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Compressible Euler Equations with General Pressure Law

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Abstract

We study the hyperbolic system of Euler equations for an isentropic, compressible fluid governed by a general pressure law. The existence and regularity of the *entropy kernel* that generates the family of weak entropies is established by solving a new Euler-Poisson-Darboux equation, which is *highly singular* when the density of the fluid vanishes. New properties of *cancellation of singularities* in combinations of the entropy kernel and the associated entropy-flux kernel are found.

We prove the *strong compactness* of any sequence that is uniformly bounded in L^∞ and whose corresponding sequence of weak entropy dissipation measures is locally H^{-1} compact. The *existence* and *large-time behavior* of L^∞ entropy solutions of the Cauchy problem are established. This is based on a reduction theorem for Young measures, whose proof is new even for the polytropic perfect gas. The existence result also extends to the *p-system* of fluid dynamics in Lagrangian coordinates.

1. Introduction

The Euler equations for an isentropic compressible fluid read

$$\begin{aligned} \partial_t \rho + \partial_x m &= 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p(\rho) \right) &= 0, \end{aligned} \tag{1.1}$$

where $\rho \geqq 0$ denotes the density, m the momentum, and $p(\rho) \geqq 0$ the pressure. As far as the well-posedness of the Cauchy problem for (1.1) is concerned, the previous research was restricted to the polytropic perfect gas (see (1.6)). This paper stems from a renewed interest in the applications toward real gases and other complex fluids governed by various pressure laws [9, 33]. One of the main difficulties for the mathematical analysis of (1.1) is the singularity at vacuum $\rho = 0$. The physical region for (1.1) is $\{(\rho, m) |\ |m| \leqq C\,\rho\}$, for some $C > 0$, in which the term m^2/ρ in the flux function is only Lipschitz continuous near the vacuum. For $\rho > 0$, $v = m/\rho$ represents the velocity of the fluid. Another difficulty is the development of shock waves in solutions of the Cauchy problem:

$$(\rho, m)|_{t=0} = (\rho_0, m_0) \tag{1.2}$$

for (1.1), no matter how smooth the initial data (ρ_0, m_0) is.

This system is an archetype of nonlinear hyperbolic systems of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u \in \mathbb{R}^N, \quad f : \mathbb{R}^N \to \mathbb{R}^N. \tag{1.3}$$

For background on conservation laws, we refer to LAX [20, 21]. Strict hyperbolicity and genuine nonlinearity away from the vacuum for (1.1) require that

$$p'(\rho) > 0, \qquad 2\,p'(\rho) + \rho\,p''(\rho) > 0 \quad \text{for} \quad \rho > 0. \tag{1.4}$$

At the vacuum, the two characteristic speeds of (1.1) may coincide and the system be nonstrictly hyperbolic.

An entropy-entropy flux pair (η, q), by definition, provides the additional conservation law

$$\partial_t \eta(\rho, m) + \partial_x q(\rho, m) = 0,$$

for any smooth solution (ρ, m). A *weak entropy* is an entropy that vanishes at the vacuum. An entropy solution is determined by the entropy inequality

$$\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \leqq 0 \tag{1.5}$$

in the sense of distributions, for any weak entropy pair (η, q) with convex η.

The so-called polytropic perfect gas is described by the equation of state

$$p_*(\rho) = \kappa\,\rho^\gamma, \qquad \gamma > 1. \tag{1.6}$$

One may assume $\kappa = (\gamma - 1)^2/(4\gamma)$, which is a convenient normalization. For early results on the existence of entropy solutions, we refer to [29] for the RIEMANN problem, [34, 14] for a special class of initial data with bounded variation, and [28] for large total variation with small $\gamma - 1$ or vice versa by using the GLIMM scheme [18].

The first global existence for (1.1) with large initial data in L^∞ was established in DIPERNA [16] for the case $\gamma = 1 + 2/N$ ($N \geqq 5$ odd) by the vanishing viscosity method. The existence problem for general values $\gamma \in (1, 5/3]$ was solved in CHEN [2] and DING, CHEN & LUO [13]. The case $\gamma \geqq 3$ was treated by LIONS, PERTHAME, & TADMOR [23]. LIONS, PERTHAME & SOUGANIDIS [24] dealt with the interval $(5/3, 3)$ and simplified the proof for the whole interval.

The present paper is devoted to the compressible fluids governed by a general pressure law that has singularities near $\rho = 0$. We assume that the pressure law $p = p(\rho)$ is smooth away from the vacuum but very singular near the vacuum: The principal singular part of $p(\rho)$ coincides with (1.6) for some $\gamma \in (1, 3)$, but additional singularities not accounted for in (1.6) are allowed. See the precise statement (2.1) in Section 2.

We will prove the following result announced in [7].

Main Theorem. *Consider the compressible Euler system* (1.1) *under assumptions* (1.4) *and* (2.1).

(1) *Given any measurable and bounded initial data* (ρ_0, m_0) *satisfying*

$$0 \leqq \rho_0(x) \leqq C_0, \quad |m_0(x)| \leqq C_0\, \rho_0(x) \quad \textit{for a.e. } x \textit{ and some } C_0 > 0,$$

there exists an entropy solution (ρ, m) *of the Cauchy problem* (1.1), (1.2) *satisfying*

$$0 \leqq \rho(t,x) \leqq C, \qquad |m(t,x)| \leqq C\, \rho(t,x), \quad \textit{for a.e. } (t,x), \tag{1.7}$$

where $C > 0$ *depends only on* C_0.

(2) *Let* $(\rho^\varepsilon, m^\varepsilon)$ *be a sequence of functions, satisfying* (1.7) *uniformly in* ε, *such that, for any weak entropy pair* (η, q),

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \quad \textit{is compact in } H^{-1}_{\mathrm{loc}}.$$

Then the sequence $(\rho^\varepsilon, m^\varepsilon)$ *is compact in* L^r_{loc}, $1 \leqq r < \infty$.

The asymptotic decay of L^∞ entropy solutions and the convergence of the Lax-Friedrichs scheme are also established below. For the proof of Main Theorem, we develop new techniques to handle the difficulties that arise with the general pressure law. In particular, in contrast with case (1.6), no explicit formula is available for the entropies of (1.1). Our approach turns out to simplify further the proofs for case (1.6).

When (1.6) holds, the weak entropies of (1.1) are given by a convolution product between an arbitrary smooth function $\psi = \psi(s)$ and the fundamental kernel of a linear wave equation, χ_*, defined by

$$\chi_*(\rho, v, s) = M_* \left[\rho^{2\theta} - (v-s)^2\right]_+^\lambda \quad \text{for } \rho > 0. \tag{1.8}$$

Here $[y]_+ = \max(0, y)$, and θ, λ, M_* are constants depending on γ (see (2.2) and (2.11)). The weak entropies have the form

$$\eta(\rho, v) = \int_{\mathbb{R}} \chi_*(\rho, v, s)\, \psi(s)\, ds. \tag{1.9}$$

We refer to χ_* as the *entropy kernel* of the γ-law gas. The singularities of χ_* are easily read on the explicit formula. One of the main difficulties for the general pressure law is to identify the singularities of different orders of the entropy kernel, denoted by χ, when an explicit formula is not available.

The general strategy for proving the existence of entropy solutions is as follows. One first constructs approximate solutions, $(\rho^\varepsilon, m^\varepsilon)$, by adding a higher-order regularization term to (1.1) or by using a finite difference scheme. As the parameter ε converges to zero, the functions $(\rho^\varepsilon, m^\varepsilon)$ *formally* converge to an entropy solution of (1.1). However, carrying out this approach rigorously is very challenging. In general, only L^∞ bounds on $(\rho^\varepsilon, m^\varepsilon)$ are available and a weakly convergent subsequence can be extracted. System (1.1) contains nonlinear composite functions that are not continuous in the weak topology, and additional information on the approximate solutions is needed.

TARTAR [30] first used Young measures to describe oscillating solutions to nonlinear partial differential equations. A Young measure $\nu_{(t,x)}$ is a weakly-star measurable mapping from $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}$ to the set of all probability measures. For hyperbolic systems of conservation laws, the so-called Tartar commutation relations constrain the Young measure:

$$\big\langle \nu_{(t,x)}, \eta_1\, q_2 - \eta_2\, q_1 \big\rangle = \big\langle \nu_{(t,x)}, \eta_1 \big\rangle \big\langle \nu_{(t,x)}, q_2 \big\rangle - \big\langle \nu_{(t,x)}, \eta_2 \big\rangle \big\langle \nu_{(t,x)}, q_1 \big\rangle \qquad (1.10)$$

for a.e. (t, x) and for any two (suitably restricted) entropy pairs (η_i, q_i), $i = 1, 2$. These conditions are derived by the method of compensated compactness, especially the div-curl lemma (see TARTAR [30] and MURAT [26]). To this end, one needs certain uniform bounds on the approximate solutions and, in particular, the H^{-1}_{loc} compactness of the entropy dissipation measures, for which Murat's lemma is useful [27,30].

If any measure satisfying (1.10) reduces to a Dirac mass for a.e. (t, x), then the sequence of approximate solutions converges in the strong topology and, for appropriate approximations, toward an entropy solution. For the Euler equations, to show that the Young measure $\nu_{(t,x)}$ is a Dirac mass in the (ρ, m)-plane, it suffices to prove that the measure in the (ρ, v)-plane, still denoted by $\nu_{(t,x)}$, is either a single point or a subset of the vacuum line

$$\{(\rho, v) \,|\, \rho = 0,\ |v| \leqq \sup_{\varepsilon>0} \|m^\varepsilon/\rho^\varepsilon\|_{L^\infty}\}.$$

The main difficulty is that only *weak* entropy pairs can be used, because of the vacuum problem.

In the proof of [2, 13, 16] (also cf. [3]), the heart of the matter is to construct special functions ψ in (1.9) in order to exploit the form of the set of constraints (1.10). These test-functions are suitable approximations of high-order derivatives of the Dirac measure. Use is made of the fact that (1.10) represents an *imbalance of regularity:* the operator on the left is more regular than the one on the right due to cancellation. DIPERNA [16] considered the case where $\lambda \geqq 2$ is an integer so that the weak entropies are polynomial functions of the Riemann invariants. The novel idea of applying the technique of fractional derivatives was introduced in [2, 13] to deal with real values of λ.

A new analysis of (1.10) was proposed by LIONS, PERTHAME & TADMOR [23] for $\gamma \in [3, \infty)$ and by LIONS, PERTHAME & SOUGANIDIS [24] for $\gamma \in (1, 3)$. Motivated by a kinetic formulation of (1.1) and (1.6), they made the crucial observation that the use of the test-functions ψ could in fact be bypassed, and (1.10) be directly expressed with the entropy kernel χ_*. Namely, (1.10) holds for all s_1 and s_2 by replacing $\eta_j := \chi_*(s_j)$ and $q_j := \sigma_*(s_j)$ for $j = 1, 2$. Here σ_* is the entropy flux kernel defined as

$$\sigma_*(\rho, v, s) = \big(v + \theta\,(s - v)\big)\, \chi_*(\rho, v, s).$$

In [24], the commutation relations are differentiated in s, by using the fractional derivative operator $\partial_s^{\lambda+1}$, so that singularities arise by differentiation of χ_*. This approach relies on the lack of balance in regularity of the two sides of (1.10) and on the observation that $< \nu_{(t,x)}, \chi_*(s) >$ is smoother than the kernel $\chi_*(s)$ itself, due to the average by the Young measure.

Many of the previous arguments do not carry over to the general pressure law. Our first aim is to construct all of the weak entropy pairs of (1.1). Sections 2, 3 contain an extensive discussion of the entropy and entropy flux kernels, denoted $\chi(\rho, v, s)$ and $\sigma(\rho, v, s)$ respectively. The existence and uniqueness of the kernels are established in Theorem 2.1. This allows us to generalize (1.9) and obtain the family of weak entropy pairs. In Theorems 2.2, 2.3, we determine the singularities of different orders arising in fractional derivatives of the kernels. Specifically, we decompose the kernels into a sum of the most singular part, the singular part of the next order, and the remainder, the former given by an explicit formula which involves the pressure law $p(\rho)$. The proofs are postponed until Section 3. A connection between the entropy kernel and the entropy flux-splittings will be discussed in [8] (also see [6, 7]).

In Section 4, we study the compactness of a sequence of approximate solutions to the Euler equations. In Theorem 4.1, for a sequence with a uniform L^∞ bound and the H^{-1}_{loc} compactness of its weak entropy dissipation measures, we prove that the sequence is compact in L^r_{loc} for all $r \in [1, \infty)$. The main point is to establish the reduction theorem: a Young measure satisfying the commutation relations (1.10) for all weak entropy pairs is a Dirac mass (Theorem 4.2). Our proof is based on new properties of *cancellation of singularities* of the kernels χ and σ in the following combination

$$E(\rho, v; s_1, s_2) := \chi(\rho, v, s_1)\,\sigma(\rho, v, s_2) - \chi(\rho, v, s_2)\,\sigma(\rho, v, s_1).$$

Then we observe that the following identity is an elementary consequence of the symmetric form of (1.10):

$$\begin{aligned}\left\langle\chi(s_1)\right\rangle\left\langle\partial_{s_2}^{\lambda+1}\partial_{s_3}^{\lambda+1}E(s_2,s_3)\right\rangle+\left\langle\partial_{s_2}^{\lambda+1}\chi(s_2)\right\rangle\left\langle\partial_{s_3}^{\lambda+1}E(s_3,s_1)\right\rangle&\\+\left\langle\partial_{s_3}^{\lambda+1}\chi(s_3)\right\rangle\left\langle\partial_{s_2}^{\lambda+1}E(s_1,s_2)\right\rangle=0&\quad(1.11)\end{aligned}$$

for all s_1, s_2, and s_3, where for instance $\left\langle\chi(s_1)\right\rangle := \left\langle\nu_{(t,x)}, \chi(s_1)\right\rangle$, and the derivatives are understood in the sense of distributions. We prove that, when $s_2, s_3 \to s_1$, the second and third terms converge in the weak-star sense of measures to the *same* term but with opposite sign. The first term is more *singular* and contains the products of functions of bounded variation by bounded measures, which are known to depend upon regularization (see DAL MASO, LEFLOCH & MURAT [10]). The first term in (1.11) converges to a non-trivial limit which is determined explicitly. Finally, the genuine nonlinearity on $p(\rho)$ is required to conclude that the Young measure ν either reduces to a Dirac mass or is supported on the vacuum line.

In Section 5, we prove the convergence of the Lax-Friedrichs scheme for the general pressure law, extending the approach in [2, 13] for $\gamma \in (1, 2]$. The same approach applies when showing the strong convergence of the approximate solutions

$(\rho^\varepsilon, m^\varepsilon)$ constructed by the vanishing viscosity method, i.e.,

$$\partial_t \rho^\varepsilon + \partial_x m^\varepsilon = \varepsilon\, \partial_{xx} \rho^\varepsilon,$$
$$\partial_t m^\varepsilon + \partial_x \left(\frac{(m^\varepsilon)^2}{\rho^\varepsilon} + p(\rho^\varepsilon) \right) = \varepsilon\, \partial_{xx} m^\varepsilon.$$

The existence, compactness, and asymptotic decay of L^∞ entropy solutions of the Cauchy problem then follow, relying on the compactness framework in Section 4.

We point out that the approach developed in this paper is very general and applies to other hyperbolic systems as long as the singularities of the entropy and entropy flux kernels are determined. See CHEN & LEFLOCH [8] for the details.

We remark that all of the results in this paper can be extended to the p-system of fluid dynamics in Lagrangian coordinates

$$\begin{aligned} \partial_t \tau - \partial_y v = 0, \\ \partial_t v + \partial_y \tilde{p}(\tau) = 0, \end{aligned} \tag{1.12}$$

where τ is the specific volume and v the velocity of the fluid. The system is hyperbolic under the condition $\tilde{p}'(\tau) < 0$ for all $\tau > 0$ and is genuinely nonlinear when $\tilde{p}''(\tau) > 0$. Observe that, when the density vanishes, the specific volume is unbounded and should be understood as a distribution.

There is a one-to-one correspondence between entropies and entropy solutions of systems (1.1) and (1.12) (WAGNER [32], also se DAFERMOS[11]). Denote by χ^{E} and σ^{E} the entropy and entropy flux kernels for the Euler equations (1.1). The p-system (1.12) admits an entropy kernel, χ^{L}, and a corresponding entropy flux kernel, σ^{L}, that generate the family of weak entropy pairs. Indeed, setting $\rho = 1/\tau$,

$$\chi^{\mathrm{L}}(\tau, v, s) = \frac{\chi^{\mathrm{E}}(\rho, v, s)}{\rho}, \quad \sigma^{\mathrm{L}}(\tau, v, s) = \big(\sigma^{\mathrm{E}} - v\, \chi^{\mathrm{E}}\big)(\rho, v, s).$$

Observe that χ^{L} blows up when $\tau \to \infty$.

