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Invariant Manifolds for Parabolic Partial Differential Equations on Unbounded Domains

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Abstract

In this paper finite-dimensional invariant manifolds for nonlinear parabolic partial differential equations of the form

$$\frac{\partial u}{\partial \tau} = \Delta_\xi u + F(u), \quad u = u(\xi, \tau),\ \xi \in \mathbb{R}^d,\ \tau \geqq 1$$

are constructed. Such results are somewhat surprising because of the continuous spectrum of the linearized equation. These manifolds control the long-time behavior of solutions of these equations and can be used to construct systematic, rigorous expansions of the long-time asymptotics in inverse powers of τ. They also give a new perspective on the change in the long-time asymptotics of the equation with nonlinear term $F(u) = |u|^{p-1}u$, when p passes through the critical value $p_c = 1 + 2/d$.

1. Introduction and Statement of Results

Invariant manifold theorems have found numerous applications in partial differential equations. (Carr (1983), Carr & Muncaster (1983), Eckmann & Wayne (1991), Foias & Saut (1984), Henry (1981), Kirchgässner (1988), and Mielke (1986) provide some examples, and Vanderbauwhede & Iooss (1992) list many more in the bibliography of their paper.) These applications, however, have been limited to situations in which either the time or space variables have varied over a bounded domain, or to examples like traveling waves which could be reduced to one of these cases. This restriction results in a linearized problem with pure point spectrum which permits one to identify modes associated with a center, or center stable manifold. In equations of the form

$$\frac{\partial u}{\partial \tau} = \Delta_{\xi} u + F(u), \qquad u = u(\xi, \tau), \quad \xi \in \mathbb{R}^d, \ \tau \geqq 1 \tag{1.1}$$

with $F(u)$ nonlinear, the linearized problem is the heat equation which clearly has continuous spectrum. (By nonlinear, I mean that $|F(u)| \approx |u|^p$ as $|u| \to 0$ for some $p > 1$. In particular, $F(0) = F'(0) = 0$.) Nonetheless, (1.1) does possess finite-dimensional invariant manifolds which control the long-time asymptotics of solutions near the origin.

It will become apparent in the course of the proof that it is more natural to construct these invariant manifolds in the extended phase space. That is, given a Hilbert space $\mathscr{H}$ with $u \in \mathscr{H}$, the invariant manifold will be constructed in the space $\mathscr{P} = \mathscr{H} \times \{\tau \in \mathbb{R} \mid \tau \geqq 1\}$. (I choose the initial time to be $\tau = 1$, rather than $\tau = 0$, to simplify a point that arises later in the paper.) One also requires a certain degree of smoothness in the nonlinear term, which again will be made precise below, but note that one of the examples that will be considered below is $F(u) = |u|^{p-1}u$, and this does have sufficient smoothness provided p is sufficiently large (depending on d). Under these conditions, one has

Theorem 1.1. *Suppose that the nonlinear term F in* (1.1) *is sufficiently smooth. Fix* $n \geqq 0$. *There exists a Hilbert space* $\mathscr{H}(n)$, *such that in the extended phase space* $\mathscr{P}(n) = \mathscr{H}(n) \times \{\tau \in \mathbb{R} \mid \tau \geqq 1\}$, *there exists a* $1 + \sum_{j=0}^{n} \binom{j+d-1}{d-1}$-*dimensional invariant manifold defined in a neighborhood of the origin, and left invariant by the semi-flow defined by* (1.1). *Furthermore, solutions on the invariant manifold control the long-time asymptotics of "small" solutions of* (1.1) *up to a fixed* (*but n-dependent*) *inverse power of* τ.

Remark. The norm in which the solution must be "small" will be defined below, but the point I wish to emphasize about this theorem is that if one wishes to compute the long-time asymptotics of (1.1) up to some given inverse power of τ, then by choosing n appropriately, this calculation can be reduced to the study of the asymptotics of a finite-dimensional system of ordinary differential equations. The algorithm that implements this reduction, along with an example, is explained in more detail in Section 4.

In the study of ordinary differential equations, invariant-manifold theorems have many applications. (See, e.g., RUELLE (1989)). In this paper, I exhibit two uses for them in the present context. First, as I mentioned in the previous remark, they can be used to give a systematic, rigorous expansion of the long-time asymptotics of solutions of (1.1). The second use I will illustrate is to give an explanation based on bifurcation theory of the changes in the long-time asymptotics of the equation with nonlinear term $F(u) = -|u|^{(p-1)}u$, when p passes through its critical value of $p_c = 1 + 2/d$.

My approach to this problem was inspired by BRICMONT & KUPIAINEN (1996), who study the stability of certain non-Gaussian solutions of (1.1) with $F(u) = -|u|^{(p-1)}u$, and $1 < p < p_c$. Their analysis was based on a change of variables to what might be called "similarity variables". This same change of variables was also used by Escobedo & Kavian (1988) in their study of this problem, and it is used here in a slightly different context to construct the invariant manifolds. Related changes of variables have found use in other questions connected with the asymptotics of parabolic partial differential equations. For instance, Giga & Kohn (1987) apply a similar change of variables to study the blow-up of solutions. Indeed, Bressan (1992), Filippas & Kohn (1992), and Berbenes & Bricher (1992) all appeal to invariant-manifold theorems as motivation for their results on blow-up in nonlinear heat equations, but technical difficulties prevented the actual construction of such manifolds in those problems.

Another point of view is obtained if one notes that in order to obtain successively higher orders in the asymptotic expansion for the long-time behavior of the solution, the Hilbert spaces in $\mathscr{H}(n)$ in Theoerm 1.1 are chosen so that they impose increasingly stringent decay properties on $u(x)$ as $|x| \to \infty$. (However, $|u(x)|$ is required to decay only as an inverse power of $|x|$; it is not necessary to impose exponential decay.) The use of weighted spaces to analyze the stability properties of solutions of partial differential equations on unbounded domains has been exploited by a number of authors including Henry (1981), Sattinger (1976), Bates & Jones (1989), and Kapitula (1994), who analyze the stability of traveling waves in parabolic partial differential equations, and by Soffer & Weinstein (1991), and Pego & Weinstein (1994) who apply these ideas to study the asymptotics of solutions of Hamiltonian partial differential equations. The novelty of the present work lies in the fact that while the works mentioned above concentrate just on the decay rate of the solutions (effectively, the lowest-order asymptotics), the existence of the invariant manifolds described in Theorem 1.1 allows one to systematically compute higher-order asymptotics.

For parabolic partial differential equations on bounded domains, the use of invariant-manifold theorems to determine long-time asymptotics is much more fully developed. (See Vishik (1992) and the references therein.) In that case, however, the situation is more reminiscent of the problem in finite dimensions since the linearized operator has point spectrum, which gives a natural and obvious separation of the phase space of the linear problem into stable, unstable, and central subspaces. Furthermore, the asymptotics in that case is characterized by exponential decay in time, as it is for ordinary differential equations, rather than the power-law decay that one encounters in the present context. In the case of bounded spatial domains, however, invariant manifold theorems have been proved for classes of equations much more general than (1.1) and it will be interesting to see to what extent the present theory can be extended to encompass such examples.

The remainder of the paper is organized as follows. In the next section I show how invariant manifolds can be constructed in a relatively "small" Hilbert space. In Section 3, I then extend these results to a class of Sobolev spaces with polynomial weights. In Section 4, these results are used to give a systematic expansion of the long-time asymptotics of (1.1) in inverse powers of t and to analyze the behavior of the equation with $F(u) = -|u|^{(p-1)}u$ near the critical power of the nonlinearity. Finally, Section 5 is devoted to conclusions and possible extensions.

Notation. Throughout this paper I work with L^2-based Sobolev spaces. Define

$$H^{q,2} = \left\{ u \mid \|u\|^2_{H^{q,2}} \equiv \sum_{|\alpha| \leqq q} \|D^\alpha u\|^2_{L^2} < \infty \right\}.$$

Here, as usual, α is a multi-index. In Section 3, the polynomially weighted spaces $H^{q,2}_r = \{v \mid (1+|x|^2)^{r/2} v \in H^{q,2}\}$ play a central role.

2. The Existence of Invariant Manifolds

Begin by making the change of variables

$$u(\xi, \tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log \tau). \tag{2.1}$$

Recall that p is determined by the behavior of the nonlinear term near zero – it need not be an integer. In terms of the new variables, $x = \xi/\sqrt{\tau}$ and $t = \log \tau$, (1.1) becomes

$$\frac{\partial v}{\partial t} = \Delta_x v + \frac{1}{2} x \cdot \nabla_x v + \frac{1}{p-1} v + \tau^{\frac{p}{p-1}} F(\tau^{\frac{-1}{p-1}} v). \tag{2.2}$$

Now make a further change of variables, and set $w(x,t) = \exp\left(\frac{1}{8}x^2\right) v(x,t)$. When expressed in terms of these variables, the equation becomes

$$\frac{\partial w}{\partial t} = H_0 w + e^{\frac{x^2}{8}} \tau^{\frac{p}{p-1}} F(\tau^{\frac{-1}{p-1}} e^{\frac{-x^2}{8}} w), \tag{2.3}$$

where the linear operator $H_0 = \Delta_x - \frac{x^2}{16} + (\frac{1}{p-1} - \frac{d}{4})$. Noting that H_0 is (up to the additive constant) just the Hamiltonian of the quantum mechanical harmonic oscillator, we can conclude (see, e.g., Glimm & Jaffe (1981)) that its spectrum, considered as an operator on $L^2(\mathbb{R}^d)$, is $\sigma(H_0) = \{\frac{1}{p-1} - \frac{d}{2} - \frac{m}{2} | m = 0, 1, 2, \ldots\}$. Furthermore, its eigenfunctions can be explicitly computed in terms of Hermite polynomials, a fact that I use later. It is not convenient to study (2.3) on $L^2(\mathbb{R}^d)$, because the nonlinear term in the equation has too little regularity when considered as a function on this space to apply the version of the invariant-manifold theorem used below. However, if we choose a constant $c(p)$ sufficiently large, the operator $(c(p)\mathbf{1} - H_0)$ is positive-definite. Thus one can take its square root. This allows us to define a family of Sobolev-like Hilbert spaces via

$$\tilde{H}^{q,2} = \{w | (c(p)\mathbf{1} - H_0)^{q/2} w \in L^2(\mathbb{R}^d)\}.$$

Remark. While it is not necessary to introduce the "w" variables, the fact that H_0 is self-adjoint on $\tilde{H}^{q,2}$ will simplify many of the computations in later sections.

Remark. Another useful fact that is easy to prove is that for q a non-negative integer, the $\tilde{H}^{q,2}$ norm is equivalent to the norm

$$|||w|||_q^2 \equiv \sum_{|\alpha|+|\beta| \leqq q} \|x^\alpha D^\beta w\|_{L^2}^2.$$

These norms control the "ordinary" Sobolev norms. For instance, just from the definition of the norm, one has

Lemma 2.1. *For any non-negative integer q, there exists a constant $c(q)$ such that $\|w\|_{H^{q,2}} \leqq c(q)\|w\|_{\tilde{H}^{q,2}}$.*

As a corollory of this lemma, one sees immediately that if $w \in \tilde{H}^{q,2}$ and $q > \frac{1}{2}\, d$, then w can be assumed to be continuous.

