Arch. Rational Mech. Anal. 138 (1997) 1–35. © Springer-Verlag 1997

On The Concepts of State and Free Energy in Linear Viscoelasticity

Dedicated to the memory of Ignace Kolodner
"... ben tetragono ai colpi di ventura" (*Par.*, *XVII*, *24*)

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Communicated by D. Owen

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1. Introduction

The constitutive equation of linear viscoelasticity is the Boltzmann-Volterra equation

$$T(t) = \boldsymbol{G}_0 E(t) + \int_0^{+\infty} \dot{\boldsymbol{G}}(s) E(t-s)\, ds, \tag{1.1}$$

in which the stress at the time t in a material element subject to the *deformation process* E is determined by the *current deformation* $E(t)$ and by the *past history* E^t of E at t:

$$E^t(s) := E(t-s), \qquad s \in (0, +\infty). \tag{1.2}$$

If we fix a time $t > 0$ and if we consider the restriction $E_{(0,t)}$ of E to the time interval $(0, t)$ as a *continuation* of the history E^0, we can separate the influence of $E_{(0,t)}$ and E^0 on the stress response by rewriting the equation (1.1) in the form

$$T(t) = \boldsymbol{G}_0 E(t) + \int_0^t \dot{\boldsymbol{G}}(t-r)E(r)\,dr + \int_0^{+\infty} \dot{\boldsymbol{G}}(t+r)E^0(r)\,dr. \tag{1.3}$$

This equation tells us that two histories E^0, E'^0 have the same influence on the stress response if and only if their difference $H := E^0 - E'^0$ satisfies the condition

$$\int_0^{+\infty} \dot{\boldsymbol{G}}(t+r)H(r)\,dr = O \qquad \text{for all } t \geqq 0. \tag{1.4}$$

The third axiom of Noll's new theory of simple materials [23] says that "if two states are different ... then there must be some process which produces different stresses with the two states as initial states". If we accept this axiom, and if we agree that in our case the *processes* are the continuations, we may conclude that two histories whose difference satisfies the condition (1.4) must correspond to the same state. If we assume that the current deformation is independent of the past history, then we are led to define a *state* as a pair whose entries are an equivalence class of histories and a deformation. The deformation is the current deformation, and two histories are equivalent if their difference satisfies the condition (1.4).

With these definitions of process and state, we identify a *system* in the sense of the theory of Coleman & Owen [5], and we use the general results of that theory to study some basic questions of linear viscoelasticity which have long been debated by several authors. One such question is the characterization of the state space. Usually, states are identified, at least implicitly, with history-deformation pairs [3, 10, 13, 16]. There is, however, an important exception, that of the *viscoelastic materials of rate type,* for which the relaxation function is a linear combination of exponentials. For such materials, a state is usually identified with a finite array of internal variables [2, 18]. It is not clear, however, whether this choice is a matter of convenience or is dictated by some sort of general requirements. In the approach that we present here, the possible definitions of a state are strictly limited by the structure of the solution set of equation (1.4): Indeed, a state can be represented by a history-deformation pair if and only if the solution set reduces to the null history alone, so that the equivalence classes which constitute the first entry of a state reduce to singletons.

For a relaxation function of exponential type, we show in Section 6 that the finite dimensional characterization of a state is compatible with our definition, while the characterization as a history-deformation pair is not. This result can be easily extended to all viscoelastic materials of rate type. We also produce an example of a class of completely monotonic relaxation functions for which the equivalence classes are singletons, and therefore the states are correctly described by history-deformation pairs.

Another question which we consider here is that of the topology of the state space. When a state is defined as a history-deformation pair, it is natural to define the state space as the product of the space of histories and the space of deformations, and to endow it with the product norm of the two spaces [13]. The norm chosen for the space of histories is usually the *fading memory norm* of COLEMAN & NOLL [4], suggested by the physical consideration that the response of a material with memory is more influenced by the deformations undergone in the recent past than by those that occurred in the far past. In effect, as shown by the weaker fading memory assumptions made by VOLTERRA [24], GRAFFI [17], and DAY [11], a fading memory effect is implicit in the constitutive equation (1.1), provided that the relaxation function decays to its equilibrium value sufficiently fast. The main reason for the success of the approach of COLEMAN & NOLL lies in the far-reaching consequences of the *principle of the fading memory*, which is an assumption of continuity of the constitutive functionals in the topology induced by the fading memory norm. Under this assumption, many general properties of materials with memory have been proved, such as some restrictions and interrelations for the constitutive functionals, and the minimality of the equilibrium free energy in the set of all states having the same current deformation. In this paper, in the more limited context of linear viscoelasticity, we obtain the same results in a more direct way. We endow the space of histories with a seminorm, which is a norm for the set of the equivalence classes determined by the solution set of the equation (1.4). The sum of this seminorm and the norm of the space of deformations is a norm for the state space, and we use the topology induced by that norm.

This choice plays an important role in the definition of the free energy, which is the central subject of the paper. Among the definitions present in the literature, we focus our attention on the definition given by COLEMAN & OWEN, who define the free energy as a lower potential for the work [6, Sec. 5]. The general results of their theory are then used to prove the existence of a maximal and of a minimal free energy, characterized as the minimum work done to approach a state starting from the natural state, and as the maximum work which can be recovered from a given state, respectively. In the special case of linear viscoelasticity, we find two additional properties beyond those shared by all systems and by all free energies. Namely, we prove that every state can be approached from every other state by a sequence of processes with the property that the sequence of the works done in these processes is convergent, and we prove that the minimal free energy is lower semicontinuous with respect to the topology that we have adopted for the state space.

The last two sections are devoted to the study of two particular classes of viscoelastic material elements, characterized by relaxation functions of exponential type and by completely monotonic relaxation functions, respectively. For the first class, we generalize a result of GRAFFI & FABRIZIO [19], which asserts that there is just one free energy, whose explicit expression was determined by BREUER & ONAT [2]. We also show that some other functions, which are usually considered as appropriate to describe the free energy, are indeed not acceptable because they do not define a function of state for this specific class of relaxation functions.

For completely monotonic relaxation functions, we discuss here the characterization of the state space, while the characterization of the free energy is still under study. With the aid of Bernstein's representation formula for completely monotonic functions and of Müntz's theorem on the completeness of powers, we determine a class of completely monotonic functions for which states are characterized by history-deformation pairs. We also determine a class, consisting of completely monotonic functions of exponential type, for which the state space is finite-dimensional. These two classes are not exhaustive, and a comprehensive classification of completely monotonic functions based on the structure of their state space is not yet available.

We conclude this introduction with some remarks on notation. We denote by Sym the set of all symmetric linear transformations on the vectors, equipped with the inner product $A \cdot B := \operatorname{tr}(AB^T)$ and, consequently, with the norm

$$|A| := (A \cdot A)^{1/2}, \tag{1.5}$$

and by LinSym we denote the set of all linear transformations on Sym, equipped with the norm

$$\|\boldsymbol{C}\| := \sup_{A \in \mathrm{Sym} \setminus \{O\}} \frac{|\boldsymbol{C}A|}{|A|}. \tag{1.6}$$

By I and O we denote the identity and the zero mapping in Sym, and by $\boldsymbol{I}$ and $\boldsymbol{O}$ we denote the corresponding mappings in LinSym. For each pair $\boldsymbol{C}$, $\boldsymbol{D}$ of elements of LinSym, the notations

$$\boldsymbol{C} > \boldsymbol{D}, \qquad \boldsymbol{C} \geqq \boldsymbol{D}, \tag{1.7}$$

mean that $(\boldsymbol{C} - \boldsymbol{D})$ is positive-definite and positive-semidefinite, respectively. We also denote by $\mathscr{B}(A, \delta)$ the open ball of radius δ centered at A, and for every function H defined over an interval of the real line we denote by $(\operatorname{var} H)$ the variation of H in its domain of definition [12, Eq. 2.1].

2. Histories and segments

In the constitutive equation (1.1), E is a function from the reals into Sym, and $\dot{\boldsymbol{G}}$ is a function from the non-negative reals into LinSym. For these functions, we keep the regularity assumptions made in [12. Sec. 2]; namely, for E we assume that (i) for each $t \in \mathbb{R}$, the history E^t defined by equation (1.2) is a function of bounded variation, and that (ii) E is continuous from the right. For $\dot{\boldsymbol{G}}$ we assume that (iii) $\dot{\boldsymbol{G}}$ is Lebesgue integrable. This assumption implies that $\dot{\boldsymbol{G}}$ admits a primitive $\boldsymbol{G}$, determined to within an additive constant, which we fix by setting

$$\boldsymbol{G}(0) := \boldsymbol{G}_0, \tag{2.1}$$

where $\boldsymbol{G}_0$ is the tensor appearing in equation (1.1). It also implies that $\boldsymbol{G}$ is absolutely continuous and bounded, and that the limit

$$\boldsymbol{G}_\infty := \lim_{s\to+\infty} \boldsymbol{G}(s) \tag{2.2}$$

exists. The tensor $\boldsymbol{G}_0$ is the *instantaneous elastic modulus*, and $\boldsymbol{G}_\infty$ is the *equilibrium elastic modulus*. We also assume that (iv) both $\boldsymbol{G}_0$ and $\boldsymbol{G}_\infty$ are symmetric, and

$$\boldsymbol{G}_0 > \boldsymbol{G}_\infty \geqq \boldsymbol{O}. \tag{2.3}$$

The regularity assumptions made on E lead to the following definition of a history.

2.1. Definition. A *history* is a function from the positive reals into Sym, of bounded variation and continuous from the left.

It can be shown that every history H is bounded and has right and left limits $H(s^+)$, $H(s^-)$ at every $s > 0$, with $H(s^-) = H(s)$ by assumption (ii). Moreover, a history has continuous extensions to zero and to infinity:

$$H(0^+) := \lim_{s\to0^+} H(s), \quad H(\infty) := \lim_{s\to+\infty} H(s). \tag{2.4}$$

A *segment of duration* d is a function $K : (0, d] \to \mathrm{Sym}$, of bounded variation and continuous from the left. The *truncation* of a history H at $r > 0$ is the segment H_r of duration r defined by

$$H_r(s) := H(s), \qquad 0 < s \leqq r, \tag{2.5}$$

and the *r-section* of H is the history H^r given by

$$H^r(s) := H(r + s) \qquad \text{for } s > 0. \tag{2.6}$$

The *continuation* $K * H$ of a history H by a segment K of duration d is the history

$$(K * H)(s) := \begin{cases} K(s) & \text{for } 0 < s \leqq d, \\ H(s - d) & \text{for } s > d. \end{cases} \tag{2.7}$$

Notice that

$$H_r * H^r = H \tag{2.8}$$

for all histories H and for all $r > 0$.

In quite a similar way, one can define the *truncation* K_r of a segment K to be its restriction to $(0, r]$, and the *r-section* K^r of K to be the segment of duration $(d-r)$ defined by $K^r(s) = K(r + s)$. The *continuation* of a segment K of duration d by a segment K' of duration d' is the segment $K' * K$ taking the value $K'(s)$ for $0 < s \leqq d'$ and $K(s - d')$ for $d' < s \leqq d + d'$. Note that, since histories and segments need not be continuous, any history and any segment can be continued by any segment.

The *constant history* associated with the deformation A is the history $A^\dagger$ with $A^\dagger(s) = A$ for all $s > 0$, and a *finite history* is a continuation $K * A^\dagger$ of a constant history. The collection of all histories obeying Definition 2.1 is a vector space which is denoted by Γ. The set of all constant histories and the set of all finite histories are subspaces of Γ, and the null element of Γ is the constant history $O^\dagger$ associated with the null deformation.

Another distinguished subspace of Γ is the set Γ_0 of all solutions of equation (1.4). The function

$$\|H\|_{\Gamma} := \sup_{t \geqq 0} \left| \int_0^{+\infty} \dot{\boldsymbol{G}}(t+s)H(s)\, ds \right| \tag{2.9}$$

is a seminorm for Γ and a norm for the quotient space Γ/Γ_0. Let us prove some relevant properties of this seminorm.

2.2. Proposition. *For every pair of histories* H, H' *the following properties hold*:

(i) *For every segment* K,

$$\|K * H' - K * H\|_{\Gamma} \leqq \|H' - H\|_{\Gamma}. \tag{2.10}$$

(ii) *For each* $\varepsilon > 0$ *there is a positive real* m *such that, for all segments* K *of duration greater than* m,

$$\|K * H' - K * H\|_{\Gamma} < \varepsilon. \tag{2.11}$$

(iii) *If* H_d *is the truncation of* H *at* d, *then*

$$\lim_{d \to +\infty} \|H_d * H' - H\|_{\Gamma} = 0. \tag{2.12}$$

(iv) *For every history* H *and for every segment* K *of duration* d,

$$\lim_{r \to d} \|K^r * H - H\|_{\Gamma} = 0, \tag{2.13}$$

where K^r *is the* r*-section of* K.