2. Entropy and Entropy Flux Kernels: Main Results

Throughout this paper, besides the hyperbolicity and genuine nonlinearity (1.4) of system (1.1) away from the vacuum, it is assumed that $p(\rho)$ is a function of class $\mathcal{C}^4(0, \infty)$ and that there exist $\gamma \in (1, 3)$ and $C > 0$ such that, when ρ is sufficiently small,

$$p(\rho) = \kappa\, \rho^\gamma (1 + P(\rho)), \qquad \left| P^{(n)}(\rho) \right| \leqq C\, \rho^{1-n}, \qquad 0 \leqq n \leqq 4. \tag{2.1}$$

The solutions under consideration will remain in a bounded subset of $\{\rho \geqq 0\}$ so that the behavior of $p(\rho)$ for large ρ is irrelevant. In this paper the notation C represents a generic constant which need not be the same at each occurrence.

Remark. The pressure law $p(\rho)$ has the same principal singularity as the γ-law gas, but (2.1) allows additional singularities in the derivatives when $\rho \to 0$. Indeed observe that, for $n > \gamma + 1$, $\rho^\gamma P^{(n)}(\rho)$ is unbounded when $\rho \to 0$. Observe also that $p(0) = p'(0) = 0$, but, for $n > \gamma$, the higher derivative $p^{(n)}(\rho)$ is *unbounded* near the vacuum.

Denote the sound speed by

$$c(\rho) = \sqrt{p'(\rho)}.$$

Condition (1.4) ensures that, away from the vacuum, (1.1) is strictly hyperbolic and admits two genuinely nonlinear characteristic fields associated with two distinct wave speeds, $v \pm c(\rho)$. At the vacuum, $c(0) = 0$, and the wave speeds coincide. Consider also the function

$$k(\rho) = \int_0^\rho \frac{c(y)}{y}\, dy,$$

in which the integral is finite in view of (2.1).

Define the constants $\theta \in (0, 1)$ and $\lambda > 0$ by

$$\theta = \frac{\gamma - 1}{2}, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}. \tag{2.2}$$

For the polytropic gas,

$$c(\rho) = \theta\, \rho^\theta, \quad k(\rho) = \rho^\theta.$$

Observe that $2\,\lambda + 1 = 1/\theta$ and $2\,\lambda\,\theta = 1 - \theta$. We have $\gamma \in (1, 3)$ if and only if $\theta \in (0, 1)$ if and only if $\lambda > 0$. On the other hand, $\gamma \in (1, 5/3]$ if and only if $\theta \in (0, 1/2]$ if and only if $\lambda \geqq 1$.

Introduce the Riemann invariants

$$w = v + k(\rho), \qquad z = v - k(\rho),$$

which satisfy $w > z$ except at the vacuum where $w = z$. In the special case

$$k''(\rho) < 0 \qquad \text{or, equivalently} \qquad 2\,p'(\rho) - \rho\, p''(\rho) > 0,$$

which is a *stronger* condition than (1.4), the Riemann invariants w and z are concave and convex functions of ρ, respectively. This is the case of the γ-law gas, but is not necessarily true for a real gas satisfying solely (1.4) and (2.1).

For smooth solutions away from the vacuum, (1.1) is equivalent to

$$\partial_t w + (v + c)\,\partial_x w = 0, \quad \partial_t z + (v - c)\,\partial_x z = 0.$$

The equation

$$\partial_t v + v\,\partial_x v + \rho\, k'(\rho)^2\,\partial_x \rho = 0$$

is a consequence of (1.1), which is convenient for deriving the following equations satisfied by an entropy-entropy flux pair (η, q):

$$q_\rho = v\,\eta_\rho + \rho\, k'(\rho)^2\,\eta_v, \qquad q_v = \rho\,\eta_\rho + v\,\eta_v.$$

Eliminating q yields the following second-order linear hyperbolic partial differential equation for entropy η:

$$\eta_{\rho\rho} - k'(\rho)^2\, \eta_{vv} = 0. \tag{2.3}$$

In the variables (w, z),

$$\eta_{wz} + \frac{\Lambda(w-z)}{w-z}\big(\eta_w - \eta_z\big) = 0, \tag{2.4}$$

where $\Lambda(w-z) = -k(\rho)\, k'(\rho)^{-2}\, k''(\rho)$ with $\rho = k^{-1}(\frac{w-z}{2})$. For the γ-law gas, $\Lambda(w-z) = \lambda$ is a constant, the simplest case.

Equations (2.3) and (2.4) belongs to the *class of Euler-Poisson-Darboux equations*. The main difficulty comes from the *singular* behavior of $\Lambda(w-z)$ near the vacuum. In view of (2.1), the derivative $\Lambda'(w-z)$ blows up like $(w-z)^{-(\gamma-1)/2}$ when $w - z \to 0$ in general, and its higher derivatives are more singular, which is one of the essential differences from the γ-law case. The classical theory of Euler-Poisson-Darboux equations does not apply (cf. [1, 12, 31]). In the present section, we establish the existence of a fundamental solution to (2.3) and study its regularity.

By definition, the *entropy kernel* is the solution $\chi(\rho, v, s)$ of the problem

$$\begin{aligned} &\text{(i)} \quad \chi_{\rho\rho} - k'(\rho)^2\, \chi_{vv} = 0,\\ &\text{(ii)} \quad \chi(0, v, s) = 0,\\ &\text{(iii)} \quad \chi_\rho(0, v, s) = \delta_{v=s}, \end{aligned} \tag{2.5}$$

in the sense of distributions, where s plays the role of a parameter and $\delta_{v=s}$ denotes the Dirac measure at $v = s$. By definition, $\chi(\rho, v, s)$ satisfies

$$\int_0^\infty \int_{-\infty}^\infty \chi(\rho, v, s)\big(\varphi_{\rho\rho}(\rho, v) - k'(\rho)^2\, \varphi_{vv}(\rho, v)\big)\, d\rho\, dv - \varphi(0, s) = 0 \tag{2.6}$$

for every test-function $\varphi(\rho, v)$ with compact support in $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}$.

Since the support of the initial data is the point $(\rho, v) = (0, s)$, χ should be supported in the domain of dependence of $(0, s)$,

$$\mathcal{K} := \big\{\rho \geqq 0,\ |s - v| \leqq k(\rho)\big\} = \big\{(w, z) \mid \quad w \geqq s,\ z \leqq s\big\}.$$

Indeed the curves $\big\{w = \text{const.}\big\}$ and $\big\{z = \text{const.}\big\}$ are the characteristics of the hyperbolic equation (2.5i). Invariance under the transformation $v \mapsto \pm(v - s)$, $\chi(\rho, v, s) = \chi(\rho, |v - s|, 0) = \chi(\rho, 0, |s - v|)$, means that it suffices to study (2.5) when $s = 0$.

The entropy flux kernel σ, by definition, satisfies

$$\begin{aligned} &\text{(i)} \quad \sigma_{\rho\rho} - k'(\rho)^2 \sigma_{vv} = \frac{p''(\rho)}{\rho}\, \chi_v,\\ &\text{(ii)} \quad \sigma(0, v, s) = 0,\\ &\text{(iii)} \quad \sigma_\rho(0, v, s) = v\, \delta_{v=s}, \end{aligned} \tag{2.7}$$

for each value of s. In contrast with problem (2.5), condition (2.7iii) above depends upon v, and $\sigma(\rho, v, s) \neq \sigma(\rho, v-s, 0)$. However, in terms of the function $\sigma - v\,\chi$, condition (2.7iii) reads

$$\big(\sigma_\rho - v\,\chi_\rho\big)(0, v, s) = 0,$$

and $\sigma - v\,\chi$ depends upon $v - s$ only, as χ does. The γ-law gas is much simpler since σ_* is determined explicitly from χ_*; see (1.8).

In Section 3, we prove the following theorem.

Theorem 2.1 (Existence and uniqueness**).** *Problem* (2.5) *admits a unique Hölder continuous solution* $\chi(\rho, v, s) = \chi(\rho, v - s)$*, supported in the set* $\mathcal{K}$ *and positive in the interior of* $\mathcal{K}$*.*

Problem (2.7) *admits a unique Hölder continuous solution* $\sigma(\rho, v, s)$ *supported in the set* $\mathcal{K}$ *with* $\sigma - v\,\chi$ *depending only on* $(\rho, v - s)$*.*

From Theorem 2.1, we deduce

Corollary 2.1. *The family of weak entropies for the compressible Euler equations is described by*

$$\eta(\rho, v) = \int_{\mathbb{R}} \chi(\rho, v, s)\,\psi(s)\,ds,$$

where $\psi(v)$ *is an arbitrary function. By construction,* $\eta(0, v) = 0$*,* $\eta_\rho(0, v) = \psi(v)$*. The corresponding entropy flux is*

$$q(\rho, v) = \int_{\mathbb{R}} \sigma(\rho, v, s)\,\psi(s)\,ds.$$

We now determine the singularities arising in the derivatives of χ and σ. Without loss of generality, we assume here $s = 0$ and set $\chi = \chi(\rho, v)$. The singularities of the kernels should be localized on the characteristic curves which form the boundary of $\mathcal{K}$:

$$\partial\mathcal{K} = \big\{(\rho, v)\mid v \pm k(\rho) = 0\big\}.$$

Measure terms on $\partial\mathcal{K}$ arise when differentiating the kernel with respect to v (or equivalently s).

To state the results, we use the following notation. For any real $\alpha > 0$, the fractional derivative $\partial_s^\alpha g$ of a function $g = g(s)$ with compact support is

$$\partial_s^\alpha g \;= \Gamma(-\alpha)\, g \star \big[s\big]_+^{-\alpha-1},$$

where the convolution product is defined in the sense of distributions and Γ is the classical gamma function. Observe that the formula

$$\partial_s^{\alpha+1}(s\,g) = s\,\partial_s^{\alpha+1} g + (\alpha + 1)\,\partial_s^\alpha g$$

still holds for fractional derivatives.

All of the following properties are uniform for $\rho \geqq 0$ and v in a bounded set.

Theorem 2.2 (Asymptotic expansion for χ**).** *The entropy kernel admits the expansion*

$$\chi(\rho, v) = a_\sharp(\rho) G_\lambda(\rho, v) + a_\flat(\rho) G_{\lambda+1}(\rho, v) + g(\rho, v), \tag{2.8}$$

where

$$G_\lambda(\rho, v) = \left[k(\rho)^2 - v^2\right]_+^\lambda,$$

and the coefficients $a_\sharp(\rho)$ and $a_\flat(\rho)$ are explicitly determined and satisfy

$$\begin{aligned} &a_\sharp(\rho) = M_\lambda\, k(\rho)^{-\lambda}\, k'(\rho)^{-1/2} > 0 \qquad \text{for } \rho > 0, \\ &a_\sharp(\rho) + \frac{k(\rho)^2}{\rho}\, |a_\flat(\rho)| \leqq C, \end{aligned} \tag{2.9}$$

for some constant M_λ. The remainder $g(\rho, v)$ and its derivative $\partial_v^{\lambda+1} g(\rho, v)$ are Hölder continuous in (ρ, v) with

$$|g(\rho, v)| \leqq C\, G_{\lambda+1+\alpha_0}(\rho, v) \quad \text{for some } \alpha_0 \in (0, 1). \tag{2.10}$$

In (2.9), M_λ is given by

$$\frac{1}{M_\lambda} = \frac{2\,\lambda}{\sqrt{2\,\lambda + 1}} \int_{-1}^{1} (1 - z^2)^\lambda dz.$$

For the γ-law gas, we have

$$a_\sharp = M_* = \sqrt{2\lambda + 1}\, M_\lambda, \quad a_\flat \equiv 0, \quad g \equiv 0. \tag{2.11}$$

Similarly, we have

Theorem 2.3 (Asymptotic expansion for σ**).** *The entropy flux kernel admits the expansion*

$$\big(\sigma - v\,\chi\big)(\rho, v) = -v\,\big(b_\sharp(\rho) G_\lambda(\rho, v) + b_\flat(\rho) G_{\lambda+1}(\rho, v)\big) + h(\rho, v), \tag{2.12}$$

where the coefficients $b_\sharp(\rho)$ and $b_\flat(\rho)$ satisfy

$$\begin{aligned} &b_\sharp(\rho) = M_\lambda\, \rho\, k(\rho)^{-\lambda-1}\, k'(\rho)^{1/2} > 0 \qquad \text{for } \rho > 0, \\ &b_\sharp(\rho) + \frac{k(\rho)^2}{\rho}\, |b_\flat(\rho)| \leqq C. \end{aligned} \tag{2.13}$$

The remainder $h(\rho, v)$ and its derivative $\partial_v^{\lambda+1} h(\rho, v)$ are Hölder continuous in (ρ, v), and

$$|h(\rho, v)| \leqq C\, G_{\lambda+1+\alpha_0}(\rho, v) \quad \text{for some } \alpha_0 \in (0, 1). \tag{2.14}$$

For the γ-law gas, we have

$$b_\sharp = \frac{M_\lambda}{\sqrt{2\lambda + 1}}, \quad b_\flat \equiv 0, \quad h \equiv 0.$$

The singularities in the derivatives of order $\lambda + 1$ of the kernels are explicitly computable.

Proposition 2.4 (Explicit singularities**).** *The distributions $\partial_v^{\lambda+1}\chi$ and $\partial_v^{\lambda+1}\sigma$ decompose into two Dirac masses plus an integrable function, i.e.,*

$$\partial_v^{\lambda+1}\chi = k'(\rho)^{-1/2} \sum_{\pm} K^{\pm}\delta_{v=\pm k(\rho)} + e^{\mathrm{I}}, \tag{2.15}$$

$$\partial_v^{\lambda+1}\big(\sigma - v\,\chi\big) = -v\,\rho\,k(\rho)\,k'(\rho)^{1/2} \sum_{\pm} K^{\pm}\,\delta_{v=\pm k(\rho)} + e^{\mathrm{II}}, \tag{2.16}$$

where $K^{\pm}$ are some constants, and e^{I}, e^{II} are Hölder continuous functions in the interior of $\mathcal{K}$ such that

$$\begin{aligned} |e^{\mathrm{I}}(\rho, v)| &\leqq C\,k(\rho)^{\lambda-1+2\alpha}\,G_{-\alpha}(\rho, v), \\ |e^{\mathrm{II}}(\rho, v)| &\leqq C\,k(\rho)^{\lambda+2\alpha}\,G_{-\alpha}(\rho, v) \end{aligned} \tag{2.17}$$

for all $\alpha \in (0, 1]$.

Observe that, in (2.15)–(2.17), the coefficient $k'(\rho)^{-1/2}$ is unbounded when $\rho \to 0$. It will be convenient to use the notation $f_\lambda(y) = \big[1 - y^2\big]_+^{\lambda}$ so that

$$G_\lambda(\rho, v) = k(\rho)^{2\lambda}\, f_\lambda\left(\frac{v}{k(\rho)}\right). \tag{2.18}$$

Proof. Consider first the function $f_\lambda(y)$. Its Fourier transform $\hat{f}_\lambda(\xi)$ is a smooth, real-valued function of the Fourier variable ξ, and

$$\hat{f}_\lambda(\xi) = \int_{-1}^{1} \cos(\xi\, y)\,\big[1 - y^2\big]_+^{\lambda}\, dy = C_0\,|\xi|^{-\lambda-1/2}\, J_{\lambda+1/2}(|\xi|)$$

for all real ξ, where the classical Bessel function $J_{\lambda+1/2}(y)$ admits the asymptotic expansion

$$J_{\lambda+1/2}(y) = C_1\, y^{-1/2}\, \cos\big(y - (\lambda+1)\,\pi/2\big) + O\big(y^{-3/2}\big)$$

as $y \to +\infty$ (e.g. Gelfand & Shilov [17]). We deduce that

$$\hat{f}_\lambda(\xi) = C_2\,|\xi|^{-\lambda-1}\, \cos\big(|\xi| - (\lambda+1)\,\pi/2\big) + O\big(|\xi|^{-\lambda-2}\big). \tag{2.19}$$

On the other hand,

$$|\hat{f}_\lambda(\xi)| \leqq C_3 \tag{2.20}$$

for all ξ. The constants C_j, $0 \leqq j \leqq 3$, may depend on λ.

Using (2.19), (2.20), and $\lambda > 0$, we find that $\hat{f}_\lambda(\xi)$ is integrable in $\xi \in \mathbb{R}$. From the inverse Fourier transform in the sense of distributions, we obtain (see [17])

$$\partial_y^{\lambda+1} f_\lambda(y) = K_\lambda^{+}\,\delta_{y=1} + K_\lambda^{-}\,\delta_{y=-1} + Q_\lambda(y), \tag{2.21}$$

where $K_\lambda^{\pm}$ are constants, and Q_λ is supported on $[-1, 1]$ satisfying

$$|Q_\lambda(y)| \leqq C\,|\log(1 - y^2)| \qquad \text{for all } y \in (-1, 1). \tag{2.22}$$

We also have

$$\partial_y^\lambda f_\lambda(y) = K_\lambda^+ H(y-1) + K_\lambda^- H(y+1) + \int_{-1}^y Q_\lambda(y)\,dy, \tag{2.23}$$

where H is the Heaviside function.

