot{\xi}_j\left(\psi_{h,j}^\xi,\psi_i^\xi\right) = -\sum_j\dot{\xi}_j\left(\psi_h^\xi,\ \psi_{i,j}^\xi\right),$$

it follows that

$$\dot{\sigma} = K^{\xi,\dot{\xi}}\sigma + l \tag{7.38}$$

where

$$l = -\sum_i \sum_j \bar{\alpha}_i \dot{\xi}_j \left[\psi^{\xi}_{i,j} - \sum_h \left(\psi^{\xi}_{i,j}, \psi^{\xi}_h \right) \psi^{\xi}_h \right] \in V^{\xi}.$$

From this expression for l and from the estimate (6.8) of $\psi^{\xi}_{i,j}$ we derive

$$\|l\| \leqq C\varepsilon^{-1}|\bar{\alpha}|\ |\dot{\xi}|. \tag{7.39}$$

From the variation-of-constants formula, applied to (7.38) it follows for $t \geqq \tau$ that

$$\|\sigma(t)\| \leqq \|\Phi(t,\tau,\xi)\bar{\sigma}\| + \int_\tau^t \|\Phi(t,s,\xi)l(s)\| ds, \tag{7.40}$$

with $\bar{\sigma} = \sigma(\tau)$.

This and (ii), since $\bar{\sigma} \in V^{\xi(\tau)}$ and $l(s) \in V^{\xi(s)}$, imply that

$$\|\sigma(t)\| \leqq \|\bar{\sigma}\| + \int_\tau^t \|l(s)\| ds \leqq \|\bar{\sigma}\| + C\varepsilon^{-1}|\bar{\alpha}| \int_\tau^t |\dot{\xi}| \tag{7.41}$$

where in the last inequality we have used the estimate (7.39). Therefore, we have

$$\begin{aligned} \|\phi(t)\|^2 = |\bar{\alpha}|^2 + \|\sigma(t)\|^2 &\leqq |\bar{\alpha}|^2 + \left(\|\bar{\sigma}\| + C\varepsilon^{-1}|\bar{\alpha}| \int_\tau^t |\dot{\xi}| \right)^2 \\ &\leqq \|\phi(\tau)\|^2 \left(1 + C\varepsilon^{-1} \int_\tau^t |\dot{\xi}| \right)^2. \end{aligned} \tag{7.42}$$

The differential equation (7.31) implies that

$$\phi(t) - \phi(\tau) = -\int_\tau^t \sum_i \sum_j \dot{\xi}_j(s) \left(\phi(s), \psi^{\xi(s)}_{i,j} \right) \psi^{\xi(s)}_i ds. \tag{7.43}$$

From this, (7.42) with $t = s$, the estimate (6.8) and the fact that

$$\|\psi^{\xi}_i\|_\alpha \leqq C\varepsilon^{-b} \tag{7.44}$$

for some number b independent of ε, it follows that

$$\|\phi(t) - \phi(\tau)\|_\alpha \leqq C\varepsilon^{-b} \left[\int_\tau^t |\dot{\xi}(s)|(1 + C\varepsilon^{-1} \int_\tau^s |\dot{\xi}|) \right] \|\phi(\tau)\|, \tag{7.45}$$

that is, (iv). With a different value of b, the estimate (7.44) is also valid for the $C^k(\bar{\Omega})$ norm. Therefore in (7.45) the X^α norm can be replaced by the $C^k(\bar{\Omega})$ norm.

To see (7.32), let $\Phi^*(\tau, t, \xi)$ be the adjoint of $\Phi(t,\tau,\xi)$. Then by definition,

$$(\Phi(t,\tau,\xi)\phi, z) = \left(\phi, \Phi^*(\tau,t,\xi) z \right) \ \forall \phi, z \in H^{-1}_0. \tag{7.46}$$

Differentiating this identity with respect to t, setting $t = \tau$ and using the definition of $K^{\xi,\dot{\xi}}$ and equation (7.31) yield

$$\left(\phi, -\sum_i \sum_j \dot{\xi}_j (z, \psi^{\xi}_i) \psi^{\xi}_{i,j} \right) = \left(\phi, \frac{d}{dt} \left(\Phi^*(\tau,t,\xi) z \right)_{t=\tau} \right). \tag{7.47}$$

From this identity it follows that $\Phi^*(t,\tau,\xi)$ can be identified with the solution operator of the equation

$$\frac{dz}{dt} = \sum_i \sum_j \dot{\xi}_j (z, \psi_i^\xi)\psi_{i,j}^\xi \, , \tag{7.48}$$

which is (7.32).

