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Isoperimetric Residues and a Mesoscale Flatness Criterion for Hypersurfaces with Bounded Mean Curvature

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Abstract

We obtain a full resolution result for minimizers in the exterior isoperimetric problem with respect to a compact obstacle in the large volume regime $v \to \infty$. This is achieved by the study of a Plateau-type problem with a free boundary (both on the compact obstacle and at infinity), which is used to identify the first obstacle-dependent term (called *isoperimetric residue*) in the energy expansion, as $v \to \infty$, of the exterior isoperimetric problem. A crucial tool in the analysis of isoperimetric residues is a new "mesoscale flatness criterion" for hypersurfaces with bounded mean curvature, which we obtain as a development of ideas originating in the theory of minimal surfaces with isolated singularities.

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1. Introduction

1.1. Overview

Given a compact set $W \subset \mathbb{R}^{n+1}$ ($n \geq 1$), we consider the classical **exterior isoperimetric problem** associated to W, namely,

$$\psi_W(v) = \inf \left\{ P(E;\Omega) : E \subset \Omega = \mathbb{R}^{n+1} \setminus W \,, |E| = v \right\}, \qquad v > 0 \,, \tag{1.1}$$

in the large volume regime $v \to \infty$. Here $|E|$ denotes the volume (Lebesgue measure) of E, and $P(E;\Omega)$ the (distributional) perimeter of E relative to Ω, so that $P(E;\Omega) = \mathcal{H}^n(\Omega \cap \partial E)$ whenever ∂E is locally Lipschitz. Relative isoperimetric problems are well-known for their analytical [28, Sections 6.4–6.6] and geometric [6, Chapter V] relevance. They are also important in physical applications: beyond the obvious example of capillarity theory [19], exterior isoperimetry at large volumes provides an elegant approach to the Huisken–Yau theorem in general relativity, see [15].

When $v \to \infty$, we expect minimizers E_v in (1.1) to closely resemble balls of volume v. Indeed, by minimality and isoperimetry, denoting by $B^{(v)}(x)$ the ball of center x and volume v, and with $B^{(v)} = B^{(v)}(0)$, we find that

$$\lim_{v \to \infty} \frac{\psi_W(v)}{P(B^{(v)})} = 1 \,. \tag{1.2}$$

Additional information can be obtained by combining (1.2) with quantitative isoperimetry [22,23]: if $0 < |E| < \infty$, then

$$P(E) \geq P(B^{(|E|)}) \left\{ 1 + c(n) \inf_{x \in \mathbb{R}^{n+1}} \left(\frac{|E \Delta B^{(|E|)}(x)|}{|E|} \right)^2 \right\}. \tag{1.3}$$

The combination of (1.2) and (1.3) shows that minimizers E_v in $\psi_W(v)$ are close in L^1-distance to balls. Based on that, a somehow classical argument exploiting the local regularity theory of perimeter minimizers shows the existence of $v_0 > 0$ and of a function $R_0(v) \to 0^+$, $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, both depending on W, such that, if E_v is a minimizer of (1.1) with $v > v_0$, then (see Fig. 1)

$$\begin{aligned} &(\partial E_v) \setminus B_{R_0\, v^{1/(n+1)}} \subset \text{ a } C^1\text{-small normal graph over } \partial B^{(v)}(x), \\ &\text{for some } x \in \mathbb{R}^{n+1} \text{ with } |x| = (v/\omega_{n+1})^{1/(n+1)} + \mathrm{o}(v^{1/(n+1)}) \text{ as } v \to \infty; \end{aligned} \tag{1.4}$$

here ω_m stands for the volume of the unit ball in $\mathbb{R}^m$, $B_r(x)$ is the ball of center x and radius r in $\mathbb{R}^{n+1}$, and $B_r = B_r(0)$. The picture of the situation offered by (1.2) and (1.4) is thus incomplete under one important aspect: it offers no information related to the specific "obstacle" W under consideration—in other words, *two different obstacles are completely unrecognizable from* (1.2) *and* (1.4) *alone*.

The first step to obtain obstacle-dependent information on ψ_W is studying L^1_{loc}-subsequential limits F of exterior isoperimetric sets E_v as $v \to \infty$. Since the mean curvature of ∂E_v has order $v^{-1/(n+1)}$ as $v \to \infty$ in Ω, each ∂F is easily seen

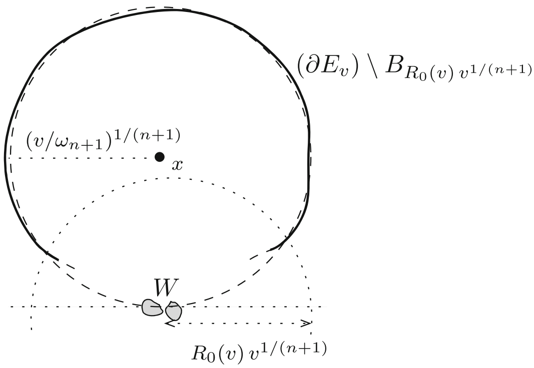

Fig. 1. Quantitative isoperimetry gives no information on how W affects $\psi_W(v)$ for v large

to be a minimal surface in Ω. A finer analysis leads to establish a more useful characterization of such limits F as minimizers in a "Plateau's problem with free boundary on the obstacle and at infinity", whose negative is precisely defined in (1.10) below and denoted by $\mathcal{R}(W)$. We call $\mathcal{R}(W)$ the **isoperimetric residue of** W because it captures the "residual effect" of W in (1.2), as expressed by the limit identity

$$\lim_{v\to\infty} \psi_W(v) - P(B^{(v)}) = -\mathcal{R}(W)\,. \tag{1.5}$$

The study of the geometric information about W stored in $\mathcal{R}(W)$ is particularly interesting: roughly, $\mathcal{R}(W)$ is close to an n-dimensional sectional area of W, although its precise value is elusively determined by the behavior of certain "plane-like" minimal surfaces with free boundary on W. The proof of (1.5) itself requires proving a blowdown result for such exterior minimal surfaces, and then extracting sharp decay information towards hyperplane blowdown limits. In particular, in the process of proving (1.5), we shall prove the existence of a positive R_2 (depending on n and W only) such that for every maximizer F of $\mathcal{R}(W)$, $(\partial F)\setminus B_{R_2}$ is the graph of a smooth solution to the minimal surfaces equation. An application of Allard's regularity theorem [3] leads then to complement (1.4) with the following "local" resolution formula: for every $S > R_2$ and large v in terms of n, W and S,

$$\begin{aligned}&\text{if } E_v \text{ minimizes (1.1), then } (\partial E_v)\cap\big(B_S\setminus B_{R_2}\big)\subset \text{ a } C^1\text{-small}\\&\text{normal graph over } \partial F, \text{ where } F \text{ is optimal for the isoperimetric}\\&\text{residue } \mathcal{R}(W) \text{ of } W.\end{aligned} \tag{1.6}$$

Interestingly, this already fine analysis gives no information on ∂E_v in the *mesoscale* region $B_{R_0(v)\,v^{1/(n+1)}}\setminus B_S$ between the resolution formulas (1.4) and (1.6). To address this issue, we are compelled to develop what we have called a **mesoscale flatness criterion** for hypersurfaces with bounded mean curvature. This kind of statement is qualitatively novel with respect to the flatness criteria typically used in the study of blowups and blowdowns of minimal surfaces—although it is clearly related to those tools at the mere technical level—and holds promise for applications to other geometric variational problems. In the study of the exterior isoperimetric problem, it allows us to prove the existence of positive constants v_0 and R_1, depending on n and W only, such that if $v > v_0$ and E_v is a minimizer of $\psi_W(v)$, then

$$(\partial E_v) \cap \big(B_{R_1\, v^{1/(n+1)}} \setminus B_{R_2}\big) \subset \text{ a } C^1\text{-small normal graph over } \partial F, \\ \text{where } F \text{ is optimal for the isoperimetric residue } \mathcal{R}(W) \text{ of } W. \tag{1.7}$$

The key difference between (1.6) and (1.7) is that the domain of resolution given in (1.7) *overlaps* with that of (1.4): indeed, $R_0(v) \to 0^+$ as $v \to \infty$ implies that $R_0\, v^{1/(n+1)} < R_1\, v^{1/(n+1)}$ for $v > v_0$. As a by-product of this overlapping and of the graphicality of ∂F outside of B_{R_2}, we deduce that *boundaries of exterior isoperimetric sets, outside of B_{R_2}, are diffeomorphic to n-dimensional disks*. Finally, when $n \le 6$, and maximizers F of $\mathcal{R}(W)$ have locally smooth boundaries in Ω, (1.7) can be propagated up to the obstacle itself; see Remark 1.7 below.

Concerning the rest of this introduction: In Sect. 1.2 we present our analysis of isoperimetric residues, see Theorem 1.1. In Sect. 1.3 we gather all our results concerning exterior isoperimetric sets with large volumes, see Theorem 1.6. Finally, we present the mesoscale flatness criterion in Sect. 1.4 and the organization of the paper in Sect. 1.5.

1.2. Isoperimetric Residues

To define $\mathcal{R}(W)$ we introduce the class

$$\mathcal{F}$$

of those pairs (F, ν) with $\nu \in \mathbb{S}^n$ (= the unit sphere of $\mathbb{R}^{n+1}$) and $F \subset \mathbb{R}^{n+1}$ a set of locally finite perimeter in Ω (i.e., $P(F;\Omega') < \infty$ for every $\Omega' \subset\subset \Omega$), with boundary ∂F contained in a slab around $\nu^\perp = \{x : x \cdot \nu = 0\}$ and projecting fully over $\nu^\perp$ itself (see Remark 1.5 below): i.e., for some $\alpha, \beta \in \mathbb{R}$,

$$\partial F \subset \big\{x : \alpha < x \cdot \nu < \beta\big\}, \tag{1.8}$$

$$\mathbf{p}_{\nu^\perp}(\partial F) = \nu^\perp := \big\{x : x \cdot \nu = 0\big\}, \tag{1.9}$$

where $\mathbf{p}_{\nu^\perp}(x) = x - (x \cdot \nu)\,\nu$, $x \in \mathbb{R}^{n+1}$. In correspondence to W compact, we define the **residual perimeter functional**, $\mathrm{res}_W : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$, by

$$\mathrm{res}_W(F,\nu) = \varlimsup_{R\to\infty} \omega_n\, R^n - P(F; \mathbf{C}_R^\nu \setminus W), \qquad (F,\nu) \in \mathcal{F},$$

where $\mathbf{C}_R^\nu = \{x \in \mathbb{R}^{n+1} : |\mathbf{p}_{\nu^\perp}(x)| < R\}$ denotes the (unbounded) cylinder of radius R with axis along ν—and where the limsup is actually a monotone decreasing limit thanks to (1.8) and (1.9) (see (4.7) below for a proof). For a reasonably "well-behaved" F, e.g. if ∂F is the graph of a Lipschitz function over $\nu^\perp$, $\omega_n\, R^n$ is the (obstacle-independent) leading order term of the expansion of $P(F; \mathbf{C}_R^\nu \setminus W)$ as $R \to \infty$, while $\mathrm{res}_W(F,\nu)$ is expected to capture the first obstacle-dependent "residual perimeter" contribution of $P(F; \mathbf{C}_R^\nu \setminus W)$ as $R \to \infty$. The **isoperimetric residue** of W is then defined by maximizing res_W over $\mathcal{F}$, so that

$$\mathcal{R}(W) = \sup_{(F,\nu)\in\mathcal{F}} \mathrm{res}_W(F,\nu)\,; \tag{1.10}$$

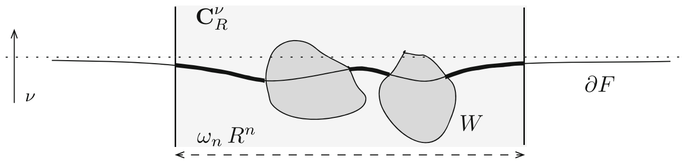

Fig. 2. If $(F, \nu) \in \mathcal{F}$ then ∂F is contained in a slab around $\nu^\perp$ and is such that ∂F has full projection over $\nu^\perp$. Only the behavior of ∂F outside W matters in computing $\mathrm{res}_W(F, \nu)$. The perimeter of F in $\mathbf{C}_R^\nu \setminus W$ (depicted as a bold line) is compared to $\omega_n R^n$ (=perimeter of a half-space orthogonal to ν in $\mathbf{C}_R^\nu$); the corresponding "residual" perimeter as $R \to \infty$, is $\mathrm{res}_W(F, \nu)$

see Fig. 2. Clearly $\mathcal{R}(\lambda W) = \lambda^n \mathcal{R}(W)$ if $\lambda > 0$, and $\mathcal{R}(W)$ is trapped between the areas of the largest hyperplane section and directional projection of W, see (1.11) below. In the simple case when $n = 1$ and W is connected, $\mathcal{R}(W) = \mathrm{diam}\,(W)$ by (1.17) and (1.18) below, although, in general, $\mathcal{R}(W)$ does not seem to admit a simple characterization, and it is finely tuned to the near-to-the-obstacle behavior of "plane-like" minimal surfaces with free boundary on W. Our first main result collects these (and other) properties of isoperimetric residues and of their maximizers.

Theorem 1.1. (Isoperimetric residues) *If $W \subset \mathbb{R}^{n+1}$ is compact, then there are R_2 and C_0 positive and depending on W with the following property.*
(i): *If $\mathcal{S}(W) = \sup\{\mathcal{H}^n(W \cap \Pi) : \Pi$ is a hyperplane in $\mathbb{R}^{n+1}\}$ and $\mathcal{P}(W) = \sup\{\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)) : \nu \in \mathbb{S}^n\}$, then we have*

$$\mathcal{S}(W) \le \mathcal{R}(W) \le \mathcal{P}(W). \tag{1.11}$$

(ii): *The family* $\mathrm{Max}[\mathcal{R}(W)]$ *of maximizers of $\mathcal{R}(W)$ is non-empty. If $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$, then F is a* **perimeter minimizer with free boundary in** $\Omega = \mathbb{R}^{n+1} \setminus W$, *i.e.*

$$P(F; \Omega \cap B) \le P(G; \Omega \cap B)\,, \qquad \forall F \Delta G \subset\subset B,\ B \text{ a ball}\,; \tag{1.12}$$

and if $\mathcal{R}(W) > 0$, then ∂F is contained in the smallest slab $\{x : \alpha \le x \cdot \nu \le \beta\}$ containing W, and there are $a, b \in \mathbb{R}$, $c \in \nu^\perp$ with $\max\{|a|, |b|, |c|\} \le C_0$ and $f \in C^\infty(\nu^\perp)$ such that

$$(\partial F) \setminus \mathbf{C}_{R_2}^\nu = \big\{x + f(x)\,\nu : x \in \nu^\perp\,, |x| > R_2\big\}\,, \tag{1.13}$$

$$\begin{aligned}
& f(x) = a, && (n = 1)\\
& \Big|f(x) - \Big(a + \frac{b}{|x|^{n-2}} + \frac{c \cdot x}{|x|^n}\Big)\Big| \le \frac{C_0}{|x|^n}, && (n \ge 2)\\
& \max\big\{|x|^{n-1}\,|\nabla f(x)|, |x|^n\,|\nabla^2 f(x)|\big\} \le C_0, && \forall x \in \nu^\perp, |x| > R_2.
\end{aligned} \tag{1.14}$$

(iii): *At fixed diameter, isoperimetric residues are maximized by balls, i.e.*

$$\mathcal{R}(W) \le \omega_n \big(\mathrm{diam}\, W/2\big)^n = \mathcal{R}\big(\mathrm{cl}\big(B_{\mathrm{diam}\, W/2}\big)\big)\,, \tag{1.15}$$

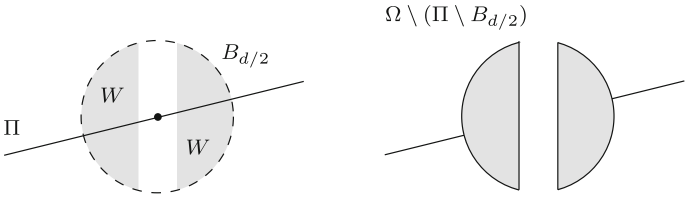

Fig. 3. The obstacle W (depicted in grey) is obtained by removing a cylinder C_r^{en+1} from a ball $B_{d/2}$ with $d/2 > r$. In this way $d = \text{diam}\,(W)$ and $B_{d/2}$ is the only ball such that (1.17) can hold. Hyperplanes Π satisfying (1.17) are exactly those passing through the center of $B_{d/2}$, and intersecting W on a $(n-1)$-dimensional sphere of radius $d/2$. For every such Π, $\Omega \setminus (\Pi \setminus B_{d/2})$ has exactly one unbounded connected component, and (1.18) does not hold

where $\text{cl}\,(X)$ *denotes topological closure of* $X \subset \mathbb{R}^{n+1}$*. Moreover, if equality holds in* (1.15) *and* $(F, \nu) \in \text{Max}[\mathcal{R}(W)]$*, then* (1.14) *holds with* $b = 0$ *and* $c = 0$*, and setting* $\Pi = \{y : y \cdot \nu = a\}$*, we have*

$$(\partial F) \setminus W = \Pi \setminus \text{cl}\left(B_{\text{diam}\,W/2}(x)\right), \tag{1.16}$$

for some $x \in \Pi$*. Finally, equality holds in* (1.15) *if and only if there are a hyperplane* Π *and a point* $x \in \Pi$ *such that*

$$\partial B_{\text{diam}\,W/2}(x) \cap \Pi \subset W, \tag{1.17}$$

i.e., W contains an $(n-1)$*-dimensional sphere of diameter* $\text{diam}\,(W)$*, and*

$$\Omega \setminus \left(\Pi \setminus \text{cl}\left(B_{\text{diam}\,W/2}(x)\right)\right) \text{ has exactly two unbounded connected components.} \tag{1.18}$$

Remark 1.2. The assumption $\mathcal{R}(W) > 0$ is quite weak: indeed, **if** $\mathcal{R}(W) = 0$, **then** W **is purely** $\mathcal{H}^n$**-unrectifiable**; see Proposition C.1 in the Appendix. For the role of the topological condition (1.18); see Fig. 3.

Remark 1.3. (Regularity of isoperimetric residues) In the physical dimension $n = 2$, and provided Ω has boundary of class $C^{1,1}$, maximizers of $\mathcal{R}(W)$ are $C^{1,1/2}$-regular up to the obstacle, and smooth away from it. More generally, condition (1.12) implies that $M = \text{cl}\,(\Omega \cap \partial F)$ is a smooth hypersurface with boundary in $\Omega \setminus \Sigma$, where Σ is a closed set such that $\Sigma \cap \Omega$ is empty if $1 \le n \le 6$, is locally discrete in Ω if $n = 7$, and is locally $\mathcal{H}^{n-7}$-rectifiable in Ω if $n \ge 8$; see, e.g. [27, Part III], [30]. Of course, by (1.13), $\Sigma \setminus B_{R_2} = \emptyset$ in every dimension. Moreover, justifying the initial claim concerning the case $n = 2$, if we assume that Ω is an open set with $C^{1,1}$-boundary, then M is a $C^{1,1/2}$-hypersurface with boundary in $\mathbb{R}^{n+1} \setminus \Sigma$, with boundary contained in $\partial\Omega$, $\Sigma \cap \partial\Omega$ is $\mathcal{H}^{n-3+\varepsilon}$-negligible for every $\varepsilon > 0$, and Young's law $\nu_F \cdot \nu_\Omega = 0$ holds on $(M \cap \partial\Omega) \setminus \Sigma$; see, e.g. [13,14,24,25].

Remark 1.4. An interesting open direction is finding additional geometric information on $\mathcal{R}(W)$, e.g. in the class of convex obstacles. It would also be interesting to quantify more precisely in terms of W some of the other quantities appearing in Theorem 1.1. For instance, it could be that $R_2 \le C(n)\text{diam}\,W$.

Remark 1.5. (Normalization of competitors) We adopt the convention that any set of locally finite perimeter F in Ω open is tacitly modified on and by a set of zero Lebesgue measure so to entail $\Omega \cap \partial F = \Omega \cap \mathrm{cl}\,(\partial^* F)$, where $\partial^* F$ is the reduced boundary of F in Ω; see [27, Proposition 12.19]. Under this normalization, local perimeter minimality conditions like (1.12) (or (3.1) below) imply that $F \cap \Omega$ is open in $\mathbb{R}^{n+1}$; see, e.g. [13, Lemma 2.16].

1.3. Resolution of Exterior Isoperimetric Sets

Denoting the family of minimizers of $\psi_W(v)$ by $\mathrm{Min}[\psi_W(v)]$ and the annulus $B_s \setminus \mathrm{cl}\, B_r$ by A_r^s for $0 < r < s$, our second main result is as follows:

Theorem 1.6. (Resolution of exterior isoperimetric sets) *If $W \subset \mathbb{R}^{n+1}$ is compact, then* $\mathrm{Min}[\psi_W(v)] \neq \emptyset \ \forall v > 0$. *Moreover, if $\mathcal{R}(W) > 0$, then*

$$\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) = -\mathcal{R}(W)\,, \tag{1.19}$$

and, depending on n and W only, there are v_0, C_0, R_1, and R_2 positive, and $R_0(v)$ with $R_0(v) \to 0^+$, $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, such that, if $E_v \in$ $\mathrm{Min}[\psi_W(v)]$ *and $v > v_0$, then:*

(i)*: There exist $x \in \mathbb{R}^{n+1}$ and $u \in C^\infty(\partial B^{(1)})$ such that*

$$\frac{|E_v \Delta B^{(v)}(x)|}{v} \leq \frac{C_0}{v^{1/[2(n+1)]}}, \tag{1.20}$$

$$(\partial E_v) \setminus B_{R_0(v)\, v^{1/(n+1)}} = \Big\{ y + v^{1/(n+1)}\, u\Big(\frac{y-x}{v^{1/(n+1)}}\Big)\, \nu_{B^{(v)}(x)}(y) : y \in \partial B^{(v)}(x) \Big\} \setminus B_{R_0(v)\, v^{1/(n+1)}}, \tag{1.21}$$

where, for any $G \subset \mathbb{R}^{n+1}$ with locally finite perimeter, ν_G is the outer unit normal to G;

(ii): *There exist $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$ and $f \in C^\infty((\partial F) \setminus B_{R_2})$ with*

$$(\partial E_v) \cap A_{R_2}^{R_1\, v^{1/(n+1)}} = \big\{ y + f(y)\, \nu_F(y) : y \in \partial F \big\} \cap A_{R_2}^{R_1\, v^{1/(n+1)}}\,; \tag{1.22}$$

(iii): $(\partial E_v) \setminus B_{R_2}$ *is diffeomorphic to an n-dimensional disk;*

(iv): *Finally, with (x, u) as in (1.21) and (F, ν, f) as in (1.22),*

$$\lim_{v \to \infty} \sup_{E_v \in \mathrm{Min}[\psi_W(v)]} \Big\{ \Big| \frac{|x|}{v^{1/(n+1)}} - \frac{1}{\omega_{n+1}^{1/(n+1)}} \Big|, \Big| \nu - \frac{x}{|x|} \Big|, \|u\|_{C^1(\partial B^{(1)})} \Big\} = 0,$$

$$\lim_{v \to \infty} \sup_{E_v \in \mathrm{Min}[\psi_W(v)]} \|f\|_{C^1(B_M \cap \partial F)} = 0, \qquad \forall M > R_2.$$

Remark 1.7. (Resolution up to the obstacle) By Remark 1.3 and a covering argument, if $n \leq 6$, $\delta > 0$, and $v > v_0(n, W, \delta)$, then (1.22) holds with $B_{R_1\, v^{1/(n+1)}} \setminus I_\delta(W)$ in place of $B_{R_1\, v^{1/(n+1)}} \setminus B_{R_2}$, where $I_\delta(W)$ is the open δ-neighborhood of W. Similarly, when $\partial\Omega \in C^{1,1}$ and $n = 2$ (and thus $\Omega \cap \partial F$ is regular up to the obstacle), we can find v_0 (depending on n and W only) such that (1.22) holds with $B_{R_1\, v^{1/(n+1)}} \cap \Omega$ in place of $B_{R_1\, v^{1/(n+1)}} \setminus B_{R_2}$, that is, graphicality over ∂F holds up to the obstacle itself.

Remark 1.8. If W is convex and J is an half-space, then $\psi_W(v) \ge \psi_J(v)$ for every $v > 0$, with equality for $v > 0$ if and only if ∂W contains a flat facet supporting an half-ball of volume v; see [5,21]. Since $\psi_J(v) = P(B^{(v)})/2^{1/(n+1)}$ and $\psi_W(v) - P(B^{(v)}) \to -\mathcal{R}(W)$ as $v \to \infty$, the bound $\psi_W(v) \ge \psi_J(v)$ is far from optimal if v is large. Are there stronger global bounds than $\psi_W \ge \psi_J$ on convex obstacles? Similarly, it would be interesting to quantify the convergence towards $\mathcal{R}(W)$ in (1.19), or even that of ∂E_v towards $\partial B^{(v)}$ and ∂F (where (1.20) should not to be sharp).

1.4. The Mesoscale Flatness Criterion

We work with with hypersurfaces M whose mean curvature is bounded by $\Lambda \ge 0$ in an annulus $B_{1/\Lambda} \setminus \overline{B}_R$, $R \in (0, 1/\Lambda)$. Even without information on M inside B_R (where M could have a non-trivial boundary, or topology, etc.) the classical proof of the monotonicity formula can be adapted to show the monotone increasing character on $r \in (R, 1/\Lambda)$ of

$$\Theta_{M,R,\Lambda}(r) = \frac{\mathcal{H}^n\big(M \cap (B_r \setminus B_R)\big)}{r^n} + \frac{R}{n\, r^n} \int_{M \cap \partial B_R} \frac{|x^{TM}|}{|x|}\, d\mathcal{H}^{n-1} + \Lambda \int_R^r \frac{\mathcal{H}^n\big(M \cap (B_\rho \setminus B_R)\big)}{\rho^n}\, d\rho, \tag{1.23}$$

(here $x^{TM} = \mathrm{proj}_{T_x M}(x)$); moreover, if $\Theta_{M,R,\Lambda}$ is constant over $(a, b) \subset (R, 1/\Lambda)$, then $M \cap (B_b \setminus \overline{B}_a)$ is a cone. Since the constant density value corresponding to $M = H \setminus B_R$, H an hyperplane through the origin, is ω_n (as a result of a double cancellation which also involves the "boundary term" in $\Theta_{H\setminus B_R,R,0}$), we consider the **area deficit**

$$\delta_{M,R,\Lambda}(r) = \omega_n - \Theta_{M,R,\Lambda}(r)\,, \qquad r \in (R, 1/\Lambda)\,, \tag{1.24}$$

which defines a decreasing quantity on $(R, 1/\Lambda)$. Here we use the term "deficit", rather than the more usual term "excess", since $\delta_{M,R,\Lambda}$ does not necessarily have non-negative sign (which is one of the crucial property of "excess quantities" typically used in ε-regularity theorems, see, e.g., [27, Lemma 22.11]). Recalling that $A_r^s = B_s \setminus \mathrm{cl}\,(B_r)$ if $s > r > 0$, we are now ready to state the following "smooth version" of our mesoscale flatness criterion (see Theorem 2.1 below for the varifold version):

Theorem 1.9. (Mesoscale flatness criterion (smooth version)) *If $n \ge 2$, $\Gamma \ge 0$, and $\sigma > 0$, then there are M_0 and ε_0 positive and depending on n, Γ and σ only, with the following property. Let $\Lambda \ge 0$, $R \in (0, 1/\Lambda)$, and M be a smooth hypersurface with mean curvature bounded by Λ in $A_R^{1/\Lambda}$, and with*

$$\mathcal{H}^{n-1}\big(M \cap \partial B_R\big) \le \Gamma\, R^{n-1}\,, \qquad \sup_{\rho \in (R, 1/\Lambda)} \frac{\mathcal{H}^n\big(M \cap (B_\rho \setminus B_R)\big)}{\rho^n} \le \Gamma\,. \tag{1.25}$$

If there is $s > 0$ *such that*

$$\max\{M_0, 64\}\, R < s < \frac{\varepsilon_0}{4\,\Lambda}, \tag{1.26}$$

and

$$|\delta_{M,R,\Lambda}(s/8)| \le \varepsilon_0\,, \tag{1.27}$$

and if, setting,

$$R_* = \sup\left\{\rho \ge \frac{s}{8} : \delta_{M,R,\Lambda}(\rho) \ge -\varepsilon_0\right\}, \qquad S_* = \min\left\{R_*, \frac{\varepsilon_0}{\Lambda}\right\},$$

we have $R_* > 4\,s$ *(and thus* $S_* > 4\,s$*), then*

$$\begin{aligned} M \cap A^{S_*/16}_{s/32} &= \left\{x + f(x)\,\nu_K : x \in K\right\} \cap A^{S_*/16}_{s/32}, \\ \sup\left\{|x|^{-1}\,|f(x)| + |\nabla f(x)| : x \in K\right\} &\le C(n)\,\sigma \end{aligned} \tag{1.28}$$

for a hyperplane K *with* $0 \in K$ *and unit normal* ν_K*, and for* $f \in C^1(K)$.

Remark 1.10. (Structure of the statement) The first condition in (1.26) implicitly requires R to be sufficiently small in terms of $1/\Lambda$, as it introduces a mesoscale s which is both small with respect to $1/\Lambda$ and large with respect to R. The condition in (1.27) expresses the flatness of M at the mesoscale s in terms of its area deficit. The final key assumption, $R_* > 4\,s$, expresses the requirement that the area deficit does not decrease too abruptly, and stays above $-\varepsilon_0$ at least up to the scale $4\,s$. Under these assumptions, graphicality with respect to a hyperplane K is inferred on an annulus whose lower radius $s/32$ has the order of the mesoscale s, and whose upper radius $S_*/16$ can be as large as the decay of the area deficit allows (potentially up to $\varepsilon_0/16\,\Lambda$ if $R_* = \infty$), but in any case not too large with respect to $1/\Lambda$.

Remark 1.11. (Relationship to other flatness criteria) If M is a hypersurface containing the origin, so that, formally speaking, $R = 0$, and the tangent cone of M there is a plane, Theorem 1 reduces to Allard's theorem [3]. Similarly, if $\Lambda = 0$ and the exterior minimal hypersurface M has a planar tangent cone at infinity, we recover the exterior blow-down results stated in [35,36]. In particular, although the motivation for Theorem 1 comes from scenarios where both R and Λ are positive, it can also be viewed as a general framework containing as special cases the blow-up and blow-down flatness criteria for hypersurfaces with planar tangent cones.

Remark 1.12. (Sharpness of the statement) The statement is sharp in the sense that for a surface "with bounded mean curvature and non-trivial topology inside a hole", flatness can only be established on a mesoscale which is both large with respect to the size of the hole and small with respect to the size of the inverse mean curvature. An example is provided by unduloids M_ε with waist size ε and mean curvature n in $\mathbb{R}^{n+1}$; see Fig. 4. A "half-period" of M_ε is the graph $\{x + f_\varepsilon(x)\,e_{n+1} : x \in \mathbb{R}^n, \varepsilon < |x| < R_\varepsilon\}$ of

$$f_\varepsilon(x) = \int_\varepsilon^{|x|} \left\{\left(\frac{r^{n-1}}{r^n - \varepsilon^n + \varepsilon^{n-1}}\right)^2 - 1\right\}^{-1/2} dr\,, \qquad \varepsilon < |x| < R_\varepsilon\,, \tag{1.29}$$

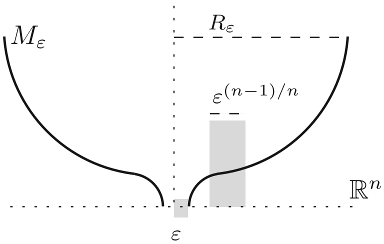

Fig. 4. A half-period of an unduloid with mean curvature n and waist size ε in $\mathbb{R}^{n+1}$. By (1.29), the flatness of M_ε is no smaller than $\mathrm{O}(\varepsilon^{2(n-1)/n})$, and is exactly $\mathrm{O}(\varepsilon^{2(n-1)/n})$ on an annulus sitting in the mesoscale $\mathrm{O}(\varepsilon^{(n-1)/n})$. This mesoscale is both very large with respect to waist size ε, and very small with respect to the size of the inverse mean curvature, which is order one

where ε and R_ε are the only solutions of $r^{n-1} = r^n - \varepsilon^n + \varepsilon^{n-1}$. Clearly f_ε solves $-\mathrm{div}\,(\nabla f_\varepsilon/\sqrt{1+|\nabla f_\varepsilon|^2}) = n$ with $f_\varepsilon = 0$, $|\nabla f_\varepsilon| = +\infty$ on $\{|x| = \varepsilon\}$, and $|\nabla f_\varepsilon| = +\infty$ on $\{|x| = R_\varepsilon\}$, where $R_\varepsilon = 1 - \mathrm{O}(\varepsilon^{n-1})$; moreover, $\min|\nabla f_\varepsilon|$ is achieved at $r = \mathrm{O}(\varepsilon^{(n-1)/n})$, and if $r \in (a\,\varepsilon^{(n-1)/n}, b\,\varepsilon^{(n-1)/n})$ for some $b > a > 0$, then $|\nabla f_\varepsilon| = \mathrm{O}_{a,b}(\varepsilon^{2(n-1)/n})$. Thus, the horizontal flatness of M_ε is no smaller than $\mathrm{O}(\varepsilon^{2(n-1)/n})$, and has that exact order on a scale which is both very large with respect to the hole ($\varepsilon^{(n-1)/n} \gg \varepsilon$) and very small with respect to the inverse mean curvature ($\varepsilon^{(n-1)/n} \ll 1$).

Remark 1.13. (On the application to $\psi_W(v)$) Exterior isoperimetric sets E_v with large volume v have small constant mean curvature of order $\Lambda = \Lambda_0(n, W)/v^{1/(n+1)}$. We will work with "holes" of size $R = R_3(n, W)$, for some R_3 sufficiently large with respect to the radius R_2 appearing in Theorem 1.1–(ii), and determined through the sharp decay rates (1.14). The decay properties of F towards $\{x : x \cdot \nu = a\}$ when (F, ν) is a maximizer of $\mathcal{R}(W)$, the C^1-proximity of ∂E to $\partial B^{(v)}(x)$ for $|x| \approx (\omega_{n+1}/v)^{1/(n+1)}$, and the C^1-proximity of ∂E to ∂F for some optimal (F, ν) on bounded annuli of the form $A^{2R_3}_{R_2}$ are used in checking that (1.25) holds with $\Gamma = \Gamma(n, W)$, that E_v is flat in the sense of (1.27), and, most importantly, that the area deficit $\delta_{M,R,\Lambda}$ of $M = (\partial E_v)\setminus B_{R_3}$ lies above $-\varepsilon_0$ up to scale $r = \mathrm{O}(v^{1/(n+1)})$ (which is the key information to deduce $R_* \approx 1/\Lambda$), and thus obtain overlapping domains of resolutions in terms of $\partial B^{(v)}(x)$ and ∂F.

Remark 1.14. While Theorem 1.9 seems clearly applicable to other problems, there are situations where one may need to develop considerably finer "mesoscale flatness criteria". For example, consider the problem of "resolving" almost CMC boundaries undergoing bubbling [9,11,12]. When the oscillation of the mean curvature around a constant Λ is small, such boundaries are close to finite unions of mutually tangent spheres of radius n/Λ, and can be covered by C^1-small normal graphs over such spheres away from their tangency points up to distance ε/Λ, with $\varepsilon = \varepsilon(n)$, and provided the mean curvature oscillation is small in terms of ε. For propagating flatness up to a distance directly related to the oscillation of the mean curvature, one would need a version of Theorem 1.9 for "double" spherical graphs; in the setting of blowup/blowdown theorems, this would be similar to passing to the harder case of multiplicity larger than one.

Remark 1.15. (Comparison with blowup/blowdown results) From the technical viewpoint, Theorem 1.9 fits into the framework set up by Allard and Almgren in [1] for the study of blowups and blowdowns of minimal surfaces with tangent integrable cones. At the same time, as exemplified by Remark 1.12, Theorem 1.9 really points in a different direction, since it pertains to situations where neither blowup or blowdown limits make sense. Another interesting point is that, in [1], the area deficit $\delta_{M,R,\Lambda}$ is considered with a sign, non-positive for blowups, and non-negative for blowdowns, see [1, Theorem 5.9(4), Theorem 9.6(4)]. A key insight here is that for hypersurfaces where the deficit changes sign, graphicality obtained through small negative (or positive) deficit nevertheless persists *past* the scale where $\delta_{M,R,\Lambda}$ vanishes, and possibly much farther depending on the surface in question; this is actually *crucial* for obtaining overlapping domains of resolutions in statements like (1.4) and (1.7).

Remark 1.16. (Extension to general minimal cones) Proving Theorem 1.9 in higher codimension and with arbitrary *integrable* minimal cones should be possible with essentially the same proof presented here. We do not pursue this extension because, first, only the case of hypersurfaces and hyperplanes is needed in studying $\psi_W(v)$; and, second, in going for generality, one should work in the framework set up by Simon in [33,35,37], which, at variance with the simpler Allard–Almgren's framework used here, allows one to dispense with the integrability assumption. In this direction, we notice that Theorem 1.9 with $\Lambda = 0$ and $R_* = +\infty$ is a blowdown result for exterior minimal surfaces (see also Theorem 2.1–(ii), (iii)). A blowdown result for exterior minimal surfaces is outside the scope of [1, Theorem 9.6] which pertains to *entire* minimal surfaces, but it is claimed, with a sketch of proof, on [35, Page 269] as a modification of [35, Theorem 5.5, $m < 0$]. It should be mentioned that, to cover the case of exterior minimal surfaces, an additional term of the form $C\int_\Sigma (\dot{u}(t))^2$ should be added on the right side of assumption [35, 5.3, $m < 0$]. This additional term seems not to cause difficulties with the rest of the arguments leading to [35, Theorem 5.5, $m < 0$]. Thus Simon's approach, in addition to giving the blowdown analysis of exterior minimal surfaces, should also be viable for generalizing our mesoscale flatness criterion.