Proof. If K is a segment of duration d, it follows from the definition of $\|\cdot\|_{\Gamma}$ that

$$\begin{aligned} \|K * H' - K * H\|_{\Gamma} &= \sup_{t \geqq 0} \left| \int_d^{+\infty} \dot{\boldsymbol{G}}(t+s)(H'(s-d) - H(s-d))\, ds \right| \\ &= \sup_{t' \geqq d} \left| \int_0^{+\infty} \dot{\boldsymbol{G}}(t'+s')(H'(s') - H(s'))\, ds' \right|, \end{aligned} \tag{2.14}$$

with $s' = s - d$ and $t' = t + d$. The inequality (2. 10) then follows by taking the supremum over $t' \geqq 0$ instead of $t' \geqq d$. To prove the second assertion, we set

$$M := \sup_{s>0} |H(s)| + \sup_{s>0} |H'(s)| \tag{2.15}$$

and note that $M < +\infty$ because both H and H' are bounded. Thus, from equation (2.14),

$$\|K * H' - K * H\|_{\Gamma} \leqq M \sup_{t \geqq 0} \int_d^{+\infty} \|\dot{\boldsymbol{G}}(t+s)\|\, ds = \boldsymbol{M} \int_d^{+\infty} \|\dot{\boldsymbol{G}}(s)\|\, ds, \tag{2.16}$$

and, because $\dot{\boldsymbol{G}}$ is integrable over $(0, +\infty)$, there is a positive real m such that for all $d > m$ the last integral is less than εM^{-1}. This proves the inequality (2.11). If we now take $K = H_d$, from the identity (2.8) and from the inequality (2.16) we have

$$\|H_d * H' - H\|_{\boldsymbol{\Gamma}} = \|H_d * H' - H_d * H^d\|_{\boldsymbol{\Gamma}} \leqq M \int_d^{+\infty} \|\dot{\boldsymbol{G}}(s)\| \, ds, \tag{2.17}$$

and (2.12) follows. The last statement asserts that the supremum of the integral

$$\int_0^{+\infty} \dot{\boldsymbol{G}}(t+s)(H(s) - (K^r * H)(s)) \, ds$$

$$= \int_0^{+\infty} (\dot{\boldsymbol{G}}(t+s) - \dot{\boldsymbol{G}}(t+s+d-r))H(s) \, ds + \int_0^{d-r} \dot{\boldsymbol{G}}(t+s)K(r+s) \, ds, \tag{2.18}$$

taken over all non-negative t, converges to zero when $r \to d$. By setting $r' = d - r$ and integrating by parts, the same integral is transformed into

$$- \int_0^{+\infty} (\boldsymbol{G}(t+s) - \boldsymbol{G}(t+s+r')) \, dH(s) - (\boldsymbol{G}(t) - \boldsymbol{G}(t+r'))H(0^+)$$

$$- \int_0^{r'} \boldsymbol{G}(t+s) \, dK(d-r'+s) + \boldsymbol{G}(t+r')K(d) - \boldsymbol{G}(t)K(d-r'), \tag{2.19}$$

and therefore it is less than

$$(\mathrm{var} H) \sup_{s \geqq 0} \|\boldsymbol{G}(t+s) - \boldsymbol{G}(t+s+r')\| + |H(0^+)| \, \|\boldsymbol{G}(t) - \boldsymbol{G}(t+r')\|$$

$$+ (\mathrm{var} K) \int_0^{r'} \|\boldsymbol{G}(t+s)\| \, ds + \|\boldsymbol{G}(t+r')\| |K(d) - K(d-r')|$$

$$+ \|\boldsymbol{G}(t+r') - \boldsymbol{G}(t)\| |K(d-r')|. \tag{2.20}$$

By the absolute continuity of $\boldsymbol{G}$, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $r' < \delta$ implies that

$$\|\boldsymbol{G}(s) - \boldsymbol{G}(s+r')\| < \varepsilon, \qquad \int_s^{s+r'} \|\boldsymbol{G}(s')\| \, ds' < \varepsilon \tag{2.21}$$

for all $s \geqq 0$, and the fact that K is continuous from the left implies that there is a $\delta' > 0$ such that

$$|K(d) - K(d - r')| < \varepsilon \tag{2.22}$$

for all $r' < \delta'$. Thus, for sufficiently small r' the expression (2.20) is less than

$$((\mathrm{var}\ H) + |H(0^+)| + (\mathrm{var}\ K) + \sup_{s \geqq 0} \|\boldsymbol{G}(s)\| + \sup_{d>s \geqq 0} |K(s)|)\varepsilon, \tag{2.23}$$

and because this bound is independent of t we conclude that the supremum of the integral (2.18) taken over all $t \geqq 0$ converges to zero when $r' \to 0$. □

The constitutive equation (1.1) defines the stress reached by the material when subjected to the history E^t and to the current deformation $E(t)$; because E is only continuous from the right, $E(t)$ need not coincide with the final value $E(t^+)$ of E^t, and therefore history and current deformation are unrelated. Thus, the *response functional* can be defined as the function $\tilde{T} : \Gamma \times \mathrm{Sym} \to \mathrm{Sym}$ which with every history-deformation pair (H, A) associates the stress

$$\tilde{T}(H, A) := \boldsymbol{G}_0 A + \int_0^{+\infty} \dot{\boldsymbol{G}}(s) H(s)\, ds. \tag{2.24}$$

The response functional is linear. Moreover, the inequality

$$|\tilde{T}(H, A)| \leqq |\boldsymbol{G}_0 A| + \|H\|_{\Gamma}, \tag{2.25}$$

which follows from the definition (2.9) of the seminorm $\|\cdot\|_{\Gamma}$, shows that $\tilde{T}$ is continuous in the topology induced by $\|\cdot\|_{\Gamma}$. It also follows from the property (2.11) of $\|\cdot\|_{\Gamma}$ that, if we continue two histories H' and H with the same segment K, the difference between $\tilde{T}(K * H', A)$ and $\tilde{T}(K * H, A)$ can be made arbitrarily small by taking K of sufficiently long duration. This property of linear viscoelastic materials was called the *principle of dissipation of hereditary influence* by Volterra [24]. As shown in the proof of Proposition 2.2, this property is a consequence of the assumed integrability of $\dot{\boldsymbol{G}}$. In fact, the integrability of $\dot{\boldsymbol{G}}$ was taken by Graffi as a weak form of the *principle of fading memory* [17].

3. States and processes

On the basis of the motivations discussed in the Introduction, we identify the *states* of the material with the elements of the Cartesian product

$$\Sigma := (\Gamma/\Gamma_0) \times \mathrm{Sym}, \tag{3.1}$$

i.e., with ordered pairs whose first item is an equivalence class of histories modulo Γ_0, and whose second item is a deformation. We use the notation

$$\sigma \sim (H, A) \tag{3.2}$$

to mean that the state σ is represented by the history-deformation pair (H, A). Of course, σ may be represented as well by any other pair (H', A)

such that $H' - H \in \Gamma_0$. In particular, $\sigma_A \sim (A^\dagger, A)$ is the *equilibrium state* associated with the deformation A, and $\sigma_0 \sim (O^\dagger, O)$ is the *natural state*. The *state space* Σ is a vector space, and can be made a normed space by setting

$$\|\sigma\|_\Sigma := |\boldsymbol{G}_0 A| + \|H\|_\Gamma \tag{3.3}$$

whenever $\sigma \sim (H, A)$.

It is clear from the linearity of $\tilde{T}$ and from the inequality (2.25) that, if (H, A) and (H', A) represent the same state, then the stresses $\tilde{T}(H, A)$ and $\tilde{T}(H', A)$ are the same. This fact tells us that the response functional $\tilde{T}$ is indeed a function of state, and we may define a function $\hat{T} : \Sigma \to \mathrm{Sym}$ such that

$$\hat{T}(\sigma) = \tilde{T}(H, A) \tag{3.4}$$

when $\sigma \sim (H, A)$. The function $\hat{T}$ is linear, and the inequality (2.25) tells us that it is continuous:

$$|\hat{T}(\sigma)| \leqq \|\sigma\|_\Sigma. \tag{3.5}$$

We define a *process of duration d* as a segment-deformation pair

$$P := (K,\ B), \tag{3.6}$$

where B is a deformation and K is a segment of duration d with $K(d) = O$. For a state $\sigma \sim (H, A)$ and a process $P = (K, B)$, the *state reached from σ when subjected to P* is, by definition,

$$P\sigma \sim (K_A * H, B + A), \tag{3.7}$$

where K_A is the segment

$$K_A(s) := K(s) + A. \tag{3.8}$$

We denote by Π the set of all processes, and for each state σ we denote by $\Pi\sigma$ the set of all states which can be reached from σ:

$$\Pi\sigma := \{P\sigma \,|\, P \in \Pi\}. \tag{3.9}$$

We prove below that for every $\sigma \in \Sigma$ the set $\Pi\sigma$ is dense in Σ. This is done by constructing, for each $\sigma' \in \Sigma$, a family $r \mapsto P_r$ of processes such that the states $P_r\sigma$ converge to σ' when $r \to +\infty$.

3.1. Proposition. *Given two states $\sigma \sim (H, A)$ and $\sigma' \sim (H', A')$, for each $r > 0$ consider the process*

$$P_r := (K_r * K', A' - A), \tag{3.10}$$

where K' is a segment of arbitrary duration d with $K'(d) = O$, and K_r is the segment of duration r given by

$$K_r(s) := H'_r(s) - A, \tag{3.11}$$

with H'_r the truncation of H' at r. Then the family $r \mapsto P_r\sigma$ converges to σ' in the norm $\|\cdot\|_\Sigma$ when $r \mapsto +\infty$.

Proof. The definition (3.7) tells us that $P_r\sigma \sim (H'_r * K'_A * H, A')$. Thus, by the definition (3.3) of $\|\cdot\|_\Sigma$,

$$\|P_r\sigma - \sigma'\|_\Sigma = \|H'_r * K'_A * H - H'\|_\Gamma, \tag{3.12}$$

and the third item in Proposition 2.2 shows that the right-hand side converges to zero when $r \to +\infty$. $\square$

We now fix a $P \in \Pi$, and consider the map $\sigma \mapsto P\sigma$ given by the definition (3.7). We prove that this map is Lipschitz continuous.

3.2. Proposition. *There is a positive constant m, depending only upon the relaxation function* $\boldsymbol{G}$, *such that*

$$\|P\sigma' - P\sigma\|_\Sigma \leqq m\|\sigma' - \sigma\|_\Sigma \tag{3.13}$$

for every process P and for every pair of states σ, σ'.

Proof. Let $\sigma \sim (H,A), \sigma' \sim (H',A')$, and $P = (K,B)$ with K a segment of duration d. Then by the definitions (3.3), (2.9) of $\|\cdot\|_\Sigma$ and $\|\cdot\|_\Gamma$

$$\begin{aligned}\|P\sigma' - P\sigma\|_\Sigma &= \|K_{A'} * H' - K_A * H\|_\Gamma + |G_0(A' - A)| \\ &= \sup_{t \geqq 0}\Big| \int_0^d \dot{\boldsymbol{G}}(t+s)(K_{A'}(s) - K_A(s))\, ds \\ &\quad + \int_d^{+\infty} \dot{\boldsymbol{G}}(t+s)(H'(s-d) - H(s-d))\, ds\Big| \\ &\quad + |\boldsymbol{G}_0(A' - A)|, \end{aligned} \tag{3.14}$$

with $K_{A'}(s) - K_A(s) = A' - A$ by the definition (3.8), so that the first integral reduces to

$$\int_0^d \dot{\boldsymbol{G}}(t+s)(A' - A)\, ds = (\boldsymbol{G}(t+d) - \boldsymbol{G}(t))(A' - A). \tag{3.15}$$

Because the second integral is less than or equal to $\|H' - H\|_\Gamma$, we have

$$\|P\sigma' - P\sigma\|_\Sigma \leqq (1 + 2\sup_{r \geqq 0} \|\boldsymbol{G}_0^{-1}\boldsymbol{G}(r)\|)|\boldsymbol{G}_0(A' - A)| + \|H' - H\|_\Gamma, \tag{3.16}$$

and the inequality (3.13) follows, with

$$m = 1 + 2\sup_{r \geqq 0} \|\boldsymbol{G}_0^{-1}\boldsymbol{G}(r)\|. \tag{3.17}$$

Note that $m = 1$ if $A' = A$. $\square$

Given two processes P, P', the composition of the maps $\sigma \mapsto P\sigma$, $\sigma \mapsto P'\sigma$ is, by definition, the map $\sigma \mapsto P'P\sigma$ which with each state σ associates the state $P'P\sigma$ reached from $P\sigma$ when subjected to P'. Thus, for $\sigma \sim (H,A)$, $P = (K,B)$, $P' = (K',B')$, we have

$$P'P\sigma \sim (K'_{B+A} * K_A * H, B' + B + A). \tag{3.18}$$

Note that $P'P\sigma$ is the state reached from σ when subjected to the process $(K'_B * K, B' + B)$. This process is denoted by $P'P$ and is called the *continuation* of P with P'.

The density of the sets $\Pi\sigma$ in Σ and the definition of the composition $\sigma \mapsto P'P\sigma$ characterize the pair (Σ, Π) as a *system* in the sense of COLEMAN & OWEN [5, Def. 2.1]. The further specification of the response functional $\hat{T}$ completes the description of what is called here a *linearly viscoelastic material element*. In the following, the adverb *linearly* is omitted for brevity. This definition parallels the definition given in [5, Sec. 9], which includes the thermal variables.