One can now apply the invariant-manifold theorem to (2.3). First convert it to an autonomous equation by a standard trick. Set $\eta = \tau^{-1/(p-1)} = \exp(-t/(p-1))$. If a " $\cdot$ " denotes differentiation with respect to t, then (2.3) is equivalent to the system of equations

$$\begin{aligned} \dot{w} &= H_0 w + \eta^{-p} e^{x^2/8} F(\eta e^{-x^2/8} w), \\ \dot{\eta} &= -\left(\frac{1}{p-1}\right)\eta. \end{aligned} \tag{2.4}$$

Remark. Define $f(\eta, w) = \eta^{-p} e^{x^2/8} F(\eta e^{-x^2/8} w)$. In order to apply the following version of the invariant-manifold theorem, f must be a C^k function on $\mathbb{R} \times \tilde{H}^{q,2}$ for some $k > 1$ (not necessarily integral). I assume that $F(u)$ is smooth enough that this is the case, and then verify that it is true in each of the examples considered in Section 4.

One can now construct an invariant manifold for (2.4) corresponding to the $n+1$ largest eigenvalues of H_0, $\lambda_m = \frac{1}{p-1} - \frac{d}{2} - \frac{m}{2}$, $m = 0, 1, \ldots, n$. Let $\mathscr{P}^c$ be the projection (in $\tilde{H}^{q,2}$) onto the subspace spanned by the eigenvectors corresponding to these eigenvalues. (Note that the dimension of this space is $\sum_{j=0}^{n} \binom{j+d-1}{d-1}$.) Let $\mathscr{P}^s = (\mathscr{P}^c)^\perp$. I refer to this manifold as the "pseudo-center manifold" in analogy with the more common pseudo-stable manifolds (see, e.g., De La Llave & Wayne (1995)). Assume that n is so large that $\lambda_n < -\frac{1}{p-1}$. In this case the η direction is included in the "pseudo-center" direction. The case in which $\lambda_n > -\frac{1}{p-1}$ is an easy modification of this argument, because the dynamics in the η direction are trivial in either case. Let $\mathscr{E}^c = \mathbb{R} \oplus \text{range}(\mathscr{P}^c)$ and $\mathscr{E}^s = \text{range}(\mathscr{P}^s)$, where the summand $\mathbb{R}$ in $\mathscr{E}^c$ corresponds to the η direction. Let $\mathscr{B}^{c,(s)}(\rho)$ be the ball of radius ρ in $\mathscr{E}^{c,(s)}$, $\mathscr{B}(\rho)$ the ball of radius ρ in $\mathbb{R} \oplus \tilde{H}^{q,2}$, and let $A^c = -\frac{1}{p-1} \oplus \mathscr{P}^c H_0 \mathscr{P}^c$ and $A^s = \mathscr{P}^s H_0 \mathscr{P}^s$. By the spectral theorem, there exist constants $\lambda_{n+1} < \lambda_s < \lambda_c < \lambda_n$, such that

$$\sup_{t \geqq 0} \|e^{A^s t}\| e^{-\lambda_s t} < \infty, \quad \sup_{t \in \mathbb{R}} \|e^{A^c t}\| e^{\lambda_c |t|} < \infty. \tag{2.5}$$

We can now apply Theorem 4.1 of GALLAY (1993) to obtain

Theorem 2.2. *If the nonlinear term F in* (1.1) *is such that $f(\eta, w)$ is C^k, with $k > 1$, then there exists $\alpha > 0$ and $\rho > 0$, and a $C^{1+\alpha}$ function $h : \mathscr{B}^c(\rho) \to \mathscr{E}^s$, such that the graph of h is invariant under the* (*semi-*) *flow of* (2.3). *Furthermore, $h(0) = 0$ and $Dh(0) = 0$.*

Proof. Let x^c and x^s be coordinates on $\mathscr{E}^{c,s}$ respectively. Then (2.3) can be rewritten as

$$\dot{x}^c = A^c x^c + f^c(x^c, x^s),\qquad (2.6)$$
$$\dot{x}^s = A^s x^s + f^s(x^c, x^s).$$

By the hypotheses of the theorem, the nonlinear terms f^c and f^s are C^k functions for some $k > 1$, so the hypotheses of Theorem 4.1 of GALLAY (1993) are satisfied, and the theorem follows. □

Remark. Note that while Theorem 2.2 only guarantees the existence of the invariant manifold in a neighborhood of the origin, this neighborhood can be chosen to include the interval $0 \leqq \eta \leqq 1$, and hence, since $\eta = \tau^{-1/(p-1)}$, this includes all values of the temporal variable τ in which we are interested. The reason that the whole interval $0 \leqq \eta \leqq 1$ can be included in the domain of existence of the invariant manifold is that GALLAY's theorem constructs an invariant manifold on a set whose size is determined by the requirement that the Lipschitz constant of the nonlinear term be small on that neighborhood. In the case of the equations (2.4), the "η"-equation has no nonlinear term, and hence its Lipschitz constant may be taken to be zero, while in the "w"-equation, the Lipschitz constant can be made uniformly small for all $0 \leqq \eta \leqq 1$ by making the neighborhood in the "w"-variables small enough.

3. Attractivity of the Pseudo-Center Manifold

In this section I prove that the pseudo-center manifold attracts all solutions that remain in some neighborhood of the origin. What is somewhat surprising is that this is true for solutions in a much larger space than the space in which this manifold was constructed. For the present construction we revert to the "v" variables, and consider the equation in the form (2.2)

$$\frac{\partial v}{\partial t} = \Delta_x v + \frac{1}{2} x \cdot \nabla_x v + \frac{1}{p-1} v + \tau^{p/(p-1)} F(\tau^{-1/(p-1)} v).\qquad (3.1)$$

It is assumed that v lies in the weighted Sobolev spaces, $H_r^{q,2} = \{v \mid (1+|x|^2)^{r/2} v \in H^{q,2}\}$. Note that it may seem that the spaces $H_r^{q,2}$ are actually smaller than the space $\tilde{H}^{q,2}$ used in the previous section, but recall that the "w"-variables are defined by $v(x,t) = \exp(-x^2/8) w(x,t)$, so that the "$v$"-variables correspond to solutions of our original equation with polynomial decay at infinity, while the "w"-variables correspond to solutions with Gaussian decay. It is in this sense that I refer to $\tilde{H}^{q,2}$ as being a "smaller" space than $H_r^{q,2}$.

If $\{\varphi_n\}_{n \geqq 0}$ are the eigenfunctions of the linear operator H_0 in the previous section, then $\{\psi_n\}_{n \geqq 0}$, with $\psi_j(x) = \exp(-x^2/8)\varphi_j(x)$ are eigenfunctions of $\mathscr{L}_0 = \Delta_x + \frac{1}{2}x \cdot \nabla_x + 1/(p-1)$, with the same eigenvalues.

Define a projection operator

$$(P_n v)(x) = \sum_{j=0}^{n} \psi_j(x) \langle\langle \psi_j, v \rangle\rangle_q,$$

where the inner product

$$\langle\langle v, \tilde{v} \rangle\rangle_q \equiv \int (e^{x^2/8}\tilde{v}(x))(e^{x^2/8}(c(p) - \mathscr{L}_0)^q \overline{v(x)})\, dx$$

is obtained formally from the inner product $\langle w, \tilde{w} \rangle_q$ in the Hilbert space $\tilde{H}^{q,2}$ considered in the previous section upon substituting $w(x) = \exp(x^2/8)v(x)$, and $\tilde{w}(x) = \exp(x^2/8)\tilde{v}(x)$. While the computation leading to the form of $\langle\langle \cdot, \cdot \rangle\rangle_s$ is formal, the following lemma shows that P_n is well defined.

Lemma 3.1. *If* $r > n + \frac{1}{2}(d+1)$, *then there exists* $C > 0$ *such that for any* $v \in H_r^{q,2}$,

$$\|P_n v\|_{H_r^{q,2}} \leqq C \|v\|_{H_r^{q,2}}.$$

Proof. The $H_r^{q,2}$ norm of ψ_j is finite so the lemma follows from the estimate $|\langle\langle \psi_j, v \rangle\rangle_q| \leqq C\|v\|_{H_r^{q,2}}$ for $j = 0, \ldots, n$. This in turn follows if we note that

$$\begin{aligned} |\langle\langle \psi_j, v \rangle\rangle_q| &= \left| \int e^{x^2/8} v(x) e^{x^2/8} (c(p) - \mathscr{L}_0)^q \psi_j(x)\, dx \right| \\ &\leqq (c(p) + |\lambda_j|)^q \int e^{x^2/4} |v(x)| |\psi_j(x)|\, dx. \end{aligned}$$

Now recall that $\psi_j(x) = e^{-x^2/8}\varphi_j(x) = e^{-x^2/8}(e^{-x^2/8}h(x))$, where $h(x)$ is (up to normalization) a product of Hermite polynomials the sum of whose orders is j. (In particular, if $d = 1$, $h(x) = H_j(x)$.) This implies that $|h(x)| \leqq C(1 + |x|)^j$. Thus,

$$\begin{aligned} |\langle\langle \psi_j, v \rangle\rangle_q| &\leqq C(c(p) + |\lambda_j|)^q \int (1 + |x|)^j |v(x)|\, dx \\ &\leqq C(c(p) + |\lambda_j|)^q \left(\int (1+|x|)^{-(d+1)}\, dx \right)^{1/2} \left(\int (1+|x|)^{2j+(d+1)} |v(x)|^2\, dx \right)^{1/2}. \end{aligned}$$

Thus, if $2r > 2j + d + 1$, this expression is bounded by $C\|v\|_{H_r^{q,2}}$ and the lemma follows. □

Remark. The projection operator P_n is the analogue of $\mathscr{P}^c$, defined in Section 2. However, at different points in the argument it will be convenient to allow n to vary, and for this reason this new notation is useful.

Remark. Note that an additional property of the inner product $\langle\langle\cdot,\cdot\rangle\rangle_q$ is that $\langle\langle\psi_j,\psi_{j'}\rangle\rangle_q=\delta_{j,j'}$. This follows from the orthonormality of the eigenfunctions φ_j.

The key estimate for the present section is the following estimate on the linear evolution operator acting on $H_r^{q,2}$. Let $Q_n=1-P_n$.

Proposition 3.2. *Fix* $n>0$ *such that* $\lambda_{n+1}<0$. *For* r *sufficiently large*, *there exists* $C(r,q,n)>0$ *such that*

$$\|e^{t\mathscr{L}_0}Q_n v\|_{H_r^{q,2}}\leqq C(r,q,n)e^{-t|\lambda_{n+1}|}\|v\|_{H_r^{q,2}}.$$

Proof. See Appendix A.

Note that this is exactly the sort of decay estimate we could expect for vectors in the "stable-subspace" $\mathscr{E}^s\subset\tilde{H}^{q,2}$ — what is surprising it that it also holds in the much larger space $H_r^{q,2}$.

