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Enhanced Dissipation for Two-Dimensional Hamiltonian Flows

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Abstract

Let $H \in C^1 \cap W^{2,p}$ be an autonomous, non-constant Hamiltonian on a compact 2-dimensional manifold, generating an incompressible velocity field $b = \nabla^\perp H$. We give sharp upper bounds on the enhanced dissipation rate of b in terms of the properties of the period $T(h)$ of the closed orbit $\{H = h\}$. Specifically, if $0 < \nu \ll 1$ is the diffusion coefficient, the enhanced dissipation rate can be at most $O(\nu^{1/3})$ in general, the bound improves when H has isolated, non-degenerate elliptic points. Our result provides the better bound $O(\nu^{1/2})$ for the standard cellular flow given by $H_{\mathsf{c}}(x) = \sin x_1 \sin x_2$, for which we can also prove a new upper bound on its mixing rate and a lower bound on its enhanced dissipation rate. The proofs are based on the use of action-angle coordinates and on the existence of a good invariant domain for the regular Lagrangian flow generated by b.

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Contents

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1. Advection–Diffusion Equations

Let (M, g) be a compact 2-dimensional manifold, possibly with boundary, and consider an autonomous, non-constant Hamiltonian H which generates a velocity field $b := \nabla^{\perp} H$, tangent to the boundary whenever $\partial M \neq \emptyset$. We are interested in the long-time dynamics of the scalar function $\rho^{\nu} : [0, \infty) \times M \to \mathbb{R}$ subject to the advection–diffusion equation

$$\begin{cases} \partial_t \rho^{\nu} + b \cdot \nabla \rho^{\nu} = \nu \Delta \rho^{\nu}, \\ \rho^{\nu}(0, \cdot) = \rho_0. \end{cases} \tag{A-D}$$

Here, $\rho_0 : M \to \mathbb{R}$ is an assigned mean-free initial datum and $\nu > 0$ is the diffusivity parameter. When $\partial M \neq \emptyset$, we prescribe homogeneous Neumann conditions $\partial_n \rho^{\nu} = 0$, where n is the outer normal to the boundary.

In the non-diffusive case, i.e. when $\nu = 0$, (A-D) reduces to the standard transport equation

$$\begin{cases} \partial_t \rho + b \cdot \nabla \rho = 0, \\ \rho(0, \cdot) = \rho_0. \end{cases} \tag{T}$$

The goal of this article is to study the mixing and diffusive properties of (A-D) and (T) in terms of sharp decay rates for ρ^{ν} and ρ, under general assumptions on the Hamiltonian H.

1.1. Mixing and Enhanced Dissipation

Enhanced dissipation typically refers to the accelerated decay of solutions to (A-D) due to the interaction of transport and diffusion. We are interested in putting on sound mathematical grounds the following statement from [29]:

> The homogenization of a passive tracer in a flow with closed mean streamlines occurs in two stages: first, a rapid phase dominated by shear-augmented diffusion over a time $\sim \nu^{-1/3}$, in which initial values of the tracer are replaced by their (generalized) average about a streamline; second, a slow phase requiring the full diffusion time $\sim \nu^{-1}$.

The above statement can be interpreted in terms of the behavior of the L^2 norm of the solution ρ^{ν} to (A-D). In view of the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho^{\nu}\|^2 + 2\nu \|\nabla \rho^{\nu}\|^2 = 0, \qquad \forall t \geq 0,$$

all mean-free solutions decay exponentially to zero as $\mathrm{e}^{-c_{\mathsf{p}}\nu t}$, where $c_{\mathsf{p}} > 0$ is related to the Poincaré constant. Hence, the natural diffusive time-scale $O(\nu^{-1})$ appears trivially, and no role is played by the velocity field b. Now, the above-mentioned *slow phase* refers to such diffusive behavior for the average of ρ^ν on the streamlines of the Hamiltonian H: indeed, if ρ^ν were constant on the streamlines, it would then follow that $b \cdot \nabla \rho^\nu = 0$, implying that the diffusive behavior is the only possible one. On the contrary, the rest of the solution is conjectured to undergo the *rapid phase*, in which decay happens on a much faster time-scale $O(\nu^{-1/3})$. These considerations are at the heart of the concept of *enhanced dissipation*, formalized in the definition below.

Definition 1.1. Let $\nu_0 \in (0, 1)$ and $\lambda : (0, \nu_0) \to (0, 1)$ be a continuous increasing function such that

$$\lim_{\nu \to 0} \frac{\nu}{\lambda(\nu)} = 0.$$

The velocity field b is *dissipation enhancing* at rate $\lambda(\nu)$ if there exists $A \geq 1$ only depending on b such that if $\nu \in (0, \nu_0)$ then for every $\rho_0 \in L^2$ with zero streamlines-average we have the enhanced dissipation estimate

$$\|\rho^\nu(t)\|_{L^2} \leq A\mathrm{e}^{-\lambda(\nu)t}\|\rho_0\|_{L^2},$$

for every $t \geq 0$.

While the above concept has been more or less informally studied in the physics literature since the late nineteenth century, it has received much attention by the mathematical community only recently, starting with the seminal article [12]. In this work, enhanced dissipation has been proven to be equivalent to the non-existence of non-trivial H^1-eigenfunctions of the transport operator $b \cdot \nabla$: in particular, functions that are constant on streamlines are eigenfunctions and hence have to be excluded when studying enhanced dissipation in the two-dimensional, autonomous setting. From a quantitive point of view, the picture is now quite clear in the context of shear flows [2,4,6,11,15,16,32] and radial flows [14,15]. For more general velocity fields, there are only some results linking mixing rates and enhanced dissipation time-scales [13,20], and others that study the interplay between regularity and dissipation [9]. We also mention the interesting work [31], which deals with enhanced dissipation for an averaged equation stemming from general hamiltonians. However, a precise quantitative picture is still missing.

The goal of this article is to analyze enhanced dissipation in the case of velocity fields originating from general (regular) Hamiltonians. According to Definition 1.1, the case $\lambda(\nu) = c_{\mathsf{p}}\nu^{1/3}$ is precisely the one described in [29]. One of our main results is that the exponent $1/3$ is the best possible in the autonomous setting.

Theorem 1. *Let $p \geq 2$ and $H \in C^1 \cap W^{2,p}(M)$ be such that $b := \nabla^\perp H$ is dissipation enhancing with rate $\lambda(\nu)$. Then*

$$\lambda(\nu) \leq C_0 \nu^{1/3}, \quad \textit{for every } \nu \in (0, 1), \tag{1.1}$$

for some positive constant $C_0 = C_0(H)$.

In the language of [29], we prove that the rapid phase cannot happen before a time-scale $O(\nu^{-1/3})$. The exponent 1/3 is not always the correct one, at least in the sense of Definition 1.1. It is achieved, for instance, by the shear flow $b = (x_2, 0)$ on $\mathbb{S}^1 \times [0, 1]$, namely the Couette flow [24]. However, it is well-known that the presence of critical points can slow down the dissipation [6]. This particular exponent is due to dimensionality, regularity and the autonomous nature of our problem. Indeed:

- Contact Anosov flows on smooth odd dimensional connected compact Riemannian manifolds have an enhanced dissipation time-scale $O(|\ln \nu|^2)$, see [13];
- There exist Hölder continuous shear flows on the two-dimensional torus with an enhanced dissipation time-scale $O(\nu^{-\gamma})$, for any $\gamma \in (0, 1)$, see [11,32]
- Non-autonomous velocity fields generated as solutions to the stochastic Navier–Stokes equations on the two-dimensional torus have an enhanced dissipation time-scale $O(|\ln \nu|)$, see [5].

The proof of Theorem 1 is based on the following key observation: any Hamiltonian $H \in C^1 \cap W^{2,p}(M)$ has an invariant domain Ω where the gradient of the flow map grows at most linearly in time. It follows that

$$\|\rho^\nu(t) - \rho(t)\|_{L^2}^2 \lesssim \nu t^3 \qquad \forall t \geq 1\,, \tag{1.2}$$

provided ρ_0 is concentrated in Ω. In view of the inviscid conservation $\|\rho(t)\|_{L^2} = \|\rho_0\|_{L^2}$, estimate (1.2) provides the desired upper bound on $\lambda(\nu)$ by choosing $t \sim \nu^{-1/3}$.

As mentioned earlier, the enhanced dissipation properties of b are closely linked to its mixing features, as defined below.

Definition 1.2. Let $\gamma : [0, +\infty) \to [0, +\infty)$ be a continuous and decreasing function vanishing at infinity. The velocity field b is *mixing* with rate $\gamma(t)$ if for every $\rho_0 \in H^1$ with zero streamlines-average we have the following estimate

$$\|\rho(t)\|_{H^{-1}} \leq \gamma(t)\|\rho_0\|_{H^1}\,, \tag{1.3}$$

for every $t \geq 0$.

Due to the conservation of the L^2 norm in the transport equation (T), the mixing estimate (1.3) implies that the H^1 norm of ρ has to grow at least as $1/\gamma(t)$. This has been already observed in the case of shear flow with critical points [6], and we will provide another example when dealing with the standard cellular flow below (see Sect. 1.3).