A different definition of a linearly viscoelastic material element in the framework of COLEMAN & OWEN's theory was given by FABRIZIO & MORRO in the paper [13]. In accordance with the *principle of fading memory* of COLEMAN & NOLL [4], these authors assume that the relaxation function satisfies the condition

$$\int_0^{+\infty} \|\dot{\boldsymbol{G}}(s)\|^2 h^{-1}(s)\, ds \;<\; +\infty, \tag{3.19}$$

where h is a positive, non-increasing function decaying to zero at a prescribed rate when $s \to +\infty$. They also define the fading memory space of histories as the set Γ^* of all functions $H : (0, +\infty) \to \mathrm{Sym}$ for which the *fading memory norm*

$$\|H\|_{\Gamma^*} := \left(\int_0^{+\infty} |H(s)|^2 h(s)\, ds \right)^{1/2} \tag{3.20}$$

is finite. The states are then identified *with* history-deformation pairs, and the state space Σ^* is the product space $\Gamma^* \times \mathrm{Sym}$, endowed with the norm

$$\|\sigma\|_{\Sigma^*} := (|A|^2 + \|H\|^2_{\Gamma^*})^{1/2} \tag{3.21}$$

with $\sigma \sim (H, A)$. If we assume that h is integrable, then we see that the history space Γ is included in Γ^* and, consequently, that the state space Σ is included in Σ^*. Indeed, since every history $H \in \Gamma$ is bounded, its norm (3.20) is finite if h is integrable. Moreover, if we use the definition (2.9) of $\|H\|_\Gamma$ and the Cauchy-Schwarz inequality, we get

$$\|H\|_\Gamma \leqq \sup_{t \geqq 0} \left(\int_0^{+\infty} \|\dot{\boldsymbol{G}}(t+s)\|^2 h^{-1}(s)\, ds \right)^{1/2} \left(\int_0^{+\infty} |H(s)|^2 h(s)\, ds \right)^{1/2}. \tag{3.22}$$

The supremum is attained at $t = 0$ because h^{-1} is positive and non-decreasing. Thus, if we denote by m^2 the integral in inequality (3.19), we conclude that

$$\|H\|_\Gamma \leqq m \|H\|_{\Gamma^*} \qquad \forall\, H \in \Gamma, \tag{3.23}$$

and the definitions (3.3) and (3.21) allow us to conclude that there is a positive constant m' such that

$$\|\sigma\|_{\Sigma} \leqq m'\|\sigma\|_{\Sigma^*} \qquad \forall\, \sigma \in \Sigma. \tag{3.24}$$

This shows that the norm of Σ is continuous with respect to the norm of Σ^*. Consequently, every open set in Σ is an open set in Σ^*, and therefore the topology induced in Σ by the norm $\|\cdot\|_{\Sigma}$ is coarser than the one induced by the norm $\|\cdot\|_{\Sigma^*}$. The inequality (3.23) cannot be reversed in general. Indeed, for every history $H \in \Gamma_0\backslash\{O^\dagger\}$ we have $\|H\|_{\Gamma^*} > 0$ and $\|H\|_{\Gamma} = 0$.

4. Work

The work done by a viscoelastic material element subject to a history H is

$$w(H) := -\int_{0^+}^{+\infty} \tilde{T}(H^r, H(r)) \cdot dH(r) \tag{4.1}$$

with H^r and $\tilde{T}$ defined by equations (2.6) and (2.24), respectively, and with the integral over $(0^+, +\infty)$ defined as the limit of the integral over $(a, +\infty)$ when $a \to 0$ from the right. Notice that the Stieltjes integral on the right side is well defined only if $\tilde{T}$ is continuous, and this is not the case for discontinuous histories. Indeed, it follows from the constitutive equation (2.24) that the stress has a jump exactly at those points at which H has a jump. Nevertheless, if the instantaneous modulus $\boldsymbol{G}_0$ is symmetric as we have assumed from the beginning, then it is possible to define, by a limit procedure, the work done in a discontinuous history. Indeed, it is shown by Proposition 3.5 in [12] that the work done in a history H having a jump at a can be expressed again by equation (4.1), with the convention that the integral over $(0^+, +\infty)$ is now the sum of the integrals over $(0^+, a)$ and $(a^+, +\infty)$, plus the product of the jump $(H(a^+) - H(a))$ of H by the arithmetic mean of the stresses $\tilde{T}(H^a, H(a^+))$ and $\tilde{T}(H^a, H(a))$. In particular, if we denote by $\tilde{w}(H, A)$ the work done in the history H followed by the deformation A, then $\tilde{w}(H, H(0^+)) = w(H)$ and

$$\tilde{w}(H, A) = w(H) + \tfrac{1}{2}(\tilde{T}(H, A) + \tilde{T}(H, H(0^+))) \cdot (A - H(0^+)). \tag{4.2}$$

We find it convenient to consider A as the extension of H to $s = 0$, and to use the notation

$$-\int_{0}^{+\infty} \tilde{T}(H^r, H(r)) \cdot dH(r) := \tilde{w}(H, A) \tag{4.3}$$

to include in the integral the work done in the jump $(H(0^+) - A)$ at $s = 0$.

If H and H' are histories corresponding to the same state, the works $w(H)$ and $w(H')$ need not be equal, as will be shown by an example in Section 6. In other words, the work is not a function of state whenever Γ_0 contains histories other than $O^\dagger$. On the contrary, the work done in a given process is always a function of the initial state. Indeed, take a history H and a process $P = (K, B)$ of duration d. Then the work done in the process is

$$\tilde{w}(K_A * H, B) - \tilde{w}(H, A) = -\int_0^d \tilde{T}(K_A^r * H, K_A(r)) \cdot dK_A(r), \qquad (4.4)$$

with K_A^r the truncation of K_A at r. Take another history H' such that $H' - H \in \Gamma_0$. Then the history $(K_A^r * H' - K_A^r * H)$ also belongs to Γ_0 by Proposition 2.2, and therefore the pairs $(K_A^r * H, K_A(r))$ and $(K_A^r * H', K_A(r))$ represent the same state. Because $\tilde{T}$ is a function of state, it follows that the right-hand side of equation (4.4) does not change if H is replaced by H'. Thus, we may conclude that the work done in a process is a function of the process of the initial state only, and we may introduce the notation

$$\hat{w}(P, \sigma) := \tilde{w}(K_A * H, B + A) - \tilde{w}(H, A) \qquad (4.5)$$

to denote the work done in the process $P = (K, B)$ starting from the state $\sigma \sim (H, A)$.

In this way, we have defined a function $\hat{w}$ from the set $\Pi \times \Sigma$ into the reals. According to [5, Def. 2.2], $\hat{w}$ is an *action* if it enjoys the following two properties:

(i) Additivity with respect to continuations:

$$\hat{w}(P'P, \sigma) = \hat{w}(P', P\sigma) + \hat{w}(P, \sigma) \quad \forall P, P' \in \Pi,\ \forall \sigma \in \Sigma. \qquad (4.6)$$

(ii) Continuity with respect to the states: For each $P \in \Pi$, the function $\hat{w}(P, \cdot)$ is continuous in Σ.

We prove below that $\hat{w}$ satisfies these conditions, and that the continuity with respect to the states is indeed Lipschitz continuity.

4.1. Proposition. *The function* $\hat{w} : \Pi \times \Sigma \to \mathbb{R}$ *defined by equation* (4.4) *is an action. Moreover, it satisfies the inequality*

$$|\hat{w}(P, \sigma') - \hat{w}(P, \sigma)| \leqq m \,(\mathrm{var}\, K) \|\sigma' - \sigma\|_\Sigma \qquad (4.7)$$

for all $P = (K, B)$ *in* Π *and for all* σ', σ *in* Σ, *with m the constant appearing in equation* (3.13).

Proof. Take a state $\sigma \sim (H, A)$ and two processes $P = (K, B)$, $P' = (K', B')$. The definitions (4.5) and (3.18) tell us that

$$\hat{w}(P', P\sigma) = \tilde{w}(K'_{B+A} * K_A * H,\ B' + B + A) - \tilde{w}(K_A * H, B + A), \qquad (4.8)$$

$$\hat{w}(P'P, \sigma) = \tilde{w}(K'_{B+A} * K_A * H, B' + B + A) - \tilde{w}(H, A), \qquad (4.9)$$

and the equality (4.6) is obtained by subtraction and substitution into equation (4.5). Now take a process $P = (K, B)$ of duration d and two states $\sigma \sim (H, A), \sigma' \sim (H', A')$. From the definitions (4.4) and (4.5) and from the fact that $dK_{A'}(r) = dK_A(r) = dK(r)$ it follows that

$$|\hat{w}(P,\sigma') - \hat{w}(P,\sigma)| = \Big|\int_0^d (\tilde{T}(K^r_{A'} * H', K_{A'}(r)) - \tilde{T}(K^r_A * H, K_A(r))) \cdot dK(r)\Big|$$
$$\leqq \operatorname{var} K) \sup_{r\in(0,d]} |\tilde{T}(K^r_{A'} * H', K_{A'}(r)) - \tilde{T}(K^r_A * H, K_A(r))| \tag{4.10}$$

If we denote by P^r the process $(K^r, K(r))$, then $(K^r_{A'} * H', K_{A'}(r)) \sim P^r\sigma'$ and $(K^r_A * H, K_A(r)) \sim P^r\sigma$, and from the definition (3.4), the linearity of $\hat{T}$, and the inequalities (3.5), (3.13) it follows that

$$\begin{aligned}&|\tilde{T}(K^r_{A'} * H', K_{A'}(r)) - \tilde{T}(K^r_A * H, K_A(r))|\\ &\quad = |\hat{T}(P^r\sigma') - \hat{T}(P^r\sigma)| = |\hat{T}(P^r\sigma' - P^r\sigma)|\\ &\quad \leqq \|P^r\sigma' - P^r\sigma\|_\Sigma \leqq m\|\sigma' - \sigma\|_\Sigma.\end{aligned} \tag{4.11}$$

Substitution into inequality (4.10) then proves the inequality (4.7). □

According to [5, Def 3.3], a state σ' is said to be $\hat{w}$-*approachable* from another state σ if there is a sequence $n \mapsto P_n$ in Π such that (i) the sequence $n \mapsto P_n\sigma$ converges to σ', and (ii) the sequence $n \mapsto \hat{w}(P_n,\sigma)$ converges. In the next Proposition 4.3, for every pair of states σ,σ' we construct a one-parameter family of processes with the above properties, showing thereby that in a viscoelastic material element every state is $\hat{w}$-approachable from every other state. The proof is based upon the following preliminary result, essentially due to Day[1].

4.2. Lemma. *For every pair of histories H, H' and for every deformation A,*

$$\lim_{p\to+\infty} \tilde{w}(H'_p * H, A) = \tilde{w}(H, H'(\infty)) + \tilde{w}(H', A). \tag{4.12}$$

Proof. Let H'^s be the s-section of H' and let H'^s_r be the truncation of H'^s at r. For fixed values of s and r, the identity $H'_s * H'^s_r = H'_{r+s}$ and the definition (4.5) of $\hat{w}$ tell us that

$$\tilde{w}(H'_{r+s} * H, A) = \tilde{w}(H, H'(r+s)) + \hat{w}(P^s_r, \sigma(r+s)) + \hat{w}(P_s, \sigma_r(s)), \tag{4.13}$$

where $\sigma(r+s) \sim (H, H'(r+s)), \sigma_r(s) \sim (H'^s_r * H, H'(s))$, and P^s_r, P_s are the processes of duration r and s defined by

$$P^s_r := (K^s_r, H'(s) - H'(r+s)), \qquad K^s_r(r') := H'(r'+s) - H'(r+s), \tag{4.14}$$

$$P_s := (K_s, A - H'(s)), \qquad K_s(s') := H'(s') - H'(s), \tag{4.15}$$

[1] [11, p. 63]. Note that, while Day's proof relies upon the fading memory property implicit in the assumption that $\dot{G}$ is Lebesgue integrable, the proof given here is based on the assumption that every history has bounded variation.

respectively. If we keep r fixed and let s grow to $+\infty$, the work $\tilde{w}(H, H'(r+s))$ converges to $\tilde{w}(H, H'(\infty))$ by its definition (4.2). Moreover,

$$|\hat{w}(P_r^s, \sigma(r+s))| = \left| \int_0^r T(r, s+r') \cdot dH'(s+r) \right| \leqq \sup_{r' \epsilon (0,r]} |T(r, s+r')| (\mathrm{var}\ \mathrm{H}'^s_r), \tag{4.16}$$

with

$$T(r, s+r') := \tilde{T}(H, H'(r+s)) + \int_{r'}^r \dot{\boldsymbol{G}}(r'') H'(s+r'')\, dr''. \tag{4.17}$$

It is not difficult to prove that $|T(r, s+r')|$ has a finite upper bound, independent of s, r, and r'. On the other hand, $(\mathrm{var}\ H'^s_r) \leqq (\mathrm{var}\ H'^s)$ and $(\mathrm{var}\ H'^s) \to 0$ when $s \to +\infty$ because H' has bounded variation. Thus, we may conclude that for every fixed $r > 0$ the work $\hat{w}(P_r^s, \sigma(r+s))$ converges to zero when $s \to +\infty$.