Given such an estimate on the linearized evolution, results about the attractivity of the invariant manifold follow very much as in the finite-dimensional case. We include proofs of Proposition 3.3 and Theorem 3.5 for completeness, but they are essentially taken verbatim from CARR (1983, pp. 20–25). If we let η be as in the previous section, $v^c=P_nv$ and $v^s=Q_nv$, then we can rewrite (2.2) as

$$\begin{aligned}\dot\eta&=-\left(\frac{1}{p-1}\right)\eta,\\ \dot{v^c}&=\mathscr{L}_0^c v^c+F^c(\eta,v^c,v^s),\\ \dot{v^s}&=\mathscr{L}_0^s v^s+F^s(\eta,v^c,v^s).\end{aligned}\tag{3.2}$$

Here $\mathscr{L}_0^c=P_n\mathscr{L}_0P_n$, $\mathscr{L}_0^s=Q_n\mathscr{L}_0Q_n$, while $F^c(\eta,v^c,v^s)=P_n(\eta^pF(\eta^{-p}v))$ and $F^s(\eta,v^c,v^s)=Q_n(\eta^pF(\eta^{-p}v))$. We require that F^c and F^s be Lipschitz functions on $H_r^{q,2}$, a fact which we again check in each application. In addition we assume that q, the index on the Sobolev space in which we work, is greater than $\frac{1}{2}d$ so that our functions are all continuous. The results of the previous section imply that there exists an invariant manifold tangent at the origin to $\mathbb{R}\oplus\operatorname{range}(P_n)$. This manifold is given (locally) as the graph of a function $v^s=h(\eta,v^c)$. The following results show first that solutions near this manifold are attracted to it, and second that one can approximate such solutions, up to an exponentially small error, by the finite-dimensional system of ordinary differential equations corresponding to motion on the manifold.

Proposition 3.3. *Let* $(v^c(t), v^s(t))$ *be a solution of* (3.2) *which remains for all time in a sufficiently small neighborhood of the origin.* (*Note that the evolution of* η *is trivial.*) *Then there exist positive constants* C_1 *and* μ *such that*

$$\|v^s(t) - h(\eta(t), v^c(t))\|_{\mathscr{E}^s} \leqq C_1 e^{-\mu t}\|v^s(0) - h(\eta(0), v^c(0))\|_{\mathscr{E}^s}.$$

Proof. Following Carr, we let $z(t) = v^s(t) - h(\eta(t), v^c(t))$. Then

$$\dot{z}(t) = \mathscr{L}_0^s z + N(\eta, v^c, z), \tag{3.3}$$

where

$$N(\eta, v^c, z) = F^s(\eta, v^c, h(v^c) + z) - F^s(\eta, v^c, h(v^c)) + D_{v^c}h(\eta, v^c)\{F^c(\eta, v^c, h(v^c)) - F^c(\eta, v^c, h(v^c) + z)\}.$$

Note that $D_{v^c}h(\eta, v^c)$ is a linear operator from range (P_n) to range (Q_n). If we rewrite (3.3) in integral form we obtain

$$z(t) = e^{t\mathscr{L}_0^s}z(0) + \int_0^t e^{(t-s)\mathscr{L}_0^s}N(\eta(s), v^c(s), z(s))\, ds. \tag{3.4}$$

From the definitions of F^c and F^s, there exists a constant $\delta(\rho)$ which goes to zero as $\rho \to 0$, and such that if $\|v^c\|_{\mathscr{E}^c} \leqq \rho$, $\|N(\eta, v^c, z)\|_{\mathscr{E}^c} \leqq \delta(\rho)\|z\|_{\mathscr{E}^s}$. Here, $\|\cdot\|_{\mathscr{E}^c}$ is the ordinary finite-dimensional Euclidean norm on range (P_n), while $\|\cdot\|_{\mathscr{E}^s}$ is the $H_r^{q,2}$-norm, restricted to $\mathscr{E}^s$.

Applying this estimate of N, and the estimate of Proposition 3.2 to (3.4) one sees that

$$\|z(t)\|_{\mathscr{E}^s} \leqq Ce^{-t|\lambda_{n+1}|}\|z(0)\|_{\mathscr{E}^s} + C\delta(\rho)\int_0^t e^{-(t-s)|\lambda_{n+1}|}\|z(s)\|_{\mathscr{E}^s}\, ds. \tag{3.5}$$

The proposition then follows immediately from Gronwall's Lemma. □

Remark. Note that the constant $\delta(\rho)$ can be chosen to be $CL(\rho)$ where $L(\rho)$ is the Lipschitz constant of the nonlinearity $F(v)$ on a ball of radius ρ in $H_r^{q,2}$. This coupled with (3.5) implies that the constant μ in Proposition 3.3 can be chosen to be any number less than $||\lambda_{n+1}| - CL(\rho)|$. In particular, by taking ρ sufficiently small, we can make $L(\rho)$ arbitrarily small, and μ arbitrarily close to $|\lambda_{n+1}|$. Thus, one has

Corollary 3.4. *There exist* $C > 0$ *and* $\rho_0 > 0$, *such that if the solution of* (2.2) *remains in a ball of radius* $\rho < \rho_0$ *centered at the origin for all* $t \geqq 0$, *then* $v(t)$ *approaches the invariant manifold with a rate* $\mathcal{O}(e^{-\mu t})$, *where* $\mu = |\lambda_{n+1}| - CL(\rho)$.

We now turn to the second important result, namely, that we can approximate any trajectory of (2.2) near the origin by a solution of a finite-dimensional system of ordinary differential equations. Let

$$\begin{aligned} \dot{\eta} &= -\left(\frac{1}{p-1}\right)\eta, \\ \dot{\psi} &= \mathscr{L}_0^c \psi + F^c(\eta, \psi, h(\eta, \psi)). \end{aligned} \tag{3.6}$$

Theorem 3.5. *Let* $(v^c(t), v^s(t))$ *be a solution of* (3.2) *which remains in a sufficiently small neighborhood of the origin for all* $t \geqq 0$. *Then there exists a solution* $\psi(t)$ *of* (3.6), *and* $\mu > 0$, *such that as* $t \to \infty$,

$$\begin{aligned} v^c(t) &= \psi(t) + \mathcal{O}(e^{-\mu t}), \\ v^s(t) &= h(\eta(t), \psi(t)) + \mathcal{O}(e^{-\mu t}). \end{aligned}$$

Remark. As above, we can choose $\mu = |\lambda_{n+1}| - CL(\rho)$.

Proof. We follow Carr's proof [1983, pp. 21–25], merely indicating the differences. We use his notation without comment. Define $z(t) = v^s(t) - h(v^c(t))$ and $\varphi(t) = v^c(t) - \psi(t)$. Then z and φ satisfy

$$\begin{aligned} \dot{z} &= \mathscr{L}_0^s z + N(\eta, \varphi + \psi, z), \\ \dot{\varphi} &= \mathscr{L}_0^c \varphi + R(\eta, \varphi, z), \end{aligned} \tag{3.7}$$

where N is as in (3.3) and $R(\eta, \varphi, z) = F^c(\eta, \varphi + \psi, z + h(\eta, \varphi + \psi)) - F^c(\eta, \psi, h(\eta, \psi))$. Consider the Banach space of functions

$$X_a = \{\varphi \in C([0, \infty), \mathbb{R}^{n+1}) \mid |||\varphi|||_a = \sup_{t \geqq 0} \|\varphi(t) e^{at}\|_{\mathscr{E}^s} < \infty\}.$$

Note that we can rewrite the second equation in (3.7) in integral form as

$$\varphi(t) = e^{(t-t_0)\mathscr{L}_0^c} \varphi(t_0) - \int_t^{t_0} e^{(t-s)\mathscr{L}_0^c} R(\eta(s), \varphi(s), z(s))\, ds.$$

If $\varphi \in X_a$, and $a > |\lambda_n|$, then $e^{(t-t_0)\mathscr{L}_0^c} \varphi(t_0) \to 0$ as $t_0 \to \infty$. This motivates the definition of the transformation

$$(T\varphi)(t) = -\int_t^{\infty} e^{(t-s)\mathscr{L}_0^c} R(\eta(s), \varphi(s), z(s))\, ds.$$

By Proposition 3.3, we know that for any $0 < \beta_1 \leqq (|\lambda_{n+1}| - CL(\rho))$, $\|z(s)\|_{\mathscr{E}^s} \leqq C\|z(0)\| e^{-\beta_1 s}$. Using this, Carr shows that for any $|\lambda_n| < a < \beta_1$, T has a unique fixed point in X_a for any $(\psi(0), z(0))$ sufficiently small. Furthermore (possibly by shrinking the size of the neighborhood), the fixed point depends continuously on $(\psi(0), z(0))$. (In Carr's case, $|\lambda_n| = 0$; however, the case $|\lambda_n| \neq 0$ produces no essential change in the argument.)

Now define $S(\psi(0), z(0)) = (v^c(0), z(0))$, where $v^c(0) = \varphi(0) + \psi(0)$. We want to prove that S is one-to-one and hence invertible on a neighborhood of the identity. This will then establish the theorem, with the constant $\mu = a$. As Carr notes, proving that S is one-to-one is equivalent to showing that if $\psi(0) + \varphi(0) = \tilde{\psi}(0) + \tilde{\varphi}(0)$, then $\varphi(0) = \tilde{\varphi}(0)$ and $\psi(0) = \tilde{\psi}(0)$. By the uniqueness of solutions of (3.2) we see that $\psi(t) + \varphi(t) = \tilde{\psi}(t) + \tilde{\varphi}(t)$ for all $t \geqq 0$, where $\psi(t)$ and $\tilde{\psi}(t)$ are the solutions of (3.6) with initial conditions $\psi(0)$ and $\tilde{\psi}(0)$, respectively. This equation is equivalent to the equation

$$\varphi(t) - \tilde{\varphi}(t) = \psi(t) - \tilde{\psi}(t). \tag{3.8}$$

Choose β_2 such that $|\lambda_n| < \beta_2 < a \leqq \beta_1$, where a is the constant which determines the exponential decay rate of the space X_a. Then from (3.6) we see that

$$\lim_{t\to\infty} e^{\beta_2 t}\|\psi(t) - \tilde{\psi}(t)\|_{\mathscr{E}^c} = \infty,$$

unless $\psi(0) = \tilde{\psi}(0)$. On the other hand, from the definition of X_a, $\lim_{t\to\infty} e^{\beta_2 t}\| \varphi(t) - \tilde{\varphi}(t)\|_{\mathscr{E}^c} = 0$. Thus, (3.8) implies $\psi(0) = \tilde{\psi}(0)$, and hence $\varphi(0) = \tilde{\varphi}(0)$.

This implies that S is invertible, or that given any $v^c(0)$ and $z(0)$ sufficiently small, we can find $z(0)$ and $\psi(0)$ such that if $v^c(t)$ and $\psi(t)$ are the solutions of (3.2) and (3.6) with these initial conditions and if $\varphi(t) = v^c(t) - \psi(t)$, then $\|\varphi(t)\|_{\mathscr{E}^c} \leqq Ce^{-at}$ and $\|z(t)\|_{\mathscr{E}^s} \leqq Ce^{-at}$. □

If we combine Theorems 2.2 and 3.5, we see that any solution of (2.2) in a sufficiently small neighborhood of the origin in $H_r^{q,2}$ approaches a solution on the invariant manifold at a rate $\mathcal{O}(\exp(-|\lambda_{n+1}|t))$. To prove Theorem 1.1, we revert to the original variables

$$u(\xi, \tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log\tau).$$

Thus, if we consider the evolution of (1.1) in its extended phase space $H_r^{q,2} \times \{\tau \mid \tau \geqq 1\}$, the invariant manifold for (2.2) gets mapped to an invariant manifold for (1.1). Note further that the exponential approach toward the manifold in the "t" variable becomes an algebraic approach in τ, in fact an approach like $\mathcal{O}(\tau^{-|\lambda_{n+1}|}) \approx \mathcal{O}(\tau^{-\frac{d}{2}-\frac{n+1}{2}+\frac{1}{p-1}})$.