From (2.18), we have

$$\partial_v^{\lambda+1} G_\lambda = k(\rho)^{2\lambda} \partial_v^{\lambda+1} f_\lambda \left(\frac{v}{k(\rho)} \right).$$

Hence we deduce from (2.23) that

$$\begin{aligned}
\partial_v^{\lambda+1}\big(a_\sharp G_\lambda + a_\flat G_{\lambda+1}\big) &= a_\sharp\, k^\lambda \Big(K_\lambda^+\,\delta_{v=k} + K_\lambda^-\,\delta_{v=-k}\Big) + a_\sharp\, k^{\lambda-1}\, Q_\lambda\left(\frac{v}{k}\right) \\
&+ a_\flat\, k^{\lambda+2}\Big(K_{\lambda+1}^+\, H(v-k) + K_{\lambda+1}^-\, H(v+k)\Big) + a_\flat\, k^{\lambda+1} \int_{-1}^{v/k} Q_{\lambda+1}\,dy.
\end{aligned}$$

By Theorem 2.2,

$$\partial_v^{\lambda+1}\chi = \partial_v^{\lambda+1}\big(a_\sharp G_\lambda + a_\flat G_{\lambda+1}\big) + \partial_v^{\lambda+1} g,$$

where $\partial_v^{\lambda+1} g$ is Hölder continuous. Thus the above formula implies (2.15) with $K^\pm := M_\lambda\, K_\lambda^\pm$. The proof of (2.16) is similar. Estimate (2.17) for e^{I} follows from (2.10) and (2.22). □

In Section 4, we use the results in Proposition 2.4 formulated on the functions $\chi(\rho, v-s)$ and $\sigma(\rho, v, s)$. That is,

$$\begin{aligned}
\partial_s^{\lambda+1}\chi(\rho, v-s) &= k'(\rho)^{-1/2} \sum_\pm K^\pm \delta_{s=v\pm k(\rho)} + e^{\mathrm{I}}(\rho, v-s), \\
\partial_s^{\lambda+1}\Big(\sigma(\rho, v, s) - v\,\chi(\rho, v-s)\Big) &= (s-v)\,\rho k(\rho)^{-1}\, k'(\rho)^{1/2} \\
&\quad \times \sum_\pm K^\pm\, \delta_{s=v\pm k(\rho)} + e^{\mathrm{II}}(\rho, v-s).
\end{aligned}$$

Integrating in s, we get

$$\begin{aligned}
\partial_s^{\lambda}\chi(\rho, v-s) &= k'(\rho)^{-1/2} \\
&\quad \times \sum_\pm K^\pm H(s-v\mp k(\rho)) + \tilde e^{\mathrm{I}}(\rho, v-s), \\
\partial_s^{\lambda}\Big(\sigma(\rho, v, s) - v\,\chi(\rho, v-s)\Big) &= (s-v)\,\rho k(\rho)^{-1}\, k'(\rho)^{1/2} \\
&\quad \times \sum_\pm K^\pm\, H(s-v\mp k(\rho)) + \tilde e^{\mathrm{II}}(\rho, v-s),
\end{aligned}$$

where $\tilde e^{\mathrm{J}}(\rho, v) := \int_{-k(\rho)}^v e^{\mathrm{J}}(\rho, v')\,dv'$, J = I, II.

Finally we record a technical property needed in Section 4, which follows by a direct calculation based on expressions (2.9) and (2.13).

Proposition 2.5. *The coefficients of the asymptotic expansions* (2.8) *and* (2.12) *satisfy*

$$\begin{aligned} D(\rho) &:= a_\sharp(\rho)\, b_\sharp(\rho) - 2\, k(\rho)^2 \Big(a_\sharp(\rho) b_\flat(\rho) - a_\flat(\rho) b_\sharp(\rho) \Big) \\ &= \frac{M_\lambda^2}{4(\lambda+1)} \frac{k(\rho)}{\rho^2 k'(\rho)^3} \Big((\rho k'(\rho))' + k'(\rho) \Big) > 0 \quad \textit{for} \quad \rho > 0. \end{aligned}$$

3. Entropy and Entropy Flux Kernels: Proofs

This section contains the proofs of Theorems 2.1–2.3 and Proposition 2.5. We first state and prove three lemmas. First of all, we study the singular behavior of the function

$$\alpha_\sharp(\rho) = M_\lambda\, k^{\lambda+1}(\rho) k'(\rho)^{-1/2}$$

near the vacuum $\rho = 0$ when the pressure law satisfies (2.1). This result plays an important role in identifying the singularities of the entropy kernel χ. There is an extra singularity in $\alpha_\sharp(\rho)$ which is not seen in the γ-law case for which $\alpha_\sharp(\rho) = M_\lambda \rho$ has no singularity. The notation $C > 0$ represents a constant depending only on $\gamma \in (1, 3)$ and a fixed upper-bound $\rho_M > 0$ for the density.

Lemma 3.1. *The function $\alpha_\sharp(\rho)$ satisfies*

$$|\alpha_\sharp(\rho)| \leqq C\rho, \quad |\alpha_\sharp'(\rho)| + |\alpha_\sharp''(\rho)| \leqq C, \quad |\alpha_\sharp'''(\rho)| \leqq C\rho^{-1}, \quad \textit{for} \ \rho \in (0, \rho_M]. \tag{3.1}$$

This fact can be seen from assumption (2.1) that

$$k(\rho) = \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau}\, d\tau = \rho^\theta \left(1 + H(\rho)\right), \tag{3.2}$$

where

$$|H^{(m)}(\rho)| \leqq C\rho^{1-m}, \quad 0 \leqq m \leqq 3. \tag{3.3}$$

It is then elementary to deduce (3.1) from (3.2) and (3.3).

The second lemma provides us with an *a priori* energy estimate for

$$\begin{aligned} &\text{(i)} \quad \mu_{\rho\rho}(\rho, \xi) + k'(\rho)^2 \xi^2\ \mu(\rho, \xi) = r(\rho, \xi), \\ &\text{(ii)} \quad \mu(\varepsilon, \xi) = 0, \\ &\text{(iii)} \quad \mu_\rho(\varepsilon, \xi) = 0, \end{aligned} \tag{3.4}$$

where $\varepsilon > 0$ is a constant, the function $r = r(\rho, \xi) \in C^1[\varepsilon, \infty)$ is given, and $\xi \in \mathbb{R}$ is a parameter.

Lemma 3.2 (Energy estimates I**).** *Let* $\mu(\rho,\xi)$ *be a* C^2 *solution of* (3.4) *defined in* (ε,∞) *for any fixed* $\xi\in\mathbb{R}$. *Then we have*

$$\mu_\rho(\rho,\xi)^2+k'(\rho)^2\xi^2\mu(\rho,\xi)^2\leqq C\sum_{i=1}^{3}I_i(\rho,\xi)\quad\text{for any } \rho\geqq\varepsilon, \xi\neq 0, \tag{3.5}$$

where

$$I_1(\rho,\xi):=k'(\rho)^{-2}\,\xi^{-2}\,r(\rho,\xi)^2,\quad I_2(\rho,\xi)\quad=\xi^{-2}\int_\varepsilon^\rho k'(\tau)^{-2}\,r(\tau,\xi)^2\,d\tau,$$
$$I_3(\rho,\xi):=\xi^{-2}\int_\varepsilon^\rho\frac{r_\tau(\tau,\xi)^2}{\left|k'(\tau)k''(\tau)\right|+k'(\tau)^2}\,d\tau. \tag{3.6}$$

Furthermore, when $|k(\rho)\xi|\leqq 1$,

$$\mu_\rho(\rho,\xi)^2+k'(\rho)^2\xi^2\mu^2(\rho,\xi)\leqq C\rho\int_\varepsilon^\rho r(\tau,\xi)^2d\tau. \tag{3.7}$$

Proof. Multiply (3.4i) by $2\,\mu_\rho$, integrate over (ε,ρ), and finally integrate by parts. This gives

$$\mu_\rho(\rho,\xi)^2+k'(\rho)^2\,\xi^2\,\mu(\rho,\xi)^2=2\bigg(r(\rho,\xi)\,\mu(\rho,\xi)$$
$$-\int_\varepsilon^\rho r_\tau(\tau,\xi)\,\mu(\tau,\xi)\,d\tau+\int_\varepsilon^\rho k'(\tau)\,k''(\tau)\,\xi^2\,\mu(\tau,\xi)^2\,d\tau\bigg).$$

Using the inequality $\alpha\,\beta\leqq\delta\alpha^2+\frac{1}{4\delta}\beta^2$ with suitably chosen weights δ, we find that

$$\begin{aligned}\mu_\rho(\rho,\xi)^2&+k'(\rho)^2\,\xi^2\,\mu(\rho,\xi)^2\\&\leqq C\,k'(\rho)^{-2}\,\xi^{-2}r(\rho,\xi)^2\\&\quad+C\int_\varepsilon^\rho\left\{\left|k'(\tau)k''(\tau)\right|+k'(\tau)^2\right\}^{-1}\xi^{-2}\,r_\tau(\tau,\xi)^2\,d\tau\\&\quad+\tfrac{1}{2}\,k'(\rho)^2\,\xi^2\,\mu(\rho,\xi)^2\\&\quad+\int_\varepsilon^\rho\left\{2\,k'(\tau)\,k''(\tau)+\left|k'(\tau)\,k''(\tau)\right|+k'(\tau)^2\right\}\xi^2\,\mu(\tau,\xi)^2\,d\tau.\end{aligned} \tag{3.8}$$

In view of (2.2) and for all $\rho\geqq 0$ in a bounded subset, we get

$$2\,k'(\tau)\,k''(\tau)+\left|k'(\tau)\,k''(\tau)\right|+k'(\tau)^2\leqq C\tau^{2\theta-2}+k'(\tau)^2\leqq C\,k'(\tau)^2. \tag{3.9}$$

Indeed the principal term in the expansion of $k'(\tau)\,k''(\tau)$, $-\theta^2\,(1-\theta)\,\tau^{2\theta-3}$, is a *singular* term with a *negative* coefficient and does not contribute to the upper bound in (3.9).

Estimate (3.9) allows us to apply Gronwall's inequality to (3.8) and obtain

$$k'(\rho)^2\,\xi^2\,\mu(\rho,\xi)^2\leqq C\left(G(\rho,\xi)+\int_\varepsilon^\rho G(\tau,\xi)\,d\tau\right)$$

for all $\rho \geqq \varepsilon$, where

$$G(\rho) := k'(\rho)^{-2}\,\xi^{-2} r(\rho,\xi)^2 + \xi^{-2}\int_\varepsilon^\rho \frac{r_\tau(\tau,\xi)^2}{|k'(\tau)k''(\tau)| + k'(\tau)^2}d\tau.$$

Since the double integral involved in this upper-bound is bounded by the single integral, we arrive at

$$k'(\rho)^2\xi^2\mu(\rho,\xi)^2 \leqq C\left(I_1(\rho,\xi) + I_2(\rho,\xi) + I_3(\rho,\xi)\right).$$

Returning to (3.8), we also obtain

$$\mu_\rho(\rho,\xi)^2 \leqq C\,(I_1(\rho,\xi) + I_2(\rho,\xi) + I_3(\rho,\xi)).$$

We now derive (3.7) when $|k(\rho)\xi| \leqq 1$. Multiplying (3.4i) by $2\mu_\rho$, we obtain

$$\begin{aligned}(\mu_\rho^2 + k'(\rho)^2\xi^2\mu^2)_\rho &= 2r\,\mu_\rho + 2k'(\rho)k''(\rho)\xi^2\mu^2\\ &\leqq \frac{\mu_\rho^2}{\rho} + C\rho\, r^2 + 2\,k'(\rho)k''(\rho)\xi^2\mu^2.\end{aligned}$$

There exists $\rho_1 > 0$ such that

$$2k'(\rho)k''(\rho) < 0 < \frac{k'(\rho)^2}{\rho}, \qquad \text{for } 0 \leqq \rho \leqq \rho_1.$$

Therefore, we have

$$(\mu_\rho^2 + k'(\rho)^2\xi^2\mu^2)_\rho \leqq \frac{1}{\rho}(\mu_\rho^2 + k'(\rho)^2\xi^2\mu^2) + C(\rho\, r^2 + X_{[\rho_1,\infty)}(\rho)\mu^2),$$

since $|k(\rho)\xi| \leqq 1$, where $X_{[\rho_1,\infty)}$ is the characteristic function. Then

$$(\rho^{-1}(\mu_\rho^2 + k'(\rho)^2\xi^2\mu^2))_\rho \leqq C(r^2 + \chi_{[\rho_1,\infty)}(\rho)\mu^2),$$

that is,

$$\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2 \leqq C\rho\left(\int_\varepsilon^\rho r(\tau,\xi)^2 d\tau + \int_{\rho_1}^{\max(\rho,\rho_1)} \mu(\tau,\xi)^2 d\tau\right). \tag{3.10}$$

First of all, for $\rho \leqq \rho_1$,

$$\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2 \leqq C\rho\int_\varepsilon^\rho r(\tau,\xi)^2\,d\tau. \tag{3.11}$$

Second, for $\rho \geqq \rho_1$, we have $|\xi| \leqq C(\rho_1)$ since $|k(\rho)\xi| \leqq 1$. Note that

$$\begin{aligned}\mu(\rho,\xi)^2 &\leqq \int_\varepsilon^\rho \mu_\tau(\tau,\xi)^2 d\tau + \int_{\rho_1}^\rho \mu_\tau(\tau,\xi)^2\,d\tau\\ &\leqq C\left(\int_\varepsilon^\rho r(\tau,\xi)^2 d\tau + \int_{\rho_1}^\rho \mu_\tau(\tau,\xi)^2\,d\tau\right),\end{aligned}$$

so that, from (3.10),

$$\mu_\rho(\rho,\xi)^2 \leqq C\rho \int_\varepsilon^\rho r(\tau,\xi)^2 d\tau + C\int_{\rho_1}^\rho \mu_\tau(\tau,\xi)^2\,d\tau.$$

Gronwall's inequality implies

$$\mu_\rho(\rho,\xi)^2 \leqq C\int_\varepsilon^\rho r(\tau,\xi)^2 d\tau.$$

Hence, when $\rho \geqq \rho_1$,

$$\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2 \leqq C\rho\int_\varepsilon^\rho r(\tau,\xi)^2\,d\tau. \tag{3.12}$$

Estimates (3.11) and (3.12) yield (3.7). This completes the proof of Lemma 3.2. □

Lemma 3.3 (Energy estimates II)**.** *Let* $\mu = \mu(\rho,\xi)$ *be a* C^2 *solution of* (3.4) *defined for* $\rho \in (\varepsilon,\infty)$. *Let* $r(\rho,\xi)$ *be such that*

$$|\partial_\rho^j r(\rho,\xi)| \leqq \frac{C\rho^{p-j}}{(1+|k(\rho)\xi|)^{q-j}}, \qquad j = 0,1 \tag{3.13}$$

for $q \leqq (p+1)(2\lambda+1)$. *Then we have*

$$|\partial_\rho^m \mu(\rho,\xi)| \leqq \frac{C\rho^{2-m+p}}{(1+|k(\rho)\xi|)^{\beta+1-m}} \qquad \text{for } m = 0,1,2, \tag{3.14}$$

where $\beta = \min(q, \lambda+1+(p+1)\lambda_0)$ *and* $0 < \lambda_0 < \min(1,\lambda)$.