To derive the estimate (v) we write $z = \sum_i \alpha_i \psi_i^\xi + \sigma$ with $(\sigma, \psi_i^\xi) = 0$ as before. Then (7.48) implies that

$$\sum_i \dot{\alpha}_i \psi_i^\xi + \dot{\sigma} = 0. \tag{7.49}$$

From this, and $(\sigma, \psi_i^\xi) = 0$, which implies that

$$\left(\dot{\sigma}, \psi_i^\xi\right) + \sum_j \dot{\xi}_j \left(\sigma, \psi_{i,j}^\xi\right) = 0,$$

it follows that

$$\dot{\alpha}_i = \sum_j \dot{\xi}_j (\sigma, \psi_{i,j}^\xi), \tag{7.50}$$

and therefore

$$\dot{\sigma} = -\sum_i \sum_j \dot{\xi}_j (\sigma, \psi_{i,j}^\xi)\psi_i^\xi = K^{\xi,\dot{\xi}}\sigma. \tag{7.51}$$

This and (ii) imply $\|\sigma(t)\| = \|\sigma(\tau)\|$ and thus from (7.50), and (7.51), we obtain

$$|\alpha_i(t) - \alpha_i(\tau)| < C\varepsilon^{-1} \left(\int_\tau^t |\dot{\xi}|\right) \|\sigma(\tau)\|, \tag{7.52}$$

$$\|\sigma(t) - \sigma(\tau)\|_{C^0(\bar{\Omega})} \leqq C\varepsilon^{-2} \left(\int_\tau^t |\dot{\xi}|\right) \|\sigma(\tau)\|, \tag{7.53}$$

From these estimates, since the H_0^{-1} norm is bounded by the C^0 norm: $\|z\| < C\|z\|_{C^0(\bar{\Omega})}$, it follows that

$$\|z(t)\|_{C^0(\bar{\Omega})} \leqq C \left[1 + C\varepsilon^{-2} \left(\int_\tau^t |\dot{\xi}|\right)\right] \|z(\tau)\|_{C^0(\bar{\Omega})}. \tag{7.54}$$

Applying this inequality to $\psi_{i,j}^{\xi(\tau)}$ and using the fact that

$$\|\psi_{i,j}^\xi\|_{C^0(\bar{\Omega})} = O(\varepsilon^{-2}) \tag{7.55}$$

(which was already used in (7.53)) yields (v). □

Lemma 7.5. *Assume that* $\xi : [a,b] \to \Omega_{\rho+\delta}$ *is a smooth function and let* $S(t,s,\xi) : X^\alpha \to X^\alpha$, *with* α *as in statement of Theorem 7.2, be the solution operator of equation* (7.15). *Then*

$$v \in V^{\xi(s)} \quad \Rightarrow \quad S(t,s,\xi)v \in V^{\xi(t)}.$$

Moreover, there exist numbers $\bar{a}, \bar{\varepsilon} > 0$ *such that* $0 < \varepsilon < \bar{\varepsilon},\ a > \bar{a}$ *and*

$$\left|\dot{\xi}(t)\right| < C\varepsilon^{a}, \quad t \in [a,b], \tag{7.56}$$

imply that the operator $S(t,s,\xi) : V^{\xi(s)} \to V^{\xi(t)}$ *satisfies*

$$\begin{aligned} \|S(t,s,\xi)v\|_{\alpha} &\leqq e^{-\varepsilon\beta(t-s)}\|v\|_{\alpha}, \\ \|S(t,s,\xi)v\|_{\alpha} &\leqq M(t-s)^{-\alpha}e^{-\varepsilon\beta(t-s)}\|v\|, \end{aligned} \tag{7.57}$$

where $M, \beta > 0$ *are independent of* ε, a.

Proof. From (7.15), and the definition (7.9) of $K^{\xi,\dot{\xi}}$, it is seen that

$$\begin{aligned} \frac{d}{dt}(v,\psi_i^{\xi}) &= (L^{\xi}v,\psi_i^{\xi}) + (K^{\xi,\dot{\xi}}v,\psi_i^{\xi}) + \textstyle\sum_j \dot{\xi}_j(v,\psi_{i,j}^{\xi}) \\ &= (L^{\xi}v,\psi_i^{\xi}) = 0, \end{aligned} \tag{7.58}$$

which proves (i). Now, we regard (7.15) as a differential equation on the fibration $\xi \to V^{\xi}$, and transform it into an equation on a fixed fiber $V^{\xi(s)}$. We do this by introducing a new variable $w \in V^{\xi(s)}$ through the time-dependent change of variables

