Arch. Rational Mech. Anal. 157 (2001) 165–191
Digital Object Identifier (DOI) 10.1007/s002050100126

On a Crystalline Variational Problem, Part I: First Variation and Global L^∞ Regularity

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Communicated by L. Ambrosio

Abstract

Let $\phi : \mathbb{R}^n \to [0, +\infty[$ be a given positively one-homogeneous convex function, and let $\mathcal{W}_\phi := \{\phi \leqq 1\}$. Pursuing our interest in motion by crystalline mean curvature in three dimensions, we introduce and study the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of "smooth" boundaries in the relative geometry induced by the ambient Banach space $(\mathbb{R}^n, \phi)$. It can be seen that, even when $\mathcal{W}_\phi$ is a polytope, $\mathcal{R}_\phi(\mathbb{R}^n)$ cannot be reduced to the class of polyhedral boundaries (locally resembling $\partial\mathcal{W}_\phi$). Curved portions must be necessarily included and this fact (as well as the nonsmoothness of $\partial\mathcal{W}_\phi$) is the source of several technical difficulties related to the geometry of Lipschitz manifolds. Given a boundary ∂E in the class $\mathcal{R}_\phi(\mathbb{R}^n)$, we rigorously compute the first variation of the corresponding anisotropic perimeter, which leads to a variational problem on vector fields defined on ∂E. It turns out that the minimizers have a uniquely determined (intrinsic) tangential divergence on ∂E. We define such a divergence to be the ϕ-mean curvature κ_ϕ of ∂E; the function κ_ϕ is expected to be the initial velocity of ∂E, whenever ∂E is considered as the initial datum for the corresponding anisotropic mean curvature flow. We prove that κ_ϕ is bounded on ∂E and that its sublevel sets are characterized through a variational inequality.

1. Introduction

Motion by crystalline mean curvature describes the interface evolution obtained as gradient flow of a surface energy functional P_ϕ having a crystalline density with respect to the usual perimeter. This means that, with $t \to \partial E(t)$ denoting the evolving front, the flow tries to reduce as fast as possible the value of $P_\phi(E(t))$, where

$$P_\phi(E(t)) := \int_{\partial E(t)} \phi^o(\nu^{E(t)})\, d\mathcal{H}^{n-1}.$$

Here $\nu^{E(t)}$ is the outward Euclidean unit normal to $\partial E(t)$ and ϕ^o (the surface tension) is a positively one-homogeneous function such that $\mathcal{W}^o_\phi := \{\phi^o \leqq 1\}$ is a convex *polytope* of $\mathbb{R}^n$ containing the origin in its interior. The term crystalline refers to the fact that the convex body $\mathcal{W}_\phi^o$ is faceted.

This evolution provides a simplified model for describing several phenomena in material science and crystal growth, see for instance [12, 13, 28, 23, 32]. It also represents an extreme example of anisotropic geometric flow, being the set $\mathcal{W}_\phi^o$ neither strictly convex nor of class C^2. The mathematical analysis of this problem began with the works of TAYLOR [28, 27, 30, 31] and has received a certain amount of attention in the last few years [2, 11, 1, 20, 17, 33]. In comparison with more familiar geometric evolutions (such as motion by mean curvature) it presents additional difficulties, due to the lack of regularity both of the involved operators and of the flowing interfaces. These obstructions, which at a first sight are of technical nature, reveal that the study of the geometric properties of hypersurfaces in a finite dimensional Banach space endowed with a crystalline metric cannot easily be reduced to more regular situations.

The simplest (even if not realistic) model is in $n = 2$ space dimensions, when the interface is a closed curve. In this case several results have been proved: in particular, the class of curves which are admissible as "regular" initial data for the evolution is characterized. The structure of a curve in this class is the following: if we denote by $\mathcal{W}_\phi$ the Wulff shape (that is, $\mathcal{W}_\phi := \{\phi \leqq 1\}$, where ϕ is the dual function of ϕ^o), a closed Lipschitz curve is admissible if it is a sequence of segments (with a precise order) which are parallel to some edge of $\partial\mathcal{W}_\phi = \{\phi = 1\}$ and of segments or arcs which correspond to vertices of $\partial\mathcal{W}_\phi$ [31, 25, 21, 26, 22, 18, 19, 4]. In addition, a local-in-time existence theorem for "regular" evolutions holds: each segment corresponding to an edge of $\partial\mathcal{W}_\phi$ translates parallel to itself (with a suitable velocity), while the remaining segments or arcs have zero velocity. It is important to remark that the admissibility properties of the curve, in relation with the geometry of $\partial\mathcal{W}_\phi$, remain unchanged during this evolution. Finally, a comparison principle holds, and therefore the flow is uniquely determined [19].

The physical case is however in three space dimensions, where the situation is much more complicated. In this case, one of the crucial mathematical problems, which to the best of our knowledge is still open, is the short-time existence of a "smooth" flow. Some examples can enlighten the difficulties related to such a result. In [6] two explicit crystalline evolutions of surfaces are constructed. In both the examples, the initial surfaces are polyhedral and their facets correspond to facets of the Wulff shape. The first example is completely rigorous, and shows that, at the initial time of the evolution, some facet can subdivide into smaller facets (facet-breaking phenomenon) and therefore new facets may appear. By means of a comparison argument [5], one can also show that the computed evolution is the unique crystalline evolution of the given initial surface. The second example, which is not completely rigorous, but is confirmed by several numerical simulations performed with different methods [24, 29] suggests that some facets can instantly curve. This unexpected phenomenon has some interesting consequences, which influence the approach to the crystalline evolution problem in three dimensions. For instance, it shows that the preferred set of directions, corresponding to the orientation of the facets of $\partial\mathcal{W}_\phi$, is not preserved during the evolution: indeed, new directions outside the preferred set may appear. In addition, one infers that the class of polyhedral surfaces (compatible with the geometry of $\partial\mathcal{W}_\phi$) is too restricted a class when one is looking for an existence result for crystalline evolutions. We remark that, in any case, the evolutions of the two examples should be regarded as “regular” crystalline evolutions. Summing up, in order to study crystalline motion by mean curvature, it seems that some preliminary steps are necessary. A first step is to have a reasonable definition of the class of crystalline “regular” surfaces. Given a set E in this class, the next step is to understand which is the initial velocity field on ∂E, in particular which are its singularities (which could be interpreted as the breaking regions). Finally, it is natural to investigate whether the class of regular surfaces is stable or not under motion by crystalline mean curvature.

The above arguments are the motivations for studying the *stationary* problem considered in the present paper. We first introduce the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of compact admissible interfaces, called Lipschitz ϕ-regular sets, which should be considered as the analogue of smooth boundaries in the case of crystalline geometry. The boundary ∂E of a Lipschitz ϕ-regular set E may be polyhedral (with a structure locally resembling the structure of $\partial\mathcal{W}_\phi$) but may also have curved portions. In addition, we impose the existence of a family of normal convex cones $x \to K(x)$ (with varying dimensions, in connection with the geometry of the Wulff shape) which contains a *continuous* section on ∂E. The precise requirement is that ∂E admits a vector field $n_\phi : \partial E \to \mathbb{R}^n$ which is a *Lipschitz* selection in this family of cones, i.e., $n_\phi \in \mathrm{Lip}(\partial E; \mathbb{R}^n)$ and $n_\phi(x) \in K(x)$ for any $x \in \partial E$. If $\partial\mathcal{W}_\phi$ were smooth and strictly convex, the vector field n_ϕ would be uniquely determined (since each cone reduces to a vector at any point of ∂E) and is usually called the Cahn-Hoffman vector field. A similar class (the so-called ϕ-regular sets) has been introduced in [5,6]. Such a class may be, in principle, larger than $\mathcal{R}_\phi(\mathbb{R}^n)$, since in that case the selection n_ϕ is required only to be bounded with bounded divergence. Even if the local existence theorem is still missing, the class of ϕ-regular sets supports a uniqueness result, which is obtained as a by-product of an Allen-Cahn type approximation argument [5].

Several technical reasons in the present paper impose the requirement that the selection n_ϕ be Lipschitz: one of these is related to the weak definition of a suitable divergence operator (see (1)) on a Lipschitz manifold (Remark 4.2). We stress once again that the definition of Lipschitz ϕ-regularity is a regularity property in the relative sense of the geometry induced on $\mathbb{R}^n$ by the Finsler metric ϕ. For instance, it is easy to see that, in $n = 2$ space dimensions, if we choose $\phi(\xi) := |\xi_1| + |\xi_2|$, the Euclidean ball $B_1(0) := \{x \in \mathbb{R}^2 : |x| \leqq 1\}$ does not admit any vector field $\eta : \partial B_1(0) \to \mathbb{R}^2$ such that $(B_1(0), \eta)$ becomes Lipschitz ϕ-regular.

Fix now a Lipschitz ϕ-regular set $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$. We introduce a variational problem on vector fields defined on ∂E, whose solution gives the definition of ϕ-mean curvature of ∂E and is expected to describe the initial velocity field of the evolution problem having ∂E as initial datum. More precisely, we propose to study the following minimum problem:

$$\inf\left\{\int_{\partial E} (\mathrm{div}_{\phi,n_\phi,\tau}\, N)^2\, \phi^o(\nu^E)\, d\mathcal{H}^{n-1} : N \in H(\partial E; \mathbb{R}^n)\right\}. \tag{1}$$

The symbol $H(\partial E;\mathbb{R}^n)$ denotes the class of all vector fields $N \in L^2(\partial E;\mathbb{R}^n)$ such that $\mathrm{div}_{\phi,n_\phi,\tau} N \in L^2(\partial E)$ and which satisfy the constraint $N(x) \in K(x)$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. We refer to Section 3 for the precise definition of the tangential divergence operator appearing in (1), which must be intended in a weak sense and in the relative geometry induced on ∂E by ϕ. Any solution of problem (1) has the same divergence: if $N_{\min}$ denotes a solution of (1), we are therefore led in a natural way to define the ϕ-mean curvature κ_ϕ of ∂E as $\kappa_\phi := \mathrm{div}_{\phi,n_\phi,\tau}\, N_{\min}$. Our conjecture is that κ_ϕ represents the effective velocity of the initial set E (recall that there is uniqueness of the evolution). To substantiate this conjecture we consider the following arguments. It is well known (see for instance [10]) that the solution of a parabolic partial differential inclusion of the form $u_t \in Au$ (where A is a maximal monotone multivalued operator) selects, at each time, a particular element (the so-called canonical element) which minimizes $\|Au\|_{L^2}$, i.e., u actually solves $u_t = A_m u$, where $A_m x$ realizes the minimum in $\min\{\|y\|^2_{L^2} : y \in Ax\}$. This idea has been applied to crystalline evolutions of graphs by FUKUI & Y. GIGA in [16]. It is not difficult to see [24] that the analogue of this minimum property in our geometric framework is given by (1).

Another remarkable argument which leads us to consider problem (1) comes from the expression of the first variation of the functional P_ϕ. This computation shows that the intrinsic perimeter P_ϕ is reduced with maximal speed when the variation is performed along the field $-\kappa_\phi\, N_{\min}$.

If κ_ϕ gives really the initial velocity, its jump set should correspond to the “fractures” along which new facets appear in the subsequent evolution; moreover, the regions where κ_ϕ is continuous but not constant should represent the regions of ∂E where curving is expected. This is in accordance with the examples computed in [6].