1.5. Organization of the Paper

In Sect. 2 we prove Theorem 1.9 (actually, its generalization to varifolds, i.e. Theorem 2.1). In Sect. 3 we prove those parts of Theorem 1.6 which follow simply by quantitative isoperimetry (i.e., they do not require isoperimetric residues nor our mesoscale flatness analysis); see Theorem 3.1. Section 4 is devoted to the study of isoperimetric residues and of their maximizers, and contains the proof Theorem 1.1. We also present there a statement, repeatedly used in our analysis, which summarizes some results from [32]; see Proposition 4.1. Finally, in Sect. 5, we prove the energy expansion (1.19) and those parts of Theorem 1.6 left out in Sect. 3 (i.e., statements (ii, iii, iv)). This final Section is, from a certain viewpoint, the most interesting part of the paper: indeed, it is only the detailed examination of those arguments that clearly illustrates the degree of fine tuning of the preliminary analysis of exterior isoperimetric sets and of maximizers of isoperimetric residues which is needed in order to allow for the application of the mesoscale flatness criterion.

2. A Mesoscale Flatness Criterion for Varifolds

In Sect. 2.1 we introduce the class $\mathcal{V}_n(\Lambda, R, S)$ of varifolds used to reformulate Theorem 1.9, see Theorem 2.1. In Sects. 2.2–2.3 we present two reparametrization lemmas (2.3, 2.5) and some "energy estimates" (Theorem 2.6) for spherical graphs; in Sect. 2.4 we state the monotonicity formula in $\mathcal{V}_n(\Lambda, R, S)$ and some energy estimates involving the monotonicity gap; in Sect. 2.5, we prove Theorem 2.1.

2.1. Statement of the Criterion

Given an n-dimensional integer rectifiable varifold $V = \mathbf{var}\,(M, \theta)$ in $\mathbb{R}^{n+1}$, defined by a locally $\mathcal{H}^n$-rectifiable set M, and by a multiplicity function $\theta : M \to \mathbb{N}$ (see [34]), we denote by $\|V\| = \theta\,\mathcal{H}^n \llcorner M$ the weight of V, and by δV the first variation of V, so that $\delta V(X) = \int \mathrm{div}^T X(x)\, dV(x, T) = \int_M \mathrm{div}^M X(x)\,\theta\, d\mathcal{H}^n_x$ for every $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$. Given $S > R > 0$ and $\Lambda \ge 0$, we consider the family

$$\mathcal{V}_n(\Lambda, R, S),$$

of those n-dimensional integral varifolds V with $\mathrm{spt}\, V \subset \mathbb{R}^{n+1} \setminus B_R$ and

$$\delta V(X) = \int X \cdot \boldsymbol{H}\, d\|V\| + \int X \cdot \nu_V^{\mathrm{co}}\, d\,\mathrm{bd}_V, \qquad \forall X \in C^1_c(B_S; \mathbb{R}^{n+1}),$$

holds for a Radon measure bd_V in $\mathbb{R}^{n+1}$ supported in ∂B_R, and Borel vector fields $\boldsymbol{H} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $|\boldsymbol{H}| \le \Lambda$ and $\nu_V^{\mathrm{co}} : \partial B_R \to \mathbb{R}^{n+1}$ with $|\nu_V^{\mathrm{co}}| = 1$. We let $\mathcal{M}_n(\Lambda, R, S) = \{V \in \mathcal{V}_n(\Lambda, R, S) : V = \mathbf{var}\,(M, 1) \text{ for } M \text{ smooth}\}$, that is, $M \subset \mathbb{R}^{n+1} \setminus B_R$ is a smooth hypersurface with boundary in A_R^S, $\mathrm{bdry}\,(M) \subset \partial B_R$, and $|H_M| \le \Lambda$. If $V \in \mathcal{M}_n(\Lambda, R, S)$, then $\boldsymbol{H}$ is the mean curvature vector of M, $\mathrm{bd}_V = \mathcal{H}^{n-1} \llcorner \mathrm{bdry}\,(M)$, and ν_V^{co} is the outer unit conormal to M along ∂B_R. Given $V \in \mathcal{V}_n(\Lambda, R, S)$, we define

$$\Theta_{V,R,\Lambda}(r) = \frac{\|V\|(B_r \setminus B_R)}{r^n} - \frac{1}{n\, r^n} \int x \cdot \nu_V^{\mathrm{co}} d\,\mathrm{bd}_V + \Lambda \int_R^r \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} d\rho.$$

$\Theta_{V,R,\Lambda}(r)$ is increasing for $r \in (R, S)$ (Theorem 2.7–(i) below), and equal to (1.23) when $V \in \mathcal{M}_n(\Lambda, R, S)$. The **area deficit** of V is then defined as in (1.24), while given a hyperplane H in $\mathbb{R}^{n+1}$ with $0 \in H$ we call the quantity

$$\int_{A_r^s} \omega_H(y)^2\, d\|V\|_y\,, \qquad \omega_H(y) = \arctan\Big(\frac{|y \cdot \nu_H|}{|\mathbf{p}_H y|}\Big)\,,$$

the **angular flatness of** V **on the annulus** $A_r^s = B_s \setminus \mathrm{cl}\,(B_r)$ **with respect to** H. (See (2.8) for the notation concerning H.)

Theorem 2.1. (Mesoscale flatness criterion) *If $n \ge 2$, $\Gamma \ge 0$, and $\sigma > 0$ then there are positive constants M_0 and ε_0, depending on n, Γ and σ only, with the property that: $\Lambda \ge 0$, $R \in (0, 1/\Lambda)$, $V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda)$,*

$$\|\mathrm{bd}_V\|(\partial B_R) \le \Gamma\, R^{n-1}\,, \qquad \sup_{\rho\in(R,1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \le \Gamma\,, \tag{2.1}$$

and, for some $s > 0$, we have that

$$\frac{\varepsilon_0}{4\,\Lambda} > s > \max\{M_0, 64\}\, R, \tag{2.2}$$

$$|\delta_{V,R,\Lambda}(s/8)| \le \varepsilon_0, \tag{2.3}$$

$$R_* := \sup\left\{\rho \ge \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \ge -\varepsilon_0\right\} \ge 4\,s, \tag{2.4}$$

then
(i): *if $S_* = \min\{R_*, \varepsilon_0/\Lambda\} < \infty$, then there is an hyperplane $K \subset \mathbb{R}^{n+1}$ with $0 \in K$ and $u \in C^1((K \cap \mathbb{S}^n) \times (s/32, S_*/16))$ with*

$$(\mathrm{spt}\, V) \cap A^{S_*/16}_{s/32} = \left\{r\, \frac{\omega + u(r,\omega)\,\nu_K}{\sqrt{1+u(r,\omega)^2}} : \omega \in K \cap \mathbb{S}^n,\, r \in \big(s/32, S_*/16\big)\right\}$$
$$\sup_{(K\cap\mathbb{S}^n)\times\big(s/32,S_*/16\big)} \left\{|u| + |\nabla^{K\cap\mathbb{S}^n} u| + |r\,\partial_r u|\right\} \le C(n)\,\sigma; \tag{2.5}$$

(ii): *if $\Lambda = 0$ and $\delta_{V,R,0} \ge -\varepsilon_0$ on $(s/8, \infty)$, then $\delta_{V,R,0} \ge 0$ on $(s/8, \infty)$, (2.5) holds with $S_* = \infty$, and one has decay estimates, continuous in the radius, of the form*

$$\delta_{V,R,0}(r) \le C(n) \left(\frac{s}{r}\right)^\alpha \delta_{V,R,0}\Big(\frac{s}{8}\Big), \qquad \forall r > \frac{s}{4}, \tag{2.6}$$

$$\frac{1}{r^n} \int_{A_r^{2r}} \omega_K^2\, d\|V\| \le C(n)\,(1+\Gamma) \left(\frac{s}{r}\right)^\alpha \delta_{V,R,0}\Big(\frac{s}{8}\Big), \qquad \forall r > \frac{s}{4}, \tag{2.7}$$

for some $\alpha(n) \in (0, 1)$.

Remark 2.2. In Theorem 2.1, graphicality is formulated in terms of the notion of *spherical graph* (see Sect. 2.2) which is more natural than the usual notion of "cylindrical graph" in setting up the iteration procedure behind Theorem 2.1. Spherical graphicality in terms of a C^1-small u as in (2.5) translates into cylindrical graphicality in terms of f as in (1.28) with $f(x)/|x| \approx u(|x|, \hat{x})$ and $\nabla_{\hat{x}} f(x) - (f(x)/|x|) \approx |x|\,\partial_r\, u(|x|, \hat{x})$ for $x \ne 0$ and $\hat{x} = x/|x|$; see, in particular, Lemma B.1 in "Appendix B".

2.2. Spherical Graphs

We start setting up some notation. We denote by

$$\mathcal{H}$$

the family of the oriented hyperplanes $H \subset \mathbb{R}^{n+1}$ with $0 \in H$, so that for any $H \in \mathcal{H}$ a unit normal vector ν_H to H is defined. Given $H \in \mathcal{H}$, we set

$$\Sigma_H = H \cap \mathbb{S}^n \,, \qquad \mathbf{p}_H : \mathbb{R}^{n+1} \to H \,, \qquad \mathbf{q}_H : \mathbb{R}^{n+1} \to H^\perp \,, \tag{2.8}$$

for the equatorial sphere defined by H on $\mathbb{S}^n$ and for the orthogonal projections of $\mathbb{R}^{n+1}$ onto H and onto $H^\perp = \{t\, \nu_H : t \in \mathbb{R}\}$. We set

$$\mathcal{X}_\sigma(\Sigma_H) = \big\{u \in C^1(\Sigma_H) : \|u\|_{C^1(\Sigma_H)} < \sigma\big\}, \qquad \sigma > 0.$$

Clearly there is $\sigma_0 = \sigma_0(n) > 0$ such that if $H \in \mathcal{H}$ and $u \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, then

$$f_u(\omega) = \frac{\omega + u(\omega)\, \nu_H}{\sqrt{1 + u(\omega)^2}}, \qquad \omega \in \Sigma_H,$$

defines a diffeomorphism of Σ_H into an hypersurface $\Sigma_H(u) \subset \mathbb{S}^n$, namely

$$\Sigma_H(u) = f_u(\Sigma_H) = \left\{\frac{\omega + u(\omega)\, \nu_H}{\sqrt{1 + u(\omega)^2}} : \omega \in \Sigma_H\right\}. \tag{2.9}$$

We call $\Sigma_H(u)$ a **spherical graph** over Σ_H. Exploiting the fact that Σ_H is a minimal hypersurface in $\mathbb{S}^n$ and that if $\{\tau_i\}_i$ is a local orthonormal frame on Σ_H then $\nu_H \cdot \nabla_{\tau_i} \tau_j = 0$, a second variation computation (see, e.g., [16, Lemma 2.1]) gives, for $u \in \mathcal{X}_\sigma(\Sigma_H)$,

$$\left|\mathcal{H}^{n-1}(\Sigma_H(u)) - n\, \omega_n - \frac{1}{2} \int_{\Sigma_H} |\nabla^{\Sigma_H} u|^2 - (n-1)\, u^2\right| \le C(n)\, \sigma \int_{\Sigma_H} u^2 + |\nabla^{\Sigma_H} u|^2,$$

(where $n\, \omega_n = \mathcal{H}^{n-1}(\Sigma_H) = \mathcal{H}^{n-1}(\Sigma_H(0))$). We recall that $u \in L^2(\Sigma_H)$ is a unit norm Jacobi field of Σ_H (i.e., a zero eigenvector of $\Delta^{\Sigma_H} + (n-1)\,\mathrm{Id}$ with unit $L^2(\Sigma_H)$-norm) if and only if there is $\tau \in \mathbb{S}^n$ with $\tau \cdot \nu_H = 0$ and $u(\omega) = c_0(n)\, (\omega \cdot \tau)$ $(\omega \in \Sigma_H)$ for $c_0(n) = (n/\mathcal{H}^{n-1}(\Sigma_H))^{1/2}$. We denote by $E^0_{\Sigma_H}$ the orthogonal projection operator of $L^2(\Omega)$ onto the span of the Jacobi fields of Σ_H. The following lemma provides a way to reparameterize spherical graphs over equatorial spheres so that the projection over Jacobi fields is annihilated.

Lemma 2.3. *There exist constants C_0, ε_0 and σ_0, depending on the dimension n only, with the following properties:*
(i): *if $H, K \in \mathcal{H}$, $|\nu_H - \nu_K| \le \varepsilon < \varepsilon_0$, and $u \in \mathcal{X}_\sigma(\Sigma_H)$ for $\sigma < \sigma_0$, then the map $T^K_u : \Sigma_H \to \Sigma_K$ defined by*

$$T^K_u(\omega) = \frac{\mathbf{p}_K(f_u(\omega))}{|\mathbf{p}_K(f_u(\omega))|} = \frac{\mathbf{p}_K \omega + u(\omega)\, \mathbf{p}_K \nu_H}{|\mathbf{p}_K \omega + u(\omega)\, \mathbf{p}_K \nu_H|}, \qquad \omega \in \Sigma_H,$$

is a diffeomorphism between Σ_H and Σ_K, and $v^K_u : \Sigma_K \to \mathbb{R}$ defined by

$$v^K_u(T^K_u(\omega)) = \frac{\mathbf{q}_K(f_u(\omega))}{|\mathbf{p}_K(f_u(\omega))|} = \frac{\nu_K \cdot (\omega + u(\omega)\, \nu_H)}{|\mathbf{p}_K \omega + u(\omega)\, \mathbf{p}_K \nu_H|}, \qquad \omega \in \Sigma_H \,, \tag{2.10}$$

is such that

$$v^K_u \in \mathcal{X}_{C(n)\,(\sigma+\varepsilon)}(\Sigma_K), \quad \Sigma_H(u) = \Sigma_K(v^K_u), \tag{2.11}$$

$$\Big|\int_{\Sigma_K}(v_u^K)^2-\int_{\Sigma_H}u^2\Big|\le C(n)\Big\{|\nu_H-\nu_K|^2+\int_{\Sigma_H}u^2\Big\}. \tag{2.12}$$

(ii): *if $H\in\mathcal{H}$ and $u\in\mathcal{X}_{\sigma_0}(\Sigma_H)$, then there exist $K\in\mathcal{H}$ with $|\nu_H-\nu_K|<\varepsilon_0$ and $v\in\mathcal{X}_{C_0\,\sigma_0}(\Sigma_K)$ such that*

$$\Sigma_H(u)=\Sigma_K(v), \tag{2.13}$$

$$E^0_{\Sigma_K}[v]=0, \tag{2.14}$$

$$|\nu_K-\nu_H|^2\le C_0(n)\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u]\big)^2, \tag{2.15}$$

$$\Big|\int_{\Sigma_K}v^2-\int_{\Sigma_H}u^2\Big|\le C_0(n)\int_{\Sigma_H}u^2. \tag{2.16}$$

Remark 2.4. It may seem unnecessary to present a detailed proof of Lemma 2.3, as we are about to do, given that, when Σ_H is replaced by a generic integrable minimal surface Σ in $\mathbb{S}^n$, similar statements are found in the first four sections of [1, Chapter 5]. However, two of those statements, namely [1, 5.3(4), 5.3(5)], seem not to be correct; and the issue requires clarification, since those statements are used in the iteration arguments for the blowup/blowdown theorems [1, Theorem 5.9/Theorem 9.6]; see, e.g., the second displayed chain of inequalities on [1, Page 254]. To explain this issue **we momentarily adopt the notation of** [1]. In [1, Chapter 5] they consider a family of minimal surfaces $\{M_t\}_{t\in U}$ in $\mathbb{S}^n$ obtained as diffeomorphic images of a minimal surface $M=M_0$. The parameter t ranges in an open ball $U\subset\mathbb{R}^j$, where j is the dimension of the space of Jacobi fields of M. Given a vector field Z in $\mathbb{S}^n$, defined on and normal to M_t, they denote by $F_t(Z)$ the diffeomorphism of M_t into $\mathbb{S}^n$ obtained by combining Z with the exponential map of $\mathbb{S}^n$ (up to lower than second order corrections in Z, this is equivalent to taking the graph of Z over M_t, and then projecting it back on $\mathbb{S}^n$, which is what we do, following [33], in (2.9)). Then, in [1, 5.2(2)], they define Λ_t as the family of those Z such that $\mathrm{Image}(F_t(Z))=\mathrm{Image}(F_0(W))$ for some vector field W normal to M, and, given $t,u\in U$ and $Z\in\Lambda_t$, they define $F_t^u:\Lambda_t\to\Lambda_u$ as the map between such classes of normal vector fields with the property that $\mathrm{Image}(F_t(Z))=\mathrm{Image}(F_u(F_t^u(Z)))$: in particular, $F_t^u(Z)$ is the vector field that takes M_u to the same surface to which Z takes M_t. With this premise, in [1, 5.3(5)] they say that if $t,u\in U$, and $Z\in\Lambda_t$, then

$$\Big|\int_{M_u}|F_t^u(Z)|^2-\int_{M_t}|Z|^2\Big|\le C\,|t-u|\int_{M_t}|Z|^2\,, \tag{2.17}$$

for a constant C depending on M only. Testing this with $Z=0$ (notice that $0\in\Lambda_t$ by [1, 5.3(1)]) one finds $F_t^u(0)=0$, and thus $M_t=\mathrm{Image}(F_t(0))=\mathrm{Image}(F_u(F_t^u(0)))=\mathrm{Image}(F_u(0))=M_u$. In particular, $M_u=M_t$ for every $t,u\in U$, that is, $\{M_t\}_{t\in U}$ consists of a single surface, M itself. But this is never the case since $\{M_t\}_{t\in U}$ always contains, to the least, every sufficiently small rotation of M in $\mathbb{S}^n$. An analogous problem is contained in [1, 5.3(4)]. Coming back to our notation, the analogous estimate to (2.17) in our setting would mean that, for every

$H, K \in \mathcal{H}$ with $|\nu_K - \nu_H| < \varepsilon_0$ and $u \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, v_u^K defined in (2.10) satisfies

$$\Big| \int_{\Sigma_K} (v_u^K)^2 - \int_{\Sigma_H} u^2 \Big| \le C(n)\, |\nu_H - \nu_K| \int_{\Sigma_H} u^2 \,, \tag{2.18}$$

which again gives a contradiction if $u = 0$. A correct estimate, analogous in spirit to (2.18) and still sufficiently precise to be used in iterations, is (2.12) in Lemma 2.3. There should be no obstruction[1] in adapting our proof to the more general context of integrable cones, and then in using the resulting generalization of (2.12) to implement the iterations needed in [1, Theorem 5.9, Theorem 9.6].

Proof of Lemma 2.3. The constants ε_0 and σ_0 in the statement will be such that $\sigma_0 = \varepsilon_0 / C_*$ for a sufficiently large dimension dependent constant C_*.
Step one: To prove statement (i), let $H, K \in \mathcal{H}$, $|\nu_H - \nu_K| \le \varepsilon < \varepsilon_0$ and $u \in \mathcal{X}_\sigma(\Sigma_H)$ with $\sigma < \sigma_0$. Setting (for $\omega \in \Sigma_H$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$)

$$g_u^K(\omega) = \mathbf{p}_K \omega + u(\omega)\, \mathbf{p}_K \nu_H, \qquad \Phi(x) = x/|x|,$$

we have $T_u^K = \Phi \circ g_u^K$, and, if u is identically 0,

$$g_0^K(\omega) = \mathbf{p}_K \omega, \qquad T_0^K(\omega) = \frac{\mathbf{p}_K \omega}{|\mathbf{p}_K \omega|}, \qquad \forall \omega \in \Sigma_H.$$

By $|\mathbf{p}_K \nu_H|^2 = 1 - (\nu_H \cdot \nu_K)^2 \le 2\,(1 - (\nu_H \cdot \nu_K)) = |\nu_H - \nu_K|^2$,

$$\begin{aligned} |g_u^K - g_0^K| &= |u|\, |\mathbf{p}_K \nu_H| \le |u|\, |\nu_H - \nu_K|, \\ |\nabla^{\Sigma_H} g_u^K - \nabla^{\Sigma_H} g_0^K| &\le |\nabla^{\Sigma_H} u|\, |\nu_H - \nu_K|. \end{aligned}$$

In particular, $|g_u^K| \ge 1 - \sigma_0\, \varepsilon_0 \ge 1/2$, and since Φ and $\nabla \Phi$ are Lipschitz continuous on $\{|x| \ge 1/2\}$, we find

$$\max \left\{ \|g_u^K - g_0^K\|_{C^1(\Sigma_H)}, \|T_u^K - T_0^K\|_{C^1(\Sigma_H)} \right\} \le C(n)\, \|u\|_{C^1(\Sigma_H)}\, |\nu_H - \nu_K| \,. \tag{2.19}$$

Similarly, since $\omega \cdot \nu_K = \omega \cdot (\nu_K - \nu_H)$ for $\omega \in \Sigma_H$, we find that

$$\|g_0^K - \mathrm{id}\|_{C^1(\Sigma_H)} \le C(n)\, |\nu_H - \nu_K| \,, \qquad \|T_0^K - \mathrm{id}\|_{C^1(\Sigma_H)} \le C(n)\, |\nu_H - \nu_K| \,, \tag{2.20}$$

and we thus conclude that T_u^K is a diffeomorphism between Σ_H and Σ_K. As a consequence, the definition (2.10) of v_u^K is well-posed, and (2.11) immediately follows (in particular, $\Sigma_H(u) = \Sigma_K(v_u^K)$ is deduced easily from (2.10) and (2.9)). Finally, if we set $F_u^K(\omega) = v_u^K(T_u^K(\omega))^2\, J^{\Sigma_H}\, T_u^K(\omega)$ ($\omega \in \Sigma_H$), then

$$\int_{\Sigma_K} (v_u^K)^2 - \int_{\Sigma_H} u^2 = \int_{\Sigma_H} \Big(\frac{\nu_K \cdot (\omega + u\, \nu_H)}{|g_u^K(\omega)|} \Big)^2 J^{\Sigma_H}\, T_u^K(\omega) - u^2,$$

where, using again $|\omega \cdot \nu_K| \le |\nu_H - \nu_K|$ for every $\omega \in \Sigma_H$, we find

$$|J^{\Sigma_H} T_u^K(\omega) - 1| \le C(n)\, \|T_u^K - \mathrm{id}\|_{C^1(\Sigma_H)} \le C(n)\, |\nu_H - \nu_K|,$$

[1] At the time of publication of this paper, Allard has published a corrigendum to [1], see [4].

$$\begin{aligned}\big|1-|g_u^K(\omega)|^2\big| &\le \big|1-|\mathbf{p}_K\omega|^2\big| + |\mathbf{p}_K\nu_H|\,u^2 + 2\,|u|\,|\mathbf{p}_K\nu_H|\,|\mathbf{p}_K\omega|\\ &\le C\,\big(|\nu_H-\nu_K|^2+u^2\big),\end{aligned}$$

$$\begin{aligned}&\big|(\nu_K\cdot(\omega+u\,\nu_H))^2-u^2\big|\\ &\le |\nu_K\cdot\omega|^2 + u^2\,(1-(\nu_H\cdot\nu_K)^2) + 2\,|u|\,|\nu_H\cdot\nu_K|\,|\omega\cdot\nu_K|\\ &\le |\nu_K-\nu_K|^2 + 2\,u^2\,|\nu_H-\nu_K| + 2\,|u|\,|\nu_H-\nu_K| \le C\,\big(|\nu_H-\nu_K|^2+u^2\big)\end{aligned}$$

and thus, (2.12), thanks to

$$\begin{aligned}\Big|\int_{\Sigma_K}(v_u^K)^2-\int_{\Sigma_H}u^2\Big| &\le \int_{\Sigma_H}|J^{\Sigma_H}\,T_u^K-1|\,u^2 + 2\,\frac{|(\nu_K\cdot(\omega+u\,\nu_H))^2-u^2|}{|g_u^K|^2}\\ &\quad +2\int_{\Sigma_H}\Big|1-\frac{1}{|g_u^K|^2}\Big|\,u^2 \le C(n)\Big(|\nu_H-\nu_K|^2+\int_{\Sigma_H}u^2\Big).\end{aligned}$$

Step two: We prove (ii). If $E^0_{\Sigma_H}[u]=0$, then we conclude with $K=H$, $v=u$. We thus assume $\gamma^2=\int_{\Sigma_H}(E^0_{\Sigma_H}[u])^2>0$, and pick an orthonormal basis $\{\phi_H^i\}_{i=1}^n$ of $L^2(\Sigma_H)\cap\{E^0_{\Sigma_H}=0\}$ with $E^0_{\Sigma_H}[u]=\gamma\,\phi_H^1$ and $\gamma=\int_{\Sigma_H}u\,\phi_H^1\neq 0$. This corresponds to choosing an orthonormal basis $\{\tau_H^i\}_{i=1}^n$ of H such that

$$\phi_H^i(\omega)=c_0(n)\,\omega\cdot\tau_H^i\,,\qquad \omega\in\Sigma_H\,,$$

for $c_0(n)=(n/\mathcal{H}^{n-1}(\Sigma_H))^{1/2}$. For each $K\in\mathcal{H}$ with $\mathrm{dist}_{\mathbb{S}^n}(\nu_H,\nu_K)<\varepsilon_0$ we define an orthonormal basis $\{\tau_K^i\}_{i=1}^n$ of K by parallel transport of $\{\tau_H^i\}_{i=1}^n\subset H\equiv T_{\nu_H}\mathbb{S}^n$ to $K\equiv T_{\nu_K}\mathbb{S}^n$. The maps $\nu\mapsto\tau^i(\nu):=\tau^i_{K(\nu)}$ define an orthonormal frame $\{\tau^i\}_{i=1}^n$ of $\mathbb{S}^n$ on the open set $A=B^{\mathbb{S}^n}_{\varepsilon_0}(\nu_H)=\{\nu\in\mathbb{S}^n:\mathrm{dist}_{\mathbb{S}^n}(\nu,\nu_H)<\varepsilon_0\}$. We denote by ρ_H^K the rotation of $\mathbb{R}^{n+1}$ which takes H into K by setting $\rho_H^K(\tau_H^i)=\tau_K^i$ and $\rho_H^K(\nu_H)=\nu_K$. By the properties of parallel transport we have that

$$\|\rho_H^K-\mathrm{Id}\|_{C^0(\Sigma_K)}\le C(n)\,\mathrm{dist}_{\mathbb{S}^n}(\nu_H,\nu_K)\le C(n)\,\varepsilon_0\,. \tag{2.21}$$

Finally, we define an $L^2(\Sigma_K)$-orthonormal basis $\{\phi_K^i\}_{i=1}^n$ of $L^2(\Sigma_K)\cap\{E^0_{\Sigma_K}=0\}$ by setting $\phi_K^i(\omega)=c_0(n)\,\omega\cdot\tau_K^i$ ($\omega\in\Sigma_K$), and correspondingly consider the map $\Psi_u:A\to\mathbb{R}^n$ defined by setting

$$\Psi_u(\nu)=\Big(\int_{\Sigma_{K(\nu)}}v_u^{K(\nu)}\,\phi^1_{K(\nu)},\dots,\int_{\Sigma_{K(\nu)}}v_u^{K(\nu)}\,\phi^n_{K(\nu)}\Big),\qquad \nu\in A,$$

where $v_u^{K(\nu)}$ is well-defined for every $\nu\in A$ thanks to step one. **We now claim** the existence of $\nu_*\in A$ such that

$$\Psi_u(\nu_*)=0\,. \tag{2.22}$$

Before proving (2.22), we use it to deduce (2.13)–(2.16), thus finishing the proof of (ii) and the lemma modulo (2.22). With $K=K(\nu_*)$ and $v=v_u^K$ we deduce (2.13) from (2.11) and (2.14) from $\Psi_u(\nu_*)=0$. By (2.26) and (2.27), if $\eta=\mathrm{dist}_{\mathbb{S}^n}(\nu_*,\nu_H)$, then

$$\Big(\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u]\big)^2\Big)^{1/2}=|\gamma|=|\Psi_u(\nu_H)|=|\Psi_u(\nu_H)-\Psi_u(\nu_*)|$$

$$= \Big| \int_0^\eta \frac{d}{ds}\,\Psi_u([\nu_H,\nu_*]_s)\,ds \Big| \geq \Big(\frac{1}{c_0(n)} - C(n)\,(\varepsilon_0+\sigma_0)\Big)\,\eta \geq \frac{|\nu_*-\nu_H|}{2\,c_0(n)}\,,$$

that is (2.15). Finally, (2.16) follows from (2.15) and (2.12).

Turning now towards proving (2.22), by the area formula, (2.10), and $\mathbf{q}_{K(\nu)}[e] = \nu\cdot e$, we find that

$$\begin{aligned}(e_j\cdot\Psi_u)(\nu) &:= \int_{\Sigma_{K(\nu)}} v_u^{K(\nu)}\,\phi^j_{K(\nu)} = \int_{\Sigma_H} v_u^{K(\nu)}(T_u^{K(\nu)})\,\phi^j_{K(\nu)}(T_u^{K(\nu)})\,J^{\Sigma_H}T_u^{K(\nu)}\\ &= c_0(n)\int_{\Sigma_H}\nu\cdot(\omega+u\,\nu_H)\Big(\rho_H^{K(\nu)}[\tau_H^j]\cdot\frac{\mathbf{p}_K(\omega+u\,\nu_H)}{|\mathbf{p}_K(\omega+u\,\nu_H)|^2}\Big)J^{\Sigma_H}T_u^{K(\nu)}d\mathcal{H}^{n-1}_\omega,\end{aligned}$$

so that (2.19) gives that

$$\begin{aligned}&\|\Psi_u-\Psi_0\|_{C^1(A)}\leq C(n)\,\sigma_0,\qquad\text{where}\\ &e_j\cdot\Psi_0(\nu)=c_0(n)\int_{\Sigma_H}(\nu\cdot\omega)\,\Big(\rho_H^{K(\nu)}[\tau_H^j]\cdot\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|^2}\Big)\,J^{\Sigma_H}\Big[\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|}\Big]\,d\mathcal{H}^{n-1}_\omega.\end{aligned}\tag{2.23}$$

By definition of A and by (2.20) and (2.21),

$$\begin{aligned}&\sup_{\nu\in A}\sup_{\omega\in\Sigma_H}\Big|\tau_H^j\cdot\omega-\Big(\rho_H^{K(\nu)}[\tau_H^j]\cdot\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|^2}\Big)\,J^{\Sigma_H}\Big[\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|}\Big]\Big|\leq C(n)\,\varepsilon_0,\\ &\text{and thus }\|\Psi_0-\Psi_*\|_{C^1(A)}\leq C(n)\,(\sigma_0+\varepsilon_0),\end{aligned}\tag{2.24}$$

where $\Psi_*:A\to\mathbb{R}^n$ is defined by $e_j\cdot\Psi_*(\nu)=c_0(n)\int_{\Sigma_H}(\nu\cdot\omega)\,(\tau_H^j\cdot\omega)\,d\mathcal{H}^{n-1}_\omega$ $(\nu\in A)$. Recalling that $\{\tau^i\}_{i=1}^n$ is an orthonormal frame of $\mathbb{S}^n$ on A, with $\nabla_{\tau^i}\nu = \tau^i(\nu)=\tau^i_{K(\nu)}=\rho_H^{K(\nu)}[\tau^i_H]$, we find that

$$\begin{aligned}e_j\cdot\nabla_{\tau^i}\Psi_*(\nu)&=c_0(n)\int_{\Sigma_H}(\rho_H^{K(\nu)}[\tau_H^i]\cdot\omega)\,(\tau_H^j\cdot\omega)\,d\mathcal{H}^{n-1}_\omega,\\ e_j\cdot\nabla_{\tau^i}\Psi_*(\nu_H)&=c_0(n)\int_{\Sigma_H}(\tau_H^i\cdot\omega)\,(\tau_H^j\cdot\omega)\,d\mathcal{H}^{n-1}_\omega=\delta_{ij}/c_0(n).\end{aligned}$$

By (2.21), (2.23) and (2.24) we conclude that

$$\|\Psi_u-\Psi_*\|_{C^1(A)}\leq C(n)\,(\sigma_0+\varepsilon_0),\tag{2.25}$$

$$\Big\|\nabla^{\mathbb{S}^n}\Psi_u-c_0(n)^{-1}\sum_{j=1}^n e_j\otimes\tau^j\Big\|_{C^0(A)}\leq C(n)\,(\sigma_0+\varepsilon_0).\tag{2.26}$$

Let us finally consider the map $h:A\times[0,1]\to\mathbb{R}^n$,

$$h(\nu,t)=h_t(\nu)=t\,\Psi_*(\nu)+(1-t)\,\Psi_u(\nu),\qquad(\nu,t)\in A\times[0,1],$$

which defines an homotopy between Ψ_* and Ψ_u. By (2.25) and (2.26) we see that if $\nu\in\partial A$, that is, if $\mathrm{dist}_{\mathbb{S}^n}(\nu,\nu_H)=\varepsilon_0$, then, denoting by $[\nu_H,\nu]_s$ the unit-speed length minimizing geodesic from ν_H to ν, considering that $[\nu_H, \nu]_s \in A$ for every $s \in (0, \varepsilon_0)$, and that $\mathbb{S}^n$ is close to be flat in A, we find

$$\begin{aligned}|h_t(\nu)| &\geq \Big|\int_0^{\varepsilon_0} \frac{d}{ds}\, h_t([\nu_H, \nu]_s)\, ds\Big| - |h_t(\nu_H)| \\ &\geq \Big(\frac{1}{c_0(n)} - C(n)\,(\varepsilon_0 + \sigma_0)\Big)\,\varepsilon_0 - C(n)\,\sigma_0 \geq \frac{\varepsilon_0}{2\,c_0(n)},\end{aligned}$$

provided $\sigma_0 = \varepsilon_0/C_*$ is small enough with respect to ε_0 (i.e., provided C_* is large), ε_0 is small in terms of c_0, and where we have used $\Psi_*(\nu_H) = 0$ and

$$|\Psi_u(\nu_H)| = |\gamma| = \Big|\int_{\Sigma_H} u\,\phi_H^1\Big| \leq C(n)\,\sigma_0\,, \tag{2.27}$$

to deduce $|h_t(\nu_H)| \leq C(n)\,\sigma_0$. This proves that $0 \notin \partial\, h_t(\partial A)$ for every $t \in [0, 1]$, so that $\deg(h_t, A, 0)$ is independent of $t \in [0, 1]$. In particular, $h_0 = \Psi_u$ and $h_1 = \Psi_*$ give $\deg(\Psi_u, A, 0) = \deg(\Psi_*, A, 0) = 1$, where we have used $\Psi_*(\nu_H) = 0$ and the fact that, up to decreasing the value of ε_0, Ψ_* is injective on A. By $\deg(\Psi_u, A, 0) = 1$, there is $\nu_* \in A$ such that $\Psi_u(\nu_*) = 0$, **as claimed** in (2.22). □

2.3. Energy Estimates for Spherical Graphs Over Annuli

Given $H \in \mathcal{H}$ and $0 < r_1 < r_2$ we let $\mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$ be the class of those $u \in C^1(\Sigma_H \times (r_1, r_2))$ such that, setting $u_r = u(\cdot, r)$, one has $u_r \in \mathcal{X}_\sigma(\Sigma_H)$ for every $r \in (r_1, r_2)$ and $|r\,\partial_r u| \leq \sigma$ on $\Sigma_H \times (r_1, r_2)$. If $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$, then the spherical graph of u over $A_{r_1}^{r_2} \cap H$, given by

$$\Sigma_H(u, r_1, r_2) = \Big\{ r\,\frac{\omega + u_r(\omega)\,\nu_H}{\sqrt{1 + u_r(\omega)^2}} : \omega \in \Sigma_H, r \in (r_1, r_2)\Big\},$$

is an hypersurface in $A_{r_1}^{r_2}$. It is useful to keep in mind that $\Sigma_H(0, r_1, r_2) = \{r\,\omega : \omega \in \Sigma\,, r \in (r_1, r_2)\} = H \cap A_{r_1}^{r_2}$ is a flat annular region of area $\omega_n\,(r_2^n - r_1^n)$, and that if $\sigma < \sigma_1 = \sigma_1(n)$, then

$$\frac{1}{C(n)} \int_{\Sigma_H(u, r_1, r_2)} \omega_H^2\, d\mathcal{H}^n \leq \int_{\Sigma_H \times (r_1, r_2)} r^{n-1}\, u^2 \leq C(n) \int_{\Sigma_H(u, r_1, r_2)} \omega_H^2\, d\mathcal{H}^n\,. \tag{2.28}$$

Lemma 2.5. *There are ε_0, σ_0, C_0 positive, depending on n only, such that:*
(i): *if $H, K \in \mathcal{H}$, $\nu_H \cdot \nu_K > 0$, $|\nu_H - \nu_K| = \varepsilon < \varepsilon_0$, $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$, and $\sigma < \sigma_0$, then there is $v \in \mathcal{X}_{C_0(\sigma+\varepsilon)}(\Sigma_H, r_1, r_2)$ such that $\Sigma_K(v, r_1, r_2) = \Sigma_H(u, r_1, r_2)$.*
(ii): *if $H \in \mathcal{H}$, $u \in \mathcal{X}_{\sigma_0}(\Sigma_H, r_1, r_2)$, and $(a, b) \subset\subset (r_1, r_2)$, then there exist $K \in \mathcal{H}$, $v \in \mathcal{X}_{C_0\,\sigma_0}(\Sigma_K, r_1, r_2)$, and $r_* \in [a, b]$ such that*

$$\begin{aligned}&\Sigma_H(u, r_1, r_2) = \Sigma_K(v, r_1, r_2),\\ &E^0_{\Sigma_K}\big(v_{r_*}\big) = 0,\end{aligned}$$

$$|\nu_H - \nu_K|^2 \le C_0(n) \min_{\rho\in[a,b]} \int_{\Sigma_H} \big(E^0_{\Sigma_H}[u_\rho]\big)^2. \tag{2.29}$$

Moreover, for every $r \in (r_1, r_2)$,

$$\Big|\int_{\Sigma_K} (v_r)^2 - \int_{\Sigma_H} (u_r)^2\Big| \le C_0(n) \Big\{ \min_{\rho\in[a,b]} \int_{\Sigma_H} (u_\rho)^2 + \int_{\Sigma_H} (u_r)^2 \Big\}. \tag{2.30}$$

Proof. We prove statement (i). If $|\nu_H - \nu_K| = \varepsilon < \varepsilon_0$, since $u_r \in \mathcal{X}_\sigma(\Sigma_H)$ for every $r \in (r_1, r_2)$, by Lemma 2.3–(i) we see that $T_r : \Sigma_H \to \Sigma_K$,

$$T_r(\omega) = |\mathbf{p}_K[\omega + u_r(\omega)\,\nu_H]|^{-1}\,\mathbf{p}_K[\omega + u_r(\omega)\,\nu_H] \qquad \omega \in \Sigma_H\,, \tag{2.31}$$

is a diffeomorphism between Σ_H and Σ_K, and $v_r : \Sigma_K \to \mathbb{R}$,

$$v_r(T_r(\omega)) = \frac{\nu_K \cdot (\omega + u_r(\omega)\,\nu_H)}{|\mathbf{p}_K[\omega + u_r(\omega)\,\nu_H]|}\,, \qquad \omega \in \Sigma_H\,, \tag{2.32}$$

satisfies $v_r \in \mathcal{X}_{C_0\,(\sigma+\varepsilon)}(\Sigma_K)$, $\Sigma_H(u_r) = \Sigma_K(v_r)$ for every $r \in (r_1, r_2)$, and

$$\Big|\int_{\Sigma_K} (v_r)^2 - \int_{\Sigma_H} (u_r)^2\Big| \le C(n) \Big\{|\nu_H - \nu_K|^2 + \int_{\Sigma_H} (u_r)^2\Big\}. \tag{2.33}$$

Since $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$, and T_r and v_r depend smoothly on u_r, setting $v(\omega, r) := v_r(\omega)$ we have $\Sigma_H(u, r_1, r_2) = \Sigma_K(v, r_1, r_2)$ (by $\Sigma_H(u_r) = \Sigma_K(v_r)$ for every $r \in (r_1, r_2)$), and $v \in \mathcal{X}_{C_0\,(\sigma+\varepsilon)}(\Sigma_H, r_1, r_2)$ ($|r\,\partial_r v_r| \le C_0(\sigma + \varepsilon)$ is deduced by differentiation in (2.31) and (2.32), and by $|u_r|$, $|r\,\partial_r u_r| < \sigma$).