$$v(t) = \Phi(t,s,\xi)w(t), \tag{7.59}$$

where Φ is the operator discussed in Lemma 7.4. The equation for $w(t)$ is

$$w_t = L^{\xi(s)}w + Bw \tag{7.60}$$

with

$$B = \Phi(s,t,\xi)L^{\xi(t)}\Phi(t,s,\xi) - L^{\xi(s)}. \tag{7.61}$$

To simplify the notation we write $\Phi(s,t),\ L^t$ and K^t instead of $\Phi(s,t,\xi),\ L^{\xi(t)}$ and $K^{\xi(t),\dot{\xi}(t)}$, etc. By definition, Φ satisfies

$$\Phi(t,\tau)\phi(\tau) = \phi(\tau) + \int_{\tau}^{t} K^{s}\Phi(s,\tau)\phi(\tau)\,ds;$$

therefore from (7.61) we get, assuming w is sufficiently smooth, that

$$\begin{aligned} Bw &= L^t\Phi(t,s)w + \int_t^s K^r\Phi(r,t)L^t\Phi(t,s)\,w\,dr - L^s w \\ &= (L^t - L^s)w + L^t\int_s^t K^r\Phi(r,s)\,w\,dr + \int_t^s K^r\Phi(r,t)L^t w\,dr \\ &\quad + \int_t^s K^r\Phi(r,t)L^t\left(\int_s^t K^{\tau}\Phi(\tau,s)\,w\,d\tau\right)ds. \end{aligned} \tag{7.62}$$

We estimate only the third term, which is the most singular; the remaining terms can be estimated by the same method and satisfy similar estimates. Let $I(w)$ be the third term in (7.62); then recalling the definition (7.9) of $K^{\xi,\dot{\xi}}$ we have

$$I(w) = \int_t^s \sum_i \sum_j \dot{\xi}_j(r)(\Phi(r,t)L^t w, \psi^r_{i,j})\psi^r_i \, dr, \tag{7.63}$$

which implies that

$$\begin{aligned} \|I(w)\| &\leqq C\|\dot{\xi}\|_0 \left| \int_t^s \sum_i \sum_j \left| \left(\Phi(r,t)L^t w, \psi^r_{i,j}\right) \right| dr \right| \\ &= C\|\dot{\xi}\|_0 \left| \int_t^s \sum_i \sum_j \left| \left(L^t w, \Phi^*(t,r)\psi^r_{i,j}\right) \right| dr \right| \end{aligned} \tag{7.64}$$

where $\|\dot{\xi}\|_0 = \max_{s\in[a,b]} |\dot{\xi}(s)|$.

From (6.2) it follows that

$$(L^t w, \Phi^*(t,r)\psi^r_{i,j}) = -\langle \varepsilon^2 \Delta w - F''(\tilde{u}^t)w, \Phi^*(t,r)\psi^r_{i,j}\rangle. \tag{7.65}$$

From this, the assumption $X^\alpha \hookrightarrow W^{2,2}(\Omega)$ which implies that

$$\|\varepsilon^2 \Delta w - F''(\tilde{u}^t)w\|_{L^2} \leqq C\varepsilon^{-b}\|w\|_\alpha$$

for some real number b, and the estimates (v) in Lemma 7.4, it follows that

$$\left|(L^t w, \Phi^*(t,r)\psi^r_{i,j})\right| \leqq C\varepsilon^{-(b+2)}\left(1 + \varepsilon^{-2}\|\dot{\xi}\|_0 |t-r|\right)\|w\|_\alpha. \tag{7.66}$$

Therefore (7.64) yields

$$\|I(w)\| \leqq C\varepsilon^{-(b+2)}\|\dot{\xi}\|_0 \left(|t-s| + \varepsilon^{-2}\|\dot{\xi}\|_0 \left| \int_t^s |t-r|\, dr \right| \right) \|w\|_\alpha. \tag{7.67}$$

As mentioned above, the other terms in (7.62) satisfy similar estimates, and as a result we can conclude that there is a number b' such that

$$\|Bw\| \leqq C\varepsilon^{-b'}\|\dot{\xi}\|_0 \left(|t-s| + \varepsilon^{-b'}\|\dot{\xi}\|_0 \left| \int_t^s |t-r|\, dr \right| \right) \|w\|_\alpha. \tag{7.68}$$

Next, we note that the invariance of V^ξ under L^ξ implies that L^ξ can be considered as a self-adjoint operator on V^ξ. For the restricted operator we have by Theorem 6.1 that

$$\sigma(L^\xi) < -C'\varepsilon.$$

It follows that L^s generates an analytic semigroup on V^ξ and

$$\|e^{tL^s} w\|_\alpha \leqq e^{-\varepsilon C' t}\|w\|_\alpha, \quad \|e^{tL^s} w\|_\alpha \leqq M' t^{-\alpha} e^{-\varepsilon C' t}\|w\| \tag{7.69}$$

for some constant M' independent of ε.