It remains to consider the last term in equation (4.13). Consider the state $\sigma'(s) \sim (H'^s, H'(s))$ and the work

$$\hat{w}(P_s, \sigma'(s)) = \tilde{w}(H'_s * H'^s, A) - \tilde{w}(H'^s, H'(s)). \tag{4.18}$$

The first term on the right side is equal to $\tilde{w}(H', A)$ by the identity (2.8), and the second converges to zero when $s \to +\infty$ because the total variation of H'^s converges to zero. If we now compare this work with the last term in equation (4.13), with the aid of inequality (4.7) and of the definition (3.3) of $\|\cdot\|_\Sigma$ we find that

$$\begin{aligned} |\hat{w}(P_s, \sigma_r(s)) - \hat{w}(P_s, \sigma'(s))| &\leqq (\mathrm{var}\ K)\ \| \sigma_r(s) - \sigma'(s) \|_\Sigma \\ &= (\mathrm{var}\ K)\ \| H'^s_r * H - H'^s \|_\Gamma, \end{aligned} \tag{4.19}$$

and therefore, when $s \to +\infty$,

$$\left| \lim_{s \to +\infty} \hat{w}(P_s, \sigma_r(s)) - \tilde{w}(H', A) \right| \leqq (\mathrm{var}\ K)\ \| H'_r * H - H' \|_\Gamma\ . \tag{4.20}$$

Equation (4.13) then implies the inequality

$$\left| \lim_{s \to +\infty} \tilde{w}(H'_{r+s} * H, A) - \tilde{w}(H, H'(\infty)) - \tilde{w}(H', A) \right| \leqq (\mathrm{var}\ K) \| H'_r * H - H' \|_\Gamma\ . \tag{4.21}$$

By item (iii) in Proposition 2.2, for every $\varepsilon > 0$ there is an $r_\varepsilon > 0$ such that $(\mathrm{var}\ K)\ \| H'_r * H - H' \|_\Gamma < \varepsilon$ for all $r > r_\varepsilon$. The arbitrariness of ε then leads to the equality (4.12). □

We are now ready to prove that every state σ' is $\hat{w}$-approachable from every other state σ. Because we have already constructed a family of processes starting at σ and converging to σ', it is now sufficient to prove that the work done in the same family of processes converges.

4.3. Proposition. *Each state* $\sigma' \sim (H', A')$ is $\hat{w}$*-approachable from every state* $\sigma \sim (H, A)$. *Indeed, not only does the family* $r \mapsto P_r \sigma$ *defined by equations* (3.10), (3.11) *converge to* σ' *as stated in Proposition 3.1*, *but also*

$$\lim_{r \to +\infty} \hat{w}(P_r, \sigma) = \tilde{w}(H', A') + \hat{w}(P, \sigma) \tag{4.22}$$

with $P = (K', H'(\infty) - A)$.

Proof. By equations (3.10), (3.11) and by the definition (4.5) of $\hat{w}$,

$$\hat{w}(P_r, \sigma) = \tilde{w}(H'_r * K'_A * H, A') - \tilde{w}(H, A), \tag{4.23}$$

and, by the preceding lemma,

$$\lim_{r \to +\infty} \tilde{w}(H'_r * K'_A * H, A') = \tilde{w}(K'_A * H, H'(\infty)) + \tilde{w}(H', A'). \tag{4.24}$$

Again by the definition (4.5),

$$\tilde{w}(K'_A * H, H'(\infty)) = \tilde{w}(H, A) + \hat{w}(P, \sigma), \tag{4.25}$$

and the desired result (4.22) follows. □

5. Free energy

In the more general context of the theory of simple materials with memory, Coleman [3] defines the Helmholtz free energy as a function $\tilde{\psi} : \Gamma^* \times \mathrm{Sym} \to \mathbb{R}$, continuous and continuously differentiable with respect to both of its arguments. Here Γ^* denotes the fading memory space of histories defined in Sec. 3, and continuity and differentiability with respect to Γ^* are referred to the fading memory norm (3.20). It is also proved in [3] that, as a consequence of the second law of thermodynamics, $\tilde{\psi}$ has the following properties:[2]

(P1) $\tilde{\psi}$ satisfies the *integrated dissipation inequality*

$$\tilde{\psi}(K_A * H, B) - \tilde{\psi}(H, A) \leqq \tilde{w}(K_A * H, B) - \tilde{w}(H, A) \tag{5.1}$$

for every pair of deformations A, B, for every history H, and for every segment K of duration d with $K(d) = O$.

(P2) For every deformation A and for every history H,

$$\tilde{\psi}(A^\dagger, A) \leqq \tilde{\psi}(H, A). \tag{5.2}$$

(P3) For every deformation A and for every history H, the derivative of $\tilde{\psi}(H, \cdot)$ at A is equal to the stress $\tilde{T}(H, A)$.

(P4) For every deformation A,

$$\tilde{\psi}(A^\dagger, A) - \tilde{\psi}(O^\dagger, O) = \tfrac{1}{2} \boldsymbol{G}_\infty A \cdot A. \tag{5.3}$$

[2]The property (P4), implicit in [3, Remark 11], is stated explicitly in [8, Sec. 8].

In linear viscoelasticity, several attempts have been made to define the free energy under assumptions weaker than continuity with respect to the fading memory norm. For instance, VOLTERRA [25], DAY [8, 9], and GRAFFI [15] proposed some explicit forms of the free energy, satisfying the properties listed above. Later, GRAFFI [16], MORRO & VIANELLO [22], and FABRIZIO, GIORGI & MORRO [14] took a more general viewpoint: They defined a free energy to be any function which satisfies those properties.

It is common, in thermodynamics, to assume that the free energy is a function of state. The notion of state introduced in Section 3 supplies the following restriction that $\tilde{\psi}$ has to satisfy to be a function of state:

$$\tilde{\psi}(H',A) = \tilde{\psi}(H,A) \tag{5.4}$$

for all $A \in \mathrm{Sym}$ and for all H', $H \in \Gamma$ with $H' - H \in \Gamma_0$. If this condition is satisfied, then it is possible to define a function $\psi : \Sigma \to \mathbb{R}$ such that

$$\psi(\sigma) = \tilde{\psi}(H,A) \tag{5.5}$$

whenever $\sigma \sim (H,A)$. In terms of the function ψ, the integrated dissipation inequality (5.1) takes the form

$$\psi(P\sigma) - \psi(\sigma) \leqq \hat{w}(P,\sigma) \tag{5.6}$$

for all $P \in \Pi$ and for all $\sigma \in \Sigma$.

In their theory of thermodynamic systems, COLEMAN & OWEN [5, 6] define the free energy as a lower potential for the work. In the present context, a function $\psi : \Sigma \to \mathbb{R}$ is called a *lower potential* for the action $\hat{w}$ if for every $\varepsilon > 0$ and for every $\sigma, \sigma' \in \Sigma$ there is a $\delta > 0$ such that

$$\psi(\sigma') - \psi(\sigma) < \hat{w}(P,\sigma) + \varepsilon \tag{5.7}$$

for every process P such that $P\sigma \in \mathscr{B}(\sigma',\delta)$. If ψ is a lower potential for $\hat{w}$, then it satisfies the inequality (5.6), and therefore it has the property (P1). Indeed, if $\sigma' = P\sigma$, then $P\sigma$ belongs to $\mathscr{B}(\sigma',\delta)$ for all $\delta > 0$, and therefore the inequality (5.7) is satisfied with $\sigma' = P\sigma$ for every $\varepsilon > 0$. We now prove

5.1. Proposition. *Every lower potential for $\hat{w}$ satisfies the properties* (P2)–(P4).

Proof. Consider a state $\sigma \sim (H,A)$, the equilibrium state $\sigma_A \sim (A^\dagger, A)$, and the process $P_r := (O_r^\dagger, O)$. By Proposition 3.1, $P_r\sigma \to \sigma_A$ when $r \to +\infty$. Thus, for each $\delta > 0$ there are values of r sufficiently large to ensure that $P_r\sigma \in \mathscr{B}(\sigma_A,\delta)$. If ψ is a lower potential for $\hat{w}$, then for each $\varepsilon > 0$ the inequality (5.7) can be written for $\sigma' = \sigma_A$, for the given σ, and for $P = P_r$ with sufficiently large r. Keeping in mind that $\hat{w}(P_r\sigma) = 0$ because P_r has constant values, we get $\psi(\sigma_A) - \psi(\sigma) < \varepsilon$, and the property (5.2) follows from the arbitrariness of ε. To prove (P3), take two states $\sigma \sim (H,A), \sigma' \sim (H,A')$ corresponding to the same history H, and the process $P'_r := (O_r^\dagger, A' - A)$. The equality

$$\| P'_r\sigma - \sigma' \|_\Sigma = \| A_r^\dagger * H - H \|_\Gamma \tag{5.8}$$

and the last item in Proposition 2.2 tell us that $P'_r\sigma \to \sigma'$ when $r \to 0$. Therefore, for every $\delta > 0$ we have $P'_r\sigma \in \mathscr{B}(\sigma', \delta)$ for sufficiently small values of r. If ψ is a lower potential for $\hat{w}$, the inequality (5,7) yields

$$\tilde{\psi}(H, A') - \tilde{\psi}(H, A) < \hat{w}(P'_r, \sigma) + \varepsilon \tag{5.9}$$

for every fixed $\varepsilon > 0$ and for sufficiently small $r > 0$. The work $\hat{w}(P'_r, \sigma)$ is concentrated at the jump from A to A'; according to equation (4.2), it is given by

$$\hat{w}(P'_r, \sigma) = \tfrac{1}{2}(\tilde{T}(A^\dagger_r * H, A') + \tilde{T}(A^\dagger_r * H, A)) \cdot (A' - A), \tag{5.10}$$

and, in view of the identity

$$\tilde{T}(H, A') - \tilde{T}(H, A) = \boldsymbol{G}_0(A' - A), \tag{5.11}$$

it can be given the form

$$\hat{w}(P'_r, \sigma) = \tilde{T}(A^\dagger_r * H, A) \cdot (A' - A) + o(A' - A). \tag{5.12}$$

The linearity of $\tilde{T}$, the inequality (2.25) and the convergence of $r \mapsto A^\dagger_r * H$ to H imply the convergence of $r \mapsto \tilde{T}(A^\dagger_r * H, A)$ to $\tilde{T}(H, A)$ when $r \to 0$. Thus, for $r \to 0$ the inequality (5.9) yields

$$\tilde{\psi}(H, A') - \tilde{\psi}(H, A) \; < \; \tilde{T}(H, A) \cdot (A' - A) + o(A' - A) + \varepsilon, \tag{5.13}$$

and, by the arbitrariness of ε,

$$\tilde{\psi}(H, A') - \tilde{\psi}(H, A) \leqq \tilde{T}(H, A) \cdot (A' - A) + o(A' - A). \tag{5.14}$$

If we now repeat the whole procedure with A and A' interchanged, we get

$$\begin{aligned} \tilde{\psi}(H, A) - \tilde{\psi}(H, A') &\leqq \tilde{T}(H, A') \cdot (A - A') + o(A' - A) \\ &= -\tilde{T}(H, A) \cdot (A' - A) + o(A' - A), \end{aligned} \tag{5.15}$$

with the last step following from the identity (5.11). We then conclude that the equality sign holds in inequality (5.14). This proves that $\tilde{\psi}(H, \cdot)$ is differentiable and that its derivative is $\tilde{T}(H, \cdot)$.

To prove the property (P4), we consider a segment K of duration d with $K(d) = O$, and for every $\alpha > 1$ we define the *α-retardation* of K as the segment K_α of duration (αd) given by

$$K_\alpha(\alpha s) := K(s), \quad s \in (0, d]. \tag{5.16}$$

It is known (see, e.g., [12, Sec. 3]) that, if σ_B and σ_A are the equilibrium states associated with B and with $A := B + K(0^+)$, and if $P_\alpha = (K_\alpha, K(0^+))$, then

$$\lim_{\alpha \to +\infty} P_\alpha \sigma_B = \sigma_A, \tag{5.17}$$

$$\lim_{\alpha \to +\infty} \hat{w}(P_\alpha, \sigma_B) = \tfrac{1}{2}\, \boldsymbol{G}_\infty A \cdot A - \tfrac{1}{2} \boldsymbol{G}_\infty B \cdot B. \tag{5.18}$$

If we fix $\delta > 0$, equation (5.17) tells us that $P_\alpha \sigma_B$ belongs to $\mathscr{B}(\sigma_A, \delta)$ for sufficiently large values of α. Then we can use the inequality (5.7) and the property (5.18) of the retardations to get

$$\psi(\sigma_A) - \psi(\sigma_B) < \hat{w}(P_\alpha, \sigma_B) + \varepsilon < \tfrac{1}{2}\boldsymbol{G}_\infty A \cdot A - \tfrac{1}{2}\boldsymbol{G}_\infty B \cdot B + 2\varepsilon \tag{5.19}$$

for every pair of deformations A, B. In particular, for $B = O$ and ε arbitrarily small we have

$$\psi(\sigma_A) - \psi(\sigma_0) \leqq \tfrac{1}{2}\boldsymbol{G}_\infty A \cdot A \tag{5.20}$$

where $\sigma_0 \sim (O^\dagger, O)$ is the natural state, and for $A = O$ we get the opposite inequality

$$\psi(\sigma_B) - \psi(\sigma_0) \geqq \tfrac{1}{2}\boldsymbol{G}_\infty B \cdot B. \tag{5.21}$$

The equality (5.3) follows from the arbitratiness of A and B. $\square$

In this paper we adopt Coleman & Owen's definition of free energy, with the requirement that the domain of the free energy be all of Σ rather than merely a dense subset of Σ as required in [6]. Moreover, we add for convenience the normalization condition $\psi(\sigma_0) = 0$.