Unfortunately, because of the way that the change of variables (2.1) mixes the time and space coordinates, we do not obtain a simple estimate of the asymptotics of $u(\xi, \tau)$ in terms of the $H_r^{q,2}$ norm in (ξ, τ), but rather in a more complicated space-time dependent norm. (The invariant manifold of course exists in the extended phase space, $H_r^{q,2} \times \{\tau \mid \tau \geqq 1\}$; it is just that it is not convenient to study convergence toward it in this norm.) More precisely, one has

Theorem 3.6. *Fix $n \geqq 0$ and $\varepsilon > 0$, and suppose that the nonlinear term F in* (1.1) *is so smooth that Theorems* 2.2 *and* 3.5 *apply. Then for r sufficiently large, there exists a $1 + \sum_{j=0}^{n}\binom{j+d-1}{d-1}$-dimensional invariant manifold for* (1.1) *in the extended phase space $H_r^{q,2} \times \{\tau \mid \tau \geqq 1\}$. If, in addition, the solution $u(\xi, \tau)$ of* (1.1) *is such that when expressed in the (x, t) variables through* (2.1) *it remains in a sufficiently small neighborhood of the origin in $H_r^{q,2}$, then there exists a solution $\bar{u}(\xi, \tau)$ on the invariant manifold and a constant $C > 0$, such that*

$$\left(\int (1 + |\xi|^2/\tau)^r |u(\xi,\tau) - \overline{u}(\xi,\tau)|^2 \, d\xi \right)^{1/2} \leqq C\tau^{-\left(\frac{d}{4} + \frac{n+1}{2} - \varepsilon\right)}. \tag{3.9}$$

Remark. It may not be clear from the statement of the theorem what determines the value of the index q in the definition of $H_r^{q,2}$. In general (and in particular, in the applications in the next section) one finds that the requirement that the nonlinear term be smooth enough to apply Theorems 2.2 and 3.5 determine the allowed values of q.

Proof. The existence of the invariant manifold was discussed in the preceding paragraph so we need only verify (3.9). To see that this holds, note that if one rewrites $u(\xi,\tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log\tau)$, then Theorem 3.5 guarantees that if v remains in a sufficiently small neighborhood of the origin, then there exists a solution $\overline{v}(x,t)$ on the invariant manifold such that

$$\left(\int (1 + |x|^2)^r |v(x,t) - \overline{v}(x,t)|^2 \, dx \right)^{1/2} \leqq Ce^{-(|\lambda_{n+1}| - \varepsilon)t}. \tag{3.10}$$

Letting $t = \log\tau$, changing variables in the integral to $\xi = x\sqrt{\tau}$, and defining $\overline{u}(\xi,\tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log\tau)$ result immediately in (3.9). □

Note that Theorem 1.1 follows immediately from Theorem 3.6 if we define the requirement in Theorem 1.1 that solutions be "small" to mean that they are small in $H_r^{q,2}$ when expressed in the variables given by (2.1).

Remark. Since Theorem 3.5 implies that convergence toward a solution on the invariant manifold actually occurs in the $H_r^{q,2}$ norm, we could replace (3.10) by an estimate not just on $v - \overline{v}$, but also on derivatives of this difference. This would in turn give a strengthened version of (3.9), but one which is somewhat complicated to write because the various derivatives involve different powers of τ.

Remark. The estimate of the rate of convergence can also be written as

$$\left(\int (1 + |x|^2)^r |u(x\sqrt{\tau},\tau) - \overline{u}(x\sqrt{\tau},\tau)|^2 \, dx \right)^{1/2} \leqq C\tau^{-(n+1+d-\varepsilon)/2},$$

again, simply by changing variables in (3.10).

Remark. While it is somewhat awkward to bound the asymptotics in terms of norms depending on both space and time, such estimates are not uncommon in the study of the asymptotics of solutions of parabolic partial differential equations (Bricmont, Kupiainen, & Lin (1994), Filippas & Kohn (1992)). On the other hand, if one wishes to work with ordinary Sobolev norms, rather than these space-time norms, one can infer from (3.9) that

$$\|u(\tau) - \overline{u}(\tau)\|_{H_s^{0,2}} \leqq C\tau^{-\left(\frac{d}{4} + \frac{n+1}{2} - \frac{s}{2} - \varepsilon\right)}, \quad 0 \leqq s \leqq r, \tag{3.11}$$

by expanding the term $(1 + |\xi|^2/\tau)^r$ in the integrand. This estimate is only interesting for those values of s for which $\frac{d}{4} + \frac{n+1}{2} - \frac{s}{2} > 0$, but it emphasizes that if one wants to study the long-time asymptotics in a particular weighted Sobolev space, rather than the space-time norms in (3.9), then one can fix s, and then choose n so large that $\frac{d}{4}+\frac{n+1}{2}-\frac{s}{2}>0$. Constructing the invariant manifold corresponding to this n guarantees that there is a solution on the invariant manifold which captures the long-time behavior in the $H_s^{0,2}$ norm up to correction terms of $\mathcal{O}(\tau^{-(\frac{d}{4}+\frac{n+1}{2}-\frac{s}{2}-\varepsilon)})$.

4. Applications

In the present section two applications of the preceding theory are given. The first is the computation of higher-order asymptotics for equations like (1.1). I work through an example in some detail to illustrate the differences between these sorts of corrections and the lowest-order asymptotics discussed in Bricmont, Kupiainen, & Lin (1994). The second example shows that these results, when combined with bifurcation theory, give a new perspective on the change in the nature of the long-time behavior of solutions of

$$\frac{\partial u}{\partial \tau}=\Delta_\xi u-|u|^{p-1}u, \quad u=u(\xi,\tau),\ \xi\in\mathbb{R}^d,\ \tau\geqq 1, \tag{4.1}$$

when p passes through its critical value of $p_c=1+2/d$.

4.1. Higher-order asymptotics

In general, if one wishes to compute the asymptotics of solutions of an equation of the form (1.1) to $\mathcal{O}(t^{-\alpha})$, one chooses n so that $n+1+\frac{1}{2}d>2\alpha$. One then computes the pseudo-center manifold corresponding to the n lowest eigenvalues of the linear operator H_0 in (2.3). By Theorem 3.6, this manifold has dimension $1+\sum_{j=0}^{n}\binom{j+d-1}{d-1}$, and all trajectories in a neighborhood of the origin approach this manifold at a rate $\mathcal{O}(\tau^{-(n+1+(d/2)-\varepsilon)/2})<\mathcal{O}(t^{-\alpha})$ if ε is sufficiently small. Thus, the problem of computing the asymptotics of (1.1) has been reduced to the problem of computing the asymptotics of this finite system of ordinary differential equations, for which well-understood techniques exist.

As an example, let us consider the initial-value problem

$$\frac{\partial u}{\partial \tau}=\Delta_\xi u-u^4, \quad u(\xi,1)=u_0(\xi),\ \xi\in\mathbb{R}^2, \quad \tau\geqq 1. \tag{4.2}$$

By considering the second-order asymptotics, one can prove

Proposition 4.1. *Fix* $\varepsilon>0$. *For* $u_0(\xi)$ *in a sufficiently small neighborhood of the origin, there exist constants* $\mathbf{z}(0)=(z_{00},z_{10},z_{01})$, *and* $K>0$, *and functions* $\zeta_{00},\zeta_{10},\zeta_{01}$, *such that*

$$\Bigg\| u(\xi_1\sqrt{\tau}, \xi_2\sqrt{\tau}, \tau) - \Bigg\{ \frac{1}{\tau}\frac{1}{2\sqrt{\pi}}\left(\frac{3}{5}\right)^2 (z_{00} + \zeta_{00}(\mathbf{z}(0)))e^{-(\xi_1^2+\xi_2^2)/4}$$
$$+ \frac{1}{\tau^{3/2}}\frac{1}{2\sqrt{2\pi}}\left(\frac{6}{13}\right)^2 (z_{10} + \zeta_{10}(\mathbf{z}(0)))H_1(\xi_1)e^{-(\xi_1^2+\xi_2^2)/4}$$
$$+ \frac{1}{\tau^{3/2}}\frac{1}{2\sqrt{2\pi}}\left(\frac{6}{13}\right)^2$$
$$\times (z_{01} + \zeta_{01}(\mathbf{z}(0)))H_1(\xi_2)e^{-(\xi_1^2+\xi_2^2)/4} \Bigg\} \Bigg\|_{H_r^{q,2}} \leqq K\tau^{2-\varepsilon}.$$

Remark. The function H_1 is the first Hermite polynomial. The constants z_{00}, z_{10}, z_{01} can be computed in terms of the initial condition u_0, while the functions $\zeta_{00}, \zeta_{10}, \zeta_{01}$ are determined by the flow on the pseudo-center manifold – a system of three ordinary differential equations in this case.

Proof. Transformed to the "w"-variables used in Section 2, the equation becomes

$$\frac{\partial w}{\partial t} = H_0 w + e^{-3|x|^2/8} w^4, \quad w(x,0) = u_0(x),\ t \geqq 0, \tag{4.3}$$

where $H_0 = \Delta_x - \frac{x^2}{16} - \frac{1}{6}$. The spectrum of H_0 is $\sigma(H_0) = \{-\frac{2}{3} - \frac{m}{2} \mid m = 0, 1, 2, \ldots\}$ and we construct the pseudo-center manifold corresponding to the two lowest eigenvalues. In order to apply Theorem 2.2, Proposition 3.3, and Theorem 3.5, we must verify that the nonlinear term in the equation is sufficiently smooth. We begin by showing that $f(w) = \exp(-3|x|^2/8)w^4$ is in C^k with $k > 1$ on $\tilde{H}^{q,2}$ for any $q > \frac{1}{2}d = 1$. (Note that because the nonlinear term is a homogeneous function, f does not depend on the auxiliary variable η.) If one defines $g(x,y) = \exp(-3|x|^2/8)y^4$, then $f(w)(x) = g(x, w(x))$. By Theorem 4.3 of VALENT (1988), if $g \in C^{m+k}$, and $w \in H^{m,2}$, with $m > \frac{1}{2}d$, then the composition $f(w)(x) = g(x, w(x))$ is a C^k function from $H^{m,2}$ to itself. The proof in VALENT (1988) may be readily modified to show that the result is also true in the weighted Sobolev spaces $\tilde{H}^{q,2}$, and $H_r^{q,2}$. We can also extend the theorem to allow for functions defined on unbounded domains in $\mathbb{R}^d$, provided that g and its derivatives vanish at $y = 0$ as is the case here. Thus, since $g \in C^\infty$, we conclude that $f(w)$ is a C^k function on $\tilde{H}^{q,2}$ for any $q > 1$, and any k. Similarly, if we rewrite (4.2) in terms of the "v"-variables, the nonlinearity becomes simply $\tilde{f}(v) = v^4$, and again the results of VALENT (1989) imply that the nonlinearity is C^k on $H_r^{q,2}$ for any k. Thus, we may apply all of the results of Sections 2 and 3 to this example, provided that we choose $q > 1$. For concreteness choose $q = 4$. Proposition (3.3) then implies that all solutions in a sufficiently small neighborhood of the origin, approach this manifold at a rate $\mathcal{O}(1/t^{\frac{5}{3}-\varepsilon})$, where we can make ε as small as we like by taking the neighborhood on which we work sufficiently small.