Proof. We first derive the estimates for the case $|k(\rho)\xi| \geqq 1$. Using (3.13), we have

$$I_1(\rho,\xi) \leqq C\rho^{2-2\theta+2p}\xi^{-2}\left|\rho^\theta\xi\right|^{-2q} \leqq C\rho^{2(p+1)}\left|k(\rho)\xi\right|^{-2(q+1)}. \tag{3.15}$$

To estimate I_2, we decompose it into two terms, $I_2 = I_{2,1} + I_{2,2}$, with

$$I_{2,1}(\rho,\xi) := \xi^{-2}\int_{|\xi|^{-1/\theta}}^\rho k'(\tau)^{-2}\,|r(\tau,\xi)|^2\,d\tau \leqq C\,\rho^{3+2p}\left|k(\rho)\,\xi\right|^{-2q}, \tag{3.16}$$

where we have used $q < (p+\frac{3}{2})(2\lambda+1)$. On the other hand,

$$I_{2,2}(\rho,\xi) = \xi^{-2}\int_\varepsilon^{|\xi|^{-1/\theta}} k'(\tau)^{-2} r(\tau,\xi)^2\,d\tau \leqq C\,\rho^{3+2p}\left|k(\rho)\,\xi\right|^{-(3+2p)(2\lambda+1)}. \tag{3.17}$$

Finally, we estimate $I_3 = I_{3,1} + I_{3,2}$ with

$$I_{3,1}(\rho,\xi) := \xi^{-2}\int_{|\xi|^{-1/\theta}}^\rho \frac{r_\tau(\tau,\xi)^2}{\left|k'(\tau)k''(\tau)\right| + k'(\tau)^2}\,d\tau \leqq C\rho^{2(p+1)}\left|k(\rho)\xi\right|^{-2q}, \tag{3.18}$$

where we have used $q < (p+1)(2\lambda+1)$. Similarly, we have

$$I_{3,2}(\rho,\xi) \leqq C\rho^{2+2p}|k(\rho)\xi|^{-2(p+1)(2\lambda+1)}. \tag{3.19}$$

Combining (3.15)–(3.19), we conclude that, when $|k(\rho)\xi| \geqq 1$,

$$\mu_\rho(\rho,\xi)^2 \leqq C\,\rho^{2(p+1)}\left|k(\rho)\,\xi\right|^{-2\beta}.$$

Returning to the energy estimates (3.5) and using $k'(\rho)^2\xi^2 = \rho^{-2}|k(\rho)\xi|^2$ and (3.4), we can also bound

$$\mu(\rho,\xi)^2 + \rho^4|k(\rho)\xi|^{-4}\mu_{\rho\rho}(\rho,\xi)^2 \leqq C\,\rho^{2(p+2)}\left|k(\rho)\,\xi\right|^{-2(\beta+1)}.$$

When $|k(\rho)\xi| \leqq 1$, we conclude from (3.7) that

$$|\mu(\rho,\xi)| + \rho\,|\mu_\rho(\rho,\xi)| + \rho^2\,|\mu_{\rho\rho}(\rho,\xi)| \leqq C\rho^{p+2}.$$

Then (3.14) follows. The proof of Lemma 3.3 is completed. □

Lemma 3.4 (Energy estimate III). *Let, for any fixed $\xi \in \mathbb{R}$, $\mu(\rho,\xi)$ be a C^2 solution of the problem*

$$\mu_{\rho\rho}(\rho,\xi) + k'(\rho)^2\xi^2\mu(\rho,\xi) = r(\rho,\xi), \quad 0 < \rho \leqq \rho_M,$$
$$\mu(\rho_M,\xi) = 0, \qquad \mu_\rho(\rho_M,\xi) = 0.$$

Then, for all (ρ,ξ), we have

$$\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2 \leqq C\int_\rho^{\rho_M} r(\tau,\xi)^2 d\tau. \tag{3.20}$$

Proof. Multiplying both sides of the equation by $2\mu_\rho(\rho,\xi)$, we have

$$(\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2)_\rho = r(\rho,\xi)\,\mu_\rho(\rho,\xi) + 2k'(\rho)k''(\rho)\,\xi^2\mu(\rho,\xi)^2,$$

so that

$$\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2$$
$$\leqq C\int_\rho^{\rho_M}\{\mu_\rho(\tau,\xi)^2 + k'(\tau)^2\xi^2\mu(\tau,\xi)^2\}\,d\tau + C\int_\rho^{\rho_M} r(\tau,\xi)^2\,d\tau.$$

Gronwall's inequality implies

$$\mu_\rho(\rho,\xi)^2 + k'(\rho)^2\xi^2\mu(\rho,\xi)^2 \leqq C\int_\rho^{\rho_M} r(\tau,\xi)^2 d\tau. \qquad \square$$

Proof of Theorems 2.1 and 2.2. Without loss of generality, we view the entropy kernel as a function of two variables, $\chi(\rho, s - v)$, and set $s = 0$ to simplify the notation. We first establish certain properties of the Fourier transform of χ in the variable v and determine the singularities of χ. We prove that $\hat{\chi}$ has the form

$$\begin{aligned} \hat{\chi}(\rho,\xi) &= \hat{\chi}^\sharp(\rho,\xi) + \hat{\chi}^\flat(\rho,\xi) + \hat{g}(\rho,\xi), \\ \hat{\chi}^\sharp(\rho,\xi) &= a_\sharp(\rho) k(\rho)^{2\lambda+1} \hat{f}_\lambda(k(\rho)\xi), \\ \hat{\chi}^\flat(\rho,\xi) &= a_\flat(\rho) k(\rho)^{2\lambda+3} \hat{f}_{\lambda+1}(k(\rho)\xi), \end{aligned} \tag{3.21}$$

where the above coefficients will be explicitly determined (see also (2.9)–(2.18)) and

$$\begin{aligned} |a_\sharp(\rho) - M_\lambda| + \rho^2|a_\sharp'(\rho)| + \rho^2|a_\sharp''(\rho)| + \rho^3|a_\sharp'''(\rho)| &\leqq C\,\rho, \\ |a_\flat(\rho)| + \rho^2|a_\flat'(\rho)| + \rho^2|a_\flat''(\rho)| + \rho^3|a_\flat'''(\rho)| &\leqq C\,\rho, \end{aligned} \tag{3.22}$$

and

$$|\partial_\rho^m \hat{g}(\rho,\xi)| \leqq \frac{C\rho^{2-m}}{\big(1 + |k(\rho)\xi|\big)^{\lambda+\lambda_0+2-m}}, \quad m = 0, 1, 2. \tag{3.23}$$

Problem (2.5) becomes

$$\begin{aligned} &\text{(i)} \quad \chi_{\rho\rho} - k'(\rho)^2\,\chi_{vv} = 0, \\ &\text{(ii)} \quad \chi(0,v) = 0, \\ &\text{(iii)} \quad \chi_\rho(0,v) = \delta_{v=0}. \end{aligned} \tag{3.24}$$

Using the Fourier transform in the v variable, (3.24) is equivalent to

$$\begin{aligned} &\text{(i)} \quad \hat{\chi}_{\rho\rho} + k'(\rho)^2\,\xi^2\,\hat{\chi} = 0, \\ &\text{(ii)} \quad \hat{\chi}(0,\xi) = 0, \\ &\text{(iii)} \quad \hat{\chi}_\rho(0,\xi) = 1, \end{aligned} \tag{3.25}$$

which is a family of second-order differential equations in ρ, the Fourier variable $\xi \in \mathbb{R}$ playing the role of a parameter. Observe that $\hat{\chi}$ is real-valued and (3.25i) contains a singular coefficient at the "initial time" $\rho = 0$.

Step 1: Equation for the remainder function $\hat{g}(\rho,\xi)$. Note that, in (3.21),

$$\begin{aligned} \hat{\chi}^\sharp(\rho,\xi) &= a_\sharp(\rho)k(\rho)^{2\lambda+1} \int \cos(k(\rho)\xi z)\,\big[1 - z^2\big]_+^\lambda\,dz \\ &=: \alpha_\sharp(\rho)\,\hat{f}_\lambda(k(\rho)\xi), \quad \text{with} \;\; \alpha_\sharp(\rho) = a_\sharp(\rho)k(\rho)^{2\lambda+1}. \end{aligned}$$

Similarly, we have

$$\hat{\chi}^\flat(\rho,\xi) = \alpha_\flat(\rho)\,\hat{f}_{\lambda+1}(k(\rho)\xi), \quad \text{with} \quad \alpha_\flat(\rho) = a_\flat(\rho)\,k(\rho)^{2\lambda+3}.$$

Using the identities $\hat{f}_\lambda(y) + \hat{f}_\lambda''(y) = \hat{f}_{\lambda+1}(y) = -\frac{2(\lambda+1)}{y}\hat{f}_\lambda'(y)$, we obtain

$$\hat{\chi}^\sharp_{\rho\rho} + k'(\rho)^2\xi^2\hat{\chi}^\sharp = \alpha_\sharp''(\rho)\,\hat{f}_\lambda(k(\rho)\xi) \tag{3.26}$$

provided that

$$\frac{\alpha_\sharp'(\rho)}{\alpha_\sharp(\rho)} = (\lambda+1)\frac{k'(\rho)}{k(\rho)} - \frac{k''(\rho)}{2k'(\rho)}.$$

Clearly $a_\sharp(\rho)$ determined by (2.9) satisfies the equation, and the constant of normalization M_λ given in (2.11) is chosen to ensure that (3.25iii) holds.

Similarly, we get

$$\begin{aligned}\hat{\chi}^\flat_{\rho\rho} + k'(\rho)^2\xi^2\hat{\chi}^\flat =& \left(\alpha_\flat'' - \frac{(2\lambda+3)}{k}\left(2k'\alpha_\flat' - 2(\lambda+2)\frac{k'^2}{k}\alpha_\flat + k''\alpha_\flat\right)\right)\hat{f}_{\lambda+1}\\ &+ \frac{2(\lambda+1)}{k}\left(2k'\alpha_\flat' - 2(\lambda+2)\frac{k'^2}{k}\alpha_\flat + k''\alpha_\flat\right)\hat{f}_\lambda, \end{aligned} \quad (3.27)$$

where we used the identity $\hat{f}'_{\lambda+1}(y) = -\frac{2\lambda+3}{y}\hat{f}_{\lambda+1}(y) + \frac{2(\lambda+1)}{y}\hat{f}_\lambda(y)$.

We obtain the following equation from (3.26) and (3.27) for $\hat{g}$:

$$\begin{aligned}\hat{g}_{\rho\rho} + k'(\rho)^2\xi^2\hat{g} =& -\left(\alpha_\flat'' - \frac{2\lambda+3}{k}\left(2k'\alpha_\flat' - 2(\lambda+2)\frac{k'^2}{k}\alpha_\flat + k''\alpha_\flat\right)\right)\hat{f}_{\lambda+1}\\ &- \left(\alpha_\sharp'' + \frac{2(\lambda+1)}{k}\left(2k'\alpha_\flat' - 2(\lambda+2)\frac{k'^2}{k}\alpha_\flat + k''\alpha_\flat\right)\right)\hat{f}_\lambda\\ =& -\left(\alpha_\flat'' + \frac{2\lambda+3}{2(\lambda+1)}\alpha_\sharp''\right)\hat{f}_{\lambda+1} \equiv A(\rho)\hat{f}_{\lambda+1}, \end{aligned} \quad (3.28)$$

provided that

$$\alpha_\flat'(\rho) + \left(-(\lambda+2)\frac{k'(\rho)}{k(\rho)} + \frac{k''(\rho)}{2k'(\rho)}\right)\alpha_\flat(\rho) = -\frac{k(\rho)}{4(\lambda+1)k'(\rho)}\alpha_\sharp''(\rho). \quad (3.29)$$

We choose $\alpha_\flat(\rho)$ to be the less singular solution to this singular equation, that is,

$$\alpha_\flat(\rho) = -\frac{1}{4(\lambda+1)}k(\rho)^{\lambda+2}k'(\rho)^{-1/2}\int_0^\rho k(\tau)^{-(\lambda+1)}k'(\tau)^{-1/2}\alpha_\sharp''(\tau)\,d\tau.$$

Note that

$$|\alpha_\sharp''(\rho)| + \rho|\alpha_\sharp'''(\rho)| + |\alpha_\flat''(\rho)| + |\alpha_\flat'''(\rho)| \leqq C.$$

Therefore, $\hat{g}$ satisfies

$$\hat{g}_{\rho\rho} + k'(\rho)^2\xi^2\hat{g} = A(\rho)\hat{f}_{\lambda+1}(k(\rho)\xi), \quad (3.30)$$

where

$$|A(\rho)| + \rho|A'(\rho)| \leqq C. \quad (3.31)$$

Step 2: Existence of the entropy kernel and estimates for $\hat{g}(\rho,\xi)$. For every $\varepsilon > 0$ and $\xi \in \mathbb{R}$, we consider (3.30) with

$$\hat{g}^\varepsilon(\varepsilon,\xi) = 0, \qquad \hat{g}^\varepsilon_\rho(\varepsilon,\xi) = 0.$$

This problem admits a smooth solution $\hat{g}^\varepsilon$ defined for $\rho \geqq \varepsilon$.

Using Lemma 3.3 with $p = 0$ and $q = 2\lambda + 1$, we have

$$|\partial_\rho^m \hat{g}^\varepsilon(\rho, \xi)| \leqq \frac{C\rho^{2-m}}{(1 + |k(\rho)\xi|)^{\lambda+\lambda_0+2-m}}, \qquad m = 0, 1, 2, \tag{3.32}$$

where $C > 0$ is a constant independent of $\varepsilon > 0$.

By the Cauchy-Arzela theorem, it follows from (3.32) that, as $\varepsilon \to 0$, the functions $\hat{g}^\varepsilon(\rho, \xi)$ converge uniformly to a limiting function $\hat{g}(\rho, \xi)$ that is a solution of (3.30) (on every compact subset of $\{\rho \geqq 0\}$) with the initial data:

$$\hat{g}(0, \xi) = 0, \qquad \hat{g}_\rho(0, \xi) = 0. \tag{3.33}$$

Moreover, $\hat{g}$ satisfies

$$|\partial_\rho^m \hat{g}(\rho, \xi)| \leqq \frac{C\rho^{2-m}}{(1 + |k(\rho)\xi|)^{\lambda+\lambda_0+2-m}}, \qquad m = 0, 1, 2. \tag{3.34}$$

In particular, $\hat{g}(\cdot, \xi)$ and $\partial_\rho \hat{g}(\cdot, \xi)$ are continuous at $\rho = 0$, uniformly in all ξ. This shows that the initial conditions (3.33) are satisfied in a classical sense.

This completes the proof for the existence and asymptotic behavior of $\hat{g}$, as a function of $\rho \geqq 0$, in which $\xi \in \mathbb{R}$ plays the role of a parameter. The uniqueness of $\hat{g}$ follows easily from the energy estimates derived in Lemma 3.2, by using $\varepsilon = 0$, $q = 0$, and $r = 0$. Then, using the inverse Fourier transform, we conclude that there exists a solution $\chi(\rho, v)$ of problem (3.24) understood in the sense of (2.6) and defined globally.

Step 3: Hölder continuity of χ. It suffices to show that there exists $\delta > 0$ such that

$$|\partial_\rho^\delta \partial_v^{\lambda+1+\delta} g(\rho, v)| \leqq C, \tag{3.35}$$

which implies that $\partial_v^{\lambda+1} g \in C^{0,\delta}(\mathbb{R}_+^2)$. In turn, since $\chi \equiv 0$ outside the region $\mathcal{K}$, (3.35) gives (2.10). Estimate (3.35) is proved as follows:

$$|\partial_\rho^\delta \partial_v^{\lambda+1+\delta} g(\rho, v)| \leqq C \left(1 + \int_{|\xi| \geqq 1} |\xi|^{2\lambda+1+\delta} |\partial_\rho^\delta \hat{g}(\rho, \xi)|^2 d\xi\right)^{1/2}.$$

Since $\partial_\rho^\delta \hat{g}(0, \xi) = 0$, we can extend $\partial_\rho^\delta \hat{g}(\rho, \xi)$ to the half-space $\rho \leqq 0$ by simply setting

$$\partial_\rho^\delta \hat{g}(\rho, \xi) \equiv 0, \qquad \rho \leqq 0.$$

Then we obtain

$$\begin{aligned} |\partial_\rho^\delta \hat{g}(\rho, \xi)|^2 &= C \left| \int |\tau|^\delta \hat{\hat{g}}(\tau, \xi) e^{-i\rho\tau} \, d\tau \right|^2 \leqq C \int |\tau|^{2\delta} |\hat{\hat{g}}(\tau, \xi)|^2 \, d\tau \\ &\leqq C \left(\int |\hat{g}(\tau, \xi)|^2 \, d\tau \right)^{1-\delta} \left(\int |\partial_\tau \hat{g}(\tau, \xi)|^2 \, d\tau \right)^\delta, \end{aligned}$$

where we have used the Parseval identity and the interpolation inequality

$$\int |\tau|^{2\delta} |f(\tau)|^2 \, d\tau \leqq \left(\int |f(\tau)|^2 \, d\tau \right)^{1-\delta} \left(\int |\tau f(\tau)|^2 \, d\tau \right)^{\delta}.$$

On the other hand, for $|\xi| \geqq 1$,

$$\begin{aligned}\int |\hat{g}(\tau,\xi)|^2 \, d\tau &\leqq C \left(\int_0^{|\xi|^{-1/\theta}} + \int_{|\xi|^{-1/\theta}}^{\rho_M} \right) \frac{\tau^4}{(1+|k(\tau)\xi|)^{2(\lambda+\lambda_0+2)}} \, d\tau \\ &\leqq C|\xi|^{-2(\lambda+\lambda_0+2)},\end{aligned}$$

where we have used $2(\lambda+\lambda_0+2) \leqq 5(2\lambda+1)$. Similarly, we obtain

$$\begin{aligned}\int |\partial_\tau \hat{g}(\tau,\xi)|^2 \, d\tau &\leqq C \left(\int_0^{|\xi|^{-1/\theta}} + \int_{|\xi|^{-1/\theta}}^{\rho_M} \right) \frac{\tau^2}{(1+|k(\tau)\xi|)^{2(\lambda+\lambda_0+1)}} \, d\tau \\ &\leqq C|\xi|^{-2(\lambda+\lambda_0+1)}\end{aligned}$$

when we use $2(\lambda+\lambda_0+1) \leqq 3(2\lambda+1)$. Therefore, we have

$$\begin{aligned}|\partial_\rho^\delta \hat{g}(\rho,\xi)|^2 &\leqq \left(\int |\hat{g}(\tau,\xi)|^2 \, d\tau \right)^{1-\delta} \left(\int |\partial_\tau \hat{g}(\tau,\xi)|^2 \, d\tau \right)^{\delta} \\ &\leqq C|\xi|^{-2(\lambda+\lambda_0+2-\delta)}.\end{aligned}$$

Then we obtain

$$|\partial_\rho^\delta \partial_v^{\lambda+1+\delta} g(\rho,v)| \leqq C \left(1 + \int_{|\xi| \geqq 1} |\xi|^{-2\lambda_0-3+3\delta} d\xi \right)^{1/2} \leqq C,$$

provided that $\delta < \min\left(2(1+\lambda_0)/3, 1\right)$.

