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On the Generation of Discontinuous Shearing Motions of a Non-Newtonian Fluid

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In memory of Charles Conley and Wolfgang Wasow

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Abstract

The system under study models unsteady, one-dimensional shear flow of a highly elastic and viscous incompressible non-Newtonian fluid with fading memory under isothermal conditions. The flow, in a channel, is driven by a constant pressure gradient, is symmetric about the center line, and satisfies a no-slip boundary condition at the wall. The non-Newtonian contribution to the stress is assumed to obey a differential constitutive law (due to Oldroyd, Johnson & Segalman), the key feature of which is a non-monotone relation between the total steady shear stress and strain rate. In a regime in which the Reynolds number is much smaller than the Deborah (or Weissenberg) number, one obtains a degenerate, singularly perturbed system of nonlinear reaction-diffusion equations. It is shown that if the driving pressure gradient exceeds a critical value (the local shear stress maximum of the steady stress vs. strain rate relation), then the solution to the governing system, starting from rest at $t = 0$, tends as $t \to \infty$ to a particular discontinuous steady state solution (the "top-jumping" steady state), except in a small neighborhood of the discontinuity. This discontinuous steady state is shown to be nonlinearly stable in a precise sense with respect to perturbations yielding smooth initial data. Such discontinuous steady states have been proposed to explain "spurting" flows, which exhibit a large increase in mean flow rate when the driving pressure is raised above a critical value.

1. Introduction

In this paper, we complete the study initiated in [8] concerning the nonlinear stability and generation of certain discontinuous steady states which model the spurt phenomenon observed in experiments on pressure-driven shear flows of a highly elastic and viscous, incompressible, non-Newtonian fluid through a capillary [11]. The term "spurt" refers to the rapid and dramatic increase in flow rate encountered as the driving pressure gradient exceeds a certain threshold value. As in earlier studies [5, 6, 8], we model the capillary as a channel in the plane between $-1 \leqq x \leqq 1$, aligned in the direction of the y-axis, and we assume that the flow is symmetric about the center line $x = 0$. We assume that the stress in the fluid has a polymer contribution which obeys the Johnson-Segalman-Oldroyd (J-S-O) differential constitutive equations [4, 9], which satisfy the principle of material objectivity. And we assume that the flow takes place under isothermal conditions. Since the fluid is undergoing simple shearing, the flow variables are functions of position x in the channel and time t. In dimensionless units, the flow is governed by a degenerate system of reaction-diffusion equations for the unknown functions S, σ, Z:

$$\begin{aligned} \alpha S_t &= \varepsilon S_{xx} + \alpha[-\sigma + (Z+1)(S + \bar{T} - \sigma)/\varepsilon], \\ \sigma_t &= -\sigma + (Z+1)(S + \bar{T} - \sigma)/\varepsilon, \\ Z_t &= -Z - \sigma(S + \bar{T} - \sigma)/\varepsilon \end{aligned} \tag{1.1}$$

for $-1 \leqq x \leqq 0$, $t > 0$, together with the boundary conditions

$$S_x(-1,t) = 0, \quad S(0,t) = 0, \quad \sigma(0,t) = 0, \tag{1.2}$$

and the initial conditions

$$S(x,0) = S_0(x), \quad \sigma(x,0) = \sigma_0(x), \quad Z(x,0) = Z_0(x). \tag{1.3}$$

To enforce consistency of the initial and boundary conditions, the compatibility conditions $S_0'(-1) = 0$ and $S_0(0) = 0$ and the continuity condition $\sigma_0(0) = 0$ are assumed to hold. We refer to [6] for a derivation of the system governing the flow starting from three-dimensional balance laws and constitutive equations, and to [7, 8] for the recasting of the initial-boundary-value problem into the equivalent form (1.1)–(1.3). In terms of physical quantities, the unknown function S is defined by

$$S := \varepsilon v_x + \sigma - \bar{T}, \qquad \bar{T} := -\bar{f}x, \tag{1.4}$$

reflecting the fact that the total stress on the fluid is the sum of three contributions: a Newtonian stress, a non-Newtonian extra stress and an isotropic pressure, respectively. Here, v represents the fluid velocity, v_x the strain rate, σ the polymer contribution to the non-Newtonian extra shear stress, and $\bar{f} > 0$ the constant pressure gradient driving the flow. The quantity Z is proportional to the first normal stress difference. The second and third equations in (1.1) stem directly from the J-S-O constitutive equations with a single relaxation time that are assumed to govern the class of fluids under study. The parameter ε is the ratio of Newtonian viscosity to shear viscosity (scaled by relaxation time), and α is the ratio of Reynolds number to Deborah number. For highly elastic and viscous fluids such as those in the experiments of Vinogradov *et al.* [11], α and ε are both small, with $\varepsilon \sim 10^{-2}$ or 10^{-3}, and α is

7 to 10 orders of magnitude smaller than ε (see [6]). Consequently, in much of the analysis the governing system is regarded as a singular perturbation of the system having $\alpha = 0$.

System (1.1) admits steady states $(\bar{S}(x), \bar{\sigma}(x), \bar{Z}(x))$ that satisfy the following relations (see [6, 8]):

$$\bar{\sigma}(x) = \frac{\bar{v}_x}{1+\bar{v}_x^2}, \qquad \bar{Z}(x) + 1 = \frac{1}{1+\bar{v}_x^2}, \tag{1.5}$$

$$\bar{S}(x) = \varepsilon\bar{v}_x + \frac{\bar{v}_x}{1+\bar{v}_x^2} - \bar{T} \equiv 0. \tag{1.6}$$

Thus the steady strain rate $\bar{v}_x(x)$ satisfies the equation $\omega(\bar{v}_x) = \bar{T} := -\bar{f}x$, for $-1 \leqq x \leqq 0$, where $\bar{T}$ represents the total steady shear stress at position x, and where

$$\omega(\xi) := \frac{\xi}{1+\xi^2} + \varepsilon\xi, \quad -\infty < \xi < \infty. \tag{1.7}$$

For $\varepsilon < \frac{1}{8}$, the function ω is not monotone (see Fig. 1). Consequently, if $\bar{T}$ is sufficiently large, there are multiple steady states $\bar{v}_x(x)$ satisfying (1.6), which are discontinuous in x. The resulting steady velocity profiles $\bar{v}(x)$ have kinks at points x_* where $\bar{v}_x$ is discontinuous. (For an example with one kink, see Fig. 2.) Formulae (1.5) imply that the corresponding steady states $(\bar{\sigma}, \bar{Z})$ have discontinuities at the same points x_* where $\bar{v}_x$ is discontinuous. Such discontinuous steady states stemming from the non-monotone constitutive function ω play a key role in the dynamics.

If $\alpha > 0$ and $\varepsilon > 0$, the initial-boundary-value problem (1.1)–(1.3) is globally well-posed in time. In what follows, the function spaces H^s, C, C^1,

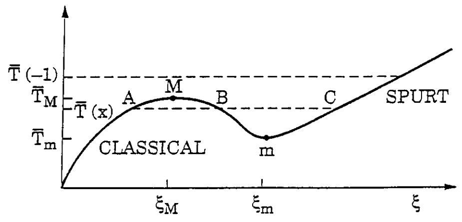

Figure 1. ω vs. ξ.

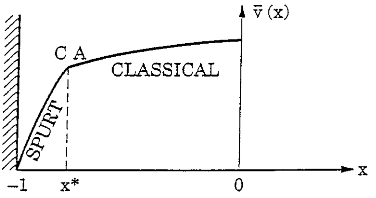

Figure 2. Velocity profile with a kink; $\omega(\bar{v}_x(x)) = \bar{T}$.