1.2. Elliptic Points

In the presence of elliptic points, mixing rates can be slower and in turn affect the enhanced dissipation time-scale. This is explicit in the case of shear flows $b = (v(x_2), 0)$ on $\mathbb{T}^2$, where the crucial role is played by the order of vanishing of derivatives of v at critical points [6]. The case of a velocity field generated by

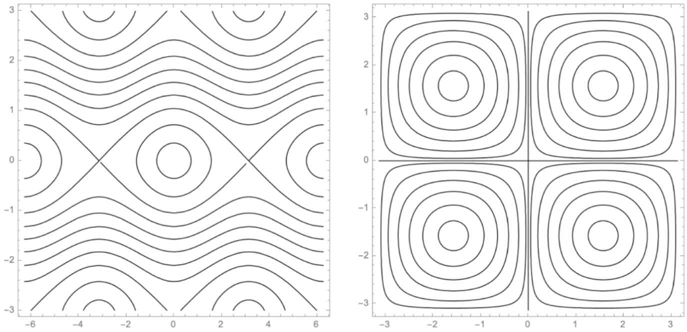

Fig. 1. **A** A general hamiltonian on a periodic domain. **B** The standard cellular flow

a Hamiltonian is in general much more complicated, as level sets of H are not, as for shear flows, simply horizontal lines (see Fig. 1A).

By compactness, H always admits a minimum point in $p \in M$. To avoid degeneracies, we assume that $p \notin \partial M$ is an isolated elliptic point, i.e. $\nabla^2 H(p)$ is positive definite. To study the system around $p \in M$, we use local coordinates such that $p = (0, 0)$.

Let $r_0 > 0$ be fixed (possibly small), and let $r \in (0, r_0)$. Consider the unique closed orbit $\ell_r = \{x \in M : H(x) = H(re_1)\} \subset M$ containing the point $re_1 = (r, 0)$, and denote by $T(r)$ its period, which is a C^1 function since $r \to H(re_1)$ is C^1 and T is a C^1 function of H. By using that $H(re_1) \sim r^2$, $\|b\|_{L^\infty(\ell_r)} \lesssim r$, and (2.4) we immediately get $T(0) > 0$.

Theorem 2. *Let $H \in C^3(M)$ have an isolated, non-degenerate elliptic point $p \in M \setminus \partial M$. Assume that $T'(r) \sim r^\beta$ for some $\beta \geq 0$. Assume that $b := \nabla^\perp H$ is mixing (resp. dissipation enhancing) with rate $\gamma(t)$ (resp. $\lambda(\nu)$). Then*

$$\gamma(t) \geq \frac{C_0}{\langle t\rangle^{\frac{2}{\beta+1}}} \quad \text{and} \quad \lambda(\nu) \leq C_0 \nu^{\frac{1+\beta}{3+\beta}}, \quad \text{for every } \nu \in (0, 1),$$

for some positive constant $C_0 = C_0(H)$.

These estimates are obtained by studying the dynamics around the elliptic point of H. This local analysis improves the available bound on the mixing rate for Hamiltonian flows when $\beta > 1$ (see [7]), as well as the bound on the dissipation enhancing rate as soon as $\beta > 0$, compare with Theorem 1.

The cellular flow $H_{\mathsf{c}}(x) = \sin x_1 \sin x_2$ satisfies the assumption of the theorem with $\beta = 1$.

Remark 1.3. The heuristic built on the case of shear flows, which allows to deduce the dissipation enhancing rate $\lambda(\nu) \sim \nu^{\frac{1}{3}}$ from the mixing rate $\gamma(t) \sim t^{-1}$ does not match with the previous result: this difference can be attributed to the different geometry of the level sets of H in a shear flow and around an elliptic point.

1.3. Cellular Flows

Cellular flows (along with shear and radial flows) are perhaps the most studied two-dimensional flows, especially from the point of view of fluid dynamics, homogenization and as random perturbations of dynamical systems [10,18,21,25,28,29]. Strictly related to the idea of dissipation enhancement, there has been a number of articles [19,22,23] dealing with suitable rescalings of cellular flows (which create small scales, and hence roughness in the velocity field) and proving that the dissipation time can be made arbitrarily small by taking the rescaling parameter small.

On the contrary, we here consider the standard cellular flow $H_{\mathsf{c}}(x) = \sin x_1 \, \sin x_2$ on $\mathbb{T}^2$, as depicted in Fig. 1b, and prove a direct estimate on the mixing and enhanced dissipation rates.

Theorem 3. *Consider the standard cellular flow* $b_{\mathsf{c}} = \nabla^{\perp} H_{\mathsf{c}}$ *with* $H_{\mathsf{c}}(x) = \sin x_1 \, \sin x_2$ *on* $\mathbb{T}^2$*. Then for every* $\varepsilon > 0$ *the vector field* b_{c} *is mixing with rate*

$$\frac{1}{\langle t \rangle} \lesssim \gamma_{\mathsf{c}}(t) \lesssim_{\varepsilon} \frac{1}{\langle t \rangle^{\frac{1}{3}-\varepsilon}}, \tag{1.4}$$

and dissipation enhancing with rate

$$\lambda_{\mathsf{c}}(\nu) \lesssim \nu^{1/2}. \tag{1.5}$$

We stress that in the statement above $\lesssim$ and $\lesssim_{\varepsilon}$ denote that the inequalities hold up to a universal constant and up to a constant depending only on ε respectively.

The lower bound on the mixing rate and the upper bound on the enhanced dissipation rate are a consequence of Theorem 2. As we shall see, our approach is based on a careful study of the action-angle coordinates near the elliptic and hyperbolic points of H_{c}, *without* making use of the well-known estimates of the coefficient of effective diffusivity of H_{c} from homogenization theory [10,18,25]. We stress that the rates (1.4) and (1.5) are *global* in nature, and do not distinguish where the initial datum is supported. In fact (see Section 4.4), the upper bound in (1.4) comes from data that are supported near the hyperbolic points of H_{c}, as these are the hardest to control with our methods. Nonetheless, Proposition 4.6 below provides an algebraic bound on the $W^{1,1}$ norm of the associated Lagrangian flow and implies that no datum can mix faster than algebraic. An extensive study of such $W^{1,1}$ bounds for Hamiltonian flows in the presence of hyperbolic points will be the subject of a forthcoming work.

Near elliptic points, the solution is mixed at the faster rate $\langle t \rangle^{-1+\varepsilon}$, where $\varepsilon > 0$ can be taken arbitrarily small, while away from both elliptic and hyperbolic points, the mixing rate improves to $\langle t \rangle^{-1}$. In particular, the lower bound in (1.4) is sharp.

A natural question concerns lower bounds for the enhanced dissipation rate in (1.5). Our Definition 1.1 requires initial data that have zero-average over streamlines. While this property that is preserved by the viscous evolution in the setting of shear and radial flows, for the complicated geometry of streamlines of cellular flows it is no longer the case. In particular, [13, Theorem 2.1] does not apply directly. However, we can prove that the system loses a fraction of its initial energy at a time-scale shorter than the diffusive one.

Theorem 4. *Consider the standard cellular flow* $b_{\mathsf{C}} = \nabla^{\perp} H_{\mathsf{C}}$ *with* $H_{\mathsf{C}}(x) = \sin x_1 \sin x_2$ *on* $\mathbb{T}^2$ *and let* ρ^ν *the solution of (A-D). Then for every* $q \in \left(\frac{24}{25}, 1\right)$ *there is* $\nu_0 = \nu_0(q) > 0$ *such that*

$$\|\rho^\nu(\nu^{-q})\|_{L^2}^2 \leq \frac{7}{8}\|\rho_0\|_{L^2}^2 \tag{1.6}$$

for every $\nu \in (0, \nu_0)$ *and every* $\rho_0 \in L^2(\mathbb{T}^2)$ *with zero-average over streamlines. Further, if* ρ_0 *is supported away from the hyperbolic points, then we can take any* $q \in \left(\frac{2}{3}, 1\right)$.

In the case of shear flows, the estimate (1.6) can be iterated to produce exponential decay at a rate proportional to ν^q. For cellular flows, this procedure cannot be implemented since ρ^ν at $t = \nu^{-q}$ cannot be guaranteed to have zero-average over streamlines. In fact, we cannot expect that the whole L^2 norm decays at an enhanced rate, since streamline average can be dynamically created. At this stage, it is even unclear that one can obtain enhanced dissipation by simply removing the streamline average from the whole solution.

When the initial data are supported away from hyperbolic points (say, in one of the cells), then they are mixed at the faster rate $\langle t\rangle^{-1+\varepsilon}$, yielding the improvement in the range of q above (see Remark 4.8). However, due to diffusion, the support of the solution will immediately reach the hyperbolic point, so that (again) the estimate cannot be iterated.

2. Gradient Estimates for Transport Equations

In this section, we focus on the transport equation (T) and its associated flow $X : \mathbb{R} \times M \to M$ defined as the solution to the ODE

$$\begin{cases} \partial_t X(t,x) = b(t, X(t,x)), \\ X(0,x) = x. \end{cases} \tag{2.1}$$

Besides some regularity assumptions, which will be specified later, we assume that $H = 0$ on ∂M whenever $\partial M \neq \emptyset$. We will call a set $E \subset M$ *invariant* (under the flow X) if $X(t, E) = E$ for every $t \geq 0$. The key step in the proof of Theorem 1 is the existence of a good invariant set for X.