5.2. Definition. A *free energy* for the viscoelastic material element is a function $\psi : \Sigma \to \mathbb{R}$ which is a lower potential for $\hat{w}$ and satisfies $\psi(\sigma_0) = 0$.

The set of all free energies is denoted by $\mathscr{F}$. It has been proved above that all functions in $\mathscr{F}$ have the properties (P1)–(P4). In particular, a consequence of (P4) and of the positive-semidefiniteness of $\boldsymbol{G}_\infty$ is that the free energy of every equilibrium state is non-negative; moreover, the property (P2) tells us that the free energy of every state is non-negative. It is also known that $\mathscr{F}$ is convex, i.e., that every convex combination of free energies is a free energy [6, Sec. 3].

We now discuss the question of the existence of free energies. More precisely, we wish to provide necessary and sufficient conditions under which the set $\mathscr{F}$ is not empty. We observe that, if ψ is a lower potential for $\hat{w}$, then the inequality (5.7) implies that

$$\hat{w}(P, \sigma) > -\varepsilon \tag{5.22}$$

for all $P \in \Pi$ with $P\sigma \in \mathscr{B}(\sigma, \delta)$. This property of $\hat{w}$ is called the *dissipation property at* σ [6].

Thus, if $\hat{w}$ has a lower potential, it has the dissipation property at all states in Σ. It has been proved in [6, Theorem 3.3] that if $\hat{w}$ is an action with the dissipation property at some $\sigma \in \Sigma$, then it has a lower potential whose domain is the set of all states which are $\hat{w}$-approachable from σ. By Proposition 4.3, this set is here the whole state space Σ. Therefore, for a viscoelastic element, the assertions

- $\hat{w}$ has the dissipation property at some $\sigma \in \Sigma$,
- $\hat{w}$ has the dissipation property at all $\sigma \in \Sigma$,
- $\mathscr{F}$ is not empty

are equivalent. Moreover, the first of them is equivalent to

- the relaxation function $\boldsymbol{G}$ is dissipative.

Indeed, we recall that $\boldsymbol{G}$ is *dissipative* if the work done in any process starting from the natural state is non-negative. Thus, if $\boldsymbol{G}$ is dissipative the inequality (5.22) with $\sigma = \sigma_0$ is satisfied for all $\varepsilon > 0$, and therefore $\hat{w}$ has the dissipation property at σ_0. Conversely, if $\hat{w}$ has the dissipation property at σ_0, then $\mathscr{F}$ is not empty, and the inequality (5.6) together with the condition $\psi(\sigma_0) = 0$ and the fact that ψ takes non-negative values tell us that

$$\hat{w}(P, \sigma_0) \geqq \psi(P\sigma_0) - \psi(\sigma_0) \geqq 0 \tag{5.23}$$

for all $P \in \Pi$, i.e., that $\boldsymbol{G}$ is dissipative.

Consider the function ψ^0 defined over Σ by

$$\psi^0(\sigma) := \lim_{\delta \to 0^+} \inf \{\hat{w}(P, \sigma_0) \mid P \in \Pi, P\sigma_0 \in \mathscr{B}(\sigma, \delta)\}. \tag{5.24}$$

This is the *minimum work expended* to approach σ by a process starting from the natural state. It follows from Theorem 3.3 in [6] that, if $\hat{w}$ has the dissipation property at σ_0, then ψ^0 is a lower potential for $\hat{w}$ and is lower semicontinuous. Moreover, Theorem 3.5 of [6] implies that ψ^0 is the *maximal* free energy in the sense that, for every function ψ in $\mathscr{F}$,

$$\psi(\sigma) \leqq \psi^0(\sigma) \quad \forall \sigma \in \Sigma. \tag{5.25}$$

The same theorem also shows that, under more restricted conditions, the function

$$\psi_0(\sigma) := - \lim_{\delta \to 0^+} \inf \{\hat{w}(P, \sigma) \mid P \in \Pi, P\sigma \in \mathscr{B}(\sigma_0, \delta)\} \tag{5.26}$$

is the minimal element of $\mathscr{F}$. In the present setting of viscoelastic elements, this part of the theorem can be restated as

5.3. Proposition. *Assume that $\hat{w}$ has the dissipation property at σ_0, and that*

(i) *σ_0 is $\hat{w}$-approachable from every other state in Σ,*
(ii) *$\psi_0(\sigma) > -\infty$ for all $\sigma \in \Sigma$.*

Then ψ_0 belongs to $\mathscr{F}$, and every other function ψ in $\mathscr{F}$ satisfies

$$\psi_0(\sigma) \leqq \psi(\sigma) \quad \forall \sigma \in \Sigma. \tag{5.27}$$

For viscoelastic material elements, property (i) has been proved in Proposition 4.3, and (ii) is a consequence of (i). Indeed, it is sufficient to take a family $\delta \mapsto P_\delta$ of processes such that $\| P_\delta\sigma - \sigma_0 \|_\Sigma < \delta$ and $\delta \mapsto \hat{w}(P_\delta, \sigma)$ converges when $\delta \to 0^+$, to get from inequality (5.6) that

$$-\psi_0(\sigma) \leqq \lim_{\delta \to 0^+} \hat{w}(P_\delta, \sigma) < +\infty. \tag{5.28}$$

Therefore, the hypotheses (i), (ii) in Proposition 5.3 are satisfied whenever $\hat{w}$ has the dissipation property at σ_0. As a complement to the same proposition we prove now that, if $\hat{w}$ has the dissipation property at σ_0, then the function ψ_0 is lower semicontinuous. We begin with a preliminary result, which characterizes $\psi_0(\sigma)$ as the *maximum recoverable work* from σ.

5.4. Lemma. *If $\hat{w}$ has the dissipation property at σ_0, then for each $\delta > 0$ and for each $\sigma \in \Sigma$,*

$$\inf\left\{\hat{w}(P,\sigma)\,\middle|\,P \in \Pi, P\sigma \in \mathscr{B}(\sigma_0,\delta)\right\} = \inf\left\{\hat{w}(P,\sigma)\,\middle|\,P \in \Pi\right\}. \tag{5.29}$$

Proof. The above relation is satisfied trivially if the equality sign is replaced by $\geqq$; it is then sufficient to prove the same relation with $\leqq$ in place of $=$. For every process P and for every state σ, the non-negativity of the free energy and inequality (5.6) imply that

$$\hat{w}(P,\sigma) \geqq \psi^0(P\sigma) - \psi^0(\sigma) \geqq -\psi^0(\sigma), \tag{5.30}$$

and therefore that $\{\hat{w}(P,\sigma)\,|\,P \in \Pi\}$ has a finite lower bound. Then, for every $\varepsilon > 0$ there is a process P such that

$$\hat{w}(P,\sigma) \leqq \inf\{\hat{w}(P,\sigma)\mid P \in \Pi\} + \tfrac{1}{3}\,\varepsilon. \tag{5.31}$$

Let $\sigma \sim (H,A)$ and $P = (K,B)$. For each $r > 0$, consider the process $P_r := (B_r^\dagger * K, B)$ obtained by continuing K with a constant segment of duration r. Because no work is done in the continuation, the work $\hat{w}(P_r,\sigma)$ is equal to $\hat{w}(P,\sigma)$. Moreover, $P_r\sigma \sim ((B+A)_r^\dagger * K_A * H, B+A)$, so that

$$\|\, P_r\sigma - \sigma_{B+A}\,\|_\Sigma = \|\,(B+A)_r^\dagger * K_A * H - (B+A)^\dagger\,\|_\Gamma, \tag{5.32}$$

and the family $r \mapsto P_r\sigma$ converges to σ_{B+A} when $r \to +\infty$ by item (iii) in Proposition 2.2. Take another segment K' of duration d', with $K'(d') = 0$ and $K'(0^+) = -B-A$, and let P'_α be the process $(K'_\alpha, -B-A)$, with K'_α the α- retardation of K'. Then the work done in the process $P'_\alpha P_r$ from σ is

$$\hat{w}(P'_\alpha P_r,\sigma) = \hat{w}(P_\alpha{}',P_r\sigma) + \hat{w}(P_r,\sigma) = \hat{w}(P_\alpha{}',P_r\sigma) + \hat{w}(P,\sigma). \tag{5.33}$$

By proposition 4.1, the convergence of $r \mapsto P_r\sigma$ to σ_{B+A} implies the convergence of $r \mapsto \hat{w}(P_\alpha{}',P_r\sigma)$ to $\hat{w}(P_\alpha{}',\sigma_{B+A})$. Moreover, by the properties (5.17), (5.18) of α-retardations, the states $\alpha \mapsto P_\alpha{}'\sigma_{B+A}$ converge to σ_0 when $\alpha \to +\infty$, and the works $\alpha \mapsto \hat{w}(P_\alpha{}',P_r\sigma)$ converge to $-\frac{1}{2}\,\boldsymbol{G}_\infty(B+A)\cdot(B+A)$, which is a non-positive real by the positive definiteness of $\boldsymbol{G}_\infty$. It is then possible to select r and α sufficiently large to have

$$\hat{w}(P_\sigma{}',P_r\sigma) \leqq \hat{w}(P'_\alpha,\sigma_{B+A}) + \tfrac{1}{3}\,\varepsilon \leqq \tfrac{2}{3}\,\varepsilon\,, \tag{5.34}$$

and therefore, by (5.31) and (5.33),

$$\hat{w}(P'_\alpha,P_r\sigma) \leqq \inf\{\hat{w}(P,\sigma)|P \in \Pi\} + \varepsilon. \tag{5.35}$$

The convergence of $r \mapsto P_r\sigma$ to σ_{B+A} and that of $\alpha \mapsto P_\alpha{}'\sigma_{B+A}$ to σ_0 also imply that, for sufficiently large values of r and α, the state $P_\alpha{}'\,P_{\mathrm{r}}\sigma$ belongs to $\mathscr{B}(\sigma_0,\delta)$ for each fixed $\delta > 0$. For this choice of α and r,

$$\inf\{\hat{w}(P,\sigma)\mid P \in \Pi, P\sigma \in \mathscr{B}(\sigma_0,\delta)\} \leqq \hat{w}(P_\alpha{}'P_r,\sigma), \tag{5.36}$$

and the combination with the preceding inequality and the arbitrariness of ε lead to the relation (5.29) with $\leqq$ in place of $=$. $\square$

5.5. Proposition. *If $\hat{w}$ has the dissipation property at σ_0, then the function ψ_0 is lower semicontinuous.*

Proof. Select an $\varepsilon > 0$, a state σ', and a process P such that

$$\hat{w}(P, \sigma') \leqq \inf \{\hat{w}(P, \sigma') \mid P \in \Pi\} + \tfrac{1}{2}\,\varepsilon \tag{5.37}$$

By the continuity of $\hat{w}$ with respect to the states, there is a $\delta > 0$ such that

$$\hat{w}(P, \sigma) \leqq \hat{w}(P, \sigma') + \tfrac{1}{2}\,\varepsilon \tag{5.38}$$

for all σ in $\mathscr{B}(\sigma', \delta)$. By the preceding lemma, the infimum in inequality (5.37) is equal to $-\psi_0(\sigma')$, and $\hat{w}(P, \sigma)$ is not less than $-\psi_0(\sigma)$. Thus,

$$-\psi_0(\sigma) \leqq -\psi_0(\sigma') + \varepsilon \tag{5.39}$$

for all σ in $\mathscr{B}(\sigma', \delta)$. This proves the semicontinuity of ψ_0 at each σ' in Σ. □

Consider the function $\tilde{\psi}_{MV} : \Gamma \times \mathrm{Sym} \to \mathbb{R}$ defined by

$$\tilde{\psi}_{MV}(H, A) := \tilde{w}(H, A) + \tfrac{1}{2}\boldsymbol{G}_\infty H(\infty) \cdot H(\infty). \tag{5.40}$$

It has been proved by Morro & Vianello [22] that $\tilde{\psi}_{MV}$ has the properties $(P_1) - (P_4)$, and that it is the maximal element in the class of all functions which satisfy the same properties. Thus, if the free energy is defined as a function which satisfies the properties $(P_1)-(P_4)$, as was done in [22], then $\tilde{\psi}_{MV}$ is the maximal free energy.

This conclusion is no longer true for free energies defined according to Definition 5.2. Indeed, $\tilde{\psi}_{MV}$ need not satisfy the condition (5.4), and therefore need not be a function of state. This certainly occurs when Γ_0, the set of all solutions of equation (1.4), consists of the null history alone. There is an explicit example for this situation, which we discuss in Section 7 in the context of completely monotonic relaxation functions. There is another example in which $\tilde{\psi}_{MV}$ is not a function of state, the case of a relaxation function of exponential type, which we discuss in the next section.

Even if $\tilde{\psi}_{MV}$ is a function of state, it need not be a lower potential for the action $\hat{w}$, and therefore it need not belong to the set $\mathscr{F}$ of all free energies according to Definition 5.2. The conditions under which $\tilde{\psi}_{MV} \in \mathscr{F}$ are not known. However, the fact that $\tilde{\psi}_{MV}$ is the maximal element in the class of all functions which satisfy the properties $(P_1)-(P_4)$ and that all functions in $\mathscr{F}$ satisfy the same properties tells us that, whenever $\tilde{\psi}_{MV}$ belongs to $\mathscr{F}$, it coincides with the maximal element ψ^0 of $\mathscr{F}$.