The plan of the paper is the following. In Section 2 we give some notation, we recall the main properties of the duality mappings and we define what we mean by a facet of ∂E. In Section 3 we introduce the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of Lipschitz ϕ-regular sets (Definition 3.1). Relations between different definitions of ϕ-regular sets are briefly illustrated in Remark 3.2. Given $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$, one of the first technical difficulties is that we need to extend Lipschitz vector fields, originally defined on ∂E, in an open neighbourhood of ∂E, in a Lipschitz way. This is the content of Lemma 3.3. The properties of the level sets of the intrinsic oriented distance function from ∂E and the connection with n_ϕ are considered in Lemma 3.4. These preliminary results are used to define the tangential divergence of a vector field defined on ∂E with respect to ϕ (Definition 4.1), and to exploit some of its properties. The weak operator of ϕ-tangential divergence is denoted by $\mathrm{div}_{\phi,n_\phi,\tau}$. In principle, it may depend on the choice of the field n_ϕ, but it turns out that, for all vector fields of interest in this paper, this is not the case. It must be noted that, on flat portions of ∂E, this divergence coincides with the usual (weak) tangential divergence div_τ. In Proposition 4.6 and Corollary 4.7 we prove that, even if for the same set E there are infinitely many possible choices of the Cahn-Hoffman vector field n_ϕ so that (E, n_ϕ) becomes Lipschitz ϕ-regular, the definition of $\mathrm{div}_{\phi,n_\phi,\tau}$ as operator on ϕ-normal vector fields is intrinsic, i.e., does not depend on the choice of

n_ϕ. In Section 5 we compute the first variation of the functional P_ϕ, thus relating the minimum problem (1) with motion by crystalline mean curvature. The first variation of P_ϕ turns out also to be strictly related to the operator $\mathrm{div}_{\phi,n_\phi,\tau}$. In Section 6 we are concerned with a minimum problem of the form (1), with an additional term depending on a given function g. In the evolution problem g plays the rôle of the forcing term. In Lemma 6.1 we prove the existence of minimizers; it is easily seen that their ϕ-tangential divergence is uniquely determined. The corresponding Euler-Lagrange inequality is computed in (6.3). We denote by $d_{\min}$ the tangential divergence of a minimizer ($d_{\min}$ reduces to κ_ϕ in the case $g = 0$). The crucial result proved in Theorem 6.5 is a reformulation of the Euler-Lagrange inequality for $d_{\min}$, by means of an inequality on each of its level sets. A direct consequence of Theorem 6.5 is that $d_{\min}$ is a bounded function (Theorem 6.7). In particular, Lipschitz ϕ-regular sets have bounded ϕ-mean curvature.

In a forthcoming paper [7] we continue the analysis on the structure of Lipschitz ϕ-regular sets and, using the result $\kappa_\phi \in L^\infty(\partial E)$, we prove that κ_ϕ has bounded variation on suitable facets F. We also investigate further properties of the level sets of κ_ϕ on F, in connection with the geometry of the facet of $\mathcal{W}_\phi$ parallel to F.

2. Notation

In the following we denote by $\cdot$ the standard Euclidean scalar product in $\mathbb{R}^n$ and by $|\cdot|$ the Euclidean norm of $\mathbb{R}^n$, $n \geqq 2$. If $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B_\rho(x) := \{y \in \mathbb{R}^n : |y - x| < \rho\}$.

Given two vectors $a, b \in \mathbb{R}^n$ we let $a \otimes b$ be the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. If M is a $(n \times n)$ matrix, by Ma (or aM) we denote the vector of components $(Ma)_i = M_{ij} a_j$ (resp., $(aM)_i = M_{ji} a_j$), and $\mathrm{tr} M$ is the trace of M. Note that $a \otimes b\, c = ab \cdot c$ for any $c \in \mathbb{R}^n$. With the notation $A \Subset B$ we mean that the set A is compactly contained in B.

The symbol $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^n$, $k \in [0, n]$.

If $E \subset \mathbb{R}^n$, we denote by 1_E the characteristic function of E and by ∂E the topological boundary of E. By $\mathrm{Lip}(\partial E)$ (or $\mathrm{Lip}(\partial E; \mathbb{R}^n)$) we denote the class of all Lipschitz functions (resp., vector fields) defined on ∂E.

We say that E is Lipschitz if E is open and, for any $x \in \partial E$, there exists $\rho > 0$ such that $B_\rho(x) \cap \partial E$ is the graph of a Lipschitz function f and $B_\rho(x) \cap E$ is the subgraph of f (with respect to a suitable orthogonal coordinate system). If E is Lipschitz, for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$ we denote by $\nu^E(x)$ the outward unit Euclidean normal to ∂E at x, and by $T_x \partial E$ the tangent hyperplane to ∂E at x.

Finsler metrics on $\mathbb{R}^n$. We indicate by $\phi : \mathbb{R}^n \to [0, +\infty[$ a convex function satisfying the properties

$$\phi(\xi) \geqq \Lambda|\xi|, \qquad \phi(a\xi) = a\phi(\xi), \qquad \xi \in \mathbb{R}^n,\ a \geqq 0, \tag{2}$$

for a suitable constant $\Lambda \in\ ]0, +\infty[$. The function $\phi^o : \mathbb{R}^n \to [0, +\infty[$ is defined as

$$\phi^o(\xi^*) := \sup\left\{\xi^* \cdot \xi \ : \ \phi(\xi) \leqq 1\right\},$$

and is the dual of ϕ; ϕ and ϕ^o are sometimes called Finsler metrics on $\mathbb{R}^n$. We set

$$\mathcal{W}_\phi^o := \{\xi^* \in \mathbb{R}^n : \phi^o(\xi^*) \leqq 1\}, \qquad \mathcal{W}_\phi := \{\xi \in \mathbb{R}^n : \phi(\xi) \leqq 1\}.$$

Clearly $\mathcal{W}_\phi^o$ and $\mathcal{W}_\phi$ are compact convex sets whose interior parts contain the origin. By a facet of $\partial\mathcal{W}_\phi$ (or of $\partial\mathcal{W}_\phi^o$) we always mean a $(n-1)$-dimensional facet.

We say that ϕ is crystalline if $\mathcal{W}_\phi$ is a convex polytope. If ϕ is crystalline, then also $\mathcal{W}_\phi^o$ is a convex polytope. The set $\mathcal{W}_\phi^o$ is sometimes called the Frank diagram and $\mathcal{W}_\phi$ the Wulff shape.

Duality mappings. By T and T^o we denote the possibly multivalued mappings defined by

$$\begin{aligned} T(\xi) &:= \{\xi^* \in \mathbb{R}^n : \xi^* \cdot \xi = \phi(\xi)^2 = (\phi^o(\xi^*))^2\}, \qquad \xi \in \mathbb{R}^n, \\ T^o(\xi^*) &:= \{\xi \in \mathbb{R}^n : \xi \cdot \xi^* = (\phi^o(\xi^*))^2 = \phi(\xi)^2\}, \qquad \xi^* \in \mathbb{R}^n, \end{aligned} \tag{3}$$

which are called the duality mappings. One can check that

$$T(\xi) = \tfrac{1}{2} D^-(\phi(\xi))^2 = \phi(\xi) D^-\phi(\xi), \qquad T(a\xi) = aT(\xi), \qquad \xi \in \mathbb{R}^n, a \geqq 0,$$

and similarly for T^o and ϕ^o, where D^- denotes the subdifferential. Moreover T, T^o are maximal monotone operators, T (or T^o) takes $\partial\mathcal{W}_\phi$ (resp., $\partial\mathcal{W}_\phi^o$) onto $\partial\mathcal{W}_\phi^o$ (resp., onto $\partial\mathcal{W}_\phi$).

Notice that, if $\xi \in \partial\mathcal{W}_\phi$, $T(\xi)$ is the intersection of the closed outward normal cone to $\partial\mathcal{W}_\phi$ with $\partial\mathcal{W}_\phi^o$.

The ϕ-distance function. Given a nonempty set $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we set

$$\operatorname{dist}_\phi(x, E) := \inf_{y \in E} \phi(x - y), \qquad \operatorname{dist}_\phi(E, x) := \inf_{y \in E} \phi(y - x),$$

$$d_\phi^E(x) := \operatorname{dist}_\phi(x, E) - \operatorname{dist}_\phi(\mathbb{R}^n \setminus E, x).$$

The function d_ϕ^E is therefore the oriented intrinsic distance function negative inside E; note that, since in general ϕ is not symmetric, $-d_\phi^E$ does not necessarily coincide with $d_\phi^{\mathbb{R}^n \setminus E}$.

At each point x where d_ϕ^E is differentiable, we have $\nabla d_\phi^E(x) \in \partial\mathcal{W}_\phi^o$, hence

$$\phi^o(\nabla d_\phi^E) = 1 \qquad \text{at } x; \tag{4}$$

we set $\nu_\phi^E(x) := \nabla d_\phi^E(x) = \frac{\nu^E(x)}{\phi^o(\nu^E(x))}$. As a consequence of (3), at $\mathcal{H}^{n-1}$-almost every $x \in \partial E$ we have

$$\nu_\phi^E(x) \cdot p = 1 \qquad \forall\, p \in T^o(\nu_\phi^E(x)). \tag{5}$$

If E is Lipschitz we define

$$\begin{aligned} &\operatorname{Nor}_\phi(\partial E; \mathbb{R}^n) := \{N : \partial E \to \mathbb{R}^n : N(x) \in T^o(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E\}, \\ &\operatorname{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n) := \operatorname{Lip}(\partial E; \mathbb{R}^n) \cap \operatorname{Nor}_\phi(\partial E; \mathbb{R}^n). \end{aligned}$$

Note that if $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, then $\phi(N(x)) = 1$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. Moreover, if $N_1, N_2 \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, then $N_1 - N_2$ is tangent to ∂E, since $N_1 \cdot \nu_\phi^E = 1 = N_2 \cdot \nu_\phi^E$.

We let dP_ϕ be the measure supported on ∂E with density $\phi^o(\nu^E)$ with respect to $\mathcal{H}^{n-1}$, i.e.,

$$dP_\phi(B) := \int_B \phi^o(\nu^E)\, d\mathcal{H}^{n-1}, \quad B \text{ Borel set}, \ B \subseteq \partial E.$$

If E is Lipschitz and $\psi \in \mathrm{Lip}(\partial E)$ we denote by $\nabla_\tau \psi$ the Euclidean tangential gradient of ψ on ∂E, which is defined at $\mathcal{H}^{n-1}$-almost every point of ∂E. If $v \in \mathrm{Lip}(\partial E;\mathbb{R}^n)$ we denote by $\mathrm{div}_\tau v$ the Euclidean tangential divergence of v, which is defined (at $\mathcal{H}^{n-1}$-almost every point of ∂E).

In the following, whenever there is no risk of confusion, we often do not indicate the dependence on E of the unit normals ν^E and ν_ϕ^E, i.e., we set $\nu := \nu^E$, $\nu_\phi := \nu_\phi^E$.

3. Lipschitz ϕ-regular sets

Definition 3.1. Let $E \subseteq \mathbb{R}^n$ be a Lipschitz set with compact boundary. We say that E is *Lipschitz ϕ-regular* if there exists a vector field $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. We denote by $\mathcal{R}_\phi(\mathbb{R}^n)$ the class of all Lipschitz ϕ-regular sets.

With a little abuse of notation, we shall sometimes write $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$, and we shall say that (E, n_ϕ) is Lipschitz ϕ-regular.

Observe that if (E, n_ϕ) is Lipschitz ϕ-regular, then $\phi(n_\phi) = 1$ everywhere on ∂E. The canonical example of Lipschitz ϕ-regular set is given by $(\mathcal{W}_\phi, x)$.

Remark 3.2. In [6] is introduced the class of ϕ-regular sets, whose definition is different from Definition 3.1, since n_ϕ is required to belong to $L^\infty(\partial E;\mathbb{R}^n)$ and to admit an extension on an open set containing ∂E which is bounded and has bounded divergence. In view of Lemmas 3.4 and 4.5 below, it follows that $\mathcal{R}_\phi(\mathbb{R}^n)$ is contained in the class of ϕ-regular sets. One can also prove that, if ϕ is crystalline and E is a ϕ-regular polyhedron, then E is Lipschitz ϕ-regular.

