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A Homogenized Model for Vortex Sheets

Yann Brenier

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1. Introduction

The motion of an incompressible inviscid fluid moving in a bounded domain of Euclidean space can be described, in a classical way, by the evolution of vortex sheets separating areas where the velocity field is smooth. In this paper, we consider an ideal situation when these areas are so intricate that, at a macroscopical level, each of them can be seen as a phase occupying a positive portion of the physical space. The corresponding homogenized model can be described as follows. A finite number $M > 1$ of fluid 'phases', labelled by $\alpha = 1, \ldots, M$, moving in a bounded domain D of $\mathbb{R}^d$ (or in the periodic box $D = \mathbb{R}^d/\mathbb{Z}^d$) are considered. Each 'phase' has a density field $\rho_\alpha(t,x) > 0$ (where $t \geqq 0$ and $x \in D$ denote the time and space variables) and a velocity field $v_\alpha(t,x)$. All phases have the same pressure field $p(t,x)$ and they obey the following set of equations

$$\sum_\alpha \rho_\alpha = 1,$$

$$\partial_t \rho_\alpha + \nabla . m_\alpha = 0,$$

where

$$m_\alpha = \rho_\alpha v_\alpha$$

is supposed to be parallel to the boundary ∂D, and

$$\partial_t v_\alpha + (v_\alpha . \nabla) v_\alpha + \nabla p = 0.$$

Here we use the notations

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \nabla = (\partial_1, \ldots, \partial_d), \quad a.b = \sum_{j=1}^d a_j b_j$$

for $a, b \in \mathbb{R}^d$. In the present paper, we consider only *potential* velocity fields

$$v_\alpha = \nabla \Phi_\alpha(t,x),$$

and we substitute Bernoulli's law

$$\partial_t \Phi_\alpha + \tfrac{1}{2}|\nabla \Phi_\alpha|^2 + p = 0,$$

for the momentum equation. This choice is consistent, since curl-free velocity fields are preserved by the original equations. Indeed, in the case $d = 3$ for example, the vorticity field ω_α satisfies a conservation law

$$\partial_t \omega_\alpha + \nabla.(\omega_\alpha \otimes v_\alpha - v_\alpha \otimes \omega_\alpha) = 0,$$

which enforces, at least formally, that $\omega_\alpha = 0$ at all positive time t once it holds true at time $t = 0$. These 'homogenized vortex sheet equations' (HVSEs) partly correspond to the 'équations relaxées' due to Duchon & Robert [DR] (and coincide, in the special case $d = M = 2$). The HVSEs can also be considered as a crude model for multiphase flows or quasineutral plasmas, for which we refer respectively to [CP, Appendix A] and [Gr]. They also appear in a natural way in the variational theory of the Euler equations as discussed in [Br3]. Notice that the classical Euler equations formally correspond to the limit case when each 'phase density' ρ_α takes values in $\{0, 1\}$ and is the characteristic function of one of the M simply-connected components of the portion of D delimited by the vortex sheets of the fluid.

In the present paper, there is no attempt to provide a rigorous derivation of the HVSEs from the Euler equations (this is only briefly discussed in Appendix 3 at the end of the paper). The main focus of interest is the mathematical analysis of the HVSEs, and the word 'homogenized' is loosely used, without technical implication. Let us consider the *initial-value problem*, when ρ_α and $\nabla \Phi_\alpha$ are prescribed at $t = 0$. The (formal) conservation in time of the kinetic energy, defined by

$$K = \frac{1}{2} \sum_\alpha \int_D \rho_\alpha |\nabla \Phi_\alpha|^2 dx,$$

directly follows from the HVSEs. One of our results will show that a uniform bound on the kinetic energy is sufficient to get a kind of weak non-linear stability provided each ρ_α stays bounded away from zero. However, from a more classical point of view, the initial-value problem can be (and usually is) linearly ill posed. For instance, assume that $M = 2$ and that the initial conditions depend only on the first spatial coordinate $s = x_1$. We get from the HVSEs that $m_1 + m_2$ is a constant and can be supposed to be 0, which implies that $m_1 m_2 \leqq 0$. Then, we can reduce the system to

$$\partial_t v_1 + \partial_s\left(\tfrac{1}{2} v_1^2 + v_1 v_2\right) = 0,$$
$$\partial_t v_2 + \partial_s\left(\tfrac{1}{2} v_2^2 + v_1 v_2\right) = 0,$$

where

$$m_\alpha = \rho_\alpha v_\alpha,$$

which is hyperbolic when $v_1 v_2 > 0$, but elliptic when $v_1 v_2 < 0$. Therefore, from the classical point of view, one cannot expect much more than a local existence theorem for analytic initial data. Such a result can be found in [Gr] as a byproduct of Grenier's work on quasineutral plasmas, in the case when $D = \mathbb{R}^d/\mathbb{Z}^d$. More precisely,

Proposition 1.1. *Assume that the values of $\rho_\alpha > 0$ and Φ_α at time $t = 0$ are prescribed analytic functions on $\mathbb{R}^d/\mathbb{Z}^d$. Then, there is a positive time T such that the initial-value problem for the HVSEs has a continuous solution in time on $[0, T]$ with values in a suitable set of analytic functions and ρ_α stay bounded away from zero.*

Due to their ill-posedness, the HVSEs do not provide a fully satisfactory homogenized model for vortex sheet motions. However, they have interesting mathematical properties that will be discussed in the present paper. In particular, we will obtain *a priori* results on solutions such that

$$r(t) = \inf_{x\alpha} \rho_\alpha(t, x)$$

stay bounded away from 0 on a time interval $[0, T]$. We then say that the solution is 'strongly homogenized', in complete contrast with the classical vortex sheet solutions, when each ρ_α takes values in $\{0, 1\}$ (as discussed in Section 1). Our main result reads as follows (in the case $D = \mathbb{R}^d/\mathbb{Z}^d$):

Theorem 1.2. *Let (ρ_α, v_α) be a strongly homogenized solution of the HVSEs on the time interval $[0, T]$. Then the kinetic energy K is constant in time, KT^2 is bounded by a constant depending only on $D = \mathbb{R}^d/\mathbb{Z}^d$ and for each $\delta \in]0, T/2[$, there is a constant C_δ such that*

$$\int_\delta^{T-\delta}\int_D \left[|\nabla p| + \sum_\alpha \rho_\alpha \Big(|\partial_t v_\alpha|^2 + |\nabla v_\alpha|^2 \Big) \right] \leqq (1 + K) C_\delta.$$

In particular, the strongly homogenized solutions have a finite life span and, therefore, the local analytic solutions mentioned above cannot be global in time. We believe that this phenomenon is of some interest. Our results are indirectly obtained through an analysis of the *time boundary-value problem*, when a final time $T > 0$ is prescribed as well as the value of ρ_α at time $t = 0$ and $t = T$. For this problem, we show a general existence result with a particular concept of so-called 'variational' solutions. They are obtained by minimizing the kinetic energy in space time, which means that, just as the Euler equations [AK, CM, MP, MT, Se, Shn, Br2, Shn2, ...], the HVSEs can be obtained from a variational principle. Then, the strongly homogenized solutions of the HVSEs are characterized as those variational solutions for which the lower bound $r(t)$ stay bounded away from 0. For these solutions, we show various properties, including Sobolev time and space regularity and, with the help of recent tools in transport theory [DL], uniqueness and stability for the time boundary-value problem.

The paper is organized as follows. In the second section, we provide different definitions of solutions. In the third section, we prove the existence of variational solutions for the time boundary-value problem by using classical tools of convex analysis and previous results of mine. In the four last sections, Sobolev regularity, uniqueness and stability results are established for the strongly homogenized solutions.

2. Homogenized, Variational and Strongly Homogenized Solutions

In this section, different notions of solutions are introduced for the HVSEs. We denote by Q the cylinder $]0,T[\times D$, where $T>0$ is fixed. Typically, D is the periodic cube $\mathbb{R}^d/\mathbb{Z}^d$, but some of our proofs are also valid for (or can be extended to) piecewise smooth bounded domains in $\mathbb{R}^d$, which will be explicitly mentioned.

Definition 2.1. (ρ_α, m_α) is a *homogenized solution* of the HVSEs, if

$$0 \leqq \rho_\alpha \in L^\infty(Q), \quad m_\alpha = \rho_\alpha v_\alpha, \quad v_\alpha \in L^2(Q, \rho_\alpha dt\, dx), \tag{1}$$

$$\sum_a \rho_\alpha = 1, \quad \partial_t \rho_\alpha + \nabla . m_\alpha = 0, \tag{2}$$

and there are locally integrable functions Φ_α and p such that

$$\partial_t \Phi_\alpha,\ |\nabla \Phi_\alpha|^2 \ \in L^1(Q, \rho_\alpha\ dt\ dx), \quad p \in L^1(Q,\ dt\ dx), \tag{3}$$

$$\rho_\alpha > 0 \text{ a.e.}, \quad v_\alpha = \nabla \Phi_\alpha \text{ a.e.}, \tag{4}$$

$$\partial_t \Phi_\alpha + \tfrac{1}{2}|\nabla \Phi_\alpha|^2 + p = 0 \text{ a.e.} \tag{5}$$

Remark. Notice that (5) holds in the almost-everywhere sense (as in Kruzhkov's theory of convex Hamilton-Jacobi equations, for which we refer to [Li]). From (1), (2), it follows that $m_\alpha \in L^2(Q)$ and

$$\rho_\alpha \in C^{1/2}([0,T], H^{-1}(D)). \tag{6}$$

Thus the value of $\rho_\alpha(t,.)$ is well defined for all $t \in [0,T]$, in particular for $t=0$ and $t=T$, and time boundary conditions for ρ_α are meaningful.

Definition 2.2. (ρ_α, m_α) is a *variational solution* of the HVSEs if (1), (2) are satisfied and if, for each $\varepsilon > 0$, there are functions $\Phi_{\alpha\varepsilon} \in C^1(\bar{Q}),\ p_\varepsilon \in C^0(\bar{Q})$ such that

$$\int_Q \rho_\alpha |v_\alpha - \nabla \Phi_{\alpha\varepsilon}|^2\ dx\ dt \leqq \varepsilon, \tag{7}$$

$$\int_Q \rho_\alpha |\partial_t \Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla \Phi_{\alpha\varepsilon}|^2 + p_\varepsilon|^2\ dx\ dt \leqq \varepsilon, \tag{8}$$

$$\partial_t \Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla \Phi_{\alpha\varepsilon}|^2 + p_\varepsilon \leqq 0. \tag{9}$$

Remark. This definition looks awkward since a limiting process involving an ε-dependent family is used. We could have introduced ad hoc weighted Sobolev spaces, according to the measure $\rho_\alpha dt\, dx$, but since nothing special is assumed about the vanishing properties of ρ_α, such spaces would have been essentially useless. Anyway, our definition turns out to be very natural in the framework of time boundary-value problems.