We only prove the estimate $(7.57)_1$. The proof of the other estimate is similar. To prove $(7.57)_1$ it suffices to consider the case $t - s \leqq 1$. In fact, the general case follows from this special case and the semigroup property of the operator S. If $t-s \leqq 1$ and (7.56) holds with $a > b'$, the estimate (7.68) for ε sufficiently small yields

$$\|Bw\| \leqq C\varepsilon^{a'}\,|t-s|\;\|w\|_\alpha, \tag{7.70}$$

for some $a' > 0$.

From this, the variation-of-constants formula applied to (7.60) and the estimates (7.69) it follows for $t > s$ that

$$\begin{aligned}\|w(t)\|_\alpha &\leqq e^{-\varepsilon C'(t-s)}\|w(s)\|_\alpha \\ &\quad + C\varepsilon^{a'}\int_s^t e^{-\varepsilon C'(t-r)}(t-r)^{-\alpha}(r-s)\|w(r)\|_\alpha dr.\end{aligned} \tag{7.71}$$

This inequality implies the existence of an interval $[s, s+h]$ such that $\|w(r)\|_\alpha \leqq 2\|w(s)\|_\alpha$ for $r \in [s, s+h]$ and $\varepsilon < \bar{\varepsilon}$. For $t \leqq s+h$ we have

$$\int_s^t e^{-\varepsilon C'(t-r)}(t-r)^{-\alpha}(r-s)\|w(r)\|_\alpha dr \leqq \frac{2}{1-\alpha}\|w(s)\|_\alpha (t-s)^{2-\alpha},$$

and therefore (7.71) implies for $s \leqq t \leqq s+h$ that

$$\|w(t)\|_\alpha \leqq \left[e^{-\varepsilon C'(t-s)} + \varepsilon^{a'}\frac{2(t-s)^{2-\alpha}}{1-\alpha}\right]\|w(s)\|_\alpha. \tag{7.72}$$

We can take h so small (depending on ε) that the coefficient of $\|w(s)\|_\alpha$ in (7.72) satisfies

$$e^{-\varepsilon C'(t-s)} + \varepsilon^{a'}\frac{2(t-s)^{2-\alpha}}{1-\alpha} \leqq e^{-\varepsilon(C'/2)(t-s)},$$

for $t \in [s, s+h]$. Therefore (7.72) implies that

$$\|w(t)\|_\alpha \leqq e^{-\varepsilon(C'/2)(t-s)}\|w(s)\|_\alpha\;,\; t \in [s, s+h] \tag{7.73}$$

for some $h > 0$.

If a in (7.56) is taken sufficiently large, then for $\varepsilon > 0$ small, the estimate (iv) in Lemma 7.4 (since $|t-s| \leqq 1$) implies that

$$\|\Phi(t,s,\xi)w(t) - w(t)\|_\alpha \leqq C\varepsilon^{a'}|t-s|\|w(t)\| \tag{7.74}$$

for some $a' > 0$. From this estimate, (7.73), and the definition (7.59) of w it follows that

$$\begin{aligned}\|v(t)\|_\alpha &\leqq \|\Phi(t,s,\xi)w(t) - w(t)\|_\alpha + \|w(t)\|_\alpha \\ &\leqq (1 + C\varepsilon^{a'}|t-s|)\|w(t)\|_\alpha \\ &\leqq (1 + C\varepsilon^{a'}|t-s|)e^{-\varepsilon(C'/2)(t-s)}\,\|v(s)\|_\alpha\,, \quad t \in [s, s+h],\end{aligned} \tag{7.75}$$

where in the last inequality we have used $w(s) = v(s)$. By taking a sufficiently large in (7.56), we can assume that $a' > 1$. Then, for $\varepsilon > 0$ small, we obtain

$$\left(1 + C\varepsilon^{a'}|t-s|\right)e^{-\varepsilon(C'/2)(t-s)} \leqq e^{-\varepsilon(C'/4)(t-s)}, \quad t \in [s, s+h'],$$

for some $h' > 0$ depending on ε. Therefore we conclude that under the same assumptions,

$$\|v(t)\|_\alpha \leqq e^{-\varepsilon(C'/4)(t-s)}\|v(s)\|_\alpha, \quad t \in [s, s+h'].$$

As remarked earlier, a repeated application of this argument proves the first of the estimates (7.57) with $\beta = \frac{1}{4}C'$. The proof of Lemma 7.5 is complete. □