Step two: We prove (ii). Let $\gamma = \min_{\rho\in[a,b]} \int_{\Sigma_H} \big(E^0_{\Sigma_H}[u_\rho]\big)^2$, and let $r_* \in [a, b]$ be such that the minimum γ is achieved at $r = r_*$. If $\gamma = 0$, then we set $K = H$ and $v = u$. If $\gamma > 0$, then we apply Lemma 2.3–(ii) to $u_{r_*} \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, and find $K \in \mathcal{H}$ with $|\nu_K - \nu_H| < \varepsilon_0$ and $v_{r_*} \in \mathcal{X}_{C_0\,s_0}(\Sigma_K)$ such that $\Sigma_H(u_{r_*}) = \Sigma_K(v_{r_*})$ and

$$E^0_{\Sigma_K}[v_{r_*}] = 0, \tag{2.34}$$

$$|\nu_K - \nu_H|^2 \le C_0(n) \int_{\Sigma_H} \big(E^0_{\Sigma_H}[u_{r_*}]\big)^2 = C_0(n)\,\gamma,$$

$$\Big|\int_{\Sigma_K} (v_{r_*})^2 - \int_{\Sigma_H} (u_{r_*})^2\Big| \le C_0(n) \int_{\Sigma_H} (u_{r_*})^2. \tag{2.35}$$

Since $v_{r_*} = v(\cdot, r_*)$ for v constructed in step one starting from u, H and K, we deduce (2.30) by (2.33) and (2.35), while (2.34) is (2.29). □

We will use two basic "energy estimates" for spherical graphs over annuli. To streamline the application of these estimates to diadic families of annuli we consider intervals (r_1, r_2) and (r_3, r_4) are (η, η_0)-**related**, meaning that

$$r_2 = r_0(1 + \eta_0)\,, \quad r_1 = r_0(1 - \eta_0)\,, \quad r_4 = r_0(1 + \eta)\,, \quad r_3 = r_0(1 - \eta)\,, \tag{2.36}$$

for some $\eta_0 > \eta > 0$, and with $r_0 = (r_1 + r_2)/2 = (r_3 + r_4)/2$; in particular, (r_3, r_4) is contained in, and concentric to, (r_1, r_2). The case $\Lambda = 0$ of the following statement is the codimension one, equatorial spheres case of [1, Lemma 7.14, Theorem 7.15].

Theorem 2.6. (Energy estimates for spherical graphs) *If $n \geq 2$ and $\eta_0 > \eta > 0$, then there are $\sigma_0 = \sigma_0(n, \eta_0, \eta)$ and $C_0 = C_0(n, \eta_0, \eta)$ positive, with the following property. If $H \in \mathcal{H}$, $\Lambda \geq 0$, and $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$ is such that* $\max\{1, \Lambda\, r_2\}\, \sigma \leq$ *σ_0 and $\Sigma_H(u, r_1, r_2)$ has mean curvature bounded by Λ in $A_{r_1}^{r_2}$, then, whenever (r_1, r_2) and (r_3, r_4) are (η, η_0)-related as in (2.36),*

$$\left|\mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))\right| \leq C_0 \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} \left(u^2 + \Lambda\, r\, |u|\right).$$

Moreover, if there is $r \in (r_1, r_2)$ s.t. $E^0_{\Sigma_H} u_r = 0$ on Σ_H, then we also have

$$\int_{\Sigma_H \times (r_3, r_4)} r^{n-1}\, u^2 \leq C(n)\, \Lambda\, r_2\, (r_2^n - r_1^n) + C_0 \int_{\Sigma_H \times (r_1, r_2)} r^{n-1}\, (r\, \partial_r u)^2.$$

Proof. Since this proof is quite long and the arguments are not needed to understand the rest of the paper, we postpone it to "Appendix A". □

2.4. Monotonicity for Exterior Varifolds with Bounded Mean Curvature

The following theorem states the monotonicity of $\Theta_{V,R,\Lambda}$ for $V \in \mathcal{V}_n(\Lambda, R, S)$, and provides, when V corresponds to a spherical graph, a quantitative lower bound for the gap in the associated monotonicity formula; the case $\Lambda = 0$, $R = 0$ is contained in [1, Lemma 7.16, Theorem 7.17].

Theorem 2.7. **(i):** *If $V \in \mathcal{V}_n(\Lambda, R, S)$, then*

$$\Theta_{V,R,\Lambda} \text{ is increasing on } (R, S)\,.$$

(ii): *There is $\sigma_0(n)$ such that, if $V \in \mathcal{V}_n(\Lambda, R, S)$ and, for some $H \in \mathcal{H}$, $u \in \mathcal{X}_\sigma(\Sigma, r_1, r_2)$ with $\sigma \leq \sigma_0(n)$, and $(r_1, r_2) \subset (R, S)$, we have*

$$V \text{ corresponds to } \Sigma_H(u, r_1, r_2) \text{ in } A_{r_1}^{r_2}, \tag{2.37}$$

then

$$\int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r\, \partial u_r)^2 \leq C(n)\, r_2^n \left\{\Theta_{V,R,\Lambda}(r_2) - \Theta_{V,R,\Lambda}(r_1)\right\}. \tag{2.38}$$

(iii): *Finally, given $\eta_0 > \eta > 0$, there exist σ_0 and C_0 depending on n, η_0, and η only, such that if the assumptions of part (i) and part (ii) hold and, in addition to that, we also have* $\max\{1, \Lambda\, r_2\}\, \sigma \leq \sigma_0$ *and*

$$\exists\, r \in (r_1, r_2) \text{ s.t. } E^0_{\Sigma_H} u_r = 0 \text{ on } \Sigma_H\,, \tag{2.39}$$

then, whenever (r_1, r_2) and (r_3, r_4) are (η, η_0)-related as in (2.36), we have

$$\begin{aligned} &\left|\mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))\right| \\ &\qquad \leq C_0\, r_2^n \left\{\Theta_{V,R,\Lambda}(r_2) - \Theta_{V,R,\Lambda}(r_1) + (\Lambda\, r_2)^2\right\}. \end{aligned} \tag{2.40}$$

Proof. We give details of the proof of (i) when $V \in \mathcal{M}_n(\Lambda, R, S)$ (whereas the general case is addressed as in [34, Section 17]). By the coarea formula, the divergence theorem and $|\boldsymbol{H}| \le \Lambda$, for a.e. $\rho > R$,

$$\begin{aligned}
\frac{d}{d\rho}\,\frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} &= \frac{1}{\rho^n}\int_{M\cap\partial B_\rho} \frac{|x|\,d\mathcal{H}^{n-1}}{|x^{TM}|} - \frac{n\,\mathcal{H}^n(M\cap(B_\rho\setminus B_R))}{\rho^{n+1}} \\
&= \frac{1}{\rho^n}\int_{M\cap\partial B_\rho} \frac{|x|\,d\mathcal{H}^{n-1}}{|x^{TM}|} - \frac{1}{\rho^n}\int_{M\cap(B_\rho\setminus B_R)} \frac{x}{\rho}\cdot \boldsymbol{H}\,d\mathcal{H}^n \\
&\quad -\frac{1}{\rho^{n+1}}\Big\{\int_{M\cap\partial B_\rho} \nu_M^{\rm co}\cdot x\,d\mathcal{H}^{n-1} + \int_{M\cap\partial B_R} \nu_M^{\rm co}\cdot x\,d\mathcal{H}^{n-1}\Big\} \\
&\ge \frac{1}{\rho^n}\int_{M\cap\partial B_\rho}\Big(\frac{|x|}{|x^{TM}|} - \frac{|x^{TM}|}{|x|}\Big)\,d\mathcal{H}^{n-1} \\
&\quad -\frac{1}{\rho^{n+1}}\int_{M\cap\partial B_R} \nu_M^{\rm co}\cdot x\,d\mathcal{H}^{n-1} - \Lambda\,\frac{\mathcal{H}^n(M\cap(B_\rho\setminus B_R))}{\rho^n} \\
&= \mathrm{Mon}(V,\rho) + \frac{d}{d\rho}\,\frac{1}{n\,\rho^n}\int x\cdot\nu_V^{\rm co}\,d\,\mathrm{bd}_V - \Lambda\,\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n} \qquad (2.41)
\end{aligned}$$

where $\mathrm{Mon}(V,\rho) = (d/d\rho)\int_{B_\rho\setminus B_R} |x^\perp|^2\,|x|^{-n-2}\,d\|V\|$. Since $\mathrm{Mon}(V,\rho) \ge 0$, this proves (i). Assuming now (2.37), a straightforward computation which we omit (c.f. for example in [1, Lemma 3.5(6), Lemma 7.16]), we see that, under (2.37),

$$C(n)\,r_2^n \int_{r_1}^{r_2} \mathrm{Mon}(V,\rho)\,d\rho \ge \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2,$$

thus proving (ii). To prove (iii), we set $a = r_0\,(1-(\eta+\eta_0)/2)$ and $b = r_0\,(1+(\eta+\eta_0)/2)$, so that (a,b) and (r_3,r_4) are $(\eta, (\eta+\eta_0)/2)$-related, and (r_1,r_2) and (a,b) are $((\eta+\eta_0)/2, \eta_0)$-related (in particular, $(r_3,r_4)\subset(a,b)\subset(r_1,r_2)$). By suitably choosing σ_0 in terms of n, η and η_0, we can apply Theorem 2.6 with (r_3,r_4) and (a,b), so to find (with $C = C(n,\eta_0,\eta)$)

$$\begin{aligned}
\Big|\mathcal{H}^n(\Sigma(u,r_3,r_4)) - \mathcal{H}^n(\Sigma(0,r_3,r_4))\Big| &\le C\int_{\Sigma_H\times(a,b)} r^{n-1}\big(u^2 + \Lambda\,r\,|u|\big) \\
&\le C\,\Big\{(\Lambda\,b)^2\,(b^n - a^n) + \int_{\Sigma_H\times(a,b)} r^{n-1}\,u^2\Big\}.
\end{aligned}$$

Thanks to (2.39) we can apply Theorem 2.6 with (a,b) and (r_1,r_2) to find

$$\int_{\Sigma_H\times(a,b)} r^{n-1}\,u^2 \le C\,\Big\{(\Lambda\,r_2)^2\,(r_2^n - r_1^n) + \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2\Big\}.$$

We find (2.40) by (2.38) and $(\Lambda\,b)^2\,(b^n - a^n) \le (\Lambda\,r_2)^2\,r_2^n$. □

2.5. *Proof of the Mesoscale Flatness Criterion*

As a final preliminary result to the proof of Theorem 2.1, we prove the following lemma, where Allard's regularity theorem is combined with a compactness argument to provide the basic graphicality criterion used throughout the iteration. The statement should be compared to [1, Lemma 5.7].

Lemma 2.8. (Graphicality lemma) *Let $n \ge 2$. For every $\sigma > 0$, $\Gamma \ge 0$, $(\lambda_3, \lambda_4) \subset\subset (\lambda_1, \lambda_2) \subset\subset (0,1)$, and $(\eta_1, \eta_2) \subset\subset (0,1)$, there are positive constants ε_1 and M_1, depending only on n, σ, Γ, (λ_1, λ_2), (λ_3, λ_4), and (η_1, η_2), and ε_2 and M_2, depending only on n, σ, Γ, λ_1, and (η_1, η_2), with the following properties.*
(i): *If $\Lambda \ge 0$, $R \in (0, 1/\Lambda)$, $V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda)$,*

$$\|\mathrm{bd}_V\|(\partial B_R) \le \Gamma\, R^{n-1}\,, \qquad \sup_{\rho \in (R, 1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \le \Gamma\,, \tag{2.42}$$

there exists $r > 0$ such that

$$\max\{M_1, 64\}\, R \le r \le \frac{\varepsilon_1}{\Lambda}, \tag{2.43}$$

$$|\delta_{V,R,\Lambda}(r)| \le \varepsilon_1, \tag{2.44}$$

$$\|V\|(A^{\lambda_4 r}_{\lambda_3 r}) > 0, \tag{2.45}$$

and if, for some $K \in \mathcal{H}$, we have

$$\frac{1}{r^n} \int_{A^{\lambda_2 r}_{\lambda_1 r}} \omega_K^2\, d\|V\| \le \varepsilon_1\,, \tag{2.46}$$

then there exists $u \in \mathcal{X}_\sigma(\Sigma_K, \eta_1\, r, \eta_2\, r)$ such that

$$V \text{ corresponds to } \Sigma_K(u, \eta_1\, r, \eta_2\, r) \text{ on } A^{\eta_2 r}_{\eta_1 r}.$$

(ii): *If Λ, R, and V are as in (i), (2.42) holds, and there exists r such that*

$$\max\{M_2, 64\}\, R \le r \le \frac{\varepsilon_2}{\Lambda}, \tag{2.47}$$

$$\max\{|\delta_{V,R,\Lambda}(\lambda_1\, r)|, |\delta_{V,R,\Lambda}(r)|\} \le \varepsilon_2, \tag{2.48}$$

then there exists $K \in \mathcal{H}$ and $u \in \mathcal{X}_\sigma(\Sigma_K, \eta_1\, r, \eta_2\, r)$ such that

$$V \text{ corresponds to } \Sigma_K(u, \eta_1\, r, \eta_2\, r) \text{ on } A^{\eta_2 r}_{\eta_1 r}.$$

Proof. Step one: As a preliminary, we first show that if V is a stationary, n-dimensional, integer rectifiable varifold in B_1 such that

$$\|V\|(B_1) \le \omega_n\,, \quad \operatorname{spt} V \cap A^{\beta_2}_{\beta_1} \subset K\,, \quad \text{and} \quad \operatorname{spt} V \cap A^{\beta_2}_{\beta_1} \neq \emptyset\,, \tag{2.49}$$

for some $K \in \mathcal{H}$ and $0 < \beta_1 < \beta_2 \le 1$, then $V = \mathbf{var}\,(K \cap B_1,\ 1|_{K \cap B_1})$.

Let $\beta' \in (\beta_1, \beta_2)$ and $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^{n+1}; [0,1])$ be such that $\operatorname{spt} \varphi_1 \subset B_{\beta_2}$, $\varphi_1|_{B_{\beta'}} \equiv 1$, and $\varphi_1 + \varphi_2 \equiv 1$. As a consequence of (2.49) and the stationarity of V in B_{β_2}, for $X \in C^1_c(\mathbb{R}^{n+1} \setminus (K \cap (\overline{B}_{\beta_2} \setminus B_{\beta'}))$, we have

$$\begin{aligned} \delta(V \llcorner B_{\beta'})(X) &= \int_{B_{\beta'}} \operatorname{div}^M(\varphi_1 X) + \operatorname{div}^M(\varphi_2 X)\, d\|V\| \\ &= \int_{B_{\beta_2}} \operatorname{div}^M(\varphi_1 X)\, d\|V\| = 0. \end{aligned}$$

Then by the convex hull property [34, Theorem 19.2], $\mathrm{spt}\,(V \llcorner B_{\beta'}) \subset K$. By the constancy theorem [34, Theorem 41.1], $V \llcorner B_{\beta_2} = \mathbf{var}\,(K \cap B_{\beta_2}, \theta)$ for some constant θ. Furthermore, since V assigns non-trivial mass to B_{β_2} by (2.49) and is integer rectifiable, $\theta \geq 1$. Therefore $0 \in \mathrm{spt}\|V\|$, and the monotonicity formula gives $\omega_n \leq \lim_{r\to 0^+} \|V\|(B_r) r^{-n} \leq \|V\|(B_1) \leq \omega_n$. Thus V is a stationary, n-dimensional, integer rectifiable varifold in B_1 with constant area ratios ω_n and $\mathrm{spt} V \cap A_{\beta_1}^{\beta_2} \subset K$, so $V = \mathbf{var}\,(K \cap B_1, 1|_{K\cap B_1})$.

Step two: We prove item (i) by contradiction. If it were false, we could find $\sigma > 0$, $\Gamma \geq 0$, $(\lambda_3, \lambda_4) \subset\subset (\lambda_1, \lambda_2) \subset\subset (0,1)$, $(\eta_1, \eta_2) \subset (0,1)$, with $K_j \in H$, positive numbers R_j, $\Lambda_j < 1/R_j$, r_j, and $W_j \in \mathcal{V}_n(\Lambda_j, R_j, 1/\Lambda_j)$ such that $\|W_j\|\big(A_{\lambda_3 r_j}^{\lambda_4 r_j}\big) > 0$, $\|\mathrm{bd}_{W_j}\|(\partial B_{R_j}) \leq \Gamma\, R_j^{n-1}$, $\|W_j\|(B_\rho \setminus B_{R_j}) \leq \Gamma\, \rho^n$ for every $\rho \in (R_j, 1/\Lambda_j)$, and $\rho_j = R_j/r_j \to 0$, $r_j\,\Lambda_j \to 0$, $\delta_{W_j,R_j,\Lambda_j}(r_j) \to 0$, and $r_j^{-n} \int_{B_{\lambda_2 r_j}\setminus B_{\lambda_1 r_j}} \omega_{K_j}^2 \, d\|W_j\| \to 0$, but there is no $u \in \mathcal{X}_\sigma(\Sigma_{K_j}, \eta_1 r_j, \eta_2 r_j)$ with the property that W_j corresponds to $\Sigma_{K_j}(u, \eta_1 r_j, \eta_2 r_j)$ on $A_{\eta_1 r_j}^{\eta_2 r_j}$. Hence, setting $V_j = W_j / r_j$, no $u \in \mathcal{X}_\sigma(\Sigma_{K_j}, \eta_1, \eta_2)$ can exist such that V_j corresponds to $\Sigma_{K_j}(u, \eta_1, \eta_2)$ on $A_{\eta_1}^{\eta_2}$, despite the fact that each V_j belongs to $\mathcal{V}_n(r_j\,\Lambda_j, \rho_j, 1/(r_j\,\Lambda_j))$ and satisfies

$$\|V_j\|(A_{\lambda_3}^{\lambda_4}) > 0, \quad \frac{\|\mathrm{bd}_{V_j}\|(\partial B_{\rho_j})}{\rho_j^{n-1}} \leq \Gamma, \quad \sup_{\rho \in (\rho_j, 1/(\Lambda_j r_j))} \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^n} \leq \Gamma,$$

$$\lim_{j\to\infty} \max\Big\{\delta_{V_j,\rho_j,r_j\,\Lambda_j}(1),\ \int_{A_{\lambda_1}^{\lambda_2}} \omega_{K_j}^2 \, d\|V_j\|\Big\} = 0. \tag{2.50}$$

Clearly we can find $K \in \mathcal{H}$ such that, up to extracting subsequences, $K_j \cap B_1 \to K \cap B_1$ in $L^1(\mathbb{R}^{n+1})$. Similarly, by (2.50), we can find an n-dimensional integer rectifiable varifold V such that $V_j \rightharpoonup V$ as varifolds in $B_1 \setminus \{0\}$. Since the bound on the distributional mean curvature of V_j on $B_{1/(\Lambda_j r_j)} \setminus \overline{B}_{\rho_j}$ is $r_j\,\Lambda_j$, and since $\rho_j \to 0^+$ and $r_j\,\Lambda_j \to 0^+$, it also follows that V is stationary in $B_1 \setminus \{0\}$, and thus, by a standard argument and since $n \geq 2$, on B_1. By $\|V_j\|(A_{\lambda_3}^{\lambda_4}) > 0$, for every j there is $x_j \in A_{\lambda_3}^{\lambda_4} \cap \mathrm{spt}\, V_j$, so that, up to extracting subsequences, $x_j \to x_0$ for some $x_0 \in \overline{A}_{\lambda_3}^{\lambda_4} \cap \mathrm{spt}\, V$. By $(\lambda_3, \lambda_4) \subset\subset (\lambda_1, \lambda_2)$, there is $\rho > 0$ such that $B_\rho(x_0) \subset A_{\lambda_1}^{\lambda_2}$, hence

$$\|V\|(A_{\lambda_1}^{\lambda_2}) \geq \|V\|(B_\rho(x_0)) \geq \omega_n\, \rho^n > 0\,, \tag{2.51}$$

thus proving $V \llcorner A_{\lambda_1}^{\lambda_2} \neq \emptyset$. By this last fact, by $\omega_K = 0$ on $(\mathrm{spt}\, V) \cap A_{\lambda_1}^{\lambda_2}$, and by the constancy theorem [34, Theorem 41.1], we have

$$A_{\lambda_1}^{\lambda_2} \cap \mathrm{spt}\, V = A_{\lambda_1}^{\lambda_2} \cap K\,.$$

At the same time, since $\|\mathrm{bd}_{V_j}\|(\partial B_{\rho_j}) \leq \Gamma\, \rho_j^{n-1}$ and $\|V_j\|(B_\rho \setminus B_{\rho_j}) \leq \Gamma\, \rho^n$ for every $\rho \in (\rho_j, 1/(\Lambda_j\, r_j)) \supset (\rho_j\, 1)$, by (2.50),

$$\omega_n = \lim_{j\to\infty} \|V_j\|(B_1 \setminus B_{\rho_j}) - \frac{\rho_j}{n} \|\delta V_j\|(\partial B_{\rho_j}) + \Lambda_j\, r_j \int_{\rho_j}^1 \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^n}\, d\rho$$

$$\geq \|V\|(B_1) - \Gamma \overline{\lim_{j\to\infty}} \left(\rho_j^n + \Lambda_j r_j\right) = \|V\|(B_1). \tag{2.52}$$

Since V is stationary in B_1 and integer rectifiable, and since (2.51) and (2.52) imply (2.49) with $\lambda_1 = \beta_1$ and $\lambda_2 = \beta_2$, the first step yields $V = \mathbf{var}\,(K \cap B_1,\, 1|_{K\cap B_1})$. By Allard's regularity theorem and by $V_j \rightharpoonup V$ as $j \to \infty$ we deduce the existence of a sequence $\{u_j\}_j$, with $u_j \in \mathcal{X}_{\sigma_j}(\Sigma_K, \eta_1, \eta_2)$ for some $\sigma_j \to 0$ as $j \to \infty$, such that V_j corresponds to $\Sigma_K(u_j, \eta_1, \eta_2)$ in $A_{\eta_1}^{\eta_2}$ for j large enough. As soon as j is large enough to give $\sigma_j < \sigma$, we have reached a contradiction.

Step three: For item (ii), we again argue by contradiction. Should the lemma be false, then we could find $\sigma > 0$, $\Gamma \geq 0$, $\lambda_1 \in (0, 1)$, $(\eta_1, \eta_2) \subset (0, 1)$, positive numbers R_j, $\Lambda_j < 1/R_j$, r_j, and, by the same rescaling as in step two, $V_j \in \mathcal{V}_n(r_j\,\Lambda_j, \rho_j, 1/(r_j\,\Lambda_j))$ with

$$\frac{\|\mathrm{bd}_{V_j}\|(\partial B_{\rho_j})}{\rho_j^{n-1}} \leq \Gamma\,, \qquad \sup_{\rho\in(\rho_j,1/(\Lambda_j r_j))} \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^n} \leq \Gamma\,, \tag{2.53}$$

$$\lim_{j\to\infty} \max\left\{\rho_j = \frac{R_j}{r_j}\,,\; r_j\,\Lambda_j\,,\; |\delta_{V_j,\rho_j,r_j\,\Lambda_j}(1)|\,,\; |\delta_{V_j,\rho_j,r_j\,\Lambda_j}(\lambda_1)|\right\} = 0\,, \tag{2.54}$$

such that there exists no $u \in \mathcal{X}_\sigma(\Sigma_{K_j}, \eta_1, \eta_2)$ with the property that V_j corresponds to $\Sigma_{K_j}(u, \eta_1, \eta_2)$ on $A_{\eta_1}^{\eta_2}$. As in step two, we can find an n-dimensional integer rectifiable varifold $V = \mathbf{var}\,(M, \theta)$ such that $V_j \rightharpoonup V$ as varifolds in $B_1\backslash\{0\}$ and V is stationary on B_1. If for some $K \in \mathcal{H}$, $V = \mathbf{var}\,(K \cap B_1,\, 1|_{K\cap B_1})$, then using Allard's theorem as in the proof of (i), we have a contradiction. So we prove $V = \mathbf{var}\,(K \cap B_1,\, 1|_{K\cap B_1})$.

For every $r \in [\lambda_1, 1]$, using $\rho_j \to 0^+$ and $r_j\Lambda_j \to 0^+$ in conjunction with (2.53), and then the monotonicity of $\delta_{V_j,\rho_j,r_j\,\Lambda_j}$ and (2.54), we have

$$\begin{aligned}\overline{\lim_{j\to\infty}} \left|\omega_n - \frac{\|V_j\|(B_r \setminus B_{\rho_j})}{r^n}\right| &= \overline{\lim_{j\to\infty}} \left|\delta_{V_j,\rho_j,r_j\,\Lambda_j}(r)\right| \\ &\leq \lim_{j\to\infty} \max_{r\in\{\lambda_1,1\}} \left\{|\delta_{V_j,\rho_j,r_j\,\Lambda_j}(r)|\right\} = 0.\end{aligned}$$

Thus the convergence $V_j \rightharpoonup V$ and the monotonicity of $\|V\|(B_r)/r^n$ yield

$$\|V\|(B_r) = \omega_n r^n \quad \forall r \in (\lambda_1, 1) \quad \text{and} \quad \|V\|(B_1) = \omega_n\,. \tag{2.55}$$

By (2.55), $V \llcorner (B_1\backslash\overline{B}_{\lambda_1}) = \mathbf{var}\,(C, \theta_C) \llcorner (B_1\backslash\overline{B}_{\lambda_1})$ for some locally $\mathcal{H}^n$-rectifiable cone $C \subset \mathbb{R}^{n+1}$ and zero homogeneous $\theta_C : C \to \mathbb{N}$. Now since the integer rectifiable varifold cone $\mathbf{var}\,(C, \theta_C)$ is stationary in $B_1 \setminus \overline{B}_{\lambda_1}$, it is stationary in $\mathbb{R}^{n+1}$ by $n \geq 2$, and due to (2.55), it satisfies $\int_{C\cap B_1} \theta_C\, d\mathcal{H}^n = \omega_n$. Therefore $C = K$ for some $K \in \mathcal{H}$, and $\theta_C \equiv 1$. From the definition of C, it follows that

$$\mathrm{spt}\, V \cap (B_1 \setminus \overline{B}_{\lambda_1}) \subset K\,. \tag{2.56}$$

Finally, (2.55) and (2.56) give (2.49) with $\beta_1 = \lambda_1$, $\beta_2 = 1$. The result of step one then completes the proof that $V = \mathbf{var}\,(K \cap B_1,\, 1|_{K\cap B_1})$. □

Proof of Theorem 2.1. The proof proceeds in four steps, which we outline here. Precise statements can be found at the beginning of each step. First, we assume that $\delta_{V,R,\Lambda}(s/8) \geq 0$, and prove that C^1-graphicality can be propagated from $s/32$ to an upper radius $S_+/16 \leq S_*/16$ as long as $\delta_{V,R,\Lambda}(S_+)$ remains non-negative and $S_+ \leq \varepsilon_0/\Lambda$. This is then enough to prove the exterior blow-down result in part (ii) of Theorem 2.1 in step two. In the third step, we argue that if $\delta_{V,R,\Lambda}(s/8) \leq 0$, then C^1-graphicality can be propagated inwards from $S_*/2$ down to $s/32$. The details in this step are quite similar to the first, so we summarize them. Finally, the first and third steps are combined in step four to conclude the proof Theorem 2.1–(i), in which there are no sign restrictions on the deficit.

Step one: In this step, given $n \geq 2$, $\Gamma \geq 0$, and $\sigma > 0$, we prove the existence of ε_0 and M_0 (specified below in (2.65) and (2.66), and depending on n, Γ, and σ) such that if (2.1), (2.2), (2.3) and (2.4) hold with ε_0 and M_0, and in addition

$$0 \leq \delta_{V,R,\Lambda}(s/8) \leq \varepsilon_0\,, \tag{2.57}$$

then there exist $K_+ \in \mathcal{H}$ and $u_+ \in \mathcal{X}_\sigma(\Sigma_{K_+}, s/32, S_+/16)$ such that

$$V \text{ corresponds to } \Sigma_{K_+}(u_+, s/32, S_+/16) \text{ on } A_{s/32}^{S_+/16}, \tag{2.58}$$

where

$$R_+ = \max\Big\{\sup\Big\{\rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq 0\Big\}, 4\,s\Big\}, \quad S_+ = \min\Big\{R_+, \frac{\varepsilon_0}{\Lambda}\Big\} \geq 4\,s\,. \tag{2.59}$$

We start by imposing some constraints on the constants ε_0 and M_0. For the finite set

$$J = \Big\{\Big(\frac{1}{3}, \frac{1}{6}\Big), \Big(\frac{2}{3}, \frac{1}{3}\Big)\Big\} \subset \big\{(\eta_0, \eta) : \eta_0 > \eta > 0\big\}\,, \tag{2.60}$$

we let $\sigma_0 = \sigma_0(n)$ be such that Lemma 2.5–(ii), Theorems 2.6, and 2.7–(ii), (iii) hold for every $(\eta_0, \eta) \in J$, Lemma 2.5–(i) holds for $\sigma < \sigma_0$, and

$$\sigma_0 \leq \frac{\sigma_1}{C_0} \qquad \text{for } \sigma_1(n) \text{ as in (2.28), and } C_0(n) \text{ as in Lemma 2.5-(ii)}\,; \tag{2.61}$$

we shall henceforth assume, without loss of generality, that

$$\sigma < \sigma_0.$$

Moreover, for ε_1 and M_1 as in Lemma 2.8–(i) and C_0 as in Lemma 2.5, we let

$$\begin{aligned} M_0' \geq \max\Big\{&M_1\Big(n, \frac{\sigma}{2\,C_0}, \Gamma, \Big(\frac{1}{8}, \frac{1}{2}\Big), \Big(\frac{1}{6}, \frac{1}{4}\Big), \Big(\frac{1}{32}, \frac{1}{2}\Big)\Big), \\ &M_1\Big(n, \frac{\sigma}{2\,C_0}, \Gamma, \Big(\frac{1}{16}, \frac{1}{8}\Big), \Big(\frac{3}{32}, \frac{7}{64}\Big), \Big(\frac{1}{32}, \frac{1}{2}\Big)\Big)\Big\}, \end{aligned}$$

$$\begin{aligned} \varepsilon_0' \leq \min\Big\{&\varepsilon_1\Big(n, \frac{\sigma}{2\,C_0}, \Gamma, \Big(\frac{1}{8}, \frac{1}{2}\Big), \Big(\frac{1}{6}, \frac{1}{4}\Big), \Big(\frac{1}{32}, \frac{1}{2}\Big)\Big), \\ &\varepsilon_1\Big(n, \frac{\sigma}{2\,C_0}, \Gamma, \Big(\frac{1}{16}, \frac{1}{8}\Big), \Big(\frac{3}{32}, \frac{7}{64}\Big), \Big(\frac{1}{32}, \frac{1}{2}\Big)\Big)\Big\}. \end{aligned} \tag{2.62}$$

We also assume that

$$C(n,\Gamma)(\varepsilon_0')^{1/2}\le\min\Big\{\varepsilon_0,\frac{\sigma}{2\,C_0}\Big\}\,, \tag{2.63}$$

where $C(n,\Gamma)$ will be specified in (2.96)–(2.97), C_0 is as in Lemma 2.5, and ε_0 is smaller than both of the n-dependent ε_0's appearing in Lemmas 2.3 and 2.5. Lastly, we choose $\overline{\sigma}>0$ such that

$$\overline{\sigma}\le\min\Big\{\frac{\sigma}{2\,C_0},\sqrt{\varepsilon_0'/\omega_n}\Big\}\,, \tag{2.64}$$

and then, for ε_2, M_2 as in Lemma 2.8–(ii), we choose ε_0 and M_0 so that

$$\varepsilon_0\le\min\Big\{\varepsilon_0',\,\varepsilon_2\Big(n,\overline{\sigma},\Gamma,\frac18,\Big(\frac{1}{32},\frac12\Big)\Big)\Big\} \tag{2.65}$$

$$M_0\ge\max\Big\{M_0',\,M_2\Big(n,\overline{\sigma},\Gamma,\frac18,\Big(\frac{1}{32},\frac12\Big)\Big)\Big\}\,. \tag{2.66}$$

Let us now recall that, by assumption, $V\in\mathcal{V}_n(\Lambda,R,1/\Lambda)$ is such that

$$\|\mathrm{bd}_V\|(\partial B_R)\le\Gamma\,R^{n-1}\,,\qquad\sup_{\rho\in(R,1/\Lambda)}\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n}\le\Gamma\,; \tag{2.67}$$

in particular, by Theorem 2.7–(i),

$$\delta_{V,R,\Lambda}\text{ is decreasing on }(R,1/\Lambda)\,. \tag{2.68}$$

Moreover, we are assuming the existence of s with $\max\{64,M_0\}\,R<s<\varepsilon_0/4\,\Lambda$ such that

$$\begin{aligned}&|\delta_{V,R,\Lambda}(s/8)|\le\varepsilon_0,\\&R_*=\sup\Big\{\rho\ge\frac{s}{8}:\delta_{V,R,\Lambda}(\rho)\ge-\varepsilon_0\Big\}\ge4\,s,\end{aligned} \tag{2.69}$$

so that the latter inequality, together with (2.59), implies

$$R_*\ge R_+\,. \tag{2.70}$$

By (2.68), (2.69) and (2.70) we have

$$|\delta_{V,R,\Lambda}(r)|\le\varepsilon_0\,,\qquad\forall r\in[s/8,R_+]\,. \tag{2.71}$$

By (2.67), the specification of s satisfying (2.2), and (2.71), the assumptions (2.42), (2.47), and (2.48), respectively, of Lemma 2.8–(ii) with $r=s$, $\lambda_1=1/8$, and $(\eta_1,\eta_2)=(1/32,1/2)$ are satisfied due to our choices (2.65) and (2.66). Setting $H_0=H$, where $H\in\mathcal{H}$ is from the application of Lemma 2.8–(ii), we thus find $u_0\in\mathcal{X}_{\overline{\sigma}}(\Sigma_{H_0},s/32,s/2)$ such that

$$V\text{ corresponds to }\Sigma_{H_0}(u_0,s/32,s/2)\text{ on }A^{s/2}_{s/32}. \tag{2.72}$$

If it is the case that $S_+ = 4\,s$, we are in fact done with the proof of (2.58), since then $s/2 \geq S_+/16$. We may for the rest of this step assume then that $S_+ > 4\,s$, so that

$$R_+ = \sup\left\{\rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq 0\right\} \geq S_+ > 4\,s\,. \tag{2.73}$$

First, we observe that thanks to (2.72) and then (2.64),

$$T_0 := \frac{1}{(s/4)^n} \int_{s/8}^{s/4} r^{n-1}\,dr \int_{\Sigma_{H_0}} [u_0]_r^2 \leq \omega_n\,\overline{\sigma}^2 \leq \varepsilon_0'. \tag{2.74}$$

We let $s_j = 2^{j-3}\,s$ for $j \in \mathbb{Z}_{\geq -1}$. By (2.73) and by $s < \varepsilon_0/4\,\Lambda \leq \varepsilon_0'/4\,\Lambda$ there exists $N \in \{j \in \mathbb{N} : j \geq 2\} \cup \{+\infty\}$ such that

$$\{0, 1, .., N\} = \left\{j \in \mathbb{N} : 8\,s_j \leq S_+ = \min\left\{R_+, \frac{\varepsilon_0'}{\Lambda}\right\}\right\}. \tag{2.75}$$

Notice that if $\Lambda > 0$ then it must be $N < \infty$. We are now in the position to make the following:
Claim: There exist $\tau = \tau(n) \in (0, 1)$ and $\{(H_j, u_j)\}_{j=0}^{N-2}$ with $H_j \in \mathcal{H}$ such that, setting

$$T_j = \frac{1}{s_{j+1}^n} \int_{s_j}^{s_{j+1}} r^{n-1}\,dr \int_{\Sigma_{H_j}} [u_j]_r^2,$$

for every $j = 0, ..., N-2$,

$$u_j \in \mathcal{X}_\sigma(\Sigma_{H_j}, s/32, 4\,s_{j-1}) \cap \mathcal{X}_{\sigma/2\,C_0}(\Sigma_{H_j}, s_j/4, 4\,s_j), \tag{2.76}$$

$$V \text{ corresponds to } \Sigma_{H_j}(u_j, s/32, 4\,s_j) \text{ on } A_{s/32}^{4\,s_j}, \tag{2.77}$$

where C_0 is from Lemma 2.5, and

$$|\delta_{V,R,\Lambda}(s_j)| \leq \varepsilon_0', \tag{2.78}$$

$$T_j \leq C(n)\,\varepsilon_0'; \tag{2.79}$$

additionally, for every $j = 1, ..., N-2$,

$$|\nu_{H_j} - \nu_{H_{j-1}}|^2 \leq C(n)\,T_{j-1}, \tag{2.80}$$

$$\delta_{V,R,\Lambda}(s_j) \leq \tau\left\{\delta_{V,R,\Lambda}(s_{j-1}) + (1+\Gamma)\,\Lambda\,s_{j-1}\right\}, \tag{2.81}$$

$$T_j \leq C(n)\left\{\delta_{V,R,\Lambda}(s_{j-1}) - \delta_{V,R,\Lambda}(s_{j+2}) + \Lambda s_{j-1}\right\}. \tag{2.82}$$

Proof of the claim: We argue by induction. Clearly $(2.76)_{j=0}$, $(2.77)_{j=0}$, $(2.78)_{j=0}$ and $(2.79)_{j=0}$ are, respectively, (2.72), (2.69) and (2.74). This concludes the proof of the claim if $N = 2$, therefore we shall assume $N \geq 3$ for the rest of the argument. To set up the inductive argument, we consider $\ell \in \mathbb{N}$ such that: either $\ell = 0$; or $1 \leq \ell \leq N-3$ and (2.76), (2.77), (2.78), and (2.79) hold for $j = 0, ..., \ell$, and (2.80), (2.81) and (2.82) hold for $j = 1, ..., \ell$; and prove that all the conclusions of the claim hold with $j = \ell + 1$.