Proposition 2.3. *If there is a constant $r > 0$ such that*

$$\rho_\alpha(t,x) \geqq r \ \ a.e.\ \ (t,x) \in Q, \tag{10}$$

for all $\alpha = 1, \ldots, M$, then the notions of homogenized and variational solutions coincide. In that case, (ρ_α, m_α) is called a strongly homogenized solution (SHS) of the HVSEs.

Proof. Let us show that a homogenized solution is a variational solution if (10) holds. The converse part will follow from the results of Section 4. Here D can be either the torus $\mathbb{R}^d/\mathbb{Z}^d$ or a convex domain in $\mathbb{R}^d$ containing a ball of radius R centered at the origin. Let us introduce a nonnegative mollifier ζ on $\mathbb{R} \times \mathbb{R}^d$, with unit mass and support included in $]-1, +1[\, \times B$, where B denotes the unit open ball in $\mathbb{R}^d$. Then, for each $\varepsilon > 0$, we set

$$\Phi_{\alpha\varepsilon}(t,x) = \iint \Phi_\alpha((1-\eta_1)t + \eta_1 T\tau, (1-\eta_2)x + \eta_2 R\xi)\zeta(\tau,\xi)\ d\tau\ d\xi, \tag{11}$$

where $\eta_1, \eta_2 \in]0,1[$ will be defined later. Since D is convex (or is the torus), $\Phi_{\alpha\varepsilon}$ is well defined. By Jensen's inequality, this smooth function satisfies

$$\begin{aligned}&|\nabla \Phi_{\alpha\varepsilon}(t,x)|^2 \\ &\quad \leqq (1-\eta_2)^2 \iint |\nabla \Phi_\alpha((1-\eta_1)t + \eta_1 T\tau, (1-\eta_2)x + \eta_2 R\xi)|^2 \zeta(\tau,\xi)\ d\tau\ d\xi.\end{aligned}$$

Thus, if we choose $0 < \eta_2 < 1,\ 1 - \eta_1 = (1-\eta_2)^2$, and set

$$p_\varepsilon(t,x) = (1-\eta_1) \iint p((1-\eta_1)t + \eta_1 T\tau, (1-\eta_2)x + \eta_2 R\xi)\zeta(\tau,\xi)\ d\tau\ d\xi, \tag{12}$$

we deduce (9), pointwise, after convolution of the almost-everywhere equality

$$\partial_t \Phi_\alpha + \tfrac{1}{2}|\nabla \Phi_\alpha|^2 + p = 0,$$

which follows from (5) because of assumption (10). Finally, we choose η_2 so that

$$\int_Q |\nabla \Phi_\alpha - \nabla \Phi_{\alpha\varepsilon}|^2\ dx\ dt \leqq \varepsilon,$$

$$\int_Q |\partial_t \Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla \Phi_{\alpha\varepsilon}|^2 + p_\varepsilon|^2 \, dx \, dt \leqq \varepsilon.$$

This is possible since p, $\partial_t \Phi_\alpha$ and $|\nabla \Phi_\alpha|^2$ belong to $L^1(Q)$ (by assumption (3)) and $\Phi_{\alpha\varepsilon}$ and p_ε are obtained by convolution of Φ_α and p. This completes the proof.

3. Existence of Variational Solutions for the Time Boundary-Value Problem

In this section we show that the time boundary-value problem always has variational solutions, characterized as minimizers for the kinetic energy.

Let us consider time boundary data $(\rho_{\alpha 0}, \rho_{\alpha T})$ defined a.e. on D, with values in $[0, 1]$ and satisfying the compatibility conditions

$$\sum_\alpha \rho_{\alpha 0} = \sum_\alpha \rho_{\alpha T} = 1, \quad \int_D \rho_{\alpha T} = \int_D \rho_{\alpha 0} \tag{13}$$

and set

$$I(\rho_0, \rho_T) = \inf \frac{1}{2} \sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \tag{14}$$

for (ρ_α, v_α) satisfying (1), (2) and

$$\rho_\alpha(0, .) = \rho_{\alpha 0}, \quad \rho_\alpha(T, .) = \rho_{\alpha T}. \tag{15}$$

The variational theory of Euler equations [Br1, Br2] provides the following result.

Theorem 3.1. *There is a constant $C = C(D)$ depending only on D such that, for any time boundary data $(\rho_{\alpha 0}, \rho_{\alpha T})$, with values in* [0,1] *and compatible with condition* (13),

$$I(\rho_0, \rho_T) \leqq CT^{-1} \tag{16}$$

and the infimum is always achieved.

The proof relies on the concept of generalized flows and is postponed until Appendix 1. The main result of this section is

Theorem 3.2. *The time boundary-value problem always has a variational solution. Moreover, a family (ρ_α, m_α) satisfying* (1), (2) *is a variational solution* (*i.e., satisfies* (7), (8), (9)) *if and only if*

$$\frac{1}{2} \sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 = I(\rho_0, \rho_T), \tag{17}$$

where

$$\rho_{\alpha 0} = \rho_\alpha(0, .), \quad \rho_{\alpha T} = \rho_\alpha(T, .).$$

An interesting consequence is

Proposition 3.3. *There is no non-trivial global variational solution for the initial-value problem corresponding to the HVSEs. In particular, there is no global strongly homogenized solution.*

Indeed, by Theorem 3.1, such a solution would satisfy, for any $T > 0$,

$$\frac{1}{2}\sum_{\alpha}\int_0^T\int_D \rho_\alpha |v_\alpha|^2 dt\, dx \leqq CT^{-1}$$

and therefore $\rho_\alpha |v_\alpha|^2 = 0$ almost everywhere. The proof of Theorem 3.2. is split into the next two subsections.

First step: a precise definition of kinetic energy. Let (ρ, m) be a pair of Borel bounded measures on $\bar{Q}$ respectively taking values in $\mathbb{R}$ and $\mathbb{R}^d$ and define

$$K(\rho, m) = \sup \int_Q (\theta\rho + w.m), \tag{18}$$

where (θ, w) is any pair of continuous function on $\bar{Q}$ respectively taking values in $\mathbb{R}$ and $\mathbb{R}^d$, subject to

$$\theta + \tfrac{1}{2}|w|^2 \leqq 0,$$

everywhere in $\bar{Q}$.

Proposition 3.4. *If ρ is a nonnegative measure and m is absolutely continuous with respect to ρ with a Radon-Nikodym derivative $v \in L^2(Q, \rho)$, then*

$$K(\rho, m) = \int_Q \tfrac{1}{2}|v|^2 \rho. \tag{19}$$

Otherwise $K(\rho, m) = +\infty$.

Proof. (i) If ρ is not nonnegative, there is $\theta_0 \in C^0(\bar{Q})$ such that $\theta_0 \geqq 0$ and $\int_Q \theta_0 \rho < 0$. Then, for each $n > 0$, we set $\theta = -n\theta_0$ and $w = 0$. Thus

$$K(\rho, m) \geqq \int_Q \theta\rho = -n \int_Q \theta_0 \rho \to +\infty$$

when $n \to +\infty$.

(ii) If m is not absolutely continuous with respect to ρ, then, for every $n > 0$, there is $w \in C^0(\bar{Q}; \mathbb{R}^d)$ such that

$$\int_Q |w|^2 \rho \leqq 1, \quad \int_Q w.m \geqq n.$$

Then, with $\theta = -\frac{1}{2}|w|^2$,

$$K(\rho, m) \geqq n - \tfrac{1}{2} \to +\infty.$$

(iii) If $m = v\rho$ for some $v \in L^1(Q, \rho)$, then

$$K(\rho, m) = \sup \int_Q (w.v - \tfrac{1}{2}|w|^2)\rho,$$

for all $v \in C^0(\bar{Q}; \mathbb{R}^d)$ with values in $\mathbb{R}^d$, and

$$K(\rho, m) = \int_Q \tfrac{1}{2}|v|^2 \rho \ \in \ [0, +\infty]$$

immediately follows, which completes the proof.

Second step: the duality principle. In this subsection we prove Theorem 3.2 by using classical tools of convex analysis. Let us assume that $(\rho_{\alpha 0}, \rho_{\alpha T})$ are fixed, For any family $(\rho_\alpha,\ m_\alpha)$, the constraints (1), (2) and the boundary conditions (15) can be expressed in integral form by

$$\int_Q (\partial_t f \rho_\alpha + \nabla f . m_\alpha) = \int_D (\rho_{\alpha T}(x) f(T, x) - \rho_{\alpha 0}(x) f(0, x))\, dx, \tag{20}$$

for all $f \in C^1(\bar{Q})$ and

$$\int_Q \Big(\sum_\alpha \rho_\alpha - 1 \Big) q = 0 \tag{21}$$

for all $q \in C^0(\bar{Q})$.

Notice that the simple combination of $\rho_\alpha \geqq 0$ (which automatically follows from $K(\rho_\alpha, m_\alpha) < +\infty$) and (21) (which means that $\sum_\alpha \rho_\alpha$ is the Lebesgue measure) implies that each ρ_α is absolutely continuous with respect to the Lebesgue measure and can be considered as a Lebesgue nonnegative measurable function with values in [0,1].

According to Theorem 3.1, there is a family $(\overline{\rho}_\alpha, \overline{m}_\alpha)$ such that (20) and (21) are satisfied together with $K(\overline{\rho}_\alpha, \overline{m}_\alpha) < +\infty$. (Otherwise $I(\rho_0, \rho_T)$ would be infinite.) Thus, we can rephrase (20) and (21) as

$$\int_Q [(\rho_\alpha - \overline{\rho}_\alpha)\partial_t f + (m_\alpha - \overline{m}_\alpha).\nabla f] = 0, \tag{22}$$

for all $f \in C^1(\bar{Q})$ and

$$\sum_\alpha \int_Q (\rho_\alpha - \overline{\rho}_\alpha) q = 0 \tag{23}$$

for all $q \in C^0(\bar{Q})$, where $(\rho_\alpha,\ m_\alpha)$ belongs to the dual space E' of

$$E = (C^0(\bar{Q}; R) \times C^0(\bar{Q}; \mathbb{R}^d))^M.$$

So we are looking for a solution $(\rho_\alpha, m_\alpha) \in E'$ of

$$\sum_\alpha K(\rho_\alpha, m_\alpha) = I(\rho_0, \rho_T), \tag{24}$$

subject to (22) and (23).