Proposition 2.1. *Let* $H \in C^1 \cap W^{2,p}(M)$ *for some* $p \geq 1$*. There exists an invariant open set* $\Omega \subset M$ *such that for any* $\rho_0 \in C^1(M)$ *with* $\operatorname{spt}(\rho_0) \subset \Omega$*, the corresponding solution* ρ *of (T) satisfies*

$$\|\nabla\rho(t)\|_{L^p} \leq C(\Omega, H)(1+t)\|\nabla\rho_0\|_{L^\infty}, \tag{2.2}$$

for all $t \geq 0$*.*

Despite the possible presence of hyperbolic points in H, the set Ω is one where only shearing is possible. As a consequence, the growth of $\nabla\rho$ is limited to be linear in time, excluding for instance exponential growth.

2.1. Construction of a Good Invariant Domain

To build a suitable invariant set, the idea is to employ a variation of the classical Morse-Sard lemma [27,30] proven in [8].

Lemma 2.2. *Let $H \in C^1 \cap W^{2,p}(M)$ for some $p \geq 1$. There exist a constant $c_0 > 0$ and an interval $(h_0, h_1) \subset H(M)$ such that $|b(x)| \geq c_0$ for any x in the invariant set $\Omega := H^{-1}((h_0, h_1))$. Moreover, $\Omega \cap \partial M = \emptyset$.*

Proof. Define the set of critical values of H by

$$\mathcal{S} := \{h \in \mathbb{R} : \exists\, x \in M \text{ with } H(x) = h \text{ and } b(x) = 0\},$$

and by $\mathcal{R} := H(M) \setminus \mathcal{S}$ the set of regular values. Since $H \in C^1(M)$ and M is compact, the set $\mathcal{S}$ is closed and therefore $\mathcal{R}$ is open in the range of H. As shown in [8], such H enjoys the Sard property, namely $\mathcal{S}$ has zero 1-dimensional Lebesgue measure on $\mathbb{R}$. The latter can be reduced to a statement on a local chart, hence the fact that M is not a subset of $\mathbb{R}^2$ is irrelevant for the sake of applying [8]. In particular, the sets $\mathcal{R}$ and $\Omega_* := H^{-1}(\mathcal{R})$ are nonempty open subsets of $H(M)$ and M, respectively. It is straightforward to check that the map $c_S : \mathcal{R} \to \mathbb{R}$ defined by

$$c_S(h) = \min_{x \in H^{-1}(h)} |b(x)|$$

is continuous on $\mathcal{R}$. Being H non-constant, there exists $c_0 > 0$ such that $c_S(h) > 2c_0$ for some $h \in \mathcal{R}$. Hence, we can find and interval $(h_0, h_1) \subset \mathcal{R}$ with $h \in (h_0, h_1)$ such that $c_S > c_0$ on (h_0, h_1). We now set $\Omega := H^{-1}((h_0, h_1))$. If $\partial M \cap \Omega = \emptyset$, we are done. Otherwise, since H vanishes on the boundary, $0 \in (h_0, h_1)$. Replacing (h_0, h_1) with smaller interval (h_0', h_1') not containing 0, we can redefine $\Omega = H^{-1}((h_0', h_1'))$. The fact that Ω is invariant follows from its definition, and hence the proof is over. □

Remark 2.3. At this stage, the above lemma holds true for more general Lipschitz Hamiltonians whose gradient is a function of bounded variation, see [8].

2.2. Action-Angle Coordinates for C^2 Hamiltonians

Assume that $H \in C^2(M)$, and let Ω and $c_0 > 0$ as in Lemma 2.2 so that in particular $\nabla H \neq 0$ in Ω. Let $x_0 \in M$ be such that $H(x_0) = h_0$ and denote by Ω_0 the connected component of Ω such that $x_0 \in \partial\Omega_0$. Given $h \in (h_0, h_1)$, we denote the period (relative to the flow map X in (2.1)) of the closed orbit $\{H = h\}$ by $T(h)$, while $x = x(h) : (h_0, h_1) \to M$ stands for the solution to the ODE

$$\begin{cases} x'(h) = \frac{\nabla H}{|\nabla H|^2}(x(h)), \\ x(h_0) = x_0. \end{cases}$$

Using, the flow map X in (2.1), we define the coordinates $\Phi : \mathbb{S}^1 \times (h_0, h_1) \to \Omega$ by

$$\Phi(\theta, h) := X(\theta T(h), x(h)).$$

Here $\mathbb{S}^1 = [0, 1)$. Notice that $H(x(h)) = h$ for any $h \in (h_0, h_1)$. The key properties of Φ are contained in the following proposition.

Lemma 2.4. *The map $\Phi : \mathbb{S}^1 \times (h_0, h_1) \to \Omega_0$ and its inverse are C^1 functions. Moreover, $\Phi^{-1}(x) = (\Psi(x), H(x))$ where $\Psi : \Omega_0 \to \mathbb{S}^1$ is a C^1 function satisfying*

$$\Psi(X(t, x)) = \Psi(x) + \frac{t}{T(H(x))}, \tag{2.3}$$

for any $x \in \Omega$, $t \geq 0$.

Proof. We begin by proving that $T : (h_0, h_1) \to (0, \infty)$ is C^1. It holds

$$\begin{aligned} T(h) = \int_{\{H=h\}\cap\Omega_0} \frac{1}{|b|} \mathrm{d}\mathscr{H}^1 &= \int_{\{h_0 \leq H \leq h\}\cap\Omega_0} \nabla \cdot \left(\frac{\nabla H}{|\nabla H|^2} \right) \mathrm{d}\mathscr{H}^2 + T(h_0) \\ &= \int_{h_0}^{h} \int_{\{H=r\}\cap\Omega_0} \nabla \cdot \left(\frac{\nabla H}{|\nabla H|^2} \right) |\nabla H|^{-1} \mathrm{d}\mathscr{H}^1 \, \mathrm{d}r + T(h_0), \end{aligned} \tag{2.4}$$

where we used the divergence theorem and the coarea formula. Using that $|\nabla H| \geq c_0$ in Ω, it is now immediate to see that $T \in C^1((h_0, h_1))$.

The fact that $b \in C^1(M)$ implies that Φ is C^1-regular. More precisely, pointwise in $(\theta, h) \in \mathbb{S}^1 \times (h_0, h_1)$ we have

$$\partial_h \Phi(\theta, h) = T'(h) \nabla^\perp H(X(\theta T(h), x(h))) + D_x X(\theta T(h), x(h)) \frac{\nabla H}{|\nabla H|^2}(x(h))$$

and

$$\partial_\theta \Phi(\theta, h) = T(h) \nabla^\perp H(\Phi(\theta, h)), \tag{2.5}$$

so that standard regularity estimates for flow maps of C^1 velocity fields imply

$$|\partial_h \Phi(\theta, h)| \leq |T'(h)||b|(\Phi(\theta, h)) + \frac{\mathrm{e}^{T(h)\|H\|_{C^2}}}{|b|(x(h))}$$

and

$$|\partial_\theta \Phi(\theta, h)| \leq T(h)|b|(\Phi(\theta, h)).$$

It is simple to see that Φ is injective and surjective. To prove that the inverse is C^1 we show that $D\Phi$ is invertible at any point. Now, from the identity

$$H(\Phi(\theta, h)) = H(x(h)) = h,$$

and (2.5) we deduce that

$$\begin{cases} \partial_h \Phi \cdot \nabla H(\Phi) = 1\,, \\ \partial_\theta \Phi = T(h) \nabla^\perp H(\Phi), \end{cases} \tag{2.6}$$

for any $\theta \in \mathbb{S}^1$ and $h \in (h_0, h_1)$. Since $(\nabla H, \nabla^\perp H)$ is an orthogonal and non-degenerate frame, the C^1 property of Φ^{-1} follows. Specifically, from (2.6) we find

$$|\det D\Phi(\theta, h)| = T(h), \tag{2.7}$$

and therefore

$$|D\Phi^{-1}|(\Phi(\theta, h)) \leq \frac{|D\Phi|(\theta, h)}{|\det D\Phi(\theta, h)|} \leq \left(\frac{|T'(h)|}{T(h)} + 1\right)|b|(\Phi(\theta, h)) + \frac{1}{T(h)} \frac{\mathrm{e}^{T(h)\|H\|_{C^2}}}{|b|(x(h))}. \tag{2.8}$$

In order to prove (2.3), observe that $x = \Phi(\Psi(x), H(x)) = X(\Psi(x)T(H(x)), x(H(x)))$, and therefore

$$X(t, x) = X(\Psi(x)T(H(x)) + t, x(H(x))), \qquad \forall t \geq 0.$$

On the other hand

$$\begin{aligned} X(t, x) &= \Phi(\Psi(X(t, x)), H(X(t, x))) \\ &= \Phi(\Psi(X(t, x)), H(x)) = X(\Psi(X(t, x))T(H(x)), x(H(x))). \end{aligned}$$

Comparing the two expressions above and recalling that $s \to X(sT(h), h)$ is 1-periodic and injective on $[0, 1)$ we obtain (2.3) and complete the proof. □

With the above lemma at hand, and in particular the explicit formula (2.3), the proof of Proposition 2.1 follows immediately.