6. Relaxation functions of exponential type

Let us consider a relaxation function of the type

$$\bar{\boldsymbol{G}}(s) = \bar{\boldsymbol{G}}_0\, e^{-s\boldsymbol{B}}, \tag{6.1}$$

where $\bar{\boldsymbol{G}}(s) := \boldsymbol{G}(s) - \boldsymbol{G}_\infty$. We assume that the tensor $\bar{\boldsymbol{G}}_0 := \bar{\boldsymbol{G}}(0)$ is symmetric and positive-definite, and that all eigenvalues of $\boldsymbol{B}$ have a positive real part. It has been proved in [12, Sec. 7] that this condition on $\boldsymbol{B}$ is necessary and sufficient to ensure that $\dot{\boldsymbol{G}}$ is Lebesgue integrable over $(0, +\infty)$, as assumed here in Section 2.

6.1. Definition. A *relaxation function of exponential type* is a function $\boldsymbol{G}$ which admits the representation (6.1), with $\bar{G}_0$ symmetric and positive definite and with all eigenvalues of $\boldsymbol{B}$ having a positive real part.

For this class of relaxation functions, the constitutive equation (2.24) takes the form

$$T(r) = \boldsymbol{G}_0 H(r) - \bar{\boldsymbol{G}}_0 \boldsymbol{B} \int_0^{+\infty} e^{-s\boldsymbol{B}} H(r+s)\, ds, \tag{6.2}$$

where by $T(r)$ we have denoted the stress $\tilde{T}(H^r, H(r))$. An important property of exponential relaxation functions, already studied by Graffi & Fabrizio [18], is that the states can be identified with the pairs formed by the current stress and the current deformation. In our approach, this is a consequence of the fact that for an exponential relaxation function the number $|T(0^+) - \boldsymbol{G}_0 H(0^+)|$ is a seminorm for the history space Γ, and that this seminorm is equivalent to $\|\cdot\|_\Gamma$.

6.2. Proposition. *If $\boldsymbol{G}$ is a relaxation function of exponential type, then there is a positive constant m such that*

$$|T(0^+) - \boldsymbol{G}_0 H(0^+)| \leqq \|H\|_\Gamma \leqq m|T(0^+) - \boldsymbol{G}_0 H(0^+)| \tag{6.3}$$

for all histories $H \in \Gamma$.

Proof. By the definition (2.9) of $\|\cdot\|_\Gamma$ and by equations (6.1) and (6.2), we have

$$\|H\|_\Gamma = \sup_{t \geqq 0} \left| \int_0^{+\infty} \bar{\boldsymbol{G}}_0 \boldsymbol{B} e^{-(t+s)\boldsymbol{B}} H(s)\, ds \right| = \sup_{t \geqq 0} |\bar{\boldsymbol{G}}_0 e^{-t\boldsymbol{B}} \bar{\boldsymbol{G}}_0^{-1} (T(0^+) - \boldsymbol{G}_0 H(0^+))|, \tag{6.4}$$

where in the last step we have taken advantage of the fact that $\boldsymbol{B}$ commutes with $e^{-t\boldsymbol{B}}$. If we take $t = 0$, we get the first of the inequalities (6.3). To prove the second inequality, we use the identity

$$\bar{\boldsymbol{G}}_0\, e^{-t\boldsymbol{B}}\, \bar{\boldsymbol{G}}_0^{-1} = e^{-t\bar{\boldsymbol{G}}_0 \boldsymbol{B} \bar{\boldsymbol{G}}_0^{-1}} \tag{6.5}$$

and the fact that the eigenvalues of $\bar{\boldsymbol{G}}_0 \boldsymbol{B} \bar{\boldsymbol{G}}_0^{-1}$ are the same as those of $\boldsymbol{B}$, and therefore have a positive real part. Under these conditions, it has been proved in [12, Prop. 7.2] that $\|e^{-t\bar{\boldsymbol{G}}_0 \boldsymbol{B} \bar{\boldsymbol{G}}_0^{-1}}\|$ is bounded by a positive constant m, and the second of inequalities (6.3) follows. □

This result tells us that, for a relaxation function of exponential type, the equivalence classes modulo Γ_0 in the history space Γ can be characterized by the tensor $(T(0^+) - \boldsymbol{G}_0 H(0^+))$. Consider now a state $\sigma \sim (H, A)$ and denote by T the current stress $\tilde{T}(H, A)$ given by equation (2.24). The same equation tells us that

$$T - \boldsymbol{G}_0 A = T(0^+) - \boldsymbol{G}_0 H(0^+). \tag{6.6}$$

Consequently, a state $\sigma \sim (H, A)$ can be identified with the pair (T, A) formed by the current stress and the current deformation, and the state space Σ can be identified with the product space Sym^2, which is finite-dimensional. Thus, the definition (3.3) of the norm $\|\cdot\|_\Sigma$ and the equivalence result (6.3) tell us that $\|\cdot\|_\Sigma$ is equivalent to any norm of Sym^2. In particular, we find it convenient to identify the state σ with the pair (T, A), and to take

$$|T - \boldsymbol{G}_\infty A| + |A| \tag{6.7}$$

as the norm in the state space.

For relaxation functions of exponential type, the action $\hat{w}$ takes a special form that we wish to determine now. Let us begin with a preliminary result.

6.3. Proposition. *Let* $\boldsymbol{G}$ *be of exponential type. Then for every history* $H \in \Gamma$ *the function* $r \rightarrow (T(r) - \boldsymbol{G}_0 H(r))$ *is differentiable, and its derivative is*

$$(T(r) - \boldsymbol{G}_0 H(r))^{\cdot} = -\dot{\boldsymbol{G}}(0)\bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_\infty H(r)), \tag{6.8}$$

with

$$\dot{\boldsymbol{G}}(0) = -\bar{\boldsymbol{G}}_0 \boldsymbol{B}. \tag{6.9}$$

Proof. With the change of variable $r + s = t$, equation (6.2) takes the form

$$T(r) - \boldsymbol{G}_0 H(r) = -\bar{\boldsymbol{G}}_0 \boldsymbol{B} \int_r^{+\infty} e^{-(t-r)\boldsymbol{B}} H(t)\, dt, \tag{6.10}$$

which shows that $r \mapsto (T(r) - \boldsymbol{G}_0\, H(r))$ is differentiable and that its derivative is

$$(T(r) - \boldsymbol{G}_0 H(r))^{\cdot} = \bar{\boldsymbol{G}}_0 \boldsymbol{B}\, H(r) - \bar{\boldsymbol{G}}_0 \boldsymbol{B}^2 \int_r^{+\infty} e^{-(t-r)\boldsymbol{B}} H(t)\, dt. \tag{6.11}$$

The desired equality (6.8) follows after subtracting from equation (6.11) the equation (6.10) premultiplied by $\bar{\boldsymbol{G}}_0\ \boldsymbol{B}\ \bar{\boldsymbol{G}}_0.^{-1}$ □

Let us now multiply both sides of equation (6.8) by $\bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_\infty H(r))$ and integrate over $(0, s)$:

$$\int_0^s (T(r) - \boldsymbol{G}_0 H(r))^{\cdot} \cdot \bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_\infty H(r))\, dr$$

$$= -\int_0^s \bar{\boldsymbol{G}}_0^{-1} \dot{\boldsymbol{G}}(0) \bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_\infty H(r)) \cdot (T(r) - \boldsymbol{G}_\infty H(r))\, dr. \tag{6.12}$$

The left-hand side is equal to

$$\left[\tfrac{1}{2} \bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_0 H(r)) \cdot (T(r) - \boldsymbol{G}_0 H(r))\right]_0^s + \int_0^s (T(r) - \boldsymbol{G}_0 H(r))^{\cdot} \cdot H(r)\, dr, \tag{6.13}$$

and can be further transformed by integration by parts, to get

$$\Big[\tfrac{1}{2} \bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_0 H(r)) \cdot (T(r) - \boldsymbol{G}_0 H(r)) + T(r) \cdot H(r) - \tfrac{1}{2} \boldsymbol{G}_0 H(r) \cdot H(r)\Big]_0^s - \int_0^s T(r) \cdot dH(r). \tag{6.14}$$

If we now set

$$\bar{\psi}(T, A) := \tfrac{1}{2} \bar{\boldsymbol{G}}_0^{-1}(T - \boldsymbol{G}_\infty A) \cdot (T - \boldsymbol{G}_\infty A) + \tfrac{1}{2} \boldsymbol{G}_\infty A \cdot A, \tag{6.15}$$

we see after some computation that the term in brackets is equal to $\bar{\psi}(T(r),\ H(r))$. Therefore, equation (6.12) takes the form

$$-\int_0^s T(r) \cdot dH(r) = -\left[\bar{\psi}(T(r), H(r))\right]_0^s - \int_0^s \bar{\boldsymbol{G}}_0^{-1} \dot{\boldsymbol{G}}(0) \bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_\infty H(r)) \cdot (T(r) - \boldsymbol{G}_\infty H(r))\, dr. \tag{6.16}$$

By the definitions (4.4), (4.5), the integral on the left-hand side is the work done in the process $(H_s - (H(s))_s^\dagger,\ A - H(s))$ starting from the state $(T(s), H(s)) \sim (H^s, H(s))$. More generally, if we consider a state $\sigma = (T, A)$ and a process $P = (K, B)$ of duration d, then equation (6.16) tells us that

$$\hat{w}(P, \sigma) = \bar{\psi}(P\sigma) - \bar{\psi}(\sigma) - \int_0^d \bar{\boldsymbol{G}}_0^{-1} \dot{\boldsymbol{G}}(0) \bar{\boldsymbol{G}}_0^{-1}(T(r) - \boldsymbol{G}_\infty K_A(r)) \cdot (T(r) - \boldsymbol{G}_\infty K_A(r))\, dr. \tag{6.17}$$

This is the expression of the action $\hat{w}$ for a relaxation function of exponential type. From it, the following result on the existence of a free energy in the sense of Definition 5.2 can be deduced.

6.4. Proposition. *Let* $\boldsymbol{G}$ *be a relaxation function of exponential type. Then the function* $\bar{\psi}$ *defined by equation* (6.15) *is a free energy in the sense of Definition* 5.2 *if and only if* $\dot{\boldsymbol{G}}(0)$ *is negative-semidefinite.*

Proof. It has already been proved in Section 5 that if there is a free energy, then the material element is dissipative. The negative semidefiniteness of $\dot{\boldsymbol{G}}(0)$ is a well-known property of the dissipative viscoelastic material element; see, e.g., [11, Sec. 6.1].

Conversely, if we assume that $\dot{\boldsymbol{G}}(0)$ is negative-semidefinite, then equation (6.17) shows that $\bar{\psi}$ satisfies the integrated dissipation inequality (5.6). Because $\bar{\psi}$ is continuous with respect to the norm (6.7) and this norm is equivalent to $\|\cdot\|_\Sigma$, we deduce that the inequality (5.7) is satisfied, and therefore $\bar{\psi}$ is a lower potential for $\hat{w}$. Because the normalization condition $\bar{\psi}(O,O)=0$ is also satisfied, we conclude that $\bar{\psi}$ is a free energy in the sense of Definition 5.2. □

A relaxation function with a scalar exponent

$$\bar{\boldsymbol{G}}(s) := \bar{\boldsymbol{G}}_0 e^{-\alpha s} \tag{6.18}$$

is a particular case of equation (6.1), obtained by taking $\boldsymbol{B}=\alpha\boldsymbol{I}$. For relaxation functions of this type, it has been proved by Graffi & Fabrizio [18][3] that there is just one function of state which satisfies the properties $(P_1)-(P_3)$ listed at the beginning of Sec. 5, and that this is the function $\bar{\psi}$ defined by equation (6.15). It follows then from Proposition 5.1 that $\bar{\psi}$ is the only possible free energy according to Definition 5.2. Here we extend this result to all relaxation functions of exponential type with $\dot{\boldsymbol{G}}(0) \leqq \boldsymbol{O}$. We recall that by Proposition 6.4 the assumption $\dot{\boldsymbol{G}}(0) \leqq \boldsymbol{O}$ implies that $\bar{\psi}$ belongs to the set $\mathscr{F}$ of all free energies, and that the functions ψ^0, ψ_0 defined by equations (5.24) and (5.26) are the maximal and the minimal element of $\mathscr{F}$, respectively.

6.5. Proposition. *Let* $\boldsymbol{G}$ *be a relaxation function of exponential type, with* $\dot{\boldsymbol{G}}(0) \leqq \boldsymbol{O}$. *Let* $\bar{\psi}$ *be the element of* $\mathscr{F}$ *defined by equation* (6.15) *and let* ψ^0 *and* ψ_0 *be the maximal and the minimal elements of* $\mathscr{F}$. *Then* $\psi^0=\psi_0=\bar{\psi}$.

Proof. With every state $\sigma=(T,A)$, let us associate the deformation

$$B := \bar{\boldsymbol{G}}_0^{-1}(\boldsymbol{G}_0 A - T). \tag{6.19}$$

Consider a segment K of duration d, with $K(d)=0$ and $K(0^+)=B$, and the process $P_\alpha=(K_\alpha,B)$, with K_α the α-retardation of K. If σ_0 is the natural state, it follows from the properties (5.17), (5.18) of the α-retardations that

$$\lim_{\alpha\to+\infty} P_\alpha\sigma_0 = (\boldsymbol{G}_\infty\, B, B), \tag{6.20}$$

$$\lim_{\alpha\to+\infty} \hat{w}(P_\alpha,\sigma_0) = \tfrac{1}{2}\,\boldsymbol{G}_\infty B\cdot B. \tag{6.21}$$

[3]See also Breuer & Onat [2].