The validity of $(2.78)_{j=\ell+1}$ is of course immediate from (2.71) and (2.75). Also, after proving $(2.82)_{j=\ell+1}$, we will be able to combine it with $(2.78)_{j=\ell+1}$ and (2.75) to deduce $(2.79)_{j=\ell+1}$. We now prove, in order, (2.80), (2.76), (2.77), (2.81), and (2.82) with $j = \ell + 1$.

To prove $(2.80)_{j=\ell+1}$: Let $[a, b] \subset\subset (s_\ell, s_{\ell+1})$ with $(b - a) = (s_{\ell+1} - s_\ell)/2$, so that

$$\frac{1}{C(n)} \min_{r\in[a,b]} \int_{\Sigma_{H_\ell}} [u_\ell]_r^2 \le \frac{1}{s_{\ell+1}^n} \int_{s_\ell}^{s_{\ell+1}} r^{n-1}\, dr \int_{\Sigma_{H_\ell}} [u_\ell]_r^2 = T_\ell\,. \tag{2.83}$$

Keeping in mind $(2.76)_{j=\ell}$, $(2.77)_{j=\ell}$, we can apply Lemma 2.5–(ii) with $(r_1, r_2) = (s/32, 4\, s_\ell)$ and $[a, b] \subset (s_\ell, s_{\ell+1})$ to find $H_{\ell+1} \in \mathcal{H}$,

$$u_{\ell+1} \in \mathcal{X}_{C_0\, \sigma_0}(\Sigma_{H_{\ell+1}}, s/32, 4\, s_\ell) \tag{2.84}$$

(with C_0 as in Lemma 2.5–(ii)) and

$$s_\ell^* \in [a, b] \subset (s_\ell, s_{\ell+1})\,,$$

such that, thanks also to (2.83),

$$\Sigma_{H_\ell}(u_\ell, s/32, 4\, s_\ell) = \Sigma_{H_{\ell+1}}(u_{\ell+1}, s/32, 4\, s_\ell), \tag{2.85}$$

$$E^0_{\Sigma_{H_{\ell+1}}}\big([u_{\ell+1}]_{s_\ell^*}\big) = 0, \tag{2.86}$$

$$|\nu_{H_\ell} - \nu_{H_{\ell+1}}|^2 \le C(n)\, T_\ell, \tag{2.87}$$

$$\int_{\Sigma_{H_{\ell+1}}} [u_{\ell+1}]_r^2 \le C(n) \Big(T_\ell + \int_{\Sigma_{H_\ell}} [u_\ell]_r^2\Big), \qquad \forall r \in (s/32, 4\, s_\ell). \tag{2.88}$$

In particular, (2.87) is $(2.80)_{j=\ell+1}$.

To prove $(2.76)_{j=\ell+1}$ *and* $(2.77)_{j=\ell+1}$: Notice that (2.84), (2.85) do not imply $(2.76)_{j=\ell+1}$ and $(2.77)_{j=\ell+1}$, since, in $(2.77)_{j=\ell+1}$, we are claiming the graphicality of V inside $A^{4\, s_{\ell+1}}_{s/32}$ (which is strictly larger than $A^{4\, s_\ell}_{s/32}$), and in $(2.76)_{j=\ell+1}$ we are claiming that $u_{\ell+1}$ has C^1-norm bounded by σ or $\sigma/2\, C_0$ (depending on the radius), and not just by $C_0\, \sigma_0$ (with C_0 as in Lemma 2.5–(ii)).

We want to apply Lemma 2.8–(i) with $K = H_{\ell+1}$ and

$$r = 8\, s_{\ell+1}\,,\ \ (\lambda_1, \lambda_2) = \Big(\frac{1}{16}, \frac{1}{8}\Big),\ \ (\lambda_3, \lambda_4) = \Big(\frac{3}{32}, \frac{7}{64}\Big),\ \ (\eta_1, \eta_2) = \Big(\frac{1}{32}, \frac{1}{2}\Big). \tag{2.89}$$

We check the validity of (2.43), (2.44), (2.45), and (2.46) with $\varepsilon_1 = \varepsilon_0'$ and $M_1 = M_0'$ for these choices of r, λ_1, λ_2, λ_3, λ_4, η_1, η_2, and K. Since $r = 8\, s_{\ell+1} \ge s \ge \max\{M_0, 64\, R\} \ge \max\{M_0', 64\, R\}$, and since (2.75) and $\ell + 1 \le N$ give $r = 8\, s_{\ell+1} \le \varepsilon_0/\Lambda \le \varepsilon_0'/\Lambda$, we deduce the validity of (2.43) with $r = 8\, s_{\ell+1}$. The validity of (2.44) with $r = 8\, s_{\ell+1}$ is immediate from (2.71) by our choice (2.62) of ε_0'. Next we notice that

$$\|V\|(A^{\lambda_4 r}_{\lambda_3 r}) = \|V\|(A^{7\,[8\, s_{\ell+1}]/64}_{3\,[8\, s_{\ell+1}]/32}) = \|V\|(A^{7\, s_\ell/4}_{3\, s_\ell/2}) > 0$$

thanks to $(2.77)_{j=\ell}$, so that (2.45) holds for r, λ_3 and λ_4 as in (2.89). Finally, by (2.28) (which can be applied to $u_{\ell+1}$ thanks to (2.61)), (2.85) and $(2.76)_{j=\ell}$, and, then by (2.88), we have

$$\begin{aligned}\frac{1}{r^n}\int_{A_{\lambda_1 r}^{\lambda_2 r}}\omega_{H_{\ell+1}}^2\,d\|V\| &\le \frac{C(n)}{s_{\ell+1}^n}\int_{s_\ell}^{s_{\ell+1}} r^{n-1}\,dr\int_{\Sigma_{H_{\ell+1}}}[u_{\ell+1}]_r^2\\ &\le C(n)\,T_\ell+\frac{C(n)}{s_{\ell+1}^n}\int_{s_\ell}^{s_{\ell+1}} r^{n-1}\,dr\int_{\Sigma_{H_\ell}}[u_\ell]_r^2\\ &\le C(n)\,T_\ell\le C(n)\,\varepsilon_0',\end{aligned}$$

where in the last inequality we have used $(2.79)_{j=\ell}$. Again by our choice (2.62) of ε_0', we deduce that (2.46) holds with r, λ_1 and λ_2 as in (2.89). We can thus apply Lemma 2.8–(i), and find $v\in\mathcal{X}_{\sigma/2\,C_0}(\Sigma_{H_{\ell+1}}, s_{\ell+1}/4, 4\,s_{\ell+1})$ such that

$$V \text{ corresponds to } \Sigma_{H_{\ell+1}}(v, s_{\ell+1}/4, 4\,s_{\ell+1}) \text{ on } A_{s_{\ell+1}/4}^{4\,s_{\ell+1}}. \tag{2.90}$$

By (2.85), $(2.77)_{j=\ell}$, and (2.90), $v=u_{\ell+1}$ on $\Sigma_{H_{\ell+1}}\times(s_{\ell+1}/4, 4\,s_\ell)$. We can thus use v to extend $u_{\ell+1}$ from $\Sigma_{H_{\ell+1}}\times(s/32, 4\,s_\ell)$ to $\Sigma_{H_{\ell+1}}\times(s/32, 4\,s_{\ell+1})$, and, thanks to (2.85), $(2.77)_{j=\ell}$ and (2.90), the resulting extension is such that

$$u_{\ell+1}\in\mathcal{X}_{\sigma/2\,C_0}(\Sigma_{H_{\ell+1}}, s_{\ell+1}/4, 4\,s_{\ell+1}) \quad\text{and} \tag{2.91}$$

$$V \text{ corresponds to } \Sigma_{H_{\ell+1}}(u_{\ell+1}, s/32, 4\,s_{\ell+1}) \text{ on } A_{s/32}^{4\,s_{\ell+1}}. \tag{2.92}$$

The bound (2.91) is part of $(2.76)_{j=\ell+1}$, and (2.92) is $(2.77)_{j=\ell+1}$, so in order to complete the proof of $(2.76)_{j=\ell+1}$ and $(2.77)_{j=\ell+1}$, it remains to show that the C^1-norm of u is bounded by σ in between $s/32$ and $4\,s_\ell$.

Towards this end, we record the following consequence of taking square roots in $(2.81)_{j=m}$ (using $\delta_{V,R,\Lambda}\ge 0$ from (2.75)) and summing over $m=1,\dots,i$ for any $1\le i\le\ell$: for $\alpha=\sum_{k=0}^\infty 2^{-k/2}$ and $\tilde C(n,\Gamma)=\tau^{1/2}(1+\Gamma)$,

$$\begin{aligned}S_i:=\sum_{m=0}^{i}\delta_{V,R,\Lambda}(s_m)^{1/2} &\le \tau^{1/2}\sum_{m=0}^{i-1}\delta_{V,R,\Lambda}(s_m)^{1/2}+(1+\Gamma)(\Lambda\,s_m)^{1/2}\\ &\quad+\delta_{V,R,\Lambda}(s_0)^{1/2}\\ &\le \tau^{1/2}S_{i-1}+\alpha\,\tilde C(n,\Gamma)(\Lambda\,s_{i-1})^{1/2}+\delta_{V,R,\Lambda}(s_0)^{1/2}\\ &\le \tau^{1/2}S_{i-1}+(1+\alpha\,\tilde C(n,\Gamma))(\varepsilon_0')^{1/2},\end{aligned} \tag{2.93}$$

where in the last line we have used (2.75) and (2.71). By induction, utilizing (2.57), (2.65) for the base case and (2.93) for the induction step we have

$$S_i\le\frac{(1+\alpha\,\tilde C(n,\Gamma))(\varepsilon_0')^{1/2}}{1-\tau^{1/2}}\quad\forall\,0\le i\le\ell\,. \tag{2.94}$$

Now by the positivity of $\delta_{V,R,\Lambda}$ and $(2.82)_{j=\ell}$, for all $m=1,\dots,\ell$,

$$T_m^{1/2}\le C(n)\delta_{V,R,\Lambda}(s_{m-1})^{1/2}+C(n)(\Lambda\,s_{m-1})^{1/2}\,. \tag{2.95}$$

In turn, by $(2.80)_{j=\ell+1}$, (2.74) and (2.95), then (2.75) and $(2.94)_{i=\ell-1}$,

$$\frac{1}{C(n)}\sum_{m=1}^{\ell+1}|\nu_{H_m}-\nu_{H_{m-1}}| \le \sum_{m=0}^{\ell} T_m^{1/2} \le (\varepsilon_0')^{1/2}+C(n)S_{\ell-1}+\alpha\, C(n)(\Lambda\, s_{\ell-1})^{1/2} \le C(n,\Gamma)(\varepsilon_0')^{1/2}/C(n) \tag{2.96}$$

for a suitable $C(n,\Gamma)$. We use (2.96) to see

$$|\nu_{H_i}-\nu_{H_{\ell+1}}| \le C(n,\Gamma)(\varepsilon_0')^{1/2} \quad \forall i=0,...,\ell. \tag{2.97}$$

Now $u_i \in \mathcal{X}_{\sigma/2\,C_0}(\Sigma_{H_j}, s_i/4, 4\,s_i)$ by $(2.76)_{j=i}$, and $\sigma/2\,C_0$ and $|\nu_{H_i}-\nu_{H_{\ell+1}}|$ are small enough to apply Lemma 2.5–(i) by our choice of σ above (2.61) and (2.97) with (2.63), respectively. Then we obtain w_i corresponding to V on $A^{4\,s_i}_{s_i/4}$ and in $\mathcal{X}_{\sigma/2+C_0|\nu_{H_i}-\nu_{H_{\ell+1}}|}(\Sigma_{H_{\ell+1}}, s_i/4, 4\,s_i)$, and by (2.97), (2.63),

$$\frac{\sigma}{2}+C_0\,|\nu_{H_i}-\nu_{H_{\ell+1}}| \le \frac{\sigma}{2}+C_0\,\frac{\sigma}{2\,C_0}=\sigma,$$

so $w_i \in \mathcal{X}_\sigma(\Sigma_{H_{\ell+1}}, s_i/4, 4\,s_i)$. Finally, since they represent the same surface over $\Sigma_{H_{\ell+1}}$, $w_i = u_{\ell+1}$ on $A^{4\,s_i}_{s_i/4}$. Gathering these estimates for $i=0,...,\ell$, we have $u_{\ell+1} \in \mathcal{X}_\sigma(\Sigma_{H_{\ell+1}}, s/32, 4\,s_\ell)$, which finishes the proof of $(2.76)_{j=\ell+1}$.
To prove $(2.81)_{j=\ell+1}$: We set $r_0=(s_\ell+s_{\ell+1})/2$ and notice that for $\eta_0=1/3$,

$$r_1=r_0\,(1-\eta_0)=s_\ell\,, \qquad r_2=r_0\,(1+\eta_0)=s_{\ell+1}\,. \tag{2.98}$$

For $\eta=1/6$ we correspondingly set

$$r_3=r_0\,(1-\eta)=:s_\ell^-\,, \qquad r_4=r_0\,(1+\eta)=:s_\ell^+\,, \tag{2.99}$$

and notice that $(\eta_0,\eta)\in J$, see (2.60). With the aim of applying Theorem 2.7–(iii) to these radii, we notice that $(2.77)_{j=\ell+1}$ implies that assumption (2.37) holds with $H=H_{\ell+1}$ and $u=u_{\ell+1}$, while, by (2.86), $r=s_\ell^*\in(s_\ell,s_{\ell+1})$ is such that (2.39) holds. By $\Lambda\, s_{\ell+1}\le\varepsilon_0\le 1$, (2.75), and (2.40), with $C(n)=C_0(n,1/6,1/3)$ for C_0 as in Theorem 2.7–(iii), we have

$$\begin{aligned} &s_{\ell+1}^{-n}\,\left|\|V\|\big(B_{s_\ell^+}\setminus B_{s_\ell^-}\big)-\omega_n\left((s_\ell^+)^n-(s_\ell^-)^n\right)\right| \\ &=s_{\ell+1}^{-n}\left|\mathcal{H}^n(\Sigma_{H_{\ell+1}}(u_{\ell+1},s_\ell^-,s_\ell^+))-\mathcal{H}^n(\Sigma_{H_{\ell+1}}(0,s_\ell^-,s_\ell^+))\right| \\ &\le C(n)\left\{(\Lambda\,s_{\ell+1})^2\;+\Theta_{V,R,\Lambda}(s_{\ell+1})-\Theta_{V,R,\Lambda}(s_\ell)\right\}. \end{aligned}$$

Setting for brevity $\delta=\delta_{V,R,\Lambda}$ and $\Theta=\Theta_{V,R,\Lambda}$, and recalling that

$$\begin{aligned} r^n\,\delta(r)&=\omega_n\,r^n-\Theta(r)\,r^n \\ &=\omega_n\,r^n-\|V\|(B_r\setminus B_R)-\Lambda\,r^n\int_R^r\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n}\,d\rho+\frac{R\,\|\delta V\|(\partial B_R)}{n}, \end{aligned}$$

we have

$$s_\ell^{-n}\,\left|(s_\ell^-)^n\,\delta(s_\ell^-)-(s_\ell^+)^n\,\delta(s_\ell^+)\right|\le C(n)\left\{(\Lambda\,s_\ell)^2+\Theta(s_{\ell+1})-\Theta(s_\ell)\right\}$$

$$+C(n)\,\Lambda\,s_\ell^{-n}\Big\{(s_\ell^+)^n\int_R^{s_\ell^+}\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n}\,d\rho-(s_\ell^-)^n\int_R^{s_\ell^-}\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n}\,d\rho\Big\}$$
$$\le C(n)\,\Big\{(\Lambda\,s_\ell)^2+\Theta(s_{\ell+1})-\Theta(s_\ell)\Big\}+C(n)\,\Lambda\int_R^{s_\ell^+}\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n}\,d\rho.$$

By $\Lambda\,s_\ell\le 1$ and since $s_\ell^+\le s_\ell\le\varepsilon_0/8\,\Lambda$ thanks to $\ell<N$, we can use the upper bound $\|V\|(B_\rho\setminus B_R)\le\Gamma\,\rho^n$ with $\rho\in(R,s_\ell^+)\subset(R,1/\Lambda)$, to find that

$$\Big|\frac{(s_\ell^-)^n}{s_\ell^n}\,\delta(s_\ell^-)-\frac{(s_\ell^+)^n}{s_\ell^n}\,\delta(s_\ell^+)\Big|\le C_*(n)\,\big\{\delta(s_\ell)-\delta(s_{\ell+1})\big\}+C_*(n)\,(\Gamma+1)\,\Lambda\,s_\ell,$$

for a constant $C_*(n)$. By rearranging terms and using the monotonicity of δ on (R,∞) and $(s_\ell^-,s_\ell^+)\subset(s_\ell,s_{\ell+1})$ we find that

$$\begin{aligned}&\big(C_*(n)\,+(s_\ell^+)^n/(s_\ell^n)\big)\,\delta(s_{\ell+1})\le C_*(n)\,\delta(s_{\ell+1})+\big((s_\ell^+)^n/(s_\ell^n)\big)\,\delta(s_\ell^+)\\&\le C_*(n)\,\delta(s_\ell)+\big((s_\ell^-)^n/(s_\ell^n)\big)\,\delta(s_\ell^-)+C_*(n)\,(1+\Gamma)\,\Lambda\,s_\ell\\&\le\big(C_*(n)\,+(s_\ell^-)^n/(s_\ell^n)\big)\,\delta(s_\ell)+C_*(n)\,(1+\Gamma)\,\Lambda\,s_\ell.\end{aligned}$$

We finally notice that by (2.98), (2.99), $\eta_0=1/3$, and $\eta=1/6$, we have

$$\frac{s_\ell^-}{s_\ell}=\frac{r_0\,(1-\eta)}{r_0\,(1-\eta_0)}=\frac{5}{4},\qquad\frac{s_\ell^+}{s_\ell}=2\,\frac{s_\ell^+}{s_{\ell+1}}=2\,\frac{1+\eta}{1+\eta_0}=\frac{7}{4},$$

so that we find that $\delta(s_{\ell+1})\le\tau\{\delta(s_\ell)+(1+\Gamma)\,\Lambda\,s_\ell\}$ (i.e. $(2.81)_{j=\ell+1}$) with

$$\tau=\tau(n)=\frac{C_*(n)+(5/4)^n}{C_*(n)+(7/4)^n},\qquad\tau_*=\tau_*(n)=\frac{C_*(n)}{C_*(n)+(7/4)^n}<\tau.$$

To prove $(2.82)_{j=\ell+1}$: We finally prove $(2.82)_{j=\ell+1}$, i.e.

$$\frac{1}{s_{j+1}^n}\int_{s_{\ell+1}}^{2\,s_{\ell+1}}r^{n-1}\int_{\Sigma_{H_{\ell+1}}}[u_{\ell+1}]_r^2\le C(n)\big\{\delta_{V,R,\Lambda}(s_\ell)-\delta_{V,R,\Lambda}(s_{\ell+3})+\Lambda\,s_\ell\big\}.\tag{2.100}$$

By $(2.77)_{j=\ell+1}$ we know that

$$V\text{ corresponds to }\Sigma_{H_{\ell+1}}(u_{\ell+1},s/32,4\,s_{\ell+1})\text{ on }A_{s/32}^{4\,s_{\ell+1}}\,.\tag{2.101}$$

Now, (2.36) holds with $r_0=3\,s_\ell$ and $(\eta_0,\eta)=(2/3,1/3)\in J$, see (2.60), if

$$\begin{aligned}&r_1=s_\ell=3\,s_\ell-2\,s_\ell,\qquad r_2=5\,s_\ell=3\,s_\ell+2\,s_\ell,\\&r_3=s_{\ell+1}=3\,s_\ell-s_\ell,\qquad r_4=2\,s_{\ell+1}=3\,s_\ell+s_\ell.\end{aligned}$$

Since $s_\ell^*\in(s_\ell,s_{\ell+1})\subset(r_1,r_2)$, by (2.101), (2.86) and $(r_1,r_2)\subset(s/32,4\,s_{\ell+1})$ we can apply Theorem 2.6 to deduce that

$$\int_{s_{\ell+1}}^{2\,s_{\ell+1}}r^{n-1}\int_{\Sigma_{H_{\ell+1}}}[u_{\ell+1}]_r^2\le C(n)\int_{s_\ell}^{5\,s_\ell}r^{n+1}\int_{\Sigma_{H_{\ell+1}}}(\partial_r u_{\ell+1})_r^2+C(n)\,\Lambda\,(s_\ell)^{n+1}.$$

Again by (2.101), Theorem 2.7–(ii) with $(r_1, r_2) = (s_\ell, 8\,s_\ell)$ gives

$$s_\ell^{-n}\int_{s_\ell}^{5\,s_\ell} r^{n+1}\int_{\Sigma_{H_{\ell+1}}}(\partial_r[u_{\ell+1}])_r^2 \le s_\ell^{-n}\int_{s_\ell}^{8\,s_\ell} r^{n+1}\int_{\Sigma_{H_{\ell+1}}}(\partial_r[u_{\ell+1}])_r^2$$
$$\le C(n)\left\{\Theta_{V,R,\Lambda}(8\,s_\ell)-\Theta_{V,R,\Lambda}(s_\ell)\right\} \le C(n)\left\{\delta_{V,R,\Lambda}(s_\ell)-\delta_{V,R,\Lambda}(s_{\ell+3})\right\}.$$

The last two estimates combined give (2.100), which finishes the **claim**.

Proof of (2.58): We assume $S_+ < \infty$ (that is either $\Lambda > 0$ or $R_+ < \infty$), and recall that we have already proved (2.58) if $S_+ = 4\,s$. Otherwise, N (as defined in (2.75)) is finite, with $2^N \le \frac{S_+}{s} < 2^{N+1}$. By $(2.76)_{j=N-2}$ and $(2.77)_{j=N-2}$, we have that $u_{N-2} \in \mathcal{X}_\sigma(\Sigma_{H_{N-2}}, s/32, 4\,s_{N-2})$ and V corresponds to $\Sigma_{H_{N-2}}(u_{N-2}, s/32, 4\,s_{N-2})$ on $A^{4\,s_{N-2}}_{s/32}$. Since $4\,s_{N-2} = 2^{N+1}\,s/16 > S_+/16$, we deduce (2.58) with $K_+ = H_{N-2}$ and $u_+ = u_{N-2}$.

Step two: In this step we prove statement (ii) in Theorem 2.1. We assume that $\Lambda = 0$ and that

$$\delta(r) \ge -\varepsilon_0 \qquad \forall r \ge \frac{s}{8}\,, \tag{2.102}$$

where we have set for brevity $\delta = \delta_{V,R,0}$. We must first show that

$$\delta(r) \ge 0 \quad \forall r \ge \frac{s}{8}\,. \tag{2.103}$$

Since δ is decreasing in r, it has a limit $\lim_{r\to\infty}\delta(r) =: \delta_\infty \ge -\varepsilon_0$, and we want to show that $\delta_\infty = 0$. Next, we know that for any sequence $R_i \to \infty$, V/R_i converges locally in the varifold sense to a limiting integer rectifiable varifold cone W. By the local varifold convergence and $n \ge 2$, W is stationary in $\mathbb{R}^{n+1}$, and it is the case that

$$\delta_{W,0,0}(r) = \delta_\infty \ge -\varepsilon_0 \quad \forall r > 0\,.$$

Up to decreasing ε_0 if necessary (and recalling that $\delta_{W,0,0}$ is the usual area excess multiplied by -1), Allard's theorem and the fact that $W/r = W$ imply that W corresponds to a multiplicity one plane. In particular, it must be that $\delta_\infty = 0$, which together with the monotonicity of δ yields (2.103).

By (2.103), $S_+ = S_* = \infty$, and so by (2.76) and (2.77), there is a sequence $\{(H_j, u_j)\}_{j=0}^N$ but with $N = \infty$ now, satisfying

$$V \text{ corresponds to } \Sigma_{H_j}(u_j, s/32, 4\,s_j) \text{ on } A^{4\,s_j}_{s/32} \quad \forall j \ge 0, \tag{2.104}$$

$$|\nu_{H_j} - \nu_{H_{j-1}}|^2 \le C(n)\,T_{j-1}, \qquad \text{if } j \ge 1, \tag{2.105}$$

$$\delta(s_j) \le \begin{cases} \varepsilon_0, & \text{if } j = 0,\\ \tau\,\delta(s_{j-1}), & \text{if } j \ge 1,\end{cases} \tag{2.106}$$

$$T_j \le \begin{cases} C(n)\,\varepsilon_0, & \text{if } j = 0,\\ C(n)\,\delta(s_{j-1}), & \text{if } j \ge 1.\end{cases} \tag{2.107}$$

Notice that, in asserting the validity of (2.107) with $j \geq 1$, we have used (2.103) to estimate $-\delta(s_{j+2}) \leq 0$ in $(2.82)_j$. By iterating (2.106) we find

$$\delta(s_j) \leq \tau^j\, \delta(s/8) \leq \tau^j\, \varepsilon_0\,, \qquad \forall j \geq 1\,, \tag{2.108}$$

which, combined with (2.107) and (2.105), gives, for every $j \geq 1$,

$$T_j \leq C(n)\ \min\{1, \tau^{j-1}\}\, \delta(s/8) \leq C(n)\, \tau^j\, \delta(s/8), \tag{2.109}$$

$$|\nu_{H_j} - \nu_{H_{j-1}}|^2 \leq C(n)\ \min\{1, \tau^{j-2}\}\, \delta(s/8) \leq C(n)\, \tau^j\, \delta(s/8), \tag{2.110}$$

thanks also to $\tau = \tau(n)$ and, again, to (2.103). By (2.110), for every $j \geq 0$, $k \geq 1$, we have $|\nu_{H_{j+k}} - \nu_{H_j}| \leq C(n)\, \sqrt{\delta(s/8)}\, \sum_{h=1}^{k+1} \big(\sqrt{\tau}\big)^{j-1+h}$, so that there exists $K \in \mathcal{H}$ such that

$$\varepsilon_j^2 := |\nu_K - \nu_{H_j}|^2 \leq C(n)\, \tau^j\, \delta(s/8)\,, \qquad \forall j \geq 1\,, \tag{2.111}$$

In particular, for j large enough, we have $\varepsilon_j < \varepsilon_0$, and thus, by Lemma 2.5–(i) and by (2.104) we can find $v_j \in \mathcal{X}_{C(n)\,(\sigma+\varepsilon_j)}(\Sigma_K, s/32, 4\, s_j)$ such that

$$V \text{ corresponds to } \Sigma_K(v_j, s/32, 4\, s_j) \text{ on } A_{s/32}^{4\, s_j}\,. \tag{2.112}$$

By (2.112), $v_{j+1} = v_j$ on $\Sigma_K \times (s/32, 4\, s_j)$. Since $s_j \to \infty$ we have thus found $u \in \mathcal{X}_{C(n)\,\sigma}(\Sigma_K; s/32, \infty)$ such that

$$V \text{ corresponds to } \Sigma_K(u, s/32, \infty) \text{ on } A_{s/32}^{\infty}\,, \tag{2.113}$$

which corresponds to (2.5) with ∞ in place of S_*.

To prove (2.6), we notice that if $r \in (s_j, s_{j+1})$ for some $j \geq 1$, then, setting $\tau = (1/2)^\alpha$ (i.e., $\alpha = \log_{1/2}(\tau) \in (0, 1)$) and noticing that $r/s \leq 2^{j+1-3}$, by (2.68) and (2.108) we have

$$\begin{aligned} \delta(r) &\leq \delta(s_j) \leq \tau^j\, \delta(s/8) = 2^{-j\,a}\, \delta(s/8) = 4^{-\alpha}\, 2^{-(j-2)\alpha}\, \delta(s/8) \\ &\leq C(n)\, (s/r)^\alpha\, \delta(s/8), \end{aligned}$$

where in the last inequality (2.102) was used again; this proves (2.6). To prove (2.7), we recall that $\omega_K(y) = \arctan(|\nu_K \cdot \hat{y}|/|\mathbf{p}_K\, \hat{y}|)$, provided arctan is defined on $\mathbb{R} \cup \{\pm\infty\}$, and where $\hat{y} = y/|y|$, $y \neq 0$. Now, by (2.113),

$$y = |y|\, \frac{\mathbf{p}_K\, \hat{y} + u(\mathbf{p}_K\, \hat{y}, |y|)\, \nu_K}{\sqrt{1 + u(\mathbf{p}_K\, \hat{y}, |y|)^2}}, \qquad \forall y \in (\operatorname{spt} V) \setminus B_{s/32},$$

so that $|\mathbf{p}_K\, \hat{y}| \geq 1/2$ for $y \in (\operatorname{spt} V)\setminus B_{s/32}$; therefore, by (2.111), up to further decreasing the value of ε_0, and recalling $\delta(s/8) \leq \varepsilon_0$, we conclude

$$|\mathbf{p}_{H_j}\, \hat{y}| \geq \frac{1}{3}\,, \qquad \forall y \in (\operatorname{spt} V) \setminus B_{s/32}\,, \tag{2.114}$$

for every $j \in \mathbb{N} \cup \{+\infty\}$ (if we set $H_\infty = K$). By (2.114) we easily find

$$|\omega_K(y) - \omega_{H_j}(y)| \leq C\, |\nu_{H_j} - \nu_K|, \qquad \forall y \in (\operatorname{spt} V) \setminus B_{s/32}, \forall j \geq 1,$$

from which we deduce that, if $j \geq 1$ and $r \in (s_j, s_{j+1})$, then

$$\begin{aligned}\frac{1}{r^n}\int_{A_r^{2r}} \omega_K^2 \, d\|V\| &\leq C(n)\left\{\frac{1}{s_j^n}\int_{A_{s_j}^{s_{j+1}}} \omega_K^2 \, d\|V\| + \frac{1}{s_{j+1}^n}\int_{A_{s_{j+1}}^{s_{j+2}}} \omega_K^2 \, d\|V\|\right\} \\ &\leq C(n)\left\{\frac{1}{s_j^n}\int_{A_{s_j}^{s_{j+1}}} \omega_{H_j}^2 \, d\|V\| + \frac{1}{s_{j+1}^n}\int_{A_{s_{j+1}}^{s_{j+2}}} \omega_{H_{j+1}}^2 \, d\|V\|\right\} \\ &\quad + C(n)\,\Gamma\left(|\nu_K - \nu_{H_j}|^2 + |\nu_K - \nu_{H_{j+1}}|^2\right),\end{aligned}$$

where (2.67) was used to bound $\|V\|(A_\rho^{2\rho}) \leq \Gamma\,(2\,\rho)^n$ with $\rho = s_j, s_{j+1} \in (R, 1/\Lambda)$. By (2.104) we can exploit (2.28) on the first two integrals, so that taking (2.111) into account we find that, if $j \geq 1$ and $r \in (s_j, s_{j+1})$, then $r^{-n}\int_{A_r^{2r}} \omega_K^2 \, d\|V\| \leq C(n)\{T_j + T_{j+1}\} + C(n)\,\Gamma\,\tau^j\,\delta(s/8) \leq C(n)\,(1+\Gamma)\,\tau^j\,\delta(s/8)$, where in the last inequality we have used (2.109). Since $\tau^j \leq C(n)\,(s/r)^\alpha$, we conclude the proof of (2.7), and thus, of Theorem 2.1–(ii).

Step three: In this step, given $n \geq 2$, $\Gamma \geq 0$, and $\sigma > 0$, we claim the existence of ε_0 and M_0, depending only on n, Γ, and σ, such that if (2.1), (2.2), (2.3) and (2.4) hold with ε_0 and M_0, and in addition,

$$-\varepsilon_0 \leq \delta_{V,R,\Lambda}(s/8) \leq 0\,, \tag{2.115}$$

then there exist $K_- \in \mathcal{H}$ and $u_- \in \mathcal{X}_\sigma(\Sigma_{K_-}, s/32, S_*/2)$ such that

$$V \text{ corresponds to } \Sigma_{K_-}(u_+, s/32, S_*/2) \text{ on } A_{s/32}^{S_*/2}, \tag{2.116}$$

where S_* and R_* are as in Theorem 2.1. The argument is quite similar to that of the first step, with minor differences due to the opposite sign of the deficit. The first is that the iteration instead begins at the outer radius S_* and proceeds inwards via intermediate radii $s_j = 2^{-j}S_*$, and the second is that, in the analogue of the graphicality propagation claims $(2.76)_{j=\ell+1}$ and $(2.77)_{j=\ell+1}$, the negative sign on $\delta_{V,R,\Lambda}$ is used to sum the "tilting" between successive planes H_j and H_{j+1}.

Step four: Finally, we combine steps one and three to prove statement (i) in Theorem 2.1. Before choosing the parameters ε_0 and M_0, we need a preliminary result. We claim that for any $\varepsilon' > 0$, there exists $\sigma'(\varepsilon') > 0$ such that if $r_1 < r_2$, $K_1, K_2 \in \mathcal{H}$ with $\nu_{K_1} \cdot \nu_{K_2} \geq 0$ and accompanying $u_i \in \mathcal{X}_{\sigma'}(\Sigma_{K_i}, r_1, r_2)$, and M is a smooth hypersurface such that $M \cap A_{r_1}^{r_2}$ corresponds to $\Sigma_{K_i}(u_i, r_1, r_2)$ for $i = 1, 2$, then

$$|\nu_{K_1} - \nu_{K_2}| < \varepsilon'\,. \tag{2.117}$$

It is immediate from $\nu_{K_1} \cdot \nu_{K_2} \geq 0$ and the fact that the L^∞-bounds on u_i imply that M is contained in the intersection of two cones containing K_1 and K_2, whose openings become arbitrarily narrow as $\sigma' \to 0$.