Step 4: Uniqueness of χ. We proved in Steps 1–3 that the Cauchy problem (3.24) admits a global solution $\chi \in C^{0,\delta}(\mathbb{R}^2_+)$ in the sense of (2.6). For any two solutions χ_1, χ_2 of problem (3.24), the function $\chi = \chi_1 - \chi_2$ satisfies

$$\iint \chi \left(\varphi_{\rho\rho} - k'(\rho)^2 \varphi_{vv} \right) d\rho \, dv = 0 \tag{3.36}$$

for any function $\varphi \in C_0^\infty(\mathbb{R}^2_+)$. By approximation, (3.36) also holds for any $\varphi \in C_0^{0,1}(\mathbb{R}^2_+) \cap W^{2,p}(\mathbb{R}^2_+)$ for some values of p such that $1 \leqq p < \infty$.

For any $\psi \in C_0^\infty(\mathbb{R}^2)$, consider the problem

$$\begin{aligned}\varphi_{\rho\rho} - k'(\rho)^2 \varphi_{vv} = \psi, \qquad & \rho \leqq \rho_M, \\ \varphi(\rho_M,\xi) = 0, \qquad & \varphi_\rho(\rho_M,\xi) = 0,\end{aligned} \tag{3.37}$$

where $\rho_M > 0$ such that $\psi|_{\rho > \rho_M} \equiv 0$.

Basing our arguments on those used in proving the existence of g from the energy estimates in Lemmas 3.2 and 3.3, we can also conclude from Lemma 3.4

that there exists a global solution $\varphi \in C_0^{0,1}(\mathbb{R}_+^2) \cap W^{2,p}(\mathbb{R}_+^2)$ in $\rho < \rho_M$ for $p \in [1, 1 + \frac{1}{2\lambda})$. This is checked as follows.

For any function $\psi(\rho, v)$ with $\operatorname{supp}\psi \subset (0,\infty) \times \mathbb{R}$, we have

$$|\hat{\psi}(\rho,\xi)| \leqq \frac{C\rho^m}{(1+|\xi|)^m}$$

for any $m > 0$. Then, from Lemma 3.4, we have

$$|\hat{\varphi}_\rho|^2 + k'(\rho)^2\xi^2|\hat{\varphi}|^2 \leqq C\int_\rho^{\rho_M} |\hat{\psi}|^2\,d\tau \leqq \frac{C}{(1+|\xi|)^{2m}}.$$

This means

$$|\hat{\varphi}(\rho,\xi)| \leqq \frac{C}{k'(\rho)|\xi|(1+|\xi|)^m}.$$

Then, from the equation,

$$|\hat{\varphi}_{\rho\rho}(\rho,\xi)| \leqq C\,|\hat{\psi}(\rho,\xi)| + C\,k'(\rho)^2\xi^2|\hat{\varphi}(\rho,\xi)| \leqq C\,\frac{1+\rho^{\theta-1}}{(1+|\xi|)^{m-1}}.$$

This implies that $\varphi \in C^{0,1}(\mathbb{R}_+^2) \cap W^{2,p}(\mathbb{R}_+^2)$ for $p \in [1, 1+\frac{1}{2\lambda})$. Then (3.36) holds for such functions. We have

$$\iint \chi(\rho,v)\psi(\rho,v)\,d\rho\,dv = 0,$$

for any $\psi \in C_0^\infty(\mathbb{R}_+^2)$, which implies $\chi(\rho,v) \equiv 0$.

Step 5: Compact support and positivity of χ. Problem (2.5) is hyperbolic, so the principle of propagation with finite speed applies: χ is identically zero outside the domain of dependence, $\mathcal{K} = \big\{(\rho,v) \mid |v| \leqq k(\rho)\big\}$, of the support of the initial data, i.e. the point $(\rho,v) = (0,0)$. Therefore, $\operatorname{supp}\chi \subset \mathcal{K}$ (this can be also checked from (3.38) below). We focus on the main issue that χ is *strictly* positive in $\mathcal{K}$.

Claim. For any $(\rho_0, v_0) \in \mathcal{K}$, we have

$$\begin{aligned}\chi(\rho_0,v_0) = {}& \frac{1}{2\,\rho_0 k'(\rho_0)}\int_0^{\rho_0} k'(\rho)\,d(\rho)\\ &\times\Big\{\chi\big(\rho, v_0 + k(\rho_0) - k(\rho)\big) + \chi\big(\rho, v_0 - k(\rho_0) - k(\rho)\big)\Big\}\,d\rho,\end{aligned} \tag{3.38}$$

where $d(\rho) := \frac{\rho p''(\rho) + 2\,p'(\rho)}{p'(\rho)} > 0$.

We deduce from the equation $\rho\,k'^2\,\hat{\chi} = -\rho\,\xi^{-2}\,\hat{\chi}_{\rho\rho}$ that

$$\begin{aligned}&\int_0^{\rho_0} \rho k'(\rho)^2 \sin\big((k(\rho)-k(\rho_0))\,\xi\big)\hat{\chi}(\rho,\xi)\,d\rho\\ &= -\int_0^{\rho_0} \rho\xi^{-2}\,\sin\big((k(\rho)-k(\rho_0))\,\xi\big)\hat{\chi}_{\rho\rho}(\rho,\xi)\,d\rho\\ &= \xi^{-2}\Big[\big\{\sin\big((k(\rho)-k(\rho_0))\,\xi\big) + \rho\,k'(\rho)\xi\,\cos\big((k(\rho)-k(\rho_0))\,\xi\big)\big\}\,\hat{\chi}(\rho,\xi)\Big]_0^{\rho_0}\\ &\quad - \xi^{-1}\int_0^{\rho_0} \big(k'(\rho) + (\rho k'(\rho))'\big)\,\cos\big((k(\rho)-k(\rho_0))\,\xi\big)\hat{\chi}(\rho,\xi)\,d\rho\\ &\quad + \int_0^{\rho_0} \rho^2\,k'(\rho)^2\,\sin\big((k(\rho)-k(\rho_0))\,\xi\big)\hat{\chi}(\rho,\xi)\,d\rho,\end{aligned}$$

where we have used integration by parts and the initial conditions on $\hat{\chi}$. Thus we obtain

$$\rho_0 k'(\rho_0)\,\hat{\chi}(\rho_0,\xi) = \int_0^{\rho_0} \big(k'(\rho) + (\rho k'(\rho))'\big)\,\cos\big((k(\rho)-k(\rho_0))\,\xi\big)\,\hat{\chi}(\rho,\xi)\,d\rho.$$

The desired formula follows by the inverse Fourier transform since, for instance,

$$\begin{aligned}&\int \cos\big((k(\rho)-k(\rho_0))\,\xi\big)\,\hat{\chi}(\rho,\xi)\,e^{i\,v\xi}\,d\xi\\ &\quad= \tfrac{1}{2}\int \hat{\chi}(\rho,\xi)\,e^{i\,(v+k(\rho)-k(\rho_0))\xi}\,d\xi + \tfrac{1}{2}\int \hat{\chi}(\rho,\xi)\,e^{i\,(v-k(\rho)+k(\rho_0))\xi}\,d\xi.\end{aligned}$$

This establishes the claim.

Next we recall that, by (2.8)–(2.10),

$$\chi(\rho,v) \geqq a_\sharp\,G_\lambda(\rho,v)\,\left(1 - C\rho\left[1-\frac{v^2}{k(\rho)^2}\right]_+\right).$$

Therefore, there exists $\tilde{\rho}>0$ such that, when $\rho\in(0,\tilde{\rho}]$, $\chi(\rho,v)\geqq\frac{1}{2}M_\lambda G_\lambda(\rho,v)$, which implies

$$\chi\,|_{\mathcal{K}\cap\{0\leqq\rho\leqq\tilde{\rho}\}} > 0.$$

Finally, we check that $\chi>0$ in the interior of $\mathcal{K}$ for all $\rho>0$, relying here on the maximal principle for hyperbolic equations. For contradiction, assume that $(\rho_0,v_0)\in\mathcal{K}$ is the first point where χ vanishes when ρ increases. Then identity (3.38) implies that

$$\chi(\rho_0,v_0)>0,$$

since the integrand in the right-hand side is positive for all $\rho\in(0,\tilde{\rho})$. This is a contradiction.

This completes the proof of Theorems 2.1 and 2.2 for the entropy kernel. The same arguments apply to the entropy flux σ and yield Theorems 2.1 and 2.3. □

4. Compactness Framework

In this section we consider a family of approximate solutions $(\rho^\varepsilon(t,x), m^\varepsilon(t,x))$ of (1.1) and derive a sufficient condition of its strong compactness.

Theorem 4.1 (Compactness framework**).** *Let $(\rho^\varepsilon, m^\varepsilon)$ be measurable functions such that*

$$0\leqq\rho^\varepsilon(t,x)\leqq C,\quad |m^\varepsilon(t,x)|\leqq C\,\rho^\varepsilon(t,x)\qquad a.e. \tag{4.1}$$

for some $C>0$. Assume that

$$\partial_t\eta(\rho^\varepsilon,m^\varepsilon)+\partial_x q(\rho^\varepsilon,m^\varepsilon)\ \text{ is compact in } H^{-1}_{\text{loc}}(\mathbb{R}^2_+) \tag{4.2}$$

for any weak entropy pair (η,q). Then there exists a function (ρ,m) such that

$$0\leqq\rho(t,x)\leqq C,\quad |m(t,x)|\leqq C\,\rho(t,x)\qquad a.e.$$

and, extracting a subsequence if necessary,

$$(\rho^\varepsilon(t,x), m^\varepsilon(t,x)) \to (\rho(t,x), m(t,x)) \quad \textit{in } L^r_{\mathrm{loc}}(\mathbb{R}^2_+) \textit{ for all } r \in [1,\infty).$$

Denote by $\nu = \nu_{(t,x)}(\rho, v)$ a Young measure associated with the sequence $(\rho^\varepsilon, v^\varepsilon)$. Here $v^\varepsilon := m^\varepsilon/\rho^\varepsilon$ for $\rho^\varepsilon > 0$. By the div-curl lemma [26], condition (4.2) implies that ν satisfies Tartar's commutation relations. To conclude with the strong convergence of the sequence and establish Theorem 4.1, we need the following theorem.

Theorem 4.2 (Reduction of the support of ν)**.** *Let $\nu(\rho, v)$ be a probability measure with bounded support in $\{\rho \geqq 0, v \in \mathbb{R}\}$ such that*

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle \tag{4.3}$$

for any two weak entropy pairs (η_1, q_1) and (η_2, q_2) of (1.1)*. Then the support of ν in the (ρ, v)-plane is either a single point or a subset of the vacuum line $\{\rho = 0\}$.*

In the proof of Theorem 4.2, we will use the following lemma.

Lemma 4.1. *Suppose that the Young measure has a non-trivial support away from the vacuum line, i.e.,* $\operatorname{supp} \nu \cap \{w > z\} \neq \emptyset$. *Let*

$$\{(w,z) \mid z_{\min} \leqq z \leqq w \leqq w_{\max}\} \tag{4.4}$$

be the smallest triangle containing the support of ν in the (w,z)-plane. Then its vertex $(w_{\max}, z_{\min})$ belongs to $\operatorname{supp} \nu$.

Theorem 4.2 is based on the *cancellation properties* summarized in Lemmas 4.2 and 4.3. Near to the diagonal $\{s_2 = s_3\}$, the function

$$E(\rho, v; s_2, s_3) := \chi(\rho, v - s_2)\,\sigma(\rho, v, s_3) - \chi(\rho, v - s_3)\,\sigma(\rho, v, s_2) \tag{4.5}$$

turns out to be much more regular than χ and σ themselves. For each $j = 2, 3$, consider a mollifying sequence $\varphi_j^\varepsilon(s_j) = \varepsilon^{-1}\varphi_j(s_j/\varepsilon)$, where the mollifier φ_j satisfies

$$\varphi_j(s_j) \geqq 0, \qquad \int_{\mathbb{R}} \varphi_j(s_j)\,ds_j = 1, \qquad \operatorname{supp} \varphi_j(s_j) \subset (-1, 1). \tag{4.6}$$

Set

$$\chi_j^\varepsilon(s_1) := (\chi \star \varphi_j^\varepsilon)(s_1), \qquad \sigma_j^\varepsilon(s_1) := (\sigma \star \varphi_j^\varepsilon)(s_1).$$

Consider the differential operator $P := \partial_s^{\lambda+1}$ and set $P_j := \partial_{s_j}^{\lambda+1}$.

Lemma 4.2 (Cancellation of singularities I)**.** *For $j = 2, 3$, the functions $\chi_1 P_j \sigma_j^\varepsilon - \sigma_1 P_j \chi_j^\varepsilon$ are Hölder continuous in (ρ, v, s_1), uniformly in ε. Also there exists a continuous function $X_1 = X(\rho, v, s_1)$, independent of the mollifying sequence φ_j, such that, when $\varepsilon \to 0$,*

$$\chi_1 P_j \sigma_j^\varepsilon - \sigma_1 P_j \chi_j^\varepsilon \to X_1 \qquad \textit{uniformly in } (\rho, v, s_1). \tag{4.7}$$

Lemma 4.3 (Cancellation of singularities II**).** *The functions* $P_2\chi_2^\varepsilon\, P_3\sigma_3^\varepsilon - P_3\chi_3^\varepsilon\, P_2\sigma_2^\varepsilon$ *are uniformly bounded measures and,when* $\varepsilon \to 0$,

$$P_2\chi_2^\varepsilon\, P_3\sigma_3^\varepsilon - P_3\chi_3^\varepsilon\, P_2\sigma_2^\varepsilon \rightharpoonup Y(\varphi_2,\varphi_3)\, Z(\rho) \sum_{\pm} \big(K^\pm\big)^2 \delta_{s_1=v\pm k(\rho)} \tag{4.8}$$

weakly-star in measures in s_1 *and uniformly in* (ρ, v)*, where*

$$Y(\varphi_2,\varphi_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_2} \big(\varphi_2(s_2)\,\varphi_3(s_3) - \varphi_3(s_2)\,\varphi_2(s_3)\big)\, ds_3\, ds_2,$$

and

$$Z(\rho) := (\lambda+1)\, M_\lambda^{-2}\, k(\rho)^{2\lambda} D(\rho),$$

where $D(\rho)$ *was introduced in Proposition 2.5.*

In other words, we have

$$\int_{-\infty}^{\infty} \big(P_2\chi_2^\varepsilon\, P_3\sigma_3^\varepsilon - P_3\chi_3^\varepsilon\, P_2\sigma_2^\varepsilon\big)\, \psi(s_1)\, ds_1$$
$$\to Y(\varphi_2,\varphi_3)\, Z(\rho) \sum_{\pm} \big(K^\pm\big)^2 \psi(s_1 - v \mp k(\rho))$$

uniformly in (ρ, v) for every test-function $\psi = \psi(s_1)$.

Remarks. (1) The limit X_1 in (4.7) is continuous, so is twice as regular as $\partial_s^{\lambda+1}\chi$ and $\partial_s^{\lambda+1}\sigma$. The singularities of the kernels cancel because σ and χ vanish on the singularities of $\partial_s^{\lambda+1}\chi$ and $\partial_s^{\lambda+1}\sigma$ respectively, so that the corresponding products are bounded functions in s, rather than measures. Furthermore, E has a symmetric form which provides further cancellation. The function

$$\chi\, \partial_s^{\lambda+1}\sigma \;-\; \sigma\, \partial_s^{\lambda+1}\chi$$

can be regarded as a Hölder continuous function of (ρ, v, s).

(2) The term treated in (4.8) is a product of measures. Expanding χ and σ and relying on the symmetry property of E, we obtain only the functions of bounded variation multiplied by measures. Such products depend upon regularization, as was pointed out by Dal Maso, LeFloch & Murat [10]; see Lemma 4.4 below.

Now we prove Theorem 4.2 and Lemmas 4.1–4.3.

Proof of Theorem 4.2. A general formula of the entropy pairs was derived in Section 2. Plugging the entropy-entropy flux pairs with the formulae in Corollary 2.1 into relations (4.3) and dropping the test-function ψ, we obtain

$$\big\langle \chi(s_1)\,\sigma(s_2) - \chi(s_2)\,\sigma(s_1)\big\rangle = \big\langle \chi(s_1)\big\rangle\big\langle \sigma(s_2)\big\rangle - \big\langle \chi(s_2)\big\rangle\big\langle \sigma(s_1)\big\rangle \tag{4.9}$$

for all $s_1, s_2 \in \mathbb{R}^n$. For simplicity, we set $\langle \chi_i\rangle = \langle \chi(s_i)\rangle = \langle \nu_{(t,x)}, \chi(s_i)\rangle$.