Moreover, if we consider the process $P'_\alpha := (K_\alpha, A)$, from the formula (5,11) we get

$$\lim_{\alpha\to+\infty} P'_\alpha\sigma_0 = (\boldsymbol{G}_\infty B + \boldsymbol{G}_0(A-B), A) = (T, A) = \sigma, \tag{6.22}$$

and, from (4.2) and the expression (6.19) for B,

$$\begin{aligned}\lim_{\alpha\to+\infty} \hat{w}(P'_\alpha, \sigma_0) &= \tfrac{1}{2}\,\boldsymbol{G}_\infty B\cdot B + \boldsymbol{G}_\infty B\cdot(A-B) + \tfrac{1}{2}\,\boldsymbol{G}_0(A-B)\cdot(A-B)\\ &= \tfrac{1}{2}\,\bar{\boldsymbol{G}}_0 B\cdot B - \bar{\boldsymbol{G}}_0 A\cdot B + \tfrac{1}{2}\boldsymbol{G}_0 A\cdot A\\ &= \tfrac{1}{2}\,\bar{\boldsymbol{G}}_0^{-1}(T-\boldsymbol{G}_0 A)\cdot(T-\boldsymbol{G}_0 A) + A\cdot(T-\boldsymbol{G}_0 A) + \tfrac{1}{2}\boldsymbol{G}_0 A\cdot A.\end{aligned} \tag{6.23}$$

Now we observe that the right-hand side in the last equation is equal to $\bar{\psi}(T, A)$, as can be checked by direct computation. Thus,

$$\lim_{\alpha\to+\infty} \hat{w}(P'_\alpha, \sigma_0) = \bar{\psi}(T, A). \tag{6.24}$$

By equation (6.22), for every fixed $\delta > 0$ the state $P'_\alpha\sigma_0$ belongs to $\mathscr{B}(\sigma, \delta)$ for sufficiently large values of α. Therefore, for each $\delta > 0$ there is an α such that

$$\hat{w}(P'_\alpha, \sigma_0) \geqq \inf \hat{w}\{\sigma_0 \to \mathscr{B}(\sigma, \delta)\}, \tag{6.25}$$

and the fact that, when $\delta \to 0$, the left-hand side converges to $\bar{\psi}(T, A)$ and the right-hand side converges to $\psi^0(T, A)$ allows us to conclude that $\bar{\psi}(T, A) \geqq \psi^0(T, A)$ for all states (T, A) in Σ, and, therefore, that $\bar{\psi} \geqq \psi^0$. Because ψ^0 is the maximal free energy, we have proved that $\bar{\psi} = \psi^0$.

To prove that $\bar{\psi} = \psi_0$, consider a state (T, A) and suppose that the deformation undergoes a jump from A to B. Then equation (5.11) tells us that the stress jumps from T to $T' = T + \boldsymbol{G}_0(B - A)$, and if we take B as in equation (6.19), we see that $T' = \boldsymbol{G}_\infty B$. This characterizes (T', B) as the equilibrium state associated with B; indeed, the constitutive equation (2.24) shows that $\boldsymbol{G}_\infty B$ is the stress associated with the history-deformation pair $(B^\dagger, B)$.

By equation (4.2), the work done in the jump from A to B is $T\cdot(B-A) + \frac{1}{2}\,\boldsymbol{G}_0(B-A)\cdot(B-A)$. If we now subject the equilibrium state (T', B) to the retarded processes $P_\alpha = (K_\alpha, O)$, the work done in those processes converges to $-\frac{1}{2}\boldsymbol{G}_\infty B\cdot B$ when $\alpha \to +\infty$ by equation (5.18). Thus, the total work done is

$$T\cdot(B-A) + \tfrac{1}{2}\,\boldsymbol{G}_0(B-A)\cdot(B-A) - \tfrac{1}{2}\,\boldsymbol{G}_\infty B\cdot B, \tag{6.26}$$

and substitution of the expression (6.19) of B shows that this work is equal to $-\bar{\psi}(T, A)$. In this way, we have proved that it is possible to recover the work $\bar{\psi}(T, A)$ from the state (T, A). Recalling the characterization (5.29) of ψ_0 as the maximum recoverable work, we conclude that $\psi_0(T, A) \geqq \bar{\psi}(T, A)$, for all states (T, A), and therefore $\psi_0 \geqq \bar{\psi}$. The equality $\psi_0 = \bar{\psi}$ then follows from the fact that ψ_0 is the minimal element of $\mathscr{F}$. $\square$

We now prove that the function $\bar{\psi}_{MV}$ defined by equation (5.40) is not a function of state when the relaxation function is of exponential type. Consider the history

$$H(s) := A\,f(s), \tag{6.27}$$

where $A \neq O$ is a fixed deformation and f is the piecewise constant function defined over $(0,\infty)$ by

$$f(s) := \begin{cases} 1 & \text{for } a < s \leqq b, \\ -1 & \text{for } b < s \leqq c, \\ 0 & \text{everywhere else,} \end{cases} \tag{6.28}$$

with $0 < a < b < c$. It is convenient to express the stress in the form

$$T(r) = \boldsymbol{G}_\infty\; Af(r) - \bar{\boldsymbol{G}}_0 \int_r^\infty e^{-(s-r)\boldsymbol{B}} A\; df(s), \tag{6.29}$$

obtained from equation (6.2) by integration by parts. In particular,

$$\begin{aligned} T(c^+) &= O, \\ T(b^+) &= \boldsymbol{G}_\infty A + \bar{\boldsymbol{G}}_0 e^{-(c-b)\boldsymbol{B}} A, \\ T(a^+) &= -\boldsymbol{G}_\infty A + \bar{\boldsymbol{G}}_0 e^{-(c-a)\boldsymbol{B}} A - 2\bar{\boldsymbol{G}}_0 e^{-(b-a)\boldsymbol{B}} A, \\ T(0^+) &= \bar{\boldsymbol{G}}_0 e^{-a\boldsymbol{B}} A - 2\bar{\boldsymbol{G}}_0 e^{-b\boldsymbol{B}} A + \bar{\boldsymbol{G}}_0 e^{-c\boldsymbol{B}} A. \end{aligned} \tag{6.30}$$

The last equation shows that, if a, b and c satisfy

$$e^{-(c-a)\boldsymbol{B}} - 2e^{-(b-a)\boldsymbol{B}} + 1 = 0, \tag{6.31}$$

then $T(0^+) = O$. Because $H(0^+) = O$, the history-deformation pair (H, O) with H defined by equations (6.27), (6.28) and (6.31) represents the natural state, just as done by the pair $(O^\dagger, O)$. We wish to show that $\bar{\psi}_{MV}(H, O)$ is different from $\bar{\psi}_{MV}(O^\dagger, O)$. Because the first is equal to $\bar{w}(H, O)$ and the second is zero, it is sufficient to show that $\bar{w}(H, O)$ is not identically zero. The history H being piecewise constant, the formulae (4.3) and (5.11) yield

$$\begin{aligned} \bar{w}(H,O) &= \tfrac{1}{2}(T(c^+) + T(c))\cdot A + \tfrac{1}{2}(T(b^+) + T(b))\cdot(-2A) + \tfrac{1}{2}(T(a^+) + T(a))\cdot A \\ &= T(c^+)\cdot A - 2T(b^+)\cdot A + T(a^+)\cdot A + 3\boldsymbol{G}_0 A\cdot A, \end{aligned} \tag{6.32}$$

and substitution of the expressions (6.30) for the stresses yields

$$\bar{w}(H,O) = \bar{\boldsymbol{G}}_0(-2e^{-(c-b)\boldsymbol{B}} + e^{-(c-a)\boldsymbol{B}} - 2e^{-(b-a)\boldsymbol{B}} + 3\,I)A\cdot A. \tag{6.33}$$

Finally, using the condition (6.31) we get

$$\bar{w}(H,O) = 2\bar{\boldsymbol{G}}_0(\boldsymbol{I} - e^{-(c-b)\boldsymbol{B}})A\cdot A. \tag{6.34}$$

That $\bar{w}(H, O)$ is not identically zero follows from the fact that the right-hand side is a continuous function of $(c-b)$, which takes the value zero when $(c-b) = 0$ and approaches the positive value $2\bar{\boldsymbol{G}}_0 A\cdot A$ when $(c-b) \to +\infty$. $\square$

6.6. Remark. A similar argument shows that the free energy of single-integral type proposed by GURTIN & HRUSA [20]:

$$\bar{\psi}_{G^-}(H,A) := \frac{1}{2}\boldsymbol{G}_0 A\cdot A + A\cdot\int_0^{+\infty}\dot{\boldsymbol{G}}(s)H(s)\,ds - \frac{1}{2}\int_0^{+\infty}\dot{\boldsymbol{G}}(s)H(s)\cdot H(s)\,ds \quad (6.35)$$

is not a free energy according to Definition 5.2 if the relaxation function is of exponential type. Indeed, for $A = O$ and for the history H defined by equation (6.27) with A replaced by B we get

$$\begin{aligned}\bar{\psi}_{\boldsymbol{GH}}(H,O) &= -\tfrac{1}{2}\int_0^{+\infty}\dot{\boldsymbol{G}}(s)B\cdot B\,f^2(s)\,ds = -\tfrac{1}{2}\int_a^{c}\dot{\boldsymbol{G}}(s)B\cdot B\,ds\\ &= \tfrac{1}{2}\,(\bar{\boldsymbol{G}}(a)-\bar{\boldsymbol{G}}(c))B\cdot B,\end{aligned} \quad (6.36)$$

and the last term is not identically zero.

7. Completely monotonic relaxation functions

We recall that a relaxation function $\boldsymbol{G}$ is *completely monotonic* if it has derivatives $\boldsymbol{G}^{(p)}$ of every order p and if

$$(-1)^p\boldsymbol{G}^{(p)}(s) \geqq \boldsymbol{O} \quad (7.1)$$

for all $s \geqq 0$ and for all $p \in \mathbb{N}$. It has been proved in [12, Sec. 6] that $\boldsymbol{G}$ is completely monotonic if and only if its symmetric part $\boldsymbol{G}^s$ admits the representation

$$\boldsymbol{G}^s(s) = \int_0^{+\infty} e^{-\omega s}\,d\boldsymbol{K}(\omega), \quad (7.2)$$

with $\boldsymbol{K}: [0,+\infty) \to \mathrm{LinSym}$, symmetric, bounded and non-decreasing with respect to the order relation (1.7).

The problem that we consider here is the characterization of the state space for completely monotonic relaxation functions. As we shall see, there are different characterizations for different subclasses of such functions. In particular, we are interested in determining a subclass for which states can be identified with history-deformation pairs. We know that this occurs whenever the solution set Γ_0 of equation (1.4) reduces to the null history. Let us take, then, a history $H \in \Gamma_0$. The equation (1.4), multiplied by $H(t)$ and integrated over $(0,\infty)$, yields

$$\int_0^{+\infty}\int_0^{+\infty}\dot{\boldsymbol{G}}(t+s)H(s)\cdot H(t)\,ds\,dt = 0. \quad (7.3)$$

In this equation, $\boldsymbol{G}$ can be replaced by its symmetric part, and we can use the representation (7.2) to obtain

$$\int_0^{+\infty} d\boldsymbol{K}(\omega)(\omega^{1/2}\hat{H}(\omega)) \cdot (\omega^{1/2}\hat{H}(\omega)) = 0, \tag{7.4}$$

where $\hat{H}(\omega)$ is the Laplace integral

$$\hat{H}(\omega) := \int_0^{+\infty} e^{-\omega s} H(s)\, ds, \tag{7.5}$$

with ω real and non-negative. That $\boldsymbol{K}$ is non-decreasing implies that the integrand in equation (7.4) vanishes at almost every ω. In particular, $\hat{H}(\omega) = O$ at all those ω at which $\boldsymbol{K}$ is continuous and has a right derivative which is positive-definite, and at all those ω at which $\boldsymbol{K}$ has a jump and $\boldsymbol{K}(\omega^+) > \boldsymbol{K}(\omega^-)$. When either of these conditions is satisfied, we say that $\boldsymbol{K}$ is *strictly increasing* at ω. Thus, we may state

7.1. Proposition. Let $H \in \Gamma_0$. *Then* $\hat{H}(\omega) = O$ *at all those* ω *at which* $\boldsymbol{K}$ *is strictly increasing*.

It is known that if $\hat{H}(\omega) = O$ for all $\omega > 0$, then H is the null history $O^\dagger$, and, therefore, that $\Gamma_0 = \{O^\dagger\}$. But there are weaker conditions on $\hat{H}$ leading to the same conclusion. We begin with the following result [26, Sec 2.6].