Proof of Proposition 2.1 when $H \in C^2$. We denote by $\mathrm{d}_M(\cdot, \cdot)$ the Riemannian distance on M. Invoking (2.3), for any $x, y \in \Omega_0$ and $t \geq 0$ it holds

$$\begin{aligned} &\mathrm{d}_M(X(t, x), X(t, y)) \\ &\quad = \mathrm{d}_M\left((\Phi\left(\Psi(X(t, x))\right), H(X(t, x))), \Phi\left((\Psi(X(t, y))\right), H(X(t, y)))\right) \\ &\quad = \mathrm{d}_M\left(\left(\Phi\left(\Psi(x) + \frac{t}{T(H(x))}\right), H(x)\right), \Phi\left(\left(\Psi(y) + \frac{t}{T(H(y))}\right), H(y)\right)\right) \\ &\quad \lesssim |\Psi(x) - \Psi(y)| + t\frac{|T(H(x)) - T(H(y))|}{T(H(x))T(H(y))} + |H(x) - H(y)| \\ &\quad \lesssim (1 + t)\mathrm{d}(x, y), \end{aligned}$$

where we used that the period T is bounded below away from zero on (h_0, h_1) thanks to (2.4). In particular, if $\rho_0 \in C^1$ is supported in Ω, then $\rho(t, x) = \rho_0(X(-t, x))$ and

$$\|\nabla\rho(\cdot, t)\|_{L^p} \lesssim \|\nabla\rho_0\|_{L^\infty}(1 + t),$$

for any $p \in [1, \infty]$, thereby concluding the proof. □

2.3. Less Regular Hamiltonians

When $H \in C^1 \cap W^{2,p}(M)$ we cannot appeal to action-angle variables to study the flow map X on good invariant domains. To be precise, we cannot even appeal to classical notions of flow since $b \in C^0 \cap W^{1,p}(M)$ is not regular enough.

In this setting, we understand X as the unique *regular Lagrangian flow* (RLF in short) associated to b in the sense of [3,17]. The latter is by definition a measurable map $X : \mathbb{R} \times M \to M$ satisfying the following properties:

(i) $X(t, \cdot)$ conserves the volume measure of M for any $t \geq 0$;
(ii) there exists a negligible set (with respect to the volume measure) $N \subset M$ such that $t \to X(t, x)$ is absolutely continuous for any $x \in M \setminus N$ and solves (2.1).

Under our assumptions $b \in C^0 \cap W^{1,p}$, $\nabla \cdot b = 0$, there exists a unique RLF associated to b. Uniqueness is understood in the following weak sense: if X_1 and X_2 are two RLF, then there exists a negligible set $N \subset M$ such that $X_1(t, x) = X_2(t, x)$ for any $x \in M \backslash N$ and $t \geq 0$.

The crucial point in our analysis is to estimate the rate of separation of trajectories, as in the following proposition.

Proposition 2.5. *[Linear growth of the H^1 norm of the flow]*
Let $b = \nabla^{\perp} H \in C^0 \cap W^{1,p}(M)$ and $c_0 > 0$, $\Omega := H^{-1}((h_0, h_1))$ be as in Lemma 2.2. Then there are two constants $r, C > 0$ and a function $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ such that for every $\bar{z} \in \Omega$ and $z \in B_r(\bar{z})$ and every $t > 0$ it holds

$$\mathrm{d}_M(X(t, z), X(t, \bar{z})) \leq C(1 + t)\Big[\mathrm{d}_M(z, \bar{z}) + |g(H(z)) - g(H(\bar{z}))|\Big],$$

where X is a suitable representative of the unique RLF associated to b. In particular there is $C' = C'(b, c_0, p) > 0$ such that

$$\|X(t, \cdot)\|_{W^{1,p}(\Omega)} \leq C'(1 + t),$$

for every $t > 0$.

The proof of Proposition 2.5 follows from the work [26]. For the reader's convenience we outline the main steps. By combining [26, Lemma 3.2] and [26, Remark 3.3], we get the following result.

Lemma 2.6. *Let $b = \nabla^{\perp} H$, $c_0 > 0$ and $\Omega := H^{-1}((h_0, h_1))$ as in Lemma 2.2. Then there exist a representative of the regular Lagrangian flow X, a function $g \in W^{1,p}_{\mathrm{loc}} \cap C^0(\mathbb{R})$ and $r > 0$ such that for every $t > 0$ the following holds: there exist $c_1, c_2 > 0$ such that $\forall \bar{z} \in \Omega$ and every $z \in B_r(\bar{z})$ there exists $s > 0$ such that*

(1) $\mathrm{d}_M(X(t, \bar{z}), X(s, z)) \leq c_1 |H(\bar{z}) - H(z)|$,
(2) $|t - s| \leq c_2 \left(|g(H(\bar{z})) - g(H(z))| + \mathrm{d}_M(\bar{z}, z)\right)$.

Remark 2.7. The constants c_1, c_2 are explicitly chosen at the end of the proof of [26, Lemma 3.2] as

$$c_1 = 2(c_0^{-1} + 1), \qquad c_2 = \tilde{N}(c_0^{-1} + 1)^2(1 + 2\|b\|_{L^\infty}) + 2(c_0^{-1} + 1),$$

where $\tilde{N} = \left\lceil \frac{t\|b\|_{L^\infty}}{\bar{r}} \right\rceil$ and $\bar{r} > 0$ is the size of a suitable covering of Ω depending only on c_0 and H. In particular c_1 is independent on t and there is $\tilde{c} = \tilde{c}(H, c_0) > 0$ such that $c_2 \leq \tilde{c}(1 + t)$.

Remark 2.8. The function g can be chosen as

$$g(h) = \int_{\{H \le h\}} |Db|(z)\mathrm{d}z,$$

in particular if $b \in W^{1,p}(\Omega)$, then $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$, see [26].

Remark 2.9. The proof of Lemma 2.6 (see [26]) can be interpreted in terms of the action-angle variables in the smooth setting: the time s in the statement of Lemma 2.6 can be chosen as the time needed by the trajectory starting at z to run across the same number of periods as the trajectory starting at $\bar{z}$ at time t and reach the same angular variable of the point $X(t, \bar{z})$.

Proof of Proposition 2.5. The statement immediately follows from Lemma 2.6 and Remark 2.7 by estimating

$$\begin{aligned} \mathrm{d}_M(X(t,z), X(t,\bar{z})) &\le \mathrm{d}_M(X(t,z), X(s,z)) + \mathrm{d}_M(X(s,z), X(t,\bar{z})) \\ &\le \|b\|_{L^\infty}|t-s| + \mathrm{d}_M(X(s,z), X(t,\bar{z})) \end{aligned}$$

and using that H is Lipschitz. □

Proof of Proposition 2.1 in the general case. We appeal to the following fundamental property of RLF [17]:

$$\rho(t,x) = \rho_0(X(-t,x)), \quad \text{for a.e. } x \in M .$$

In particular, the regularity of X proved in Proposition 2.5 immediately implies Proposition 2.1. □

3. Mixing, Dissipation, and the Role of Elliptic Points

This section is dedicated to the proofs of the main results of this paper. These concern the general bound on the dissipation rate of Theorem 1, a sharp treatment of the possible elliptic points in H (cf. Theorem 2), and a detailed analysis of the mixing properties of a standard cellular flow as in Theorem 3.

3.1. General Upper Bounds on the Dissipation Rate

First we notice that to prove the upper bound (1.1), it is enough to exhibit an initial datum ρ_0 for which the corresponding solution ρ^ν of (A-D) cannot diffuse at a time-scale faster than $O(\nu^{-1/3})$. For this purpose, we will choose $\rho_0 \in C^1(M)$ supported in the good invariant set Ω of Proposition 2.1. The idea is to turn (2.2) into a quantitative vanishing viscosity bound, as stated in the result below.

Lemma 3.1. *Let $\nu \in [0, 1)$ and $H \in C^1 \cap W^{2,p}(M)$ for some $p \ge 2$. For any $\rho_0 \in C^1(M)$ with $\mathrm{spt}(\rho_0) \subset \Omega$, there holds the estimate*

$$\|\rho^\nu(t) - \rho(t)\|^2_{L^2} \le C\nu(1+t)^3\|\nabla\rho_0\|^2_{L^\infty}, \qquad \forall t \ge 0,$$

where ρ^ν, ρ solve (A-D) and (T), respectively, with the same initial datum ρ_0 and $C = C(H, \Omega) > 0$.