The Lipschitz regularity requirement on n_ϕ in Definition 3.1 allows to define the ϕ-tangential divergence $\mathrm{div}_{\phi,n_\phi,\tau}$ on the elements of $\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$; see Definition 4.1. Assuming less regularity on n_ϕ should require a non-trivial modification of Definition 4.1, see Remark 4.2.

The following lemma is well known in the smooth case. We need its version in our nonsmooth case in order to prove Corollary 4.7 and to compute the first variation of P_ϕ (see Section 5).

Lemma 3.3. *Let $E \subseteq \mathbb{R}^n$ be a Lipschitz set with compact boundary. Let $\eta \in \mathrm{Lip}(\partial E;\mathbb{R}^n)$ be a vector field with the property that there is a constant $c > 0$ verifying*

$$|\eta(x) \cdot \nu^E(x)| \geqq c \qquad \text{for } \mathcal{H}^{n-1} \text{ -a.e. } x \in \partial E. \tag{6}$$

Then there exist $\varepsilon > 0$ and an open set $U(\partial E)$ containing ∂E such that the following properties hold.

(i) *The map* $F_\eta : \partial E \times\ ]-\varepsilon,\varepsilon[\ \to U(\partial E)$ *defined by* $F_\eta(x,t) := x + t\eta(x)$ *is bi-Lipschitz.*

(ii) *Set* $G_\eta := F_\eta^{-1}$, $G_\eta(\cdot) := (\pi_\eta(\cdot), t_\eta(\cdot)) \in \partial E \times\ ]-\varepsilon,\varepsilon[$ *on* $U(\partial E)$. *Let* $z \in U(\partial E)$. *If* η *is differentiable at* $\pi_\eta(z)$ *and there exists* $T_{\pi_\eta(z)}\partial E$, *then* G_η *is differentiable at* z.

Proof. Let $x_0 \in \partial E$. Up to a rotation of coordinates, a neighbourhood of x_0 in ∂E can be written as $(s, f(s))$, for $s \in \Omega \subseteq \mathbb{R}^{n-1}$ and $f : \Omega \to \mathbb{R}$ a Lipschitz function. Write $\eta(s, f(s)) = (\eta_1(s, f(s)), \eta_2(s, f(s)) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and set $\widehat{F}(s,t) := F_\eta(x,t) = (s + t\eta_1(s, f(s)), f(s) + t\eta_2(s, f(s)))$. We want to apply the Implicit Function Theorem in the Lipschitz case to the function $\widehat{F}$, see [14]. In order to fulfil all the assumptions, we need to prove that, if $\{(s_n, t_n)\}$ is a sequence of points in the domain of $\widehat{F}$ converging to (s,t) as $n \to \infty$ such that $\widehat{F}$ is differentiable at each (s_n, t_n) and there is the limit M of $\nabla\widehat{F}(s_n, t_n)$, then M is nonsingular. Observe that

$$\nabla\widehat{F}(s_n, t_n) = \begin{pmatrix} \mathrm{Id} + O(t_n) & \eta_1(s_n, f(s_n)) \\ \nabla f(s_n) + O(t_n) & \eta_2(s_n, f(s_n)) \end{pmatrix}.$$

Using (6) we can check that $\left|\det\left(\nabla\widehat{F}(s_n, t_n)\right)\right| \geqq c/2$, for any $n \in \mathbb{N}$ and $\varepsilon > 0$ small enough, and therefore M is nonsingular. By [14] it follows that $\widehat{F}$ is locally invertible with a Lipschitz inverse $\widehat{G}$. Let us verify that F_η (hence $\widehat{F}$) is globally injective for $\varepsilon > 0$ small enough. Assume by contradiction that $F_\eta(x,t) = x + t\eta(x) = y + r\eta(y) = F_\eta(y,r)$ for some $x, y \in \partial E$ and $t, r \in\ ]-\varepsilon,\varepsilon[$, $(x,t) \neq (y,r)$. Then $|x - y| \leqq 2\varepsilon\|\eta\|_{L^\infty(\partial E;\mathbb{R}^n)}$, which contradicts the local invertibility of F_η. Using the compactness of ∂E, property (i) follows.

Let us prove (ii). Notice that if η is differentiable at $\pi_\eta(z)$ and there exists the tangent hyperplane to ∂E at $\pi_\eta(z)$ (i.e., f is differentiable at the point of Ω corresponding to $\pi_\eta(z)$), then $\widehat{F}$ is differentiable at $\widehat{G}(z)$. Differentiating the identity $\widehat{F}(\widehat{G}) = \mathrm{Id}$ we get that $\widehat{G}$ is differentiable at the point corresponding to z, which implies that G_η is differentiable at z.

Lemma 3.4. *Let* $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ *and let* $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. *Let* $\varepsilon > 0$ *be given by Lemma 3.3. Then for any* $t \in\ ]-\varepsilon,\varepsilon[$,

$$d_\phi^E(x + t\eta(x)) = t, \qquad x \in \partial E. \tag{7}$$

In particular,

$$\{z \in \mathbb{R}^n : d_\phi^E(z) = t\} = \{x + t\eta(x) : x \in \partial E\}, \qquad t \in\ ]-\varepsilon,\varepsilon[. \tag{8}$$

Moreover,

$$\nabla d_\phi^E(x + t\eta(x)) = \nu_\phi(x), \qquad \nabla d_\phi^E(x + t\eta(x)) \cdot \eta(x) = 1, \tag{9}$$

for any $t \in\ ]-\varepsilon,\varepsilon[$ *and for* $\mathcal{H}^{n-1}$*-almost every* $x \in \partial E$. *Finally, if* π_η *is as in (ii) of Lemma 3.3, and if we define* $\eta^e(z) := \eta(\pi_\eta(z))$ *for* $z \in U(\partial E)$, *then* $(\{z \in U(\partial E) : d_\phi^E(z) = t\}, \eta^e)$ *is Lipschitz* ϕ*-regular, for any* $t \in\ ]-\varepsilon,\varepsilon[$.

Proof. Let us consider the case $t > 0$, since the case $t < 0$ is similar. Since $d_\phi^E(x+t\eta(x)) \leqq \phi(x+t\eta(x)-x) = t$, in order to prove (7) we need to show that $d_\phi^E(x+t\eta(x)) \geqq t$. Set for simplicity $\pi := \pi_\eta$, and let $U(\partial E)$ be as in Lemma 3.3. Fix $z \in U(\partial E) \cap (\mathbb{R}^n \setminus E)$ such that there exists $T_{\pi(z)}\partial E$ and η is differentiable at $\pi(z)$ (hence the function F_η is differentiable at $(\pi(z), 0)$). By (ii) of Lemma 3.3, it follows that $t(\cdot) := t_\eta(\cdot)$ is differentiable at z. We need two intermediate steps.

Step 1. Let us prove that

$$\nabla t(z) \perp T_{\pi(z)}\partial E.$$

Fix $\tau \in T_{\pi(z)}\partial E$ and set $z_\varepsilon := z + \varepsilon\tau$, for $\varepsilon \in \mathbb{R}$ small enough. From the relation $\pi(z) + t(z)\eta(\pi(z)) = z$ we get

$$\begin{aligned}\big(t(z_\varepsilon) - t(z)\big)\eta(\pi(z)) &= t(z_\varepsilon)\eta(\pi(z_\varepsilon)) - t(z)\eta(\pi(z)) \\ &\quad + t(z_\varepsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\varepsilon))\big] \\ &= z_\varepsilon - z + \pi(z) - \pi(z_\varepsilon) + t(z_\varepsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\varepsilon))\big].\end{aligned} \tag{10}$$

Taking the scalar product of both sides of (10) with $\nu_\phi(\pi(z))$, using (5) and recalling that $z_\varepsilon - z$ is a tangent vector to ∂E at $\pi(z)$, we obtain

$$\begin{aligned}t(z_\varepsilon) - t(z) = &\ (\pi(z) - \pi(z_\varepsilon)) \cdot \nu_\phi(\pi(z)) \\ &+ t(z_\varepsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\varepsilon))\big] \cdot \nu_\phi(\pi(z)).\end{aligned} \tag{11}$$

The existence of $T_{\pi(z)}\partial E$ and the fact that $z_\varepsilon - z$ is a tangent vector imply

$$\big(\pi(z) - \pi(z_\varepsilon)\big) \cdot \nu_\phi(\pi(z)) = o(\varepsilon), \tag{12}$$

so that (11) becomes

$$t(z_\varepsilon) - t(z) = t(z_\varepsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\varepsilon))\big] \cdot \nu_\phi(\pi(z)) + o(\varepsilon). \tag{13}$$

Let us now prove that

$$\big[\eta(\pi(z)) - \eta(\pi(z_\varepsilon))\big] \cdot \nu_\phi(\pi(z)) = o(\varepsilon). \tag{14}$$

If $\eta = (\eta^1, \ldots, \eta^n)$, $\nu_\phi = (\nu_{\phi_1}, \ldots, \nu_{\phi_n})$ and $\pi = (\pi_1, \ldots, \pi_n)$, summing over repeated indices and using (12) we get

$$\big[\eta(\pi(z)) - \eta(\pi(z_\varepsilon))\big] \cdot \nu_\phi(\pi(z)) = \nabla_j \eta^i(\pi(z))\nu_{\phi_i}(\pi(z))(\pi_j(z) - \pi_j(z_\varepsilon)) + o(\varepsilon). \tag{15}$$

Recalling that $\phi(\eta(\pi(z)))^2 = 1$, we can check that

$$0 = D^-(\phi(\eta(\pi(z)))^2) = \nu_\phi(\pi(z))\nabla\eta(\pi(z)) \tag{16}$$

and therefore (14) follows from (15).

We conclude, using (13) and (14) that $t(z_\varepsilon) - t(z) = o(\varepsilon)$. Recalling that $\varepsilon\nabla t(z) \cdot \tau = t(z_\varepsilon) - t(z) + o(\varepsilon)$, we get $\varepsilon\nabla t(z) \cdot \tau = o(\varepsilon)$, which proves Step 1.

Step 2. Let us prove that

$$\nabla t(z) = \nu_\phi(\pi(z)).$$

From Step 1 it follows that $\nabla t(z) = \lambda \nu_\phi(\pi(z))$ for some $\lambda \in \mathbb{R}$. Precisely, $\lambda = \nabla t(z) \cdot \eta(\pi(z))$. Differentiating with respect to r the relation $t(\pi(z) + r\eta(\pi(z))) = r$, we get $\lambda = 1$ and Step 2 is proved.

By Step 2 we deduce $\phi^o(\nabla t(z)) = 1$, which implies

$$t(z_2) - t(z_1) \leqq \phi(z_2 - z_1) \qquad \text{for any } z_1, z_2 \in U(\partial E) \cap (\mathbb{R}^n \setminus E). \tag{17}$$

Taking $z_2 := z$ and $z_1 \in \partial E$ such that $d_\phi^E(z) = \phi(z - z_1)$, inequality (17) becomes

$$d_\phi^E(z) \geqq t(z) - t(z_1) = t(z),$$

and (7) follows.

Equality (8) is a direct consequence of (7). Finally, the set $\{z \in U(\partial E) : d_\phi^E(z) = t\}$ is the image of ∂E through a bi-Lipschitz map, η^e is a Lipschitz vector field on $U(\partial E)$ and $\eta^e(z) = \eta(\pi(z)) \in T^o(\nu_\phi(\pi(z))) = T^o(\nu_\phi(z))$.