Fix $n \geq 2$, $\Gamma \geq 0$, and $\sigma > 0$; we assume without loss of generality that $\sigma < \sigma_0$, where σ_0 is the dimension-dependent constant from Lemma 2.5. We choose ε' with corresponding σ' according to (2.117) such that, up to decreasing σ' if necessary,

$$\varepsilon' < \varepsilon_0\,, \quad C_0(\sigma' + \varepsilon') \leq \sigma\,, \tag{2.118}$$

where ε_0, C_0 are as in Lemma 2.5. Next, we choose $\varepsilon_0 = \varepsilon_0(n, \Gamma, \sigma)$ and $M_0 = M_0(n, \Gamma, \sigma)$ to satisfy several restrictions: first, ε_0 is smaller than the ε_0 from Lemma 2.5 and each $\varepsilon_0(n, \Gamma, \sigma')$ from steps one and three, and M_0 is larger than $M_0(n, \Gamma, \sigma')$ from those steps; second, with ε_2 and M_2 as in Lemma 2.8–(ii), we also assume that

$$\varepsilon_0 \le \min\Big\{\varepsilon', \varepsilon_2\Big(n, \sigma', \Gamma, \frac{1}{16}, \Big(\frac{1}{128}, \frac{1}{2}\Big)\Big)\Big\}, \ M_0 \ge M_2\Big(n, \sigma', \Gamma, \frac{1}{16}, \Big(\frac{1}{128}, \frac{1}{2}\Big)\Big) \tag{2.119}$$

In the remainder of this step, we suppose that

$$V \text{ satisfies (2.1), (2.2), (2.3) and (2.4) at mesoscale } s\,. \tag{2.120}$$

In proving Theorem 2.1–(i), there are three cases depending on whether $\delta_{V,R,\Lambda}$ changes sign on $[s/8, S_*]$.
Case one: $\delta_{V,R,\Lambda}(r) \ge 0$ for all $r \in [s/8, S_*]$. If the deficit is non-negative, then in particular

$$0 \le \delta_{V,R,\Lambda}(s/8) \le \varepsilon_0 \tag{2.121}$$

and $S_* = S_+$, where S_+ was defined in (2.59). By our choice of ε_0 and M_0 at the beginning of this step and the equivalence of (2.121) and (2.57), step one applies and the conclusion (2.58) is (2.5). Thus Theorem 2.1–(i) is proved.
Case two: $\delta_{V,R,\Lambda}(r) \le 0$ for all $r \in [s/8, S_*]$. Should the deficit be non-positive in this interval, then in particular, (2.115) holds in addition to (2.1), (2.2), (2.3) and (2.4). Therefore, by our choice of ε_0 and M_0, step three applies. The conclusion (2.116) is (2.5) (in fact with larger upper radii $S_*/2$), and Theorem 2.1–(i) is proved.
Case three: $\delta_{V,R,\Lambda}$ changes sign in $[s/8, S_*]$. By the monotonicity of $\delta_{V,R,\Lambda}$,

$$\delta_{V,R,\Lambda}(s/8) > 0 > \delta_{V,R,\Lambda}(S_*)\,. \tag{2.122}$$

First, by (2.122), (2.57) is satisfied, so (2.58) gives $K_+ \in \mathcal{H}$ and $u_+ \in \mathcal{X}_{\sigma'}(\Sigma_{K_+}, s/32, S_+/16)$ such that

$$V \text{ corresponds to } \Sigma_{K_+}(u_+, s/32, S_+/16) \text{ on } A_{s/32}^{S_+/16}, \tag{2.123}$$

where

$$R_+ = \max\Big\{\sup\Big\{\rho \ge \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \ge 0\Big\}, 4\,s\Big\}, \quad S_+ = \min\Big\{R_+, \frac{\varepsilon_0}{\Lambda}\Big\}. \tag{2.124}$$

If $S_+ = S_*$, then (2.123) is (2.5) and we are done. So we assume for the rest of this case that $S_+ < S_*$, which implies $S_+ \neq \varepsilon_0/\Lambda$ and thus

$$4\,s \le R_+ = S_+ < S_*\,. \tag{2.125}$$

Next, we make the following
Claim: There exists $K_- \in \mathcal{H}$ and $u_- \in \mathcal{X}_{\sigma'}(\Sigma_{K_-}, R_+/2, S_*/2)$ such that

$$V \text{ corresponds to } \Sigma_{K_-}(u_-, R_+/2, S_*/2) \text{ on } A_{R_+/2}^{S_*/2}\,. \tag{2.126}$$

Proof of the claim: There are two subcases.
Subcase one: $16\,R_+ < \varepsilon_0/4\,\Lambda$ and $64\,R_+ < R_*$. We claim the conditions of step three are verified at $s' = 16\,R_+$. First, (2.1) holds from (2.120), and

$$\max\{64, M_0\}\,R < 16\,R_+ < \frac{\varepsilon_0}{4\,\Lambda}$$

(which is (2.2)) holds due to the assumption of the subcase and $16R_+ \geq s > \max\{64, M_0\}R$. Next, $2\,R_+ < R_*/4$ by the assumption of the subcase, which combined with the monotonicity of $\delta_{V,R,\Lambda}$ and (2.124) gives $-\varepsilon_0 \leq \delta_{V,R,\Lambda}(2\,R_+) \leq 0$. This implies (2.115) and (2.3) with $s' = 16\,R_+$. Lastly, (2.4) holds at $s' = 16\,R_+$ since $64\,R_+ < R_*$. Thus we apply (2.116) at $s' = 16\,R_+$, finding (2.126).
Subcase two: One or both of $16\,R_+ \geq \varepsilon_0/4\,\Lambda$, $64\,R_+ \geq R_*$ hold. In this case,

$$64\,R_+ \geq \min\{\varepsilon_0/\Lambda, R_*\} = S_*\,. \tag{2.127}$$

We wish to apply Lemma 2.8–(ii) with $r = S_*$, $\lambda_1 = \frac{1}{16}$, $(\eta_1, \eta_2) = (\frac{1}{128}, \frac{1}{2})$. By (2.120), (2.42) holds for V, and by (2.119), (2.120), and $S_* \geq 4\,s$,

$$\max\{M_2, 64\}\,R \leq s \leq \frac{S_*}{4} \leq S_* \leq \frac{\varepsilon_2}{\Lambda}\,,$$

which is (2.47). Finally, we have $R_* \geq S_*/16 \geq s/8$, so that by the definition of R_*, (2.3), the monotonicity of $\delta_{V,R,\Lambda}$, and (2.119),

$$\max\left\{\left|\delta_{V,R,\Lambda}\left(\frac{S_*}{16}\right)\right|, |\delta_{V,R,\Lambda}(S_*)|\right\} \leq \varepsilon_0 \leq \varepsilon_2\,,$$

which is (2.48). By the choices (2.119), Lemma 2.8–(ii) applies and yields the existence of $K_- \in \mathcal{H}$ and $u_- \in \mathcal{X}_{\sigma'}(\Sigma_{K_-}, S_*/128, S_*/2)$ such that

$$V \text{ corresponds to } \Sigma_{K_-}(u_+, S_*/128, S_*/2) \text{ on } A^{S_*/2}_{S_*/128}, \tag{2.128}$$

By (2.127), $S_*/128 \leq R_+/2$, so (2.128) implies (2.126). The proof of the **claim** is complete.

Returning to the proof of Theorem 2.1–(i) under the assumption (2.122), we recall (2.125) and choose $R' \in (R_+, \min\{2\,R_+, S_*\})$. Again, we want to apply Lemma 2.8–(ii), this time with $r = R'$, $\lambda_1 = \frac{1}{16}$, and $(\eta_1, \eta_2) = (1/128, 1/2)$. To begin with, V satisfies (2.42) as usual from (2.120). Second, (2.47) holds at R' by $s \leq R' \leq S_*$, (2.120), and the choices (2.119). By the monotonicity of $\delta_{V,R,\Lambda}$ and $[R'/16, R'] \subset [R_+/16, S_*] \subset [s/4, S_*]$, (2.48) is valid by our choice (2.119) of ε_0. The graphicality result from Lemma 2.8–(ii) therefore yields $K \in \mathcal{H}$ and $u \in \mathcal{X}_{\sigma'}(\Sigma_K, R'/128, R'/2)$ such that

$$V \text{ corresponds to } \Sigma_K(u, R'/128, R'/2) \text{ on } A^{R'/2}_{R'/128}\,. \tag{2.129}$$

Now $s/32 \leq R'/128 < R_+/64 < S_+/16$ by $R' < 2\,R_+$ and (2.125), and $R_+/2 < R'/2 < S_*/2$, so by (2.123) and (2.126), respectively, we have

$$V \text{ corresponds to } \Sigma_{K_+}(u_+, R'/128, S_+/16) \text{ on } A^{S_+/16}_{R'/128} \tag{2.130}$$

$$V \text{ corresponds to } \Sigma_{K_-}(u_-, R_+/2, R'/2) \text{ on } A_{R_+/2}^{R'/2}, \tag{2.131}$$

where $u_+ \in \mathcal{X}_{\sigma'}(\Sigma_{K_+}, R'/128, S_+/16)$ and $u_- \in \mathcal{X}_{\sigma'}(\Sigma_{K_-}, R_+/2, R'/2)$. Furthermore, up multiplying ν_{K_+} or ν_{K_-} by minus one, we may assume $\nu_K \cdot \nu_{K_\pm} \ge 0$. Thus V is represented by multiple spherical graphs on nontrivial annuli. By combining (2.129), (2.131) and (2.131), $\nu_K \cdot \nu_{K_\pm} \ge 0$ and $\sigma' = \sigma'(\varepsilon')$, (2.117) applies and gives

$$|\nu_K - \nu_{K_+}| < \varepsilon', \quad |\nu_K - \nu_{K_-}| < \varepsilon'.$$

But ε' was chosen according to (2.118) so that Lemma 2.5–(i) is applicable; that is, since $\varepsilon' < \varepsilon_0$ and $\sigma' < \sigma_0$ from that lemma, we may reparametrize (2.123) and (2.126), respectively, as

$$V \text{ corresponds to } \Sigma_K(w_+, s/32, S_+/16) \text{ on } A_{s/32}^{S_+/16} \tag{2.132}$$

$$V \text{ corresponds to } \Sigma_K(w_-, R_+/2, S_*/2) \text{ on } A_{R_+/2}^{S_*/2}, \tag{2.133}$$

where

$$w_+ \in \mathcal{X}_{C_0(\sigma'+\varepsilon')}(\Sigma_K, s/32, S_+/16), \quad w_- \in \mathcal{X}_{C_0(\sigma'+\varepsilon')}(\Sigma_K, R_+/2, S_*/2).$$

By (2.118), $C_0(\sigma' + \varepsilon') \le \sigma$, and by $R'/128 < S_+/16 < R_+/2 < R'/2$, (2.133) and (2.133), we may extend the u defined in (2.129) onto $\Sigma_K \times (s/32, S_*/2)$ using w_+ and w_- with C^1-norm bounded by σ. The resulting extension is such that (2.5) holds, so the proof of Theorem 2.1 is finished. □

3. Application of Quantitative Isoperimetry

Here we apply quantitative isoperimetry to prove Theorem 1.6–(i) and parts of Theorem 1.6–(iv).

Theorem 3.1. *If $W \subset \mathbb{R}^{n+1}$ is compact, $v > 0$, then $\mathrm{Min}[\psi_W(v)] \ne \emptyset$. Moreover, depending on n and W only, there are v_0, C_0, Λ_0 positive, $s_0 \in (0,1)$, and $R_0(v)$ with $R_0(v) \to 0^+$ and $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, such that, if $v > v_0$ and E_v is a minimizer of $\psi_W(v)$, then:*
(i): *E_v is a $(\Lambda_0/v^{1/(n+1)}, s_0\, v^{1/(n+1)})$*-**perimeter minimizer with free boundary in** *Ω, that is*

$$P(E_v; \Omega \cap B_r(z)) \le P(F; \Omega \cap B_r(z)) + \frac{\Lambda_0}{v^{1/(n+1)}} \left| E_v \Delta F \right|, \tag{3.1}$$

for every $F \subset \Omega = \mathbb{R}^{n+1} \setminus W$ with $E_v \Delta F \subset\subset B_r(z)$ and $r < s_0\, v^{1/(n+1)}$; **(ii):** *There exists $x \in \mathbb{R}^{n+1}$ such that*

$$|E_v \Delta B^{(v)}(x)| \le C_0\, v^{-1+1/[2(n+1)]}; \tag{3.2}$$

if $\mathcal{R}(W) > 0$, then there also exists $u \in C^\infty(\partial B^{(1)})$ such that

$$(\partial E_v) \setminus B_{R_0\, v^{1/(n+1)}}$$

$$= \Big\{y + v^{1/(n+1)} u\Big(\frac{y-x}{v^{1/(n+1)}}\Big)\, \nu_{B^{(v)}(x)}(y) : y \in \partial B^{(v)}(x)\Big\} \setminus B_{R_0\, v^{1/(n+1)}}; \tag{3.3}$$

(iii): *if* $\mathcal{R}(W) > 0$ *and x and u depend on E_v as in* (3.2) *and* (3.3)*, then*

$$\lim_{v\to\infty} \sup_{E_v \in \mathrm{Min}[\psi_W(v)]} \max\Big\{\big||x|\, v^{-1/(n+1)} - \omega_{n+1}^{-1/(n+1)}\big|, \|u\|_{C^1(\partial B^{(1)})}\Big\} = 0 . \tag{3.4}$$

Remark 3.2. (Improved convergence) We will repeatedly use the following fact (see, e.g. [7,8,18,20]): If Ω is an open set, $\Lambda \ge 0$, $s > 0$, if $\{F_j\}_j$ are (Λ, s)-**perimeter minimizers in** Ω, i.e. if it holds that

$$P(F_j; B_r(x)) \le P(G_j; B_r(x)) + \Lambda\, |F_j \Delta G_j| , \tag{3.5}$$

whenever $G_j \Delta F_j \subset\subset B_r(x) \subset\subset \Omega$ and $r < s$, and if F is an open set with smooth boundary in Ω such that $F_j \to F$ in $L^1_{\mathrm{loc}}(\Omega)$ as $j \to \infty$, then for every $\Omega' \subset\subset \Omega$ there is $j(\Omega')$ such that

$$(\partial F_j) \cap \Omega' = \Big\{y + u_j(y)\, \nu_F(y) : y \in \Omega \cap \partial F\Big\} \cap \Omega', \qquad \forall j \ge j(\Omega'),$$

for a sequence $\{u_j\}_j \subset C^1(\Omega \cap \partial F)$ with $\|u_j\|_{C^1(\Omega\cap\partial F)} \to 0$.

Compare the terminology used in (3.1) and (3.5): when we add “with free boundary”, the “localizing balls” $B_r(x)$ are not required to be compactly contained in Ω, and the perimeters are computed in $B_r(x) \cap \Omega$.

Proof of Theorem 3.1. **Step one:** We prove $\mathrm{Min}[\psi_W(v)] \ne \emptyset$ for all $v > 0$. Since W is compact, $B^{(v)}(x) \subset\subset \Omega$ for $|x|$ large. Hence there is $\{E_j\}_j$ with

$$E_j \subset \Omega\,, \quad |E_j| = v\,, \quad P(E_j; \Omega) \le \min\Big\{P(B^{(v)}), P(F; \Omega)\Big\} + (1/j)\,, \tag{3.6}$$

for every $F \subset \Omega$ with $|F| = v$. Hence, up to extracting subsequences, $E_j \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ with $P(E; \Omega) \le \underline{\lim}_{j\to\infty} P(E_j; \Omega)$, where $E \subset \Omega$ and $|E| \le v$. We now make three remarks concerning E:

(a): *If* $\{\Omega_i\}_{i\in I}$ *are the connected components of* Ω, *then* $\Omega \cap \partial^* E = \emptyset$ *if and only if* $E = \bigcup_{i\in I_0} \Omega_i$ ($I_0 \subset I$). Indeed, $\Omega \cap \partial^* E = \emptyset$ implies $\mathrm{cl}\,(\partial^* E) \cap \Omega = \partial E \cap \Omega$, hence $\partial E \subset \partial\Omega$ and $E = \bigcup_{i\in I_0} \Omega_i$. The converse is immediate.

(b): *If* $\Omega \cap \partial^* E \ne \emptyset$, *then we can construct a system of “volume–fixing variations” for* $\{E_j\}_j$. Indeed, if $\Omega \cap \partial^* E \ne \emptyset$, then there are $B_{S_0}(x_0) \subset\subset \Omega$ with $P(E; \partial B_{S_0}(x_0)) = 0$ and a vector field $X \in C^\infty_c(B_{S_0}(x_0); \mathbb{R}^{n+1})$ such that $\int_E \mathrm{div}\, X = 1$. By [27, Theorem 29.14], there are constants $C_0, c_0 > 0$, depending on E itself, with the following property: whenever $|(F\Delta E) \cap B_{S_0}(x_0)| < c_0$, then there is a smooth function $\Phi^F : \mathbb{R}^n \times (-c_0, c_0) \to \mathbb{R}^n$ such that, for each $|t| < c_0$, the map $\Phi^F_t = \Phi^F(\cdot, t)$ is a smooth diffeomorphism with $\{\Phi^F_t \ne \mathrm{id}\} \subset\subset B_{S_0}(x_0)$, $|\Phi^F_t(F)| = |F| + t$, and $P(\Phi^F_t(F); B_{S_0}(x_0)) \le (1 + C_0\,|t|)\, P(F; B_{S_0}(x_0))$. For j large enough, we evidently have $|(E_j\Delta E) \cap B_{S_0}(x_0)| < c_0$, and thus we can construct smooth functions $\Phi^j : \mathbb{R}^n \times (-c_0, c_0) \to \mathbb{R}^n$ such that, for each $|t| < c_0$, the map $\Phi^j_t = \Phi^j(\cdot, t)$ is a smooth diffeomorphism with $\{\Phi^j_t \ne \mathrm{id}\} \subset\subset B_{S_0}(x_0)$, $|\Phi^j_t(E_j)| = |E_j| + t$, and $P(\Phi^j_t(E_j); B_{S_0}(x_0)) \le (1 + C_0\,|t|)\, P(E_j; B_{S_0}(x_0))$.

(c): *If* $\Omega\cap\partial^*E\neq\emptyset$, *then* E *is bounded*. Since $|E|\le v<\infty$, it is enough to prove that $\Omega\cap\partial^*E$ is bounded. In turn, taking $x_0\in\Omega\cap\partial^*E$, and since W is bounded and $|E|<\infty$, the boundedness of $\Omega\cap\partial^*E$ descends immediately by the following density estimate: there is $r_1>0$ such that

$$\begin{aligned}&|E\cap B_r(x)|\ge c(n)\,r^{n+1}\\&\forall\ x\in\Omega\cap\partial^*E\,,\ \ r<r_1\,,\ \ B_r(x)\subset\subset\mathbb{R}^{n+1}\setminus\big(I_{r_1}(W)\cup B_{S_0}(x_0)\big)\,.\end{aligned}\tag{3.7}$$

To prove (3.7), let $r_1>0$ be such that $|B_{r_1}|<c_0$, let x and r be as in (3.7), and set $F_j=(\Phi_t^j(E_j)\cap B_{S_0}(x_0))\cup[E_j\backslash(B_r(x)\cup B_{S_0}(x_0))]$ for $t=|E_j\cap B_r(x)|$ (which is an admissible value of t by $|B_{r_1}|<c_0$). In this way, $|F_j|=|E_j|=v$, and thus we can exploit (3.6) with $F=F_j$. A standard argument (see, e.g. [27, Theorem 21.11]) leads then to (3.7).

Now, since $\partial\Omega\subset W$ is bounded, every connected component of Ω with finite volume is bounded. Thus, by (a), (b) and (c) above, there is $R>0$ such that $W\cup E\subset\subset B_R$. Since $|E\cap[B_{R+1}\backslash B_R]|=0$, we can pick $T\in(R,R+1)$ such that $\mathcal{H}^n(E_j\cap\partial B_T)\to 0$ and $P(E_j\backslash B_T)=\mathcal{H}^n(E_j\cap\partial B_T)+P(E_j;\Omega\backslash B_T)$, and consider the sets $F_j=(E_j\cap B_T)\cup B_{\rho_j}(y)$ corresponding to $\rho_j=(|E_j\setminus B_T|/\omega_{n+1})^{1/(n+1)}$ and to $y\in\mathbb{R}^{n+1}$ which is independent from j and such that $|y|>\rho_j+T$ (notice that $\sup_j\rho_j\le C(n)\,v^{1/(n+1)}$). Since $|F_j|=|E_j|=v$, (3.6) with $F=F_j$ and $P(B_{\rho_j})\le P(E_j\setminus B_T)$ give

$$\begin{aligned}P(E_j;\Omega)-(1/j)\le P(F_j;\Omega)&\le P(E_j;\Omega\cap B_T)+\mathcal{H}^n(E_j\cap\partial B_T)+P(B_{\rho_j})\\&\le P(E_j;\Omega)+2\,\mathcal{H}^n(E_j\cap\partial B_T),\end{aligned}$$

so that, by the choice of T, $\{F_j\}_j$ is a minimizing sequence for $\psi_W(v)$, with $F_j\subset B_{T^*}$ and T^* independent of j. We conclude by the Direct Method.

Step two: We prove (3.2). If E_v a minimizer of $\psi_W(v)$ and $R>0$ is such that $W\subset\subset B_R$, then by $P(E_v;\Omega)\le P(B^{(v)})$ we have, for $v>v_0$, and v_0 and C_0 depending on n and W,

$$\begin{aligned}P(E_v\setminus B_R)&\le P(E_v;\Omega)+n\,\omega_n\,R^n\le P(B^{(v)})+C_0\\&\le(1+(C_0/v))\,P(B^{(|E_v\setminus B_R|)})+C_0,\end{aligned}\tag{3.8}$$

where we have used that, if $v>2\,b>0$ and $\alpha=n/(n+1)$, then

$$P(B^{(v)})\,P(B^{(v-b)})^{-1}-1=(v/(v-b))^\alpha-1\le\alpha\,b/(v-b)\le 2\,\alpha\,b\,v^{-1}.$$

By combining (1.3) and (3.8) we conclude that, for some $x\in\mathbb{R}^{n+1}$,

$$c(n)\left(\frac{|(E_v\setminus B_R)\Delta B^{(|E_v\setminus B_R|)}(x)|}{|E_v\setminus B_R|}\right)^2\le\frac{P(E_v\setminus B_R)}{P(B^{(|E_v\setminus B_R|)})}-1\le\frac{C_0}{v^{n/(n+1)}},$$

provided $v>v_0$. Hence we deduce (3.2) from

$$\begin{aligned}|E_v\Delta B^{(v)}(x)|&=2\,|E_v\setminus B^{(v)}(x)|\le C_0+2\,\big|(E_v\setminus B_R)\setminus B^{(v)}(x)\big|\\&\le C_0+2\,\big|(E_v\setminus B_R)\setminus B^{(|E_v\setminus B_R|)}(x)\big|\le C_0+|E_v\setminus B_R|\,C_0\,v^{-n/2\,(n+1)}.\end{aligned}$$

Step three: We prove the existence of v_0, Λ_0, and s_0 such that every $E_v \in \mathrm{Min}[\psi_W(v)]$ with $v > v_0$ satisfies (3.1). Arguing by contradiction, we assume the existence of $v_j \to \infty$, $E_j \in \mathrm{Min}[\psi_W(v_j)]$, $F_j \subset \Omega$ with $|F_j \Delta E_j| > 0$ and $F_j \Delta E_j \subset\subset B_{r_j}(x_j)$ for some $x_j \in \mathbb{R}^{n+1}$ and $r_j = v_j^{1/(n+1)}/j$, such that

$$P(E_j; \Omega \cap B_{r_j}(x_j)) \geq P(F_j; \Omega \cap B_{r_j}(x_j)) + j\, v_j^{-1/(n+1)} \left|E_j \Delta F_j\right|.$$

Denoting by E_j^*, F_j^* and Ω_j the sets obtained by scaling E_j, F_j and Ω by a factor $v_j^{-1/(n+1)}$, we find that $F_j^* \Delta E_j^* \subset\subset B_{1/j}(y_j)$ for some $y_j \in \mathbb{R}^{n+1}$, and

$$P(E_j^*; \Omega_j \cap B_{1/j}(y_j)) \geq P(F_j^*; \Omega_j \cap B_{1/j}(y_j)) + j \left|E_j^* \Delta F_j^*\right|. \tag{3.9}$$

By (3.2) there are $z_j \in \mathbb{R}^{n+1}$ such that $|E_j^* \Delta B^{(1)}(z_j)| \to 0$. We can therefore use the volume-fixing variations of $B^{(1)}$ to find diffeomorphisms $\Phi_t^j : \mathbb{R}^n \to \mathbb{R}^n$ and constants $c(n)$ and $C(n)$ such that, for every $|t| < c(n)$, one has $\{\Phi_t^j \neq \mathrm{id}\} \subset\subset U_j$ for some open ball U_j with $U_j \subset\subset \Omega_j \setminus B_{1/j}(y_j)$, $|\Phi_t^j(E_j^*) \cap U_j| = |E_j^* \cap U_j| + t$, and $P(\Phi_t^j(E_j^*); U_j) \leq (1 + C(n)\,|t|)\, P(E_j^*; U_j)$. Since $F_j^* \Delta E_j^* \subset\subset B_{1/j}(y_j)$ implies $||F_j^*| - |E_j^*|| < c(n)$ for j large, if $t = |E_j^*| - |F_j^*|$, then $G_j^* = \Phi_t^j(F_j^*)$ is such that $|G_j^*| = |E_j^*|$, and by $E_j \in \mathrm{Min}[\psi_W(v_j)]$,

$$\begin{aligned} P(E_j^*; \Omega_j) &\leq P(G_j^*; \Omega_j) \leq P\big(E_j^*; \Omega_j \setminus (U_j \cup B_{1/j}(y_j))\big) \\ &\quad + P(F_j^*; \Omega_j \cap B_{1/j}(y_j)) + P(E_j^*; U_j) + C(n)\, P(E_j^*; U_j) \left|E_j^* \Delta F_j^*\right|. \end{aligned}$$

Taking into account $P(E_j^*; U_j) \leq \psi_W(v_j)/v_j^{n/(n+1)} \leq C(n)$, we thus find

$$P(E_j^*; \Omega_j \cap B_{1/j}(y_j)) \leq P(F_j^*; \Omega_j \cap B_{1/j}(y_j)) + C(n) \left|E_j^* \Delta F_j^*\right|,$$

which, by (3.9), gives $j \left|E_j^* \Delta F_j^*\right| \leq C(n) \left|E_j^* \Delta F_j^*\right|$. Since $|E_j^* \Delta F_j^*| > 0$, this is a contradiction for j large enough.

Step four: We now prove that, if $\mathcal{R}(W) > 0$, then

$$\lim_{v \to \infty} \sup_{E_v \in \mathrm{Min}[\psi_W(v)]} \left| |x|\, v^{-1/(n+1)} - \omega_{n+1}^{-1/(n+1)} \right| = 0\,, \tag{3.10}$$

where x is related to E_v by (3.2). In proving (3.10) we will use the assumption $\mathcal{R}(W) > 0$ and the energy upper bound

$$\varlimsup_{v \to \infty} \psi_W(v) - P(B^{(v)}) \leq -\mathcal{R}(W)\,. \tag{3.11}$$

A proof of (3.11) is given in step one of the proof of Theorem 1.6, see section 5; in turn, that proof is solely based on the results from section 4, where no part of Theorem 3.1 (not even the existence of minimizers in $\psi_W(v)$) is ever used. This said, when $|W| > 0$, and thus $\mathcal{S}(W) > 0$, one can replace (3.11) in the proof of (3.10) by the simpler upper bound

$$\varlimsup_{v \to \infty} \psi_W(v) - P(B^{(v)}) \leq -\mathcal{S}(W)\,, \tag{3.12}$$

where, we recall, $\mathcal{S}(W) = \sup\{\mathcal{H}^n(W \cap \Pi) : \Pi \text{ is a hyperplane in } \mathbb{R}^{n+1}\}$. To prove (3.12), given Π, we construct competitors for $\psi_W(v)$ by intersecting Ω with balls $B^{(v')}(x_v)$ with $v' > v$ and x_v such that $|B^{(v')}(x_v)\setminus W| = v$ and $\mathcal{H}^n(W \cap \partial B^{(v')}(x_v)) \to \mathcal{H}^n(W \cap \Pi)$ as $v \to \infty$. Hence, $\overline{\lim}_{v\to\infty} \psi_W(v) - P(B^{(v)}) \le -\mathcal{H}^n(W \cap \Pi)$, thus giving (3.12). The proof of (3.11) is identical in spirit to that of (3.12), with the difference that to glue a large ball to $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$ we will need to establish the decay of ∂F towards a hyperplane parallel to $\nu^\perp$ to the high degree of precision expressed in (1.14). Now to prove (3.10): by contradiction, consider $v_j \to \infty$, $E_j \in \mathrm{Min}[\psi_W(v_j)]$, and $x_j \in \mathbb{R}^{n+1}$ with $\inf_{x\in\mathbb{R}^{n+1}} |E_j \Delta B^{(v_j)}(x)| = |E_j \Delta B^{(v_j)}(x_j)|$, such that

$$\varliminf_{j\to\infty} \big||x_j|\, v_j^{-1/(n+1)} - \omega_{n+1}^{-1/(n+1)}\big| > 0\,, \tag{3.13}$$

and set $\lambda_j = v_j^{-1/(n+1)}$, $E_j^* = \lambda_j\,(E_j - x_j)$, $W_j^* = \lambda_j\,(W - x_j)$, and $\Omega_j^* = \lambda_j\,(\Omega - x_j)$. By (3.1), each E_j^* is a (Λ_0, s_0)-perimeter minimizer with free boundary in Ω_j^*. By (3.2) and the defining property of x_j, $E_j^* \to B^{(1)}$ in $L^1(\mathbb{R}^{n+1})$. Moreover, $\mathrm{diam}\,(W_j^*) \to 0$ and, by (3.13),

$$\varliminf_{j\to\infty} \mathrm{dist}\big(W_j^*, \partial B^{(1)}\big) > 0\,. \tag{3.14}$$

Thus there is $z_0 \notin \partial B^{(1)}$ such that, for every $\rho < \mathrm{dist}(z_0, \partial B^{(1)})$, there is $j(\rho)$ such that $\{E_j^*\}_{j\ge j(\rho)}$ is a sequence of (Λ_0, s_0)-perimeter minimizers in $\mathbb{R}^{n+1}\setminus B_{\rho/2}(z_0)$. By Remark 3.2, up to increasing $j(\rho)$, $(\partial E_j^*)\setminus B_\rho(z_0)$ is contained in the normal graph over $\partial B^{(1)}$ of u_j with $\|u_j\|_{C^1(\partial B^{(1)})} \to 0$; in particular, by (3.14), $(\partial E_j^*) \setminus B_\rho(z_0)$ is disjoint from W_j^*. By the constant mean curvature condition satisfied by $\Omega \cap \partial E_j^*$, and by Alexandrov's theorem [2], $(\partial E_j^*) \setminus B_\rho(z_0)$ is a sphere M_j^* for $j \ge j(\rho)$. Let B_j^* be the ball bounded by M_j^*. Since $M_j^* \cap W_j^* = \emptyset$, we have either one of the following:

Case one: $W_j^* \subset B_j^*$. We have $\partial[B_j^* \cup E_j^*] \subset M_j^* \cup [(\partial E_j^*)\setminus \mathrm{cl}\,(B_j^*)] \subset (\partial E_j^*)\setminus W_j^*$, so that, by $|B_j^* \cup E_j^*| \ge |E_j^*| + |W_j^*| \ge 1$, we find $P(E_j^*; \Omega_j^*) \ge P(B_j^* \cup E_j^*) \ge P(B^{(1)})$, that is, $\psi_W(v_j) \ge P(B^{(1)})$, against (3.11).

Case two: $W_j^* \cap B_j^* = \emptyset$. In this case, $E_j^* = B_j^* \cup G_j^*$, where G_j^* is the union of the connected components of E_j^* whose boundaries have non-empty intersection with W_j^*: in other words, we are claiming that B_j^* is the only connected component of E_j^* whose closure is disjoint from W_j^*. Indeed, if this were not the case, we could recombine all the connected components of E_j^* with closure disjoint from W_j^* into a single ball of same total volume, centered far away from W_j^*, in such a way to strictly decrease $P(E_j^*; \Omega_j^*)$, against $E_j \in \mathrm{Min}[\psi_W(v_j)]$. Let us now set $G_j = x_j + v_j^{1/(n+1)}\, G_j^*$ and $U_j = x_j + v_j^{1/(n+1)}\, B_j^*$, so that $E_j = G_j \cup U_j$ and $\mathrm{dist}(G_j, U_j) > 0$.

If we start sliding U_j from infinity towards $G_j \cup W$ along arbitrary directions, then at least one of the resulting "contact points" z_j belongs to $\Omega \cap \partial G_j$: if this were not the case, then G_j would be contained in the convex envelope of W, so that $|B_j| =$

$|E_j| - |G_j| \geq v_j - C(W)$, and thus, by $\psi_W(v_j) = P(E_j; \Omega) \geq P(B_j; W) = P(B_j)$, and by $P(B_j) \geq P(B^{(v_j - C(W))}) \geq P(B^{(v_j)}) - C(W)\, v_j^{-1/(n+1)}$, against with (3.11) for j large.

By construction, there is a half-space H_j such that $G_j \subset H_j$, $z_j \in (\partial G_j) \cap (\partial H_j)$, and G_j is a perimeter minimizer in $B_r(z_j)$ for some small $r > 0$. By the strong maximum principle, see, e.g. [13, Lemma 2.13], G_j has $H_j - z_j$ as its unique blowup at z_j. By De Giorgi's regularity theorem, see e.g. [27, Theorem 21.8], G_j is an open set with smooth boundary in a neighborhood of z_j. Therefore, if we denote by U_j' the translation of U_j constructed in the sliding argument, then, $E_j' = G_j \cup U_j' \in \mathrm{Min}[\psi_W(v)]$ and, in a neighborhood of z_j, E_j' is the union of two disjoint sets with smooth boundary which touch tangentially at z_j. In particular, $|E_j' \cap B_r(z_j)|/|B_r| \to 1$ as $r \to 0^+$, against volume density estimates implied by (3.1), see, e.g. [27, Theorem 21.11].

Step five: We finally show the existence of v_0 and $R_0(v)$ with $R_0(v) \to 0^+$ and $R_0(v)\, v^{1/(n+1)} \to \infty$, such that each $E_v \in \mathrm{Min}[\psi_W(v)]$ with $v > v_0$ determines x and $u \in C^\infty(\partial B^{(1)})$ such that (3.3) holds and $\sup_{E_v} \|u\|_{C^1(\partial B^{(1)})} \to 0$ as $v \to \infty$. To this end, let us consider $v_j \to \infty$, $E_j \in \mathrm{Min}[\psi_W(v_j)]$, and define x_j, E_j^* and W_j^* as in step four. Thanks to (3.10), there is $z_0 \in \partial B^{(1)}$ s.t. $\mathrm{dist}(z_0, W_j^*) \to 0$. In particular, for every $\rho > 0$, we can find $j(\rho) \in \mathbb{N}$ such that if $j \geq j(\rho)$, then E_j^* is a (Λ_0, s_0)-perimeter minimizer in $\mathbb{R}^{n+1} \setminus B_\rho(z_0)$, with $E_j^* \to B^{(1)}$. By Remark 3.2, there are $u_j \in C^1(\partial B^{(1)})$ such that

$$(\partial E_j^*) \setminus B_{2\rho}(z_0) = \big\{ y + u_j(y)\, \nu_{B^{(1)}}(y) : y \in \partial B^{(1)} \big\} \setminus B_{2\rho}(z_0), \quad \forall j \geq j(\rho),$$

and $\|u_j\|_{C^1(\partial B^{(1)})} \to 0$. By the arbitrariness of ρ and by a contradiction argument, (3.3) holds with $R_0(v) \to 0^+$ such that $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, and with the uniform decay of $\|u\|_{C^1(\partial B^{(1)})}$. □

4. Properties of Isoperimetric Residues

Here we prove Theorem 1.1. It will be convenient to introduce some notation for cylinders and slabs in $\mathbb{R}^{n+1}$: precisely, given $r > 0$, $\nu \in \mathbb{S}^n$ and $I \subset \mathbb{R}$, and setting $\mathbf{p}_{\nu^\perp}(x) = x - (x \cdot \nu)\, \nu$ ($x \in \mathbb{R}^{n+1}$), we let

$$\begin{aligned}
\mathbf{D}_r^\nu &= \big\{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_{\nu^\perp} x| < r, x \cdot \nu = 0 \big\}, \\
\mathbf{C}_r^\nu &= \big\{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_{\nu^\perp} x| < r \big\}, \\
\mathbf{C}_{r,I}^\nu &= \big\{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_{\nu^\perp} x| < r, x \cdot \nu \in I \big\}, \\
\partial_\ell \mathbf{C}_{r,I}^\nu &= \big\{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_{\nu^\perp} x| = r, x \cdot \nu \in I \big\}, \\
\mathbf{S}_I^\nu &= \big\{ x \in \mathbb{R}^{n+1} : x \cdot \nu \in I \big\}.
\end{aligned} \tag{4.1}$$

Given $x \in \mathbb{R}^{n+1}$, we also set $\mathbf{D}_r^\nu(x) = x + \mathbf{D}_r^\nu$, $\mathbf{C}_r^\nu(x) = x + \mathbf{C}_r^\nu$, etc. We premise the following proposition, used in the proof of Theorem 1.1 and Theorem 1.6, and based on [32, Proposition 1 and Proposition 3].

Proposition 4.1. *Let $n \geq 2$, $\nu \in \mathbb{S}^n$, and let f be a Lipschitz solution to the minimal surface equation on $\nu^\perp \setminus \mathrm{cl}\,(\mathbf{D}_R^\nu)$. If $n = 2$, assume in addition that $M = \{x + f(x)\,\nu : |x| > R\}$ is stable and has natural area growth, i.e.*

$$\int_M |\nabla^M \varphi|^2 - |A|^2\,\varphi^2 \geq 0, \qquad \forall \varphi \in C_c^1(\mathbb{R}^3 \setminus B_R), \tag{4.2}$$

$$\mathcal{H}^2(M \cap B_r) \leq C\,r^2, \qquad \forall r > R. \tag{4.3}$$

Then there are $a, b \in \mathbb{R}$ and $c \in \nu^\perp$ such that, for every $|x| > R$,

$$\big|f(x) - \big(a + b\,|x|^{2-n} + (c \cdot x)\,|x|^{-n}\big)\big| \leq C\,|x|^{-n}, \ \ (n \geq 3) \tag{4.4}$$

$$\big|f(x) - \big(a + b\,\log|x| + (c \cdot x)\,|x|^{-2}\big)\big| \leq C\,|x|^{-2}, \ \ (n = 2)$$

$$\max\Big\{|x|^{n-1}\,|\nabla f(x)|, |x|^n\,|\nabla^2 f(x)| : |x| > R\Big\} \leq C, \ \ (\textit{every } n). \tag{4.5}$$

Proof. If $n \geq 3$, the fact that ∇f is bounded allows one to represent f as the convolution with a singular kernel which, by a classical result of Littman, Stampacchia, and Weinberger [26], is comparable to the Green's function of $\mathbb{R}^n$; (4.4) is then deduced starting from that representation formula. For more details, see [32, Proposition 3]. In the case $n = 2$, by (4.2) and (4.3), we can exploit a classical "logarithmic cut-off argument" to see that M has finite total curvature, i.e. $\int_M |K|\,d\mathcal{H}^2 < \infty$, where K is the Gaussian curvature of M. As a consequence, see, e.g. [31, Section 1.2], the compactification $\overline{M}$ of M is a Riemann surface with boundary, and M is conformally equivalent to $\overline{M}\setminus\{p_1, ..., p_m\}$, where p_i are interior points of $\overline{M}$. One can thus conclude by the argument in [32, Proposition 1] that M has m-many ends satisfying the decay (4.5), and then that $m = 1$ thanks to the fact that $M = \{x + f(x)\,\nu : |x| > R\}$.