Proof. We take the difference between (A-D) and (T) to obtain

$$\partial_t(\rho^\nu - \rho) + b \cdot \nabla(\rho^\nu - \rho) = \nu\Delta\rho^\nu. \tag{3.1}$$

Testing (3.1) with $\rho^\nu - \rho$, integrating on M, and using the antisymmetry of the transport term, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^\nu - \rho\|_{L^2}^2 = -2\nu\int_M \nabla\rho^\nu \cdot (\nabla\rho^\nu - \nabla\rho)\mathrm{d}x \leq \\ -2\nu\|\nabla\rho^\nu\|_{L^2}^2 + 2\nu\|\nabla\rho^\nu\|_{L^2}\|\nabla\rho\|_{L^2} \leq \nu\|\nabla\rho\|_{L^2}^2. \tag{3.2}$$

By Hölder's inequality and Proposition 2.1 with $p \geq 2$, we have

$$\nu\|\nabla\rho(t)\|_{L^2}^2 \lesssim \nu\|\nabla\rho(t)\|_{L^p}^2 \lesssim \nu(1+t)^2\|\nabla\rho_0\|_{L^\infty}^2.$$

Plugging this into (3.2) and integrating in time, we obtain

$$\|\rho^\nu(t) - \rho(t)\|_{L^2}^2 \lesssim \nu\|\nabla\rho_0\|_{L^\infty}^2\int_0^t(1+s)^2\mathrm{d}s,$$

and the proof is over. □

With the above proximity estimate at hand, the proof of Theorem 1 follows from a suitable lower bound on the energy dissipation rate.

Proof of Theorem 1. We take ρ_0 as in the above Lemma 3.1, and we assume that the corresponding solution ρ^ν to (A-D) experiences enhanced dissipation at rate $\lambda(\nu)$. According to Definition 1.1, this implies that

$$\left(1 - A\mathrm{e}^{-\lambda(\nu)t}\right)\|\rho_0\|_{L^2} \leq \|\rho_0\|_{L^2} - \left\|\rho^\nu(t)\right\|_{L^2} \\ = \|\rho(t)\|_{L^2} - \left\|\rho^\nu(t)\right\|_{L^2} \lesssim \sqrt{\nu(1+t)^3}\|\nabla\rho_0\|_{L^\infty}, \tag{3.3}$$

or equivalently

$$\left(1 - C(\Omega, H)\frac{\|\nabla\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^2}}\sqrt{\nu(1+t)^3}\right) \leq A\mathrm{e}^{-\lambda(\nu)t}, \qquad \forall t \geq 0. \tag{3.4}$$

Fix now $\varepsilon_0 > 0$ so that

$$\varepsilon_0 := \frac{1}{2}\left[\frac{\|\rho_0\|_{L^2}}{C(\Omega, H)\,\|\nabla\rho_0\|_{L^\infty}}\right]^{2/3}$$

and $\nu_0 \in (0, \varepsilon_0^3)$. Taking $t = \varepsilon_0\nu^{-1/3} \geq 1$ in (3.4), we end up with

$$\frac{1}{2} \leq A\mathrm{e}^{-\varepsilon_0\lambda(\nu)\nu^{-1/3}}, \qquad \forall \nu \in (0, \nu_0),$$

which readily implies the bound (1.1) and concludes the proof. □

3.2. Bounds on Mixing and Enhanced Dissipation Rates Near Elliptic Points

Let H, $p = (0, 0)$, r_0, ℓ_r as in Sect. 1.2. The first lemma crucially associates the behavior of T near $r = 0$ with the growth of the flow map of $b = \nabla^\perp H$.

Lemma 3.2. *Let $H \in C^3(M)$ and $(0, 0)$ as above. Then there exists a constant $C_1 = C_1(H) > 0$ such that*

$$|\nabla X(t, x)| \leq 1 + C_1 r|T'(r)|\frac{t}{T(r)}, \tag{3.5}$$

for every $r \in (0, r_0)$, $t \geq 1$, and $x \in \ell_r$.

Proof. We preliminary observe that it is enough to estimate the growth of $|\nabla X(t, re_1)|$, as the same estimate will hold for any point on ℓ_r. We are interested in the range $t \gg T(r) \sim 1$.

Let $k \in \mathbb{N}$, and $\delta > 0$, we have

$$\begin{aligned}
&|X(kT(r), re_1) - X(kT(r), (r+\delta)e_1)| \\
&\quad \leq |X(kT(r), re_1) - X(kT(r+\delta), (r+\delta)e_1)| \\
&\quad + |X(kT(r+\delta), (r+\delta)e_1) - X(kT(r), (r+\delta)e_1)| \\
&\quad \leq \delta + \|b\|_{L^\infty(\ell_r)} k|T(r+\delta) - T(r)|.
\end{aligned}$$

Noting that $\|b\|_{L^\infty(\ell_r)} \lesssim r$, dividing by δ and sending $\delta \to 0$ we find

$$|\nabla X(kT(r), re_1) \cdot e_1| \leq 1 + C_1 rk|T'(r)|,$$

by choosing $k \sim t/T(r)$ we deduce

$$|\nabla X(t, x) \cdot e_1| \leq 1 + C_2 r|T'(r)|\frac{t}{T(r)},$$

for every $r \in (0, r_0)$, $t \geq 1$, and $x \in \ell_r$. Here we used that

$$|\nabla X(t + T(r), x) \cdot e_1| \leq |\nabla X(t, X(T(r), x)) \cdot e_1| \mathrm{e}^{CT(r)}$$

for any $t \in \mathbb{R}$ and that $T(r) \leq C$, provided $r \leq r_0$.

To estimate $|\nabla X(t, x)|$ it is enough to observe that re_1 is almost normal to ℓ_r at any $x \in \ell_r$, provided $r \leq r_0$ is small enough. On the other hand, the tangential derivative of $X(t, x)$ along ℓ_r is bounded by $\|b\|_{L^\infty(\ell_r)} \lesssim r$. This concludes the proof. □

Next we show that an estimate of type (3.5) along with $T'(r) \sim r^\beta$ implies sharper bounds on the mixing rate and bounds on the enhanced dissipation time-scale for the Hamiltonians considered.

Lemma 3.3. *Let $H \in C^3(M)$ and $(0,0)$ as above. Assume further that the period satisfies*

$$T'(s) \sim s^{\beta}, \qquad as \quad s \to 0 \tag{3.6}$$

for some $\beta \geq 0$. Then the corresponding enhanced dissipation rate of b has the upper upper bound

$$\lambda(\nu) \leq C_2 \nu^{\frac{1+\beta}{3+\beta}}, \tag{3.7}$$

for some constant $C_2 = C_2(H) > 0$.

Proof. Fix $r \in (0, r_0)$ and consider a smooth and mean-free initial datum $\rho_{0,r}$ supported in an invariant region contained in the annulus $B_{ar}(0)\backslash B_r(0)$, where $a > 1$ is a constant depending only on H. Accordingly, we normalize the initial datum so that $\|\rho_{0,r}\|_{L^2} = 1$, implying $\|\nabla\rho_{0,r}\|_{L^2} \sim 1/r$, and denote by ρ^ν, ρ the corresponding solutions to (A-D) and (T), respectively, with the same initial datum $\rho_{0,r}$.

By (3.6) and using that the Lagrangian flow preserves the Lebesgue measure, we can estimate

$$\int_0^t \|\nabla\rho(s)\|_{L^2}^2 \mathrm{d}s \leq \|\nabla\rho_{0,r}\|_{L^2}^2 \int_0^t \sup_{B_{ar}(0)\backslash B_r(0)} |\nabla X(s,\cdot)|^2 \mathrm{d}s \lesssim t/r^2 + r^{2\beta} t^3,$$

for every $t \geq 1$, with constant depending only on H. An optimization in r then leads to the integral bound

$$\nu \int_0^t \|\nabla\rho(s)\|_{L^2}^2 \mathrm{d}s \lesssim \nu(1+t)^{\frac{3+\beta}{1+\beta}}, \qquad \forall t \geq 0.$$

Now, thanks to (3.2), the above integral controls the difference between ρ^ν and ρ in L^2. Thus, as we did for (3.3), we assume that ρ^ν is enhanced dissipated at rate $\lambda(\nu)$ and obtain the inequality

$$1 - C(H)\sqrt{\nu(1+t)^{\frac{3+\beta}{1+\beta}}} \leq A\mathrm{e}^{-\lambda(\nu)t}, \qquad \forall t \geq 0.$$

Choosing

$$\varepsilon_0 := \frac{1}{2}\left[\frac{1}{2C(\Omega, H)}\right]^{2\frac{1+\beta}{3+\beta}}, \qquad 0 < \nu_0 < \varepsilon_0^{\frac{3+\beta}{1+\beta}}, \qquad t = \varepsilon_0 \nu^{-\frac{1+\beta}{3+\beta}} \geq 1$$

eventually leads to

$$\frac{1}{2} \leq A\mathrm{e}^{-\varepsilon_0 \lambda(\nu)\nu^{-\frac{1+\beta}{3+\beta}}}, \qquad \forall \nu \in (0, \nu_0),$$

which readily implies the bound (3.7) and concludes the proof. □

Remark 3.4. In a similar fashion, we can obtain a lower bound on the mixing rate for such b. Indeed, let $\rho_0 = \rho_{0,r}$ exactly as in the proof of Lemma 3.3. If b is mixing with rate $\gamma(t)$ as in Definition 1.2, then we have

$$\|\rho(t)\|_{H^{-1}} \gtrsim \frac{\|\rho(t)\|_{L^2}^2}{\|\rho(t)\|_{H^1}} \gtrsim \|\rho_0\|_{H^1} \frac{1}{1/r^2 + r^{\beta-1}t},$$

where the constants depends only on H. Hence $\gamma(t) \gtrsim \frac{1}{1/r^2+r^{\beta-1}t}$ for every $r > 0$. We choose $r = t^{-\frac{1}{\beta+1}}$ and find the lower bound $\gamma(t) \gtrsim \frac{1}{t^{\frac{2}{\beta+1}}}$.