We can now complete the proof of Theorem 7.2. By assumption,

$$u(t) = \tilde{u}^{\xi(t)} + v(t) \tag{7.76}$$

is the solution to the Cahn-Hilliard equation with initial condition

$$u_0 = \tilde{u}^{\xi_0} + v_0 \in \mathscr{N}_1.$$

From this and Theorem 5.1(ii) it follows that, for ε small,

$$\|v_0\|_\alpha \leqq |c^{\xi_0}| < e^{-3\gamma/2\varepsilon},$$

with γ the number defined in Lemma 7.3. Therefore, there exists a number $\tau > 0$ such that

$$\|v(t)\|_\alpha < e^{-\gamma/\varepsilon}, \quad t \in [0, \tau].$$

For $t \in [0, \tau]$ we have from the variation-of-constants formula applied to equation (7.11) and from the estimates in Lemmata 7.3 and 7.5 (cf. (7.18), (7.19), (7.57))

$$\begin{aligned} \|v(t)\|_\alpha \leqq\ & e^{-\varepsilon\beta t}\|v_0\|_\alpha + e^{-c/\varepsilon}\int_0^t e^{-\varepsilon\beta(t-s)}|c^{\xi(s)}|\,ds \\ & + e^{-c/\varepsilon}\int_0^t e^{-\varepsilon\beta(t-s)}(t-s)^{-\alpha}\|v(s)\|_\alpha\,ds, \end{aligned} \tag{7.77}$$

which is equivalent to

$$\rho(t) \leqq \|v_0\|_\alpha + e^{-c/\varepsilon}\int_0^t e^{\varepsilon\beta s}|c^{\xi(s)}|\,ds + e^{-c/\varepsilon}\int_0^t (t-s)^{-\alpha}\rho(s)\,ds \tag{7.78}$$

with $\rho(t) = e^{\varepsilon\beta t}\|v(t)\|_\alpha$. Equation (7.78) is valid for $t \in [0, \tau]$. If we restrict our attention to $t \leqq 1$, then (7.78) implies that

$$\rho(t) \leqq \|v_0\|_\alpha + e^{-(c/\varepsilon - \varepsilon\beta)}\max_{s\in[0,1]}|c^{\xi(s)}| + e^{-c/\varepsilon}\int_0^t (t-s)^{-\alpha}\rho(s)\,ds. \tag{7.79}$$

Therefore, recalling that $\|v_0\|_\alpha < e^{-3\gamma/2\varepsilon}$, we have

$$\|v_0\|_\alpha + e^{-(c/\varepsilon-\varepsilon\beta)}\max_{s\in[0,1]}|c^{\xi(s)}| \leqq e^{-c'/\varepsilon}e^{-\gamma/\varepsilon} \tag{7.80}$$

for ε small and some number $c' > 0$ independent of ε. Therefore (7.79) implies that

$$\begin{aligned} \rho(t) &\leqq e^{-c'/\varepsilon}e^{-\gamma/\varepsilon} + e^{-c/\varepsilon}\int_0^t (t-s)^{-\alpha}\rho(s)\,ds \\ &\leqq e^{-c'/\varepsilon}e^{-\gamma/\varepsilon} + e^{-c/\varepsilon}\overline{\rho}(t)\int_0^t (t-s)^{-\alpha}ds \\ &\leqq e^{-c'/\varepsilon}e^{-\gamma/\varepsilon} + \frac{e^{-c/\varepsilon}}{1-\alpha}\overline{\rho}(t), \end{aligned} \tag{7.81}$$

where $\overline{\rho}(t) = \max_{s\in[0,t]} \rho(s)$. From (7.81) it follows that

$$\rho(t) \leqq \overline{\rho}(t) \leqq \frac{e^{-c'/\varepsilon}}{1 - \frac{e^{-c/\varepsilon}}{1-\alpha}} e^{-\gamma/\varepsilon}, \tag{7.82}$$

and therefore

$$\|v(t)\|_\alpha \leqq e^{-c/\varepsilon} e^{-\gamma/\varepsilon} e^{-\varepsilon\beta t}, \quad t \in [0,\tau], \quad t \leqq 1. \tag{7.83}$$

This inequality proves that we can, in fact, assume that $\tau = 1$ and therefore that $\|v(t)\|_\alpha \leqq e^{-\gamma/\varepsilon}$ for all $t \in [0,1]$. Thus the inequality (7.79) is also valid for $t \in [0,1]$. By arguing as we have done in deducing (7.82), we see that the inequality (7.79) implies that

$$\|v(t)\|_\alpha \leqq \left(\|v_0\|_\alpha + e^{-c/\varepsilon} \max_{s\in[0,1]} |c^{\xi(s)}| \right) \frac{e^{-\varepsilon\beta t}}{1 - e^{-c/\varepsilon}}, \quad t \in [0,1]. \tag{7.84}$$