Proof of Theorem 1.1. **Step one:** Given a hyperplane Π in $\mathbb{R}^{n+1}$, if F is a half-space with $\partial F = \Pi$ and ν is a unit normal to Π, then $\mathrm{res}_W(F, \nu) = \mathcal{H}^n(W \cap \Pi)$. Therefore the lower bound in (1.11) follows by

$$\mathcal{R}(W) \geq \mathcal{S}(W) = \sup\big\{\mathcal{H}^n(\Pi \cap W) : \Pi \text{ an hyperplane in } \mathbb{R}^{n+1}\big\}. \tag{4.6}$$

Step two: We notice that, if $(F, \nu) \in \mathcal{F}$, then by (1.8), (1.9), and the divergence theorem (see, e.g., [27, Lemma 22.11]), we can define a Radon measure on the open set $\nu^\perp \setminus \mathbf{p}_{\nu^\perp}(W)$ by setting

$$\mu(U) = P\big(F; (\mathbf{p}_{\nu^\perp})^{-1}(U)\big) - \mathcal{H}^n(U), \qquad U \subset \nu^\perp \setminus \mathbf{p}_{\nu^\perp}(W).$$

In particular, setting $R' = \inf\{\rho : W \subset \mathbf{C}_\rho^\nu\}$, the fact that $\mu(\mathbf{D}_R^\nu \setminus \mathbf{p}_{\nu^\perp}(W)) \geq 0$ gives

$$P(F; \mathbf{C}_R^\nu \setminus W) \geq \omega_n\,R^n - \mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)), \qquad \forall R > R',$$

while the identity

$$\omega_n\,R^n - P(F; \mathbf{C}_R^\nu \setminus W) = -\mu(\mathbf{D}_R^\nu \setminus \mathbf{D}_{R'}^\nu) + \omega_n\,(R')^n - P(F; \mathbf{C}_{R'}^\nu \setminus W)$$

(which possibly holds as $-\infty = -\infty$ if $P(F; \mathbf{C}^\nu_{R'} \setminus W) = +\infty$) gives that

$$R \in (R', \infty) \mapsto \omega_n R^n - P(F; \mathbf{C}^\nu_R \setminus W) \text{ is } \textit{decreasing} \text{ on } (R', \infty)\,. \qquad (4.7)$$

In particular, the limsup defining res_W always exists as a limit.

Step three: We prove the existence of $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$ and (1.12). We first claim that if $\{(F_j, \nu_j)\}_j$ is a maximizing sequence for $\mathcal{R}(W)$, then, in addition to $\mathbf{p}_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp$, one can modify (F_j, ν_j), preserving the optimality in the limit $j \to \infty$, so that (writing $X \subset^{\mathcal{L}^{n+1}} Y$ for $|X \setminus Y| = 0$)

$$\begin{aligned} &\partial F_j \subset \mathbf{S}^{\nu_j}_{[A_j, B_j]}, \quad \mathbf{S}^{\nu_j}_{(-\infty, A_j)} \overset{\mathcal{L}^{n+1}}{\subset} F_j, \quad \mathbf{S}^{\nu_j}_{(B_j, \infty)} \overset{\mathcal{L}^{n+1}}{\subset} \mathbb{R}^{n+1} \setminus F_j, \\ &\text{where } [A_j, B_j] = \bigcap \big\{(\alpha, \beta) : W \subset \mathbf{S}^{\nu_j}_{(\alpha,\beta)}\big\}. \end{aligned} \qquad (4.8)$$

Indeed, since $(F_j, \nu_j) \in \mathcal{F}$, for some $\alpha_j < \beta_j \in \mathbb{R}$ we have

$$\partial F_j \subset \mathbf{S}^{\nu_j}_{[\alpha_j, \beta_j]}\,, \qquad \mathbf{p}_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp\,.$$

Would it be that either $\mathbf{S}^{\nu_j}_{(-\infty, \alpha_j) \cup (\beta_j, \infty)} \subset_{\mathcal{L}^{n+1}} F_j$ or $\mathbf{S}^{\nu_j}_{(-\infty, \alpha_j) \cup (\beta_j, \infty)} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F_j$, then, by the divergence theorem and by $\mathbf{p}_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp$,

$$P(F_j; \mathbf{C}^{\nu_j}_R \cap \Omega) \geq 2\left(\omega_n R^n - \mathcal{H}^n(\mathbf{p}_{\nu_j^\perp}(W))\right), \qquad \forall R > 0,$$

and thus $\mathrm{res}_W(F_j, \nu_j) = -\infty$; in particular, $(F_j, \nu_j) \in \mathcal{F}$ being a maximizing sequence, we would have $\mathcal{R}(W) = -\infty$, against (4.6). This proves the validity (up to switching F_j with $\mathbb{R}^{n+1} \setminus F_j$), of the inclusions

$$\mathbf{S}^{\nu_j}_{(-\infty, \alpha_j)} \subset_{\mathcal{L}^{n+1}} F_j\,, \qquad \mathbf{S}^{\nu_j}_{(\beta_j, \infty)} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F_j\,. \qquad (4.9)$$

Thanks to (4.9) (and by exploiting basic set operations on sets of finite perimeter, see, e.g., [27, Theorem 16.3]), we see that

$$\begin{aligned} &F_j^* = \big(F_j \cup \mathbf{S}^{\nu_j}_{(-\infty, A_j - 1/j)}\big) \cap \mathbf{S}^{\nu_j}_{(-\infty, B_j + 1/j)} \text{ satisfies} \\ &(F_j^*, \nu_j) \in \mathcal{F}, \qquad P\big(F_j^*; \mathbf{C}^{\nu_j}_R \setminus W\big) \leq P\big(F_j; \mathbf{C}^{\nu_j}_R \setminus W\big), \qquad \forall R > 0; \end{aligned} \qquad (4.10)$$

in particular, $\{(F_j^*, \nu_j)\}_j$ is also a maximizing sequence for $\mathcal{R}(W)$. By standard compactness theorems there are F of locally finite perimeter in $\mathbb{R}^{n+1}$ and $\nu \in \mathbb{S}^n$ such that $F_j \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $\nu_j \to \nu$. If $A \subset\subset \mathbf{C}^\nu_R \setminus W$ is open, then, for j large enough, $A \subset\subset \mathbf{C}^{\nu_j}_R \setminus W$, and thus

$$P(F; \mathbf{C}^\nu_R \setminus W) = \sup_{A \subset\subset \mathbf{C}^\nu_R \setminus W} P(F; A) \leq \varliminf_{j \to \infty} P(F_j; \mathbf{C}^{\nu_j}_R \setminus W)\,. \qquad (4.11)$$

By (4.7), $R \mapsto \omega_n R^n - P(F_j; \mathbf{C}^{\nu_j}_R \setminus W)$ is decreasing on $R > R_j = \inf\{\rho : W \subset \mathbf{C}^{\nu_j}_\rho\}$. By $\sup_j R_j \leq C(W) < \infty$ and (4.11) we have

$$\omega_n R^n - P(F; \mathbf{C}^\nu_R \setminus W) \geq \varlimsup_{j \to \infty} \omega_n R^n - P(F_j; \mathbf{C}^{\nu_j}_R \setminus W) \geq \varlimsup_{j \to \infty} \mathrm{res}_W(F_j, \nu_j),$$

for every $R > C(W)$; in particular, letting $R \to \infty$,

$$\mathrm{res}_W(F,\nu) \ge \varlimsup_{j\to\infty} \mathrm{res}_W(F_j,\nu_j) = \mathcal{R}(W)\,. \tag{4.12}$$

By $F_j \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $\partial F = \mathrm{cl}\,(\partial^* F)$ is contained in the set of accumulation points of sequences $\{x_j\}_j$ with $x_j \in \partial F_j$, so that (4.8) gives

$$\partial F \subset \mathbf{S}^\nu_{[A,B]}\,, \qquad \mathbf{S}^\nu_{(-\infty,A)} \subset_{\mathcal{L}^{n+1}} F\,, \qquad \mathbf{S}^\nu_{(B,\infty)} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F\,, \tag{4.13}$$

if $[A,B] = \bigcap\{(\alpha,\beta) : W \subset \mathbf{S}^\nu_{(\alpha,\beta)}\}$. Therefore $(F,\nu) \in \mathcal{F}$, and thus, by (4.12), $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$. We now show that (4.12) implies (1.12), i.e.

$$P(F;\Omega\cap B) \le P(G;\Omega\cap B)\,, \qquad \forall F\Delta G \subset\subset B,\, B \text{ a ball}\,. \tag{4.14}$$

Indeed, should (4.14) fail, we could find $\delta > 0$ and $G \subset \mathbb{R}^{n+1}$ with $F\Delta G \subset\subset B$ for some ball B, such that $P(G; B\backslash W) + \delta \le P(F; B\backslash W)$. For R large enough to entail $B \subset\subset \mathbf{C}^\nu_R$ we would then find

$$\mathrm{res}_W(F,\nu) + \delta \le \omega_n\, R^n - P(F;\mathbf{C}^\nu_R \setminus W) + \delta \le \omega_n\, R^n - P(G;\mathbf{C}^\nu_R \setminus W),$$

which, letting $R \to \infty$, would violate the maximality of (F,ν) in $\mathcal{R}(W)$.

Step four: We show that if $\mathcal{R}(W) > 0$ and $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$, then $\partial F \subset \mathbf{S}^\nu_{[A,B]}$ for A, B as in (4.13). Otherwise, by the same truncation procedure leading to (4.10) and by $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$, we would find

$$\omega_n R^n - P\big(F^*;\mathbf{C}^{\nu_j}_R \setminus W\big) \ge \omega_n R^n - P\big(F;\mathbf{C}^{\nu_j}_R \setminus W\big) \ge \mathcal{R}(W) \qquad \forall R > 0\,,$$

so that $(F^*,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$ too. Now $P\big(F;\mathbf{C}^{\nu_j}_R\backslash W\big) - P\big(F^*;\mathbf{C}^{\nu_j}_R\backslash W\big)$ is increasing in R, and since $\mathrm{res}_W(F,\nu) = \mathrm{res}_W(F^*,\nu)$, it follows that $P\big(F;\mathbf{C}^{\nu_j}_R\backslash W\big) = P\big(F^*;\mathbf{C}^{\nu_j}_R\backslash W\big)$ for large R. But this can hold only if $\partial F \cap \Omega$ is an hyperplane disjoint from W, in which case $\mathcal{R}(W) = \mathrm{res}_W(F,\nu) = 0$.

Step five: Still assuming $\mathcal{R}(W) > 0$, we complete the proof of statement (ii) by proving (1.14). By (4.13), if $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$, then $F/R \to H^- = \{x \in \mathbb{R}^{n+1} : x\cdot\nu < 0\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ as $R \to \infty$. By (4.14) and by improved convergence (i.e., Remark 3.2—notice carefully that ∂F is bounded in the direction ν thanks to step four), we find $R_F > 0$ and functions $\{f_R\}_{R>R_F} \subset C^1(\mathbf{D}^\nu_2\backslash\mathbf{D}^\nu_1)$ such that

$$\big(\mathbf{C}^\nu_2 \setminus \mathbf{C}^\nu_1\big) \cap \partial(F/R) = \big\{x + f_R(x)\,\nu : x \in \mathbf{D}^\nu_2 \setminus \mathbf{D}^\nu_1\big\}, \qquad \forall R > R_F\,.$$

with $\|f_R\|_{C^1(\mathbf{D}^\nu_2\backslash\mathbf{D}^\nu_1)} \to 0$ as $R \to \infty$. Scaling back to F we deduce that

$$(\partial F) \setminus \mathbf{C}^\nu_{R_F} = \big\{x + f(x)\,\nu : x \in \nu^\perp \setminus \mathbf{D}^\nu_{R_F}\big\}\,, \tag{4.15}$$

for a (necessarily smooth) solution f to the minimal surfaces equation with

$$\|f\|_{C^0(\nu^\perp\backslash\mathbf{D}^\nu_{R_F})} \le B - A\,, \qquad \lim_{R\to\infty} \|\nabla f\|_{C^0(\mathbf{D}^\nu_{2\,R}\backslash\mathbf{D}^\nu_R)} = 0\,, \tag{4.16}$$

thanks to the fact that $f(x) = R\, f_R(x/R)$ if $x \in \mathbf{D}^\nu_{2\,R} \setminus \mathbf{D}^\nu_R$. **When** $n \ge 3$, (1.14) follows by (4.15) and Proposition 4.1. **When** $n = 2$, (4.2) holds by (4.14). To check (4.3), we deduce by $\mathrm{res}_W(F,\nu) \geq 0$ the existence of $R' > R_F$ such that $\omega_n R^n \geq P(F; \mathbf{C}_R^\nu \setminus W) - 1$ if $R > R'$. In particular, setting $M = (\partial F) \setminus B_{R_F}$, for $R > R'$ we have

$$\mathcal{H}^2(M \cap B_R) \leq \mathcal{H}^2(M \cap W) + P(F; \mathbf{C}_R^\nu \setminus W) \leq \omega_n R^n + 1 + \mathcal{H}^2(M \cap W) \leq C R^n,$$

provided $C = \omega_n + [(1 + \mathcal{H}^2(M \cap W))/(R')^n]$; while if $R \in (R_F, R')$, then $\mathcal{H}^2(M \cap B_R) \leq C R^n$ with $C = \mathcal{H}^2(M \cap B_{R'})/R_F^n$. This said, we can apply Proposition 4.1 to deduce (4.5). Since ∂F is contained in a slab, the logarithmic term in (4.5) must vanish (i.e. (4.5) holds with $b = 0$), and thus (1.14) is proved. **Finally, when** $n = 1$, by (4.15) and (4.16) there are $a_1, a_2 \in \mathbb{R}$, $x_1 < x_2$, $x_1, x_2 \in \nu^\perp \equiv \mathbb{R}$ such that $f(x) = a_1$ for $x \in \nu^\perp$, $x < x_1$, and $f(x) = a_2$ for $x \in \nu^\perp$, $x > x_2$. Now, setting $M_1 = \{x + a_1 \nu : x \in \nu^\perp, x < x_1\}$ and $M_2 = \{x + a_2 \nu : x \in \nu^\perp, x > x_2\}$, we have that

$$P(F; \mathbf{C}_R^\nu \setminus W) = \mathcal{H}^n\big(\mathbf{C}_R^\nu \cap (\partial F) \setminus (W \cup M_1 \cup M_2)\big) + 2R - |x_2 - x_1|;$$

while, if L denotes the line through $x_1 + a_1 \nu$ and $x_2 + a_2 \nu$, then we can find $\nu_L \in \mathbb{S}^1$ and a set F_L such that $(F_L, \nu_L) \in \mathcal{F}$ with $\partial F_L = \big[\big((\partial F) \setminus (M_1 \cup M_2)\big) \cup (L_1 \cup L_2)\big]$, where L_1 and L_2 are the two half-lines obtained by removing from L the segment joining $x_1 + a_1 \nu$ and $x_2 + a_2 \nu$. In this way, $P(F_L; \mathbf{C}_R^{\nu_L} \setminus W) = \mathcal{H}^n\big(\mathbf{C}_R^\nu \cap (\partial F) \setminus (W \cup M_1 \cup M_2)\big) + 2R - \big|(x_1 + a_1 \nu) - (x_2 + a_2 \nu)\big|$, so that $\mathrm{res}_W(F_L, \nu_L) - \mathrm{res}_W(F, \nu) = \big|(x_1 + a_1 \nu) - (x_2 + a_2 \nu)\big| - |x_2 - x_1| > 0$, against $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$ if $a_1 \neq a_2$. Hence, $a_1 = a_2$.

We are left to prove that (4.15) holds with $R_2 = R_2(W)$ in place of R_F, and the constants a, b, c and C_0 appearing in (1.14) can be bounded in terms of W only. To this end, we notice that the argument presented in step one shows that $\mathrm{Max}[\mathcal{R}(W)]$ is pre-compact in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Using this fact and a contradiction argument based on improved convergence (Remark 3.2), we conclude the proof of statement (ii).

Step six: We complete the proof of statement (i) and begin the proof of statement (iii) by showing that, setting for brevity $d = \mathrm{diam}\,(W)$, it holds

$$\mathcal{H}^n(W \cap \Pi) \leq \mathcal{R}(W) \leq \sup_{\nu \in \mathbb{S}^n} \mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)) \leq \omega_n\,(d/2)^n\,, \tag{4.17}$$

whenever Π is a hyperplane in $\mathbb{R}^{n+1}$. We have already proved the first inequality in step one. To prove the others, we notice that, if $(F, \nu) \in \mathcal{F}$, then $\mathbf{p}_{\nu^\perp}(\partial F) = \nu^\perp$ and (4.7)

give, for every $R > R'$,

$$\begin{aligned} -\mathrm{res}_W(F, \nu) &\geq P(F; \mathbf{C}_R^\nu \setminus W) - \omega_n R^n \geq \mathcal{H}^n\big(\mathbf{p}_{\nu^\perp}(\partial F \setminus W) \cap \mathbf{D}_R^\nu\big) - \omega_n R^n \\ &= -\mathcal{H}^n\big(\mathbf{D}_R^\nu \setminus \mathbf{p}_{\nu^\perp}(\partial F \setminus W)\big) \geq -\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)) \geq -\omega_n\,(d/2)^n, \end{aligned} \tag{4.18}$$

where in the last step we have used the isodiametric inequality. Maximizing over (F, ν) in (4.18) we complete the proof of (4.17). Moreover, if $W = \mathrm{cl}\,(B_{d/2})$, then, since $\mathcal{S}(\mathrm{cl}\,(B_{d/2})) = \mathcal{H}^n(\mathrm{cl}\,(B_{d/2}) \cap \Pi) = \omega_n\,(d/2)^n$ for any hyperplane Π through the origin, we find that $\mathcal{R}(\mathrm{cl}\,(B_{d/2})) = \omega_n\,(d/2)^n$; in particular, (4.17) implies (1.15).

Step seven: We continue the proof of statement (iii) by showing (1.16). Let $\mathcal{R}(W) = \omega_n\,(d/2)^n$ and let $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$. Since every inequality in (4.18) holds as an equality, we find in particular that

$$\sup_{R>R'} P(F;\mathbf{C}_R^\nu \setminus W) - \mathcal{H}^n\big(\mathbf{p}_{\nu^\perp}(\partial F \setminus W) \cap \mathbf{D}_R^\nu\big) = 0, \tag{4.19}$$

$$\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)) = \omega_n\,(d/2)^n. \tag{4.20}$$

By (4.20) and the discussion of the equality cases for the isodiametric inequality (see, e.g. [29]), we see that, for some $x_0 \in \nu^\perp$,

$$\mathbf{p}_{\nu^\perp}(W) = \mathrm{cl}\,(\mathbf{D}_{d/2}^\nu(x_0))\,, \qquad \text{so that } W \subset \mathbf{C}_{d/2}^\nu(x_0)\,.$$

Condition (4.19) implies that (1.14) holds with $u \equiv a$ for some $a \in [A,B] = \bigcap\{(\alpha,\beta) : W \subset \mathbf{S}_{(\alpha,\beta)}^\nu\}$; in particular, since $(\partial F)\setminus W$ is a minimal surface and $W \subset \mathbf{C}_{d/2}^\nu(x_0)$, by analytic continuation we find that

$$(\partial F)\setminus \mathbf{C}_{d/2}^\nu(x_0) = \Pi \setminus \mathbf{C}_{d/2}^\nu(x_0)\,, \qquad \Pi = \big\{x : x\cdot\nu = a\big\}\,. \tag{4.21}$$

By (4.21), we have that for $R > R'$,

$$P(F;\mathbf{C}_R^\nu \setminus W) - \omega_n\,R^n = P(F;\mathbf{C}_{d/2}^\nu(x_0)\setminus W) - \omega_n\,(d/2)^n.$$

Going back to (4.18), this implies $P(F;\mathbf{C}_{d/2}^\nu(x_0)\setminus W) = 0$. However, since $(\partial F)\setminus W$ is (distributionally) a minimal surface, $P(F;B_\rho(x)\setminus W) \ge \omega_n\,\rho^n$ whenever $x \in (\partial F)\setminus W$ and $\rho < \mathrm{dist}(x,W)$, so that $P(F;\mathbf{C}_{d/2}^\nu(x_0)\setminus W) = 0$ gives $((\partial F)\setminus W)\cap \mathbf{C}_{d/2}^\nu(x_0) = \emptyset$. Hence, using also (4.21), we find $(\partial F)\setminus W = \Pi\setminus\mathrm{cl}\big(B_{d/2}(x)\big)$ for some $x \in \Pi$, that is (1.16).

Step eight: We finally prove that $\mathcal{R}(W) = \omega_n\,(d/2)^n$ if and only if there are a hyperplane Π and a point $x \in \Pi$ such that

$$\Pi \cap \partial B_{d/2}(x) \subset W\,, \tag{4.22}$$

$$\Omega \setminus (\Pi \setminus B_{d/2}(x)) \text{ has two unbounded connected components}\,. \tag{4.23}$$

We first prove that the two conditions are sufficient. Let ν be a unit normal to Π and let Π^+ and Π^- be the two open half-spaces bounded by Π. The condition $\Pi \cup \partial B_{d/2}(x) \subset W$ implies $W \subset \mathbf{C}_{d/2}^\nu(x)$, and thus

$$\Omega \setminus \mathrm{cl}\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big] = (\Pi^+ \cup \Pi^-)\setminus \mathrm{cl}\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big].$$

In particular, $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ has a connected component F which contains

$$\Pi^+ \setminus \mathrm{cl}\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big];$$

and since $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ contains exactly two unbounded connected components, it cannot be that F contains also $\Pi^- \setminus \mathrm{cl}\,[\mathbf{C}_{d/2,(-d,d)}^\nu(x)]$, therefore

$$\Pi^+ \setminus \mathrm{cl}\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big] \subset F\,, \qquad \Pi^- \setminus \mathrm{cl}\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big] \subset \mathbb{R}^{n+1}\setminus \mathrm{cl}\,(F)\,. \tag{4.24}$$

As a consequence ∂F is contained in the slab $\{y : |(y-x)\cdot\nu| < d\}$, and is such that $\mathbf{p}_{\nu^\perp}(\partial F) = \nu^\perp$, that is, $(F,\nu) \in \mathcal{F}$. Moreover, (4.24) implies

$$\Pi \setminus \mathrm{cl}\,(B_{d/2}(x)) \subset \Omega \cap \partial F,$$

while the fact that F is a connected component of $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ implies $\Omega\cap\partial F \subset \Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$. In conclusion, $\Omega\cap\partial F = \Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$, hence

$$\omega_n\,(d/2)^n = \lim_{r\to\infty}\omega_n r^n - P(F;\mathbf{C}_r^\nu\setminus W) \le \mathcal{R}(W) \le \omega_n\,(d/2)^n\,,$$

and $\mathcal{R}(W) = \omega_n\,(d/2)^n$, as claimed. We prove that the two conditions are necessary. Let $(F,\nu)\in\mathrm{Max}[\mathcal{R}(W)]$. As proved in step seven, there is a hyperplane Π and $x\in\Pi$ such that $\Omega\cap\partial F = \Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$. If $z\in\Pi\cap\partial B_{d/2}(x)$ but $z\in\Omega$, then there is $\rho>0$ such that $B_\rho(z)\subset\Omega$, and since ∂F is a minimal surface in Ω, we would obtain that $\Pi\cap B_\rho(z)\subset\Omega\cap\partial F$, against $\Omega\cap\partial F = \Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$. So it must be $\Pi\cap\partial B_{d/2}(x)\subset W$, and the necessity of (4.22) is proved. To prove the necessity of (4.23), we notice that since $\Pi^+\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ and $\Pi^-\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ are both open, connected, and unbounded subsets of $\Omega\setminus(\Pi\setminus B_{d/2}(x))$, and since the complement in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ of their union is bounded, it must be that $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ has *at most* two unbounded connected components: therefore we just need to exclude that *it has only one*. Assuming by contradiction that this is the case, we could then connect any point $x^+\in\Pi^+\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ to any point $x^-\in\Pi^-\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ with a continuous path γ entirely contained in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$. Now, recalling that $\Omega\cap\partial F = \Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$, we can pick $x_0\in\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$ and $r>0$ so that

$$B_r(x_0)\cap\Pi^+\subset F\,,\qquad B_r(x_0)\cap\Pi^-\subset\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(F)\,,\tag{4.25}$$

and $B_r(x_0)\cap\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)] = \emptyset$. We can then pick $x^+\in B_r(x_0)\cap\Pi^+$, $x^-\in B_r(x_0)\cap\Pi^-$, and then connect them by a path γ entirely contained in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$. By (4.25), γ must intersect ∂F, and since γ is contained in Ω, we see that γ must intersect $\Omega\cap\partial F = \Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$, which of course contradicts the containment of γ in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$. We have thus proved that $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ has exactly two unbounded connected components. □

5. Resolution Theorem for Exterior Isoperimetric Sets

The notation set in (4.1) is in use. Given $v_j\to\infty$, we set $\lambda_j = v_j^{1/(n+1)}$.

Proof of Theorem 1.6. Theorem 1.6–(i) and the estimate for $|v^{-1/(n+1)}|x| - \omega_{n+1}^{-1/(n+1)}|$ in Theorem 1.6–(iv), have already been proved in Theorem 3.1–(ii), (iii).

Step one: We prove that

$$\varlimsup_{v\to\infty}\psi_W(v) - P(B^{(v)}) \le -\mathcal{R}(W)\,.\tag{5.1}$$

To this end, let $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$, so that by (1.13) and (1.14), we have

$$F \setminus \mathbf{C}^\nu_{R_2} = \left\{x + t\,\nu : x \in \nu^\perp\,, |x| > R_2\,, t < f(x)\right\}, \tag{5.2}$$

for a function $f \in C^1(\nu^\perp)$ satisfying

$$\begin{aligned}&\left|f(x) - \left(a + b\,|x|^{2-n} + (c\cdot x)\,|x|^{-n}\right)\right| \le C_0\,|x|^{-n},\\ &\max\left\{|x|^{n-1}\,|\nabla f(x)|, |x|^n\,|\nabla^2 f(x)|\right\} \le C_0, \qquad \forall x \in \nu^\perp, |x| > R_2,\end{aligned} \tag{5.3}$$

and for some $a, b \in \mathbb{R}$ and $c \in \nu^\perp$ such that $\max\{|a|, |b|, |c|\} \le C(W) < \infty$ (moreover, we can take $b = 0$, $c = 0$ and $C_0 = 0$ if $n = 1$). We are going to construct competitors for $\psi_W(v)$ with v large by gluing a large sphere S to ∂F along $\partial\mathbf{C}^\nu_r$ for $r > R_2$. This operation comes at the price of an area error located on the cylinder $\partial\mathbf{C}^\nu_r$. This error will remain bounded as needed thanks to the fact that (5.3) determines the distance (inside of $\partial\mathbf{C}^\nu_r$) of ∂F from a hyperplane (namely, ∂G_r for the half-space G_r defined below) up to $\mathrm{o}(r^{1-n})$ as $r \to \infty$. Thus, the asymptotic expansion (1.14) is just as precise as needed in order to perform this construction, i.e. our construction would not be possible with a less precise information.

We now discuss the construction in detail. Given $r > R_2$, we consider the half-space $G_r \subset \mathbb{R}^{n+1}$ defined by the condition that

$$G_r \cap \partial\mathbf{C}^\nu_r = \left\{x + t\,\nu : x \in \nu^\perp\,, |x| = r\,, t < a + b\,r^{2-n} + (c\cdot x)\,r^{-n}\right\}, \tag{5.4}$$

so that G_r is the "best half-space approximation" of F on $\partial\mathbf{C}^\nu_r$ according to (5.3). Denoting by $\mathrm{hd}\,(X, Y)$ the Hausdorff distance between $X, Y \subset \mathbb{R}^{n+1}$, for every $r > R_2$ and $v > 0$ we can define $x_{r,v} \in \mathbb{R}^{n+1}$ in such a way that $v \mapsto x_{r,v}$ is continuous and

$$\lim_{v\to\infty} \mathrm{hd}\,(B^{(v)}(x_{r,v}) \cap K, G_r \cap K) = 0 \qquad \forall\, K \subset\subset \mathbb{R}^{n+1}. \tag{5.5}$$

Thus, the balls $B^{(v)}(x_{r,v})$ have volume v and are locally converging in Hausdorff distance, as $v \to \infty$, to the optimal half-space G_r. Finally, we notice that by (5.3) we can find $\alpha < \beta$ such that

$$\left((\partial F) \cup (\partial G_r) \cup (G_r \Delta F)\right) \cap \mathbf{C}^\nu_r \;\subset\; \mathbf{C}^\nu_{r,(\alpha+1,\beta-1)}\,, \tag{5.6}$$

and then define $F_{r,v}$ by setting

$$F_{r,v} = \left(F \cap \mathbf{C}^\nu_{r,(\alpha,\beta)}\right) \cup \left(B^{(v)}(x_{r,v}) \setminus \mathrm{cl}\left[\mathbf{C}^\nu_{r,(\alpha,\beta)}\right]\right), \tag{5.7}$$

see Fig. 5. We claim that, by using $F_{r,v}$ as comparisons for $\psi_W(|F_{r,v}|)$, and then sending first $v \to \infty$ and then $r \to \infty$, one obtains (5.1). We first notice that by (5.5) and (5.6) (see, e.g. [27, Theorem 16.16]), we have

$$\begin{aligned}P(F_{r,v}; \Omega) = P(F; \mathbf{C}^\nu_{r,(\alpha,\beta)} \setminus W) + P\left(B^{(v)}(x_{r,v}); \mathbb{R}^{n+1} \setminus \mathrm{cl}\left[\mathbf{C}^\nu_{r,(\alpha,\beta)}\right]\right)\\ +\mathcal{H}^n\left((F \Delta B^{(v)}(x_{r,v})) \cap \partial_\ell \mathbf{C}^\nu_{r,(\alpha,\beta)}\right),\end{aligned} \tag{5.8}$$

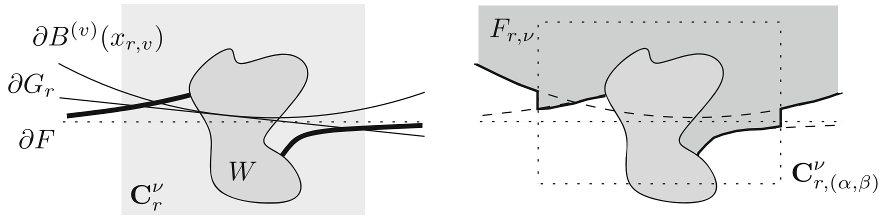

Fig. 5. The competitors $F_{r,v}$ constructed in (5.7). A maximizer F in the isoperimetric residue $\mathcal{R}(W)$ is joined to a ball of volume v, whose center $x_{r,v}$ is determined by looking at best hyperplane ∂G_r approximating ∂F on the "lateral" cylinder $\partial \mathbf{C}_r^\nu$. To ensure the area error made in joining this large sphere to ∂F is negligible, the distance between ∂F and the sphere inside $\partial \mathbf{C}_r^\nu$ must be $\mathrm{o}(r^{1-n})$ as $r \to \infty$. The asymptotic expansion (5.3) gives a hyperplane ∂G_r which is close to ∂F up to $\mathrm{O}(r^{-n})$, and is thus just as precise as needed to perform the construction

where the last term is the "gluing error" generated by the mismatch between the boundaries of ∂F and $\partial B^{(v)}(x_{r,v})$ along $\partial_\ell \mathbf{C}^\nu_{r,(\alpha,\beta)}$. Now, thanks to (5.3) we have $\mathrm{hd}\,(G_r\cap\partial\mathbf{C}^\nu_r, F\cap\partial\mathbf{C}^\nu_r)\le C_0\,r^{-n}$, so that

$$\mathcal{H}^n\big((F\Delta G_r)\cap\partial\mathbf{C}^\nu_r\big)\le n\,\omega_n\,r^{n-1}\,\mathrm{hd}\,(G_r\cap\partial\mathbf{C}^\nu_r, F\cap\partial\mathbf{C}^\nu_r)\le C(n,W)/r\,. \tag{5.9}$$

At the same time, by (5.5),

$$\lim_{v\to\infty}\mathcal{H}^n\big((G_r\Delta B^{(v)}(x_{r,v}))\cap\partial_\ell\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)=0\,,$$

and thus we have the following estimate for the gluing error,

$$\varlimsup_{v\to\infty}\mathcal{H}^n\big((F\Delta B^{(v)}(x_{r,v}))\cap\partial_\ell\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)\le\frac{C(n,W)}{r}\,,\qquad\forall r>R_2\,. \tag{5.10}$$

Again by (5.5), we find that

$$\lim_{v\to\infty}P\big(B^{(v)}(x_{r,v});\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)=P\big(G_r;\mathbf{C}^\nu_{r,(\alpha,\beta)}\big) \tag{5.11}$$

$$1\le(\omega_n\,r^n)^{-1}\,P\big(G_r;\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)=\fint_{\mathbf{D}^\nu_r}\sqrt{1+(c/r^n)^2}\le1+C_0\,r^{-2n}, \tag{5.12}$$

so that, by (5.11) and by the lower bound in (5.12), for every $r>R_2$,

$$\varlimsup_{v\to\infty}P\big(B^{(v)}(x_{r,v});\mathbb{R}^{n+1}\setminus\mathrm{cl}\,\big[\mathbf{C}^\nu_{r,(\alpha,\beta)}\big]\big)-P(B^{(v)})\le-\omega_n\,r^n\,. \tag{5.13}$$

Combining (5.10) and (5.13) with (5.8) and the fact that $\mathbf{C}^\nu_{r,(\alpha,\beta)}\cap\partial F=\mathbf{C}^\nu_r\cap\partial F$ (see (5.6)), we find that, for every $r>R_2$,

$$\begin{aligned}&\varlimsup_{v\to\infty}P(F_{r,v};\Omega)-P(B^{(v)})\le P(F;\mathbf{C}^\nu_r\setminus W)-\omega_n\,r^n+C(n,W)/r\\&\le-\mathrm{res}_W(F,\nu)+C(n,W)/r=-\mathcal{R}(W)+C(n,W)/r.\end{aligned} \tag{5.14}$$

where (4.7) has been used. Now, combining the elementary estimates

$$\max\Big\{\big||F_{r,v}|-v\big|,\,v^{-1/(n+1)}\,|P(B^{(v)})-P(B^{(|F_{r,v}|)})|\Big\}\le C(n)\,r^{n+1} \tag{5.15}$$

with (5.14), we see that

$$\varlimsup_{v\to\infty}\psi_W(|F_{r,v}|)-P(B^{(|F_{r,v}|)})\le-\mathcal{R}(W)+C(n,W)/r,\ \forall r>R_2. \tag{5.16}$$

Again by (5.15) and since $v\mapsto|F_{r,v}|$ is a continuous function, we see that $\varlimsup_{v\to\infty}\psi_W(|F_{r,v}|)-P(B^{(|F_{r,v}|)})=\varlimsup_{v\to\infty}\psi_W(v)-P(B^{(v)})$. This last identity combined with (5.16) implies (5.1) in the limit $r\to\infty$.

Step two: Now let $E_j\in\mathrm{Min}[\psi_W(v_j)]$ for $v_j\to\infty$. By (3.1) and a standard argument (see, e.g. [27, Theorem 21.14]), there is a local perimeter minimizer with free boundary F in Ω such that, up to extracting subsequences,

$$E_j\to F\text{ in }L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}),\,\mathcal{H}^n\llcorner\partial E_j\rightharpoonup\mathcal{H}^n\llcorner\partial F\text{ as Radon measures in }\Omega,$$

$$\mathrm{hd}\,(K \cap \partial E_j; K \cap \partial F) \to 0 \qquad \text{for every } K \subset\subset \Omega. \tag{5.17}$$

Notice that it is not immediate to conclude from $E_j \in \mathrm{Min}[\psi_W(v_j)]$ that (for some $\nu \in \mathbb{S}^n$) $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$ (or even that $(\nu, F) \in \mathcal{F}$), nor that $P(E_j; \Omega) - P(B^{(v_j)})$ is asymptotically bounded from below by $-\mathrm{res}_W(F, \nu)$. In this step we prove some preliminary properties of F, and in particular we exploit the blowdown result for exterior minimal surfaces contained in Theorem 2.1–(ii) to prove that F satisfies (5.2) and (5.3) (see statement (c) below). Then, in step three, we will use the decay rates (5.3) to show that E_j can be "glued" to F, similarly to the construction of step one, and then derive from the corresponding energy estimates the lower bound matching (5.1) and the optimality of F in $\mathcal{R}(W)$.

(a) $\Omega \cap \partial F \cap \partial B_\rho \neq \emptyset$ *for every* ρ *such that* $W \subset\subset B_\rho$: If not there would be $\varepsilon > 0$ such that $W \subset\subset B_{\rho-\varepsilon}$ and $\Omega \cap \partial F \cap A_{\rho-\varepsilon}^{\rho+\varepsilon} = \emptyset$ (recall that $A_r^s = \{x : s > |x| > r\}$). By (5.17) and the constant mean curvature condition satisfied by $\Omega \cap \partial E_j$, we would then find that each E_j (with j large enough) has a connected component of the form $B^{(w_j)}(x_j)$, with $B^{(w_j)}(x_j) \subset\subset \mathbb{R}^{n+1} \setminus B_{\rho+\varepsilon}$ and $w_j \geq v_j - C(n)\,(\rho+\varepsilon)^{n+1}$. In particular, against $\mathcal{R}(W) > 0$,

$$\psi_W(v_j) = P(E_j; \Omega) \geq P(B^{(v_j - C\,(\rho+\varepsilon)^{n+1})}) \geq P(B^{(v_j)}) - C\lambda_j^{-1}(\rho+\varepsilon)^{n+1}.$$

(b) *Sharp area bound*: We combine the upper energy bound (5.1) with the perimeter inequality for spherical symmetrization, to prove

$$P(F; \Omega \cap B_r) \leq \omega_n r^n - \mathcal{R}(W)\,, \qquad \text{for every } r \text{ s.t. } W \subset\subset B_r\,. \tag{5.18}$$

(Notice that (5.18) does not immediately imply the bound for $P(F; \Omega \cap \mathbf{C}_r^\nu)$ which would be needed to compare $\mathcal{R}(W)$ and $\mathrm{res}_W(F, \nu)$.) To prove (5.18) we argue by contradiction, and consider the existence of $\delta > 0$ and r with $W \subset\subset B_r$ such that $P(F; \Omega \cap B_r) > \omega_n\, r^n - \mathcal{R}(W) + \delta$. In particular, for j large enough, we would then have

$$P(E_j; \Omega \cap B_r) \geq \omega_n r^n - \mathcal{R}(W) + \delta\,. \tag{5.19}$$

Again for j large, it must be $\mathcal{H}^n(\partial E_j \cap \partial B_r) = 0$: indeed, by (3.1), $\Omega \cap \partial E_j$ has mean curvature of order $\mathrm{O}(\lambda_j^{-1})$, while of course ∂B_r has constant mean curvature equal to n/r. Thanks to $\mathcal{H}^n(\partial E_j \cap \partial B_r) = 0$,

$$P(E_j; \Omega) = P(E_j; \Omega \cap B_r) + P\big(E_j; \mathbb{R}^{n+1} \setminus \mathrm{cl}\,(B_r)\big)\,. \tag{5.20}$$

If E_j^s denotes the spherical symmetral of E_j such that $E_j^s \cap \partial B_\rho$ is a spherical cap in ∂B_ρ, centered at $\rho\, e_{n+1}$, with area equal to $\mathcal{H}^n(E_j \cap \partial B_\rho)$, then we have the perimeter inequality

$$P\big(E_j; \mathbb{R}^{n+1} \setminus \mathrm{cl}\,(B_r)\big) \geq P\big(E_j^s; \mathbb{R}^{n+1} \setminus \mathrm{cl}\,(B_r)\big)\,; \tag{5.21}$$

see [10]. Now, we can find a half-space J orthogonal to e_{n+1} and such that $\mathcal{H}^n(J \cap \partial B_r) = \mathcal{H}^n(E_j \cap \partial B_r)$. In this way, using that $|E_j^s \setminus B_r| = |E_j \setminus B_r|$ (by Fubini's theorem in spherical coordinates), and that $\mathcal{H}^n(B_r \cap \partial J) \leq \omega_n\, r^n$ (by the fact that ∂J is a hyperplane), we find

$$P\big(E_j^s; \mathbb{R}^{n+1} \setminus \mathrm{cl}\,(B_r)\big) = P\big((E_j^s \setminus \mathrm{cl}\,(B_r)) \cup (J \cap B_r)\big) - \mathcal{H}^n(B_r \cap \partial J)$$

$$\geq P\Big(B^{(|E_j|-|E_j\cap B_r|+|J\cap B_r|)}\Big) - \omega_n\, r^n$$
$$\geq P(B^{(v_j)}) - C(n)\, r^{n+1}\, \lambda_j^{-1} - \omega_n\, r^n$$

which, with (5.19), (5.20) and (5.21), finally gives $P(E_j;\Omega) - P(B^{(v_j)}) > -\mathcal{R}(W) + \delta - C(n)\, r^{n+1}\, \lambda_j^{-1}$ for j large, against (5.1).