Proof of Theorem 2. It is enough to apply Lemma 3.3 and the above remark. □

3.3. On the Behavior of T' Near the Elliptic Point

We conclude this section by showing that for every smooth Hamiltonian, T' is bounded near zero (so $\beta \geq 0$ in (3.6)).

Lemma 3.5. *Let $H \in C^3(M)$ and $(0, 0)$ be as above. Then $|T'(s)| \lesssim 1$ as $s \to 0$.*

Proof. We have $T(s) = \bar{T}(H(se_1))$, where $\bar{T}(h) = \int_{\{H=h\}} \frac{1}{|b|} \mathrm{d}\mathscr{H}^1$. In particular

$$|T'(s)| = |\bar{T}'(H(se_1)) \frac{\mathrm{d}}{\mathrm{d}s} H(se_1)| \sim |\bar{T}'(H(se_1))| s,$$

therefore it follows from (2.4) that it is sufficient to prove

$$\left| \int_{\{H=s^2\}} \nabla \cdot \left(\frac{\nabla H}{|\nabla H|^2} \right) |\nabla H|^{-1} \mathrm{d}\mathscr{H}^1 \right| \lesssim s^{-1}. \tag{3.8}$$

Observe that a trivial estimate on the size of the integrand would give a useless final upper bound of size s^{-2}. We compare H with the quadratic Hamiltonian $\tilde{H}(x) = H(0) + x^{\perp} D^2 H(0) x$ to obtain a cancellation at the first non-trivial order: observe that the period of the trajectories associated to $\tilde{H}$ is constant, in particular

$$\int_{\{\tilde{H}=s^2\}} \nabla \cdot \left(\frac{\nabla \tilde{H}}{|\nabla \tilde{H}|^2} \right) |\nabla \tilde{H}|^{-1} \mathrm{d}\mathscr{H}^1 = 0.$$

For shortness we denote by $f = \nabla \cdot \left(\frac{\nabla H}{|\nabla H|^2} \right) |\nabla H|^{-1}$ and $\tilde{f} = \nabla \cdot \left(\frac{\nabla \tilde{H}}{|\nabla \tilde{H}|^2} \right) |\nabla \tilde{H}|^{-1}$. It is straightforward to check that for every $x \in \{H = s^2\} \cup \{\tilde{H} = s^2\} \subset \{c_1 s^2 \leq H \leq c_2 s^2\}$, for appropriate constants $c_1, c_2 > 0$, it holds

$$|f(x)| \lesssim s^{-3}, \quad |\nabla f(x)| \lesssim s^{-4}, \quad |\tilde{f}(x)| \lesssim s^{-3}, \quad |\nabla \tilde{f}(x)| \lesssim s^{-4}.$$

Let $\tilde{\Omega}$ be a sufficiently small neighborhood of $(0, 0)$ and $g : \tilde{\Omega} \to \mathbb{R}^2$ be the map $g(x) = tx$ where $t = t(x)$ is the unique value $t > 0$ such that $\tilde{H}(tx) = H(x)$.

Let $g : \{H = s^2\} \to \{\tilde{H} = s^2\}$ be the map $g(x) = tx$ where $t = t(s, x)$ is the unique value $t > 0$ such that $g(x) \in \{\tilde{H} = s^2\}$. By construction we have

$$|g - \mathrm{Id}| \lesssim s^3, \quad |g^{-1} - \mathrm{Id}| \lesssim s^3, \quad |dg - \mathrm{Id}| \lesssim s, \quad |dg^{-1} - \mathrm{Id}| \lesssim s \quad \text{as } s \to 0.$$

In particular it holds

$$\left|\int_{\{H=s^2\}} f \mathrm{d}\mathscr{H}^1\right| = \left|\int_{\{H=s^2\}} f \mathrm{d}\mathscr{H}^1 - \int_{\{\tilde{H}=s^2\}} \tilde{f} \mathrm{d}\mathscr{H}^1\right|$$
$$\leq \int_{\{H=s^2\}} |f-\tilde{f}| \mathrm{d}\mathscr{H}^1 + \left|\int_{\{H=s^2\}} \tilde{f} \mathrm{d}\mathscr{H}^1 - \int_{\{\tilde{H}=s^2\}} \tilde{f} \mathrm{d}\mathscr{H}^1\right|.$$

Since on $\{H = s^2\}$ we have the estimates

$$|H-\tilde{H}| \lesssim s^3, \quad |\nabla H - \nabla \tilde{H}| \lesssim s^2, \quad |\nabla^2 H - \nabla^2 \tilde{H}| \lesssim s, \quad |\nabla H|, |\nabla \tilde{H}| \sim s$$

then $|f-\tilde{f}| \lesssim s^{-2}$. Moreover $\mathscr{H}^1(\{H=s^2\}) \sim s$, therefore we can estimate

$$\int_{\{H=s^2\}} |f-\tilde{f}| \mathrm{d}\mathscr{H}^1 \lesssim s^{-1}.$$

Now we consider

$$\begin{aligned}
&\left|\int_{\{H=s^2\}} \tilde{f} \mathrm{d}\mathscr{H}^1 - \int_{\{\tilde{H}=s^2\}} \tilde{f} \mathrm{d}\mathscr{H}^1\right| \\
&\quad = \left|\int_{\{H=s^2\}} \tilde{f} \mathrm{d}\mathscr{H}^1 - \int_{\{H=s^2\}} \tilde{f} \circ g^{-1} \mathrm{d} g_\sharp \mathscr{H}^1|_{\{\tilde{H}=s^2\}}\right| \\
&\quad \leq \int_{\{H=s^2\}} |\tilde{f} - \tilde{f} \circ g^{-1}| \mathrm{d}\mathscr{H}^1 \\
&\qquad + \left|\int_{\{H=s^2\}} \tilde{f} \circ g^{-1} \mathrm{d}\mathscr{H}^1 \right. \\
&\qquad \left. - \int_{\{H=s^2\}} \tilde{f} \circ g^{-1} \mathrm{d} g_\sharp \mathscr{H}^1|_{\{\tilde{H}=s^2\}}\right|.
\end{aligned}$$

Since $|\nabla \tilde{f}| \lesssim s^{-4}$ and $|g^{-1} - \mathrm{Id}| \lesssim s^3$, we can estimate

$$\int_{\{H=s^2\}} |\tilde{f} - \tilde{f} \circ g^{-1}| \mathrm{d}\mathscr{H}^1 \lesssim 1.$$

Moreover, by the estimates on dg we obtain that $g_\sharp \mathscr{H}^1|_{\{\tilde{H}=s^2\}} = \psi \mathscr{H}^1|_{\{H=s^2\}}$ with $|\psi - 1| \lesssim s$. Since $|\tilde{f} \circ g^{-1}| \lesssim s^{-3}$, then we deduce that

$$\left|\int_{\{H=s^2\}} \tilde{f} \circ g^{-1} \mathrm{d}\mathscr{H}^1 - \int_{\{H=s^2\}} \tilde{f} \circ g^{-1} \mathrm{d} g_\sharp \mathscr{H}^1|_{\{\tilde{H}=s^2\}}\right| \lesssim s^{-1}.$$

Combining the previous estimates we obtain (3.8) and therefore the claim. □

4. The Case of a Cellular Flow

We consider the 2π-periodic Hamiltonian $H_{\mathsf{c}}(x) = H_{\mathsf{c}}(x_1, x_2) = \sin x_1 \, \sin x_2$ and the associated velocity field

$$b_{\mathsf{c}}(x) = b_{\mathsf{c}}(x_1, x_2) = (-\sin x_1 \, \cos x_2, \cos x_1 \, \sin x_2) \, .$$

We restrict our analysis to the domain $[0,\pi]^2$ and observe that $H_{\mathsf{c}} \geq 0$ on $[0,\pi]$, $H_{\mathsf{c}} = 0$ at $\partial[0,\pi]^2$ and $(\pi/2, \pi/2)$ is the only maximum point. For any $h \in (0,1)$ we denote by $T(h)$ the period of the closed trajectory supported in $\{H_{\mathsf{c}} = h\} \cap [0,\pi]^2$.

Lemma 4.1. *For any $h \in (0,1)$ we have*

$$2h(1-h) \leq |b_{\mathsf{c}}(x)|^2 \leq 2(1-h^2) \, , \quad \textit{for any } x \in \{H_{\mathsf{c}} = h\} \cap [0,\pi]^2 \, .$$

Proof. Fix $x \in \{H_{\mathsf{c}} = h\} \cap [0,\pi]^2$, and compute

$$|b_{\mathsf{c}}(x)|^2 = -2(\sin x_1 \, \sin x_2)^2 + (\sin x_1)^2 + (\sin x_2)^2 = -2h^2 + (\sin x_1)^2 + (\sin x_2)^2.$$

This yields

$$-2h^2 + (\sin x_1)^2 + (\sin x_2)^2 \leq -2h^2 + 2 = 2(1-h^2),$$

and

$$-2h^2 + (\sin x_1)^2 + (\sin x_2)^2 = -2h^2 + 2h + (\sin x_1 - \sin x_2)^2 \geq 2h(1-h),$$

as we wanted. □

4.1. Estimates on $T(h)$

We provide an explicit formula for $T(h)$ and we show that $T(h) \sim 1 + \ln(1/h)$ up to the second order.