We conclude this section with a result which will be useful in the computation of the first variation of the functional P_ϕ. Let $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$, let $U(\partial E)$ and π_η be as in Lemma 3.3. Let $\psi \in \mathrm{Lip}(\partial E)$ and define $\psi^e \in \mathrm{Lip}(U(\partial E))$, $\eta^e \in \mathrm{Lip}(U(\partial E); \mathbb{R}^n)$ as $\psi^e(z) := \psi(\pi_\eta(z))$, $\eta^e(z) := \eta(\pi_\eta(z))$. For $t \in \mathbb{R}$ with $|t| < \varepsilon$, $\varepsilon > 0$ small enough, define

$$\widetilde{F} \in \mathrm{Lip}(U(\partial E) \times \,]-\varepsilon, \varepsilon[; \mathbb{R}^n), \qquad \widetilde{F}(z,t) := z + t\psi^e(z)\eta^e(z), \tag{18}$$

and set $\widetilde{F}^t(\cdot) := \widetilde{F}(\cdot, t)$. Finally, let $E_t := \widetilde{F}^t(E)$.

Lemma 3.5. *There exists $\varepsilon > 0$ such that for $|t| < \varepsilon$ the following properties hold.*

(P1) *The set E_t is Lipschitz continuous, and $\partial E_t = \{z : z = x + t\psi(x)\eta(x), x \in \partial E\}$.*

(P2) *For $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists $\nabla d_\phi^{E_t}(x + t\psi(x)\eta(x))$. We define*

$$\nu_t(x) := \frac{\nabla d_\phi^{E_t}(x + t\psi(x)\eta(x))}{|\nabla d_\phi^{E_t}(x + t\psi(x)\eta(x))|}. \tag{19}$$

(P3) *For $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists $\frac{d}{dt}\nu_t(x)_{|t=0}$, and*

$$\frac{d}{dt}\nu_{t\,|t=0} = -\nu^E \nabla(\psi^e\eta^e) + (\nu^E \cdot \nu^E \nabla(\psi^e\eta^e))\nu^E \qquad \mathcal{H}^{n-1} - \text{a.e. on } \partial E. \tag{20}$$

(P4) *For $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists the right derivative $\frac{d^+}{dt}\phi^o(\nu_t(x))$ at $t = 0$, and there holds*

$$\frac{d^+}{dt}\phi^o(\nu_t(x))_{|t=0} = \max_{p \in T^o(\nu_\phi(x))} p \cdot \frac{d}{dt}\nu_t(x)_{|t=0}. \tag{21}$$

(P5) $\nu_\phi(x)\nabla\eta^e(x) = 0$ *for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$.*

Proof. The field $\psi\eta$ does not verify property (6), since ψ may vanish somewhere on ∂E. However, write $t\psi\eta = -n_\phi + (n_\phi + t\psi\eta)$, where $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$; for $|t|$ small enough, both $-n_\phi$ and $n_\phi + t\psi\eta$ satisfy property (6), and therefore ∂E_t is a bi-Lipschitz image of ∂E from Lemma 3.3. In addition, it is clear that $\partial E_t := \{x + t\psi(x)\eta(x) : x \in \partial E\}$.

Notice that $d_\phi^{E_t} = d_\phi^E - t$, and therefore, for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists $\nabla d_\phi^{E_t}(x) = \nabla d_\phi^E(x) = \nu_\phi^E(x)$, and $\nu_\phi^E(x) = \nabla d_\phi^E(x + t\psi(x)\eta(x))$ by (9). This proves (P2).

Let us prove (P3). The equality $d_\phi^E(z) = d_\phi^{E_t}(\widetilde{F}^t(z))$ for $z \in U(\partial E)$ yields, by differentiation,

$$\nabla d_\phi^{E_t}(\widetilde{F}^t(z)) = \nabla d_\phi^E(z)\Big(\mathrm{Id} + t\nabla(\psi^e\eta^e)(z) + o(t)\Big)^{-1}$$

for almost every $z \in U(\partial E)$ (precisely for any $z \in U(\partial E)$ such that $\nu^E(\pi_\eta(z))$ exists and such that η is differentiable at $\pi_\eta(z)$). Therefore

$$\nabla d_\phi^{E_t}(\widetilde{F}^t(z)) = \nabla d_\phi^E(z) - t\nabla d_\phi^E(z)\nabla(\psi^e\eta^e)(z) + o(t).$$

Then (P3) follows by a direct computation and using the definition of ν_t.

Property (P4) follows using (P3) and the properties of subdifferential of convex functions, and (P5) follows by differentiating the equality $\nu_\phi \cdot \eta^e = 1$; see (16).

4. The ϕ-tangential divergence

The definition of weak ϕ-tangential divergence on a Lipschitz ϕ-regular set E with respect to a nonsmooth Finsler metric ϕ is quite involved. To justify our definition we start from the smooth case, i.e., for strictly convex smooth ϕ^2 and ϕ^{o2} and smooth sets E. We recall from [8] and [3] that, in the smooth case,

$$\int_{\partial E} \mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]\, dP_\phi = \int_{\partial E} v \cdot \nu_\phi \,\mathrm{div}\, n_\phi \, dP_\phi \tag{22}$$

for any $v \in C^1(U(\partial E); \mathbb{R}^n)$, where $U(\partial E)$ is a suitable open neighbourhood of ∂E and $n_\phi := T^o(\nu_\phi^E)$ on $U(\partial E)$. We can check that $\mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]$ depends only on the values of v on ∂E. If $\psi \in C^1(U(\partial E); \mathbb{R}^n)$, then $\mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla(\psi v)\big] = \psi\,\mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big] + \big[(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla\psi\big] \cdot v$ (notice the switch of the rôle of ν_ϕ and n_ϕ). Therefore, from (22) we get

$$\int_{\partial E} \psi \;\mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]\, dP_\phi \tag{23}$$

$$= \int_{\partial E} \psi\; v \cdot \nu_\phi \,\mathrm{div}\, n_\phi \, dP_\phi - \int_{\partial E} \big[(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla\psi\big] \cdot v \, dP_\phi. \tag{24}$$

Formula (23) will be the starting point for our definition of ϕ-tangential divergence in the nonsmooth case. Let us introduce the ϕ-tangential divergence for vector fields $v \in L^2(\partial E; \mathbb{R}^n)$ as a bounded linear operator on $\mathrm{Lip}(\partial E)$.

Definition 4.1. Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $v \in L^2(\partial E; \mathbb{R}^n)$. We define the function $\mathrm{div}_{\phi,n_\phi,\tau}\, v : \mathrm{Lip}(\partial E) \to \mathbb{R}$ as follows: for any $\psi \in \mathrm{Lip}(\partial E)$ we set

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}\ v, \psi\rangle := \int_{\partial E} \psi\ v \cdot \nu_\phi\ \mathrm{div}_\tau n_\phi\ dP_\phi - \int_{\partial E} \big[(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla_\tau \psi\big] \cdot v\ dP_\phi. \tag{25}$$

Remark 4.2. Definition 4.1 cannot be easily adapted when (E, n_ϕ) is a ϕ-regular set in the sense of [6] (see Remark 3.2). For instance, if we assume that n_ϕ is only bounded, the definition of $\mathrm{div}_\tau n_\phi$ seems to require an integration by parts formula involving the Euclidean mean curvature of ∂E, which cannot be in principle computed, since ∂E is only Lipschitz continuous.

The following observations are immediate.

(i) If ψ is extended out of ∂E by a differentiable function ψ^e, then $(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla_\tau \psi = (\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla \psi^e$ on ∂E, i.e., any Euclidean normal component of the gradient of ψ is killed in formula (25) by the operator $\mathrm{Id} - \nu_\phi \otimes n_\phi$. This is a consequence of (5).

(ii) The operator $\mathrm{div}_{\phi,n_\phi,\tau} v$ is a linear continuous map on $\mathrm{Lip}(\partial E)$.

(iii) In the smooth case, i.e., for strictly convex smooth ϕ^2 and $(\phi^o)^2$ and smooth sets E, $\mathrm{div}_{\phi,n_\phi,\tau} v = \mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]$, for any $v \in C^1(U(\partial E); \mathbb{R}^n)$.

The operator $\mathrm{div}_{\phi,n_\phi,\tau}$ depends on ϕ and could depend, in general, on n_ϕ.

Definition 4.3. Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $v \in L^2(\partial E; \mathbb{R}^n)$. We say that $\mathrm{div}_{\phi,n_\phi,\tau} v$ is independent of the choice of n_ϕ if, given $n_\phi^* \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$, we have

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} v, \psi\rangle = \langle \mathrm{div}_{\phi,n^*_\phi,\tau} v, \psi\rangle \qquad \forall\, \psi \in \mathrm{Lip}(\partial E).$$

When $\mathrm{div}_{\phi,n_\phi,\tau} v$ is independent of the choice of n_ϕ, we simply set $\mathrm{div}_{\phi,\tau} v := \mathrm{div}_{\phi,n_\phi,\tau} v$.

If $X \in L^2(\partial E; \mathbb{R}^n)$ is tangent to ∂E and $N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$, it turns out that $\mathrm{div}_{\phi,n_\phi,\tau} X$ and $\mathrm{div}_{\phi,n_\phi,\tau} N$ are independent of the choice of n_ϕ, see (A2) of Lemma 4.4 and Corollary 4.7.

Lemma 4.4. *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$. The following assertions hold.*

(A1) *For any $\psi \in \mathrm{Lip}(\partial E)$ we have $\langle \mathrm{div}_{\phi,n_\phi,\tau} n_\phi, \psi\rangle = \int_{\partial E} \psi\ \mathrm{div}_\tau n_\phi\ dP_\phi$.*

(A2) *Let $X \in L^2(\partial E; \mathbb{R}^n)$ be tangent to ∂E. Then*

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} X, \psi\rangle = -\int_{\partial E} \nabla_\tau \psi \cdot X\ dP_\phi \qquad \forall\, \psi \in \mathrm{Lip}(\partial E).$$

In particular, $\mathrm{div}_{\phi,n_\phi,\tau} X$ *is independent of the choice of* n_ϕ.

(A3) *Let $X \in L^2(\partial E; \mathbb{R}^n)$ be tangent to ∂E. Then $\langle \mathrm{div}_{\phi,\tau} X, 1\rangle = 0$.*

(A4) *Let $N_1, N_2 \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$. Then $\langle \mathrm{div}_{\phi,n_\phi,\tau}(N_1 - N_2), 1\rangle = 0$.*

Proof. If v is such that $v \cdot n_\phi = 1$, (25) becomes

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}\ v, \psi\rangle = \int_{\partial E} \psi\ \mathrm{div}_\tau n_\phi\ dP_\phi - \int_{\partial E} \nabla_\tau \psi \cdot (v - n_\phi)\ dP_\phi.$$

Letting $v = n_\phi$, assertion (A1) follows. Assertion (A2) is immediate, and (A3) follows from (A2). Assertion (A4) follows from (A3), since $N_1 - N_2 \in L^2(\partial E; \mathbb{R}^n)$ is tangent to ∂E.

A refinement of assertion (A1) of Lemma 4.4 is given in Proposition 4.6 below.

The following lemma, which follows from (P5) of Lemma 3.5, shows that for any $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$ the Euclidean tangential divergence of η coincides with the divergence of the extension η^e of η.

Lemma 4.5. *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$. Let $U(\partial E)$ and π_η be as in Lemma 3.3, and set $\eta^e(z) := \eta(\pi_\eta(z))$ for any $z \in U(\partial E)$. Then, for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$,*

$$\mathrm{div}_\tau \eta(x) = \mathrm{div}\ \eta^e(x) = \mathrm{tr}\left(\left(\mathrm{Id} - n_\phi(x) \otimes \nu_\phi(x)\right)\nabla \eta^e(x)\right). \tag{26}$$

The following proposition shows that the operator $\mathrm{div}_{\phi,n_\phi,\tau}$ coincides with the operator div_τ on vector fields of $\mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$.