To study this minimization problem, it is convenient to introduce the following lower semicontinuous convex functions on E with values in $]-\infty, +\infty]$,

$$A(a,b) = 0 \quad \text{if} \quad a_\alpha + \tfrac{1}{2}|b_\alpha|^2 \leqq 0, \tag{25}$$

and $A(a,b) = +\infty$ otherwise, for all $(a,b) \in E$,

$$B(a,b) = \sum_\alpha \int_Q (\overline{\rho}_\alpha a_\alpha + \overline{m}_\alpha . b_\alpha) \tag{26}$$

if there are $\Phi_\alpha \in C^1(\bar{Q})$ and $p \in C^0(\bar{Q})$ such that

$$a_\alpha + \partial_t \Phi_\alpha + p = 0, \quad b_\alpha + \nabla \Phi_\alpha = 0, \tag{27}$$

and $B(a,b) = +\infty$ otherwise. The Legendre-Fenchel-Moreau transforms of A and B are respectively given by

$$\begin{aligned} A^*((\rho_\alpha, m_\alpha)) &= \sup \left\{ \sum_\alpha \int_Q (\rho_\alpha a_\alpha + m_\alpha . b_\alpha); \quad a_\alpha + \tfrac{1}{2}|b_\alpha|^2 \leqq 0 \right\} \\ &= \sum_\alpha K(\rho_\alpha, m_\alpha) \end{aligned} \tag{28}$$

(as follows from definition (18)) and

$$B^*((\rho_\alpha, m_\alpha)) = \sup \sum_\alpha \int_Q [(\rho_\alpha - \overline{\rho}_\alpha) a_\alpha + (m_\alpha - \overline{m}_\alpha) . b_\alpha] \tag{29}$$

where $(a_\alpha, b_\alpha) \in E$ are subject to (27). Thus

$$\begin{aligned} B^*((\rho_\alpha, m_\alpha)) = \sup \Bigg\{ & \sum_\alpha \int_Q [(\overline{\rho}_\alpha - \rho_\alpha)(\partial_t \Phi_\alpha + p) + (\overline{m}_\alpha - m_\alpha) . \nabla \Phi_\alpha]; \\ & p \in C^0(\bar{Q}), \quad \Phi_\alpha \in C^1(\bar{Q}) \Bigg\}, \end{aligned} \tag{30}$$

which is 0 if (22) and (23) are satisfied, and $+\infty$ otherwise. It follows from (29), (30) that

$$I(\rho_0, \rho_T) = \inf\{A^*((\rho_\alpha, m_\alpha)) + B^*((\rho_\alpha, m_\alpha)); \ (\rho_\alpha, m_\alpha) \in E'\}. \tag{31}$$

Both A and B are convex, lower semicontinuous functions from E into $]-\infty, +\infty]$. Moreover, there is a point (a_α, b_α), namely

$$a_\alpha = -1, \ b_\alpha = 0,$$

where B is continuous (for the sup norm) and A is finite. Thus, we can use Rockafellar's duality theorem, as stated in [Brez], and infer that

$$I(\rho_0, \rho_T) = \min\{A^*((\rho_\alpha, m_\alpha)) + B^*((\rho_\alpha, m_\alpha)); \quad (\rho_\alpha, m_\alpha) \in E'\}$$
$$= \sup\{-A(a, b) - B(-a, -b); \quad (a, b) \in E\}. \tag{32}$$

More concretely, we get

$$I(\rho_0, \rho_T) = \sup \sum_\alpha \int_Q (\overline{\rho}_\alpha(\partial_t \Phi_\alpha + p) + \overline{m}_\alpha . \nabla \Phi_\alpha) \tag{33}$$

where $p \in C^0(\bar{Q})$, $\Phi_\alpha \in C^1(\bar{Q})$ are subject to

$$\partial_t \Phi_\alpha + \tfrac{1}{2}|\nabla \Phi_\alpha|^2 + p \leqq 0.$$

Notice that Rockafellar's theorem ensures that the infimum in (32) is achieved by some solution (ρ_α, m_α).

Such a solution is fully characterized by (22)–(24), which precisely means (because of (33)) that, for each $\varepsilon > 0$, there are $p_\varepsilon \in C^0(\bar{Q})$ and $\Phi_{\alpha\varepsilon} \in C^1(\bar{Q})$ such that

$$\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon + \tfrac{1}{2}|\Phi_{\alpha\varepsilon}|^2 \leqq 0,$$

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \leqq \sum_\alpha \int_Q (\overline{\rho}_\alpha(\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon) + \overline{m}_\alpha . \nabla \Phi_{\alpha\varepsilon}) + \varepsilon.$$

Because of (22) and (23), the last inequality means that

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \leqq \sum_\alpha \int_Q (\rho_\alpha(\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon) + m_\alpha . \nabla \Phi_{\alpha\varepsilon}),$$

or, equivalently,

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha - \nabla \Phi_{\alpha\varepsilon}|^2 + \sum_\alpha \int_Q \rho_\alpha |\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon + \frac{1}{2}|\nabla \Phi_{\alpha\varepsilon}|^2| \leqq \varepsilon,$$

which exactly means that (ρ_α, m_α) is a variational solution. This completes the proof of Theorem 3.2.

4. Some Preliminary Properties of Variational Solutions

In this section, useful information on variational solutions is obtained through Lagrangian deformations of the density and impulsion fields. For a general discussion on variational principles and the rigorous derivation of the equations of fluid mechanics equation, we refer to [Se].

Theorem 4.1. *Let* $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ *be a variational solution. Then*

$$\sum_\alpha \left[\partial_t(\rho_\alpha v_{\alpha i}) + \sum_j \partial_j(\rho_\alpha v_{\alpha j} v_{\alpha i})\right] + \partial_i p = 0 \tag{34}$$

holds in the distributional sense on Q where p is a distribution uniquely defined (*up to a distribution depending on t only*) *by the pressure equation*

$$-\Delta p = \sum_{\alpha ij} \partial^2_{ij}(\rho_\alpha v_{\alpha j} v_{\alpha i}). \tag{35}$$

The proof is split into four subsections.

First step: Lagrangian variations. Let γ be a smooth map from $\bar{Q}$ into $\overline{D}$ such that for all $t \in [0, T],\ x \to \gamma(t,x)$ is a diffeomorphism of D, there is $\delta_0 \in]0, T/2[$ such that

$$\gamma(t,x) = x \quad \forall\ t \in [0, \delta_0] \cup [T - \delta_0, T] \tag{36}$$

and $t \to \gamma(t, .)$ takes values in a small neighborhood of the identity map for the C^0 topology. Such non-trivial maps exist and can be constructed in the following way. Let δ be a small parameter. Let h be a compactly supported smooth function on $]0, 1[$ and w be a compactly supported smooth vector field on D. Then, we set

$$\gamma(t,x) = \exp_{\delta h(t)}(w)(x), \tag{37}$$

for all $t \in [0, 1],\ x \in D$, where $t \to \exp_t(w)$ denotes the flow associated with w through the ordinary differential equation $dx/dt = w(x)$ such that $\exp_0(w)(x) = x$. We will use this type of map in some of the next subsections. Let us now consider a diffeomorphism η of $[0, T]$ such that $\eta(t) - t$ is compactly supported in $]0, T[$. We denote by Γ the pair (γ, η).

If (ρ_α, m_α) is a variational solution, we first denote

$$\begin{aligned} \rho^\eta_\alpha(t,x) &= \rho_\alpha(\eta(t),x), \\ m^\eta_\alpha(t,x) &= m_\alpha(\eta(t),x)\eta'(t), \\ v^\eta_\alpha(t,x) &= v_\alpha(\eta(t),x)\eta'(t) \end{aligned}$$

and then define a Lagrangian variation $(\rho^\Gamma_\alpha, m^\Gamma_\alpha)$ by setting, for all $f \in L^1(Q),\ g \in L^2(Q)$, respectively taking values in $\mathbb{R}$ and $\mathbb{R}^d$,

$$\int_Q f(t,x)\rho^\Gamma_\alpha(t,x)\ dx\ dt = \int_Q f(t, \gamma(t,x))\rho^\eta_\alpha(t,x)\ dx\ dt, \tag{38}$$

$$\begin{aligned} &\int_Q \sum_{j=1}^d g_j(t,x) m^\Gamma_{\alpha j}(t,x)\ dx\ dt \\ &\quad = \int_Q \sum_{j=1}^d g_j(t, \gamma(t,x))[\partial_t \gamma_j(t,x)\rho^\eta_\alpha(t,x) + \sum_{k=1}^d \partial_k \gamma_j(t,x) m^\eta_{\alpha k}(t,x)]\ dx\ dt. \end{aligned} \tag{39}$$

Notice that

Proposition 4.2. *ρ_α^Γ and m_α^Γ are well defined respectively in $L^\infty(Q)$ and $L^2(Q)$, and satisfy the continuity equation*

$$\partial_t \rho_\alpha^\Gamma + \nabla . m_\alpha^\Gamma = 0, \tag{40}$$

as well as the time boundary conditions

$$\rho_\alpha^\Gamma(t,.) = \rho_\alpha(t,.), \quad t = 0, T. \tag{41}$$

More precisely, we get, through straightforward computations, the integral formulation of (40), (41), namely

$$\begin{aligned} &\int_Q \partial_t f(t,x)\rho_\alpha^\Gamma(t,x) + \nabla f(t,x).m_\alpha^\Gamma(t,x)\, dx\, dt \\ &= \int_D (f(T,x)\rho_\alpha(T,x) - f(0,x)\rho_\alpha(0,x))\, dx \end{aligned} \tag{42}$$

for all $f \in C^1(\bar{Q})$. (Use the chain rule to compute

$$\int_Q \frac{d}{dt}[\, f(t,\gamma(t,x))]\rho_\alpha^\eta(t,x)\, dt\, dx$$

as well as definitions (38), (39).)

Second step: a variational estimate.

Proposition 4.3. *Let (ρ_α, m_α) be a variational solution. Then, for any Lagrangian variation Γ,*

$$\begin{aligned} &\int_Q (\rho_\alpha^\Gamma - \rho_\alpha)p_\varepsilon + \frac{1}{2}\int_Q \rho_\alpha^\eta |\partial_t \gamma + D\gamma.v_\alpha^\eta - \nabla \Phi_{\alpha\varepsilon} \circ \gamma|^2 \\ &\leqq \varepsilon + \frac{1}{2}\int_Q \rho_\alpha^\eta |\partial_t \gamma + D\gamma.v_\alpha^\eta|^2 - \frac{1}{2}\int_Q \rho_\alpha |v_\alpha|^2. \end{aligned} \tag{43}$$

In the special case when $\eta(t) = t$, a summation with respect to α yields

$$\begin{aligned} &\int_Q (p_\varepsilon \circ \gamma - p_\varepsilon) + \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha - \nabla \Phi_{\alpha\varepsilon} \circ \gamma|^2 \\ &\leqq \varepsilon M + \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha|^2 - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2, \end{aligned} \tag{44}$$

and, in particular,

$$\int_Q (p_\varepsilon \circ \gamma - p_\varepsilon) \leqq \varepsilon M + \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha|^2 - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2. \tag{45}$$

Proof. Using definitions (38), (39), we get the following identities, where we use the notation $(D\gamma)_{kj} = \partial_k \gamma_j$,

$$\begin{aligned}\int_Q (\rho_\alpha^\Gamma - \rho_\alpha)\partial_t \Phi_{\alpha\varepsilon} &= -\int_Q (m_\alpha^\Gamma - m_\alpha).\nabla\Phi_{\alpha\varepsilon} \\ &= -\int_Q \nabla\Phi_{\alpha\varepsilon}\circ\gamma.(\rho_\alpha^\eta \partial_t\gamma + D\gamma.m_\alpha^\eta) + \int_Q m_\alpha.\nabla\Phi_{\alpha\varepsilon} \\ &= \frac{1}{2}\int_Q \rho_\alpha^\eta|\partial_t\gamma + D\gamma.v_\alpha^\eta - \nabla\Phi_{\alpha\varepsilon}\circ\gamma|^2 - \frac{1}{2}\int_Q \rho_\alpha^\eta|\partial_t\gamma + D\gamma.v_\alpha^\eta|^2 \\ &\quad -\frac{1}{2}\int_Q \rho_\alpha^\eta|\nabla\Phi_{\alpha\varepsilon}\circ\gamma|^2 + \int_Q m_\alpha.\nabla\Phi_{\alpha\varepsilon}.\end{aligned}$$