Lemma 4.2. *For any $h \in (0,1)$ we have*

$$T(h) = 4\int_h^1 \frac{1}{\sqrt{x^2-h^2}} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = 4\int_0^1 \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-(1-h^2)x^2}} \mathrm{d}x \, . \tag{4.1}$$

Proof. We parametrize a quarter of the closed curve $\{H_{\mathsf{c}} = h\} \cap [0,\pi]^2$ with a curve $[\arcsin h, \pi/2] \ni t \to (t, \gamma(t))$, where $\sin t \, \sin \gamma(t) = h$. It is not hard to check that

$$|\gamma'(t)|^2 + 1 = \frac{h^2 - 2h^2 \sin^2 t + \sin^4 t}{\sin^2 t (\sin^2 t - h^2)} \, . \tag{4.2}$$

Moreover, from the identity $|b_{\mathsf{c}}(x_1, x_2)|^2 = \sin^2 x_1 \, \cos^2 x_2 + \cos^2 x_1 \, \sin^2 x_2$ we get

$$|b_{\mathsf{c}}(t, \gamma(t))|^2 = \frac{h^2 - 2h^2 \sin^2 t + \sin^4 t}{\sin^2 t} \, . \tag{4.3}$$

By combining (4.2) and (4.3) we can compute the period

$$T(h) = 4\int_{\arcsin h}^{\pi/2} \frac{\sqrt{|\gamma'(t)|^2+1}}{|b_{\mathsf{c}}(t,\gamma(t))|}\mathrm{d}t = 4\int_{\arcsin h}^{\pi/2} \frac{1}{\sqrt{\sin^2 t - h^2}}\mathrm{d}t,$$

and the first identity in (4.1) follows from the change of variables $x = \sin t$. To prove the second identity in (4.1) we employ the change of variables $y = \sqrt{\frac{1-x^2}{1-h^2}}$.
□

Lemma 4.3. *The function T is smooth in $(0, 1)$, and there exists $C > 1$ such*

$$\frac{T(h)}{1+\ln(1/h)} + h|T'(h)| + h^2|T''(h)| \leq C,$$
$$-T'(h) \geq \frac{1}{Ch}, \tag{4.4}$$

for any $h \in (0, 1)$.

Proof. From the second identity in (4.1) it is immediate to verify that $T \in C^\infty((0, 1))$. Let us estimate $T(h)$. We first consider the case $h \in (1/2, 1)$:

$$\begin{aligned}
T(h) &\leq 10\int_h^1 \frac{1}{\sqrt{x-h}\sqrt{1-x}}\mathrm{d}x \\
&= 10\int_h^{\frac{h+1}{2}} \frac{1}{\sqrt{x-h}\sqrt{1-x}}\mathrm{d}x + 10\int_{\frac{h+1}{2}}^1 \frac{1}{\sqrt{x-h}\sqrt{1-x}}\mathrm{d}x \\
&\leq \frac{10\sqrt{2}}{\sqrt{1-h}}\int_h^{\frac{h+1}{2}} \frac{1}{\sqrt{x-h}}\mathrm{d}x + \frac{10\sqrt{2}}{\sqrt{1-h}}\int_{\frac{h+1}{2}}^1 \frac{1}{\sqrt{1-x}}\mathrm{d}x \\
&= 40.
\end{aligned}$$

Let us now assume that $h \in (0, 1/2)$, we have

$$\begin{aligned}
T(h) &\leq 4\int_h^1 \frac{1}{\sqrt{x^2-h^2}}\mathrm{d}x = 4\int_1^{1/h} \frac{1}{\sqrt{z^2-1}}\mathrm{d}z \\
&= 4\int_1^2 \frac{1}{\sqrt{z^2-1}}\mathrm{d}z + 4\int_2^{1/h} \frac{1}{\sqrt{z^2-1}}\mathrm{d}z \\
&\leq 10 + 10\ln(1/h),
\end{aligned}$$

where we used the change of variables $z = x/h$.

To study $T'(h)$, we differentiate the second identity in (4.1) and obtain

$$-T'(h) = 4\int_0^1 \frac{1}{(1-x^2)^{1/2}} \frac{hx^2}{(1-(1-h^2)x^2)^{3/2}}\mathrm{d}x.$$

We can split the integral into

$$-hT'(h) = 4\int_0^{1-h^2} \frac{1}{(1-x^2)^{1/2}} \frac{h^2x^2}{(1-(1-h^2)x^2)^{3/2}}\mathrm{d}x$$

$$
\begin{aligned}
&+4\int_{1-h^2}^{1}\frac{1}{(1-x^2)^{1/2}}\,\frac{h^2x^2}{(1-(1-h^2)x^2)^{3/2}}\mathrm{d}x\\
&=: I+II\,,
\end{aligned}
$$

and estimate

$$
\begin{aligned}
I &\le \frac{4h^2(1-h^2)}{(1-(1-h^2)^2)^{1/2}}\int_0^{1-h^2}\frac{x}{(1-(1-h^2)x^2)^{3/2}}\mathrm{d}x\\
&\le 4h\int_0^{1-h^2}\frac{\mathrm{d}}{\mathrm{d}x}(1-(1-h^2)x^2)^{-1/2}\mathrm{d}x\\
&\le 4.
\end{aligned}
$$

From

$$
\frac{4h^2(1-h^2)}{(1-(1-h^2)^3)^{3/2}}\int_{1-h^2}^{1}\frac{x}{(1-x^2)^{1/2}}\mathrm{d}x \le II \le \frac{4}{h}\int_{1-h^2}^{1}\frac{x}{(1-x^2)^{1/2}}\mathrm{d}x
$$

we deduce that

$$
1-h\le II\le 8.
$$

Since $I\ge 0$, T' is continuous in $h=1$ and $-T'(1)=\pi$ we also deduce (4.4).

To conclude the proof we have to show the upper bound on $h^2|T''(h)|$. Let us introduce the auxilary function

$$
G(s):=4\int_0^1\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-sx^2}}\,\mathrm{d}x\qquad s\in(0,1).
$$

Observe that $T(h)=G(1-h^2)$, hence

$$
h^2T''(h)=-2h^2G'(1-h^2)+4h^4G''(1-h^2)=hT'(h)+4h^4G''(1-h^2)\,.
$$

We already know that $h|T'(h)|\le 12$. To estimate the second term we use the identity

$$
\begin{aligned}
h^4|G''(1-h^2)| &= 3\int_0^1\frac{1}{(1-x^2)^{1/2}}\,\frac{h^4x^4}{(1-(1-h^2)x^2)^{\frac{5}{2}}}\mathrm{d}x\\
&= 3\int_0^{1-h^2}\frac{1}{(1-x^2)^{1/2}}\,\frac{h^4x^4}{(1-(1-h^2)x^2)^{\frac{5}{2}}}\mathrm{d}x\\
&\quad+3\int_{1-h^2}^{1}\frac{1}{(1-x^2)^{1/2}}\,\frac{h^4x^4}{(1-(1-h^2)x^2)^{\frac{5}{2}}}\mathrm{d}x\\
&=: I+II
\end{aligned}
$$

and estimate I and II exactly as we did for $-hT'(h)$. □

Remark 4.4. The expression for $T(h)$ in (4.1) is in fact a complete elliptic integral of the first kind. Its asymptotics are well-known [1, Ch. 17]: as $h \to 0^+$ we have

$$T(h) \sim 4\ln(4/h) + O(h), \qquad T'(h) \sim -\frac{4}{h} + O(1) \qquad T''(h) \sim \frac{4}{h^2} + O(1/h),$$

while as $h \to 1^-$ it holds

$$T(h) \sim 2\pi + O(h-1), \qquad T'(h) \sim -\pi + O(h-1) \qquad T''(h) \sim \frac{5\pi}{4} + O(h-1).$$

However, all we need are the global bounds contained in Lemma 4.3.