Proposition 4.6. *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$. Then*

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}\eta, \psi\rangle = \int_{\partial E} \psi\ \mathrm{div}_\tau\ \eta\ dP_\phi \qquad \forall\, \psi \in \mathrm{Lip}(\partial E). \tag{27}$$

In particular, $\mathrm{div}_{\phi,n_\phi,\tau}\eta$ *is independent of the choice of n_ϕ.*

Proof. Fix $\psi \in \mathrm{Lip}(\partial E)$, and write

$$\int_{\partial E} \psi\ \mathrm{div}_\tau\ \eta\ dP_\phi = \int_{\partial E} \psi\ \mathrm{div}_\tau\ n_\phi\ dP_\phi + \int_{\partial E} \psi\ \mathrm{div}_\tau\ (\eta - n_\phi)\ dP_\phi. \tag{28}$$

Set $\zeta := \eta - n_\phi$. Let $U(\partial E)$ and $\pi := \pi_\eta$ be as in Lemma 3.4, and set $\psi^e(z) := \psi(\pi(z))$, $\eta^e(z) := \eta(\pi(z))$, $\zeta^e(z) := \zeta(\pi(z))$ for $z \in U(\partial E)$. Let $\varepsilon > 0$ be small enough. Recalling (4), the coarea formula for Lipschitz maps [15], the one-homogeneity of ϕ^o and (9), we have

$$\begin{aligned}\int_{\{-\varepsilon<d_\phi^E<\varepsilon\}} \psi^e \mathrm{div}\zeta^e\ dz &= \int_{\{-\varepsilon<d_\phi^E<\varepsilon\}} \psi^e \mathrm{div}\zeta^e\ \phi^o(\nabla d_\phi^E)\ dz \\ &= \int_{-\varepsilon}^{\varepsilon}\int_{\{d_\phi^E=t\}} \psi^e \mathrm{div}\zeta^e\ \phi^o\left(\frac{\nabla d_\phi^E}{|\nabla d_\phi^E|}\right)\ d\mathcal{H}^{n-1}(z)\ dt \\ &= \int_{-\varepsilon}^{\varepsilon}\int_{\{d_\phi^E=t\}} \psi^e \mathrm{div}\zeta^e\ dP_\phi\ dt.\end{aligned} \tag{29}$$

Recalling (8) we have

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}\int_{\{d_\phi^E=t\}} \psi^e \mathrm{div}\zeta^e \, dP_\phi \, dt &= \int_{-\varepsilon}^{\varepsilon}\int_{\{x+t\eta(x):x\in\partial E\}} \psi^e \mathrm{div}\zeta^e \, dP_\phi \, dt \\ &= \int_{-\varepsilon}^{\varepsilon}\int_{\partial E} \psi^e(x+t\eta(x))\mathrm{div}\zeta^e(x+t\eta(x)) \\ &\quad \cdot \phi^o(\nu(x+t\eta(x)))\, d\mathcal{H}^{n-1}(x+t\eta(x))\, dt.\end{aligned} \tag{30}$$

From (9) we have $\phi^o(\nu(x+t\eta(x))) = \phi^o(\nu(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. Moreover $\psi^e(x+t\eta(x)) = \psi(x)$ by definition, and $d\mathcal{H}^{n-1}(x+t\eta(x)) = d\mathcal{H}^{n-1}(x) + O(t)$ by the area formula [15]. Finally

$$\mathrm{div}\zeta^e(x+t\eta(x)) = \mathrm{div}_\tau\zeta(x) + O(t). \tag{31}$$

Indeed, letting $z := x + t\eta(x)$, since $\zeta^e(z) = \zeta^e(z - d_\phi^E(z)\eta^e(z))$, it follows that

$$\begin{aligned}\nabla\zeta^e(z) &= \nabla\zeta^e(z - d_\phi^E(z)\eta^e(z))(\mathrm{Id} - \nabla d_\phi^E(z)\otimes\eta^e(z) - d_\phi^E(z)\nabla\eta^e(z)) \\ &= \nabla\zeta^e(x)(\mathrm{Id} - \nabla d_\phi^E(x)\otimes\eta(x) - d_\phi^E(z)\nabla\eta^e(z)),\end{aligned}$$

where the last equality follows from (9). Then (31) is a consequence of (26).

From (29), (30) and the above considerations, we then get

$$\int_{\partial E} \psi \, \mathrm{div}_\tau\zeta \, dP_\phi = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\{-\varepsilon<d_\phi^E<\varepsilon\}} \psi^e \mathrm{div}\zeta^e \, dz. \tag{32}$$

Applying the Gauss-Green Theorem for fixed positive ε, we obtain

$$\begin{aligned}\int_{\{-\varepsilon<d_\phi^E<\varepsilon\}} \psi^e \mathrm{div}\zeta^e \, dz = &-\int_{\{-\varepsilon<d_\phi^E<\varepsilon\}} \zeta^e\cdot\nabla\psi^e \, dz \\ &+ \int_{\partial\{-\varepsilon<d_\phi^E<\varepsilon\}} \psi^e\zeta^e\cdot\nu_\varepsilon \, d\mathcal{H}^{n-1},\end{aligned}$$

where ν_ε is the Euclidean outward unit normal to $\partial\{-\varepsilon < d_\phi^E < \varepsilon\}$.

Observe now that for $\mathcal{H}^{n-1}$-almost every $z \in \partial\{-\varepsilon < d_\phi^E < \varepsilon\}$, if $z = x + t\eta(x)$ for $x \in \partial E$, by (9) we have $\zeta^e(z)\cdot\nu_\varepsilon(z) = \eta(x)\cdot\nu^E(x) - n_\phi(x)\cdot\nu^E(x) = 0$. Hence from (32),

$$\int_{\partial E} \psi \, \mathrm{div}_\tau\zeta \, dP_\phi = -\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\{-\varepsilon<d_\phi^E<\varepsilon\}} \nabla\psi^e\cdot\zeta^e \, dz = -\int_{\partial E}\nabla_\tau\psi\cdot\zeta \, dP_\phi, \tag{33}$$

which gives

$$\int_{\partial E} \psi \, \mathrm{div}_\tau \, \eta \, dP_\phi = \int_{\partial E} \psi \, \mathrm{div}_\tau \, n_\phi \, dP_\phi - \int_{\partial E} \nabla_\tau\psi\cdot(\eta - n_\phi)\, dP_\phi,$$

which is (27).

We can now prove that the operator $\mathrm{div}_{\phi,n_\phi,\tau}$ is independent of the choice of n_ϕ on the whole of $\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$.

Corollary 4.7. *Let* $(E,n_\phi)\in\mathcal{R}_\phi(\mathbb{R}^n)$. *If* $N\in\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, *it follows that* $\mathrm{div}_{\phi,n_\phi,\tau}N$ *does not depend on the choice of* n_ϕ.

Proof. By (5) we have $N\cdot\nu_\phi=1$, hence for $\psi\in\mathrm{Lip}(\partial E)$ formula (25) reduces to

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}\, N,\psi\rangle=\int_{\partial E}\psi\ \mathrm{div}_\tau\, n_\phi\, dP_\phi-\int_{\partial E}\nabla_\tau\psi\cdot(N-n_\phi)\,dP_\phi. \tag{34}$$

Let now $n_\phi^*\in\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$ and set $\zeta:=n_\phi-n_\phi^*$. From (34) we deduce

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}N-\mathrm{div}_{\phi,n^*_\phi,\tau}\, N,\psi\rangle=\int_{\partial E}\psi\ \mathrm{div}_\tau\,\zeta\, dP_\phi+\int_{\partial E}\nabla_\tau\psi\cdot\zeta\,dP_\phi. \tag{35}$$

Since $n_\phi,n_\phi^*\in\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$ we have that $\zeta\in\mathrm{Lip}(\partial E;\mathbb{R}^n)$ is tangent to ∂E, hence from (A2) of Lemma 4.4 we get

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}N-\mathrm{div}_{\phi,n^*_\phi,\tau}\, N,\psi\rangle=\int_{\partial E}\psi\ \mathrm{div}_\tau\,\zeta\, dP_\phi-\langle\mathrm{div}_{\phi,n_\phi,\tau}\zeta,\psi\rangle.$$

Recalling the definition of ζ and applying Proposition 4.6, it follows that the right-hand side of the above equality vanishes, and the assertion of the corollary follows.

5. The first variation of the functional P_ϕ

In this section we compute the first variation of the functional P_ϕ. This computation is quite delicate, because neither the integrand $\phi^o(\nu)$ nor the manifold ∂E are smooth. A partial result in this direction can be found in [6].

Let $(E,n_\phi)\in\mathcal{R}_\phi(\mathbb{R}^n)$. The set $\left\{v\in L^2(\partial E;\mathbb{R}^n)\ :\ \mathrm{div}_{\phi,n_\phi,\tau}\, v\in L^2(\partial E)\right\}$ is nonempty and is a Hilbert space endowed with the norm

$$\left(\|v\|^2_{L^2(\partial E;\mathbb{R}^n)}+\|\mathrm{div}_{\phi,n_\phi,\tau}\, v\|^2_{L^2(\partial E)}\right)^{1/2}.$$

We define

$$H(\partial E;\mathbb{R}^n):=\left\{N\in\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n):\mathrm{div}_{\phi,\tau}N\in L^2(\partial E)\right\}.$$

Let $v\in\mathrm{Lip}(\partial E;\mathbb{R}^n)$. We first compute the first variation of the functional $E\to P_\phi(E)$ along the field v. As in the smooth case, P_ϕ does not change under (infinitesimal) tangential variations, and therefore it is enough to consider ϕ-normal fields, i.e., we can assume that v can be written as $v=\psi\eta$, where $\psi\in\mathrm{Lip}(\partial E)$ and $\eta\in\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$.

Let $U(\partial E)$ and π_η be as in Lemma 3.3, and define $\psi^e\in\mathrm{Lip}(U(\partial E))$, $\eta^e\in\mathrm{Lip}(U(\partial E);\mathbb{R}^n)$ as $\psi^e(z):=\psi(\pi_\eta(z))$, $\eta^e(z):=\eta(\pi_\eta(z))$, and set $v^e:=\psi^e\eta^e$. For $t\in\mathbb{R}$ with $|t|<\varepsilon$, $\varepsilon>0$ small enough, define $\widetilde{F}$ as in (18), and set $\widetilde{F}^t(\cdot):=\widetilde{F}(\cdot,t)$, $E_t:=\widetilde{F}^t(E)$. Define also

$$\mathrm{Var}(P_\phi,E)(\psi\eta):=\liminf_{t\to0^+}\frac{P_\phi(E_t)-P_\phi(E)}{t}.$$

Theorem 5.1. *The following equalities hold:*

$$\mathrm{Var}(P_\phi, E)(\psi\eta) = \lim_{t\to 0^+} \frac{P_\phi(E_t) - P_\phi(E)}{t} = \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\eta,\tau}\, N, \psi\rangle. \tag{36}$$

Proof. By (P1) of Lemma 3.5 E_t is Lipschitz; therefore, using the area formula we have

$$P_\phi(E_t) = \int_{\partial E} \phi^o(\nu_t)\, d\mathcal{H}^{n-1} + \int_{\partial E} \phi^o(\nu)\, \mathrm{div}_\tau v\, d\mathcal{H}^{n-1} + o(t),$$

where ν_t is defined in (19). Hence, by (21) and (20), $\mathrm{Var}(P_\phi, E)(\psi\eta)$ is equal to

$$\begin{aligned} &\int_{\partial E} \lim_{t\to 0^+} \frac{\phi^o(\nu_t) - \phi^o(\nu)}{t}\, d\mathcal{H}^{n-1} + \int_{\partial E} \phi^o(\nu)\, \mathrm{div}_\tau v\, d\mathcal{H}^{n-1} \qquad (37)\\ &\quad = \int_{\partial E} \Big\{ \max_{p\in T^o(\nu_\phi(x))} p\cdot \Big(-\nu_\phi(x)\nabla v^e(x) \\ &\qquad\qquad + \big[\nu(x)\cdot\nu(x)\nabla v^e(x)\big]\nu_\phi(x)\Big) + \mathrm{div}_\tau v \Big\}\, dP_\phi \\ &\quad = \int_{\partial E} \Big\{ \max_{p\in T^o(\nu_\phi(x))} -p\cdot\nu_\phi(x)\nabla v^e(x) + \nu(x)\cdot\nu(x)\nabla v^e(x) + \mathrm{div}_\tau v \Big\}\, dP_\phi, \end{aligned}$$

where the last equality follows from $p\cdot\nu_\phi(x) = 1$. It is not difficult to prove now that the map $x \to T^o(\nu_\phi(x))$, defined for $\mathcal{H}^{n-1}$-almost every $x\in\partial E$, is the smallest closed-valued $\mathcal{H}^{n-1}$-measurable multifunction with the property that, for any $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, $N(x) \in T^o(\nu_\phi(x))$ for $\mathcal{H}^{n-1}$-almost every $x\in\partial E$. Using this observation, a commutation argument between supremum and integral (see [9], Lemma 4.3) allows us to prove that the last member of (37) equals

$$\begin{aligned} &\sup_{N\in\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \int_{\partial E} \Big\{ -N\cdot\nu_\phi\nabla v^e + \nu\cdot\nu\nabla v^e + \mathrm{div}_\tau v \Big\}\, dP_\phi \\ &\qquad\qquad =: \sup_{N\in\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \int_{\partial E} I_N\, dP_\phi. \end{aligned}$$