Given $s_1, s_2, s_3 \in \mathbb{R}$, consider (4.9) for the pairs

$$(s_2, s_3), \quad (s_3, s_1), \quad (s_1, s_2).$$

Multiply each identity by

$$\langle \chi(s_1)\rangle, \quad \langle \chi(s_2)\rangle, \quad \langle \chi(s_3)\rangle$$

respectively, and add them up. By *symmetry*, the sum of the right-hand side vanishes identically:

$$\begin{aligned}\langle\chi_1\rangle\big(\langle\chi_2\rangle\langle\sigma_3\rangle-\langle\chi_3\rangle\langle\sigma_2\rangle\big)+\langle\chi_2\rangle\big(\langle\chi_3\rangle\langle\sigma_1\rangle-\langle\chi_1\rangle\langle\sigma_3\rangle\big)&\\ +\langle\chi_3\rangle\big(\langle\chi_1\rangle\langle\sigma_2\rangle-\langle\chi_2\rangle\langle\sigma_1\rangle\big)&=0,\end{aligned}$$

whereas the sum of the left-hand side is

$$\langle\chi_1\rangle\langle\chi_2\,\sigma_3-\chi_3\,\sigma_2\rangle+\langle\chi_2\rangle\langle\chi_3\,\sigma_1-\chi_1\,\sigma_3\rangle+\langle\chi_3\rangle\langle\chi_1\,\sigma_2-\chi_2\,\sigma_1\rangle=0. \quad (4.10)$$

Using the differential operator $P_2P_3 := \partial_{s_2}^{\lambda+1}\partial_{s_3}^{\lambda+1}$, we deduce from (4.10) that

$$\begin{aligned}\langle\chi_1\rangle\langle P_2\chi_2\,P_3\sigma_3-P_3\chi_3\,P_2\sigma_2\rangle+\langle P_2\chi_2\rangle\langle\sigma_1\,P_3\chi_3-\chi_1\,P_3\sigma_3\rangle&\\ +\langle P_3\chi_3\rangle\langle\chi_1\,P_2\sigma_2-\sigma_1\,P_2\chi_2\rangle&=0\end{aligned} \quad (4.11)$$

in the sense of distributions in s_1, s_2, s_3. For instance, the distribution $< P\chi >$ is defined by

$$\big(\langle P\chi\rangle,\psi\big) := -\Big\langle\int_{\mathbb{R}}\partial_s^{\lambda}\chi(s)\,\psi'(s)\,ds\Big\rangle$$

for any test-function ψ. Recall from Section 2 that $\partial_s^{\lambda}\chi$ is bounded in s and continuous in (ρ, v).

Our goal is to let s_2 and s_3 tend to s_1 in (4.11). For each $j = 2, 3$, consider a mollifying sequence $\varphi_j^{\varepsilon}(s_j) = \varepsilon^{-1}\varphi_j(s_j/\varepsilon)$ satisfying (4.6). From (4.11), we obtain

$$\begin{aligned}&\langle\chi_1\rangle\langle P_2\chi_2^{\varepsilon}\,P_3\sigma_3^{\varepsilon}-P_3\chi_3^{\varepsilon}\,P_2\sigma_2^{\varepsilon}\rangle\\ &\quad=\langle P_2\chi_2^{\varepsilon}\rangle\langle\chi_1\,P_3\sigma_3^{\varepsilon}-\sigma_1\,P_3\chi_3^{\varepsilon}\rangle-\langle P_3\chi_3^{\varepsilon}\rangle\langle\chi_1\,P_2\sigma_2^{\varepsilon}-\sigma_1\,P_2\chi_2^{\varepsilon}\rangle,\end{aligned} \quad (4.12)$$

in which each term is a continuous function of s_1. We now prove that, as $\varepsilon \to 0$, the right-hand side of (4.12) tends to zero, while the left-hand side converges to a non-zero limit, when the functions φ_j^{ε} are suitably chosen.

First, consider the right-hand side of (4.12). Since $P_j\chi_j$ is a bounded measure in s_j, we have

$$P_j\chi_j^{\varepsilon} = P_j\chi_j \star \varphi_j^{\varepsilon} \rightharpoonup P_1\chi_1 \quad (4.13)$$

weakly in measures in s_1 and uniformly in (ρ, v). In particular, by Fubini's theorem, we have

$$\langle P_j\chi_j^{\varepsilon}\rangle \to \langle P_1\chi_1\rangle$$

weakly in measures in s_1. Hence, using the convergence property (4.7) in Lemma 4.2, we arrive at

$$\begin{aligned}&\langle P_2\chi_2^{\varepsilon}\rangle\langle\chi_1P_3\sigma_3^{\varepsilon}-\sigma_1P_3\chi_3^{\varepsilon}\rangle-\langle P_3\chi_3^{\varepsilon}\rangle\langle\chi_1P_2\sigma_2^{\varepsilon}-\sigma_1P_2\chi_2^{\varepsilon}\rangle\\ &\qquad\to\langle X_1\rangle\langle P_1\chi_1\rangle-\langle X_1\rangle\langle P_1\chi_1\rangle\equiv 0\end{aligned} \quad (4.14)$$

weakly in measures in s_1. This shows that the right-hand side of (4.12) converges to zero.

By Lemma 4.3, the left-hand side of (4.12) satisfies

$$\begin{aligned}\langle\chi_1\rangle\langle P_2\chi_2^\varepsilon P_3\sigma_3^\varepsilon - P_3\chi_3^\varepsilon P_2\sigma_2^\varepsilon\rangle &\to \langle\chi_1\rangle\langle Y(\varphi_2,\varphi_3)\, Z(\rho) \sum_\pm \big(K^\pm\big)^2 \delta_{s_1=v\pm k(\rho)}\rangle \\ &= \langle\chi_1\rangle Y(\varphi_2,\varphi_3) \sum_\pm \big(K^\pm\big)^2 \langle Z(\rho)\delta_{s_1=v\pm k(\rho)}\rangle.\end{aligned} \tag{4.15}$$

We conclude that, for every test-function $\psi_1 = \psi(s_1)$,

$$Y(\varphi_2,\varphi_3) \sum_\pm \big(K^\pm\big)^2 \iint \langle\chi_1(v \pm k(\rho))\rangle Z(\rho)\, \psi(v \pm k(\rho))\, d\nu(\rho, v) = 0. \tag{4.16}$$

Choose the mollifying functions in such a way that

$$Y(\varphi_2,\varphi_3) \neq 0.$$

Such functions exist: for instance, choose $\varphi_2 \geqq 0$ with unit total mass, and set $\varphi_3(s_3) = \varphi_2(s_3 - \bar{s})$ for a fixed $\bar{s} \neq 0$. Observe that the trivial choice $\varphi_2 = \varphi_3$ does not work and the regularization in (s_2, s_3) should therefore be *asymmetric*.

Choose the compactly supported test-function ψ to be identically equal to 1 on the support of ν. Then

$$\langle\langle\chi(v \pm k(\rho))\rangle\, Z(\rho)\rangle = 0$$

for every test-function ψ, or equivalently,

$$\langle\langle\chi(w)\rangle\, Z(\rho)\rangle = \langle\langle\chi(z)\rangle\, Z(\rho)\rangle = 0,$$

where we regard $\nu = \nu(w, z)$ and $\rho = k^{-1}(\frac{w-z}{2})$.

Assume that $\operatorname{supp}\nu$ is not included in the vacuum line. Observe that the interior of the support of the nonnegative function $(w, z) \mapsto \chi(\rho, v - s)$ has a nonempty intersection with an open neighborhood of the point $(w_{\max}, z_{\min})$. Moreover, $Z(\rho) > 0$ for $\rho > 0$ by Proposition 2.5. Therefore, by Lemma 4.1,

$$\langle\chi(s)\rangle > 0, \qquad \text{for all } s \text{ in the open interval } (z_{\min}, w_{\max}).$$

It follows that

$$\operatorname{supp}\nu = \{w = z\} \cup \{(w_{\max}, z_{\min})\}.$$

Then, set

$$\nu = \tilde{\nu} + \omega\, \delta_{(w_{\max}, z_{\min})},$$

where ω represents the mass of the measure ν at the extremal point and $\operatorname{supp}\tilde{\nu} \subset \{w = z\}$. Returning to (4.9), we obtain, for all s_1, s_2,

$$(\omega - \omega^2)\,\{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)\} = 0,$$

where the functions are evaluated at the point $(w_{\max}, z_{\min})$. Therefore, either $\omega = 0$ ($\operatorname{supp}\nu \subset \{w = z\}$) or $\omega = 1$ ($\operatorname{supp}\nu = \{(w_{\max}, z_{\min})\}$). This completes the proof of Theorem 4.2. □

Proof of Lemma 4.1. For contradiction, assume that the point $(w_{\max}, z_{\min})$ does not belong to the support, i.e.,

$$\operatorname{supp} \nu \cap \big[w_{\max} - \alpha, w_{\max}\big] \times \big[z_{\min,} z_{\min} + \alpha\big] = \emptyset \tag{4.19}$$

for some $\alpha > 0$. Consider the commutation relation (4.9) in the form

$$\frac{\big\langle \chi(s_1)\,\sigma(s_2) - \chi(s_2)\,\sigma(s_1)\big\rangle}{\big\langle \chi(s_1)\big\rangle\big\langle \chi(s_2)\big\rangle} = \frac{\big\langle \sigma(s_2)\big\rangle}{\big\langle \chi(s_2)\big\rangle} - \frac{\big\langle \sigma(s_1)\big\rangle}{\big\langle \chi(s_1)\big\rangle}. \tag{4.20}$$

Set

$$s_- := z_{\min}, \quad s_+ := z_{\max}.$$

By (4.19), for $0 < s_+ - s_2 < \alpha$ and $0 < s_1 - s_- < \alpha$, the supports of $s_1 \mapsto \chi(s_1)$ and $s_2 \mapsto \sigma(s_2)$ are disjoint. The same is true for $\chi(s_2)$ and $\sigma(s_1)$. Therefore, the left-hand side of (4.20) vanishes identically.

Since $b_\sharp = \rho k' a_\sharp / k$, by (2.8) and (2.12), the entropy flux has the form

$$\sigma(\rho, v, s) = \big(v - (v - s)\, c\, k^{-1}\big)\chi(\rho, v - s) + \tilde{h}(\rho, v - s),$$

where $\tilde{h}$ satisfies (see (2.10) and (2.14))

$$|\tilde{h}(\rho, v - s)| \leqq C\,|k(\rho)^2 - (v - s)^2|\,\chi(\rho, v - s).$$

Thus

$$\sigma - \big(v \pm c\big)\chi = \big(\mp k + (v - s)\big) c\, k^{-1} \chi + \tilde{h}. \tag{4.21}$$

Therefore, we have

$$\frac{\big\langle \sigma(s)\big\rangle}{\big\langle \chi(s)\big\rangle} = \frac{\big\langle \big(v \pm c\big)\chi(s)\big\rangle}{\big\langle \chi(s)\big\rangle} + \frac{\big\langle \big(\mp k + (v - s)\big) c\, k^{-1} \chi(s)\big\rangle}{\big\langle \chi(s)\big\rangle} + \frac{\big\langle \tilde{h}(s)\big\rangle}{\big\langle \chi(s)\big\rangle}.$$

Define the trace measure μ_+ by

$$\frac{\big\langle j\,\chi(s_2)\big\rangle}{\big\langle \chi(s_2)\big\rangle} \to \big\langle \mu_+, j(w_{\max}, \cdot)\big\rangle := \int j(w_{\max}, \cdot)\, d\mu_+(z), \qquad \text{as } s_2 \to s_+,$$

for every continuous function $j = j(w, z)$. The measure μ_- is defined similarly as the trace on the line $\big\{z = z_{\min}\big\}$. Since $s_1 \to s_-$ and $s_2 \to s_+$ in (4.20), we use the decomposition (4.21) to obtain

$$\big\langle \mu_+, \big(v - c\big)\big\rangle - \big\langle \mu_-, \big(v + c\big)\big\rangle = 0. \tag{4.22}$$

Indeed there is no contribution to (4.22) from the remaining terms in (4.21) since, on one hand,

$$\left| \frac{\big\langle \big(k + (v - s_2)\big) c\, k^{-1} \chi(s_2)\big\rangle}{\big\langle \chi(s_2)\big\rangle} \right| \leqq C \max_{(w,z) \in \operatorname{supp} \nu} |w - s_2| \to 0$$

as $s_2 \to s_+$ (and similarly with s_-) and, on the other hand,

$$\left|\frac{\langle \tilde{h}(s)\rangle}{\langle \chi(s)\rangle}\right| \leqq C \max_{(\rho,v)\in \operatorname{supp}\nu} \big[k(\rho)^2 - (v-s)^2\big]_+ \leqq C \max_{\operatorname{supp}\nu} |w-s|\,|z-s| \to 0$$

when s tends to either s_- or s_+.

Set

$$\lambda_\pm(w,z) := v \pm c.$$

By the genuine nonlinearity, we have

$$\lambda_-(w_{\max}, z) \leqq \lambda_-(w_{\max}, z_{\min}) < \lambda_+(w_{\max}, z_{\min}) \leqq \lambda_+(w, z_{\min})$$

for all w, z between $z_{\min}$ and $w_{\max}$. This contradicts (4.22). □

Proof of Lemma 4.2. We rely on the asymptotic expansions obtained in Theorems 2.2 and 2.3 and on the explicit formulas in Proposition 2.4. Since only the first terms in expansions (2.8) and (2.12) are used here, we set

$$\tilde{g} := a_\flat G_{\lambda+1} + g, \qquad \tilde{h} := -(v-s)\, b_\flat G_{\lambda+1} + h,$$

which are Hölder continuous in (ρ, v, s) and satisfy

$$|\tilde{g}(\rho, v-s)| + |\tilde{h}(\rho, v-s)| \leqq C\left[k(\rho)^2 - (v-s)^2\right]^{\lambda+1}.$$

Also observe that $b_\sharp = \rho k'\, a_\sharp / k$.

By expanding the product, we get the decomposition

$$\begin{aligned}
&\chi_1\, P_j \sigma_j^\varepsilon - \sigma_1\, P_j \chi_j^\varepsilon \\
&\quad = \chi_1\, P_j\big(\sigma_j^\varepsilon - v\,\chi_j^\varepsilon\big) - \big(\sigma_1 - v\,\chi_1\big)\, P_j \chi_j^\varepsilon \\
&\quad = \big(a_\sharp G_{\lambda,1} + \tilde{g}_1\big)\Big(\rho k\, {k'}^{1/2} \sum_\pm K^\pm \big((s_j - v)\delta_{s_j = v\pm k}\big) \star \varphi_j^\varepsilon + e_j^{\mathrm{II}} \star \varphi_j^\varepsilon\Big) \\
&\qquad - \big((s_1 - v)\, b_\sharp G_{\lambda,1} + \tilde{h}_1\big)\big({k'}^{-1/2} \sum_\pm K^\pm \delta_{s_j = v\pm k} \star \varphi_j^\varepsilon + e_j^{\mathrm{I}} \star \varphi_j^\varepsilon\big) \\
&\quad := E^{\mathrm{I},\varepsilon} + E^{\mathrm{II},\varepsilon} + E^{\mathrm{III},\varepsilon},
\end{aligned}$$

where $G_{\lambda,j} := G_\lambda(\rho, v - s_j)$, and

$$\begin{aligned}
E^{\mathrm{I},\varepsilon} &= a_\sharp \rho k\, {k'}^{1/2}\, G_{\lambda,1} \sum_\pm K^\pm \big((s_j - s_1)\delta_{s_j = v\pm k}\big) \star \varphi_j^\varepsilon, \\
E^{\mathrm{II},\varepsilon} &= \rho k\, {k'}^{1/2} \sum_\pm K^\pm \tilde{g}_1 \big((s_j - v)\delta_{s_j = v\pm k}\big) \star \varphi_j^\varepsilon \\
&\qquad - {k'}^{-1/2} \sum_\pm K^\pm \tilde{h}_1\, \delta_{s_j = v\pm k} \star \varphi_j^\varepsilon, \\
E^{\mathrm{III},\varepsilon} &= \big(a_\sharp G_{\lambda,1} + \tilde{g}_1\big) e_j^{\mathrm{II}} \star \varphi_j^\varepsilon - \big((s_1 - v)\, b_\sharp G_{\lambda,1} + \tilde{h}_1\big) e_j^{\mathrm{I}} \star \varphi_j^\varepsilon.
\end{aligned}$$