$W^{2,\infty}$, L^∞ refer to the interval $[-1,0]$. The global existence and uniqueness, stated as Theorem 2.1 in [8] and repeated here for convenience, follow from a general result in [7]:

Theorem 1.1. (a) *For (smooth) initial data $S_0 \in H^s$, for some $s > \frac{3}{2}$, and σ_0, $Z_0 \in C^1$, there exists a unique classical solution on $[-1,0] \times [0,\infty)$ having the regularity*:

$$\begin{aligned} S &\in C([0,\infty), C^1) \cap C((0,\infty), H^2), \\ S_t &\in C((0,\infty), C^1), \\ \sigma, Z &\in C^1([0,\infty), C^1). \end{aligned} \tag{1.8}$$

(b) *For (rough) initial data $S_0 \in H^1$, and σ_0, $Z_0 \in L^\infty$, there is a unique semi-classical solution on $[-1,0] \times [0,\infty)$ (possibly having discontinuities in the stress components) having the regularity*:

$$\begin{aligned} S &\in C([0,\infty), H^1) \cap C((0,\infty), W^{2,\infty}), \\ S_t &\in C((0,\infty), H^s) \quad \text{for all } s < 2, \\ \sigma, Z &\in C^1([0,\infty), L^\infty). \end{aligned} \tag{1.9}$$

Given any bounded measurable representatives σ_{0*}, Z_{0*} of the equivalence classes σ_0, $Z_0 \in L^\infty$, there exist unique bounded measurable functions $\sigma_*(x,t)$, $Z_*(x,t)$, representing $\sigma(\cdot,t)$, $Z(\cdot,t)$ for each $t > 0$, such that the map $t \mapsto \sigma_*(x,t)$, $Z_*(x,t)$ is of class C^1 for $t \geqq 0$, and (S_*, σ_*, Z_*) satisfy the second and third equations of system (1.1) for $t > 0$, where S_* is the unique continuous representative of S.

The class of solutions in Theorem 1.1(b) notably includes the discontinuous steady states of the system (1.1). We identify a solution (S, σ, Z) in case (b) with some representative (S_*, σ_*, Z_*); then the second and third equations in (1.1) are satisfied pointwise for all $t > 0$. It is shown in [7] that discontinuities in the stress components (σ, Z) for such solutions can neither be created nor destroyed in finite time.

System (1.1) is endowed with the identity

$$\frac{d}{dt}\left[\sigma^2 + (Z+1)^2\right] = -2\left[\sigma^2 + (Z+1/2)^2 - \frac{1}{4}\right]; \tag{1.10}$$

observe that the identity is independent of α, ε, $\bar{T}$. It follows easily from (1.10) that σ and Z are globally bounded in (x,t); moreover, these bounds are independent of α and ε. But although solutions of (1.1)–(1.3) exist globally, it does not follow without further restrictions that S remains globally bounded in (x,t). Moreover, determining the asymptotic behavior of solutions as $t \to +\infty$ and understanding the dynamics of (1.1) also requires additional but physically reasonable assumptions.

The main focus of this paper is to complete the analysis initiated in [8], showing that in a regime where $\varepsilon < \frac{1}{8}$ is fixed and $\alpha > 0$ is sufficiently small, the dynamics of the full system (1.1) is similar to that generated by the approximating system obtained from (1.1) by putting $\alpha = 0$. The steady states of the system (1.1) are clearly the same for $\alpha > 0$ as for $\alpha = 0$. The dynamics of the approximating system (given by system (Q) below) may be studied independently for each $x \in [-1, 0]$, and has been completely determined by phase-plane analysis [6]. The results (summarized in Propositions 2.1 and 2.2 below) form the basis for one possible explanation of spurt and related phenomena first observed in experiments by G. Vinogradov *et al.* [11] in quasistatic loading of pressure-driven flows. As proposed by Malkus, Nohel & Plohr [6], the approximating system (Q) predicts spurt as well as latency, shape memory, and hysteresis under quasistatic loading and unloading of the driving pressure gradient.

The following three qualitative results, relating the dynamics of the full system with $\alpha > 0$ to that of the approximating system with $\alpha = 0$, have already been obtained in [8] for $\alpha > 0$ sufficiently small, under the assumption that the initial data (1.3) satisfy the assumptions (a) or (b) of Theorem 1.1. Precise statements needed in the present paper are given in Section 2.

(1) On any fixed interval $0 < t \leqq T$ and $\alpha > 0$, the solution (S, σ, Z) of (1.1)–(1.3) converges as $\alpha \to 0$ to the corresponding solution of the system in which $\alpha = 0$; but $\alpha = 0$ implies that $S(x, t) = 0$, and the second and third equations of (1.1) reduce to the one-parameter family of autonomous systems of quadratic ordinary differential equations

$$\begin{aligned} \sigma_t &= -\sigma + (Z + 1)(\bar{T} - \sigma)/\varepsilon, \\ Z_t &= -Z - \sigma(\bar{T} - \sigma)/\varepsilon, \end{aligned} \tag{Q}$$

parametrized by x (or equivalently by $\bar{T}$). (See Corollary 3.2 and Theorem 3.3 in [8].)

(2) If α is sufficiently small, every solution of (1.1)–(1.3) with initial data satisfying

$$\|S_0\|_{H^1} + \|\sigma_0\|_{L^\infty} + \|Z_0\|_{L^\infty} \leqq M, \tag{1.11}$$

where $M > 0$ is any constant, converges as $t \to \infty$ to *some* steady state $(0, \bar{\sigma}(x), \bar{Z}(x))$ (possibly discontinuous, possibly unstable) with

$$S \to 0 \text{ in } H^1(-1, 0), \quad (\sigma(x, t), Z(x, t)) \to (\bar{\sigma}(x), \bar{Z}(x)), \tag{1.12}$$

for each $x \in [-1, 0]$. (See Theorem 3.5 in [8].) For some smooth solutions, the corresponding asymptotic state is discontinuous in x, and the convergence in (1.12) is not uniform; such solutions contain "transition layers". For a given set of initial data, one cannot in general identify the limiting steady state in (1.12), and for this reason, it is important to test which steady states are stable.

(3) If α is sufficiently small, it is shown in Theorem 3.4 of [8] that any steady state $(0, \bar{\sigma}(x), \bar{Z}(x))$ of (1.1), continuous or discontinuous, for which $\omega(\bar{v}_x)$ takes values only on strictly increasing portions of the graph of $\omega(\xi)$ in Fig. 1 (*i.e.*, excluding a neighborhood of the local max and min), is nonlinearly Lyapunov stable with respect to perturbations of initial data from steady state in a precise sense discussed in Section 2. Such stable steady states

may possess any arbitrary pattern of discontinuities. Stability holds when the perturbation in S_0 is small in $H^1(-1,0)$, and the perturbations in $\sigma_0 - \bar{\sigma}$, $Z_0 - \bar{Z}$ are small in $L^1(-1,0)$, and are bounded pointwise by some large constant. Hence, there are smooth initial data in the basin of Lyapunov stability of any such discontinuous steady state. Fig. 3 illustrates such a perturbation of the discontinuous steady strain rate $\bar{v}_x$ having a single jump that is responsible for the kinked velocity profile in Fig. 2.

The stability result in (3), from Theorem 3.4 of [8], does not apply to an important steady solution which is naturally encountered in modeling the loading process performed in experiments. As is proved in [6], starting from rest with $(\sigma_0, Z_0) = (0,0)$, with the pressure gradient loaded to a value $\bar{f} > \bar{T}_M$ above the local steady shear stress maximum, the solution of the approximate problem (Q), corresponding to $\alpha = 0$ in (1.1), converges as $t \to \infty$ to the discontinuous, *top-jumping* steady state $(\bar{\sigma}(x), \bar{Z}(x))$ having a single point of discontinuity at the point x_* (typically near $x = -1$) where $\bar{T}(x_*) = \bar{T}_M$. In Fig. 1, as x varies from the center line at 0 to the wall at -1, the top-jumping steady solution traces out the path $(\xi, \omega(\xi)) = (\bar{v}_x, \bar{T})$, starting at the origin and proceeding along the stress *vs.* strain curve to the local shear stress maximum at M. At this point the path jumps horizontally from M to a point on the right-hand branch and continues along the curve to its highest point at $\bar{f} = \bar{T}(-1) > \bar{T}_M$. Corresponding to the single jump discontinuity in the steady strain rate $\bar{v}_x(x)$, the steady stresses $\bar{\sigma}(x)$ and $\bar{Z}(x)$ have discontinuities at the same point x_*.

Numerical simulation of the Vinogradov quasistatic loading experiment [11], based on a numerical algorithm developed in [5] for the full system (1.1)–(1.3), also shows that the numerical solution converges as $t \to \infty$ to the top-jumping steady solution when the final pressure gradient $\bar{f}$ exceeds the critical value $\bar{T}_M$. Similarly, numerical simulation in [5] shows that the discontinuous, "bottom-jumping" steady solution (*i.e.*, a steady solution which jumps across the local shear stress minimum at m in Fig. 1) is achieved when the driving pressure gradient is down-loaded quasistatically from a value $\bar{f} > \bar{T}_M$ to a value only slightly larger than $\bar{T}_m$, the local shear stress minimum at m.