(c) *Asymptotic behavior of* ∂F: We prove that there are $\nu \in \mathbb{S}^n$, $f \in C^\infty(\nu^\perp)$, $a, b \in \mathbb{R}$, $c \in \nu^\perp$, $R' > \sup\{\rho : W \subset \mathbf{C}^\nu_\rho\}$ and C positive, with

$$\begin{aligned}
&\partial F \setminus \mathbf{C}^\nu_{R'} = \big\{x + f(x)\,\nu : x \in \nu^\perp,\, |x| > R'\big\},\\
&f(x) = a, \qquad (n=1) \qquad (5.22)\\
&\Big|f(x) - \big(a + b\,|x|^{2-n} + (c\cdot x)\,|x|^{-n}\big)\Big| \leq C\,|x|^{-n}, \quad (n \geq 2),\\
&\max\Big\{|x|^{n-1}\,|\nabla f(x)|,\, |x|^n\,|\nabla^2 f(x)|\Big\} \leq C_0, \qquad \forall x \in \nu^\perp,\, |x| > R'. (5.23)
\end{aligned}$$

To this end, by a standard argument exploiting the local perimeter minimality of F in Ω, given $r_j \to \infty$, then, up to extracting subsequences, $F/r_j \stackrel{\text{loc}}{\to} J$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$, where J is a perimeter minimizer in $\mathbb{R}^{n+1}\setminus\{0\}$, $0 \in \partial J$ (thanks to property (a)), J is a cone with vertex at 0 (thanks to Theorem 2.7 and, in particular to (2.41)), and $P(J; B_1) \leq \omega_n$ (by (5.18)). **If** $n \geq 2$, then ∂J has vanishing distributional mean curvature in $\mathbb{R}^{n+1}$ (as points are removable singularities for the mean curvature operator when $n \geq 2$), thus $P(J; B_1) \geq \omega_n$ by upper semicontinuity of area densities, and, finally, by $P(J; B_1) = \omega_n$ and Allard's regularity theorem, J is a half-space. **If** $n = 1$, then ∂J is the union of two half-lines ℓ_1 and ℓ_2 meeting at $\{0\}$. If ℓ_1 and ℓ_2 are not opposite (i.e., if J is not a half-space), then we can find a half-space J^* such that $(J \cap J^*)\Delta J \subset\subset B \subset\subset \mathbb{R}^2\setminus\{0\}$ for some ball B, and $P(J\cap J^*; B) < P(J; B)$, thus violating the fact that J is a perimeter minimizer in $\mathbb{R}^{n+1}\setminus\{0\}$.

If $n = 1$ it is immediate from the above information that, for some $R' > 0$, $F\setminus B_{R'} = J\setminus B_{R'}$; this proves (5.22) and (5.23) in the case $n = 1$. To prove (5.22) and (5.23) when $n \geq 2$, we let M_0 and ε_0 be as in Theorem 2.1–(ii) with parameters n and $\Gamma = 2\,n\,\omega_n$, and with $\sigma = 1$. Since J is a half-space, by using Remark 3.2 and $F/r_j \stackrel{\text{loc}}{\to} J$ on the annulus $A^{2\,L}_{1/2}$, for some $L > \max\{M_0, 64\}$ to be chosen later on depending also on ε_0, we find that

$$(\partial F) \cap A^{4\,L\,r_j}_{r_j/2} = \big\{x + r_j\, f_j\big(x/r_j\big)\,\nu : x \in \nu^\perp\big\} \cap A^{4\,L\,r_j}_{r_j/2}\,, \qquad \nu^\perp = \partial J\,, \quad (5.24)$$

for $f_j \in C^1(\nu^\perp)$ with $\|f_j\|_{C^1(\nu^\perp)} \to 0$. By (5.24), $V_j = \mathbf{var}\big((\partial F)\setminus B_{r_j}, 1\big) \in \mathcal{V}_n(0, r_j, \infty)$, with (for $\text{o}(1) \to 0$ as $j \to \infty$)

$$\begin{aligned}
&r_j^{-n}\int x\cdot \nu^{\text{co}}_{V_j}\, d\text{bd}_{V_j} = -n\,\omega_n + \text{o}(1)\\
&r_j^{1-n}\,\|\text{bd}_{V_j}\|(\partial B_{r_j}) = n\,\omega_n + \text{o}(1), \qquad (5.25)\\
&\sup_{r\in(r_j, 3\,L\,r_j)}\Big|(r^n - r_j^n)^{-1}\,\|V_j\|(B_r\setminus B_{r_j}) - \omega_n\Big| = \text{o}(1). \qquad (5.26)
\end{aligned}$$

By our choice of Γ, by (5.18) and (5.25) we see that, for j large, we have

$$\|\mathrm{bd}_{V_j}\|(\partial B_{r_j}) \le \Gamma\, r_j^{n-1}\,, \qquad \|V_j\|(B_\rho \setminus B_{r_j}) \le \Gamma \rho^n\,,\ \forall \rho > r_j\,. \tag{5.27}$$

Moreover, we claim that setting

$$s_j = 2\,L\,r_j$$

(so that, in particular, $s_j > \max\{M_0, 64\}\,r_j$), then

$$|\delta_{V_j,r_j,0}(s_j/8)| \le \varepsilon_0, \qquad \inf_{r>s_j/8} \delta_{V_j,r_j,0}(r) \ge -\varepsilon_0, \tag{5.28}$$

provided j and L are taken large enough depending on ε_0. To check the first inequality in (5.28) we notice that, by (5.25) and (5.26),

$$\begin{aligned}
\delta_{V_j,r_j,0}(s_j/8) =& \omega_n - \frac{\|V_j\|(B_{s_j/8} \setminus B_{r_j})}{(s_j/8)^n} + \frac{1}{n\,(s_j/8)^n} \int x \cdot \nu^{\mathrm{co}}_{V_j}\, d\,\mathrm{bd}_{V_j} \\
=& \omega_n - \big(\omega_n + \mathrm{o}(1)\big)\,\frac{(s_j/8)^n - r_j^n}{(s_j/8)^n} - \frac{\omega_n\, r_j^n}{(s_j/8)^n}\,\big(1 + \mathrm{o}(1)\big) \\
=& \mathrm{o}(1)\,(1 + (r_j/s_j)^n) = \mathrm{o}(1),
\end{aligned}$$

so that $|\delta_{V_j,r_j,0}(s_j/8)| \le \varepsilon_0$ as soon as j is large with respect to ε_0. Similarly, if $r > s_j/8 = (L\,r_j)/4$, then by (5.25), (5.26), (5.18), and $r_j/r \le 4/L$,

$$\begin{aligned}
\delta_{V_j,r_j,0}(r) &= \omega_n - \frac{\|V_j\|(B_r \setminus B_{2\,r_j})}{r^n} - \frac{\|V_j\|(B_{2\,r_j} \setminus B_{r_j})}{r^n} - \frac{\omega_n\, r_j^n}{r^n}\,\big(1 + \mathrm{o}(1)\big) \\
&\ge \omega_n - \frac{\omega_n\, r^n - \mathcal{R}(W)}{r^n} - \big(\omega_n + \mathrm{o}(1)\big)\,\frac{(2\,r_j)^n - r_j^n}{r^n} - \frac{\omega_n\, r_j^n}{r^n}\,\big(1 + \mathrm{o}(1)\big) \\
&\ge r^{-n}\,\mathcal{R}(W) - 2\,(4/L)^n\,\big(\omega_n + \mathrm{o}(1)\big) - (4/L)^n\,\mathrm{o}(1) \ge -3\,(4/L)^n\,\omega_n,
\end{aligned}$$

provided j is large; hence the second inequality in (5.28) holds if L is large in terms of ε_0. By (5.27) and (5.28), Theorem 2.1–(ii) can be applied to $(V, R, \Lambda, s) = (V_j, r_j, 0, s_j)$ with j large. As a consequence, passing from spherical graphs to cylindrical graphs with the aid of Lemma B.1, we find that, for some large j,

$$(\partial F) \setminus B_{s_j/16} = \big\{x + f(x)\,\nu : x \in \nu^\perp\big\} \setminus B_{s_j/16}\,, \tag{5.29}$$

where $f : \nu^\perp \to \mathbb{R}$ is a smooth function which solves the minimal surfaces equation on $\nu^\perp \setminus B_{s_j/16}$. Since ∂F admits at least one sequential blowdown limit hyperplane (namely, $\nu^\perp = \partial J$), by a theorem of Simon [36, Theorem 2] we find that ∇f has a limit as $|x| \to \infty$; in particular, $|\nabla f|$ is bounded. Moreover, by (5.29) (or by the fact that F is a local perimeter minimizer in Ω), ∂F is a stable minimal surface in $\mathbb{R}^{n+1} \setminus B_{s_j/16}$, which, thanks to (5.18), satisfies an area growth bound like (4.3). We can thus apply Proposition 4.1 to deduce the validity of (5.23) when $n \ge 3$, and of $|f(x) - [a + b\,\log|x| + (c \cdot x)\,|x|^{-2}]| \le C\,|x|^{-2}$ for all $|x| > R'$ when $n = 2$ (with $R' > s_j$). Recalling that F is a local perimeter minimizer with free boundary in Ω (that is, $P(F; \Omega \cap B) \le P(F'; \Omega \cap B)$ whenever $F \Delta F' \subset\subset B \subset\subset \mathbb{R}^3$) it must be that $b = 0$, as it can be seen by comparing F

with the set F' obtained by changing F inside $\mathbf{C}_r^\nu$ ($r >> R'$) with the half-space G_r bounded by the plane $\{x + t\,\nu : x \in \nu^\perp, t = a + b\,\log(r) + c \cdot x/r^2\}$ and such that $\mathcal{H}^2((F\Delta G_r) \cap \partial\mathbf{C}_r^\nu) \le C/r^2$ (we omit the details of this standard comparison argument). Having shown that $b = 0$, the proof of (5.23) when $n = 2$ also is complete and we are finished with (c).

(d) $F \cup W$ *defines an element of* $\mathcal{F}$: With $R > R'$ as in (5.22) and (5.23), $V_R = \mathbf{var}\,((\partial F) \cap (B_R \setminus W))$ is a stationary varifold in $\mathbb{R}^{n+1} \setminus K_R$ for $K_R = W \cup \big\{x + f(x)\,\nu : x \in \nu^\perp, |x| = R\}$, and has bounded support. By the convex hull property [34, Theorem 19.2], we deduce that, for every $R > R'$, $\mathrm{spt} V_R$ is contained in the convex hull of K_R, for every $R > R'$. Taking into account that $f(x) \to a$ as $|x| \to \infty$ we conclude that $\Omega \cap \partial F$ is contained in the smallest slab $\mathbf{S}^\nu_{[\alpha,\beta]}$ containing both W and $\{x : x \cdot \nu = a\}$. Now set $F' = F \cup W$. Clearly F' is a set of locally finite perimeter in Ω (since $P(F'; \Omega') = P(F; \Omega')$ for every $\Omega' \subset\subset \Omega$). Second, $\partial F'$ is contained in $\mathbf{S}^\nu_{[\alpha,\beta]}$ (since $\partial F' \subset [(\partial F) \cap \Omega] \cup W$). Third, by (5.22) and (5.23),

$$\big\{x + t\,\nu : x \in \nu^\perp, |x| > R', t < \alpha\big\} \subset F', \tag{5.30}$$

$$\big\{x + t\,\nu : x \in \nu^\perp, |x| > R', t > \beta\big\} \subset \mathbb{R}^{n+1} \setminus F', \tag{5.31}$$

$$\big\{x + t\,\nu : x \in \nu^\perp, |x| < R', t \in \mathbb{R} \setminus [\alpha, \beta]\big\} \cap (\partial F') = \emptyset. \tag{5.32}$$

By combining (5.30) and (5.32) we see that $\{x + t\,\nu : x \in \nu^\perp, t < \alpha\} \subset F'$, and by combining (5.31) and (5.32) we see that $\{x + t\,\nu : x \in \nu^\perp, t > \beta\} \subset \mathbb{R}^{n+1} \setminus F'$: in particular, $\mathbf{p}_{\nu^\perp}(\partial F') = \nu^\perp$, and thus $(F', \nu) \in \mathcal{F}$.

Step three: We prove that

$$\varliminf_{v\to\infty}\ \psi_W(v) - P(B^{(v)}) \ge -\mathcal{R}(W)\,. \tag{5.33}$$

For $v_j \to \infty$ achieving the liminf in (5.33), let $E_j \in \mathrm{Min}[\psi_W(v_j)]$ and let F be a (sub-sequential) limit of E_j, so that properties (a), (b), (c) and (d) in step two hold for F. In particular, properties (5.22) and (5.23) from (c) are entirely analogous to properties (5.2) and (5.3) exploited in step one: therefore, the family of half-spaces $\{G_r\}_{r>R'}$ defined by (5.4) is such that

$$\big((\partial F) \cup (\partial G_r) \cup (G_r \Delta F)\big) \cap \mathbf{C}_r^\nu \ \subset\ \mathbf{C}^\nu_{r,(\alpha+1,\beta-1)},$$

$$\mathcal{H}^n\big((F \Delta G_r) \cap \partial\mathbf{C}_r^\nu\big) \le r^{-1}\,C(n, W), \tag{5.34}$$

$$\big|P\big(G_r; \mathbf{C}^\nu_{r,(\alpha,\beta)}\big) - \omega_n\,r^n\big| \le r^{-n}\,C(n, W), \tag{5.35}$$

(compare with (5.6), (5.9), and (5.12) in step one). By (5.35) we find

$$-\mathrm{res}_W(F', \nu) = \lim_{r\to\infty} P(F; \mathbf{C}_r^\nu \setminus W) - P(G_r; \mathbf{C}^\nu_{r,(\alpha,\beta)})\,. \tag{5.36}$$

In order to relate the residue of (F', ν) to $\psi_W(v_j) - P(B^{(v_j)})$ we consider the sets $Z_j = (G_r \cap \mathbf{C}^\nu_{r,(\alpha,b)}) \cup (E_j \setminus \mathbf{C}^\nu_{r,(\alpha,\beta)})$, which, by isoperimetry, satisfy

$$P(Z_j) \ge P(B^{(|E_j \setminus \mathbf{C}^\nu_{r,(\alpha,\beta)}|)}) \ge P(B^{(v_j)}) - C(n)\,r^n\,(\beta - \alpha)\,\lambda_j^{-1}. \tag{5.37}$$

Since for a.e. $r > R'$ we have

$$P(Z_j) = P(E_j; \mathbb{R}^{n+1} \setminus \mathbf{C}^\nu_{r,(\alpha,\beta)}) + P(G_r; \mathbf{C}^\nu_{r,(\alpha,b)}) + \mathcal{H}^n\big((E_j \Delta G_r) \cap \partial \mathbf{C}^\nu_{r,(\alpha,b)}\big),$$

we conclude that

$$\begin{aligned}\psi_W(v_j) - P(B^{(v_j)}) = {} & P(E_j; \mathbf{C}^\nu_{r,(\alpha,\beta)} \setminus W) + P(E_j; \mathbb{R}^{n+1} \setminus \mathbf{C}^\nu_{r,(\alpha,\beta)}) - P(B^{(v_j)}) \\ = {} & P(E_j; \mathbf{C}^\nu_{r,(\alpha,\beta)} \setminus W) + P(Z_j) - P(B^{(v_j)}) \\ & - P(G_r; \mathbf{C}^\nu_{r,(\alpha,b)}) - \mathcal{H}^n\big((E_j \Delta G_r) \cap \partial \mathbf{C}^\nu_{r,(\alpha,b)}\big)\end{aligned}$$

so that $E_j \to F$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$ and (5.37) give, for a.e. $r > R'$,

$$\begin{aligned}&\varliminf_{j\to\infty} \psi_W(v_j) - P(B^{(v_j)}) \ge P(F; \mathbf{C}^\nu_{r,(\alpha,\beta)} \setminus W) - P(G_r; \mathbf{C}^\nu_{r,(\alpha,b)}) \\ &-\mathcal{H}^n\big((F \Delta G_r) \cap \partial \mathbf{C}^\nu_{r,(\alpha,\beta)}\big) \ge P(F; \mathbf{C}^\nu_r \setminus W) - P(G_r; \mathbf{C}^\nu_r) - C(n, W)/r,\end{aligned}$$

thanks to (5.34) and $(F \Delta G_r) \cap \partial \mathbf{C}^\nu_r = (F \Delta G_r) \cap \partial \mathbf{C}^\nu_{r,(\alpha,\beta)}$. Letting $r \to \infty$, recalling (5.36), and by $(F', \nu) \in \mathcal{F}$, we find $\varliminf_{j\to\infty} \psi_W(v_j) - P(B^{(v_j)}) \ge -\mathrm{res}_W(F', \nu) \ge -\mathcal{R}(W)$. This completes the proof of (5.33), which in turn, combined with (5.1), gives (1.19), and also shows that $L^1_{\rm loc}$-subsequential limits F of $E_j \in \mathrm{Min}[\psi_W(v_j)]$ for $v_j \to \infty$ are such that, for some $\nu \in \mathbb{S}^n$, $(F \cup W, \nu) \in \mathcal{F}$ and $F' = F \cup W \in \mathrm{Max}[\mathcal{R}(W)]$.

Step four: Moving towards the proof of (1.22), we prove the validity, uniformly among varifolds associated to maximizers of $\mathcal{R}(W)$, of estimates analogous to (5.27) and (5.28). For a constant $\Gamma > 2\, n\, \omega_n$ to be determined later on (see (5.48), (5.49), and (5.50) below) in dependence of n and W, and for $\sigma > 0$, we let $M_0 = M_0(n, 2\,\Gamma, \sigma)$ and $\varepsilon_0 = \varepsilon_0(n, 2\,\Gamma, \sigma)$ be determined by Theorem 2.1. If $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$, then by Theorem 1.1–(ii) we can find $R_2 = R_2(W) > 0$, $f \in C^\infty(\nu^\perp)$ such that

$$(\partial F) \setminus \mathbf{C}^\nu_{R_2} = \big\{x + f(x)\,\nu : x \in \nu^\perp, |x| > R_2\big\}, \tag{5.38}$$

and such that (1.14) holds with $\max\{|a|, |b|, |c|\} \le C(W)$ and $|\nabla f(x)| \le C_0/|x|^{n-1}$ for $|x| > R_2$. Thus $\|\nabla f\|_{C^0(\nu^\perp \setminus \mathbf{D}^\nu_r)} \to 0$ as $r \to \infty$ uniformly on $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$, and there is $R_3 > \max\{2\,R_2, 1\}$ (depending on W) such that, if $V_F = \mathbf{var}\,((\partial F) \setminus B_{R_3}, 1)$, then $V_F \in \mathcal{V}_n(0, R_3, \infty)$, and

$$\|\mathrm{bd}_{V_F}\|(\partial B_{R_3}) \le \Gamma\, R_3^{n-1}, \qquad \|V_F\|(B_\rho \setminus B_{R_3}) \le \Gamma\, \rho^n \qquad \forall \rho > R_3, \tag{5.39}$$

(compare with (5.27)). Then, arguing as in step three–(c), or more simply by exploiting (5.38) and the decay estimates (1.14), we see that there is $L > \max\{M_0, 64\}$, depending on n, W and σ only, such that, setting

$$s_W(\sigma) = 2\,L\,R_3 \tag{5.40}$$

we have for some $c(n) > 0$ (compare with (5.28))

$$|\delta_{V_F, R_3, 0}(s_W(\sigma)/8)| \le \varepsilon_0/2, \qquad \inf_{r > s_W(\sigma)/8} \delta_{V_F, R_3, 0}(r) \ge -\varepsilon_0/2. \tag{5.41}$$

Step five: Given $E_j \in \mathrm{Min}[\psi_W(v_j)]$ for $v_j \to \infty$, we prove the existence of $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$ and $h_j \in C^\infty((\partial F)\setminus B_{R_2})$ such that

$$(\partial E_j) \cap A^{R_1\,\lambda_j}_{4\,R_2} = \Big\{ y + h_j(y)\,\nu_F(y) : y \in \partial F \Big\} \cap A^{R_1\,\lambda_j}_{4\,R_2}, \tag{5.42}$$

$$\lim_{j\to\infty} \|h_j\|_{C^1((\partial F)\cap A^M_{4\,R_2})} = 0, \qquad \forall M < \infty; \tag{5.43}$$

and that if x_j satisfies $|E_j \Delta B^{(v_j)}(x_j)| = \inf_x |E_j \Delta B^{(v_j)}(x)|$, then

$$\lim_{j\to\infty} \big||x_j|^{-1}\,x_j - \nu\big| = 0\,; \tag{5.44}$$

finally, we prove statement (iii) (i.e., $(\partial E_j)\setminus B_{R_2}$ is diffeomorphic to an n-dimensional disk). By step three, there is $(F,\nu) \in \mathrm{Max}[\mathcal{R}(W)]$ such that, up to extracting subsequences, (5.17) holds. By (5.17) and (5.38), and with $s_W(\sigma)$ defined as in step four (see (5.40)) starting from F, we can apply Remark 3.2 to find $f_j \in C^\infty(\nu^\perp)$ such that

$$(\partial E_j) \cap A^{s_W(\sigma)}_{2\,R_2} = \big\{ x + f_j(x)\,\nu : x \in \nu^\perp \big\} \cap A^{s_W(\sigma)}_{2\,R_2}\,, \tag{5.45}$$

for j large enough (in terms of σ, n, W, and F), and such that $f_j \to f$ in $C^1(\mathbf{D}^\nu_{s_W(\sigma)} \setminus \mathbf{D}^\nu_{2\,R_2})$. With R_3 as in step four and with the goal of applying Theorem 2.1 to the varifolds $V_j = \mathbf{var}\,((\partial E_j)\setminus B_{R_3}, 1)$, we notice that $V_j \in \mathcal{V}_n(\Lambda_j, R_3, \infty)$, for some $\Lambda_j \le \Lambda_0\,\lambda_j^{-1}$ (thanks to (3.1)). In particular, by (5.40), $s_W(\sigma)$ satisfies the "mesoscale bounds" (compare with (2.2))

$$\varepsilon_0\,(4\,\Lambda_j)^{-1} > s_W(\sigma) > \max\{M_0, 64\}\,R_3 \tag{5.46}$$

provided j is large. Moreover, by $R_3 > 2\,R_2$ and $s_W(\sigma)/8 > 2\,R_2$, by (5.38), (5.45) and $f_j \to f$ in C^1, we exploit (5.39) and (5.41) to deduce

$$\begin{aligned} \|\mathrm{bd}_{V_j}\|(\partial B_{R_3}) &\le (2\,\Gamma)\,R_3^{n-1}, \\ |\delta_{V_j,R_3,0}(s_W(\sigma)/8)| &\le (2/3)\,\varepsilon_0. \end{aligned} \tag{5.47}$$

We claim that, up to increasing Γ (depending on n and W), we can entail

$$\|V_j\|(B_\rho \setminus B_{R_3}) \le \Gamma\,\rho^n\,, \qquad \forall \rho > R_3\,. \tag{5.48}$$

Indeed, by Theorem 3.1–(i), for some positive Λ_0 and s_0 depending on W only, E_j is a $(\Lambda_0\,\lambda_j^{-1}, s_0\,\lambda_j)$-perimeter minimizer with free boundary in Ω. Comparing E_j to $E_j\setminus B_r$ by (3.1), for every $r < s_0\,\lambda_j$,

$$P(E_j;\Omega\cap B_r) \le C(n)\left(r^n + \Lambda_0\,\lambda_j^{-1}\,r^{n+1}\right) \le C(n,W)\,r^n\,; \tag{5.49}$$

since, at the same time, if $r > s_0\,\lambda_j$, then

$$P(E_j;\Omega\cap B_r) \le P(E_j;\Omega) = \psi_W(v_j) \le P(B^{(v_j)}) \le C(n)\,s_0^{-n}\,r^n\,, \tag{5.50}$$

by combining (5.49) and (5.50) we find (5.48). With (5.47) and (5.48) at hand, we can also show that

$$|\delta_{V_j,R_3,\Lambda_j}(s_W(\sigma)/8)| \le \varepsilon_0\,. \tag{5.51}$$

Indeed, by $s_W(\sigma) = 2\,L\,R_3$ and by $\Lambda_j \le \Lambda_0\,\lambda_j^{-1}$,

$$\begin{aligned}&\left|\delta_{V_j,R_3,\Lambda_j}(s_W(\sigma)/8) - \delta_{V_j,R_3,0}(s_W(\sigma)/8)\right|\\&\le (\Lambda_0/\lambda_j)\int_{R_3}^{s_W(\sigma)/8}\rho^{-n}\,\|V_j\|(B_\rho\setminus B_{R_3})\,d\rho \le \frac{\Lambda_0\,R_3\,\Gamma}{\lambda_j}\Big(\frac{L}{4}-1\Big)\le\frac{\varepsilon_0}{3},\end{aligned}$$

provided j is large enough. To complete checking that Theorem 2.1 can be applied to every V_j with j large enough, we now consider the quantities

$$R_{*j} = \sup\big\{\rho > s_W(\sigma)/8 : \delta_{V_j,R_3,\Lambda_j}(\rho) \ge -\varepsilon_0\big\},$$

and prove that, for a constant τ_0 depending on n and W only, we have

$$R_{*j} \ge \tau_0\,\lambda_j\,; \tag{5.52}$$

in particular, provided j is large enough, (5.52) implies immediately

$$R_{*j} \ge 4\,s_W(\sigma)\,, \tag{5.53}$$

which was the last assumption in Theorem 2.1 that needed to be checked. To prove (5.52), we pick τ_0 such that

$$\left|\tau_0^{-n}\,\mathcal{H}^n\big(B_{\tau_0}(z)\cap\partial B^{(1)}\big) - \omega_n\right| \le \varepsilon_0/2\,,\qquad \forall z\in\partial B^{(1)}\,.$$

(Of course this condition only requires τ_0 to depend on n; the dependence on W will appear later.) By definition of x_j and by (3.4), and up to extracting a subsequence, we have $x_j \to z_0$ for some $z_0 \in \partial B^{(1)}$. In particular, setting $\rho_j = \tau_0\,\lambda_j$, we find

$$\begin{aligned}\rho_j^{-n}\,\|V_j\|(B_{\rho_j}\setminus B_{R_3}) &= \tau_0^{-n}\,P\big((E_j - x_j)/\lambda_j;\ B_{\tau_0}(-x_j)\setminus B_{R_3/\rho_j}(-x_j)\big)\\&\to \tau_0^{-n}\,\mathcal{H}^n\big(B_{\tau_0}(-z_0)\cap\partial B^{(1)}\big) \le \omega_n + (\varepsilon_0/2),\end{aligned}$$

thus proving that, for j large enough,

$$\begin{aligned}\delta_{V_j,R_3,\Lambda_j}(\rho_j) &\ge -\frac{\varepsilon_0}{2} + \frac{1}{n\,\rho_j^n}\int x\cdot\nu_{V_j}^{\rm co}\,d\,{\rm bd}_{V_j} - \Lambda_j\int_{R_3}^{\rho_j}\frac{\|V_j\|(B_\rho\setminus B_{R_3})}{\rho^n}\,d\rho\\&\ge -\frac{\varepsilon_0}{2} - \frac{2\,\Gamma\,R_3^n}{n\,\tau_0^n\,\lambda_j} - \Lambda_0\,\Gamma\,\frac{(\rho_j - R_3)}{\lambda_j} \ge -\frac{\varepsilon_0}{2} - \frac{C_*(n,W)}{\tau_0^n\,\lambda_j} - C_{**}(n,W)\,\tau_0,\end{aligned}$$

where we have used (5.47), ${\rm spt}\,{\rm bd}_{V_j}\subset\partial B_{R_3}$, and (5.48). Therefore, provided we pick τ_0 depending on n and W so that $C_{**}\,\tau_0 \le \varepsilon_0/4$, and then we pick j large enough to entail $(C_*(n,W)/\tau_0^n)\lambda_j^{-1} \le \varepsilon_0/4$, we conclude that if $r\in(R_3,\rho_j]$, then $\delta_{V_j,R_3,\Lambda_j}(r) \ge \delta_{V_j,R_3,\Lambda_j}(\rho_j) \ge -\varepsilon_0$, where in the first inequality we have used Theorem 2.7–(i) and the fact that $V_j \in \mathcal{V}_n(\Lambda_j, R_3, \infty)$. In summary, by (5.47) and (5.48) (which give (2.1)), by (5.46) (which gives (2.2) with $s = s_W(\sigma)/8$), and by (5.51) and (5.53) (which imply, respectively, (2.3) and (2.4)) we see that Theorem 2.1–(i) can be applied with $V = V_j$ and $s = s_W(\sigma)/8$ provided j is large in terms of σ, n, W and the limit F of the E_j's. Thus, setting

$$S_{*j} = \min\big\{R_{*j}, \varepsilon_0/\Lambda_j\big\}\,,$$

and noticing that by (5.52) and $\Lambda_j \le \Lambda_0 \lambda_j^{-1}$ we have

$$S_{*j} \ge 16\, R_1\, \lambda_j\,,$$

(for R_1 depending on n and W only) we conclude that, for j large, there are $K_j \in \mathcal{H}$ and $u_j \in \mathcal{X}_\sigma(\Sigma_{K_j}, \sigma_W(\sigma)/32, R_1\,\lambda_j)$, such that

$$(\partial E_j) \cap A^{R_1\,\lambda_j}_{s_W(\sigma)/32} = \Sigma_{K_j}\big(u_j, s_W(\sigma)/32, R_1\,\lambda_j\big)\,. \tag{5.54}$$

Similarly, by (5.39) and (5.41), thanks to Theorem 2.1–(ii) we have

$$(\partial F) \cap \big(\mathbb{R}^{n+1} \setminus B_{s_W(\sigma)/32}\big) = \Sigma_{\nu^\perp}\big(u, s_W(\sigma)/32, \infty\big)\,, \tag{5.55}$$

for $u \in \mathcal{X}_{\sigma'}(\Sigma_{\nu^\perp}, s_W(\sigma)/32, \infty)$ for every $\sigma' > \sigma$. Now, by $E_j \to F$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$, (5.54) and (5.55) can hold only if $|\nu_{K_j} - \nu| \le \zeta(\sigma)$ for a function ζ, depending on n and W only, such that $\zeta(\sigma) \to 0$ as $\sigma \to 0^+$. In particular (denoting by σ_0^*, ε_0^* and C_0^* the dimension dependent constants originally introduced in Lemma 2.5 as σ_0, ε_0 and C_0) we can find $\sigma_1 = \sigma_1(n, W) \le \sigma_0^*$ such that if $\sigma < \sigma_1$, then $\varepsilon_0^* \ge \zeta(\sigma) \ge |\nu_{K_j} - \nu|$, and correspondingly, Lemma 2.5–(i) can be used to infer the existence of $u_j^* \in \mathcal{X}_{C_0\,(\sigma+\zeta(\sigma))}(\Sigma_{\nu^\perp}, s_W(\sigma)/32, 2\,R_1\,\lambda_j)$ such that, for j large,

$$\begin{aligned}\Sigma_{\nu^\perp}\big(u_j^*, s_W(\sigma)/32, 2\,R_1\,\lambda_j\big) &= \Sigma_{K_j}\big(u_j, s_W(\sigma)/32, 2\,R_1\,\lambda_j\big)\\ &= (\partial E_j) \cap A^{2\,R_1\,\lambda_j}_{s_W(\sigma)/32}.\end{aligned} \tag{5.56}$$

By (5.45) and Lemma B.1, (5.56) implies cylindrical graphicality: more precisely, provided σ_1 is small enough, there are $g_j \in C^1(\nu^\perp)$ such that

$$\sup_{x\in\nu^\perp} \{|g_j(x)|\,|x|^{-1}, |\nabla g_j(x)|\} \le C\,\big(\sigma + \zeta(\sigma)\big), \tag{5.57}$$

$$(\partial E_j) \cap A^{R_1\,\lambda_j}_{2\,R_2} = \big\{x + g_j(x)\,\nu : x \in \nu^\perp\big\} \cap A^{R_1\,\lambda_j}_{2\,R_2}. \tag{5.58}$$

At the same time, by (5.38), (1.14), and up to further increasing R_2 and decreasing σ_1, we can exploit Lemma B.2 in the Appendix to find $h_j \in C^1(G(f))$, $G(f) = \{x + f(x)\,\nu : x \in \nu^\perp\}$, such that

$$\big\{x + g_j(x)\,\nu : x \in \nu^\perp\big\} \setminus B_{4\,R_2} = \big\{z + h_j(z)\,\nu_F(z) : z \in G(f)\big\} \setminus B_{4\,R_2},$$

which, combined with (5.38) and (5.58) shows that

$$(\partial E_j) \cap A^{R_1\,\lambda_j}_{4\,R_2} = \big\{z + h_j(z)\,\nu_F(z) : z \in \partial F\big\} \cap A^{R_1\,\lambda_j}_{4\,R_2}\,,$$

that is (5.42). By $E_j \to F$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$, we find $h_j \to 0$ in $L^1((\partial F) \cap A^M_{4\,R_2})$ for every $M < \infty$, so that, by elliptic regularity, (5.43) follows. We now recall that, by Theorem 3.1–(ii), $(\partial E_j)\setminus B_{R_0(v_j)\,\lambda_j}$ coincides with

$$\begin{aligned}&\big\{y + \lambda_j w_j\big((y - x_j)/\lambda_j\big)\,\nu_{B^{(v_j)}(x_j)}(y) : y \in \partial B^{(v_j)}(x_j)\big\} \setminus B_{R_0(v_j)\,\lambda_j}\\ &\text{with } \|w_j\|_{C^1(\partial B^{(1)})} \to 0 \text{ and } R_0(v_j) \to 0.\end{aligned} \tag{5.59}$$

The overlapping of (5.58) and (5.59) (i.e., the fact that $R_0(v_j) < R_1$ if j is large enough) implies statement (iii). Finally, combining (5.57) and (5.58) with (5.59) and $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$ we deduce the validity of (5.44). More precisely, rescaling by λ_j in (5.57) and (5.58) and setting $E_j^* = E_j/\lambda_j$, we find $g_j^* \in C^1(\nu^\perp)$ such that, for every $j \geq j_0(\sigma)$ and $\sigma < \sigma_1$,

$$\sup_{x\in\nu^\perp} \{|g_j^*(x)||x|^{-1}, |\nabla g_j^*(x)|\} \leq C\left(\sigma + \zeta(\sigma)\right),$$
$$(\partial E_j^*) \cap A^{R_1}_{2R_2/\lambda_j} = \left\{x + g_j^*(x)\,\nu : x \in \nu^\perp\right\} \cap A^{R_1}_{2R_2/\lambda_j}, \tag{5.60}$$

while rescaling by λ_j in (5.59) and setting $z_j = x_j/\lambda_j$ we find

$$(\partial E_j^*) \setminus B_{R_0(v_j)} = \left\{z_j + z + w_j(z)\,\nu_{B^{(1)}}(z) : y \in \partial B^{(1)}(z_j)\right\} \setminus B_{R_0(v_j)}$$

where $||z_j| - \omega_{n+1}^{1/(n+1)}| \to 0$ thanks to (3.4). Up to subsequences, $z_j \to z_0$, where $|z_0| = \omega_{n+1}^{1/(n+1)}$. Should $z_0 \neq |z_0|\,\nu$, then picking σ small enough in terms of $|\nu - (z_0/|z_0|)| > 0$ and picking j large enough, we would then be able to exploit (5.60) to get a contradiction with $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$.