We also have

$$\int_Q (\rho_\alpha^\Gamma - \rho_\alpha)|\nabla\Phi_{\alpha\varepsilon}|^2 = \int_Q \rho_\alpha^\eta|\nabla\Phi_{\alpha\varepsilon}\circ\gamma|^2 - \int_Q \rho_\alpha|\nabla\Phi_{\alpha\varepsilon}|^2.$$

Thus,

$$\begin{aligned}&\int_Q \left(\rho_\alpha^\Gamma - \rho_\alpha\right)\left(\partial_t\Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla\Phi_{\alpha\varepsilon}|^2\right) \\ &\quad = \frac{1}{2}\int_Q \rho_\alpha^\eta|\partial_t\gamma + D\gamma.v_\alpha^\eta - \nabla\Phi_{\alpha\varepsilon}\circ\gamma|^2 - \frac{1}{2}\int_Q \rho_\alpha^\eta|\partial_t\gamma + D\gamma.v_\alpha^\eta|^2 \\ &\quad\quad -\frac{1}{2}\int_Q \rho_\alpha|\nabla\Phi_{\alpha\varepsilon} - v_\alpha|^2 + \frac{1}{2}\int_Q \rho_\alpha|v_\alpha|^2.\end{aligned}$$

Since (ρ_α, m_α) is a variational solution, we get from (8) and (9) that

$$\varepsilon \geqq \int_Q \left(\rho_\alpha^\Gamma - \rho_\alpha\right)\left(p_\varepsilon + \partial_t\Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla\Phi_{\alpha\varepsilon}|^2\right).$$

(Notice that we use both (8) and (9) to get this estimate.) We also have

$$\frac{1}{2}\int_Q \rho_\alpha|\nabla\Phi_{\alpha\varepsilon} - v_\alpha|^2 \leqq \varepsilon.$$

This completes the proof.

Third step: existence of a pressure field. In this subsection, we prove the weak compactness of the family (p_ε). By using a refined version of Moser's Lemma due to Dacorogna & Moser [DM], as in [Br2], it is possible to find a $C^{k,\delta}$-type norm $|||.|||$ on $\bar{Q}$ and a constant $\varepsilon_0 > 0$ so that one can associate with any real-valued function $\beta \in C_c^\infty(Q)$ satisfying $|||\beta||| \leqq \varepsilon_0$ and

$$\int_D \beta(t,x)\, dx = 0 \quad \forall\, t \in [0,T], \tag{46}$$

a map γ satisfying (36),

$$\int_Q f(t,\gamma(t,x))\, dx\, dt = \int_Q f(t,x)(1+\beta(t,x))\, dt\, dx \quad \forall\, f \in L^1(Q), \tag{47}$$

$$\|\partial_t \gamma\|_{L^\infty(Q)} + \|D\gamma - I\|_{L^\infty(Q)} \leqq C|||\beta|||,$$

where I denotes the identity matrix and $C = C(D)$ is a constant depending only on D.

The result of Proposition (4.3), namely (45), shows that

$$\int_Q p_\varepsilon \beta \leqq \varepsilon M + \tfrac{1}{2}\sum_\alpha \int_Q \rho_\alpha(|\partial_t\gamma + D\gamma.v_\alpha|^2 - |v_\alpha|^2).$$

Thus,

$$\int_Q p_\varepsilon \beta \leqq \varepsilon M + C|||\beta|||\left(1 + \sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2\right), \tag{48}$$

provided that $|||\beta|||$ is small enough. This proves that (p_ε) is bounded in the distributional sense, provided we perform the normalization

$$\int_Q p_\varepsilon(t,x)\, dx = 0 \quad \forall\, t \in [0,T], \tag{49}$$

which is possible (by subtracting out of the $\Phi_{\alpha\varepsilon}$ a fixed function of t). Thus (p_ε) is weakly compact and has a cluster point p.

Fourth step: completion of proof of Theorem 4.1. Let us consider a convergent subsequence of (p_ε) (still labelled by ε). We can now pass to the limit in inequality (45) for all maps γ satisfying (36). We have

$$\int_Q (p_\varepsilon \circ \gamma - p_\varepsilon) = \langle p_\varepsilon, \beta_\gamma \rangle$$

where $\langle .,. \rangle$ denotes the distribution bracket and $1 + \beta_\gamma$ is the (smooth) inverse Jacobian determinant of γ implicitly defined by

$$\int_Q f(t,x)(1 + \beta_\gamma(t,x))\, dt\, dx = \int_Q f \circ \gamma \tag{50}$$

for all smooth functions f. Thus, from (45) we get

$$\langle p, \beta_\gamma \rangle \leqq \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha|^2 - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2. \tag{51}$$

Let us now consider γ of the form (37) and let δ go to zero. Since $\gamma(t,x) = \exp_{\delta h(t)}(w)(x)$, we deduce, through straightforward Taylor expansions with respect to $\delta \to 0$, that

$$|\beta_\gamma - \delta h \nabla . w| \leqq C\delta^2,$$

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma . v_\alpha|^2 - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \tag{52}$$

$$\leqq \delta \sum_\alpha \int_Q \rho_\alpha [h v_\alpha . (Dw . v_\alpha) + h' w . v_\alpha] + C\delta^2 \sum_\alpha \int_Q \rho_\alpha (1 + |v_\alpha|^2),$$

where $C = C(h, w)$ depends only on h and w. It follows that

$$\langle p, h\nabla . w\rangle = \sum_\alpha \int_Q \rho_\alpha [h v_\alpha . (Dw . v_\alpha) + h' w . v_\alpha], \tag{53}$$

$$\begin{aligned} &\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma . v_\alpha|^2 - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \\ &\qquad \leqq \delta \langle p, h\nabla . w\rangle + C\delta^2 \sum_\alpha \int_Q \rho_\alpha (1 + |v_\alpha|^2). \end{aligned} \tag{54}$$

Notice that the uniqueness of p implies that the entire family (p_ε) weakly converges to p, provided that p is properly normalized. This completes the proof of Theorem 4.1.

A second variational estimate. Let us now get back to inequality (44) in the case when γ is of the form (37) and w is divergence-free. The right-hand side of (44) can be estimated thanks to (54), which yields

Proposition 4.4. *Let (ρ_α, m_α) be a variational solution and let $\gamma(t, .) = \exp_{\delta h(t)} w$ where w is divergence-free. Then*

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma . v_\alpha - \nabla \Phi_{\alpha\varepsilon} \circ \gamma|^2 \leqq \varepsilon M + C\delta^2, \tag{55}$$

where

$$C = C(h, w) \sum_\alpha \int_Q \rho_\alpha (1 + |v_\alpha|^2).$$

5. Regularity of Strongly Homogenized Solutions

In this section, it is shown, when $D = \mathbb{R}^d/\mathbb{Z}^d$, that strongly homogenized solutions (SHSs) have regularity properties, namely, that the time and space derivatives of the velocity fields belong to $L^2_{\mathrm{loc}}([0, T], L^2(D))$.

Theorem 5.1. *Let* $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ *be a variational solution of the HVSEs such that*

$$\rho_\alpha(t,x) \geqq r$$

a.e. for some constant $r > 0$. *Then the kinetic energy*

$$K = \frac{1}{2}\sum_\alpha \int_D \rho_\alpha |\nabla\Phi_\alpha|^2$$

is time independent and

$$v_\alpha = \nabla\Phi_\alpha, \quad \partial_t\Phi_\alpha + \tfrac{1}{2}|\nabla\Phi_\alpha|^2 + p = 0$$

hold a.e. in Q, *where*

$$\partial_t\Phi_\alpha, \nabla\Phi_\alpha \in L^2_{\mathrm{loc}}(]0,T[, H^1(D)), \qquad \nabla p \in L^1_{\mathrm{loc}}(]0,T[, L^1(D)). \tag{56}$$

Moreover, for each α,

$$\partial_t(\rho_\alpha v_{\alpha i}) + \sum_j \partial_j(\rho_\alpha v_{\alpha i} v_{\alpha j}) + \rho_\alpha \partial_i p = 0 \tag{57}$$

holds in the distributional sense and, for each $\delta_0 \in]0, T/2[$, *there is a constant* $C = C(\delta_0)$ *such that*

$$\int_{\delta_0}^{T-\delta_0} \int_D \left[|\nabla p| + \sum_\alpha \rho_\alpha (|\partial_t v_\alpha|^2 + |Dv_\alpha|^2) \right] \leqq (1+K)C. \tag{58}$$

Remark. This result confirms that a variational solution is 'strongly homogenized' provided that $\rho_\alpha(t,x) \geqq r$ holds a.e. for some constant $r > 0$ and completes the proof of Proposition 2.3.

Proof of Theorem 5.1.

First step: Let (ρ_α, m_α) be a variational solution. It follows from (7), (8) that

$$\nabla\Phi_{\alpha\varepsilon} \to v_\alpha \text{ in } L^2(Q),$$

$$\partial_t\Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla\Phi_{\alpha\varepsilon}|^2 + p_\varepsilon \to 0 \quad \text{in } L^1(Q).$$

Since $p_\varepsilon \to p$ in the distributional sense, as shown in the previous subsection, this shows that

$$\partial_t\Phi_{\alpha\varepsilon} \to -p - \tfrac{1}{2}|v_\alpha|^2$$

in the distributional sense.This implies the existence of a distribution Φ_α such that

$$\Phi_{\alpha\varepsilon} \to \Phi_\alpha,$$

$$\partial_t\Phi_\alpha + \tfrac{1}{2}|\nabla\Phi_\alpha|^2 + p = 0. \tag{59}$$

Second step: an estimate for the time derivatives. Let us consider a degenerate variation Γ when $\gamma(t,x) = x$. Then $\rho_\alpha^\Gamma = \rho_\alpha^\eta$, $m_\alpha^\Gamma = m_\alpha^\eta$ and

$$\langle \rho_\alpha^\eta - \rho_\alpha, p\rangle + \frac{1}{2}\int_Q \rho_\alpha^\eta |v_\alpha - v_\alpha^\eta|^2 \leqq \frac{1}{2}\int_Q \rho_\alpha^\eta |v_\alpha^\eta|^2 - \frac{1}{2}\int_Q \rho_\alpha |v_\alpha|^2$$

follows from (43), once $\varepsilon \to 0$. In particular,

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha^\eta |v_\alpha - v_\alpha^\eta|^2 \leqq \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha^\eta |v_\alpha^\eta|^2 - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2,$$

since $\sum_\alpha \rho_\alpha^\eta = \sum_\alpha \rho_\alpha = 1$ almost everywhere. From

$$\sum_\alpha \int_Q \rho_\alpha^\eta |v_\alpha^\eta|^2 = \sum_\alpha \int_Q \rho_\alpha(\eta(t),x)|v_\alpha(\eta(t),x)|^2 \eta'(t)^2 \, dt\, dx$$
$$= \sum_\alpha \int_Q \rho_\alpha(t,x)|v_\alpha(t,x)|^2 \eta'(\eta^{-1}(t), \, dt\, dx,$$

we deduce that

$$\sum_\alpha \int_Q \rho_\alpha^\eta |v_\alpha - v_\alpha^\eta|^2 \leqq \sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 (\eta' \circ \eta^{-1} - 1).$$