These results suggest that the steady top- and bottom-jumping solutions of the full system (1.1), with $\alpha > 0$, are nonlinearly stable with respect to

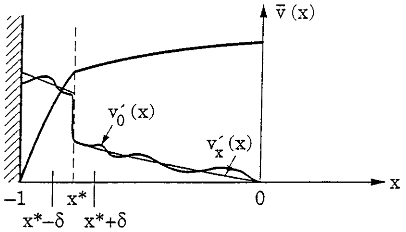

Figure 3. Smooth perturbation of velocity gradient.

perturbations of initial data. It is desirable to settle this issue in a rigorous fashion, for the problem is a delicate one, both in terms of interpreting computations of solutions which are losing their smoothness, and from the point of view of analysis. Two reasons suggest analytical difficulties: (a) Spectral analysis of the linearization about the top-jumping solution reveals that the continuous spectrum includes an interval with the origin as its right end point. (b) Linearized stability is not sufficient to imply the stability of the discontinuous steady states with respect to perturbations yielding smooth initial data.

A key result of this paper, Theorem 2.3 below, establishes the nonlinear stability of the top-jumping steady solution, in a precise sense which allows for perturbations yielding smooth initial data. Its proof, given in Sections 3 and 4, employs parabolic smoothing estimates different from those used in [8], but relies on several constructions made in [8] to analyze the dynamical behavior of approximate solutions of system (Q).

The main result of this paper, Theorem 2.6 below, is a consequence of the stability result in Theorem 2.3 and of the dynamical systems techniques developed in [8]. It shows that the top-jumping steady solution is indeed generated asymptotically in the "start-up" problem for the full system (1.1), for sufficiently small $\alpha > 0$: If the flow starts from rest with $(S_0, \sigma_0, Z_0) = (0, 0, 0)$, and if it is subjected to a driving pressure gradient $\bar{f} > \bar{T}_M$ above the local steady shear stress maximum, then the corresponding solution (S, σ, Z) of the full system (1.1) converges pointwise to the discontinuous, top-jumping steady state $(0, \bar{\sigma}(x), \bar{Z}(x))$ as $t \to \infty$ for $x \in [-1, 0]$, except perhaps in a small neighborhood of the single point of discontinuity at x_*. The size of this neighborhood shrinks to zero as $\alpha \to 0$.

As in the Vinogradov loading experiment [11], the same discontinuous, top-jumping steady state can also be generated analytically by the full system (1.1) with $\alpha > 0$ sufficiently small, by increasing the driving pressure gradient $\bar{f}$ "quasistatically" from values below to values above the local, steady shear stress maximum $\bar{T}_M$ in Fig. 1. We also discuss the nonlinear stability and generation of the discontinuous, bottom-jumping solutions.

These results, for one-dimensional shear flows governed by the initial-boundary-value problem (1.1)–(1.3) for $\alpha > 0$ sufficiently small, further support the argument given in [6] based on the inertialess flow approximation governed by system (Q) above (corresponding to $\alpha = 0$ in (1.1)), namely, that the spurt phenomenon is a consequence of material properties rather than of a failure of material to adhere to the wall of the capillary.

Further in this direction, Aarts & van de Ven [1] recently simulated and partially analyzed spurt and related phenomena, using the inertialess approximation of the governing system for three-dimensional axisymmetric shear flow in a pipe. Their treatment is based on a particular modification of the K-BKZ integral constitutive law (with a single exponentially decaying memory kernel) that exhibits non-monotone behavior in steady shear, in a manner similiar to that shown here in Fig. 1 for the J-S-O differential constitutive law. Their numerical results are similar to those described in [5] for the initial-boundary-value problem (1.1)–(1.3) with small $\alpha > 0$, confirming that non-monotonicity of the total steady shear stress as function of strain rate is key to explaining spurt.

In related recent work, Y. Y. Renardy [10] investigates the linearized stability of spurting planar Couette flows in a Johnson-Segalman fluid, with respect to two-dimensional infinitesimal perturbations of the one-dimensional base flow. Such Couette flows model spurting flows near the boundary, in situations like those that occur with the flows considered in this paper having driving pressure gradients $\bar{f}$ near that for top-jumping. Interfacial instabilities with dominant growth rate for short waves are found. These results suggest that oscillations observed in the extrusion of molten polymers ("sharkskin") may be generated by multi-dimensional flow instabilities. Another factor possibly contributing to the generation of such oscillations is that normal-stress oscillations occur with long relaxation times in one-dimensional spurting flows, as shown in [6].

2. Main Result

For $\alpha > 0$, it is convenient to follow [8] and write (1.1) in the more compact form

$$\begin{aligned} S_t &= \tfrac{\varepsilon}{\alpha} S_{xx} + a(x,u)S + b(x,u), \\ u_t &= G(x,u) + H(x,u)S, \end{aligned} \tag{2.1}$$

where $u = (\sigma, Z)$ and the components of $G = (G_1, G_2)$ and $H = (H_1, H_2)$ are given by

$$\begin{aligned} G_1(x,u) &= -\sigma + (Z+1)(\bar{T} - \sigma)/\varepsilon, \quad & H_1(x,u) &= (Z+1)/\varepsilon, \\ G_2(x,u) &= -Z - \sigma(\bar{T} - \sigma)/\varepsilon\,, & H_2(x,u) &= -\sigma/\varepsilon, \end{aligned} \tag{2.2}$$

$$a(x,u) = H_1(x,u), \qquad b(x,u) = G_1(x,u). \tag{2.3}$$

The family of approximating quadratic systems (Q) now takes the compact form

$$\dot{u} = G(x,u), \tag{2.4}$$

where the parameter $x \in [-1, 0]$ enters through $\bar{T} = -\bar{f}x$, with $\bar{f} > 0$ a constant.

In what follows, an important fact about system (2.1), implied by (1.10), is that an *a priori* bound exists for $|u(x,t)|$ independent of S, namely, for some constant M_u independent of α (and ε), depending only on the initial data $u(x,0) = u_0(x)$,

$$\sup_{\substack{-1 \leqq x \leqq 0 \\ t \geqq 0}} |u(x,t)| \leqq M_u. \tag{2.5}$$

Moreover, system (2.1) is linear in S, and the functions a, b, G, H and their Lipschitz constants with respect to u are bounded by some constant L/ε, where L is independent of α (and ε).

To state the main stability result and its consequences, we require the following facts about the dynamics of the approximating system (2.4) proved in [6]. Considering $\bar{T} = -\bar{f}x \geqq 0$ as a fixed parameter in (2.2) and (2.4), we have

(1) For each $\bar{T} \geqq 0$, system (2.4) has no periodic or homoclinic orbits, and every solution converges as $t \to \infty$ to some critical point.
(2) The critical points of (2.4) lie in the fourth quadrant of the $u = (\sigma, Z)$-phase plane, at the intersections of the circle

$$\Gamma := \left\{(\sigma, Z) \mid \sigma^2 + (Z + \tfrac{1}{2})^2 = \tfrac{1}{4}\right\} \tag{2.6}$$

and the parabola $Z = \sigma(\sigma - \bar{T})/\varepsilon$.

The character of the critical points is described as follows: In Fig. 1, denote the coordinates of the local maximum M and of the local minimum m by $(\xi_M, \bar{T}_M)$ and $(\xi_m, \bar{T}_m)$ respectively; $\bar{T}_m$ and $\bar{T}_M$ are the critical values of the function $\omega(\xi)$, with $\bar{T}_m < \bar{T}_M$. There are three cases:

(i) If $0 < \bar{T} < \bar{T}_m$, there is a single critical point $A = (\sigma_A, Z_A)$ which is a globally attracting node.
(ii) If $\bar{T} > \bar{T}_M$, there is a single, globally attracting spiral point $C = (\sigma_C, Z_C)$.
(iii) If $\bar{T}_m < \bar{T} < \bar{T}_M$, there are three critical points A, B, C (see Fig. 4). A is an attracting node, B is a saddle point and C is generally an attracting spiral point. (But for $\bar{T}$ close to $\bar{T}_m$, C is an attracting node.)