7.2 Proposition. *If* $H \in \Gamma_0$ *and if* $\hat{H}(n) = O$ *for* $n = 1, 2, \ldots,$ *then* $H = O^\dagger$.

Proof. The proof is based on the change of variable $r = e^{-s}$, which transforms equation (7.5) into

$$\hat{H}(\omega) = \int_0^1 r^{\omega-1} K(r)\, dr, \tag{7.6}$$

with $K(r) = H(\ln(r^{-1}))$. Assume that there is a sequence $n \mapsto \omega_n$ of points at which $\hat{H}(\omega_n) = O$. If we consider the linear combination

$$P_N(r) := \sum_{n=1}^{N} P_n r^{\omega_n - 1} \tag{7.7}$$

with coefficients P_n in Sym, we get

$$\int_0^1 P_N(r) \cdot K(r)\, dr = 0, \tag{7.8}$$

and therefore, by the Cauchy-Schwarz inequality,

$$\|K\|_{L^2}^2 := \int_0^1 |K(r)|^2 \, dr = \int_0^1 K(r) \cdot (K(r) - P_N(r)) \, dr \leqq \|K\|_{L^2} \|K - P_N\|_{L^2}. \tag{7.9}$$

Note that K is bounded, and therefore square integrable, because K and H take the same values and H is bounded. We may then divide by the L^2-norm of K, to obtain

$$\|K\|_{L^2} \leqq \|K - P_N\|_{L^2} \tag{7.10}$$

for every P_N of the type (7.7).

If $\omega_n = n$ as stated in the proposition, then the functions P_N are the polynomials. By the Weierstrass approximation theorem, the polynomials are dense in C^0 endowed with the supremum norm, and therefore they are dense in L^2. Consequently, for every $\varepsilon > 0$ there is a polynomial P_N such that the right-hand side of inequality (7.10) is less than ε. This implies that $\|K\|_{L^2} = 0$, i.e., that $K(r) = O$ almost everywhere.

We now observe that $K(r) = H(f(r))$, with f continuous and decreasing and H continuous from the left. Then K is continuous from the right, and a function which vanishes almost everywhere and is continuous from the right is identically zero. Because H takes the same values as K, we conclude that H is identically zero as well. □

It is clear from the preceding proof that the conclusion $H = O^\dagger$ remains valid if the sequence $n = 1, 2, \dots$ is replaced by any other sequence ω_n with the property that the linear combinations P_N are dense in C^0. The theorem of Müntz [7, p. 100] provides a class of sequences with this property. They are the sequences $n \mapsto \omega_n$ which satisfy the two conditions

$$\lim_{n \to +\infty} \omega_n = +\infty, \qquad \sum_{n=1}^{\infty} \omega_n^{-1} = +\infty. \tag{7.11}$$

It was recognized later (see [1]) that the unboundedness of the sequence is not essential. Thus, we can state the following fairly general conclusion.

7.3. Proposition. *For a completely monotonic relaxation function, the states can be identified with the history-deformation pairs if the function* $\boldsymbol{K}$ *defined by equation* (7.2) *is strictly increasing at a countable set* $n \mapsto \omega_n$ *of points, satisfying the condition* (7.11).

In particular, the identification of states with history-deformation pairs is possible whenever $\boldsymbol{K}$ is strictly increasing on a subset of $(0, +\infty)$ with positive measure. However, one cannot expect that this identification can be extended to all completely monotonic relaxation functions. Indeed, there are completely monotonic relaxation functions which are of exponential type, and it has been shown in the preceding section that for such functions the states cannot be identified with history-deformation pairs.

The question arises of which functions of exponential type are completely monotonic. In one dimension, the answer is easy: An exponential function is completely monotonic if and only if its exponent is negative. In higher dimensions, the definition (7.1) of complete monotonicity involves only the symmetric part of $\boldsymbol{G}$, and it is not so immediate to relate it with the definition (6.1), which involves the whole function. The result which we present here is obtained with the aid of the $(\boldsymbol{S}+\boldsymbol{N})$ *decomposition* of linear operators on a real finite-dimensional vector space [21, Sec. 6.2]. According to this decomposition, any $\boldsymbol{B} \in$ LinSym admits the representation

$$\boldsymbol{B} = \sum_{h=1}^{l} \lambda_h \, \boldsymbol{E}^h + \boldsymbol{N}. \tag{7.12}$$

where $\lambda_h = \alpha_h + i\beta_h$ are the eigenvalues of $\boldsymbol{B}, \boldsymbol{E}^h$ are projections of LinSym such that

$$\boldsymbol{E}^h \boldsymbol{E}^k = \delta_{hk} \boldsymbol{E}^h, \qquad \sum_{h=1}^{l} \boldsymbol{E}^h = \boldsymbol{I}, \tag{7.13}$$

and $\boldsymbol{N}$ is a nilpotent operator, i.e., an element of LinSym such that $\boldsymbol{N}^q = \boldsymbol{O}$ for some $q \in \mathbb{N}$. Moreover, $\boldsymbol{N}$ commutes with every $\boldsymbol{E}^h$:

$$\boldsymbol{N}\boldsymbol{E}^h = \boldsymbol{E}^h \boldsymbol{N} \qquad \forall\, h \in \{1, \ldots . l\}. \tag{7.14}$$

7.4. Proposition. *Let $\boldsymbol{G}$ be a relaxation function of exponential type with exponent $\boldsymbol{B}$. Then $\boldsymbol{G}$* is *completely monotonic if and only if all eigenvalues of $\boldsymbol{B}$ are real, all tensors $\boldsymbol{E}^h$ appearing in the decomposition* (7.12) *of $\boldsymbol{B}$ satisfy*

$$\bar{\boldsymbol{G}}_0 \boldsymbol{E}^h \geqq \boldsymbol{O} \qquad \forall h \in \{1, \ldots, l\}. \tag{7.15}$$

and the tensor $\boldsymbol{N}$ appearing in the same decomposition (7.12) *is zero.*

Proof. The properties (7.13) of the projections $\boldsymbol{E}^h$ and the definition of an exponential tensor function[4] imply that

$$\exp\left(\sum_{h=1}^{l} \lambda_h \boldsymbol{E}^h \right) = \sum_{h=1}^{l} e^{\lambda_h} \, \boldsymbol{E}^h. \tag{7.16}$$

Thus, by using the decomposition (7.12) of $\boldsymbol{B}$, equation (6.1) can be rewritten in the form

$$\bar{\boldsymbol{G}}(s) = \sum_{h=1}^{l} e^{-s\lambda_h} \bar{\boldsymbol{G}}_0 \, e^{-sN} \boldsymbol{E}^h. \tag{7.17}$$

Assume that $\boldsymbol{G}$ is completely monotonic. Then $\bar{\boldsymbol{G}}$ is positive-semidefinite, and this implies that

$$\sum_{h=1}^{l} e^{-s\lambda_h} \bar{\boldsymbol{G}}_0 e^{-sN} \boldsymbol{E}^h \boldsymbol{E}^p A \cdot \boldsymbol{E}^p A \geqq \boldsymbol{O} \tag{7.18}$$

[4] See, e.g., [12, Eq. 7.2].

for all $A \in \mathrm{Sym}$ and for all $p \in \{1, \ldots l\}$, and from the identity $\boldsymbol{E}^h\boldsymbol{E}^p = \delta_{hp}\boldsymbol{E}^p$ we conclude that

$$e^{-s\lambda_p}(\boldsymbol{E}^p)^T\bar{\boldsymbol{G}}_0 e^{-s\boldsymbol{N}}\boldsymbol{E}^p \geqq \boldsymbol{O}. \tag{7.19}$$

Because $\boldsymbol{N}^q = \boldsymbol{O}$, $e^{-s\boldsymbol{N}}$ is a polynomial of order $(q-1)$

$$e^{-s\boldsymbol{N}} = \sum_{r=0}^{q-1}\frac{s^r}{r!}(-\boldsymbol{N})^r, \tag{7.20}$$

and therefore the inequality (7.19) can be verified for every $s > 0$ only if λ_p is real. Thus, we have proved that $\boldsymbol{G}$ completely monotonic implies that all eigenvalues of $\boldsymbol{B}$ are real.

Now the coefficient $e^{-s\lambda_p}$ can be dropped form inequality (7.19); what remains is a polynomial of order $(q-1)$, whose term of highest order must be positive-semidefinite. Then, if we set $\boldsymbol{M} := (-\boldsymbol{N})^{q-1}$ and use the commutativity property (7.14), we get

$$(\boldsymbol{E}^p)^T\bar{\boldsymbol{G}}_0\boldsymbol{E}^p\boldsymbol{M} \geqq \boldsymbol{O} \qquad \forall p \in \{1, \ldots l\}. \tag{7.21}$$

We claim that $\boldsymbol{M} = \boldsymbol{O}$. Indeed, if $\boldsymbol{M} \neq \boldsymbol{O}$, there is a $C \in \mathrm{Sym}$ such that $\boldsymbol{M}C \neq O$, and the fact that $\sum_{p=1}^{l}\boldsymbol{E}^p\boldsymbol{M}C = \boldsymbol{M}C \neq O$ tells us that there must be some p for which $\boldsymbol{E}^p\boldsymbol{M}C \neq O$. For one such p, the inequality (7.21) yields

$$\bar{\boldsymbol{G}}_0(\boldsymbol{E}^p\boldsymbol{M}A + \boldsymbol{E}^p\boldsymbol{M}B)\cdot(\boldsymbol{E}^pA + \boldsymbol{E}^pB) \geqq 0 \tag{7.22}$$

for all $A, B \in \mathrm{Sym}$. In particular, if we take $B = \boldsymbol{M}C$, we have $\boldsymbol{M}B = \boldsymbol{M}^2C = O$ because $\boldsymbol{M}^2 = \boldsymbol{N}^{2q-2} = \boldsymbol{O}$ except for the trivial case $q = 1$. The inequality (7.22) then reduces to

$$\bar{\boldsymbol{G}}_0\boldsymbol{E}^p\boldsymbol{M}A\cdot(\boldsymbol{E}^pA + \boldsymbol{E}^p\boldsymbol{M}C) \geqq 0 \tag{7.23}$$

for all $A \in \mathrm{Sym}$, and this implies

$$\bar{\boldsymbol{G}}_0\boldsymbol{E}^p\boldsymbol{M}A\cdot\boldsymbol{E}^p\boldsymbol{M}C = 0 \tag{7.24}$$

for all $A \in \mathrm{Sym}$. It is now sufficient to take $A = C$ to get a contradiction, because $\boldsymbol{E}^p\boldsymbol{M}C \neq O$ and $\bar{\boldsymbol{G}}_0$ is positive-definite. Thus, we have proved that, if $\bar{\boldsymbol{G}}$ is positive-semidefinite, then $\boldsymbol{N}^q = \boldsymbol{O}$ implies $\boldsymbol{N}^{q-1} = \boldsymbol{O}$. By induction, we may conclude that $\boldsymbol{N} = \boldsymbol{O}$. The equation (7.17) then takes the form

$$\bar{\boldsymbol{G}}(s) = \sum_{h=1}^{l} e^{-s\alpha_h}\bar{\boldsymbol{G}}_0\boldsymbol{E}^h, \tag{7.25}$$

with α_h the real eigenvalues of $\boldsymbol{B}$. Now assume that $\boldsymbol{G}$ is completely monotonic. In view of the definition (7.1), this is equivalent to assuming that

$$\sum_{h=1}^{l}\alpha_h^p e^{-s\alpha_h}\,\bar{\boldsymbol{G}}_0\boldsymbol{E}^h \geqq \boldsymbol{O} \tag{7.26}$$

for all $s \geqq 0$ and for all $p \in \mathbb{N}$. In particular, if we set $s = 0$ and recall that the vectors $(\alpha_1^p, \alpha_2^p, \ldots, \alpha_l^p)$, $p \in \{0, 1, \ldots l-1\}$ are linearly independent, we obtain the inequalities (7.15). This completes the proof that a function of exponential type that is completely monotonic satisfies the conditions stated in the proposition. That the same conditions are sufficient for complete monotonicity is trivially verified. □

The formula (7.25) tells us that a completely monotonic relaxation function of exponential type is a linear combination of scalar exponentials, whose coefficients $\bar{\boldsymbol{G}}_0\boldsymbol{E}^h$ are positive semidefinite. Moreover, if we take the eigenvalues of $\boldsymbol{B}$ with the ordering $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_l$ and if we consider the piecewise constant function $\boldsymbol{K}$ defined by

$$\boldsymbol{K}(\omega) := \sum_{h=1}^{j} \bar{\boldsymbol{G}}_0\boldsymbol{E}^h \quad \text{for } \alpha_j < \omega < \alpha_{j+1},\ j \in \{1, \ldots, l\}, \tag{7.27}$$

we see that the formula (7.25) can be written in the form

$$\bar{\boldsymbol{G}}(s) = \int_0^{+\infty} e^{-\omega s}\, d\boldsymbol{K}(\omega), \tag{7.28}$$

and the representation (7.2) of $\boldsymbol{G}^s$ is obtained after replacing $\boldsymbol{K}$ by its symmetric part $\boldsymbol{K}^s$. Because it is constant except at a finite number of jump points, $\boldsymbol{K}^s$ does not satisfy the condition, required by Proposition 7.3, of being strictly increasing at a countable set of points. This agrees with the conclusion obtained in Section 6 that for relaxation functions of exponential type the states cannot be identified with history-deformation pairs.

Acknowledgements. We thank D. R. Owen for stimulating discussions and helpful suggestions, and V. Mizel for drawing our attention to reference [1] and for other valuable comments. We also acknowledge support by the Italian Ministry for University and Scientific Research.

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(Accepted December 7, 1995)