By choosing $\eta(t) = t + \delta\theta(t)$, where $\theta \in C_c^\infty(]0,T[)$ and letting $\delta \to 0$, we infer from a straightforward Taylor expansion that

(i)
$$\frac{1}{2}\sum_\alpha \int_D \rho_\alpha(t,x)|v_\alpha(t,x)|^2 \, dx$$

is time independent (and subsequently denoted by K),
(ii) for all $\delta_0 \in]0, T/2[$,

$$\sum_\alpha \int_{\delta_0}^{T-\delta_0} \int_D \rho_\alpha(t,x)|v_\alpha(t+\delta,x) - v_\alpha(t,x)|^2 \, dx\, dt \leqq CK\delta^2,$$

where C depends only on δ_0. So, it follows that, for all SHSs, $\partial_t v_\alpha$ belongs to $L^2_{\mathrm{loc}}(]0,T[, L^2(D))$ and

$$\sum_\alpha \int_{\delta_0}^{T-\delta_0} \int_D \rho_\alpha(t,x)|\partial_t v_\alpha(t,x)|^2 \, dx\, dt \leqq C(\delta_0)K. \tag{60}$$

Third step: an estimate for the space derivatives. Here, our proof is restricted to the case of the torus $D = \mathbb{R}^d/\mathbb{Z}^d$.We set

$$\eta(t) = t, \;\; w(x) = a,$$

where a is a fixed vector of $\mathbb{R}^d$ with length 1, and we assume that $h(t) = 1$ when $0 < \delta_0 \leqq t \leqq T - \delta_0 < T$, where δ_0 is fixed in $]0, T/2[$. If we get back to inequality (44), we can pass to the limit and obtain

$$\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha - v_\alpha \circ \gamma|^2 \leqq \sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha|^2 - \sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2.$$

Indeed, w is divergence-free; thus each $\gamma(t,.)$ is Lebesgue measure-preserving and

$$\Phi_{\alpha\varepsilon} \circ \gamma \to v_\alpha \circ \gamma$$

in $L^2(Q)$. Since, we explicitly have

$$\gamma(t,x) = x + \delta h(t) a,$$

we can directly compute the different terms of this inequality and obtain, for the right-hand side

$$\begin{aligned}
&\sum_\alpha \int_Q \rho_\alpha (|\partial_t \gamma + D\gamma.v_\alpha|^2 - |v_\alpha|^2) \\
&\quad = \sum_\alpha \int_Q \rho_\alpha(t,x)(|\delta h'(t) a + v_\alpha(t,x)|^2 - |v_\alpha(t,x)|^2)\ dt\ dx \\
&\quad = \delta^2 |a|^2 \sum_\alpha \int_Q \rho_\alpha(t,x) |h'(t)|^2\ dt\ dx \\
&\quad = \delta^2 |D| \int_0^T |h'(t)|^2\ dt\ dx,
\end{aligned}$$

where $|D| = 1$ denotes the Lebesgue measure of $D = \mathbb{R}^d/\mathbb{Z}^d$. (Notice that

$$\sum_\alpha \int_Q \rho_\alpha(t,x) v_\alpha(t,x).a h'(t)\ dt\ dx = 0$$

is necessarily null, according to Proposition 4.1 and equation (34).) For the left-hand side, we have

$$\begin{aligned}
&\sum_\alpha \int_Q \rho_\alpha |\partial_t \gamma + D\gamma.v_\alpha - v_\alpha \circ \gamma|^2 \\
&\quad = \sum_\alpha \int_Q \rho_\alpha(t,x) |\delta h'(t) a + v_\alpha(t,x) - v_\alpha(t, x + \delta h(t) a)|^2 dt\, dx.
\end{aligned}$$

Finally we get

$$\sum_\alpha \int_Q \rho_\alpha(t,x) |h(t) D v_\alpha(t,x).a - h'(t) a|^2\ dt\ dx \leqq \int_0^T |h'(t)|^2\ dt\ dx \tag{61}$$

for all a in the unit sphere and all h compactly supported in $]0,T[$, which leads to

$$\int_{\delta_0}^{T-\delta_0}\int_D \rho_\alpha |Dv_\alpha|^2 \leqq C(\delta_0),$$

and shows that $v_\alpha \in L^2_{\text{loc}}(]0,T[,H^1(D))$.

Fourth step: regularity of the pressure field. Since (ρ_α, m_α) is an SHS, we have

$$\partial_t \Phi_\alpha + \tfrac{1}{2}|\nabla\Phi_\alpha|^2 + p = 0$$

almost everywhere, which, after differentiating with respect to x, leads to

$$\partial_t v_{\alpha i} + \sum_j \partial_j(v_{\alpha i}v_{\alpha j}) = -\partial_i p \tag{63}$$

since $v_\alpha = \nabla\Phi_\alpha$. Thus, we deduce from estimates (60) and (62) that $\nabla p \in L^1_{\text{loc}}(L^1(D))$. In addition, since $\sum_\alpha \rho_\alpha = 1$, we get

$$-\partial_i p = \sum_\alpha \rho_\alpha \left(\partial_t v_{\alpha i} + \sum_j \partial_j (v_{\alpha i} v_{\alpha j})\right),$$

which shows that, for all $\delta_0 \in]0, T/2[$,

$$\int_{\delta_0}^{T-\delta_0} |\nabla p| \leqq \sum_\alpha \int_{\delta_0}^{T-\delta_0}\int_D \rho_\alpha(|\partial_t v_\alpha| + 2|v_\alpha||Dv_\alpha|)\, dx\, dt \leqq (1+K)C, \tag{64}$$

where K denotes the kinetic energy and $C = C(\delta_0)$ depend only on δ_0.

Moreover, since $\partial_t\Phi_\alpha, \nabla\Phi_\alpha \in L^2_{\text{loc}}(]0,T[,H^1(D))$; $\rho_\alpha \in C^{1/2}([0,T],H^{-1}(D))$; the following identities are meaningful:

$$\begin{aligned}\partial_t(\rho_\alpha \partial_i\Phi_\alpha) + \sum_j \partial_j(\rho_\alpha\partial_i\Phi_\alpha\partial_j\Phi_\alpha) &= \partial_i\Phi_\alpha\left(\partial_t\rho_\alpha + \sum_j \partial_j(\rho_\alpha\partial_j\Phi_\alpha)\right)\\ &\quad + \rho_\alpha\left(\partial_t\partial_i\Phi_\alpha + \sum_j \partial_j(\partial_i\Phi_\alpha\partial_j\Phi_\alpha)\right)\\ &= \rho_\alpha\left(\partial_t\partial_i\Phi_\alpha + \sum_j \partial_j(\partial_i\Phi_\alpha\partial_j\Phi_\alpha)\right).\end{aligned}$$

(since $\partial_t\rho_\alpha + \nabla.(\rho_\alpha\nabla\Phi_\alpha) = 0$)

$$= \rho_\alpha\left(\partial_t v_{\alpha i} + \sum_j \partial_j(v_{\alpha i}v_{\alpha j})\right)$$

(since $v_\alpha = \nabla\Phi_\alpha$)

$$= -\rho_\alpha\partial_i p$$

(because of (63)), which shows (57) and completes the proof.

Remarks. (i) Notice that the estimate on ∇p given by (64) does not involve the lower bound r of the density fields.
(ii) The regularity result for p can be improved by using Sobolev's embedding theorem. Indeed

$$v_\alpha = \nabla \Phi_\alpha \in L^2_{\text{loc}}(]0, T[, L^s(D)), \tag{65}$$

for $s = 2^*$, where $1 - d/2 = -d/2^*$ if $d > 2$, and for all $s < +\infty$ otherwise. Then, we find that ∇p belongs to $L^r_{\text{loc}}(Q)$ for $r = d/(d-1)$ when $d \geqq 2$ and all $r < 2$ if $d = 2$.

6. Uniqueness of Strongly Homogenized Solutions

In this section, we prove a uniqueness result for strongly homogenized solutions:

Theorem 6.1. *Let (ρ_α, m_α) be a strongly homogenized solution. Then there is no other variational solution $(\rho'_\alpha, m'_\alpha)$ such that*

$$\rho'_\alpha(0,.) = \rho_\alpha(0,.), \quad \rho'_\alpha(T,.) = \rho_\alpha(T,.). \tag{66}$$

Proof. Let $(\rho'_\alpha, m'_\alpha = \rho'_\alpha v'_\alpha)$ be a variational solution such that (66) holds. Then, by Theorem (3.2.), both $(\rho'_\alpha, m'_\alpha)$ and (ρ_α, m_α) are minimizers for $I(\rho_0, \rho_T)$. So, for each $\varepsilon > 0$, there are p_ε and $\Phi_{\alpha\varepsilon}$ such that

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 = \frac{1}{2}\sum_\alpha \int_Q \rho'_\alpha |v'_\alpha|^2 = I(\rho_0, \rho_T)$$

$$\leqq \sum_\alpha \int_Q \overline{\rho}_\alpha((\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon) + \overline{m}_\alpha . \nabla \Phi_{\alpha\varepsilon}) + \varepsilon,$$

which implies (just as in the last subsection) that

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha - \nabla \Phi_{\alpha\varepsilon}|^2 + \sum_\alpha \int_Q \rho_\alpha |\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon + \tfrac{1}{2}|\nabla \Phi_{\alpha\varepsilon}|^2| \leqq \varepsilon,$$
$$\frac{1}{2}\sum_\alpha \int_Q \rho'_\alpha |v'_\alpha - \nabla \Phi_{\alpha\varepsilon}|^2 + \sum_\alpha \int_Q \rho'_\alpha |\partial_t \Phi_{\alpha\varepsilon} + p_\varepsilon + \tfrac{1}{2}|\nabla \Phi_{\alpha\varepsilon}|^2| \leqq \varepsilon \ .$$

Since $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ is strongly homogenized, it follows that

$$\nabla \Phi_{\alpha\varepsilon} \to v_\alpha$$

in $L^2(Q)$ and, a fortiori, in $L^2(Q, \rho'_\alpha \, dt \, dx)$. Thus

$$v_\alpha = v'_\alpha \quad \rho'_\alpha\text{-a.e.} \tag{67}$$

in Q and $\rho'_\alpha v_\alpha = \rho'_\alpha v'_\alpha$ holds Lebesgue almost everywhere in Q. Therefore, ρ'_α satisfies

$$\partial_t \rho'_\alpha + \nabla.(\rho'_\alpha v_\alpha) = 0 \tag{68}$$

in the distributional sense. Now, we observe that, by Theorem 5.1., v_α belongs to $L^2_{\mathrm{loc}}(]0,T[,H^1(D))$ and $L^2(Q)$ and that

$$\partial_t \rho_\alpha + \nabla.(\rho_\alpha v_\alpha) = 0$$

holds in the distributional sense with

$$0 < r \leqq \rho_\alpha(t,x) \leqq 1$$

almost everywhere in Q. This is enough, as shown in Appendix 2, by using the tools of the DiPerna & Lions theory on ordinary differential equations [DL], to show that equation (68) has a unique solution once $\rho_\alpha(0,.)$ is prescribed. Thus,

$$\rho'_\alpha = \rho_\alpha$$

holds almost everywhere in Q, which, together with (67), completes the proof.