As shown in Fig. 4, for any value of $\bar{T}$ (positive or negative), the attractor A lies on the upper arc of the circle Γ through the origin between the points M and M'. The attractor C lies on the lower arc of Γ through the point $(0, -1)$ between the points m and m', while the saddle B lies on the remaining two arcs of Γ, either between m and M or between m' and M'. Two saddle-node

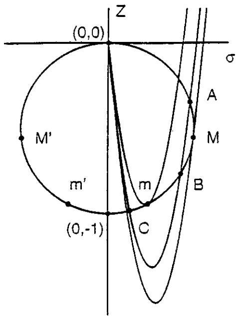

Figure 4. Critical points of (2.4) in case (iii).

bifurcations occur as $\bar{T}$ varies: As $\bar{T} \to \bar{T}_m$ from above, points B and C coalesce at m, and as $\bar{T} \to \bar{T}_M$ from below, points A and B coalesce at M. The manifold of critical points of (2.4) in the full $(\sigma, Z, \bar{T})$ parameter space is a simple smooth curve; this set is visualized in Figure 5.

The asymptotic behavior of the solutions of (2.4) is completely characterized as follows: For each $\bar{T} < \bar{T}_m$ or $\bar{T} > \bar{T}_M$, every solution tends to the unique critical point, A or C respectively, where A is a globally attracting node and C is a globally attracting spiral point. For $\bar{T}_m < \bar{T} < \bar{T}_M$, the behavior of solutions is described by Proposition 3.5 in [6] as follows:

Proposition 2.1. *The basin of attraction of A, i.e., the set of points that flow toward A as $t \to \infty$, comprises those points on the same side of the stable manifold of B as is A; points on the other side are in the basin of attraction of C. Moreover, the arc of the circle Γ in Fig. 4, through the origin, between $B = (\sigma_B, Z_B)$ and its reflection $B' = (-\sigma_B, Z_B)$, is contained in the basin of attraction of A.*

These results for the quadratic system (2.4) have the following immediate consequences for solutions $(0, \sigma, Z)$ of (1.1) with $\alpha = 0$, for $(x, t) \in [-1, 0] \times [0, \infty)$.

Proposition 2.2. *Consider system* (1.1) *with* $\alpha = 0$.

(i) *The asymptotic behavior of any given solution may be completely characterized. For each x in $[-1, 0]$, $(\sigma(x,t), Z(x,t)) \to A$, B* or *C as $t \to \infty$, according as to whether $(\sigma_0(x), Z_0(x))$ lies in the basin of attraction of A, on the stable manifold of B, or in the basin of attraction of C.*
(ii) *The asymptotically stable steady states $(0, \bar{\sigma}, \bar{Z})$ of* (1.1) *with $\alpha = 0$ are those for which*

$$(\bar{\sigma}(x), \bar{Z}(x)) = A \quad \textit{or} \quad C \quad \textit{for a.e. } x \in [-1, 0]. \tag{2.7}$$

In particular, the "top-jumping" steady state solution $(0, \bar{u}(x))$ of system (1.1) *with $\bar{u}$ given by*

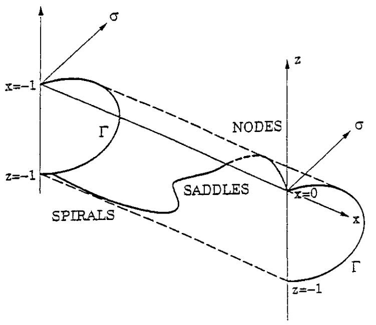

Figure 5. Manifold of equilibria of system (2.4).

$$\bar{u}(x) = (\bar{\sigma}(x), \bar{Z}(x)) = \begin{cases} A(x) & \textit{for} \;\; x_M < x \leqq 0 \; (\textit{where} \; 0 \leqq \bar{T} < \bar{T}_M), \\ C(x) & \textit{for} \;\; -1 \leqq x < x_M \; (\textit{where} \; \bar{T}_M < \bar{T}), \end{cases} \tag{2.8}$$

where each $A(x)$ *is an attracting node and each* $C(x)$ *is a globally attracting spiral point, is asymptotically stable; the function* $\bar{u}$ *has a single jump discontinuity at* $x_* := x_M = -\bar{T}_M/\bar{f}$, *where* $-1 < x_* < 0$ *and* $\bar{f} > \bar{T}_M$. *Similarly, the "bottom-jumping" steady state solution* $(0, \bar{u}(x))$ *of system* (1.1) *with* $\bar{u}$ *given by* (2.8) *with subscript M replaced by m throughout and* $x_m := -\bar{T}_m/\bar{f}$, *is asymptotically stable.*

One goal of this paper is to extend the stability result in Proposition 2.2(ii), concerning the top-jumping steady state solution (2.8) of system (1.1) when $\alpha = 0$, to the full system (2.1) with $\alpha > 0$ sufficiently small. The stability of this solution is not established by Theorem 3.4 of [8] (described in item (3) of Section 1) because of a degeneracy associated with the local maximum of the steady stress-strain relation.

Our principal result in this regard establishes the nonlinear Lyapunov stability of the top-jumping solution. We prove the following in Section 3:

Theorem 2.3. *Let* $(0, \bar{u}(x))$ *be the top-jumping steady state solution of system* (1.1), *where* $\bar{u}(x)$ *is given by* (2.8). *There exist positive constants* α_0, η_0, $\bar{C}$ *and* C_0 *such that if* $0 < \alpha < \alpha_0$, $0 < \eta < \eta_0$ *and if*

$$\|S_0\|_{H^1} \leqq \eta \quad \textit{and} \quad |u_0(x) - \bar{u}(x)| \leqq \eta \quad \textit{for} \quad x \in [-1, 0] \backslash \mathscr{U}, \tag{2.9}$$

where $\mathscr{U} = (x_* - \bar{C}\eta, x_* + \bar{C}\eta)$ *and* $x_* = x_M$, *then for all* $t > 0$, *the solution* (S, u) *to the full system* (2.1) *corresponding to the initial data* (S_0, u_0) *satisfying the hypothesis of Theorem* 1.1, *satisfies*

$$\|S(\cdot, t)\|_{H^1} \leqq C_0 \eta \quad \textit{and} \quad |u(x, t) - \bar{u}(x)| \leqq C_0 \sqrt{\eta} \quad \textit{for} \quad x \in [-1, 0] \backslash \mathscr{U}. \tag{2.10}$$

Remark 2.4. A similar stability result holds for the bottom-jumping steady solution $(0, \bar{u}(x))$ of system (1.1), where $\bar{u}$ is given by (2.8) with M replaced by m throughout.

Remark 2.5. From the proofs of Theorem 2.3 here and Theorem 3.4 of [8], it can be seen that a similar stability result holds for a steady state solution $(0, \bar{u})$ of (1.1) having an arbitrary pattern of jumps in the interval $x_M < x < x_m$, taking the form

$$\bar{u}(x) = (\bar{\sigma}(x), \bar{Z}(x)) = \begin{cases} A & \text{for } x \text{ where } 0 \leqq \bar{T} < \bar{T}_m, \\ A \text{ or } C & \text{for } x \text{ where } \bar{T}_m \leqq \bar{T} \leqq \bar{T}_M, \\ C & \text{for } x \text{ where } \bar{T}_M \leqq \bar{T} \end{cases} \tag{2.11}$$

for $-1 \leqq x \leqq 0$. This removes a nondegeneracy hypothesis in the statement of Theorem 3.4 of [8], permitting the steady state solution $\bar{u}(x)$ to take the value

$A(x)$ and $C(x)$ at points x arbitrarily close to x_M and x_m respectively, where the saddle-node bifurcation occurs. This allows top- or bottom-jumping in the presence of other discontinuities.