In order to construct the invariant manifold we need to know the eigenfunctions of H_0, which in this case are $\varphi_{jk}(x,y) = \gamma_{jk}H_j(x)H_k(y)$ $\exp(-(x^2+y^2))/8$, where the constants γ_{jk} are chosen so that $\langle \varphi_{jk}, \varphi_{\tilde{j}\tilde{k}} \rangle_q =$

$\delta_{j\tilde{j}}\delta_{k\tilde{k}}$, and H_k is the k^{th} Hermite polynomial. In particular, if we choose the constant $c(p)$ in the definition of the inner product in $\tilde{H}^{q,2}$ to be 1 and take $q = 4$, we find that $\gamma_{00} = 9/(50\sqrt{\pi})$ and $\gamma_{10} = \gamma_{01} = 18/(169\sqrt{2\pi})$. The eigenvalue corresponding to φ_{jk} is $\lambda_{jk} = -\frac{2}{3} - \frac{(j+k)}{2}$. Thus, the "central subspace" $\mathscr{E}^c = \text{span}\{\varphi_{00}, \varphi_{10}, \varphi_{01}\}$.

By Theorem 3.5, any solution of (4.3) in a sufficiently small neighborhood of the origin approaches a solution of

$$\dot{\psi} = H_0^c \psi + F^c(\psi, h(\psi)). \tag{4.4}$$

Remark. Equation (4.4) is just (3.6) rewritten in the "w"-variables. Also, since the nonlinear term in (4.1) is a homogeneous function, the nonlinear term in (4.4) is independent of η.

Let z_{jk} be coordinates in W^s in the φ_{jk} direction, and let $h_{jk}(z_{00}, z_{01}, z_{10}) \equiv h_{jk}(\mathbf{z})$, $j + k \geqq 2$ be the component of h in the φ_{jk} direction. (Recall that h is the function whose graph gives the invariant manifold.) We can compute h by standard techniques (see, e.g, Carr (1983)) and we find that

$$h_{jk}(\mathbf{z}) = \sum_{\substack{m_1+m_2+m_3=4 \\ m_j \geqq 0}} \frac{-c_{jk}(\mathbf{m})}{(\lambda_{jk} + \frac{8}{3} + \frac{m_2+m_3}{2})} \mathbf{z}^{\mathbf{m}} + \mathscr{O}(|\mathbf{z}|^5). \tag{4.5}$$

I have used the standard multi-index notation here, so that $\mathbf{z}^{\mathbf{m}} \equiv z_{00}^{m_1} z_{01}^{m_2} z_{10}^{m_3}$. Also, $c_{jk}(\mathbf{m})$ is the coefficient of $\mathbf{z}^{\mathbf{m}}$ in the eigenfunction expansion of the nonlinear term in (4.3). More precisely, it is the coefficient of $\mathbf{z}^{\mathbf{m}}$ in the expression $\langle \varphi_{jk}, \exp(-3(x^2+y^2)/8)(z_{00}\varphi_{00} + z_{01}\varphi_{01} + z_{10}\varphi_{10})^4 \rangle_4$. Again, these are straightforward to compute and one finds, for example, $c_{00}(4,0,0) = (\frac{1}{\pi})^{3/2}(\frac{1}{4})^{5/2}(\frac{3}{5})^6$, $c_{00}(3,1,0) = 0$, $c_{00}(3,0,1) = 0, \ldots$

Using this information about h, one can write (4.4) more explicitly as

$$\begin{aligned}
\dot{z}_{00} &= -\tfrac{2}{3} z_{00} + \sum_{\substack{m_1+m_2+m_3=4 \\ m_j \geqq 0}} c_{00}(\mathbf{m})\mathbf{z}^{\mathbf{m}} + \mathscr{O}(|z|^5), \\
\dot{z}_{10} &= -\tfrac{7}{6} z_{10} + \sum_{\substack{m_1+m_2+m_3=4 \\ m_j \geqq 0}} c_{10}(\mathbf{m})\mathbf{z}^{\mathbf{m}} + \mathscr{O}(|z|^5), \\
\dot{z}_{01} &= -\tfrac{7}{6} z_{01} + \sum_{\substack{m_1+m_2+m_3=4 \\ m_j \geqq 0}} c_{01}(\mathbf{m})\mathbf{z}^{\mathbf{m}} + \mathscr{O}(|z|^5).
\end{aligned} \tag{4.6}$$

We can easily derive the asymptotics of this system of equations. If we start from the point $\mathbf{z}(0) = (z_{00}(0), z_{10}(0), z_{01}(0))$, then we find

$$\begin{aligned}
z_{00}(t) &= e^{-2t/3}(z_{00}(0) + \zeta_{00}(\mathbf{z}(0)) + \mathscr{O}(e^{-\mu t}), \\
z_{10}(t) &= e^{-7t/6}(z_{10}(0) + \zeta_{10}(\mathbf{z}(0)) + \mathscr{O}(e^{-\mu t}), \\
z_{01}(t) &= e^{-7t/6}(z_{10}(0) + \zeta_{00}(\mathbf{z}(0)) + \mathscr{O}(e^{-\mu t}),
\end{aligned}$$

with $\mu \geqq 2$. The functions ζ_{jk} can be approximated by their Taylor series and we find that

$$\zeta_{jk}(\mathbf{z}(0)) = \sum_{\substack{m_1+m_2+m_3=4 \\ m_j \geqq 0}} \frac{6c_{jk}(\mathbf{m})}{(4m_1 + 7m_2 + 7m_3 - 4)} \mathbf{z}(0)^{\mathbf{m}} + \mathcal{O}(|\mathbf{z}|^5).$$

If we now return to the variables of (4.4), this implies that

$$\begin{aligned}\psi(x,y,t) = {} & e^{-2t/3}(z_{00}(0) + \zeta_{00}(\mathbf{z}(0)))\varphi_{00}(x,t) + e^{-7t/6}(z_{10}(0) + \zeta_{10}(\mathbf{z}(0)))\varphi_{10}(x,t) \\ & + e^{-7t/6}(z_{01}(0) + \zeta_{01}(\mathbf{z}(0)))\varphi_{01}(x,t) + \mathcal{O}(e^{-\mu_1 t}),\end{aligned}$$

for some $\mu_1 \geqq 2$. In light of Proposition 3.3 and the fact that $h_{jk}(\mathbf{z}) = \mathcal{O}(|\mathbf{z}|^4)$, we see that if $w(t) = (w^r(t), w^s(t))$ is a solution of (4.3) which remains in a neighborhood of the origin for all time, then there exists $\mathbf{z}(0) = (z_{00}(0), z_{10}(0), z_{01}(0))$ such that

$$w^c(t) = \psi(t), \quad w^s(t) = \mathcal{O}(e^{-8/3t}) + \mathcal{O}(e^{-\mu_2 t}),$$

where μ_2 can be made as close as we like to $\frac{5}{3}(= \lambda_{02} = \lambda_{11} = \lambda_{20})$. Thus, if we revert to our original variables, we see that for long times, the solution of (4.1) is

$$\begin{aligned}u(\xi\sqrt{\tau},\tau) = {} & \frac{1}{2\sqrt{\pi}}\left(\frac{3}{5}\right)^2 \frac{1}{\tau}(z_{00} + \zeta_{00}(\mathbf{z}(0))) \exp\left(-\tfrac{1}{4}(\xi_1^2 + \xi_2^2)\right) \\ & + \frac{1}{2\sqrt{2\pi}}\left(\frac{6}{13}\right)^2 \frac{1}{\tau^{3/2}}(z_{10} + \zeta_{10}(\mathbf{z}(0)))H_1(\xi_1) \exp\left(-\tfrac{1}{4}(\xi_1^2 + \xi_2^2)\right) \\ & + \frac{1}{2\sqrt{2\pi}}\left(\frac{6}{13}\right)^2 \frac{1}{\tau^{3/2}}(z_{01} + \zeta_{01}(\mathbf{z}(0))) \\ & \times H_1(\xi_2) \exp\left(-\tfrac{1}{4}(\xi_1^2 + \xi_2^2)\right) + \mathcal{O}\left(\frac{1}{\tau^{2-\varepsilon}}\right),\end{aligned}$$

where ε can be made arbitrarily small by choosing a sufficiently small neighborhood of the origin. This estimate completes the proof of Proposition (4.1.) □

Remark. Note that while the lowest-order asymptotics for this equation (see, e.g., BRICMONT, KUPIAINEN, & LIN (1994)) are "universal" – that is, regardless of initial conditions they approach the function $u^*(\xi\sqrt{\tau},\tau) = \frac{K}{\tau}e^{-(\xi_1^2+\xi_2^2)/4}$ for some choice of K, the higher-order asymptotics are more complicated, in particular, not universal, and depend in a nontrivial way on the initial conditions. In related equations, which model the propagation of waves on a fluid surface, such dependence may have experimentally observable consequences (see BONA, PROMISLOW, & WAYNE (1995)).

Remark. Since Theorem 2.2 only guarantees that the invariant manifold is of class $C^{1+\alpha}$, one may wonder whether expansions like (4.5) are justified. However, as CARR (1983, Section 2.5) shows, if we can find a function $\tilde{h}$

which satisfies the equation defining the invariant manifold up to terms of order $\mathcal{O}(|x|^s)$, with $s > 1$, then $\tilde{h}$ provides an approximation of the true invariant manifold up to an error of the same order. It is also worth remarking, however, that there is no guarantee that we can solve for the Taylor coefficients of h to arbitrary order, as resonances between the eigenvalues of H_0 may prevent us from continuing beyond a certain point.

Remark. While this example shows that the second-order asymptotics are more complicated than the first-order, yet more involved phenomena may occur. For examples of such problems, the reader may wish to consider the higher-order asymptotics of $\partial_\tau u = \partial_{\xi\xi} u - u^4$, or $\partial_\tau u = \partial_{\xi\xi} u - u^3$, for $\xi \in \mathbb{R}$. In both cases we find that the higher-order asymptotics contain terms involving $\log \tau$ rather than just inverse powers of τ.

4.2. Behavior of solutions for p near $p_c = 1 + 2/d$.

In this subsection we consider the behavior of solutions of (4.1), for $p \approx 1 + 2/d$. This question has already received much attention (Galaktionov, Kudymov, & Samarskii (1986), Kamin & Peletier (1985), Bricmont & Kupiainen (1996), Escobedo & Kavian (1988), Escobedo, Kavian, & Matano (1995)). I wish to illustrate a new outlook on the change in the behavior of solutions when p passes through p_c by using ideas from bifurcation theory, coupled with the invariant-manifold theorems of this paper. For reasons explained below, I restrict consideration to one spatial dimension. If we rewrite (4.1) in the "w"-variables, one finds that

$$\frac{\partial w}{\partial \tau} = H_0(p_c)w + \frac{(p_c - p)}{(p_c - 1)(p - 1)} w - e^{-\frac{1}{8}(p-1)x^2} |w|^{p-1} w.$$

Now use a standard "trick" from bifurcation theory (see, e.g., Ruelle (1989)) and introduce $\rho = p_c - p$ as a new variable. Then, using the fact that $p_c = 3$ for $d = 1$, one has:

$$\dot{\rho} = 0,$$

$$\frac{\partial w}{\partial \tau} = \left(\frac{d^2}{dx^2} - \frac{x^2}{16} + \frac{1}{4}\right) w + \frac{\rho w}{2(2-\rho)} - e^{-\frac{1}{8}(2-\rho)x^2} |w|^{2-\rho} w. \tag{4.7}$$

Noting that the spectrum of $\left(\frac{d^2}{dx^2} - \frac{x^2}{16} + \frac{1}{4}\right)$ is $\{0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\}$, we see that provided the nonlinear term in this system satisfies the hypotheses of Theorem 2.2, (4.7) has a two-dimensional center manifold, tangent at the origin to the zero eigenspace of $H_0(p_c)$ and the ρ direction.