The assumption $\|v_0\|_\alpha \leqq |c^{\xi_0}|$ implies the existence of a number $r > 0$ such that

$$\|v(t)\|_\alpha \leqq 2|c^{\xi(t)}|, \quad t \in [0,r].$$

It follows that for $t \in [0,r]$ the estimate (7.17) in Lemma 7.3 and (7.7) imply

$$|\dot{\xi}(t)| \leqq C|c^{\xi(t)}|, \quad t \in [0,r]. \tag{7.85}$$

From this, and Theorem 5.1(iii), it is seen that

$$\|c^\xi|^\cdot| \leqq |(c^\xi)^\cdot| \leqq C \sum_i \sum_j |\dot{\xi}_j|, \quad |c^\xi_{i,j}| \leqq e^{-c/\varepsilon}|c^\xi|. \tag{7.86}$$

Therefore, for $t \in [0,r]$

$$\left(1 - e^{-c/\varepsilon} t\right)|c^{\xi_0}| \leqq |c^{\xi(t)}| \leqq (1 + e^{-c/\varepsilon} t)|c^{\xi_0}|, \tag{7.87}$$

where we have also used the assumption that $t \leqq 1$. From these inequalities and (7.84) it follows that

$$\begin{aligned} \|v(t)\|_\alpha &\leqq \left(1 + e^{-c/\varepsilon}\right) e^{-\varepsilon\beta t} |c^{\xi_0}| \\ &\leqq (1 + e^{-c/\varepsilon})(1 + e^{-c/\varepsilon} t) e^{-\varepsilon\beta t} |c^{\xi(t)}|, \end{aligned} \tag{7.88}$$

which is valid for $t < 1$ and $t \in [0,r]$.

If ε is sufficiently small, this inequality implies that

$$\|v(t)\|_\alpha \leqq \left(1 + e^{-c/\varepsilon}\right)^2 |c^{\xi(t)}| \leqq 2|c^{\xi(t)}|, \tag{7.89}$$

with the sign of strict inequality if $c^{\xi(t)} \not\equiv 0$. This implies that we can take $r = 1$ and that the inequalities (7.88), (7.89) hold in the interval [0,1]. Therefore, we have $u(t) \in \mathscr{N}_2$ for $t \in [0,1]$ and $u(t)$ in the interior of $\mathscr{N}_2$ if u_0 is not an equilibrium. From equation (7.7), the inequality (7.89) and Lemma 7.3 it also follows that for $t \in [0,1]$

$$\dot{\xi}(t) = c^{\xi(t)} + O(e^{-c/\varepsilon}|c^{\xi(t)}|).$$

Having established all this, we complete the proof if we show that $u(1)$ is actually in $\mathcal{N}_1$. In fact, the theorem then follows from an obvious induction argument. From (7.88) we obtain

$$\|v(1)\|_\alpha \leqq (1 + e^{-c/\varepsilon})^2 e^{-\varepsilon\beta}|c^{\xi(1)}|$$

which implies that

$$\|v(1)\|_\alpha \leqq |c^{\xi(1)}|$$

provided $\varepsilon > 0$ is sufficiently small. The proof of Theorem 7.2 is complete. □

Acknowledgement. The work of N. D. ALIKAKOS was partially supported by NSF, DMS-9306229, the Science Alliance, and NATO Scientific Affairs Division, CRG 930791. The work of G. FUSCO was partially supported by CNR, and NATO Scientific Affairs Division, CRG 930791. We thank S. STACY for her careful typing of a very complicated manuscript.

References

[A-B-Ch] N. D. ALIKAKOS, P. W. BATES & XINFU CHEN, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model,* Arch. Rational Mech. Anal. **128** (1994), 165–205.

[A-B-Ch-F] N. D. ALIKAKOS, P. W. BATES, XINFU CHEN & G. FUSCO, *Mullins-Sekerka motion of small droplets on a fixed boundary,* preprint.

[A-Ch-F] N. D. ALIKAKOS, XINFU CHEN & G. FUSCO, *Motion of a drop by surface tension along the boundary,* to appear in Calc. Variations P.D.E.

[A-B-F] N. D. ALIKAKOS, P. W. BATES, & G. FUSCO, *Slow motion for the Cahn-Hilliard equation in one space dimension,* J. Diff. Eqs. **90** (1991), 81–135.

[A-Bro-F] N. D. ALIKAKOS, L. BRONSARD, & G. FUSCO, *Slow motion for the Cahn-Hilliard equation via energy and spectral estimates,* Calc. Variations P.D.E., in press.

[A-F1] N. D. ALIKAKOS & G. FUSCO, *Slow dynamics for the Cahn-Hilliard equation in higher space dimensions. Part I: Spectral estimates,* Comm. Part. Diff. Eqs. **19** (1994), 1397–1447.