Our primary goal in this paper is to study the long-time behavior of the solution of the start-up problem for system (1.1) for small $\alpha > 0$, corresponding to a fluid initially at rest with fully relaxed stress components, but suddenly subjected to a driving pressure gradient $\bar{f} > \bar{T}_M$. In the case $\alpha = 0$, it was shown in [6] that the solution $(0, u^0(x,t))$ to system (1.1) with $\bar{f} > \bar{T}_M$ and zero initial data tends to the top-jumping steady state solution $(0, \bar{u}(x))$ as $t \to \infty$, where $\bar{u}$ is given in (2.8). Here, we show that the asymptotic state generated by the solution of (1.1) for small $\alpha > 0$ with zero initial data, is discontinuous and approaches the top-jumping state in the limit $\alpha \to 0$. The following result is proved in Section 4:

Theorem 2.6. *Let $\bar{f} > \bar{T}_M$. There exists $\alpha_1 > 0$ such that if $0 < \alpha < \alpha_1$, then the solution (S, u) to the full system* (2.1) *corresponding to zero initial data approaches an asymptotic limit $(0, u^\infty(x))$ as $t \to \infty$, satisfying*

$$u^\infty(x) = (\bar{\sigma}(x), \bar{Z}(x)) = \begin{cases} A(x) & \textit{for} \quad 0 \geqq x > x_M + \beta, \\ C(x) & \textit{for} \quad -1 \leqq x < x_M \end{cases} \tag{2.12}$$

for some $\beta = \beta(\alpha)$, where $\beta \to 0$ as $\alpha \to 0$.

That is, the asymptotic steady-state solution $(0, u^\infty)$ may differ from the top-jumping solution $(0, \bar{u})$ of (2.8) only on an interval $[x_*, x_* + \beta]$ with $x_* = x_M$; the length of this interval shrinks to zero in the limit $\alpha \to 0$. Therefore we may conclude that a "spurting flow" is generated starting from rest with $\bar{f} > \bar{T}_M$ when $\alpha > 0$ is small; the average flow rate is much larger than for classical continuous steady states occurring when $\bar{f} < \bar{T}_M$.

Remark 2.7. The proof of Theorem 2.6 also implies the same conclusion for any smooth nonzero initial data $(0, u_0(x))$ such that $u_0(x)$ lies in the basin of attraction of the node $A(x)$ for system (2.4) whenever $x_M < x \leqq 0$, i.e., whenever $\bar{T} < \bar{T}_M$. In particular, this is known to be the case (see Propsition 2.1) if $u_0(x) = A_*(x)$ corresponds to the continuous steady state solution for some driving pressure gradient $\bar{f}_* < \bar{T}_M$. This models the process in the Vinogradov loading experiment [11] in which $\bar{f}$ is raised quasistatically from below to above $\bar{T}_M$ (also see discussion based on phase-plane analysis of system (2.4) in Section 5 of [6]).

Remark 2.8. The proof of Theorem 2.6 can also be used to show how a steady state can be generated close to the discontinuous bottom-jumping steady solution $(0, \bar{u})$, where $\bar{u}$ is given by (2.8) with M replaced by m throughout. In this case, one shows that, given initial data such that $u_0(x)$ lies in the basin of attraction of $C(x)$ for system (2.4) whenever $-1 < x < x_m$, the asymptotic limit of the solution of (2.1) for small $\alpha > 0$ agrees with the bottom-jumping steady state except in a small neighborhood of x_m. A means of generating such initial data has been discussed in [6]. One may start with a spurting flow corresponding to a driving pressure gradient $\bar{f}^* > \bar{T}_M$, near the top-jumping solution $(0, \bar{u}(x))$ except in a neighborhood of its discontinuity at $x_M^* = -\bar{T}_M/\bar{f}^*$. Then the loading is decreased to a value $\bar{f}$ only slightly larger than $\bar{T}_m$, such that $x_m = -\bar{T}_m/\bar{f} < x_M^*$. The point is that for $x < x_m$, the basin of attraction of $C(x)$ in the flow of system (2.4) includes the lower arc of the circle Γ in Fig. 4 between $C(x)$ and the point $(\sigma, Z) = (0, -1)$, hence includes the points $C^*(x)$ corresponding to $\bar{f}^*$. (See Propositions 2.1 and 2.2 above and the discussion based on phase-plane analysis of system (2.4) in 5.3 and 5.4 of [6]). However, for $x > x_m$ the only critical point in (2.4) is $A(x)$, which therefore must be the limiting stress state for those values of x. (This corresponds to a shape-memory effect, which is more fully discussed in [6].)

3. Proof of Theorem 2.3

We first prove an estimate for solutions of system (2.1) that bounds the H^1 norm of S globally in time for small α in terms of the L^1 norm of b. The corresponding estimate used in [8] bounds the H^1 norm of S for small α in terms of the L^2 norm of b; this is not adequate for our current purpose. As in [8], our method of obtaining the estimate on S uses the parabolic smoothing properties of the heat operator, and the variation of constants formula, following ideas in [3].

Lemma 3.1. *Let* (S, u) *be a solution of* (2.1), (1.2), (1.3) *with the regularity properties* (1.8) *or* (1.9) *of Theorem 1.1. Let* $\lambda := \pi^2/4$ *and let* $\widetilde{C}$, $\widetilde{K}$ *be the constants*

$$\widetilde{C} = 4\left(\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\right)^{1/2}, \quad \widetilde{K} = \int_0^{1/2\lambda} (2\mathrm{e}t)^{-1/2}\mathrm{e}^{\lambda t/2}dt + \int_{1/2\lambda}^{\infty} \lambda^{1/2}\mathrm{e}^{-\lambda t/2}dt. \tag{3.1}$$

If $\alpha < \varepsilon^2/2L\widetilde{K}$, *then*

$$\|S(\cdot,t)\|_\infty \leqq \|S(\cdot,t)\|_{H^1} \leqq 2\|S_0\|_{H^1}\mathrm{e}^{-\varepsilon\lambda t/2\alpha} + \frac{\alpha\widetilde{C}}{\varepsilon}\sup_{0\leqq\tau\leqq t}\|b(\cdot,u(\cdot,\tau))\|_{L^1}. \tag{3.2}$$

Proof. On the Hilbert space $X = L^2(-1,0)$, let Λ denote the operator with domain

$$D(\Lambda) := \{w \in H^2(-1,0) \mid w_x(-1) = 0 = w(0)\} \tag{3.3}$$

given by $\Lambda w := -w_{xx}$. The operator Λ is self adjoint and positive. The eigenvalues of Λ are $\lambda_n = ((2n-1)/2)^2\pi^2$, $n = 1, 2, \ldots$, so that $\lambda := \pi^2/4$ is its first eigenvalue. The corresponding normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\right)\pi x, \quad n = 1, 2, \ldots, \tag{3.4}$$

with $\|\phi_n\|_{L^2} = 1$ and $\|\phi_n\|_\infty = \sqrt{2}$. For what follows, define

$$H^1_{\mathrm{b}} := \{v \in H^1(-1, 0) \mid v(0) = 0\}, \tag{3.5}$$

and recall the elementary estimates

$$\|v\|_{L^2} \leqq \|v\|_\infty \leqq \|v_x\|_{L^2} \qquad \text{for } v \in H^1_{\mathrm{b}}. \tag{3.6}$$

For $v \in H^1_{\mathrm{b}}$, we define $\|v\|_{H^1} := \|v_x\|_{L^2}$. Also, if $v \in H^1$, elementary Fourier analysis yields

$$\|v\|^2_{H^1} = \sum_{n=1}^{\infty} \lambda_n |(v, \phi_n)|^2, \tag{3.7}$$

where $(\cdot, \cdot)$ denotes the L^2 inner product on the interval $-1 \leqq x \leqq 0$.