7. Stability of Strongly Homogenized Solutions

In this section, we prove that uniformly strongly homogenized solutions are stable with respect to weak convergence.

Theorem 7.1. *Let* $(\rho^n_\alpha, m^n_\alpha = \rho^n_\alpha v^n_\alpha)$ be *a sequence of strongly homogenized solutions on* Q *such that*

$$0 < r \leqq \rho^n_\alpha$$

holds a.e. on Q*, where* $r > 0$ *is a fixed constant. Then there is a strongly homogenized solution* $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ *and a subsequence, still labelled by* n*, such that*

$$\rho^n_\alpha \to \rho_\alpha, \quad v^n_\alpha \to v_\alpha,$$

respectively in $L^\infty(Q)$ *weak-* and strongly in* $L^2_{\mathrm{loc}}(L^2(D))$.

Proof. According to Theorem 5.1 and estimate (58), for all $\delta_0 \in]0, T/2[$, there is a constant $C = C(\delta_0) > 0$ such that

$$\sum_\alpha \int_{\delta_0}^{T-\delta_0} \int_D \rho^n_\alpha (|\partial_t v^n_\alpha|^2 + |Dv^n_\alpha|^2)\, dx\, dt \leqq (1 + K^n)C,$$

where the kinetic energy K^n is bounded by $C(D)T^{-1}$, according to Theorem 3.1.

Thus (up to a subsequence extraction), v^n_α strongly converges to v_α in $L^2_{\mathrm{loc}}(L^2(D))$. Since

$$r \leqq \rho^n_\alpha \leqq 1,$$

we also have

$$\rho^n_\alpha \to \rho_\alpha$$

in $L^\infty(Q)$ weak-*. Thus

$$\partial_t \rho_\alpha + \nabla.(\rho_\alpha v_\alpha) = 0, \quad \sum_\alpha \rho_\alpha = 1,$$

immediately follow from

$$\partial_t \rho^n_\alpha + \nabla.(\rho^n_\alpha v^n_\alpha) = 0, \quad \sum_\alpha \rho^n_\alpha = 1.$$

Since $(\rho^n_\alpha, m^n_\alpha)$ are strongly homogenized, and therefore variational solutions, there are p^n and Φ^n_α, respectively in $C^0(\bar{Q})$ and $C^1(\bar{Q})$ such that

$$\partial_t \Phi^n_\alpha + p^n + \tfrac{1}{2}|\Phi^n_\alpha|^2 \leqq 0,$$

$$\frac{1}{2}\sum_\alpha \int_Q \rho^n_\alpha |v^n_\alpha - \nabla\Phi^n_\alpha|^2 + \sum_\alpha \int_Q \rho^n_\alpha |\partial_t \Phi^n_\alpha + p^n + \tfrac{1}{2}|\nabla \Phi^n_\alpha|^2| \to 0,$$

which implies that

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha - \nabla\Phi^n_\alpha|^2 + \sum_\alpha \int_Q \rho_\alpha |\partial_t \Phi^n_\alpha + p^n + \tfrac{1}{2}|\nabla \Phi^n_\alpha|^2| \to 0.$$

Thus $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ is a variational solution, which completes the proof.

8. Appendix 1: Generalized Flows and Variational Solutions

In this subsection, we prove Theorem 3.1 by using some results obtained in [Br1, Br2] with the concept of generalized flows. It is shown, in addition, that the variational solutions (and, in particular, the strongly homogenized solutions) are minimizers not only in the framework of Theorem 5.1, but also in the framework of generalized flows.

8.1. A Review of Generalized Flows

The proof of Theorem 3.1 is based on the concept of generalized flow [Br1, Br2, Shn2] related to Young's measures for which we refer to [Ta, Yo]. I recently discovered that Shelukhin [She] introduced this concept earlier (in a slightly different context). A generalized flow μ is a bounded nonnegative Borel measureon the set $\overline{D}^{[0,T]}$ of all paths z with values in $\overline{D}$. This infinite product space is compact (by Tychonov's theorem) for the product topology, and the set of all bounded Borel measures does not differ from the dual space of all continuous function $z \to f(z)$ equipped with the sup-norm. In particular, for all integer k, all sequence $0 \leqq t_1 < \cdots < t_k \leqq T$, and all $f \in C^0(\overline{D}^k)$

$$\int f(z(t_1), \ldots, z(t_k))\mu(dz)$$

is well defined. Let us now introduce

$$e(z) = \frac{1}{2}\int_0^T |z'(t)|^2 \, dt,$$

if $z \in H^1([0,T];\overline{D})$, with $e(z) = +\infty$ otherwise, which defines a lower semicontinuous (and therefore Borel measurable) function $\overline{D}^{[0,T]} \to [0,+\infty]$. Only generalized flows μ such that

$$\int e(z)\mu(dz) < \infty$$

are considered. Each sequence (μ_n) of generalized flows such that

$$\sup_n \int e(z)\mu_n(dz) < \infty$$

has a subsequence, still denoted by (μ_n), converging to a generalized flow μ, for the weak-* topology, which means that for all integer k, all sequence $0 \leqq t_1 < \cdots < t_k \leqq T$, and all $f \in C^0(\overline{D}^k)$

$$\int f(z(t_1),\ldots,z(t_k))\mu_n(dz) \to \int f(z(t_1),\ldots,z(t_k))\mu(dz).$$

With each generalized flow μ, and for each $t \in [0,T]$, we can associate the projection $\mu|_t$ defined by

$$\int_{\overline{D}} f(x)\mu|_t(dx) = \int f(z(t))\mu(dz), \tag{69}$$

for all $f \in C^0(\bar{Q})$.

Subsequently, we consider only generalized flows such that μ_t is absolutely continuous with respect to Lebesgue measure; the density is denoted by $\rho(t,.)$.

Then, we associate with μ a vector-valued Borel measure m on $\bar{Q}$ defined by

$$\int_Q g.m = \int \left(\int_0^T z'(t).g(t,z(t)) \, dt\right)\mu(dz) \tag{70}$$

for all $g \in C^0(\bar{Q};\mathbb{R}^d)$, and we check (just as in Proposition 3.4) that m is absolutely continuous with respect to $\rho \, dt \, dx$ and can be written

$$m = \rho v, \quad v \in L^2(Q, \rho \, dt \, dx),$$

and

$$\frac{1}{2}\int_Q \rho|v|^2 \leqq \int e(z)\mu(dz). \tag{71}$$

(Let us sketch the proof: For each path $z \in H^1([0,T];\overline{D})$, each smooth function g and each integer $N > 0$, we introduce

$$I_N(z) = \sum_{k=1}^{N}\Big(z\Big(\frac{k}{N}T\Big) - z\Big(\frac{k-1}{N}T\Big)\Big)\cdot g\Big(\frac{k}{N}T, z\Big(\frac{k}{N}T\Big)\Big),$$

$$I(z) = \int_0^T z'(t).g(t, z(t))\, dt.$$

By the mean value theorem, we easily get

$$|I_N(z) - I(z)| \leqq CN^{-1/2}(1 + e(z)),$$

where C depends only on T and g. Thus I is (square) integrable with respect to μ and

$$\left|\int I(z)\mu(dz)\right|^2 \leqq 2\int e(z)\mu(dz)\int_Q |g|^2\rho.$$

This shows that there is a unique function $v \in L^2(Q, \rho dt\, dx)$ such that

$$\int I(z)\mu(dz) = \int_Q g.v\rho, \quad \int_Q |v|^2\rho \leqq \int 2e(z)\mu(dz).$$

An easy consequence of definitions (69), (70) is

$$\int_Q (\rho\partial_t f + m.\nabla f) = \int_D (\rho(T,.)f(T,.) - \rho(0,.)f(0,.)) \tag{72}$$

for all $f \in C^1(\bar{Q})$, since for all path $z \in H^1([0,T];\overline{D})$ (and therefore μ almost every path z),

$$f(T, z(T)) - f(0, z(0)) = \int_0^T \frac{d}{dt}[f(t, z(t))]\, dt$$
$$= \int_0^T [\partial_t f(t, z(t)) + z'(t).\nabla f(t, z(t))]\, dt.$$

In particular,

$$\partial_t\rho + \nabla.m = 0$$

holds in the distributional sense, and therefore $\rho \in C^{1/2}([0,T], H^{-1}(D))$.

8.2. The Proof of Theorem 3.1

We use two measure-theoretic lemmata and a theorem from [Br1].

Lemma 8.1. *Let* $(A^0_\alpha, \alpha = 1, ..., M)$ *and* $(A^T_\alpha, \alpha = 1, ..., M)$ *be two Borel partitions of* $\overline{D}$ *such that*

$$|A_\alpha^0| = |A_\alpha^T| \quad \forall \alpha,$$

where $|.|$ denotes the Lebesgue measure. Then there is a Lebesgue measure-preserving Borel map h such that

$$h^{-1}(A_\alpha^T) = h^{-1}(A_\alpha^0) \quad \forall \alpha,$$

where, by a Lebesgue measure-preserving map, we mean that $|h^{-1}(A)| = |A|$ for all Borel subsets of $\overline{D}$.

Lemma 8.2. *Let $\sigma_1, ..., \sigma_M \in L^\infty(D; [0,1])$ such that*

$$\sum_\alpha \sigma_\alpha = 1.$$

Then, for each positive integer n, there is a Borel partition $(A_{\alpha n}, \alpha = 1, ..., M)$ of $\overline{D}$ such that

$$|A_{\alpha n}| = \int_D \sigma_\alpha \ \ \forall \alpha, n,$$

$$1_{A_{\alpha n}} \to \sigma_\alpha,$$

in $L^\infty(D)$ weak-.*

Theorem 8.3. *Let h be a Borel map from $\overline{D}$ into itself, preserving the Lebesgue measure. Then*

$$\inf \int e(z)\mu(dz)$$

is achieved among all generalized flows μ subject to

$$\mu|_t = 1 \quad \forall\ t \in [0, T],$$

$$\int f(z(0), z(T))\mu(dz) = \int_D f(x, h(x))\ dx,$$

and is bounded by CT^{-1} where $C = C(D)$ depends only on D.