The term $E^{\mathrm{I},\varepsilon}$ is the most singular; it contains the products of Hölder continuous functions by measures. Relying on the favorable factor $s_1 - s_j$, we have

$$
\begin{aligned}
&|E^{\mathrm{I},\varepsilon}(\rho, v - s_1)| \\
&\quad \leqq C\,\rho^{\theta\lambda}\big[k(\rho)^2 - (v - s_1)^2\big]_+^\lambda \sum_\pm K^\pm |s_1 - v \mp k(\rho)|\varphi_j^\varepsilon(s_1 - v \mp k(\rho)) \\
&\quad \leqq C\,\rho^{(1-\theta)/2} \sum_\pm |s_1 - v \mp k(\rho)|^{\lambda+1}\,\varphi_j^\varepsilon(s_1 - v \pm k(\rho)) \leqq C\,\rho^{(1-\theta)/2}\,\varepsilon^\lambda \to 0
\end{aligned}
$$

uniformly in (ρ, v, s_1) in a compact set. Here we used the fact that, since φ_j is continuous,

$$
|s^{\lambda+1}\,\varphi_j^\varepsilon(s)| \leqq \varepsilon^\lambda \sup_s |s^\lambda\,\varphi_j(s)| \leqq C\,\varepsilon^\lambda.
$$

The term $E^{\mathrm{II},\varepsilon}$ contains the products of Dirac masses by Hölder continuous functions with exponent > 1. We have

$$
\begin{aligned}
|E^{\mathrm{II},\varepsilon}(\rho, v - s_1)| &\leqq C\,\rho^{3(1-\theta)/2}\big[k(\rho)^2 - (v - s_1)^2\big]^{\lambda+1} \sum_\pm K^\pm \varphi_j^\varepsilon(s_1 - v \mp k(\rho)) \\
&\leqq C\,\rho^{3(1-\theta)/2} \sum_\pm |s_1 - v \pm k(\rho)|^{\lambda+1}\varphi_j^\varepsilon(s_1 - v \pm k(\rho)) \\
&\leqq C\,\rho^{3(1-\theta)/2}\,\varepsilon^\lambda \to 0, \qquad \text{uniformly for } (\rho, v, s_1).
\end{aligned}
$$

Dealing with $E^{\mathrm{III},\varepsilon}$ is easier. For example, we treat the product

$$
\tilde{E}^{\mathrm{III},\varepsilon} := a_\sharp G_{\lambda,1}\, e_j^{\mathrm{II}} \star \varphi_j^\varepsilon.
$$

In the region $|k(\rho)^2 - (v - s_1)^2| \leqq \beta$ (with $\beta > 0$ to be determined), we use (2.17) and so

$$
|\tilde{E}^{\mathrm{III},\varepsilon}| \leqq C\,G_{\lambda,1}\,G_{-\alpha,1} = C\,G_{\lambda-\alpha,1} \leqq C\,\beta^{\lambda-\alpha},
$$

which we can make as small as we want by taking β small, provided $\alpha \in (0, 1] \cap (0, \lambda)$.

In the complement region $|k(\rho)^2 - (v - s_1)^2| \geqq \beta > 0$, each of the two functions $G_\lambda(s_1)$ and $e^{\mathrm{II}}(s_1)$ is Hölder continuous in (ρ, v, s_1). The convergence of the convolution product is uniform in this domain and the limit $a_\sharp G_\lambda(\rho, v - s_1)\, e^{\mathrm{II}}(\rho, v - s_1)$ is continuous. This shows that $\tilde{E}^{\mathrm{III},\varepsilon}$ converges uniformly in (ρ, v, s_1). This completes the proof of Lemma 4.2. □

Proof of Lemma 4.3. This proof again relies on the asymptotic expansions in Theorems 2.2 and 2.3 and on the explicit formulas obtained in Proposition 2.4.

Observe that, in the sense of distributions,

$$
\begin{aligned}
&P_2\chi_2 P_3\sigma_3 - P_3\chi_3\, P_2\sigma_2\\
&\quad = P_2\chi_2\, P_3\big(\sigma_3 - v\,\chi_3\big) - P_3\chi_3\, P_2\big(\sigma_2 - v\,\chi_2\big)\\
&\quad = \big(a_\sharp P_2 G_{\lambda,2} + a_\flat P_2 G_{\lambda+1,2} + P_2 g_2\big)\Big((s_3 - v)\,(b_\sharp P_3 G_{\lambda,3} + b_\flat P_3 G_{\lambda+1,3})\\
&\qquad\qquad + P_3 h_3 + (\lambda+1)\, b_\sharp \partial^\lambda_{s_3} G_{\lambda,3} + (\lambda+1)\, b_\flat \partial^\lambda_{s_3} G_{\lambda+1,3}\Big)\\
&\qquad + \big(a_\sharp P_3 G_{\lambda,3} + a_\flat P_3 G_{\lambda+1,3} + P_3 g_3\big)\Big((s_2 - v)\,(b_\sharp P_2 G_{\lambda,2} + b_\flat P_2 G_{\lambda+1,2})\\
&\qquad\qquad + P_2 h_2 + (\lambda+1)\, b_\sharp \partial^\lambda_{s_2} G_{\lambda,2} + (\lambda+1)\, b_\flat \partial^\lambda_{s_2} G_{\lambda+1,2}\Big)\\
&\quad =: E^{\mathrm{I}} + E^{\mathrm{II}} + E^{\mathrm{III}},
\end{aligned}
$$

where we have used the chain rule for fractional derivatives. We define

$$
\begin{aligned}
E^I &:= (s_3 - s_2)\, a_\sharp b_\sharp\, P_2 G_{\lambda,2}\, P_3 G_{\lambda,3},\\
E^{\mathrm{II}} &:= a_\sharp\, P_2 G_{\lambda,2}\Big((s_3 - v)\, b_\flat P_3 G_{\lambda+1,3} + (\lambda+1)\, b_\sharp \partial^\lambda_{s_3} G_{\lambda,3}\Big)\\
&\quad - a_\sharp\, P_3 G_{\lambda,3}\Big((s_2 - v)\, b_\flat P_2 G_{\lambda+1,2} + (\lambda+1)\, b_\sharp \partial^\lambda_{s_2} G_{\lambda,2}\Big)\\
&\quad + a_\flat b_\sharp\Big(P_2 G_{\lambda+1,2}(s_3 - v)\, P_3 G_{\lambda,3} - P_3 G_{\lambda+1,3}(s_2 - v)\, P_2 G_{\lambda,2}\Big),
\end{aligned}
$$

and E^{III} the remainder.

Consider the decomposition

$$
P_2\chi_2^\varepsilon\, P_3\sigma_3^\varepsilon - P_3\chi_3^\varepsilon\, P_2\sigma_2^\varepsilon = \big(E^{\mathrm{I}} + E^{\mathrm{II}} + E^{\mathrm{III}}\big) \star \varphi_2^\varepsilon \star \varphi_3^\varepsilon =: E^{\mathrm{I},\varepsilon} + E^{\mathrm{II},\varepsilon} + E^{\mathrm{III},\varepsilon}
$$

and determine the limit of the first two terms. Dealing with $E^{\mathrm{III},\varepsilon}$ is easy since it involves only the products of Hölder continuous functions (such as h_3) by measures (such as $a_\sharp\, P_2 G_{\lambda,2}$), or more regular products. Classical theorems on weak convergence of convolution products apply. By using symmetry, one can easily check that

$$
E^{\mathrm{III},\varepsilon} \rightharpoonup 0 \qquad \text{as } \varepsilon \to 0,
$$

weakly in measures in s_1 and uniformly in (ρ, v).

In view of Proposition 2.4 and its proof which provides the asymptotic expansion of the functions G_λ, the term

$$
E^{\mathrm{I},\varepsilon} := E^{\mathrm{I}} \star \varphi_2^\varepsilon \star \varphi_3^\varepsilon = \big((s_3 - s_2)\, a_\sharp b_\sharp\, P_2 G_{\lambda,2}\, P_3 G_{\lambda,3}\big) \star \varphi_2^\varepsilon \star \varphi_3^\varepsilon
$$

can be decomposed into the products of measures, the products of measures by L^q functions, and the products of L^q functions. We need to consider the first two cases.

Consider the product of two measures. A typical product is $k'(\rho)^{-1/2}\,\delta_{s=v+k(\rho)}$ by $k'(\rho)^{-1/2}\,\delta_{s=v-k(\rho)}$. Using the Riemann invariants $w = v + k(\rho)$ and $z =$

$v - k(\rho)$, we estimate

$$
\begin{aligned}
k'(\rho)^{-1} & \Big| \int (w-z) \varphi_2^\varepsilon(s_1 - w)\, \varphi_3^\varepsilon(s_1 - z) \psi(s_1)\, ds_1 \Big| \\
& \leqq C \left(w - z\right)^{1+2\lambda} \int \varphi_2^\varepsilon(s_1 - w)\, \varphi_3^\varepsilon(s_1 - z) |\psi(s_1)|\, ds_1 \\
& \leqq C\, \varepsilon^{2\lambda} \left(\frac{w-z}{\varepsilon} \right)^{1+2\lambda} \int_{-1}^{1} \varphi_2(s_1)\, \varphi_3\Big(s_1 + \frac{w-z}{\varepsilon}\Big) \,|\psi(w + \varepsilon s_1)|\, ds_1 \\
& \leqq C\, \varepsilon^{2\lambda} \quad \longrightarrow\ 0.
\end{aligned}
$$

We treat the product of a measure $\mu_2 = \mu(s_2)$ by an L^q function $l_3 = l(s_3)$ as follows:

$$
\begin{aligned}
& \Big| \int \iint (s_3 - s_2) \varphi_2^\varepsilon(s_1 - s_2) \varphi_3^\varepsilon(s_1 - s_3) l(s_3)\, ds_3 d\mu(s_2)\, \psi(s_1)\, ds_1 \Big| \\
& \quad = \varepsilon^2 \Big| \int \iint (s_3 - s_2) \varphi_2(s_1 - s_2) \varphi_3(s_1 - s_3) l(\varepsilon s_3)\, ds_3 d\mu(\varepsilon s_2)\, \psi(\varepsilon s_1)\, ds_1 \Big| \\
& \quad \leqq C\, \varepsilon^{1-1/p}\, \|l\|_{L^q_s}\, \|\psi\|_{\mathcal{C}_s} \int |d\mu(s_2)| \quad \longrightarrow \quad 0,
\end{aligned}
$$

uniformly in (ρ, v), where $p = q/(p-1)$.

The other terms are handled similarly. This proves

$$
E^{\mathrm{I},\varepsilon} \rightharpoonup 0, \qquad \text{as } \varepsilon \to 0,
$$

weakly in measures in s_1 and uniformly in (ρ, v).

The term $E^{\mathrm{II},\varepsilon}$ contains the products of functions of bounded variation by bounded measures. Such products converge to the limits that depend on the regularization, i.e., on φ_2, φ_3. We can replace s_2 and s_3 by s_1 in $E^{\mathrm{II},\varepsilon}$ since the remaining terms converge to zero, as can be checked by the arguments used earlier. So we now study

$$
\begin{aligned}
\tilde{E}^{\mathrm{II},\varepsilon} =: & \, a_\sharp\, P_2 G_{\lambda,2} \star \varphi_2^\varepsilon \Big((s_1 - v)\, b_\flat P_3 G_{\lambda+1,3} + (\lambda+1)\, b_\sharp \partial_{s_3}^\lambda G_{\lambda,3} \Big) \star \varphi_3^\varepsilon \\
& - a_\sharp\, P_3 G_{\lambda,3} \star \varphi_3^\varepsilon \Big((s_1 - v)\, b_\flat P_2 G_{\lambda+1,2} + (\lambda+1)\, b_\sharp \partial_{s_2}^\lambda G_{\lambda,2} \Big) \star \varphi_2^\varepsilon \\
& + a_\flat b_\sharp \Big(P_2 G_{\lambda+1,2} (s_1 - v)\, P_3 G_{\lambda,3} - P_3 G_{\lambda+1,3} (s_1 - v)\, P_2 G_{\lambda,2} \Big) \star \varphi_2^\varepsilon \star \varphi_3^\varepsilon,
\end{aligned}
$$

that is,

$$
\begin{aligned}
\tilde{E}^{\mathrm{II},\varepsilon} & = (\lambda+1)\, a_\sharp b_\sharp \big(P_2 G_{\lambda,2} \partial_{s_3}^\lambda G_{\lambda,3} - P_3 G_{\lambda,3} \partial_{s_2}^\lambda G_{\lambda,2} \big) \star \varphi_2^\varepsilon \star \varphi_3^\varepsilon \\
& \quad + (s_1 - v) \left(a_\sharp b_\flat - a_\flat b_\sharp \right) \left(P_2 G_{\lambda,2} P_3 G_{\lambda+1,3} - P_3 G_{\lambda,3} P_2 G_{\lambda+1,2} \right) \star \varphi_2^\varepsilon \star \varphi_3^\varepsilon.
\end{aligned}
$$

Since

$$
\begin{aligned}
& \partial_s^{\lambda+1} G_{\lambda+1}(\rho, v - s) \\
& \quad = \left[k^2 - (v-s)^2\right]_+ \partial_s^{\lambda+1} G_\lambda(\rho, v - s) - 2\,(\lambda+1)(s - v)\, \partial_s^\lambda G_\lambda(\rho, v - s),
\end{aligned}
$$

the weak limit of $\tilde{E}^{\mathrm{II},\varepsilon}$ is the same as the limit of

$$(\lambda+1)\left(a_\sharp\, b_\sharp - 2\,k^2\big(a_\sharp b_\flat - a_\flat b_\sharp\big)\right)\left(P_2 G_{\lambda,2}\partial_{s_3}^{\lambda} G_{\lambda,3} - P_3 G_{\lambda,3}\partial_{s_2}^{\lambda} G_{\lambda,2}\right)\star\varphi_2^\varepsilon\star\varphi_3^\varepsilon.$$

We denote by $H_{s=m}$ the Heaviside function with a jump at the point m. Using the asymptotic expansions in Section 2, we arrive at

$$Z(\rho)\Bigg(\sum_\pm K^\pm\delta_{s_2=v\pm k}\,\sum_\pm K^\pm H_{s_3=v\pm k} - \sum_\pm K^\pm\delta_{s_3=v\pm k}\,\sum_\pm K^\pm H_{s_2=v\pm k}\Bigg)\star\varphi_2^\varepsilon\star\varphi_3^\varepsilon.$$

To conclude, we observe

Lemma 4.4. *For all $m_2, m_3\in\mathbb{R}$, one has*

$$\big(H_{s_2=m_2}\star\varphi_2^\varepsilon\big)\big(\delta_{s_3=m_3}\star\varphi_3^\varepsilon\big)\rightharpoonup\Omega(m_2,m_3)\,\delta_{s_1=m_3}$$

in measures, where

$$\Omega(m_2,m_3):=\begin{cases}0, & \text{if } m_2<m_3,\\ \int_{\mathbb{R}}\varphi_2(s)\int_{-\infty}^{s}\varphi_3(t)\,dt\,ds, & \text{if } m_2=m_3,\\ 1=\int_{\mathbb{R}}\varphi_2(s)\,ds\int_{\mathbb{R}}\varphi_3(t)\,dt, & \text{if } m_2>m_3.\end{cases}$$

The proof is omitted. In view of the lemma, $E^{\mathrm{II},\varepsilon}$ converges in the weak sense in s_1 to the limit stated in (4.8). The proof of Lemma 4.3 is completed. □

5. Existence, Compactness, and Asymptotic Decay

In this section we establish the existence, compactness, and asymptotic decay of entropy solutions of the Cauchy problem (1.1), (1.2), relying on assumptions (1.4) and (2.1).

Theorem 5.1 (Existence**).** *Assume that the initial data (ρ_0, m_0) satisfy*

$$0\leqq\rho_0(x)\leqq C_0,\quad |m_0(x)|\leqq C_0\,\rho_0(x)\qquad a.e. \tag{5.1}$$

Then there exists an entropy solution (ρ, m) of the Cauchy problem (1.1), (1.2), *globally defined in time, satisfies*

$$0\leqq\rho(t,x)\leqq C,\quad |m(t,x)|\leqq C\,\rho(t,x)\qquad a.e.\ (t,x),$$

where C depends only on C_0 and the pressure function $p(\cdot)$.

The proof is postponed until the end of the section. A direct application of Theorem 4.1 then yields the following compactness theorem.