Let (S, u) be a solution of (2.1), (1.2), (1.3) satisfying the hypothesis of Lemma 3.1. We apply the variation-of-constants formula to the first equation in (2.1). Suppressing the x-dependence, we obtain

$$S(t) = \mathrm{e}^{-\varepsilon A t/\alpha} S_0 + \int_0^t \mathrm{e}^{-\varepsilon A(t-\tau)/\alpha}(aS + b)(\tau)d\tau. \tag{3.8}$$

To prove Lemma 3.1, define

$$M_S(t) = \sup_{0 \leqq \tau \leqq t} \|S(\tau)\|_{H^1} \mathrm{e}^{\varepsilon\lambda\tau/2\alpha}\mathrm{e}^{-\varepsilon\lambda t/2\alpha}, \qquad M_b(t) = \sup_{0 \leqq \tau \leqq t} \|b(\tau)\|_{L^1}. \tag{3.9}$$

As in equation (4.2) in [8], define

$$K(t) := \begin{cases} (2\mathrm{e}t)^{-1/2} & \text{for } t < 1/2\lambda, \\ \lambda^{1/2}\mathrm{e}^{-\lambda t} & \text{for } t > 1/2\lambda, \end{cases} \tag{3.10}$$

and note that

$$\int_0^\infty K(t)\mathrm{e}^{\lambda t/2}\, dt = \widetilde{K}. \tag{3.11}$$

To estimate the H_1 norm of S, we begin by estimating the H_1 norm of the integral

$$\int_0^t \mathrm{e}^{-\varepsilon A(t-\tau)/\alpha} b(\tau)\, d\tau \tag{3.12}$$

in (3.8). First, compute the inner product

$$\left(\int_0^t \mathrm{e}^{-\varepsilon A(t-\tau)/\alpha} b(\tau)\, d\tau, \phi_n\right) = \int_0^t \mathrm{e}^{-\varepsilon\lambda_n(t-\tau)/\alpha}(b, \phi_n)\, d\tau, \tag{3.13}$$

where the self-adjointness of Λ has been used. Recalling definition (3.9), we easily calculate that

$$\left|\left(\int_0^t e^{-\varepsilon\Lambda(t-\tau)/\alpha} b(\tau)\,d\tau, \phi_n\right)\right| \leqq \frac{\alpha\sqrt{2}}{\varepsilon\lambda_n} M_b(t). \tag{3.14}$$

Multiplying both sides by $\sqrt{\lambda_n}$, recalling the definition (3.1), and substituting this estimate in (3.7) yields

$$\left\| \int_0^t e^{-\varepsilon\Lambda(t-\tau)/\alpha} b(\tau)\,d\tau \right\|_{H_1} \leqq \frac{\alpha\widetilde{C}}{2\varepsilon} M_b(t). \tag{3.15}$$

Next, take the H_1 norm on both sides of (3.8). Using Lemma 4.1 in [8], recalling the bound (2.5) satisfied by u and the resulting bound satisfied by a, and using (3.15) yields

$$\|S(t)\|_{H^1} \leqq e^{-\varepsilon\lambda t/\alpha}\|S_0\|_{H^1} + L\varepsilon^{-1}\int_0^t K(\varepsilon(t-\tau)/\alpha)\|S(\tau)\|_{L^2}\,d\tau + \frac{\alpha\widetilde{C}}{2\varepsilon} M_b(t). \tag{3.16}$$

Let $T > 0$, and let $0 \leqq t \leqq T$. Repeating the argument of [8, p. 924] that uses (3.9), (3.10), (3.11), (3.1), as well as $\|S(t)\|_{L^2} \leqq \|S(t)\|_{H^1} \leqq M_S(t)$, we obtain

$$e^{\varepsilon\lambda t/2\alpha}\|S(t)\|_{H^1} \leqq \|S_0\|_{H^1} + e^{\varepsilon\lambda T/2\alpha}\left(\frac{\alpha}{\varepsilon^2} L\tilde{K} M_S(T) + \frac{\alpha\tilde{C}}{2\varepsilon} M_b(T)\right), \tag{3.17}$$

for $0 \leqq t \leqq T$. Finally, multiplying (3.17) by $e^{-\varepsilon\lambda T/2\alpha}$ and choosing $\alpha < \varepsilon^2/2L\tilde{K}$, we easily obtain

$$\|S(T)\|_{H^1} \leqq 2\left(\|S_0\|_{H^1} e^{-\varepsilon\lambda T/2\alpha} + \frac{\alpha\tilde{C}}{2\varepsilon} M_b(T)\right). \tag{3.18}$$

Since $T > 0$ is arbitrary, Lemma 3.1 follows. □

As in [8], the main point of view underpinning the analysis will be that for small α, the bound on S from Lemma 3.1 and the linearity of (2.1) in S allow us to consider the evolution of $u = (\sigma, Z)$ as given *approximately* by the quadratic system (2.4) for each $x \in [-1, 0]$ independently. We will make extensive use of the following notion of a δ-*approximate solution*. This notion was already introduced in [8] to study the general problem of convergence to equilibrium for solutions of system (1.1), by constructing a "semi-Morse decomposition" of the phase space $[-1, 0] \times R^2$, based on ideas of C. Conley [2].

Definition 3.2. For $\delta > 0$ and x fixed, a δ-*approximate solution* of the quadratic system (2.4) is a C^1-function $w : [0, \infty) \mapsto \mathscr{R}^2$ such that

$$|\dot{w} - G(x, w)| \leqq \delta, \qquad 0 \leqq t < \infty, \tag{3.19}$$

where $|\cdot|$ denotes a vector norm in $\boldsymbol{R}^2$.

An important step now is to identify suitable neighborhoods of the stable critical points $\bar{u}(x)$ given by the top-jumping solution in (2.8), which can trap δ-approximate solutions of (2.4) *uniformly in x*. Special attention is required for values of x near x_M, where the stable node $A(x)$ coalesces with the saddle $B(x)$. Our construction, based on the development in [8], is carried out in Section 4, establishing the following result.

Proposition 3.3. *Let $\bar{u}(x)$ be given by* (2.8), *so* $(0, \bar{u}(x))$ *is the top-jumping steady state solution of system* (1.1). *There exist positive constants δ_1, $\bar{K}$ and K such that if $0 < \delta < \delta_1$ and $x \in [-1, 0]$ with $x \notin [x_M, x_M + \bar{K}\delta]$, then there exists a closed neighborhood $N(x,\delta)$ of $\bar{u}(x)$ such that*

(i) *$N(x,\delta)$ is positively invariant for any δ-approximate solution of* (2.4).
(ii) *$|G(x,u)| \leqq K\delta$ for all $u \in N(x,\delta)$.*
(iii) *If $|u - \bar{u}(x)| \leqq \delta/K$, then $u \in N(x,\delta)$.*
(iv) *$|u - \bar{u}(x)| \leqq K\sqrt{\delta}$ for all $u \in N(x,\delta)$.*

Our stability proof will involve a simple continuation argument relying on the *a priori* bounds implied by the following result.

Proposition 3.4. *There exist positive constants α_0, η_0, and K_* such that the following hold. Let (S,u) be a solution of* (2.1), (1.2), (1.3) *with the regularity properties* (1.8) *or* (1.9) *of Theorem 1.1. Suppose that $0 < \alpha < \alpha_0$, $0 < \eta < \eta_0$ and $\delta = K_*\eta$. Suppose that*

$$\|S_0\|_{H_1} \leqq \eta, \quad u_0(x) \in N(x,\delta) \quad \textit{for } |x - x_M| \ \geqq \ \bar{K}\delta, \tag{3.20}$$

and further suppose that, with $t_1 \geqq 0$ given,

$$u(x,t) \in N(x, 2\delta) \quad \textit{for } |x - x_M| \ \geqq \ \bar{K}\delta, \tag{3.21}$$

for $0 \leqq t \leqq t_1$. Then

$$\|S(\cdot,t)\|_{H_1} \leqq 3\eta \quad \textit{and} \quad u(x,t) \in N(x,\delta) \quad \textit{for } |x - x_M| \ \geqq \ \bar{K}\delta, \tag{3.22}$$

for $0 \leqq t \leqq t_1$.

Proof. With reference to the definition of b in (2.3), the hypotheses imply that for $0 \leqq t \leqq t_1$,

$$\begin{aligned}\|b(\cdot,u)\|_{L^1} &\leqq \int\limits_{|x-x_M| \,\geqq\, \bar{K}\delta} |G(\cdot,u)|\,dx + \int\limits_{|x-x_M| \,\leqq\, \bar{K}\delta} |G(\cdot,u)|\,dx \\ &\leqq 2\delta K + 2\bar{K}\delta M = \tilde{K}\eta,\end{aligned} \tag{3.23}$$

where $\tilde{K} = 2(K + \bar{K}M)K_*$. Here we have used the estimate of Proposition 3.3(ii) for $G(x,u)$ in the first integral and the fact that $G(x,u)$ is bounded by a constant M for all $(x,t) \in [-1,0] \times [0,\infty)$ in the second integral. Now the result of Lemma 3.1 implies that for $0 \leqq t \leqq t_1$,

$$\|S(\cdot,t)\|_{H_1} \leqq 2\eta + \alpha\tilde{C}\tilde{K}\eta/\varepsilon. \tag{3.24}$$