Fix $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$: recalling the expression of the Euclidean tangential divergence, I_N can be written as $-N\cdot\nu_\phi\nabla v^e + \mathrm{div}\, v^e = -N\cdot\nu_\phi\nabla v^e + \psi^e \mathrm{div}\,\eta^e + \nabla\psi^e\cdot\eta^e$. Therefore, since Lemma 4.5 implies $\mathrm{div}\,\eta^e = \mathrm{div}_\tau\,\eta$ on ∂E, we obtain

$$\int_{\partial E} I_N\, dP_\phi = \int_{\partial E} \Big\{ -N\cdot\nu_\phi\nabla v^e + \psi\, \mathrm{div}_\tau\eta + \nabla\psi^e\cdot\eta \Big\}\, dP_\phi.$$

On the other hand, substituting $v^e = \psi^e\eta^e$ into $-N\cdot\nu_\phi\nabla v^e$ and using (P5) of Lemma 3.5, we get $-N\cdot\nu_\phi\nabla v^e = -\nabla\psi^e\cdot N$ on ∂E. Since $N-\eta$ is a tangent vector field, we have $\nabla\psi^e\cdot(N-\eta) = \nabla_\tau\psi\cdot(N-\eta)$ on ∂E. Therefore we conclude that

$$\int_{\partial E} I_N\, dP_\phi = \int_{\partial E} \Big\{ \psi\, \mathrm{div}_\tau\eta - \nabla_\tau\psi\cdot(N-\eta) \Big\}\, dP_\phi = \langle \mathrm{div}_{\phi,\eta,\tau} N, \psi\rangle,$$

and the theorem follows.

Recall that $\mathrm{div}_{\phi,\eta,\tau}\, N$ in (36) actually does not depend on η (see Corollary 4.7), so that $\mathrm{Var}(P_\phi, E)(\psi\eta)$ is independent of η. We define (with a little abuse of notation) the functional $\mathrm{Var}(P_\phi, E) : L^2(\partial E) \to]-\infty, +\infty]$ as

$$\mathrm{Var}(P_\phi, E)(\psi) := \begin{cases} \sup\limits_{N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle & \text{if } \psi \in \mathrm{Lip}(\partial E), \\ +\infty & \text{if } \psi \in L^2(\partial E) \setminus \mathrm{Lip}(\partial E). \end{cases}$$

Define also

$$\begin{aligned} B_\phi &:= \left\{ \psi \in \mathrm{Lip}(\partial E) : \int_{\partial E} \psi^2\, dP_\phi \leqq 1 \right\}, \\ B_\phi^n &:= \left\{ \psi\eta : \psi \in B_\phi,\ \eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n) \right\}. \end{aligned} \tag{38}$$

The next result gives, roughly speaking, the expression of minus the norm of the gradient of the functional P_ϕ.

Proposition 5.2. *We have*

$$\inf_{\psi\eta \in B_\phi^n} \mathrm{Var}(P_\phi, E)(\psi\eta) = - \min_{N \in H(\partial E;\mathbb{R}^n)} \left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi \right)^{\frac{1}{2}}.$$

Proof. Using Theorem 5.1 we get

$$\begin{aligned} \inf_{\psi\eta \in B_\phi^n} \mathrm{Var}(P_\phi, E)(\psi\eta) &= \inf_{\psi \in B_\phi} \mathrm{Var}(P_\phi, E)(\psi) \\ &= \inf_{\psi \in B_\phi} \sup_{N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau}\, N, \psi \rangle =: I. \end{aligned} \tag{39}$$

Notice that the map $(\psi, N) \in B_\phi \times \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n) \to \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle$ is bilinear. Moreover both B_ϕ and $\mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$ are convex sets, and B_ϕ (or $\mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$) is closed in the $W^{1,\infty}(\partial E)$ topology (resp., strongly closed and weakly compact in the $L^2(\partial E; \mathbb{R}^n)$ topology). Therefore, by a commutation result between sup and inf (see [10, Proposition 1.1]), we get

$$I = \sup_{N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \inf_{\psi \in B_\phi} \langle \mathrm{div}_{\phi,\tau}\, N, \psi \rangle.$$

Since $N \notin H(\partial E; \mathbb{R}^n)$ implies $\inf_{\psi \in B_\phi} \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle = -\infty$, we deduce

$$I = \sup_{N \in H(\partial E;\mathbb{R}^n)} \inf_{\psi \in B_\phi} \langle \mathrm{div}_{\phi,\tau}\, N, \psi \rangle.$$

Using the fact that $\inf_{\psi \in B_\phi} \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle = \inf_{\psi \in \overline{B_\phi}^{L^2}} \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle$, it follows that

$$\begin{aligned} I &= \sup_{N \in H(\partial E;\mathbb{R}^n)} - \frac{\int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi}{\left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi\right)^{\frac{1}{2}}} \\ &= - \inf_{N \in H(\partial E;\mathbb{R}^n)} \left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi \right)^{\frac{1}{2}}. \end{aligned}$$

Thanks to Proposition 6.1 below, the infimum in the above inequality is a minimum, and the proposition is proved.

Given $v \in L^2(\partial E;\mathbb{R}^n)$ such that $v = \psi\widetilde{N}$ for some $\psi \in L^2(\partial E)$ and $\widetilde{N} \in H(\partial E;\mathbb{R}^n)$, we define

$$V_\phi^E(v) := \sup_{N\in H(\partial E;\mathbb{R}^n)} \int_{\partial E} \psi \,\mathrm{div}_{\phi,\tau} N \, dP_\phi =: V_\phi^E(\psi),$$

where $V_\phi^E : L^2(\partial E) \to]-\infty,+\infty]$.

The following observation shows that V_ϕ^E is the lower semicontinuous envelope of $\mathrm{Var}(P_\phi, E)$, with respect to the $L^2(\partial E)$ topology.

Proposition 5.3. *The functional V_ϕ^E is the greatest $L^2(\partial E)$ lower semicontinuous functional less than or equal to* $\mathrm{Var}(P_\phi, E)$. *In particular,*

$$\inf_{\psi\in B_\phi} \mathrm{Var}(P_\phi, E)(\psi) = \inf_{\psi\in B_\phi} V_\phi^E(\psi).$$

Proof. Given $\psi \in L^2(\partial E)$ and $\varepsilon > 0$, we set

$$B_\varepsilon(\psi) := \left\{\tilde{\psi} \in \mathrm{Lip}(\partial E) : \ ||\tilde{\psi} - \psi||_{L^2(\partial E)} < \varepsilon\right\}.$$

The $L^2(\partial E)$ lower semicontinuous envelope of $\mathrm{Var}(P_\phi, E)$ is, by definition

$$\sup_{\varepsilon>0} \inf_{\tilde{\psi}\in B_\varepsilon(\psi)} \mathrm{Var}(P_\phi, E)(\tilde{\psi}).$$

By Theorem 5.1, this is equal to

$$\sup_{\varepsilon>0} \inf_{\tilde{\psi}\in B_\varepsilon(\psi)} \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau} N, \tilde{\psi}\rangle.$$

Commuting the sup and the inf [10], we then have

$$\begin{aligned}
&\sup_{\varepsilon>0} \inf_{\tilde{\psi}\in B_\varepsilon(\psi)} \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau} N, \tilde{\psi}\rangle \\
&\qquad = \sup_{\varepsilon>0} \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \inf_{\tilde{\psi}\in B_\varepsilon(\psi)} \langle \mathrm{div}_{\phi,\tau} N, \tilde{\psi}\rangle \\
&\qquad = \sup_{\varepsilon>0} \sup_{N\in H(\partial E;\mathbb{R}^n)} \inf_{\tilde{\psi}\in B_\varepsilon(\psi)} \int_{\partial E} \tilde{\psi}\,\mathrm{div}_{\phi,\tau} N\, dP_\phi \\
&\qquad = \sup_{N\in H(\partial E;\mathbb{R}^n)} \sup_{\varepsilon>0} \inf_{\tilde{\psi}\in B_\varepsilon(\psi)} \int_{\partial E} \tilde{\psi}\,\mathrm{div}_{\phi,\tau} N\, dP_\phi \\
&\qquad = \sup_{N\in H(\partial E;\mathbb{R}^n)} \int_{\partial E} \psi\,\mathrm{div}_{\phi,\tau} N\, dP_\phi = V_\phi^E(\psi),
\end{aligned}$$

which proves the assertion.

The following result, which follows from Proposition 5.3, shows that the direction of minimal slope for the functional P_ϕ is given by $N_{\min}$, and is one of the incentives for introducing and studying the functional in (1) in connection with motion by crystalline mean curvature.

Corollary 5.4. *The following equality holds:*

$$\inf_{\psi\eta \in B^n_\phi} \mathrm{Var}(P_\phi, E)(\psi\eta) = V^E_\phi(\overline{\psi} N_{\min}),$$

where $N_{\min}$ *is a minimizer of the functional* $N \to \int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2 \, dP_\phi$ *(see Proposition 6.1 below, with* $g = 0$*) and*

$$\overline{\psi} = -\frac{\mathrm{div}_{\phi,\tau} N_{\min}}{\left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N_{\min})^2 \, dP_\phi\right)^{\frac{1}{2}}}.$$

6. The minimum problem on ∂E: L^∞ regularity

Let $g \in L^2(\partial E)$ and let $\mathcal{F} : H(\partial E; \mathbb{R}^n) \to [0, +\infty[$ be the functional defined as

$$\mathcal{F}(N) := \int_{\partial E} \left(\mathrm{div}_{\phi,\tau}\, N - g\right)^2 dP_\phi. \tag{40}$$

We are interested in studying the following minimum problem on ∂E:

$$\inf \left\{\mathcal{F}(N) : N \in H(\partial E; \mathbb{R}^n)\right\}. \tag{41}$$

It is clear that problem (41) is equivalent, up to constants, to the problem

$$\inf \left\{ \int_{\partial E} (\mathrm{div}_{\phi,\tau}\, N)^2 - 2g \mathrm{div}_{\phi,\tau} N \, dP_\phi : N \in H(\partial E; \mathbb{R}^n) \right\}. \tag{42}$$

Proposition 6.1. *Problem* (41) *admits a solution. Moreover, if* N_1 *and* N_2 *are two minimizers of* (41)*, then* $\mathrm{div}_{\phi,\tau}\, N_1(x) = \mathrm{div}_{\phi,\tau}\, N_2(x)$ *for* $\mathcal{H}^{n-1}$*-almost every* $x \in \partial E$.