[A-F2] N. D. ALIKAKOS & G. FUSCO, *Equilibrium and dynamics of bubbles for the Cahn-Hilliard equation,* Int. Conf. on Diff. Equations, Barcelona 91 (C. PERELLÓ, C. SIMÓ, J. SOLÀ-MORALES, eds.), Vol. 1, World Scientific, 1993, pp. 59–67.

[Ag] S. AGMON, *Bounds on exponential decay of eigenfunctions of Schrödinger operators*, Springer Lecture Notes in Mathematics 1159, 1985, pp. 1–38.

[A-Mc] N. D. ALIKAKOS & W. R. MCKINNEY, *Remarks on the equilibrium theory for the Cahn-Hilliard equation in one space dimension,* Reaction-Diffusion Equations (K. BROWN & A. LACEY, eds.), Oxford Univ. Press, 1990.

[B] P. W. BATES, Personal communication.

[B-E1] J. F. BLOWEY & C. M. ELLIOTT, *The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy. Part I: Mathematical analysis,* Euro. J. Appl. Math. **22** (1991), 233–279.

[B-E2] J. F. Blowey & C. M. Elliott, *The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy. Part II: Numerical analysis,* Euro. J. Appl. Math., in press.

[B-F1] P. W. Bates & P. Fife, *Spectral comparison principles for the Cahn-Hilliard and phase field equations, and time scales for coarsening,* Physica D **43** (1990), 335–348.

[B-F2] P. W. Bates & P. Fife, *The dynamics of nucleation for the Cahn-Hilliard equation,* SIAM J. Appl. Math. **53** (1993), 990–1008.

[Bro-H] L. Bronsard & D. Hilhorst, *On the slow dynamics for the Cahn-Hilliard equation in one space dimension,* Proc. Roy. Soc. London A **439** (1992), 669–682.

[Bro-K1] L. Bronsard & R. V. Kohn, *On the slowness of the phase boundary motion in one space dimension,* Comm. Pure Appl. Math. **43** (1990), 983–997.

[Bro-K2] L. Bronsard & R. V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics,* J. Diff. Eqs. (1990), 211–237.

[B-X1] P. W. Bates & J. P. Xun, *Metastable patterns for the Cahn-Hilliard equation: Part I,* J. Diff. Eqs. **111** (1994), 421–457.

[B-X2] P. W. Bates & J. P. Xun, *Metastable patterns for the Cahn-Hilliard equations: Part II, Layer dynamics and slow invariant manifold,* J. Diff. Eqs. **116** (1995), 165–216.

[C1] J. W. Cahn, *Theory of crystal growth and interface motion in crystalline materials,* Acta Metallurgica **8** (1960), 554–562.

[C2] J. W. Cahn, *On spinodal decomposition,* Acta Metallurgica **9** (1961), 795–801.

[C3] J. W. Cahn, *Spinodal decomposition,* Trans. Metallurg. Soc. of AIME **242** (1968), 166–180.

[Ca] G. Caginalp, *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw and Cahn-Hilliard models as asymptotic limits,* IMA J. of Math. **44** (1989).

[Ca-F] G. Caginalp & P. C. Fife, *Phase field methods for interfacial boundaries,* Phys. Rev. B **33** (1986), 7792–7794.

[C-H] J. W. Cahn & J. E. Hilliard, *Free energy of a nonuniform system I. Interfacial energy,* J. Chemical Physics **28** (1957), 258–267.

[Ch1] Xinfu Chen, *Hele-Shaw problem and area preserved curve shortening motion,* Arch. Rational Mech. Anal. **123** (1993), 117–151.

[Ch2] Xinfu Chen, Personal communication.

[Ch-H-Y] Xinfu Chen, J. Hong & F. Yi, *Existence, uniqueness, and regularity of classical solutions of the Mulins-Sekerka problem,* preprint.

[C-G-S] J. Carr, M. E. Gurtin, & M. Slemrod, *Structured phase transitions on a finite interval,* Arch. Rational Mech. Anal. **86** (1984), 317–351.

[Ch-K] Xinfu Chen & M. Kowalczyk, *Existence of equilibria for the Cahn-Hilliard equation via local minimizers of perimeter,* Comm. Part. Diff. Eqs. **21** (1996), 1097–1123.

[C-Pe] J. Carr & R. L. Pego, *Metastable patterns in solutions of* $u_t = \varepsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math. **42** (1989), 523–576.

[C-Pu] P. Constantin & M. Pugh, *Global solutions for small data to the Hele-Shaw problem,* Nonlinearity **6** (1993), 393–415.

[E] C. M. Elliott, *The Cahn-Hilliard model for the kinetics of phase separation, in Mathematical Models for Phase Change Problems,* Int. Sem. of Numerical Math., Vol. 88 (J.F. Rodrigues, ed.) Birkhäuser, 1989, pp. 35–73.