This is less than 3η provided that $\alpha < \varepsilon/\tilde{C}\tilde{K}$. Now if $K_* > 3L/\varepsilon$, then

$$|H(x,u)S| \leqq \varepsilon^{-1}L\|S\|_\infty \leqq 3\eta L/\varepsilon < \delta,$$

for $(x,t) \in [-1,0] \times [0,t_1]$. It follows that for each $x \in [-1,0]$, $u(x,t)$ is a δ-approximate solution of (2.4) for $0 \leqq t \leqq t_1$. Hence $N(x,\delta)$ is positively invariant on $[0,t_1]$ for $x \notin [x_M, x_M + \bar{K}\delta]$, and the result of Proposition 3.4 follows immediately. $\square$

Now the proof of Theorem 2.3 may be concluded by using a continuation argument. First, note that if the hypotheses of Theorem 2.3 hold with η sufficiently small and $\delta = K_*\eta$, then the hypotheses of Proposition 3.4 hold with $t_1 = 0$. We claim that the conclusion (3.22) of Proposition 3.4 holds for all $t > 0$. It suffices to show that the set of $t_1 \geqq 0$ for which (3.22) holds for $0 \leqq t \leqq t_1$ is both open and closed; hence this set coincides with R^+. Clearly this set is closed, since the sets $N(x,\delta)$ are closed. To show that the set is open, suppose that (3.22) holds for $0 \leqq t \leqq t_1$. Then assumption (3.21) holds for $0 \leqq t \leqq \tilde{t}_1$, for some $\tilde{t}_1 > t_1$. But then Proposition 3.4 implies that (3.22) holds for $0 \leqq t \leqq \tilde{t}_1$, and the conclusion follows.

Now the conclusion of Theorem 2.3 follows from (3.22), the estimate from part (iv) of Proposition 3.3, and the definition $\delta = K_*\eta$.

4. Construction of Positively Invariant Neighborhoods

In this section we prove Proposition 3.3, constructing the neighborhoods $N(x,\delta)$ of the critical points $\bar{u}(x)$ which are positively invariant for δ-approximate solutions of (2.4) and satisfy the estimates asserted in Proposition 3.3. In fact, the appropriate neighborhoods may be found among the constructions performed in Sections 7–10 of [8] in order to study the general problem of convergence to equilibrium as $t \to \infty$. The only difference is that some of the relevant constructions near the saddle-node bifurcation were described in Section 10 of [8] for the case when $B(x)$ and $C(x)$ coalesce, which corresponds here to studying the bottom-jumping solution. Here, similar constructions are needed when $A(x)$ and $B(x)$ coalesce.

The construction of $N(x,\delta)$ breaks down into two cases, depending on whether $N(x,\delta)$ is far from or near the saddle-node bifurcation which occurs at $x = x_M$. In the first case, let $h > 0$, and suppose that

$$x \in \mathscr{I}_h := [-1, x_M) \cup [x_M + h, 0]. \tag{4.1}$$

(The number h will be fixed in analyzing the second case.) For such x, write $P(x) = \bar{u}(x) = A(x)$ or $C(x)$ as appropriate. As shown in [8], the critical point $P(x)$ is stable for system (2.4), uniformly for $x \in \mathscr{I}_h$. Indeed, in this case we define $N(x,\delta)$ exactly as in (8.3) of [8], namely as $N(x,\delta) = N_P(x,\delta)$ where $N_P(x,\delta)$ has the form

$$N_P(x,\delta) = \{u = P(x) + w \mid (w^T D w)^{1/2} \leqq K_a\delta\}. \tag{4.2}$$

Here $D = D_P(x)$ is a certain positive definite matrix, such that D and D^{-1} are uniformly bounded for $x \in \mathscr{I}_h$, and $K_a > 0$ is a suitable constant. (See Lemma 7.2 and equations (7.11) and (8.3) of [8].) From these facts, the bounds asserted in parts (ii)–(iv) of Proposition 3.3 are evident. The statement of part (i) for this case, namely that there exists $\delta_1 > 0$ such that $N(x,\delta)$ is positively invariant for δ-approximate solutions of (2.4) if $0 \leqq \delta < \delta_1$ and $x \in \mathscr{I}_h$, follows immediately from Lemma 8.1 of [8].

In the second case we have $x_M \leqq x \leqq x_M + h$ and $\bar{u}(x) = A(x)$. To analyze this case, we make a change of variables performed in [8] using the center-manifold theorem applied at the degenerate critical point $(x,u) = (x_M, A(x_M)) = (x_M, B(x_M))$ in (2.4). Summarizing the results of Lemmas 7.4 and 7.5 in [8], we have

Proposition 4.1. *On some neighborhood N_* of the point $(u,x) = (A(x_M), x_M)$, there exists a C^k ($k \geqq 3$ is fixed) invertible map $(u,x) \mapsto (w_1, w_2, y) := W(u,x)$, where $y = x - x_M$, such that*

(i) $W(A(x_M), x_M) = (0,0,0)$.
(ii) *W^{-1} is defined for $|W|_\infty := \max(|w_1|, |w_2|, |y|) \leqq h_0$, for some $h_0 > 0$.*
(iii) *There exists $\delta_2 > 0$ such that if $0 \leqq \delta < \delta_2$ and u is a δ-approximate solution of (2.4) in N_*, then the transformed variables $(w_1(t), w_2(t), y(t)) := W(u,x)$ satisfy the system of ordinary differential equations*

$$\begin{aligned} w_{1t} &= \lambda_1 w_1 + g_1(w_1, w_2, y) + f_1(t), \\ w_{2t} &= w_2^2 - a(w_2, y)y + g_2(w_1, w_2, y) + f_2(t), \\ y_t &= 0, \end{aligned} \tag{4.3}$$

where $\lambda_1 < 0$ is one eigenvalue of the linearization of (2.4) at the point $(x_M, A(x_M))$ (the other being zero), and where the functions g_1, g_2 and a are in C^k, and the following hold for $|W|_\infty \leqq h_0$, for some constant $K_0 > 0$ independent of δ:

$$\begin{aligned} |g_1(W)| &\leqq K_0 |w_1||W|_\infty, \qquad a_2 := a(0,0) > 0, \\ |g_2(W)| &\leqq K_0 |w_1||W|_\infty, \qquad |f_1(t)| + |f_2(t)| < K_0\delta. \end{aligned} \tag{4.4}$$

For convenience, the notation in (4.3) is slightly altered from that in [8]; here, g_1 replaces F_1 and $-a$ replaces a. By taking h_0 smaller if necessary, we may suppose further that

$$K_0 h_0 < \min(\tfrac{1}{2}, \tfrac{1}{2}a_2, \tfrac{1}{4}|\lambda_1|), \tag{4.5}$$

and that for $|W|_\infty \leqq h_0$ we have

$$\frac{1}{2}a_2 \leqq \frac{\partial}{\partial y}(a(w_2,y)y) \leqq 2a_2, \qquad \frac{\partial^2}{\partial w_2^2}(w_2^2 - ay) \geqq 1. \tag{4.6}$$

In the local coordinates described in Proposition 4.1, the center manifold at $W = 0$ is simply given by $w_1 = 0$. For $\delta = 0$ ($f_1 = f_2 = 0$), and for small $y > 0$, the system (4.3) has the two critical points $W^A = (0, w_2^A, y)$ and $W^B = (0, w_2^B, y)$ corresponding to the two small roots of $w_2^2 - a(w_2, y)y = 0$. We have $w_2^A < 0 < w_2^B$ and $W^A = W(A(x), x)$, $W^B = W(B(x), x)$.