Proof. Define

$$C := \{\mathrm{div}_{\phi,\tau} N : N \in H(\partial E; \mathbb{R}^n)\}.$$

Then C is a convex subset of $L^2(\partial E)$. Let us prove that C is closed in $L^2(\partial E)$. Let $f_k := \mathrm{div}_{\phi,\tau} N_k \in C$ be such that $f_k \to f$ in $L^2(\partial E)$ as $k \to \infty$. We have to prove that $f \in C$. Since $\sup_k \|N_k\|_{L^2(\partial E;\mathbb{R}^n)} < +\infty$, possibly passing to a subsequence (still denoted by (N_k)) we can assume that (N_k) converges weakly in $L^2(\partial E; \mathbb{R}^n)$ to a vector field $N \in L^2(\partial E; \mathbb{R}^n)$. Since $N_k \in T^o(\nu^E_\phi)$, by Mazur's Theorem we find that $N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$. Moreover, for any $\psi \in \mathrm{Lip}(\partial E)$ we have, using (25),

$$\begin{aligned}
\int_{\partial E} \psi f \, dP_\phi &= \lim_{k\to+\infty} \int_{\partial E} \psi \, \mathrm{div}_{\phi,\tau}\, N_k \, dP_\phi \\
&= \int_{\partial E} \psi \, \mathrm{div}_\tau n_\phi \, dP_\phi - \lim_{k\to+\infty} \int_{\partial E} \nabla_\tau \psi \cdot (N_k - n_\phi) \, dP_\phi \\
&= \int_{\partial E} \psi \, \mathrm{div}_\tau n_\phi \, dP_\phi - \int_{\partial E} \nabla_\tau \psi \cdot (N - n_\phi) \, dP_\phi.
\end{aligned}$$

It follows that $f = \mathrm{div}_{\phi,\tau} N$, hence C is closed in $L^2(\partial E; \mathbb{R}^n)$. The proof of the proposition is then is a standard consequence of minimization on convex sets of strictly convex functionals on Hilbert spaces.

Let $N_{\min} \in H(\partial E; \mathbb{R}^n)$ be a minimizer of $\mathcal{F}$. A direct computation shows that the Euler-Lagrange inequality of $\mathcal{F}$ reads as follows:

$$\int_{\partial E} (\mathrm{div}_{\phi,\tau} N_{\min} - g)\, \mathrm{div}_{\phi,\tau}(N_{\min} - N)\, dP_\phi \leqq 0 \quad \forall N \in H(\partial E; \mathbb{R}^n). \tag{43}$$

We now give the definition of mean curvature of a Lipschitz ϕ-regular set with respect to the metric ϕ.

Definition 6.2. Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. Let $N_{\min}$ be a solution of (41). We set

$$d_{\min} := \mathrm{div}_{\phi,\tau} N_{\min} \in L^2(\partial E).$$

When $g = 0$, we define the *ϕ-mean curvature* κ_ϕ of ∂E as

$$\kappa_\phi := \mathrm{div}_{\phi,\tau} N_{\min} \in L^2(\partial E).$$

We shall see in Theorem 6.7 that Lipschitz ϕ-regular sets have actually bounded ϕ-mean curvature.

Remark 6.3. Since $\mathcal{F}$ is strictly convex if considered as a function of the divergence, if f is of the form $f = \mathrm{div}_{\phi,\tau} \overline{N}$ for some $\overline{N} \in H(\partial E; \mathbb{R}^n)$ and if

$$\int_{\partial E} (f - g) \mathrm{div}_{\phi,\tau}(\overline{N} - N)\, dP_\phi \leqq 0 \qquad \forall N \in H(\partial E; \mathbb{R}^n),$$

then $\overline{N}$ is a solution of (41).

The next step is to prove that $d_{\min} \in L^\infty(\partial E)$. To this end we begin with the following auxiliary lemma, whose standard proof is omitted.

Lemma 6.4. *Let $a, b \in \mathbb{R}$, $a < b$ and let $f, \beta \in L^2(\partial E)$. Let μ be a measure on ∂E absolutely continuous with respect to the restriction of $\mathcal{H}^{n-1}$ to ∂E. Assume that $\int_{\partial E} \beta\, d\mu = 0$. Then*

$$\begin{aligned} \int_{\partial E} f\beta\, d\mu &= \int_{-\infty}^{+\infty} \int_{\{f>t\}} \beta\, d\mu\, dt, \\ \int_{\{a<f<b\}} f\beta\, d\mu &= \int_a^b \int_{\{f>t\}} \beta\, d\mu\, dt + a \int_{\{f>a\}} \beta\, d\mu - b \int_{\{f>b\}} \beta\, d\mu. \end{aligned} \tag{44}$$

The crucial result of this section is the following.

Theorem 6.5. *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. For any $t \in \mathbb{R}$, define*

$$A_t := \{d_{\min} - g > t\}, \qquad \Omega_t := \{d_{\min} - g < t\}.$$

Then

$$\int_{A_t} d_{\min}\, dP_\phi \leqq \int_{A_t} \mathrm{div}_{\phi,\tau}\, N\, dP_\phi \qquad \forall t \in \mathbb{R},\ \forall N \in H(\partial E; \mathbb{R}^n), \tag{45}$$

and

$$\int_{\Omega_t} d_{\min}\, dP_\phi \geqq \int_{\Omega_t} \mathrm{div}_{\phi,\tau} N\, dP_\phi \qquad \forall t \in \mathbb{R},\ \forall N \in H(\partial E; \mathbb{R}^n). \tag{46}$$

Proof. We shall prove only (45), since the proof of (46) is similar. For simplicity of notation, set

$$V := d_{\min} - g.$$

Moreover, if $\chi \in H(\partial E; \mathbb{R}^n)$ and $B \subseteq \partial E$ is a Borel set, we let

$$D(B, \chi) := \int_B \mathrm{div}_{\phi,\tau}(\chi - N_{\min})\, dP_\phi.$$

Assume by contradiction that there exist $\lambda \in \mathbb{R}$, $N \in H(\partial E; \mathbb{R}^n)$, and $c > 0$ such that

$$D(A_\lambda, N) = -4c < 0.$$

Since $A_\lambda = \bigcup_{t>\lambda} A_t$, we have $1_{A_t} \to 1_{A_\lambda}$ in $L^1(\partial E)$ (hence, being characteristic functions, also in $L^2(\partial E)$) as $t \downarrow \lambda$. Therefore there exists $\varepsilon > 0$ such that

$$D(A_t, N) \leqq -2c \qquad \forall\, t \in [\lambda, \lambda + \varepsilon]. \tag{47}$$

Fix $\lambda' \in \mathbb{R}$ and $\varepsilon' > 0$ with the following properties:

$$[\lambda', \lambda' + \varepsilon'] \subseteq [\lambda, \lambda + \varepsilon], \tag{48}$$

$$\mathcal{H}^{n-1}\Big(\{V = \lambda'\} \cup \{V = \lambda' + \varepsilon'\}\Big) = 0. \tag{49}$$

Clearly from (47) and (48) we get

$$\int_{\lambda'}^{\lambda'+\varepsilon'} D(A_t, N)\, dt \leqq -2c\varepsilon'. \tag{50}$$

Let $\{f_i\}$ be a sequence of functions in $\mathrm{Lip}(\partial E)$ converging to V in $L^2(\partial E)$ and almost everywhere. For any $i \in \mathbb{N}$ and $t \in \mathbb{R}$ we define

$$A_t^i := \{f_i > t\}.$$

We split the proof into three intermediate steps.

Step 1. Let us prove that there exists $i_0 \in \mathbb{N}$ such that

$$\int_{\lambda'}^{\lambda'+\varepsilon'} D(A_t^i, N)\, dt \leqq -c\varepsilon' \qquad \forall\, i \geqq i_0. \tag{51}$$

We claim that

$$\lim_{i\to+\infty} \int_{\lambda'}^{\lambda'+\varepsilon'} D(A_t^i, N)\, dt = \int_{\lambda'}^{\lambda'+\varepsilon'} D(A_t, N)\, dt. \tag{52}$$

By (A4) of Lemma 4.4 we have $D(\partial E, N) = 0$, hence applying Lemma 6.4 with f, β, $d\mu$, a, b replaced by f_i, $\mathrm{div}_{\phi,\tau}(N - N_{\min})$, dP_ϕ, λ', $\lambda' + \varepsilon'$ in that order, from (44) we find

$$\int_{\lambda'}^{\lambda'+\varepsilon'} D(A_t^i, N)\, dt = \int_{\{\lambda' < f_i < \lambda'+\varepsilon'\}} f_i \mathrm{div}_{\phi,\tau}(N - N_{\min})\, dP_\phi - \lambda' D(A_{\lambda'}^i, N)$$
$$+ (\lambda' + \varepsilon') D(A_{\lambda'+\varepsilon'}^i, N).$$

In view of this equality and of the corresponding one with f_i and A_t^i replaced by V and A_t, to prove (52) it is enough to show that $1_{A^i_{\lambda'}}$ (or $1_{\{\lambda'<f_i<\lambda'+\varepsilon'\}}$, $1_{A^i_{\lambda'+\varepsilon'}}$) converges to $1_{A_{\lambda'}}$ (resp., $1_{\{\lambda'<V<\lambda'+\varepsilon'\}}$, $1_{A_{\lambda'+\varepsilon'}}$) in $L^1(\partial E)$ as $i \to +\infty$. We show this property for $A^i_{\lambda'}$, the other cases being similar. Define $A_i := (A^i_{\lambda'} \setminus A_{\lambda'}) \cup (A_{\lambda'} \setminus A^i_{\lambda'})$. It is enough to check that $\mathcal{H}^{n-1}(\cap_i \cup_{m\geqq i} A_m) = 0$. Let $x \in \cap_i \cup_{m\geqq i} A_m$. Then there exists a subsequence (i_k) such that $f_{i_k}(x) > \lambda'$ and $V(x) \leqq \lambda'$, or $f_{i_k}(x) \leqq \lambda'$ and $V(x) > \lambda'$ for any $k \in \mathbb{N}$.

Since $f_i \to V$ almost everywhere, we get $\mathcal{H}^{n-1}(\cap_i \cup_{m\geqq i} A_m) \leqq \mathcal{H}^{n-1}(\{V = \lambda'\}) = 0$, by (49). The claim is proved. Then (51) follows from (52) and (50), and Step 1 is proved.

Step 2. Let us prove that

$$\liminf_{i\to+\infty} \int_{\partial E} f_i \operatorname{div}_{\phi,\tau} (\chi - N_{\min})\, dP_\phi \geqq 0 \qquad \forall\, \chi \in H(\partial E; \mathbb{R}^n). \tag{53}$$

Let $\chi \in H(\partial E; \mathbb{R}^n)$. Since $f_i \to V$ in $L^2(\partial E)$, we have

$$\liminf_{i\to+\infty} \int_{\partial E} f_i \operatorname{div}_{\phi,\tau} (\chi - N_{\min})\, dP_\phi = \int_{\partial E} V \operatorname{div}_{\phi,\tau}(\chi - N_{\min})\, dP_\phi \geqq 0, \tag{54}$$

where the last inequality follows from the Euler inequality (43). Step 2 is proved.