Proof. The first lemma is standard (see [Roy], for instance). The proof of Theorem 8.3. is given in [Br1] when $D = \mathbb{R}^d/\mathbb{Z}^d$, and for smooth domains in [Roe]. Let us sketch the proof of the second lemma. Let us split up D in n cells D_{jn}, $j = 1, ..., n$ such that

$$\delta_n = \sup\ \text{diameter}\ (D_{jn}) \to 0, \ \ n \to +\infty.$$

Then we can split up each D_{jn} into M subcells $D_{jn\alpha}$, $\alpha = 1, \ldots, M$, so that

$$|D_{jn\alpha}| = \int_{D_{jn}} \sigma_\alpha \qquad \forall\ \alpha, n, j.$$

Finally, set

$$A_{\alpha n} = \bigcup_j D_{jn\alpha}.$$

Let us now prove Theorem 3.1. According to Lemmata 8.1, 8.2, we can associate with each positive integer n a Lebesgue measure-preserving map h_n and two Borel partitions $(A^0_{\alpha n})$, $(A^T_{\alpha n})$ of $\overline{D}$ such that

$$h_n^{-1}(A^T_{\alpha n}) = h_n^{-1}(A^0_{\alpha n}) \ \forall\alpha,$$
$$|A^T_{\alpha n}| = |A^0_{\alpha n}| = \int_D \rho_{\alpha 0} = \int_D \rho_{\alpha T},$$
$$1_{A^0_{\alpha n}} \to \rho_{\alpha 0}, \ \ 1_{A^T_{\alpha n}} \to \rho_{\alpha T}$$

in $L^\infty(D)$ weak-*. From Theorem 8.3, we infer that there is a sequence (μ_n) of incompressible generalized flows such that

$$\int e(z)\mu_n(dz) \leqq CT^{-1},$$

where $C = C(D)$ depends only on D, and

$$\int f(z(0), z(T))\mu_n(dz) = \int_D f(x, h_n(x)) \ dx$$

for all $f \in C^0(\overline{D}^2)$. For each $\alpha = 1, \ldots, M$, we define a generalized flow $\mu_{\alpha n}$ by setting

$$\int f(z(t_1), \ldots, z(t_k))\mu_{\alpha n}(dz) = \int 1_{A^0_{\alpha n}}(z(0))f(z(t_1), \ldots, z(t_k))\mu_n(dz)$$

for all integer k, all sequence $0 \leqq t_1 < \cdots < t_k \leqq T$, and all $f \in C^0(\overline{D}^k)$. We obviously have

$$\mu_{\alpha n}|_0 = 1_{A^0_{\alpha n}}, \ \ \mu_{\alpha n}|_T = 1_{A^T_{\alpha n}},$$
$$\sum_\alpha \mu_{\alpha n}|_t = 1 \ \ \forall \ t \in [0, T].$$

Each family $(\mu_{\alpha n})$ is sequentially weakly compact and has a cluster point μ_α. These μ_α satisfy

$$\sum_\alpha \int e(z)\mu_\alpha(dz) \leqq CT^{-1},$$
$$\sum_\alpha \mu_\alpha|_t = 1 \quad \forall \ t \in [0, T].$$
$$\mu_\alpha|_0 = \rho_{\alpha 0}, \ \ \mu_\alpha|_T = \rho_{\alpha T}.$$

The corresponding density and impulsion fields ρ_α, m_α defined from μ_α by (69), (70) satisfy

$$\sum_\alpha \int_Q \tfrac{1}{2}\rho_\alpha |v_\alpha|^2 \leqq \textstyle\sum_\alpha \int e(z)\mu_\alpha(dz) \leqq CT^{-1},$$

$$\sum_\alpha \rho_\alpha = 1,$$

$$\rho_\alpha(0, .) = \rho_{\alpha 0}, \ \ \rho_\alpha(T, .) = \rho_{\alpha T},$$

which completes the proof of Theorem 3.1.

8.3. More on Variational Solutions in the Generalized Framework

The variational solutions (and, in particular, the strongly homogenized solutions) have a remarkable property. They minimize the kinetic energy not only in the framework of Theorem 5.1, but in the framework of generalized flows.

Theorem 8.4. *Let $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ be a variational solution and (μ_α) a family of generalized flows satisfying*

$$\sum_\alpha \mu_\alpha|_t = 1, \quad \forall t \in [0, T], \tag{73}$$

$$\mu_\alpha|_{t=0} = \rho_{\alpha 0}, \quad \mu_\alpha|_{t=T} = \rho_{\alpha T}. \tag{74}$$

Then

$$\frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \leqq \sum_\alpha \int e(z)\mu_\alpha(dz). \tag{75}$$

Moreover, if there is equality and if $(\rho_\alpha, m_\alpha = \rho_\alpha v_\alpha)$ is strongly homogenized, then

$$z'(t) = v_\alpha(t, z(t)) \tag{76}$$

holds a.e. on $[0, T]$ for μ_α-almost every path z.

Proof. The proof is strongly reminiscent of the proof of Theorem 3.2. To avoid any confusion, we denote

$$\rho_\alpha^\mu(t, .) = \mu_\alpha|_t \quad \forall\ t \in [0, T]$$

and, in the same way, we associate m_α^μ with μ_α by (70). Clearly,

$$\rho_\alpha^\mu(0, .) = \rho_{\alpha 0}, \quad \rho_\alpha^\mu(T, .) = \rho_{\alpha T}. \tag{77}$$

Let (ρ_α, m_α) be a variational solution (which means that (1), (2), (7)–(9) are satisfied). Since $\Phi_{\alpha\varepsilon}$ belongs to $C^1(\bar{Q})$ and (2) holds for both (ρ_α, m_α) and $(\rho_\alpha^\mu, m_\alpha^\mu)$, we have

$$\int_Q [\rho_\alpha \partial_t \Phi_{\alpha\varepsilon} + m_\alpha \nabla \Phi_{\alpha\varepsilon}] = \int_Q [\rho_\alpha^\mu \partial_t \Phi_{\alpha\varepsilon} + m_\alpha^\mu \nabla \Phi_{\alpha\varepsilon}]. \tag{78}$$

By definition of ρ_α^μ and m_α^μ, the right-hand side of (78) can be written

$$\iint_0^T [\partial_t \Phi_{\alpha\varepsilon}(t,z(t)) + z'(t).\nabla\Phi_{\alpha\varepsilon}(t,z(t))]\ dt\mu_\alpha(dz)$$

$$= \int \left\{ \int_0^T ([\partial_t \Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla\Phi_{\alpha\varepsilon}|^2 + p_\varepsilon](t,z(t)) \right.$$

$$\left. - \tfrac{1}{2}|z'(t) - \nabla\Phi_{\alpha\varepsilon}(t,z(t))|^2)\ dt + e(z) \right\} \mu_\alpha(dz)$$

$$- \iint_0^T p_\varepsilon(t,z(t))\ dt\mu_\alpha(dz).$$

(Notice that all these integrals are well defined and finite). Since $\sum_\alpha \rho_\alpha^\mu = 1 = \sum_\alpha \rho_\alpha = 1$, we have

$$\sum_\alpha \iint_0^T p_\varepsilon(t,z(t))\ dt\mu_\alpha(dz) = \int_Q p_\varepsilon = \sum_\alpha \int_Q \rho_\alpha p_\varepsilon.$$

Thus, it follows from (78) that

$$\begin{aligned} \sum_\alpha \int \left\{ \int_0^T ([\partial_t \Phi_{\alpha\varepsilon} + \tfrac{1}{2}|\nabla\Phi_{\alpha\varepsilon}|^2 + p_\varepsilon](t,z(t)) \right. & \\ \left. - \frac{1}{2}|z'(t) - \nabla\Phi_{\alpha\varepsilon}(t,z(t))|^2)\ dt + e(z) \right\} & \mu_\alpha(dz) \\ = \sum_\alpha \int_Q \rho_\alpha[\partial_t \Phi_{\alpha\varepsilon} + v_\alpha \nabla\Phi_{\alpha\varepsilon} + p_\varepsilon] & \\ = \sum_\alpha \int_Q \rho_\alpha[\partial_t \Phi_{\alpha\varepsilon} + \frac{1}{2}|\nabla\Phi_{\alpha\varepsilon}|^2 + p_\varepsilon] & \\ - \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha|v_\alpha - \nabla\Phi_{\alpha\varepsilon}|^2 + \tfrac{1}{2}\sum_\alpha \int_Q \rho_\alpha|v_\alpha|^2. & \end{aligned} \tag{79}$$

Since (ρ_α, m_α) is a variational solution, the right-hand side is bounded from below by

$$-2\varepsilon + \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2,$$

and the left-hand side is bounded from above by

$$\sum_\alpha \int \left\{ -\int_0^T \tfrac{1}{2}|z'(t) - \nabla\Phi_{\alpha\varepsilon}(t,z(t))|^2 dt + e(z) \right\} \mu_\alpha(dz).$$

Finally we get

$$\sum_\alpha \int \int_0^T \frac{1}{2}|z'(t) - \nabla\Phi_{\alpha\varepsilon}(t, z(t))|^2 dt \mu_\alpha(dz) + \frac{1}{2}\sum_\alpha \int_Q \rho_\alpha |v_\alpha|^2 \leqq \sum_\alpha \int e(z)\mu_\alpha(dz) + 2\varepsilon. \tag{80}$$

The proof then follows easily.

9. Appendix 2: On Transport Equations

Let $D = \mathbb{R}^d/\mathbb{Z}^d$ and $Q =]0, T[\times D$. Let $(t, x) \to v(t, x)$ be a vector field on Q such that $v \in L^2(Q)$ and $Dv \in L^2_{\text{loc}}(L^2(D))$. Assume there is a constant $r > 0$ and a scalar field ρ such that

$$\begin{aligned} &r \leqq \rho(t, x) \leqq r^{-1}, \\ &\partial_t \rho + \nabla \cdot (\rho v) = 0, \end{aligned} \tag{81}$$

holds in the distributional sense. Notice that $\rho \in C^{1/2}([0, T], H^{-1}(D))$ follows from (81), since $v \in L^2(Q)$. Then

Theorem 9.1. *There is no solution of* (81) *in* $L^\infty(Q)$ *other than* ρ *once* $\rho(0, .)$ *is prescribed.*

This uniqueness result follows from

Proposition 9.2. *Let* $g \in L^1_{\text{loc}}(Q)$ *and* $\sigma \in L^\infty(Q)$. *Then* σ *is a distributional solution of*

$$\partial_t(\rho\sigma) + \nabla.(\rho\sigma v) = \rho g \tag{82}$$

if and only if

$$\partial_t \sigma + \nabla.(\sigma v) - \sigma \nabla \cdot v = g \tag{83}$$

holds in the distributional sense. Moreover, σ *belongs to* $C^0([0, T], L^p(D))$ *for all* $p \in [1, +\infty[$ *and, for all locally Lipschitz continuous functions* β,

$$\partial_t(\rho\beta(\sigma)) + \nabla \cdot (\rho\beta(\sigma)v) = \rho\beta'(\sigma)g. \tag{84}$$

Proof. Let us first prove that Theorem 9.1 follows from Proposition 9.2. Let $\overline{\rho}$ be a different solution of (81) and set

$$\sigma = \frac{\overline{\rho}}{\rho} - 1 \in L^\infty(Q).$$

Then, σ is a distributional solution of (82), with $g = 0$, such that $\sigma(0, .) = 0$. Thus, by Proposition (9.2),

$$\partial_t(\rho|\sigma|) + \nabla \cdot (\rho|\sigma|v) = 0.$$

So $|\overline{\rho} - \rho| = \rho|\sigma|$ belongs to $C^{1/2}([0, T], H^{-1}(D))$ and satisfies

$$\langle|\overline{\rho}-\rho|(t,.),1\rangle = \langle|\overline{\rho}-\rho|(0,.),1\rangle,$$

where $\langle .,.\rangle$ denotes the distribution bracket on $D=\mathbb{R}^d/\mathbb{Z}^d$, which shows that $\overline{\rho}=\rho$ almost everywhere in Q and completes the proof of Theorem 9.1.