For a solution of (4.3) with $|W|_\infty \leqq h_0$, from (4.4) it follows that if $|w_1| \geqq K_1\delta$ where $\frac{1}{4}|\lambda_1|K_1 = K_0$, then

$$|w_1|_t \leqq \lambda_1|w_1| + K_0h_0|w_1| + K_0\delta \leqq \tfrac{1}{2}\lambda_1|w_1| < 0. \tag{4.7}$$

Furthermore, it is easy to verify that

$$|w_2^2 - ay| \geqq |w_1| + K_0\delta \quad \text{implies} \quad \operatorname{sgn} w_{2t} = \operatorname{sgn}(w_2^2 - ay). \tag{4.8}$$

At this point we fix $h > 0$, requiring that

$$h \leqq h_0, \quad 2a_2h \leqq \tfrac{1}{2}h_0^2, \quad \text{and} \quad h|\partial a/\partial w_2|^2 \leqq \tfrac{1}{4}a_2 \text{ for } |W|_\infty \leqq h_0. \tag{4.9}$$

Then for $0 \leqq y \leqq h$, using (4.6) we find that taking $w_2 = -h_0$ gives

$$w_2^2 - ay \geqq h_0^2 - 2a_2h \geqq \tfrac{1}{2}h_0^2 > 0, \tag{4.10}$$

and taking $w_2 = 0$ gives

$$w_2^2 - ay \leqq -\tfrac{1}{2}a_2y < 0. \tag{4.11}$$

Let $K_2 = 2(K_0 + K_1)$ and define $\bar{K} = 4K_2/a_2$. From (4.6) and the conditions on h in (4.9), it follows that provided $0 < \bar{K}\delta \leqq y \leqq h$, we have $K_2\delta \leqq \frac{1}{4}a_2y \leqq \frac{1}{2}h_0^2$, hence

$$\{w_2 \in (-h_0, 0) \mid |w_2^2 - ay| \leqq K_2\delta\} = [\gamma_1, \gamma_2] \tag{4.12}$$

for some interval $[\gamma_1, \gamma_2] \subset (-h_0, 0)$, depending on x and δ.

At this point, for $0 < \bar{K}\delta \leqq y = x - x_M \leqq h$ we define

$$N(x, \delta) = \{u \in N_* \mid \gamma_1(x, \delta) \leqq w_2 \leqq \gamma_2(x, \delta) \text{ and } \||w_1| \leqq K_1\delta\}. \tag{4.13}$$

From (4.8), if $0 < \bar{K}\delta \leqq y \leqq h$ and $|w_1| \leqq K_1\delta$, it follows that

$$w_{2t} > 0 \text{ if } w_2 = \gamma_1, \quad w_{2t} < 0 \text{ if } w_2 = \gamma_2. \tag{4.14}$$

Combined with (4.7), this implies that $N(x, \delta)$ is positively invariant for δ-approximate solutions of (2.4). The flow in the phase plane for (2.4) near the saddle-node bifurcation, for small $x - x_M > 0$, is indicated in Fig. 6.

From this construction, it is straightforward to verify the estimates asserted in parts (ii) and (iii) of Proposition 3.3. For some constant $\hat{K}$, the right-hand side of (4.3) is bounded by $\hat{K}\delta$ for all $u \in N(x, \delta)$; since the change of variables $u \mapsto W$ is in C^k, part (ii) follows. For some constant $K > 0$ we have $|w_2^2 - ay| \leqq K|w_2 - w_2^A|$ whenever $|W|_\infty \leqq h_0$, so $w_2 \in [\gamma_1, \gamma_2]$ provided $|w_2 - w_2^A| \leqq K_2\delta/K$, and part (iii) follows.

To deduce part (iv), let $F(w_2, y) = w_2^2 - a(w_2, y)y$. For $0 < \bar{K}\delta \leqq y \leqq h$, (4.6) implies that F is a convex function of w_2, so if $w_2 \in [\gamma_1, \gamma_2]$, then

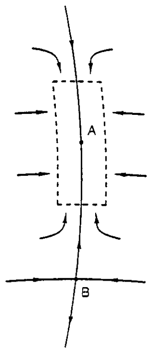

Figure 6. Phase plane of system (2.4) near the saddle-node bifurcation.

$$\frac{\partial}{\partial w_2}F(w_2,y) \leqq \frac{\partial}{\partial w_2}F(\gamma_2,y) = 2\gamma_2 - \frac{\partial}{\partial w_2}a(\gamma_2,y)y. \tag{4.15}$$

Because h satisfies (4.9), and $K_2\delta \leqq \frac{1}{4}\, a_2 y$, we infer that

$$\left|\frac{\partial}{\partial w_2}a(\gamma_2,y)y\right|^2 \leqq \tfrac{1}{4}\, a_2 y \leqq a(\gamma_2,y)y - K_2\delta = \gamma_2^2, \tag{4.16}$$

and therefore $\partial F/\partial w_2(\gamma_2,y) \leqq \gamma_2 \leqq -\sqrt{K_2\delta}$. Now it follows that

$$2K_2\delta = F(\gamma_1,y) - F(\gamma_2,y) \geqq -(\gamma_2-\gamma_1)\frac{\partial}{\partial w_2}F(\gamma_2,y) \geqq \sqrt{K_2\delta}(\gamma_2-\gamma_1), \tag{4.17}$$

and therefore

$$w_2 \in [\gamma_1,\gamma_2] \quad \text{implies} \quad |w_2 - w_2^A| \leqq |\gamma_2-\gamma_1| \leqq 2\sqrt{\delta/K_2}. \tag{4.18}$$

This implies part (iv), and finishes the proof of Proposition 3.3.

5. Proof of Theorem 2.6

Let $(0, u^0(x,t))$ be the smooth solution to system (1.1) with $\alpha = 0$ and $\bar{f} > \bar{T}_M$, corresponding to initial data $S_0(x) = 0$, $u_0(x) = 0$, for $-1 \leqq x \leqq 0$. Recall that $u^0(x,t) \to \bar{u}(x)$ as $t \to \infty$, where $\bar{u}$ is the piecewise smooth, top-jumping steady solution given in (2.8) with a single jump discontinuity at $x_* = x_M$. Furthermore, given any $\eta > 0$, there exists a time $T_1 > 0$ such that

$$|u^0(x,T_1) - \bar{u}(x)| \leqq \tfrac{1}{2}\,\eta \quad \text{for } |x - x_M| \geqq \bar{C}\eta. \tag{5.1}$$

Here the constant $\bar{C}$ is taken from the conclusion of Theorem 2.3. Let α_0, η_0, C_0 also be taken from Theorem 2.3. Given any $\beta > 0$, in what follows we fix $\eta > 0$ such that $\bar{C}\eta < \beta$ and $0 < \eta < \eta_0$.

Next, for T_1 as above, use Theorem 3.3 in [8] to fix $\alpha_1 > 0$, where $\alpha_1 \leqq \alpha_0$, such that if $0 < \alpha < \alpha_1$, the solution (S, u) to system (2.1), satisfying the boundary conditions (1.2) and initial conditions (1.3), satisfies

$$\|S(\cdot, T_1)\|_{H^1} \leqq \eta \quad \text{and} \quad |u(x, T_1) - u^0(x, T_1)| \leqq \tfrac{1}{2}\eta \quad \text{for} \quad -1 \leqq x \leqq 0. \tag{5.2}$$

Notice that (5.1) and (5.2) imply that

$$|u(x, T_1) - \bar{u}(x)| \leqq \eta \quad \text{for } |x - x_M| \geqq \bar{C}\eta. \tag{5.3}$$

Now we may apply Theorem 2.3 to the solution (S, u) for $t \geqq T_1$. With $S(\cdot, T_1)$ and $u(\cdot, T_1)$ taken as initial data, the hypotheses in (2.9) hold, and we may conclude that for all $t \geqq T_1$,

$$\|S(\cdot, t)\|_{H^1} \leqq C_0\eta \quad \text{and} \quad |u(x, t) - \bar{u}(x)| \leqq C_0\sqrt{\eta} \quad \text{for } |x - x_M| \geqq \bar{C}\eta. \tag{5.4}$$

In fact, we conclude from the proof of Theorem 2.3 that for $|x - x_M| \geqq \bar{C}\eta$, $u(x, t) \in N(x, \delta)$ for all $t \geqq T_1$, where the positively invariant sets $N(x, \delta)$ are described in Proposition 4.1 and $\delta = K_*\eta$, the constant K_* being taken from Proposition 3.4.

Finally, we apply Theorem 3.5 in [8] to conclude that (by taking α_1 smaller if necessary) for $0 < \alpha < \alpha_1$, the solution (S, u) converges asymptotically as $t \to \infty$ to an equilibrium $(0, u^\infty(x))$. Since for $-1 \leqq x < x_M$ the only critical point of (2.4) is $\bar{u}(x) = C(x)$, and for $x_M + \bar{C}\eta \leqq x \leq 0$ the only critical point in $N(x, \delta)$ is $\bar{u}(x) = A(x)$, it follows that $u^\infty(x) = \bar{u}(x)$ if $x \notin [x_M, x_M + \bar{C}\eta)$. Since $\bar{C}\eta < \beta$, this completes the proof.

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