Fix now $\delta > 0$ and define $\tilde\eta = \tilde\eta(\eta, \lambda', \varepsilon', i, \delta) : \partial E \to \mathbb{R}^n$ as follows:

$$\tilde\eta := \begin{cases} N & \text{on } \{\lambda' < f_i < \lambda' + \varepsilon'\}, \\ N_{\min} & \text{on } \{f_i < \lambda' - \delta\} \cup A^i_{\lambda'+\varepsilon'+\delta}, \end{cases}$$

and

$$\tilde\eta(x) := \begin{cases} \psi\left(\frac{f_i(x)-\lambda'+\delta}{\delta}\right) N(x) + \left(1 - \psi\left(\frac{f_i(x)-\lambda'+\delta}{\delta}\right)\right) N_{\min}(x) \\ \text{for } x \in \{\lambda' - \delta \leqq f_i \leqq \lambda'\}, \\ \\ \psi\left(\frac{f_i(x)-\lambda'-\varepsilon'}{\delta}\right) N_{\min}(x) + \left(1 - \psi\left(\frac{f_i(x)-\lambda'-\varepsilon'}{\delta}\right)\right) N(x) \\ \text{for } x \in \{\lambda' + \varepsilon' \leqq f_i \leqq \lambda' + \varepsilon' + \delta\}, \end{cases}$$

where ψ has the following properties: $\psi \in C^\infty([0,1];[0,1])$, there is $\sigma \in\ ]0, \frac{1}{2}[$ such that $\psi(s) = 0$ for $s \in [0,\sigma]$, $\psi(s) = 1$ for $s \in [1-\sigma, 1]$, and $\psi_{|[\sigma,1-\sigma]}$ is strictly increasing.

Note that $\tilde\eta \in H(\partial E; \mathbb{R}^n)$ and that $(\tilde\eta - N)$ has compact support in A^i_t, for any $t \in [\lambda', \lambda' + \varepsilon']$. It follows from (A4) of Lemma 4.4 that

$$\int_{A^i_t} \operatorname{div}_{\phi,\tau}(\tilde\eta - N)\, dP_\phi = 0 \qquad \forall\, t \in [\lambda', \lambda' + \varepsilon]. \tag{55}$$

Therefore from (55) and (51)

$$\int_{\lambda'}^{\lambda'+\varepsilon'} D(A^i_t, \tilde\eta)\, dt = \int_{\lambda'}^{\lambda'+\varepsilon'} D(A^i_t, N)\, dt \leqq -c\varepsilon' \qquad \forall\, i \geqq i_0. \tag{56}$$

Step 3. Let us prove that

$$0 = \lim_{\delta \downarrow 0} \int_{\lambda'-\delta}^{\lambda'} D(A_t^i, \tilde{\eta})\, dt = \lim_{\delta \downarrow 0} \int_{\lambda'+\varepsilon'}^{\lambda'+\varepsilon'+\delta} D(A_t^i, \tilde{\eta})\, dt \qquad \forall\, i \geqq i_0. \tag{57}$$

Since $(\tilde{\eta} - N_{\min})$ has compact support in $A^i_{\lambda'-\delta}$, using (A4) of Lemma 4.4 we have $D(A^i_{\lambda'-\delta}, \tilde{\eta}) = 0$. Therefore, by Lemma 6.4 applied with $f, \beta, d\mu, a, b$ replaced by f_i, $\mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})$, dP_ϕ, $\lambda' - \delta$, λ' in that order, we have

$$\int_{\lambda'-\delta}^{\lambda'} D(A_t^i, \tilde{\eta})\, dt = \int_{\{\lambda'-\delta < f_i < \lambda'\}} f_i \,\mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi + \lambda' D(A^i_{\lambda'}, \tilde{\eta}). \tag{58}$$

Define now

$$h_i(x) := \begin{cases} f_i(x) & \text{if } x \in \{f_i < \lambda'\}, \\ \lambda' & \text{if } x \in \{f_i \geqq \lambda'\}. \end{cases}$$

Then $h_i \in \mathrm{Lip}(\partial E)$, and recalling that $\tilde{\eta} = N_{\min}$ in a neighbourhood of $\{f_i < \lambda' - \delta\}$, we get

$$\int_{\{\lambda'-\delta < f_i < \lambda'\}} f_i \,\mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi + \lambda' D(A^i_{\lambda'}, \tilde{\eta}) = \int_{\partial E} h_i \mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi. \tag{59}$$

Hence, by (58), (59), and using (25) we deduce

$$\begin{aligned} \int_{\lambda'-\delta}^{\lambda'} D(A_t^i, \tilde{\eta})\, dt &= - \int_{\partial E} \nabla_\tau h_i \cdot (\tilde{\eta} - N_{\min})\, dP_\phi \\ &= - \int_{\{\lambda'-\delta < f_i < \lambda'\}} \nabla_\tau h_i \cdot (\tilde{\eta} - N_{\min})\, dP_\phi \\ &\leqq \|\tilde{\eta} - N_{\min}\|_{L^\infty(\partial E)} \|\nabla_\tau h_i\|_{L^\infty(\partial E)} \mathcal{H}^{n-1}(\{\lambda' - \delta < f_i < \lambda'\}). \end{aligned}$$

Now the first equality of (57) follows by observing that $\mathcal{H}^{n-1}(\{\lambda' - \delta < f_i < \lambda'\}) \downarrow 0$ as $\delta \to 0$. The other equality is similar. Step 3 is proved.

We can now conclude the proof of the theorem. Recalling Step 2, we can fix a natural number $i_1 \geqq i_0$, such that

$$\int_{\partial E} f_{i_1} \,\mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi > -c\frac{\varepsilon'}{4}.$$

We have

$$\begin{aligned} -c\frac{\varepsilon'}{4} &< \int_{\partial E} f_{i_1} \mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi \\ &= \int_{-\infty}^{\lambda'-\delta} D(A_t^{i_1}, \tilde{\eta})\, dt + \int_{\lambda'-\delta}^{\lambda'} D(A_t^{i_1}, \tilde{\eta})\, dt + \int_{\lambda'}^{\lambda'+\varepsilon'} D(A_t^{i_1}, \tilde{\eta})\, dt \\ &\quad + \int_{\lambda'+\varepsilon'}^{\lambda'+\varepsilon'+\delta} D(A_t^{i_1}, \tilde{\eta})\, dt + \int_{\lambda'+\varepsilon'+\delta}^{+\infty} D(A_t^{i_1}, \tilde{\eta})\, dt \\ &:= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}. \end{aligned}$$

By the definition of $\tilde{\eta}$ and Lemma 4.4 we have I $=$ V $= 0$. In addition III $\leqq -c\varepsilon'$ by (56), and II and IV tends to zero as $\delta \downarrow 0$ by step 3. Therefore, taking $\delta > 0$ small enough, we get II $+$ IV $\leqq c\frac{\varepsilon'}{2}$. Then

$$-\varepsilon\frac{c'}{4} < \int_{\partial E} f_{i_1} \mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi \leqq -\varepsilon' c + \varepsilon' \frac{c}{2} = -\varepsilon' \frac{c}{2},$$

which is a contradiction.

Since $\int_{\partial E} \mathrm{div}_{\phi,\tau}(N - N_{\min})\, dP_\phi = 0$, from (45) and (46) it follows that the statement of Theorem 6.5 holds also if, in the definitions of A_t and Ω_t, we write the weak inequalities in place of the strict inequalities.

The following observation, which follows from Lemma 6.4 and Remark 6.3, is a kind of converse of Theorem 6.5.

Proposition 6.6. *Let $N \in H(\partial E; \mathbb{R}^n)$ and define $B_t := \{\mathrm{div}_{\phi,\tau}\, N - g > t\}$ for any $t \in \mathbb{R}$. If*

$$\int_{B_t} \mathrm{div}_{\phi,\tau} N\, dP_\phi \leqq \int_{B_t} \mathrm{div}_{\phi,\tau}\, \chi\, dP_\phi \qquad \forall\, t \in \mathbb{R},\ \forall\, \chi \in H(\partial E; \mathbb{R}^n), \tag{60}$$

then $\mathrm{div}_{\phi,\tau}\, N = d_{\min}$.

Proof. Recalling (A4) of Lemma 4.4, we have

$$\int_{\partial E} \mathrm{div}_\tau n_\phi\, dP_\phi = \int_{\partial E} \mathrm{div}_{\phi,\tau} N\, dP_\phi =: c.$$

Set $f := \mathrm{div}_{\phi,\tau} N - g$ and $\beta := \mathrm{div}_{\phi,\tau} N - c$. Assumption (60) can be rewritten as

$$\int_{\{f>t\}} \beta\, dP_\phi \leqq \int_{\{f>t\}} (\mathrm{div}_{\phi,\tau}\chi - c)\, dP_\phi \qquad \forall\, t \in \mathbb{R},\ \forall\, \chi \in H(\partial E; \mathbb{R}^n).$$

Clearly $\int_{\partial E} \beta\, dP_\phi = 0$. Applying (44) we get

$$\begin{aligned} \int_{\partial E} f\beta\, dP_\phi &= \int_{-\infty}^{\infty} \int_{\{f>t\}} \beta\, dP_\phi\, dt \\ &\leqq \int_{-\infty}^{\infty} \int_{\{f>t\}} (\mathrm{div}_{\phi,\tau}\chi - c)\, dP_\phi\, dt = \int_{\partial E} f(\mathrm{div}_{\phi,\tau}\chi - c)\, dP_\phi. \end{aligned}$$

It follows that

$$\int_{\partial E} f(\beta - \mathrm{div}_{\phi,\tau}\chi + c)\, dP_\phi \leqq 0 \qquad \forall\, \chi \in H(\partial E; \mathbb{R}^n),$$

that is,

$$\int_{\partial E} (\mathrm{div}_{\phi,\tau} N - g)\, \mathrm{div}_{\phi,\tau}(N - \chi)\, dP_\phi \leqq 0 \qquad \forall\, \chi \in H(\partial E; \mathbb{R}^n).$$

Using Remark 6.3 the assertion follows.

Note that Proposition 6.6 still holds if, in the definition of B_t, we replace the weak inequality with the strict inequality.

We are now in a position to prove the L^∞ regularity of the divergence of solutions of (41).

Theorem 6.7. *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and assume that $g \in L^\infty(\partial E)$. Then*

$$d_{\min} \in L^\infty(\partial E). \tag{61}$$

More precisely,

$$\|d_{\min} - g\|_{L^\infty(\partial E)} \leqq \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)}. \tag{62}$$

Proof. Set $V := d_{\min} - g$. By (45) we have

$$\int_{A_t} V \, dP_\phi \leqq \int_{A_t} \Big(\mathrm{div}_\tau n_\phi - g\Big) \, dP_\phi \leqq \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)} P_\phi(A_t) \quad \forall t \in \mathbb{R},$$

so that

$$t \leqq \frac{1}{P_\phi(A_t)} \int_{A_t} V \, dP_\phi \leqq \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)} \qquad \forall t \in \mathbb{R} \text{ with } A_t \neq \emptyset,$$

which implies $\mathcal{H}^{n-1} - \mathrm{esssup}_{\partial E} V \leqq \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)}$.

Since $\int_{\partial E} \mathrm{div}_{\phi,\tau}(N_{\min} - n_\phi) \, dP_\phi = 0$, using (46) we also get

$$\int_{\Omega_t} V \, dP_\phi \geqq \int_{\Omega_t} \Big(\mathrm{div}_\tau n_\phi - g\Big) \, dP_\phi, \qquad \forall t \in \mathbb{R},$$

which implies

$$t \geqq \frac{1}{P_\phi(\Omega_t)} \int_{\Omega_t} V \, dP_\phi \geqq -\|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)} \qquad \forall t \in \mathbb{R} \text{ with } \Omega_t \neq \emptyset.$$

It follows that $\mathcal{H}^{n-1} - \mathrm{essinf}_{\partial E} V \geqq -\|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)}$. This concludes the proof of (62).

Remark 6.8. Thanks to Theorem 6.7, if $g \in L^\infty(\partial E)$ the functional $\mathcal{F}$ can be equivalently minimized on the space

$$\widehat{H}(\partial E; \mathbb{R}^n) := \big\{N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n) : \mathrm{div}_{\phi,\tau} N \in L^\infty(\partial E)\big\}.$$

Moreover,

$$\|d_{\min} - g\|_{L^\infty(\partial E)} = \min\big\{\|\mathrm{div}_{\phi,\tau} N - g\|_{L^\infty(\partial E)} : \; N \in \widehat{H}(\partial E; \mathbb{R}^n)\big\}.$$

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(*Accepted October 11, 2000*)
Published online February 14, 2001 –