To verify the hypotheses on the nonlinear term, define $g(\rho, x, y) = \exp(-\frac{1}{8}(2-\rho)x^2)|y|^{2-\rho} y$; then $g \in C^2$ (for $|\rho| < 1$), and hence by Theorem 4.1 of Valent (1988) (again, suitably modified to account for the weights in $\tilde{H}^{q,2}$ and the fact that we work with functions defined on unbounded domains), $f(\rho, x) = g(\rho, x, w(x))$ is a C^1 function of $\tilde{H}^{1,2}$ to itself.

(Here I have fixed $q = 1$. This is where the restriction to $d = 1$ arises. If one wished to work in $d = 2$ or 3, one would have to assume that $w \in \tilde{H}^{2,2}$, and then f would not be smooth enough to apply Theorem 4.1 of VALENT (1988). I believe that this is only a technical restriction which could be overcome by taking advantage of the smoothing that comes from the linear semi-group.) In fact, to apply Theorem 2.2, one requires that $f \in C^k$ with $k > 1$. To see that this is true, note that if $|\rho| < \frac{1}{2}$, then $g \in C^{2+\eta}$, for some $\eta > \frac{1}{2}$. By an extension of VALENT's argument, which is included in Appendix B, one can show that this implies that the nonlinearity is then a $C^{1+\eta}$ function from $\tilde{H}^{1,2}$ to itself. Thus, Theorem 2.2 applies to (4.7). If we now rewrite (4.7) in terms of the "v"-variables, the nonlinear term becomes $\tilde{f}(\rho, x) = |v|^{2-\rho}v$, and if $|\rho| < 1$, then $\tilde{f} \in C^1$ (as a function from $\tilde{H}^{1,2}$ to itself), and this suffices to ensure that the results of Section 3 also apply to this example.

Turning now to the actual construction of the center manifold, we take $c(p) = 1$ (in the definition of $\tilde{H}^{1,2}$), and we then find that the approximate equations in the center manifold are

$$\begin{aligned} \dot{\rho} &= 0, \\ \dot{z} &= \frac{\rho}{2(2-\rho)} z - \frac{1}{2}\frac{1}{\sqrt{3\pi}} z^3 + \mathcal{O}_4(\rho, z), \end{aligned} \tag{4.8}$$

where z is the coordinate in the direction of $\varphi_0(x)$, the eigenfunction of H_0 with zero eigenvalue, and $\mathcal{O}_4(\rho, z)$ means that the terms that have been omitted are of the form $\rho^\alpha z^\beta$ with $\alpha + \beta \geqq 4$. Note that for $\rho < 0$ (i.e., $p > p_c$), the origin is a stable fixed point of (4.8). Thus, $z(t) \approx z(0)\exp(\rho t/2(2-\rho))$, or in the original variables,

$$u(\xi\sqrt{\tau}, \tau) \approx \frac{K}{\tau^{\frac{1}{p-1}}} \tau^{\frac{\rho}{2(2-\rho)}} e^{\frac{-\xi^2}{4}} = \frac{K}{\sqrt{\tau}} e^{\frac{-\xi^2}{4}},$$

as expected.

On the other hand, as ρ passes through zero, the origin becomes unstable and a new stable fixed point of (4.8) appears at $z^* \approx \sqrt{\sqrt{3\pi}\rho/(2-\rho)}$. (This is the exact location of the fixed point if one ignores the terms $\mathcal{O}_4(\rho, z)$.) Thus in this case,

$$z(t) \approx z^* + K(z(0))e^{-\mu t},$$

where for ρ small, $\mu = -\frac{1}{2}\rho + o(\rho)$. Thus, if $w^*(x)$ is the function corresponding to z^* (the properties of w^* have been studied in BREZIS, PELETIER & TERMAN (1986), GALAKTIONOV, KUDYMOV, & SAMARSKII (1986), KAMIN & PELETIER (1985)) we find that

$$\|w(x,t) - w^*(x)\|_{\tilde{H}^{1,2}} \leqq Ke^{-\mu t},$$

or if we again revert to our original variables,

$$u(\xi\sqrt{\tau}, \tau) \approx \frac{1}{\tau^{1/(p-1)}} e^{\xi^2\tau/2} w^*(\xi\sqrt{\tau}) + \mathcal{O}\left(\frac{1}{\tau^{\mu+\frac{1}{p-1}}}\right).$$

Note that in addition to the appearance of a new limiting function w^*, the rate of decay has also changed from $t^{-1/2}$ to $t^{-1/(p-1)}$ as p passed through p_c.

This is again consistent with the results of Escobedo & Kavian (1988) and Bricmont & Kupiainen (1996), but in contrast to those results, gives an explicit estimate of the rate of decay of the higher-order terms for ρ small. Since this paper was submitted, the work of Escobedo, Kavian, & Matano (1995) has appeared in which they also compute the rate of decay of the higher-order terms, albeit by very different methods.

Finally, note that exactly at $p_c = 3$, the equations on the center manifold become

$$\dot{\rho} = 0,$$
$$\dot{z} = -\frac{1}{2}\frac{1}{\sqrt{3\pi}}z^3 + \mathcal{O}_4(\rho, z).$$

In this case, one finds that $z(t)$ approaches the origin as $z(t) \approx 1/\sqrt{t}$. Reverting to our original variables, one finds that

$$u(\xi\sqrt{\tau}, \tau) \approx \frac{K}{\sqrt{\tau \log \tau}} e^{-\xi^2/4}.$$

Thus we immediately and easily recover the logarithmic corrections to the decay rate in the critical case.

5. Conclusions and Possible Extensions

One obvious extension of this work is to expand the class of nonlinear terms allowed in (1.1). For instance, the studies of the lowest-order asymptotics of such equations (e.g., Bricmont, Kupiainen, & Lin (1994)) admit nonlinearities of the form $F(u, \nabla u)$ which depend on the gradient of u. In the present case we cannot include such terms because of the requirement of Gallay's theorem that the nonlinear term be a C^k function with $k > 1$. Such a condition seems to be necessary if one works with only very weak assumptions on the linear evolution, as Gallay does. In the present situation, however, we have quite detailed knowledge of the linear evolution, and I hope that by taking advantage of that knowledge, one can extend these "pseudo-center" manifolds to equations with derivatives in the nonlinear terms, much as Mielke (1991) was able to do for the ordinary center manifolds.

Another natural extension would be to examine the possible existence of invariant manifolds in the neighborhood of non-constant solutions of (1.1) such as fronts. As mentioned in the introduction, a number of authors have studied the stability of fronts and derived essentially the first-order asymptotics of solutions near these fronts. If the present work could be extended to such situations, it would give a much clearer picture of the invariant geometrical structures in the phase space of such partial differential equations, and also allow one to derive higher-order asymptotics for solutions near these fronts.

A third intriguing question is whether or not results analogous to those described here are also applicable to dispersive equations. Strauss (1974) has investigated the long-time behavior of nonlinear perturbations of the Schrödinger equation and linearized Korteweg-de Vries equation and found that under appropriate assumptions on the nonlinear term, solutions of such equations approach a solution of the linear equation, just as do solutions of (1.1), when $p > p_c$. It is natural to wonder whether the geometric structures underlying the long-time behavior in the dissipative case are also present in the dispersive equations.

Appendix A

In this appendix I prove Proposition (3.2), which forms the basis of the study of the attractivity of the pseudo-center manifold in Section 3. Recall that those estimates hold in the weighted Sobolev spaces

$$H_r^{q,2} = \left\{ v \,\middle|\, \sum_{|\alpha| \leqq q} \|D^\alpha (1+|x|^2)^{r/2} v\|_{L^2} < \infty \right\}.$$

The desired estimate is proved by representing the linear semi-group $\exp(t\mathscr{L}_0)$ as an integral using MEHLER's formula, and then estimating this integral by considering separately its action on functions supported inside, and outside, of a ball of radius R. Since the integral is time-dependent, the radius R is also chosen to depend on time (although, I do not make this dependence explicit in the notation). Define a smooth function $\chi_l(x)$ such that

$$\chi_l(x) = \begin{cases} 0 & \text{if } |x| \leqq R, \\ 1 & \text{if } |x| \geqq \frac{8}{7}R. \end{cases}$$

Given a positive integer N, define Q_N as in Section 3. Note that if $v \in H_r^{q,2}$ is in the range of Q_N, then $Q_N v = v$. Also, given any function f, define $f_>(x) = \chi_l(x) f(x)$ and $f_<(x) = (1-\chi_l(x)) f(x)$. The essence of the proof of Proposition (3.2) is contained in the following pair of propositions:

Proposition A.1. *If* $r \geqq \min(q, N + \frac{1}{2}(d+1))$, *then there exists* $C(r,q) > 0$, *independent of* R, *such that*

$$\|e^{t\mathscr{L}_0} Q_N v_<\|_{H_r^{q,2}} \leqq C(r,q) e^{R^2/6} e^{-|\lambda_{N+1}|t} \|v\|_{H_r^{q,2}}.$$

Proposition A.2. *If* $r \geqq \min(q, N + \frac{1}{2}(d+1))$ *and* $R \geqq 1$, *then there exists* $C(r,q) > 0$, *independent of* R, *such that*

$$\begin{aligned} \|e^{t\mathscr{L}_0} Q_N v_>\|_{H_r^{q,2}} \leqq C(r,q) \Big\{ & e^{R^2/6} e^{(\frac{1}{p-1} + \frac{q}{2} - \frac{d}{4})\frac{t}{2}} e^{-|\lambda_{N+1}|\frac{t}{2}} \\ & + e^{(\frac{1}{p-1} + \frac{q}{2} - \frac{d}{4} - \frac{r}{4})t} + e^{(\frac{1}{p-1} + \frac{q}{2} - \frac{d}{4})t} e^{-\frac{3}{16}R^2} \Big\} \|v\|_{H_r^{q,2}}. \end{aligned}$$

Before proving these two propositions, I now show that they imply Proposition 3.2. Let n be the integer fixed in Proposition 3.2. Fix

$$N > \max\left(4n + 3d - \frac{6}{p-1} + 3, \frac{142}{p-1} + 72q - 1\right).$$

Note that this definition, together with the definition of the eigenvalues λ_m, ensures that

$$|\lambda_{N+1}| > 4|\lambda_{n+1}|, \ -\tfrac{5}{18}|\lambda_{N+1}| + 2\left(\frac{1}{p-1} + \frac{q}{2}\right) < -\tfrac{1}{4}|\lambda_{N+1}|.$$