Theorem 5.2 (Compactness**).** *The solution operator $(\rho,m)(t,\cdot)=S_t(\rho_0,m_0)(\cdot)$ determined by Theorem 5.1 is compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$.*

Proof. Consider any (oscillatory) sequence of initial data $(\rho_0^\varepsilon, m_0^\varepsilon)$, $\varepsilon > 0$, satisfying

$$0 \leqq \rho_0^\varepsilon(x) \leqq C_0, \quad |m_0^\varepsilon(x)| \leqq C_0\, \rho_0^\varepsilon(x) \tag{5.2}$$

with $C_0 > 0$ independent of $\varepsilon > 0$. Then there exists $C > 0$ independent of $\varepsilon > 0$ such that the corresponding sequence $(\rho^\varepsilon, m^\varepsilon)$, determined by Theorem 5.1, satisfies

$$0 \leqq \rho^\varepsilon(t,x) \leqq C, \quad |m^\varepsilon(t,x)| \leqq C\, \rho^\varepsilon(t,x).$$

Since $(\rho^\varepsilon, m^\varepsilon)$ are entropy solutions satisfying $\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \leqq 0$ in the sense of distributions, for any $\mathcal{C}^2$ convex weak entropy pair (η, q), we deduce from Murat's lemma (see [4] for details) that

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \qquad \text{is compact in} \quad H^{-1}_{\text{loc}}(\mathbb{R}^2_+),$$

for any weak entropy pair (η, q). Combining this with Theorem 4.1 shows that $(\rho^\varepsilon, m^\varepsilon)$ is compact in $L^1_{\text{loc}}(\mathbb{R}^2_+)$, which implies our conclusion. □

Finally, based on the analytical framework for the asymptotic decay of periodic solutions established in CHEN & FRID [4], we obtain

Theorem 5.3 (Asymptotic decay**).** *Let* $(\rho, m) \in L^\infty(\mathbb{R}^2_+)$ *be a periodic entropy solution of the Cauchy problem* (1.1), (1.2) *with period* $[\alpha, \beta]$. *Then* (ρ, m) *asymptotically decays:*

$$\operatorname*{ess\,lim}_{t\to\infty} \int_\alpha^\beta \big(|\rho(t,x) - \bar\rho|^r + |m(t,x) - \bar m|^r\big)\, dx = 0 \quad \textit{for all } 1 \leqq r < \infty,$$

where $(\bar\rho, \bar m) := \frac{1}{\beta-\alpha}\int_\alpha^\beta (\rho_0(x), m_0(x))\, dx$.

Remark. The results in Theorems 5.2 and 5.3 are somewhat surprising since the flux function of (1.1) is only Lipschitz continuous. Notice that a counterexample found by GREENBERG & RASCLE [19] demonstrates that there exist certain systems with only C^1 (but not C^2) flux functions admitting time-periodic and space-periodic solutions. This example indicates that the compactness and asymptotic decay of entropy solutions are sensitive with respect to the smoothness of the flux functions.

Proof. Theorem 5.2 implies that the self-similar scaling sequence

$$u^T(t,x) \equiv (\rho^T(t,x), m^T(t,x)) = (\rho(Tt, Tx), m(Tt, Tx))$$

is compact in $L^1_{\text{loc}}(\mathbb{R}^2_+)$ as $T \to \infty$. From [4], it follows that

$$\operatorname*{ess\,lim}_{t\to\infty} \int_\alpha^\beta (\eta_*(u(t,x)) - \eta_*(\bar u) - \nabla\eta_*(\bar u)(u(t,x) - \bar u))\, dx = 0,$$

or, equivalently,

$$\operatorname*{ess\,lim}_{t\to\infty} \int_\alpha^\beta \int_0^1 (1-\tau)(u(t,x)-\bar u)^\top \nabla^2\eta_*(\bar u + \tau(u(t,x)-\bar u))(u(t,x)-\bar u)\, d\tau\, dx = 0. \tag{5.3}$$

Here $\bar{u} = (\bar{\rho}, \bar{m})$, etc., and η_* is the standard entropy, the mechanical energy of (1.1), with corresponding entropy flux q_*:

$$\eta_*(\rho, m) = \frac{m^2}{2\rho} + \rho \int_0^\rho \frac{p(r)}{r^2}\, dr, \quad q_*(\rho, m) = \frac{m^3}{2\rho^2} + m \int_0^\rho \frac{p'(r)}{r}\, dr. \quad (5.4)$$

We observe the following facts.

1. For $1 < \gamma \leqq 2$, the entropy η_* is uniformly convex, that is $\nabla^2\eta_* \geqq c_0$, for some $c_0 > 0$, and (5.3) is equivalent to

$$\operatorname{ess}\lim_{t\to\infty} \int_\alpha^\beta |u(t,x) - \bar{u}|^2 dx = 0. \quad (5.5)$$

2. For $\gamma > 2$, (5.3) means that

$$\operatorname{ess}\lim_{t\to\infty} \int_\alpha^\beta \left(\frac{1}{2}\rho(t,x) \left(\frac{m(t,x)}{\rho(t,x)} - \frac{\bar{m}}{\bar{\rho}} \right)^2 \right. \\ \left. + \int_0^1 (1-\tau) \frac{p'(\bar{\rho} + \tau(\rho(t,x) - \bar{\rho}))}{\bar{\rho} + \tau(\rho(t,x) - \bar{\rho})} d\tau (\rho(t,x) - \bar{\rho})^2 \right) dx = 0,$$

which implies

$$\operatorname{ess}\lim_{t\to\infty} \int_\alpha^\beta \left(\rho(t,x) \left(\frac{m(t,x)}{\rho(t,x)} - \frac{\bar{m}}{\bar{\rho}} \right)^2 + |\rho(t,x) - \bar{\rho}|^\gamma \right) dx = 0. \quad (5.6)$$

Note that

$$|m - \bar{m}|^2 = \left| \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right) \rho + \frac{\bar{m}}{\bar{\rho}} (\rho - \bar{\rho}) \right|^2 \leqq 2 \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right)^2 \rho^2 + 2 \left(\frac{\bar{m}}{\bar{\rho}} \right)^2 (\rho - \bar{\rho})^2 \\ \leqq C \left\{ \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right)^2 \rho + (\rho - \bar{\rho})^2 \right\}, \quad (5.7)$$

and

$$\int_\alpha^\beta |\rho - \bar{\rho}|^2 dx \leqq C \left(\int_\alpha^\beta |\rho - \bar{\rho}|^\gamma dx \right)^{1/2} \quad (5.8)$$

by Hölder's inequality. We conclude from (5.6)–(5.8) and the uniform bound on the solution (ρ, m) that, for any $1 \leqq r < \infty$,

$$\operatorname{ess}\lim_{t\to\infty} \int_\alpha^\beta (|m(t,x) - \bar{m}|^r + |\rho(t,x) - \bar{\rho}|^r)\, dx = 0. \quad (5.9)$$

Combining (5.5) with (5.9) leads to the completion of the proof. □

To establish the existence result stated in Theorem 5.1, we now apply the compactness framework established in Theorem 4.1 and prove the convergence of the Lax-Friedrichs scheme for the Cauchy problem (1.1), (1.2) satisfying (5.1) for some $C_0 > 0$.

As every difference scheme, the Lax-Friedrichs scheme satisfies the property of propagation with finite speed, which is an advantage over the vanishing viscosity method: our convergence result applies without assumption on the decay of the initial data at infinity. We now introduce the family of Lax-Friedrichs approximate solutions $(\rho^h(t,x), m^h(t,x))$. Also, we set $v^h = m^h/\rho^h$ when $\rho^h > 0$ and $v^h = 0$ otherwise. The Lax-Friedrichs scheme is based on a regular partition of the half-plane $t \geqq 0$ defined by $t_n = n\,\tau$, $x_j = j\,h$ for $n \in \mathbb{N}$, $j \in \mathbf{Z}$. Here τ and h are the lengths of time step and space step respectively. It is assumed that the ratio h/τ is constant and satisfies the Courant-Friedrichs-Lewy stability condition:

$$\frac{\tau}{h} \sup_{(t,x)} |v^h(t,x) \pm c(\rho^h(t,x))| < 1.$$

For each $n \in \mathbb{N}$, we set

$$J_n = \big\{ j \mid j \text{ integer with even } n + j \big\}.$$

In the first strip $\{(t,x) \,|\, 0 < t < t_1,\ x_{j-1} < x < x_{j+1},\ \ j \text{ odd}\}$, we define $(\rho^h(t,x), m^h(t,x))$ by solving a sequence of Riemann problems for (1.1) corresponding to the Riemann data:

$$(\rho^h, m^h)(x,0) = \begin{cases} (\rho^0_{j-1}, m^0_{j-1}), & x < x_j, \\ (\rho^0_{j+1}, m^0_{j+1}), & x > x_j, \end{cases}$$

with

$$(\rho^0_{j+1}, m^0_{j+1}) = \frac{1}{2\,h} \int_{x_j}^{x_{j+2}} (\rho_0, m_0)(x)\,dx.$$

It can be checked that the Riemann problem is uniquely solvable for the general pressure law (1.4) and (2.1).

If (ρ^h, m^h) is known for $t < t_n$, we set

$$(\rho^n_j, m^n_j) = \frac{1}{2\,h} \int_{x_{j-1}}^{x_{j+1}} (\rho^h, m^h)(t_n - 0, x)\,dx.$$

In the region $\{(t,x) \,|\, t_n < t < t_{n+1},\ x_j < x < x_{j+2},\ \ j \in J_n\}$, we define $(\rho^h(t,x), m^h(t,x))$ by solving the Riemann problems with the data

$$(\rho^h, m^h)(t_n, x) = \begin{cases} (\rho^n_j, m^n_j), & x < x_{j+1}, \\ (\rho^n_{j+2}, m^n_{j+2}), & x > x_{j+1}. \end{cases}$$

This completes the construction of the Lax-Friedrichs approximate solutions $(\rho^h(t,x), m^h(t,x))$.

The main result of this section is the following.

Theorem 5.4. *Let* (ρ_0, m_0) *be the Cauchy data satisfying* (5.1). *Extracting a subsequence if necessary, the Lax-Friedrichs approximate solutions* (ρ^h, m^h) *converge strongly to a limit* $(\rho, m) \in L^\infty(\mathbb{R}^2_+)$ *which is an entropy solution of the Cauchy problem* (1.1), (1.2).

The following two lemmas are used toward the proof of Theorem 5.4.

Lemma 5.1. *For all* $w_0 > z_0$*, the regions*

$$R(w_0, z_0) = \left\{(\rho, m) \mid w \leqq w_0, z \geqq z_0, w - z \geqq 0\right\}$$

are invariant for both the Riemann solutions and the Lax-Friedrichs approximate solutions.

Proof. The fact that $R(w_0, z_0)$ is an invariant region for the Riemann solutions can be checked directly from the explicit formulas known for the Riemann problem. Since the sets $R(w_0, z_0)$ are convex in the (ρ, m)-plane, it follows from Jensen's inequality that, for any function satisfying $\{(\rho(x), m(x)) \mid a \leqq x \leqq b\} \subset R(w_0, z_0)$ for some (w_0, z_0), we have

$$(\bar{\rho}, \bar{m}) = \frac{1}{b-a} \int_a^b (\rho(x), m(x))\, dx \in R(w_0, z_0).$$

Therefore, $R(w_0, z_0)$ is also an invariant region for the Lax-Friedrichs scheme. □

In particular, Lemma 5.1 shows that the density ρ^h remains nonnegative so that it is indeed possible to construct the approximate solutions globally, as described earlier.

Consider the entropy pair (η_*, q_*) defined from the kinetic and internal energies by (5.4).

Lemma 5.2. *For any weak entropy pair* (η, q) *and any invariant region* $R(w_0, z_0)$*, there exists a constant* $C > 0$ *such that, for any solution* $(\rho(t, x), m(t, x))$ *of the Riemann problem with initial data in* $R(w_0, z_0)$*,*

$$\left|x'(t)\,[\eta(\rho, m)](t) - [q(\rho, m)](t)\right| \leqq C\left|x'(t)\,[\eta_*(\rho, m)](t) - [q_*(\rho, m)](t)\right|,$$

where $x'(t)$ *is the speed of any shock located at* $x(t)$ *in the Riemann solution* (ρ, m) *and* $[g(\rho, m)](t) := g(\rho, m)(x(t)+, t) - g(\rho, m)(x(t)-, t)$ *for any function* $g(\rho, m)$.

The proof of Theorem 5.4 then follows similar lines to those in [2,5,13] for the γ-law case. It is not difficult to include the interval $\gamma \in (2, 3)$ for which the standard entropy (5.9) is degenerate near the vacuum.

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References

1. BETSADZE, A. V., *Equations of Mixed Type*, Macmillan, New York, 1964.
2. CHEN, G.-Q., Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III), *Acta Math. Sci.* **6** (1986), 75–120 (in English); and **8** (1988), 243–276 (in Chinese).
3. CHEN, G.-Q., *The compensated compactness method and the system of isentropic gas dynamics*, Preprint MSRI-00527-91, Math. Sci. Res. Inst., Berkeley, 1990.
4. CHEN, G.-Q. & FRID, H., Decay of entropy solutions of nonlinear conservation laws, *Arch. Rational Mech.* **146** (1999), 95–127.
5. CHEN, G.-Q. & GLIMM, J., Global solutions to the compressible Euler equations with geometrical structure, *Commun. Math. Phys.* **180** (1996), 153–193.
6. CHEN, G.-Q. & LEFLOCH, P. G., Entropy flux-splittings for hyperbolic conservation laws: General framework, *Comm. Pure Appl. Math.* **48** (1995), 691–729.
7. CHEN, G.-Q. & LEFLOCH, P. G., Entropy and weak solutions of the isentropic Euler equations, *C.R. Acad. Sci. Paris, Série 1*, t. **324** (1997), 1105–1110.
8. CHEN, G.-Q. & LEFLOCH, P. G., in preparation.
9. COURANT, R. & FRIEDRICHS, K. O., *Supersonic Flow and Shock Waves*, Wiley-Interscience: New York, 1948.
10. DAL MASO, G., MURAT, F. & LEFLOCH, P. G., Definition and weak stability of nonconservative products, *J. Math. Pures Appl.* **74** (1995), 483–548.
11. DAFERMOS, C. M., *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: Berlin, Heidelberg, New York, 2000.
12. DARBOUX, G., *Leçon sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Reprint, Chelsea Publishing Comp. (1972) (in French).
13. DING, X., CHEN, G.-Q. & LUO, P., Convergence of the Lax-Friedrichs scheme for the isentropic gas dynamics (I)–(II), *Acta Math. Sci.* **5** (1985), 483–500, 501–540 (in English); and **7** (1987), 467–480; **8** (1989), 61–94 (in Chinese).
14. DING, X., ZHANG, T., WANG, C.-H., HSIAO, L. & LI, T.-C., A study of the global solutions for quasilinear hyperbolic systems of conservation laws, *Scientica Sinica* **16** (1973), 317–335.
15. DIPERNA, R. J., Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Anal.* **82** (1983), 27–70.
16. DIPERNA, R. J., Convergence of the viscosity method for isentropic gas dynamics, *Commun. Math. Phys.* **91** (1983), 1–30.
17. GELFAND, I. M. & SHILOV, G. E., *Generalized Functions*, Vol. **I**, Academic Press, 1964.
18. GLIMM, J., Solutions in large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697–715.
19. GREENBERG, J. M. & RASCLE, M., Time-periodic solutions to systems of conservation laws, *Arch. Rational Mech. Anal.* **115** (1991), 395–407.
20. LAX, P. D., Shock Wave and Entropy, in *Contributions to Functional Analysis*, ed. E. A. ZARANTONELLO, Academic Press, New York (1971), 603–634.
21. LAX, P. D., Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, *CBMS Regional Conf. Series, 11*, SIAM, Philadelphia, 1973.
22. LIU, T.-P., Initial-boundary-value problems for gas dynamics, *Arch. Rational Mech. Anal.* **64** (1977), 137–168.
23. LIONS, P. L., PERTHAME, B. & TADMOR, E., Kinetic formulation for the isentropic gas dynamics and p-system, *Commun. Math. Phys.* **163** (1994), 415–431.
24. LIONS, P. L., PERTHAME, B. & SOUGANIDIS, P. E., Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Comm. Pure Appl. Math.* **49** (1996), 599–638.
25. MORAWETZ, C., An alternative proof of DiPerna's theorem, *Comm. Pure Appl. Math.* **44** (1991), 1081–1090.
26. MURAT, F., Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat.* **5** (1978), 489–507.
27. MURAT, F., L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* **60** (1981), 309–322.

28. NISHIDA, T. & SMOLLER, J., Solutions in large for some nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.* **26** (1973), 183–200.
29. RIEMANN, B., Uber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. *Abhandl. Koenig. Gesell. Wiss., Goettingen*, Vol. **8** (1860), 43.
30. TARTAR, L., Compensated compactness and applications to partial differential equations, in *Research Notes in Mathematics, Nonlinear Analysis and Mechanics, Herriot-Watt Symposium*, Vol. **4**, ed. R.J. KNOPS, Pitman Press, 1979.
31. YANG, G. J., *Euler-Poisson-Darboux Equations* (in Chinese), Yuannan University Press, Yuannan, 1989.
32. WAGNER, D., Equivalence of the Euler and the Lagrangian equations of gas dynamics for weak solutions, *J. Diff. Eqs.* **68** (1987), 118–136.
33. WHITHAM, G. B., *Linear and Nonlinear Waves*, Wiley-Intersciences: New York, 1974.
34. ZHANG, T. & GUO, Y.-F., A class of initial-value problems for systems of aerodynamic equations, *Acta. Math. Sinica*, **15** (1965), 386–396 (in Chinese); English transl. in *Chinese Math.* **7** (1965), 90–101.

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