Conclusion: Theorem 3.1 implies Theorem 1.6–(i), and (1.19) was proved in step three. Should Theorem 1.6–(ii), (iii), or (iv) fail, then we could find a sequence $\{(E_j, v_j)\}_j$ contradicting the conclusions of either step five or Theorem 3.1. We have thus completed the proof of Theorem 1.6. □

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix A: Proof of Theorem 2.6

We assume $H \in \mathcal{H}$, $\Lambda \geq 0$, $\eta_0 > \eta > 0$, (r_1, r_2) and (r_3, r_4) are (η, η_0)-related as in (2.36), and $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$ is such that $\Sigma_H(u, r_1, r_2)$ has mean curvature bounded by Λ in $A_{r_1}^{r_2}$. We want to find σ_0 and C_0, depending on n, η_0, and η only, such that, if $\max\{1, \Lambda\, r_2\}\,\sigma \le \sigma_0$, then

$$\left|\mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))\right| \le C_0 \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\big(u^2 + \Lambda\, r\, |u|\big); \quad (A.1)$$

and such that, if there is $r \in (r_1, r_2)$ s.t. $E^0_{\Sigma_H}[u_r] = 0$ on Σ_H, then

$$\int_{\Sigma_H\times(r_3,r_4)} r^{n-1}\, u^2 \le C(n)\,\Lambda\, r_2\,(r_2^n - r_1^n) + C_0 \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2\,. \quad (A.2)$$

We make three preliminary considerations: **(i):** By [1, 4.5(8)]

$$\Big|\mathcal{H}^n(\Sigma_H(u, r_1, r_2)) - \mathcal{H}^n(\Sigma_H(0, r_1, r_2)) - \frac{1}{2}\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\left(|\nabla^{\Sigma_H} u|^2 + (r\,\partial_r u)^2 - (n-1)\,u^2\right)\Big| \le C(n)\,\sigma \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\big(u^2 + |\nabla^{\Sigma_H} u|^2 + (r\,\partial_r u)^2\big) \quad (A.3)$$

Similarly, by the last displayed formula on [1, Page 236] and by [1, Lemma 4.9(1)], if $\varphi = \psi^2\, w$, $w \in C^1(\Sigma_H \times (r_1, r_2))$ and $\psi \in C^1(r_1, r_2)$, then

$$\Big|\frac{d}{dt}\Big|_{t=0} \mathcal{H}^n(\Sigma_H(u + t\,\varphi, r_1, r_2)) - \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\left\{\nabla^{\Sigma_H} u\cdot\nabla^{\Sigma_H}\varphi + (r\,\partial_r u)\,(r\,\partial_r\varphi) - (n-1)\,u\,\varphi\right\}\Big| \le C(n)\,\sigma \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,\psi^2\left\{|\nabla^{\Sigma_H} u|^2 + |\nabla^{\Sigma_H} w|^2 + (r\,\partial_r u)^2 + (r\,\partial_r w)^2 + u^2 + w^2 + (r\,\psi')^2\, w^2\right\}, \quad (A.4)$$

which is the second order expansion of the first variation of the area at $\Sigma_H(u, r_1, r_2)$ along outer variations in spherical coordinates of the form $\varphi = \psi^2\, w$, $\psi = \psi(r)$. **(ii):** For the sake of brevity, given $\zeta : (r_1, r_2) \to \mathbb{R}$ a radial function, $u, v : \Sigma_H \times (r_1, r_2) \to \mathbb{R}$, $X, Y : \Sigma_H \times (r_1, r_2) \to \mathbb{R}^m$, we set

$$Q_\zeta(u, v) = \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,\zeta(r)^2\, u\, v, \quad Q_\zeta(X, Y) = \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,\zeta(r)^2\, X\cdot Y,$$

and $Q_\zeta(u) = Q_\zeta(u, u)$, $Q_\zeta(X) = Q_\zeta(X, X)$. **(iii):** The following two estimates (whose elementary proof is contained in [1, Lemma 7.13]) hold: whenever $v \in C^1(\Sigma_H \times (r_1, r_2))$, we have

$$\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\, v^2 \le C(n, \eta, \eta_0)\left\{\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r v)^2 + \int_{\Sigma_H\times(r_3,r_4)} r^{n-1}\, v^2\right\}, \quad (A.5)$$

and, provided there is $r \in [r_1, r_2]$ such that $v_r = 0$ on Σ_H, we have

$$\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\, v^2 \le C(n, \eta_0) \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r v)^2\,. \quad (A.6)$$

We are now ready for the proof. Compared to [1, Chapter 4], the main difference is that we replace [1, Lemma 4.10] with (A.7).

Step one: We prove that there is $h : \Sigma_H \times (r_1, r_2) \to [-\Lambda, \Lambda]$ such that for every $w \in C^1(\Sigma_H \times (r_1, r_2))$ and $\psi \in C^1(r_1, r_2)$ we have

$$\begin{aligned}\Big|T_\psi(u, w) - \int_{\Sigma_H \times (r_1, r_2)} r^n\, \psi^2\, w\, h\Big| &\le C(n)\sigma_0\big(Q_\psi(u) + Q_\psi(w) + Q_\psi(\nabla^{\Sigma_H} u)\\ &\quad + Q_\psi(\nabla^{\Sigma_H} w) + Q_{r\,\psi}(\partial_r u) + Q_{r\,\psi}(\partial_r w) + Q_{r\,\psi'}(w)\big).\end{aligned} \tag{A.7}$$

where $T_\psi(u, w) = Q_\psi(\nabla^{\Sigma_H} u, \nabla^{\Sigma_H} w) + Q_r\big(\partial_r u, \partial_r[\psi^2 w]\big) - (n-1)\, Q_\psi(u, w)$. We start rewriting (A.4) as

$$\begin{aligned}&\Big|T_\psi(u, w) - \frac{d}{dt}\Big|_{t=0} \mathcal{H}^n(\Sigma(u + t\,\psi^2\, w, r_1, r_2))\Big|\\ &\le C(n)\,\sigma\,\big(Q_\psi(u) + Q_\psi(w) + Q_\psi(\nabla^{\Sigma_H} u) + Q_\psi(\nabla^{\Sigma_H} w)\\ &\qquad + Q_{r\,\psi}(\partial_r u) + Q_{r\,\psi}(\partial_r w) + Q_{r\,\psi'}(w)\big).\end{aligned} \tag{A.8}$$

If $F_{u+t\,\varphi} : \Sigma_H \times (r_1, r_2) \to \Sigma_H(u + t\,\varphi, r_1, r_2)$, $\varphi = \psi^2\, w$, is given by

$$F_{u+t\,\varphi}(\omega, r) = r\,\frac{\omega + (u(\omega, r) + t\varphi(\omega, r))\,\nu_H}{\sqrt{1 + (u(\omega, r) + t\varphi(\omega, r))^2}},$$

then $\{\Phi_t = F_{u+t\,\varphi} \circ (F_u)^{-1}\}_{t\in[0,1]}$ are diffeomorphisms on $\Sigma_H(u, r_1, r_2)$, with $\Phi_t(\Sigma_H(u, r_1, r_2)) = \Sigma_H(u+t\,\varphi, r_1, r_2)$ and $\dot\Phi_0 = (d/dt)_{t=0}\Phi_t$. Since $\Sigma_H(u, r_1, r_2)$ has mean curvature bounded by Λ in $A_{r_1}^{r_2}$, for some bounded function $h : \Sigma_H \times (r_1, r_2) \to [-\Lambda, \Lambda]$ we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathcal{H}^n(\Sigma(u + t\varphi, r_1, r_2)) &= \Lambda \int_{\Sigma_H(u, r_1, r_2)} h(F_u^{-1})\, \dot\Phi_0 \cdot \nu_{\Sigma_H(u, r_1, r_2)}\\ &= \Lambda \int_{\Sigma_H \times (r_1, r_2)} h\, \dot\Phi_0\big(F_u\big) \cdot \star\big(\partial_r F_u \wedge \wedge_{i=1}^{n-1} \partial_i F_u\big),\end{aligned}$$

where $\partial_i = \nabla_{\tau_i}$ for a local orthonormal frame $\{\tau_i\}_{i=1}^{n-1}$ in Σ_H, and where $\star$ is the Hodge star-operator (so that $\star\,(v_1 \wedge v_2 \ldots \wedge v_n)$ is a normal vector to the hyperplane spanned by the v_i's, with length equal to the n-dimensional volume of the parallelogram defined by the v_i's, and whose orientation depends on the ordering of the v_i's themselves). We can compute the initial velocity $\dot\Phi_0$ of $\{\Phi_t\}_{t\in[0,1]}$ by noticing that $\Phi_t\big(F_u(\omega, r)\big) = r\,(1 + (u + t\,\varphi)^2)^{-1/2}\,(\omega + (u + t\,\varphi)\,\nu_H)$, so that,

$$\begin{aligned}\dot\Phi_0(F_u) &= \frac{d}{dt}\Big|_{t=0} r\,\frac{\omega + (u + t\,\varphi)\,\nu_H}{\sqrt{1 + (u + t\,\varphi)^2}} = r\,\frac{-u\,\varphi\,\omega + \varphi\,\nu_H}{(1 + u^2)^{3/2}}\\ &= r\,\big(-u\,\varphi\,\omega + \varphi\,\nu_H\big) + r\,\sigma\,\mathrm{O}\big(\psi^2\,(u^2 + w^2)\big).\end{aligned}$$

At the same time

$$\partial_r F_u = \frac{\omega + u\,\nu_H}{\sqrt{1 + u^2}} + r\,\partial_r\Big(\frac{\omega + u\,\nu_H}{\sqrt{1 + u^2}}\Big) = \frac{\omega + u\,\nu_H}{\sqrt{1 + u^2}} - \frac{r\,u\,\partial_r u\,\omega}{(1 + u^2)^{3/2}} + \frac{r\,\partial_r u\,\nu_H}{(1 + u^2)^{3/2}}$$

$$
\begin{aligned}
&= \big(1-(u^2/2)-u\,r\,\partial_r u\big)\,\omega+\big(u+r\,\partial_r u\big)\,\nu_H+\sigma\,\mathrm{O}(u^2+(r\partial_r u)^2)\\
&= A\,\omega+B\,\nu_H+\sigma\,\mathrm{O}(u^2+(r\partial_r u)^2),\\
\frac{\partial_i F_u}{r} &= \partial_i\Big(\frac{\omega+u\,\nu_H}{\sqrt{1+u^2}}\Big)=\frac{\tau_i}{\sqrt{1+u^2}}-\frac{u\,\partial_i u}{(1+u^2)^{3/2}}\,\omega+\frac{\partial_i u}{(1+u^2)^{3/2}}\,\nu_H\\
&= \big(1-(u^2/2)\big)\,\tau_i-u\,\partial_i u\,\omega+\partial_i u\,\nu_H+\sigma\,\mathrm{O}(u^2+(\partial_i u)^2)\\
&= C\,\tau_i+E_i\,\omega+F_i\,\nu_H+\sigma\,\mathrm{O}(u^2+(\partial_i u)^2)
\end{aligned}
$$

so that, with $\Xi=\wedge_{i=1}^{n-1}\tau_i$, $\hat\tau_i=\wedge_{j\neq i}\tau_j$, and $P(u)^2=u^2+|\nabla^{\Sigma_H}u|^2+(r\partial_r u)^2$,

$$
\begin{aligned}
\frac{\partial_r F_u\wedge\wedge_{i=1}^{n-1}\partial_i F_u}{r^{n-1}} &= \big(A\,\omega+B\,\nu_H\big)\wedge\wedge_{i=1}^{n-1}\big(C\,\tau_i+E_i\,\omega+F_i\,\nu_H\big)+\sigma\mathrm{O}\big(P(u)^2\big)\\
&= A\,C^{n-1}\,\omega\wedge\Xi+B\,C^{n-1}\,\nu_H\wedge\Xi+G_i\,\big(\omega\wedge\nu_H\wedge\hat\tau_i\big)+\sigma\mathrm{O}\big(P(u)^2\big),
\end{aligned}
$$

for a coefficient G_i which we do not need to compute. Indeed, $\star(\omega\wedge\nu_H\wedge\hat\tau_i)$, being parallel to τ_i, is orthogonal to ω and ν_H, so that

$$
\begin{aligned}
& r^{-n}\,\dot\Phi_0\big(F_u(r,\omega)\big)\cdot\star\big(\partial_r F_u\wedge\wedge_{i=1}^{n-1}\partial_i F_u\big)\\
&\quad=\big[\big(-u\,\varphi\,\omega+\varphi\,\nu_H\big)+\sigma\,\mathrm{O}\big(\psi^2(u^2+v^2)\big)\big]\cdot\\
&\qquad\qquad\cdot\big[A\,C^{n-1}\,\nu_H-B\,C^{n-1}\,\omega+\sigma\,\mathrm{O}\big(P(u)^2\big)\big]\\
&=C^{n-1}\Big[\Big(1-\frac{u^2}{2}-u\,r\,\partial_r u\Big)\,\varphi+(u+r\,\partial_r u)\,u\,\varphi\Big]+\sigma\,\mathrm{O}\big(\psi^2\,(w^2+P(u)^2)\big)\\
&=\varphi+\sigma\,\mathrm{O}\big(\psi^2\,(w^2+P(u)^2)\big)
\end{aligned}
$$

In particular, since $|h|\le\Lambda$,

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\mathcal{H}^n(\Sigma(u+t\varphi,r_1,r_2))=\int_{\Sigma\times(r_1,r_2)}h\,\dot\Phi_0\big(F_u\big)\cdot\star\big(\partial_r F_u\wedge\wedge_{i=1}^{n-1}\partial_i F_u\big)\\
&=\int_{\Sigma_H\times(r_1,r_2)}r^n\,\psi^2 w\,h+\sigma\,\Lambda r_2\mathrm{O}\big(Q_\psi(u)+Q_\psi(w)+Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\big).
\end{aligned}
$$

Plugging this estimate into (A.8), and by $\max\{1,\Lambda\,r_2\}\,\sigma\le\sigma_0$, we find (A.7).

Step two: We prove that

$$
Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\le Q_\psi(|u|,\Lambda\,r)+C(n)\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)\big).\tag{A.9}
$$

By $Q_r(\psi\,\partial_r u,\psi' u)\le Q_{r\,\psi}(\partial_r u)/4+C\,Q_{r\,\psi'}(u)$ and by $(\text{A.7})_{w=u}$ we find

$$
\begin{aligned}
&Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\le Q_\psi(|u|,\Lambda\,r)+C(n)\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)\big)\\
&\quad+C(n)\,\sigma_0\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)+Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\big).
\end{aligned}
$$

which implies (A.9) provided σ_0 is small enough.

Step three: We prove that, if $w:\Sigma_H\times(r_1,r_2)\to\mathbb{R}$ is slice-wise orthogonal to $u-w$, in the sense that $\int_{\Sigma_H}w_r\,(u_r-w_r)=0$, $\int_{\Sigma_H}\partial_r w_r\,(\partial_r u_r-\partial_r w_r)=0$, and $\int_{\Sigma_H}\nabla^{\Sigma_H}w_r\cdot(\nabla^{\Sigma_H}u_r-\nabla^{\Sigma_H}w_r)=0$ for every $r\in(r_1,r_2)$, then

$$
|T_\psi(u,w)|\le Q_\psi(|w|,\Lambda\,r)+C(n)\,\sigma_0\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)+Q_\psi(|u|,\Lambda\,r)\big).\tag{A.10}
$$

Indeed, by slice-wise orthogonality, we find that $Q_\zeta(w) \le Q_\zeta(u)$, $Q_\zeta(\partial_r w) \le Q_\zeta(\partial_r u)$ and $Q_\zeta(\nabla^{\Sigma_H} w) \le Q_\zeta(\nabla^{\Sigma_H} u)$ whenever $\zeta : (r_1, r_2) \to \mathbb{R}$ is radial. Therefore (A.7) gives $|T_\psi(u, w)| \le Q_\psi(|w|, \Lambda\, r) + C(n)\,\sigma_0\, R_\psi(u)$, with $R_\psi(u) = Q_\psi(u) + Q_\psi(\nabla^{\Sigma_H} u) + Q_{r\,\psi}(\partial_r u) + Q_{r\,\psi'}(u)$. Combining this with (A.9) we get (A.10).

Step four: We prove (A.1). Let now ψ be a cut-off function between (r_3, r_4) and (r_1, r_2), so that with $Z_\psi(u) = Q_\psi(\nabla^{\Sigma_H} u) + Q_\psi(u) + Q_{r\,\psi}(\partial_r u)$,

$$\Big|\int_{\Sigma_H \times (r_3, r_4)} r^{n-1}\big\{|\nabla^{\Sigma_H} u|^2 - (n-1)\,u^2 + (r\,\partial_r u)^2\big\}\Big| \le Z_\psi(u).$$

If $A(u) = \mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))$, then by (A.3) with (r_3, r_4) in place of (r_1, r_2), we find

$$\begin{aligned}|A(u)| &\le Z_\psi(u) + C(n)\,\sigma\, Z_\psi(u) \le Q_\psi(|u|, \Lambda\, r) + C(n)\left(Q_\psi(u) + Q_{r\,\psi'}(u)\right)\\ &\quad + C(n)\,\sigma\left\{Q_\psi(u) + Q_\psi(|u|, \Lambda\, r) + C(n)\left(Q_\psi(u) + Q_{r\,\psi'}(u)\right)\right\},\end{aligned}$$

where in the last inequality we have used (A.9). We deduce

$$|A(u)| \le C(n)\left(Q_\psi(|u|, \Lambda\, r) + Q_\psi(u) + Q_{r\,\psi'}(u)\right),$$

and (A.1) follows (with $C_0 = C_0(n, \eta_0, \eta)$ by the properties of ψ).

Step five: We finally prove that, if $E^0_{\Sigma_H}[u_{r_*}] = 0$ for some $r_* \in (r_1, r_2)$, then (A.2) holds, that is

$$\int_{\Sigma_H \times (r_3, r_4)} r^{n-1}\, u^2 \le C(n)\,\Lambda\, r_2\,(r_2^n - r_1^n) + C(n, \eta_0, \eta) \int_{\Sigma \times (r_1, r_2)} r^{n-1}\,(r\,\partial_r u)^2 . \quad \text{(A.11)}$$

Define $u^+, u^-, u^0 : \Sigma_H \times (r_1, r_2) \to \mathbb{R}$ by setting, for $r \in (r_1, r_2)$, $(u^+)_r = E^+_{\Sigma_H}[u_r]$, $(u^-)_r = E^-_{\Sigma_H}[u_r]$ and $(u^0)_r = E^0_{\Sigma_H}[u_r]$, where $E^\pm_{\Sigma_H}$ denote the $L^2(\Sigma_H)$-orthogonal projections on the spaces of positive/negative eigenvectors of the Jacobi operator of Σ_H, and where $E^0_{\Sigma_H}$ is the $L^2(\Sigma_H)$-orthogonal projection onto the space of the Jacobi fields of Σ_H. Since $(u^0)_{r_*} = 0$, we can directly apply (A.6) with $v = u^0$ and deduce that

$$\int_{\Sigma_H \times (r_1, r_2)} r^{n-1}\,(u^0)^2 \le C(n, \eta_0) \int_{\Sigma_H \times (r_1, r_2)} r^{n-1}\,(r\,\partial_r u^0)^2 . \quad \text{(A.12)}$$

By the orthogonality relations between u^0_r, u^+_r and u^-_r we have that

$$\int_{\Sigma_H \times (r_3, r_4)} r^{n-1}\, u^2 = \int_{\Sigma_H \times (a, b)} r^{n-1}\left((u^0)^2 + (u^+)^2 + (u^-)^2\right) \quad \text{(A.13)}$$

$$\int_{\Sigma_H \times (r_1, r_2)} r^{n+1}\,(\partial_r u)^2 = \int_{\Sigma_H \times (r_1, r_2)} r^{n+1}\left((\partial_r u^0)^2 + (\partial_r u^+)^2 + (\partial_r u^-)^2\right). \quad \text{(A.14)}$$

By the spectral theorem, for every $r \in (r_1, r_2)$ we have $C_1(n)^{-1} \int_{\Sigma_H} (u^-)^2_r \le \int_{\Sigma_H} (n-1)\,(u^-)^2_r - |\nabla^{\Sigma_H}(u^-)_r|^2$, which, multiplied by $r^{n-1}\,\psi^2$, gives

$$C_1(n)^{-1}\, Q_\psi(u^-) \le (n-1)\, Q_\psi(u^-) - Q_\psi(\nabla^{\Sigma_H} u^-)$$

$$= (n-1)\, Q_\psi(u^-, u) - Q_\psi(\nabla^{\Sigma_H} u^-, \nabla^{\Sigma_H} u)$$
$$= -T_\psi(u^-, u) + Q_r(\partial_r u, \partial_r(\psi^2 u^-)),$$

where in the second to last identity we have used that $w = u^-$ is slice-wise orthogonal to $w - u$; in particular, by (A.10) with $w = u^-$, we find

$$C_1(n)^{-1}\, Q_\psi(u^-) \leq Q_\psi(|u^-|, \Lambda\, r) + Q_r(\partial_r u, \partial_r(\psi^2 u^-)) \\ +C(n)\, \sigma_0 \big(Q_\psi(u) + Q_{r\,\psi'}(u) + Q_\psi(|u|, \Lambda\, r)\big). \tag{A.15}$$

Again by slice-wise orthogonality of $w = u^-$ to $w - u$, we have

$$Q_r(\partial_r u, \partial_r(\psi^2 u^-)) = Q_r(\partial_r u^-, \partial_r(\psi^2 u^-)) = Q_{r\,\psi}(\partial_r u^-) \\ +2\, Q_r(\psi'\, \partial_r u^-, \psi\, u^-) \leq Q_{r\,\psi}(\partial_r u^-) + \frac{Q_\psi(u^-)}{2\, C_1(n)} + C(n)\, Q_{r\,\psi'}(\partial_r u^-),$$

which combined into (A.15) gives

$$(2\, C_1(n))^{-1}\, Q_\psi(u^-) \leq Q_\psi(|u^-|, \Lambda\, r) + Q_{r\,\psi}(\partial_r u^-) + C(n)\, Q_{r\,\psi'}(\partial_r u^-) \\ +C(n)\, \sigma_0 \big(Q_\psi(u) + Q_{r\,\psi'}(u) + Q_\psi(|u|, \Lambda, r)\big).$$

Using Hölder inequality again we have

$$Q_\psi(|u^-|, \Lambda\, r) \leq \frac{Q_\psi(u^-)}{4\, C_1(n)} + C(n)\, \Lambda\, r_2\, (r_2^n - r_1^n),$$
$$Q_\psi(|u|, \Lambda\, r) \leq 2\, Q_\psi(u) + C(n)\, \Lambda\, r_2\, (r_2^n - r_1^n),$$
$$\text{so that}\ \frac{Q_\psi(u^-)}{4\, C_1(n)} \leq Q_{r\,\psi}(\partial_r u^-) + C(n)\big(Q_{r\,\psi'}(\partial_r u^-) + \Lambda\, r_2\, (r_2^n - r_1^n)\big) \\ +C(n)\, \sigma_0 \big(Q_\psi(u) + Q_{r\,\psi'}(u) + \Lambda\, r_2\, (r_2^n - r_1^n)\big)$$

Taking ψ to be a cut-off function between (r_3, r_4) and (r_1, r_2), we find

$$\int_{\Sigma\times(r_3,r_4)} r^{n-1}\, (u^-)^2 \leq C(n)\, \Lambda\, r_2\, (r_2^n - r_1^n) \\ +C(n, \eta_0, \eta) \left\{ \int_{\Sigma\times(r_1,r_2)} r^{n-1}\, (r\, \partial_r u^-)^2 + \sigma_0 \int_{\Sigma\times(r_1,r_2)} r^{n-1}\, u^2 \right\}. \tag{A.16}$$

By combining (A.12), (A.16), and the analogous estimate to (A.16) for u^+ with (A.13) and (A.14) we find that (A.16) holds with u in place of u^-; this latter estimate, thanks to (A.5), finally gives (A.11).

Appendix B: Spherical and Cylindrical Graphs

We state here for the reader's convenience two technical lemmas concerning spherical and cylindrical graphs. They are both used in the last step of the proof of Theorem 1.6. The elementary proofs are omitted.

Lemma B.1. *(Spherical graphs as cylindrical graphs) There are dimension independent positive constants C and η_0 with the following property. If $n \geq 1$, $H \in \mathcal{H}$ and $u \in \mathcal{X}_\eta(\Sigma_H, r_1, r_2)$ with $\eta < \eta_0$, then we have*

$$\mathbf{D}^{\nu_H}_{(1-C\,\eta^2)\,r_2} \setminus \mathbf{D}^{\nu_H}_{r_1} \subset \mathbf{p}_H\big(\Sigma_H(u, r_1, r_2)\big) \subset \mathbf{D}^{\nu_H}_{r_2} \setminus \mathbf{D}^{\nu_H}_{(1-C\,\eta^2)\,r_1},$$

and there is $g \in C^1(H)$ such that $\sup\big\{|x|^{-1}\,|g(x)|+|\nabla g(x)| : x \in H\big\} \leq C\,\eta$ *and $\Sigma_H(u, r_1, r_2) = \big\{x + g(x)\,\nu_H : x \in \mathbf{p}_H\big(\Sigma_H(u, r_1, r_2)\big)\big\}$. Moreover, if $(\rho_1, \rho_2) \subset ((1+C\,\eta)\,r_1, (1-C\,\eta^2)\,r_2)$, then $\Sigma_H(u, \rho_1, \rho_2) = \big\{x+g(x)\,\nu_H : x \in H\big\} \cap A^{\rho_2}_{\rho_1}$.*

Lemma B.2. *There is $\eta \in (0, 1)$ with the following property. If $H \in \mathcal{H}$, $R > 1$, $f \in C^2(H)$, and $g \in C^1(H)$ are such that*

$$\max\big\{|f(x)|,\ |x|\,|\nabla f(x)|,\ |x|\,|\nabla^2 f(x)| : x \in H, |x| > R\big\} < \eta,$$
$$\max\big\{|x|^{-1}\,|g(x)|, |\nabla g(x)| : x \in H\big\} < \eta,$$

then there is $h \in C^1(G_H(f))$ such that

$$G_H(g) \setminus B_{4\,R} = \big\{z + h(z)\,\nu_f(z) : z \in G_H(f)\big\} \setminus B_{4\,R},$$

where $G_H(f) = \{x + f(x)\,\nu_H : x \in H\}$ and, for $z = x + f(x)\,\nu_H$, we have set $\nu_f(z) = (1 + |\nabla f(x)|^2)^{-1/2}\,(-\nabla f(x) + \nu_H)$.

Appendix C: Obstacles with Zero Isoperimetric Residue

Proposition C.1. *If W is compact and $\mathcal{R}(W) = 0$, then $\psi_W(v) - P(B^{(v)}) \to 0$ as $v \to \infty$ and W is purely $\mathcal{H}^n$-unrectifiable, in the sense that W cannot contain an $\mathcal{H}^n$-rectifiable set of $\mathcal{H}^n$-positive measure. In a partial converse, if W is purely $\mathcal{H}^n$-unrectifiable and $\mathcal{H}^n(W) < \infty$, then $\mathcal{R}(W) = 0$.*

Proof. **Step one:** Let $\mathcal{R}(W) = 0$. Comparing with balls, $\overline{\lim}_{v\to\infty}\,\psi_W(v) - P(B^{(v)}) \leq 0 = \mathcal{R}(W)$. To prove the matching lower bound, we argue by contradiction and consider $E_j \in \mathrm{Min}[\psi_W(v_j)]$ with $v_j \to \infty$ such that

$$\underline{\lim}_{v\to\infty}\ \psi_W(v) - P(B^{(v)}) = \lim_{j\to\infty} P(E_j; \Omega) - P(B^{(v_j)}) < 0\,. \tag{C.1}$$

With (C.1) replacing $\mathcal{R}(W) > 0$, one can repeat *verbatim* step two-(a) of the proof of Theorem 1.1; we thus derive the asymptotic expansion for F as in step two-(c), which is then the key fact used in step three to derive that $\lim_{j\to\infty} P(E_j; \Omega) - P(B^{(v_j)}) \geq -\mathrm{res}_W(F \cup W, v) \geq -\mathcal{R}(W)$; the latter inequality is of course in contradiction with (C.1) if $\mathcal{R}(W) = 0$. Next, arguing again by contradiction, we assume the existence of an $\mathcal{H}^n$-rectifiable set S with $\mathcal{H}^n(W \cap S) > 0$. By [34, Lemma 11.1], without loss of generality, S is a C^1-embedded hypersurface in $\mathbb{R}^{n+1}$. Let x be a point of tangential differentiability for $W \cap S$, so that $\mathcal{H}^n(W \cap S \cap B_\rho(x)) = \omega_n\,\rho^n + \mathrm{o}_x(\rho^n)$ as $\rho \to 0^+$. Since S is a C^1-embedded hypersurface, there is $\nu \in \mathbb{S}^n$ such that for every $\varepsilon > 0$ there is $\rho_* = \rho_*(x, \varepsilon) > 0$ with $S \cap \mathbf{C}^\nu_{\rho_*,\rho_*}(x) = \{y + g(y)\,\nu : y \in \mathbf{D}^\nu_{\rho_*}(x)\}$, where $g \in C^1(x + \nu^\perp)$ with $g(x) = 0$

and $\mathrm{Lip}(in) \le \varepsilon$. Denoting that $G(g) = \{y + g(y)\,\nu : y \in (x + \nu^\perp)\}$, and up to a decrease ρ_*, we can get that

$$\mathcal{H}^n\big(G(g) \cap W \cap \mathbf{C}^\nu_{\rho_*}(x)\big) \ge \mathcal{H}^n(W \cap S \cap B_{\rho_*}(x)) \ge (1-\varepsilon)\,\omega_n\,\rho_*^n. \quad \text{(C.2)}$$

Since $|g| \le \varepsilon\,\rho_*$ on $\partial\mathbf{D}^\nu_{\rho_*}(x)$, we can define $f : (x + \nu^\perp) \to \mathbb{R}$ so that $f = g$ on $\mathbf{D}^\nu_{\rho_*}(x)$, $f = 0$ on $(x + \nu^\perp) \setminus \mathbf{D}^\nu_{2\,\rho_*}(x)$, and $\mathrm{Lip}(f) \le \varepsilon$. Denoting by F the epigraph of f, we have that $(F, \nu) \in \mathcal{F}$ and we compute, for R large enough to entail that $\mathbf{C}^\nu_{2\,\rho_*}(x) \cup W \subset\subset \mathbf{C}^\nu_R$,

$$\begin{aligned}\omega_n\,R^n - P(F; \mathbf{C}^\nu_R \setminus W) &\ge \omega_n\,(2\rho_*)^n - P(F; \mathbf{C}^\nu_{2\,\rho_*}(x) \setminus W)\\ &= \int_{\mathbf{D}^\nu_{2\,\rho_*}(x)} 1 - \sqrt{1 + |\nabla f|^2} + P(F; \mathbf{C}^\nu_{2\,\rho_*}(x) \cap W)\\ &\ge -\omega_n\,(2\,\rho_*)^n\,\varepsilon^2 + (1-\varepsilon)\,\omega_n\,{\rho_*}^n,\end{aligned}$$

where we have used $f = 0$ on $\nu^\perp \setminus \mathbf{D}^\nu_{2\,\rho_*}(x)$, (C.2) and $\sqrt{1+\varepsilon^2} \le 1 + \varepsilon^2$. Up to taking $\varepsilon < \varepsilon(n)$, we thus find $\mathrm{res}_W(F, \nu) > 0$, and thus deduce $\mathcal{R}(W) > 0$.

Step two: Let W be purely $\mathcal{H}^n$-unrectifiable with $\mathcal{H}^n(W) < \infty$, and let $(F, \nu) \in \mathrm{Max}[\mathcal{R}(W)]$. Since F is a local perimeter minimizer in Ω, F is open in Ω with $\Omega \cap \partial F = \mathrm{cl}\,(\partial^* F)$, where by $\partial^* F$ we mean the reduced boundary of F as a set of locally finite perimeter in Ω. Now, $\omega_n\,R^n - P(F; \mathbf{C}^\nu_R \setminus W)$ is decreasing towards $\mathcal{R}(W) \ge \mathcal{S}(W) \ge 0$, therefore $P(F; \mathbf{C}^\nu_R \setminus W) < \infty$ for every R. In particular, $\mathcal{H}^n \llcorner (\Omega \cap \partial F)$ is a Radon measure on $\mathbb{R}^{n+1}$. Now, $\partial F \subset (\Omega \cap \partial F) \cup W$, so that $\mathcal{H}^n(W) < \infty$ implies that $\mathcal{H}^n \llcorner \partial F$ is a Radon measure on $\mathbb{R}^{n+1}$ and, since F is open, that F is a set of finite perimeter in $\mathbb{R}^{n+1}$ by [17, Theorem 4.5.11]. The pure $\mathcal{H}^n$-unrectifiability of W gives $P(F; \mathbf{C}^\nu_R \setminus W) = P(F; \mathbf{C}^\nu_R)$, where $P(F; \mathbf{C}^\nu_R) \ge \omega_n\,R^n$ by (1.8) and (1.9), and thus $\mathcal{R}(W) = \mathrm{res}_W(F, \nu) \le 0$. This proves $\mathcal{R}(W) = 0$. □

References

1. Allard, W.K., Almgren, F.J., Jr.: On the radial behavior of minimal surfaces and the uniqueness of their tangent cones. *Ann. Math.* **113**(2), 215–265, 1981
2. Alexandrov, A.D.: A characteristic property of spheres. *Ann. Mat. Pura Appl.* **4**(58), 303–315, 1962
3. Allard, W.K.: On the first variation of a varifold. *Ann. Math.* **95**, 417–491, 1972
4. Allard, W.K.: Corrections to a paper of allard and almgren on the uniqueness of tangent cones, 2024
5. Choe, J., Ghomi, M., Ritoré, M.: The relative isoperimetric inequality outside convex domains in R^n. *Calc. Var. Partial Differ. Equ.* **29**(4), 421–429, 2007
6. Chavel, I.: Isoperimetric Inequalities. Cambridge Tracts in Mathematics, vol. 145. Cambridge University Press, Cambridge (2001).**Differential geometric and analytic perspectives.**
7. Cicalese, M., Leonardi, G.P.: A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Rat. Mech. Anal.* **206**(2), 617–643, 2012
8. Cicalese, M., Leonardi, G.P., Maggi, F.: Improved convergence theorems for bubble clusters I. The planar case. *Indiana Univ. Math. J.* **65**(6), 1979–2050, 2016
9. Ciraolo, G., Maggi, F.: On the shape of compact hypersurfaces with almost-constant mean curvature. *Commun. Pure Appl. Math.* **70**(4), 665–716, 2017

10. Cagnetti, F., Perugini, M., Stöger, D.: Rigidity for perimeter inequality under spherical symmetrisation. *Calc. Var. Partial Differ. Equ.* **59**(4), 139, 53, 2020
11. Delgadino, M.G., Maggi, F.: Alexandrov's theorem revisited. *Anal. PDE* **12**(6), 1613–1642, 2019
12. Delgadino, M.G., Maggi, F., Mihaila, C., Neumayer, R.: Bubbling with L^2-almost constant mean curvature and an Alexandrov-type theorem for crystals. *Arch. Ration. Mech. Anal.* **230**(3), 1131–1177, 2018
13. De Philippis, G., Maggi, F.: Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.* **216**(2), 473–568, 2015
14. De Philippis, G., Maggi, F.: Dimensional estimates for singular sets in geometric variational problems with free boundaries. *J. Reine Angew. Math.* **725**, 217–234, 2017
15. Eichmair, M., Metzger, J.: Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions. *Invent. Math.* **194**(3), 591–630, 2013
16. Engelstein, M., Spolaor, L., Velichkov, B.: (Log-)epiperimetric inequality and regularity over smooth cones for almost area-minimizing currents. *Geom. Topol.* **23**(1), 513–540, 2019
17. Federer, H.: Geometric Measure Theory, Volume 153 of Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York Inc., New York (1969)
18. Figalli, A., Fusco, N., Maggi, F., Millot, V., Morini, M.: Isoperimetry and stability properties of balls with respect to nonlocal energies. *Commun. Math. Phys.* **336**(1), 441–507, 2015
19. Finn, R.: Equilibrium Capillary Surfaces, volume 284 of Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York Inc., New York (1986)
20. Figalli, A., Maggi, F.: On the shape of liquid drops and crystals in the small mass regime. *Arch. Rat. Mech. Anal.* **201**, 143–207, 2011
21. Fusco, N., Morini, M.: Total positive curvature and the equality case in the relative isoperimetric inequality outside convex domains. *Calc. Var. Partial Differ. Equ.* **62**(3), 102, 32, 2023
22. Fusco, N., Maggi, F., Pratelli, A.: The sharp quantitative isoperimetric inequality. *Ann. Math.* **168**, 941–980, 2008
23. Figalli, A., Maggi, F., Pratelli, A.: A mass transportation approach to quantitative isoperimetric inequalities. *Inv. Math.* **182**(1), 167–211, 2010
24. Grüter, M., Jost, J.: Allard type regularity results for varifolds with free boundaries. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **13**(1), 129–169, 1986
25. Grüter, M.: Boundary regularity for solutions of a partitioning problem. *Arch. Rat. Mech. Anal.* **97**, 261–270, 1987
26. Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **17**, 43–77, 1963
27. Maggi, F.: Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory, volume 135 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2012)
28. Maz'ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations. Grundlehren der mathematischen Wissenschaften, vol. 342. Springer, Heidelberg (2011)
29. Maggi, F., Ponsiglione, M., Pratelli, A.: Quantitative stability in the isodiametric inequality via the isoperimetric inequality. *Trans. Amer. Math. Soc.* **366**(3), 1141–1160, 2014
30. Naber, A., Valtorta, D.: The singular structure and regularity of stationary varifolds. *J. Eur. Math. Soc. (JEMS)* **22**(10), 3305–3382, 2020
31. Pérez, J., Ros, A.: Properly embedded minimal surfaces with finite total curvature. In The Global Theory of Minimal Surfaces in Flat Spaces (Martina Franca, 1999), volume 1775 of Lecture Notes in Math., pp. 15–66. Springer, Berlin (2002)
32. Schoen, R.M.: Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Differ. Geom.* **18**(4), 791–809, 1984 **1983**

33. Simon, L.: Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. *Ann. Math.* **118**(3), 525–571, 1983
34. Simon, L.: Lectures on geometric measure theory. Proceedings of the Centre for Mathematical Analysis, vol. 3. Australian National University, Centre for Mathematical Analysis, Canberra (1983)
35. Simon, L.: Isolated singularities of extrema of geometric variational problems. In Harmonic mappings and minimal immersions (Montecatini, 1984), volume 1161 of Lecture Notes in Math., pp. 206–277. Springer, Berlin, 1985
36. Simon, L.: Asymptotic behaviour of minimal graphs over exterior domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4**(3), 231–242, 1987
37. Simon, L.: Theorems on regularity and singularity of energy minimizing maps. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Based on lecture notes by Norbert Hungerbühler

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