Next choose

$$r > |\lambda_{N+1}| + \frac{4}{p-1} + 2q - d.$$

This definition ensures that $(\frac{1}{p-1} + \frac{q}{2} - \frac{d}{4} - \frac{r}{4}) < -\frac{1}{4}|\lambda_{N+1}|$. Finally, define

$$R = \max\left(1, \left\{\frac{16t}{3}\left(\frac{|\lambda_{N+1}|}{4} + \frac{1}{p-1} + \frac{q}{2} - \frac{d}{2}\right)\right\}^{1/2}\right).$$

If one combines the estimates of Propositions A.1 and A.2, one sees that

$$\begin{aligned}\|e^{t\mathscr{L}_0} Q_N v\|_{H_r^{q,2}} &\leqq \|e^{t\mathscr{L}_0} Q_N v_<\|_{H_r^{q,2}} + \|e^{t\mathscr{L}_0} Q_N v_>\|_{H_r^{q,2}} \\ &\leqq C(r,q)\{e^{\frac{R^2}{6}} e^{-|\lambda_{N+1}|t} + e^{\frac{R^2}{6}} e^{(\frac{1}{p-1}+\frac{q}{2}-\frac{d}{4})\frac{t}{2}} e^{-|\lambda_{N+1}|\frac{t}{2}} \\ &\quad + e^{(\frac{1}{p-1}+\frac{q}{2}-\frac{d}{4}-\frac{r}{4})t} + e^{(\frac{1}{p-1}+\frac{q}{2}-\frac{d}{4})t} e^{-\frac{3}{16}R^2}\}\|v\|_{H_r^{q,2}}.\end{aligned}$$

Inserting the definitions of the various constants from above, we see that this expression is bounded by $C(r,q)e^{-|\lambda_{N+1}|t/4}\|v\|_{H_r^{q,2}}$. Thus, we have

Corollary A.3. *If* $r \geqq \min(q, N + (d+1)/2)$ *and if* N *and* r *are chosen as above, there exists a constant* $C(r,q,n)$ *such that*

$$\|e^{t\mathscr{L}_0} Q_N v\|_{H_r^{q,2}} \leqq C(r,q,n) e^{-|\lambda_{N+1}|t/4}\|v\|_{H_r^{q,2}}.$$

To complete the proof of Proposition 3.2, we now introduce the projection operators P_n and Q_n in addition to P_N and Q_N. Rewrite

$$e^{t\mathscr{L}_0} Q_n v = e^{t\mathscr{L}_0}(P_N + Q_N)Q_n v = e^{t\mathscr{L}_0} Q_N Q_n v + e^{t\mathscr{L}_0} P_N Q_n v.$$

By Corollary A.3, we can bound

(A1) $$\|e^{t\mathscr{L}_0} Q_N Q_n v\|_{H_r^{q,2}} \leqq C(r,q,n)e^{-|\lambda_{N+1}|t/4}\|v\|_{H_r^{q,2}} \leqq C(r,q,n)e^{-|\lambda_{n+1}|t}\|v\|_{H_r^{q,2}}.$$

On the other hand, by the orthonormality of the ψ_j's,

$$P_N Q_n v = P_N(\mathbf{1} - P_n)v = \sum_{j=n+1}^{N} \psi_j \langle\langle \psi_j, v\rangle\rangle_q,$$

while $\exp(t\mathscr{L}_0)\psi_j = \exp(t\lambda_j)\psi_j$, so that

(A2)

$$\|e^{t\mathscr{L}_0} P_N Q_n v\|_{H_r^{q,2}} = \|\sum_{j=n+1}^{N} \exp(t\lambda_j)\psi_j\langle\langle\psi_j, v\rangle\rangle_q\|_{H_r^{q,2}} \leqq C(r,q,n)e^{-t|\lambda_{n+1}|}\|v\|_{H_r^{q,2}}.$$

Combining (A.1) and (A.2) completes the proof of Proposition 3.2. □

We now return to the proofs of Propositions A.1 and A.2. The proof of Proposition A.1 is based on the fact that since $v_<$ is zero outside a ball of radius R, it is in the Sobolev space $\tilde{H}^{q,2}$ of Section 2, and hence we can estimate the action of the linear semi-group on such functions with the aid of the spectral theorem. Begin by noting

Lemma A.4. *If $w \in \tilde{H}^{q,2}$, then there exists $C(q,r) \geqq 0$ such that*

$$\|e^{-x^2/8} w(x)\|_{H_r^{q,2}} \leqq C(q,r)\|w\|_{\tilde{H}^{q,2}}.$$

Proof. This follows from the definition of the norms and the fact that for every $r > 0$, there exists $K(r) > 0$, such that $\sup_x (1+|x|^2)^r e^{-x^2/8} \leqq K(r)$. □

Proof of Proposition A.1. By Lemma A.4,

$$\|e^{t\mathscr{L}_0} Q_N v_<\|_{H_r^{q,2}} = \|e^{-x^2/8} e^{t\mathscr{H}_0} e^{x^2/8} Q_N v_<\|_{H_r^{q,2}} \leqq C(q,r)\|e^{t\mathscr{H}_0} e^{x^2/8} Q_N v_<\|_{\tilde{H}^{q,2}}.$$

But $e^{x^2/8} Q_N v_< = \tilde{Q}_N e^{x^2/8} v_<$, where $\tilde{Q}_N$ is the projection onto the orthogonal complement of $\mathrm{span}\{\varphi_0, \dot{s}, \varphi_N\}$ in $\tilde{H}^{q,2}$. Thus,

$$\|e^{t\mathscr{H}_0} \tilde{Q}_N e^{x^2/8} v_<\|_{\tilde{H}^{q,2}} \leqq C(q) e^{-|\lambda_{N+1}|t} \|e^{x^2/8} v_<\|_{\tilde{H}^{q,2}}.$$

But now we note that since $v_<(x) = 0$ if $|x| \geqq \frac{8}{7}R$, $\|\exp(\frac{1}{8}x^2)v_<\|_{\tilde{H}^{q,2}} \leqq C(q)(1+R^r)\exp(\frac{8}{49}R^2)\|v\|_{H_r^{q,2}}$, by the definition of the norms, and Proposition A.1 follows. □

Remark. Because R depends on time, it is important to stress that the constant $C(r,q)$ in Proposition A.1 is independent of R – all of the R dependence in this estimate is captured in the exponential.

We now turn to the more difficult estimate, namely, Proposition A.2. We begin with an estimate of the action of the linear semi-group.

Proposition A.5. *There exists $C(r,q) > 0$ (independent of R) such that for $R \geqq 1$, and any $v \in H_r^{q,2}$,*

$$\|\chi_l \exp(t\mathscr{L}_0)v\|^2_{H_r^{q,2}} \leqq C(r,q) e^{\frac{2t}{p-1}} e^{-\frac{dt}{2}} \{e^{\frac{(q-r)t}{2}} + e^{\frac{tq}{2}} e^{-\frac{3}{16}R^2}\}^2 \|v\|^2_{H_r^{q,2}}.$$

Proof. Using MEHLER's formula for $\exp(t\mathscr{L}_0)$ as in BRICMONT & KUPIAINEN (1996), we find that

$$(e^{t\mathscr{L}_0}v)(x) = \frac{1}{(4\pi a(t))^{-d/2}} e^{t/(p-1)} \int e^{-z^2/(4a(t))} v(e^{t/2}(x+z))\, dz,$$

where $a(t) = 1 - \exp(-t)$. Thus,

$$\begin{aligned}
\|\chi_l e^{t\mathscr{L}_0} v\|^2_{H^{q,2}_r} &\leqq \sum_{|\alpha| \leqq q} \|D^\alpha (1+|\cdot|^2)^{r/2} \chi_l(\cdot)(e^{t\mathscr{L}_0}v)(\cdot)\|^2_{L^2} \\
&\leqq \sum_{|\alpha| \leqq q} \frac{1}{(4\pi a(t))^{-d}} e^{2t/(p-1)} \left| \int dz_1\, dz_2 e^{-z_1^2/(4a(t))} e^{-z_2^2/(4a(t))} \right. \\
&\quad \times \int (D^\alpha_x (1+|x|^2)^{r/2} \chi_l(x) v(e^{t/2}(x+z_1))) \\
&\quad \left. \times (D^\alpha_x (1+|x|^2)^{r/2} \chi_l(x) v(e^{t/2}(x+z_2)))\, dx \right| \\
&\leqq \sum_{|\alpha| \leqq q} \left(\frac{1}{(4\pi a(t))^{-d/2}} e^{t/(p-1)} \int dz\, e^{-z_1^2/(4a(t))} \right. \\
&\quad \left. \times \|D^\alpha (1+|\cdot|^2)^{r/2} \chi_l(\cdot) v(e^{t/2}(\cdot + z))\|_{L^2} \right)^2,
\end{aligned} \tag{A.3}$$

where the last inequality applied the Cauchy-Schwarz inequality to estimate the integral with respect to x.

The integral over z is estimated by breaking it up into two pieces according to whether z is less than or greater than $\frac{7}{8}R$, and then estimating each of those pieces with the aid of

Lemma A.6. *There exists* $C(r,q) > 0$ (*independent of* R) *such that for any* $|\alpha| \leqq q$,

(i) *If* $|z| \leqq \frac{7}{8}R$,

$$\|D^\alpha (1+|\cdot|^2)^{r/2} \chi_l(\cdot) v(e^{t/2}(\cdot + z))\|^2_{L^2} \leqq C(r,q) e^{t(q-r-d/2)} \|v\|^2_{H^{q,2}_r}.$$

(ii) *If* $|z| > \frac{7}{8}R$,

$$\|D^\alpha (1+|\cdot|^2)^{r/2} \chi_l(\cdot) v(e^{t/2}(\cdot + z))\|^2_{L^2} \leqq C(r,q) e^{t(q-d/2)} (1+|z|^2)^r \|v\|^2_{H^{q,2}_r}.$$

Remark. The constant $C(r,q)$, also depends on the L^∞ norm of the cutoff function χ_l, and its derivatives of order q or less. χ_l can be chosen so that these derivatives are bounded independently of R for $R \geqq 1$. We assume throughout the remainder of this appendix that $R \geqq 1$, and hence we supress this dependence.