Let us now prove Proposition 9.2. We use the following (slightly) modified Lemma from [DL].

Lemma 9.3. *Let $\theta \in L^\infty(Q)$ with values in $[0,1]$ and $b \in L^1_{\text{loc}}(Q)$ be such that*

$$\partial_t\theta + \nabla\cdot(\theta v) + b = 0 \tag{85}$$

holds in the distributional sense. Then there is θ_ε in $C^\infty(Q)$ with values in $[0,1]$ such that

$$\theta_\varepsilon \to \theta$$

almost everywhere in Q and

$$\partial_t\theta_\varepsilon + \nabla.(\theta_\varepsilon v) + b \to 0$$

in $L^1_{\text{loc}}(Q)$.

From this lemma, we get $\rho_\varepsilon \in C^\infty(Q)$, such that

$$0 < r \leqq \rho_\varepsilon \leqq r^{-1},$$

$$\rho_\varepsilon \to \rho, \quad \partial_t\rho_\varepsilon + \nabla\cdot(\rho_\varepsilon v) \to 0,$$

in $L^1_{\text{loc}}(Q)$. For each fixed $\varepsilon > 0$, there is a one-to-one correspondence in $C_c^\infty(Q)$ defined by

$$\phi = \rho_\varepsilon f, \quad f = \frac{\phi}{\rho_\varepsilon}.$$

Assume that σ satisfies (82), namely,

$$\int_Q [\rho\sigma\partial_t f + \rho\sigma v.\nabla f + \rho g f] = 0$$

for all $f \in C_c^\infty(Q)$. Then, we infer that

$$\int_Q \frac{\rho}{\rho_\varepsilon}(\sigma[\partial_t\phi + v.\nabla\phi] + g\phi) = \int_Q \frac{\rho}{\rho_\varepsilon^2}\sigma\phi[\partial_t\rho_\varepsilon + v.\nabla\rho_\varepsilon]$$

$$= -\int_Q \frac{\rho}{\rho_\varepsilon}\sigma\phi\nabla.v + \int_Q \frac{\rho}{\rho_\varepsilon^2}\sigma\phi(\partial_t\rho_\varepsilon + \nabla.(\rho_\varepsilon v)).$$

Then, when $\varepsilon \to 0$, we get

$$\int_Q (\sigma[\partial_t\phi + v.\nabla\phi + \phi\nabla.v] + g\phi) = 0,$$

which exactly means (83). Conversely, if (83) holds, then

$$0 = \int_Q (\sigma[\partial_t(\rho_\varepsilon f) + v.\nabla(\rho_\varepsilon f)] + g\rho_\varepsilon f) + \int_Q \sigma\rho_\varepsilon f\nabla.v$$
$$= \int_Q \rho_\varepsilon(\sigma[\partial_t f + v.\nabla f] + gf) + \int_Q \sigma f[\partial_t\rho_\varepsilon + v.\nabla\rho_\varepsilon + \rho_\varepsilon\nabla.v]$$
$$\to \int_Q (\sigma\rho[\partial_t f + v.\nabla f] + gf),$$

which exactly means (82). Now, by Lemma 9.3, we have $\sigma_\varepsilon \in C^\infty(Q)$ such that

$$\partial_t\sigma_\varepsilon + \nabla\cdot(v\sigma_\varepsilon) - \sigma\nabla\cdot v \to g$$

a.e. in Q and $\sigma_\varepsilon \to \sigma$ a.e. boundedly in Q. Thus

$$\beta'(\sigma_\varepsilon)[\partial_t\sigma_\varepsilon + \nabla\cdot(v\sigma_\varepsilon) - \sigma\nabla\cdot v]$$
$$= \partial_t(\beta(\sigma_\varepsilon)) + \nabla\cdot(v\beta(\sigma_\varepsilon)) - (\beta(\sigma_\varepsilon) + (\sigma-\sigma_\varepsilon)\beta'(\sigma_\varepsilon))\nabla\cdot v \to \beta'(\sigma)g$$

and therefore

$$\partial_t(\beta(\sigma)) + \nabla\cdot(v\beta(\sigma)) - \beta(\sigma)\nabla\cdot v = \beta'(\sigma)g,$$

which is equivalent, as just shown, to

$$\partial_t(\rho\beta(\sigma)) + \nabla\cdot(\rho v\beta(\sigma)) = \rho\beta'(\sigma)g.$$

Moreover, for all β, $t \to \beta(\sigma(t,.)$ is continous from $[0,T]$ into $L^\infty(D)$ for the weak-* topology, which shows, by a classical argument, that, for all $p \in [1,+\infty[$, σ is continuous from $[0,T]$ into $L^p(D)$ for the strong topology, and completes the proof.

10. Appendix 3: From the Euler Equations to the Homogenized Equations

10.1. Vortex Sheet Solutions to the Euler Equations

Let us consider an incompressible inviscid fluid moving in a bounded domain D of $\mathbb{R}^d$ (or in the periodic cube $D = \mathbb{R}^d/\mathbb{Z}^d$) and introduce the velocity and pressure fields $u(t,x)$, $p(t,x)$. These fields obey the classical Euler equations (see [Ma] for a modern review)

$$\partial_t u_i + \sum_{j=1}^d \partial_j(u_i u_j) + \partial_i p = 0, \tag{86}$$

$$\nabla.u = 0 \tag{87}$$

and u is parallel to the boundary ∂D. The pressure can be recovered from the velocity field through

$$-\Delta p = \sum_{ij} \partial^2_{ij}(u_i u_j). \tag{88}$$

We say that such a flow has a vortex sheet structure if there is a 'material partition' $(D_\alpha(t), \alpha = 1, \ldots, M)$ of D such that each 'cell' $D_\alpha(t)$ is a simply-connected open set where the velocity field is smooth. By 'material partition', we mean that the characteristic functions $\rho_\alpha(t,x) = 1_{\{x \in D_\alpha(t)\}}$ satisfy

$$\sum_\alpha \rho_\alpha = 1 \tag{89}$$

(almost everywhere in D) and the transport equation

$$\partial_t \rho_\alpha + \nabla.(\rho_\alpha u) = 0.$$

If we denote

$$m_\alpha = \rho_\alpha u,$$

and if v_α is the Radon-Nikodym derivative of m_α with respect to ρ_α so that

$$m_\alpha = \rho_\alpha v_\alpha,$$

then we see that

$$u = \sum_\alpha m_\alpha,$$

$$m_\alpha = \rho_\alpha v_\alpha, \tag{90}$$

$$\partial_t \rho_\alpha + \nabla \cdot m_\alpha = 0, \tag{91}$$

and, from (88), we get the pressure equation in a new form:

$$-\Delta p = \sum_{ij} \partial^2_{ij}(\rho_\alpha v_{\alpha i} v_{\alpha j}). \tag{92}$$

From the combination of (86) and (91), we also get, for each α, the momentum equation

$$\partial_t m_{\alpha i} + \sum_j \partial_j(\rho_\alpha v_{\alpha j} v_{\alpha i}) + \rho_\alpha \partial_i p = 0, \tag{93}$$

which is not written in conservation form. Notice that (92) can also be recovered from

$$\nabla.\left(\sum_\alpha m_\alpha\right) = 0, \tag{94}$$

$$\sum_\alpha \partial_t m_{\alpha i} + \sum_{\alpha j} \partial_j(\rho_\alpha v_{\alpha j} v_{\alpha i}) + \partial_i p = 0, \tag{95}$$

which are respectively obtained from (89), after summing up equations (91) and (93) with respect to α.

So, classical vortex sheet motions are fully described by the consistent set of equations (89)–(91) and (93), where only ρ_α, v_α and p are involved and the product $\rho_\alpha \nabla p$ is required to be well defined, for instance by assuming that ∇p is locally integrable, which can be physically interpreted as a continuity condition for p across the cell boundaries.

10.2. The Homogenized Vortex Sheet Equations

If we relax the condition that ρ_α takes values in $\{0, 1\}$ and consider solutions with values in $[0, 1]$, we obtain a self-consistent system of equations with (89)–(91) and (93) (assuming that ∇p is locally integrable). The corresponding solutions can be seen as describing generalized (or homogenized) vortex sheet motions.

Obviously, these equations cannot be derived through any reasonable homogenization process if the genuine weak convergence of the density fields (leading to values of ρ_α in the range $]0, 1[$) is not compatible with the strong convergence of the corresponding velocity fields. Let us consider a sequence of classical vortex sheet solutions $(\rho_\alpha^n, v_\alpha^n)$. If we assume that

$$\frac{1}{2} \sum_\alpha \int_Q \rho_\alpha^n |v_\alpha^n|^2$$

is uniformly bounded, then, up to a subsequence extraction, both m_α^n and ρ_α^n have weak limits m_α and ρ_α, respectively in $L^2(Q)$ and $L^\infty(Q)$ weak-*. Moreover $m_\alpha = \rho_\alpha v_\alpha$, with $v_\alpha \in L^2(Q, \rho_\alpha\, dt\, dx)$.

The situation we are interested in corresponds to the case when, for each α, there is no kinetic energy defect in the sense that

$$\int_Q \rho_\alpha^n |v_\alpha^n|^2 \to \int_Q \rho_\alpha |v_\alpha|^2.$$

Since the homogeneous convex function $(\rho, m) \to |m|^2 \rho^{-1}$ is not strictly convex, it does not follow that m_α^n and ρ_α^n strongly converge. In particular, ρ_α may have values in $]0, 1[$, the velocity field

$$u^n = \sum_\alpha m_\alpha^n$$

may not strongly converge and the weak limit of u^n may not be a weak solution to the Euler equations. However, we immediately get that

$$\int_Q \rho_\alpha^n |v_\alpha^n - v_\alpha|^2 \to 0,$$

which is compatible with the strong convergence of the velocity field v_α^n (although it is not defined almost everywhere but only where $\rho_\alpha^n > 0$!).

Remark. Notice that v_α is not necessarily curl-free when the fields v_α^n are curl-free.

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Ecole Normale Supérieure
DMI, 45 rue d'Ulm
75230 Paris Cedex

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