[E-Fr] C. M. Elliott & D. A. French, *Numerical studies of the Cahn-Hilliard equation for phase separation,* IMA Jour. Appl. Math. **38** (1987), 97–128.

[E-Si] J. ESHER & G. SIMONETT, *Classical solutions of multi-dimensional Hele-Shaw flows,* to appear in SIAM J. Math. Anal.
[Ey1] D. EYRE, *Systems of Cahn-Hilliard equations,* SIAM J. Appl. Math. **53** (1993), 1686–1712.
[Ey2] D. EYRE, *A study of two pattern forming models,* Ph.D. Thesis, Univ. of Utah (1992).
[E-Z] C. M. ELLIOTT & S. ZHENG, *On the Cahn-Hilliard equation,* Arch. Rational Mech. Anal. **96** (1986), 339–357.
[F1] P. C. FIFE, *Dynamics of internal layers and diffusive interfaces,* CBMS-NSF Regional Conference Series in Applied Mathematics **53**, SIAM, Philadelphia, 1988.
[F2] P. C. FIFE, *Diffusive waves in inhomogeneous media,* Proc. Edinburg Math. Soc. **32** (1989), 291–315.
[F3] P. C. FIFE, *Models for phase separation and their mathematics,* Nonlinear Partial Differential Equations and Applications (M. MIMURA & T. NISHIDA, eds.), Kinokuniya Pubs., Tokyo, in press.
[F4] P. C. FIFE, *Dynamical aspects of the Cahn-Hilliard equation,* Barrett Lectures, University of Tennessee, Spring 1991.
[F-Ha] G. FUSCO & J. K. HALE, *Slow motion manifolds, dormant instability and singular perturbations,* J. Dynamics Diff. Eqs. **1** (1989), 75–94.
[Fo] G. FOLLAND, *Introduction to Partial Differential Equations,* Princeton University Press and Tokyo University Press, Princeton, 1976.
[G1] C. GRANT, *Spinodal decomposition for the Cahn-Hilliard equation,* Comm. Part. Diff. Eqs. **18** (1993), 453–490.
[G2] C. GRANT, *Slow motion in one-dimensional Cahn-Morral systems,* SIAM J. Math. Anal. **20** (1995), 21–34.
[Gu1] M. E. GURTIN, *On a theory of phase transitions with interfacial energy,* Arch. Rational Mech. Anal. **87** (1985), 187–212.
[Gu2] M. E. GURTIN, *On the two-phase Stefan problem with interfacial energy and entropy,* Arch. Rational Mech. Anal. **96** (1986), 199–241.
[Gu3] M. E. GURTIN, *On phase transitions with bulk, interfacial, and boundary energy,* Arch. Rational Mech. Anal. **96** (1986), 243–264.
[H] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations,* Springer Lecture Notes in Math. **840**.
[Ha1] J. K. HALE, *Asymptotic Behavior of Dissipative Systems,* Amer. Math. Soc., Providence, 1988.
[Ha2] J. K. HALE, *Ordinary Differential Equations,* 2nd ed., Kreiger, 1978.
[K-Ste] R. V. KOHN & P. STERNBERG, *Local minimizers and singular perturbation,* Proc. Roy. Soc. Edinburgh (1989), 69–84.
[McK] W. MCKINNEY, Thesis, Department of Mathematics, University of Tennessee, Knoxville, 1989.
[M1] L. MODICA, *The gradient theory of phase transitions and the minimal interface criterion,* Arch. Rational Mech. Anal. **98** (1987), 123–142.
[M2] L. MODICA, *Gradient theory of phase transitions with boundary contact energy,* Analyse Nonlineaire **5** (1987), 453–486.
[N] J. NEU, Unpublished lecture notes.
[NC-S] A. NOVICK-COHEN & L. A. SEGEL, *Nonlinear aspects of the Cahn-Hilliard equation,* Physica **10D** (1984), 277–298.
[Pe] R. L. PEGO, *Front migration in the nonlinear Cahn-Hilliard equation,* Proc. Roy. Soc. London A **422** (1989), 261–278.
[P-F] O. PENROSE & P. C. FIFE, *Thermodynamically consistent models of phase field type for the kinetics of phase transitions,* Physica D **43** (1990), 44–62.
[Ste] P. STERNBERG, *The effect of a singular perturbation on nonconvex variational problems,* Arch. Rational Mech. Anal. **101** (1988), 209–260.

[St] B. Stoth, *Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka-Problem in spherical symmetry,* preprint (1993).

[Wa] M. J. Ward, *Metastable bubble solutions for the Allen-Cahn equation with mass conservation,* SIAM J. Appl. Math. **56** (1996), 1247–1279.

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(Accepted February 12, 1996)