Applying this lemma to (A.3), one finds

$$\|\chi_l e^{t\mathscr{L}_0} v\|^2_{H^{q,2}_r} \leqq C(q,r)(4\pi a(t))^{-d} e^{(\frac{2}{p-1}-\frac{d}{2})t} \|v\|^2_{H^{q,2}_r} \times \left\{ \int_{|z| \leqq 7R/8} e^{-z^2/(4a(t))} e^{(q-r)t/2} \, dz \times \int_{|z|>7R/8} e^{-z^2/(4a(t))} e^{tq/2} (1+|z|^2)^{r/2} \, dz \right\}^2 .$$

Elementary estimates of these integrals then complete the proof of Proposition A.5. ■

Proof of Lemma A.6. Consider first case (i).

$$\begin{aligned} &\|D^\alpha (1+|\cdot|^2)^{r/2} \chi_l(\cdot) v(e^{t/2}(\cdot + z))\|^2_{L^2} \\ &\quad = \int (D^\alpha_x (1+|x|^2)^{r/2} \chi_l(x) v(e^{t/2}(x+z)))^2 \, dx \\ &\quad \leqq C(|\alpha|, r) \sum_{|\beta|+|\gamma| \leqq |\alpha|} \int (1+|x|^2)^r |D^\beta \chi_l(x)|^2 e^{|\gamma|t} |D^\gamma v(e^{t/2}(x+z))|^2 \, dx, \end{aligned}$$

where the sum runs over all ways of distributing the derivatives over χ and v. This in turn is bounded by

$$C(|\alpha|, r) e^{|\alpha|t} \int \frac{(1+|x|^2)^r}{(1+e^t|x+z|^2)^r} |D^\beta \chi_l(x)|^2 (1+e^t|x+z|^2)^r |D^\gamma v(e^{t/2}(x+z))|^2 \, dx.$$

Since $\chi_l(x) = 0$ if $|x| \leqq R$, one sees that $|x+z| \geqq \frac{1}{8}|x|$. Thus,

$$\frac{(1+|x|^2)^r}{(1+e^t|x+z|^2)^r} \leqq C(r) e^{-tr},$$

and so the previous expression is bounded by

$$\begin{aligned} &C(|\alpha|, r) e^{|\alpha|t} e^{-tr} \int (1+e^t|x+z|^2)^r |D^\gamma v(e^{t/2}(x+z))|^2 \, dx \\ &\qquad \leqq C(|\alpha|, r) e^{|\alpha|t} e^{-tr} e^{-td/2} \|v\|_{H^{q,2}_r}, \end{aligned}$$

the last step following from the change of variables $y = e^{t/2}(x+z)$.

Now turn to the estimate in part (ii) of the lemma. As before, one has

$$\begin{aligned} &\|D^\alpha (1+|\cdot|^2)^{r/2} \chi_l(\cdot) v(e^{t/2}(\cdot + z))\|^2_{L^2} \\ &\quad \leqq C(q,r) e^{tq} \sup_{|\beta| \leqq |\alpha|} \int (1+|x|^2)^r |D^\beta_x v(e^{t/2}(x+z))|^2 \, dx \\ &\quad = C(q,r) e^{tq} e^{-td/2} \int (1+|e^{-t/2} y - z|^2)^r |D^\beta v(y)|^2 \, dy \\ &\quad \leqq C(q,r) e^{tq} e^{-td/2} (1+|z|^2)^r \|v\|^2_{\tilde{H}^{q,2}}. \end{aligned}$$

Note that if the factor of χ_l is not present in Proposition A.5, the estimates proceed in a very similar fashion except that (A.3) is replaced by

$$\|e^{t\mathscr{L}_0}v\|^2_{H_r^{q,2}}$$
$$\leqq C(q,r)(4\pi a(t))^{-d}e^{2t/(p-1)}e^{-td/2}\|v\|^2_{H_r^{q,2}}\left(\int e^{-z^2/(4a(t))}e^{tq/2}(1+|z|^2)^r dz\right)^2.$$

An elementary estimate of this integral implies

Corollary A.7. *There exists $C(r,q)>0$ such that for any $v\in H_r^{q,2}$,*

$$\|e^{t\mathscr{L}_0}v\|^2_{H_r^{q,2}}\leqq C(q,r)e^{2t/(p-1)}e^{-dt/2}e^{tq}\|v\|^2_{H_r^{q,2}}.$$

We can now prove Proposition A.2. Begin by writing

$$e^{t\mathscr{L}_0}Q_Nv_> = e^{\frac{1}{2}t\mathscr{L}_0}Q_Ne^{\frac{1}{2}t\mathscr{L}_0}v_> = e^{\frac{1}{2}t\mathscr{L}_0}Q_N\chi_le^{\frac{1}{2}t\mathscr{L}_0}v_> + e^{\frac{1}{2}t\mathscr{L}_0}Q_N(\mathbf{1}-\chi_l)e^{\frac{1}{2}t\mathscr{L}_0}v_>.$$

By Proposition A.1 and Corollary A.7,

$$\begin{aligned}\|e^{\frac{1}{2}t\mathscr{L}_0}Q_N(\mathbf{1}-\chi_l)e^{1/2t\mathscr{L}_0}v_>\|_{H_r^{q,2}} &\leqq C(r,q)e^{R^2/6}e^{-|\lambda_{N+1}|t/2}\|e^{\frac{1}{2}t\mathscr{L}_0}v_>\|_{H_r^{q,2}}\\ &\leqq C(r,q)e^{R^2/6}e^{-(|\lambda_{N+1}|+(\frac{1}{p-1}-\frac{d}{4}+\frac{q}{2}))t/2}\|v\|_{H_r^{q,2}}.\end{aligned}$$

(A.4)

On the other hand, by Corollary A.7 and Proposition A.5,

$$\begin{aligned}\|e^{1/2t\mathscr{L}_0}Q_N\chi_le^{1/2t\mathscr{L}_0}v_>\|_{H_r^{q,2}} &\leqq C(r,q)e^{(\frac{1}{p-1}+\frac{q}{2}-\frac{d}{4})t/2}\|\chi_le^{\frac{1}{2}t\mathscr{L}_0}v_>\|_{H_r^{q,2}}\\ &\leqq C(q,r)e^{(\frac{1}{p-1}+\frac{q}{2}-\frac{d}{4})t/2}e^{(\frac{1}{p-1}-\frac{d}{4})t/2}\left\{e^{(q-r)t/4}+e^{tq/4}e^{-3R^2/16}\right\}\|v\|_{H_r^{q,2}}.\end{aligned}$$

(A.5)

Combining (A.4) and (A.5) completes the proof of Proposition A.2. □

Appendix B

In this appendix I consider the smoothness of the function $w\to F(w)$, defined by $w(x)\to f(x,w(x))$, considered as a map from $\tilde{H}^{1,2}$ to itself. Since I wish specifically to verify the hypotheses made in Subsection 4.2, I restrict consideration to $d=1$ and assume that $f(x,y)\in C^{2+\eta}$, for some $0<\eta<1$. Then easy modifications of the proof of Theorem 4.1 in Valent (1988) to account for the weights in our Sobolev space, and the fact that we work on an unbounded domain, allow us to conclude that F is in C^1, and that the derivative of F is given by the linear operator

$$F'(w)\tau = F_y(w)\tau,$$

where $F_y(w)(x)=f_y(x,w(x))$. I now show that the derivative of F is itself Hölder continuous, and hence prove

Proposition B.1. *Suppose that $f \in C^{2+\eta}$, with $\frac{1}{3} < \eta < 1$, and that there exists $C > 0$, and $n > 0$ such that*

$$\sup_x |D_x^\alpha D_y^\beta f(x,y)| \leqq C|y|(1+|y|)^n \quad \text{for all } |\alpha| + |\beta| \leqq 2\ , \quad |\beta| < 2.$$

Then $w \to F(w)$ is a $C^{1+\eta}$ function from $\tilde{H}^{1,2}$ to itself.

Remark. The hypothesis on the derivatives of $f(x,y)$ is necessary because of the fact that we work on an unbounded domain.

Proof. Since we know that F is in C^1 and have a formula for the first derivative, the proposition follows from an estimate of the form

$$\|(F'(w) - F'(\tilde{w}))\|_{\tilde{H}^{1,2}} \leqq C\|w - \tilde{w}\|^\eta_{\tilde{H}^{1,2}} \quad \text{for } w \text{ and } \tilde{w} \in \tilde{H}^{1,2}.$$

Recalling the definition of the norm in $\tilde{H}^{1,2}$, we see that $\|F'(w) - F'(\tilde{w})\|_{\tilde{H}^{1,2}}$ is a sum of two types of terms:
Type I.

$$\begin{aligned}\|(1+|x|)(f_y(x,w(x)) - f_y(x,\tilde{w}(x)))\|_{L^2} &\leqq C\|(1+|x|)|w(x) - \tilde{w}(x)|\|_{L^2} \\ &\qquad (\text{since } f \in C^{2+\eta}) \\ &\leqq C\|w - \tilde{w}\|_{\tilde{H}^{q,2}}.\end{aligned}$$

Type II.

$$\begin{aligned}\|D_x(f_y(x,w(x)) - f_y(x,\tilde{w}(x)))\|_{L^2} & \\ \leqq \|f_{xy}(x,w(x)) - f_{xy}(x,\tilde{w}(x))\|_{L^2} & \\ + \|f_{yy}(x,w(x))w'(x) - f_{yy}(x,\tilde{w}(x))\tilde{w}'(x)\|_{L^2}\ . &\end{aligned} \tag{B.1}$$

Since $f \in C^{2+\eta}$, $|f_{xy}(x,w(x)) - f_{xy}(x,\tilde{w}(x))| \leqq C|w(x) - \tilde{w}(x)|^\eta$, it follows that

$$\begin{aligned}&\|f_{xy}(x,w(x)) - f_{xy}(x,\tilde{w}(x))\|^2_{L^2} \\ &\leqq C\int |w(x) - \tilde{w}(x)|^{2\eta}\, dx \\ &\leqq C\int (1+|x|)^{-2\eta}((1+|x|)|w(x) - \tilde{w}(x)|)^{2\eta}\, dx \\ &\leqq C\left\{\int (1+|x|)^{-2\eta q} dx\right\}^{1/q}\left\{\int [(1+|x|)|w(x) - \tilde{w}(x)|]^{2\eta p} dx\right\}^{1/p},\end{aligned}$$

where $1/q + 1/p = 1$. If one sets $p = 1/\eta$, then the second integral is bounded by $\|w - \tilde{w}\|^{2\eta}_{\tilde{H}^{1,2}}$, while the first integral becomes $\int (1+|x|)^{-2\eta/(1-\eta)} dx$, which is bounded by a constant if $\eta > \frac{1}{3}$. To estimate the second term in (B.1), note that

$$\begin{aligned}&\|f_{yy}(x,w(x))w'(x) - f_{yy}(x,\tilde{w}(x))\tilde{w}'(x)\|_{L^2} \\ &\qquad \leqq 2\int |f_{yy}(x,w(x)) - f_{yy}(x,\tilde{w}(x))|^2 |w'(x)|^2\, dx \\ &\qquad\quad + 2\int |f_{yy}(x,\tilde{w}(x))|^2 |w'(x) - \tilde{w}'(x)|^2\, dx.\end{aligned}$$

Since $d = 1$, it follows that $\tilde{w} \in L^\infty$, so $|f_{yy}(x,\tilde{w}(x))| \leqq C$ and

$$\int |f_{yy}(x, \tilde{w}(x))|^2 |w'(x) - \tilde{w}'(x)|^2 dx \leqq C\|w - \tilde{w}\|^2_{\tilde{H}^{1,2}}.$$

On the other hand,

$$|f_{yy}(x, w(x)) - f_{yy}(x, \tilde{w}(x))|^2 \leqq C|w(x) - \tilde{w}(x)|^{2\eta} \leqq C\|w - \tilde{w}\|^{2\eta}_{L^\infty}.$$

Thus,

$$\int |f_{yy}(x, w(x)) - f_{yy}(x, \tilde{w}(x))|^2 |w'(x)|^2 \, dx \leqq C\|w - \tilde{w}\|^{2\eta}_{\tilde{H}^{1,2}},$$

and the proposition follows. □

Remark. The restriction that $\eta > \frac{1}{3}$ arises from estimating the first term on the right-hand side of (B.1). If one assumes that $f_{xy} \in C^1$ (as is the case in Subsection 4.2), one can avoid this restriction.

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