4.2. Reparametrization of the Hamiltionian and Regularity of the Change of Variables

We consider the coordinates

$$\tilde{\Phi} : \mathbb{S}^1 \times \left(0, \frac{\pi}{2}\right) \to (0,\pi)^2 \setminus \left\{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}, \quad \tilde{\Phi}(\theta, I) := X\left(\theta \tilde{T}(I), \left(\frac{\pi}{2}, I\right)\right) \tag{4.5}$$

where $\tilde{T}(I)$ is the period of the closed orbit in $\{H_{\mathsf{c}} = \sin I\}$ passing through the point $(\pi/2, I)$. These coordinates are related to the action-angle coordinates introduced in Sect. 2.2 by

$$\tilde{\Phi}(\theta, I) = \Phi(\theta, \sin I), \qquad \tilde{T}(I) = T(\sin I). \tag{4.6}$$

Lemma 4.5. *The map* $\tilde{\Phi} : \mathbb{S}^1 \times \left(0, \frac{\pi}{2}\right) \to (0,\pi)^2 \setminus \{(\pi/2, \pi/2)\}$ *and its inverse are* C^1*. Moreover the following estimates hold:*

$$|\partial_\theta \tilde{\Phi}(\theta, I)| \lesssim |\ln I| \left(\frac{\pi}{2} - I\right), \qquad |\partial_I \tilde{\Phi}(\theta, I)| \lesssim I^{-1}.$$

Proof. The C^1-regularity of $\tilde{\Phi}$ and its inverse follows from (4.6) and the same properties proved for Φ in Lemma 2.4. The first inequality follows from Lemma 4.3:

$$|\partial_\theta \tilde{\Phi}(\theta, I)| \leq |\tilde{T}(I)||b_{\mathsf{c}}|(\tilde{\Phi}(\theta, I)) \lesssim |\ln I| \left(\frac{\pi}{2} - I\right).$$

Let us prove the second inequality: for every $\alpha \in \mathbb{S}^1$ and $I \in \left(0, \frac{\pi}{2}\right)$, let $P(\alpha, I) \in \tilde{\Phi}(\mathbb{S}^1, I)$ and $g(\alpha, I) > 0$ be such that

$$P(\alpha, I) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right) + g(\alpha, I)\mathrm{e}^{i(3\pi/2 - \alpha)}. \tag{4.7}$$

Take now $\alpha : \mathbb{S}^1 \times (0, \pi/2) \to \mathbb{S}^1$ such that

$$\mathrm{e}^{i(3\pi/2 - \alpha(\theta, I))} = \frac{\tilde{\Phi}(\theta, I) - (\pi/2, \pi/2)}{\left|\tilde{\Phi}(\theta, I) - (\pi/2, \pi/2)\right|}.$$

In particular we have

$$\tilde{\Phi}(\theta, I) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right) + g(\alpha(\theta, I), I)\mathrm{e}^{i(3\pi/2-\alpha(\theta,I))},$$

therefore

$$\begin{aligned}|\partial_I \tilde{\Phi}(\theta, I)| &\le |\partial_\alpha g(\alpha(\theta, I), I)||\partial_I \alpha(\theta, I)| \\ &+|\partial_I g(\alpha(\theta, I), I)| + |g(\alpha(\theta, I), I)||\partial_I \alpha(\theta, I)|.\end{aligned} \tag{4.8}$$

We observe that $g(\alpha, I) \sim (\pi/2 - I)$. Moreover it holds

$$\begin{aligned}|\partial_I g(\alpha, I)| &\lesssim 1, \qquad \text{as} \quad I \to \frac{\pi}{2}^-, \\ |\partial_I g(\alpha, I)| &\lesssim \frac{1}{|b_{\mathsf{c}} \circ P|} \lesssim \frac{1}{\sqrt{I}}, \qquad \text{as} \quad I \to 0^+.\end{aligned}$$

Differentiating (4.7) and observing that $g(\alpha, I), |\partial_\alpha P| \sim (\pi/2 - I)$ we obtain $|\partial_\alpha g| \lesssim \left(\frac{\pi}{2} - I\right)$. Therefore it follows by (4.8) that

$$|\partial_I \tilde{\Phi}(\theta, I)| \lesssim \left(\frac{\pi}{2} - I\right) |\partial_I \alpha(\theta, I)| + \frac{1}{\sqrt{I}}. \tag{4.9}$$

In remains to estimate $\partial_I \alpha$: we consider the angular velocity with respect to the center $(\pi/2, \pi/2)$:

$$\omega : \mathbb{S}^1 \times \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, \qquad \omega(\alpha, I) = \frac{b_{\mathsf{c}}(P(\alpha, I))\, \mathrm{e}^{i(2\pi-\alpha)}}{g(\alpha, I)}.$$

By construction it holds

$$\int_0^{\alpha(\theta,I)} \frac{1}{\omega(\alpha, I)} \mathrm{d}\alpha = \theta \tilde{T}(I).$$

Differentiating the expression above with respect to I we get

$$\partial_I \alpha(\theta, I) = \omega(\alpha(\theta, I), I) \left[\theta \tilde{T}'(I) + \int_0^{\alpha(\theta,I)} \frac{\partial_I \omega(\alpha, I)}{\omega(\alpha, I)^2} \mathrm{d}\alpha\right]. \tag{4.10}$$

Since $g(\alpha, I) \sim \left(\frac{\pi}{2} - I\right)$ and $b_{\mathsf{c}}(P(\alpha, I))\, \mathrm{e}^{i(2\pi-\alpha)} \ge \sqrt{2}/2$, then $\omega(\alpha, I) \sim \frac{|b_{\mathsf{c}} \circ P|}{g(\alpha, I)} \lesssim 1$. By Lemma 4.3 we have $\tilde{T}'(I) \lesssim I^{-1} \left(\frac{\pi}{2} - I\right)$. In order to estimate the integral, we distinguish two regimes: $I \to 0^+$ and $I \to \frac{\pi}{2}^-$, and rely on the estimate

$$|\partial_I \omega| \le \frac{|\partial_I (b_{\mathsf{c}} \circ P)|}{g} + \frac{|b_{\mathsf{c}} \circ P||\partial_I g|}{g^2}.$$

For $I \to \frac{\pi}{2}^-$ we have $g(\alpha, I) \sim \left(\frac{\pi}{2} - I\right)$ and $\left(\frac{\pi}{2} - I\right)^{-1} |b_{\mathsf{c}} \circ P| + |\partial_I (b_{\mathsf{c}} \circ P)| + |\partial_I g| \lesssim 1$, therefore

$$|\partial_I \omega| \lesssim \left(\frac{\pi}{2} - I\right)^{-1} \qquad \text{as } I \to \frac{\pi}{2}^-.$$

In particular we get

$$|\partial_I \alpha(\theta, I)| \lesssim \left(\frac{\pi}{2} - I\right)^{-1} \qquad \text{as } I \to \frac{\pi}{2}^-.$$

We now estimate the integral term in (4.10) as $I \to 0^+$: by the symmetries of the cellular flow we have that for every $\theta \in \mathbb{S}^1$ it holds

$$\left|\int_0^{\alpha(\theta,I)} \frac{\partial_I \omega(\alpha, I)}{\omega(\alpha, I)^2} \mathrm{d}\alpha\right| \leq \int_0^{2\pi} \left|\frac{\partial_I \omega(\alpha, I)}{\omega(\alpha, I)^2}\right| \mathrm{d}\alpha = 8 \int_0^{\frac{\pi}{4}} \left|\frac{\partial_I \omega(\alpha, I)}{\omega(\alpha, I)^2}\right| \mathrm{d}\alpha$$

If $\alpha, I \in (0, \pi/4)$, then, denoting by P_1 the first component of the vector P, we have

$$|b_{\mathsf{c}} \circ P| \sim P_1(\alpha, I), \qquad |\partial_I (b_{\mathsf{c}} \circ P)| \sim \frac{|(D_x b_{\mathsf{c}}) \circ P|}{|b_{\mathsf{c}} \circ P|}$$
$$\lesssim \frac{1}{|b_{\mathsf{c}} \circ P|}, \qquad |\partial_I g| \lesssim \frac{1}{|b_{\mathsf{c}} \circ P|}, \qquad g \sim 1,$$

therefore $|\partial_I \omega(\alpha, I)| \lesssim \frac{1}{P_1(\alpha, I)}$ and $|\omega(\alpha, I)| \sim P_1(\alpha, I)$. We observe that for every $I \in (0, \pi/4)$ the map $\alpha \mapsto P_1(\alpha, I)$ is bi-Lipschitz from $(0, \pi/4)$ to its image with constants independent of I.

Eventually we can estimate

$$\left|\int_0^{\alpha(\theta,I)} \frac{\partial_I \omega(\alpha, I)}{\omega(\alpha, I)^2} \mathrm{d}\alpha\right| \lesssim \left|\int_{P_1(\frac{\pi}{4}, I)}^{\frac{\pi}{2}} \frac{1}{x_1^3} \mathrm{d}x_1\right| \lesssim I^{-1} \qquad \text{as } I \to 0^+,$$

where in the last inequality we used that $P_1\left(\frac{\pi}{4}, I\right) \sim \sqrt{I}$, since $\sin^2\left(P_1\left(\frac{\pi}{4}, I\right)\right) = \sin I$. Combining the estimates in the two regimes we get

$$|\partial_I \alpha(\theta, I)| \lesssim \left(\frac{\pi}{2} - I\right)^{-1} I^{-1},$$

therefore we conclude by (4.9) that

$$|\partial_I \tilde{\Phi}(\theta, I)| \lesssim I^{-1},$$

finishing the proof. □

It is convenient to state also the estimates relative to the action-angle coordinates in Sect. 2.2: recalling (4.6), it immediately follows from Lemma 4.5 that

$$|\partial_\theta \Phi| \lesssim \ln\left(1 + \frac{1}{h}\right)\sqrt{1-h}, \qquad |\partial_h \Phi| \lesssim \frac{1}{h\sqrt{1-h}}. \tag{4.11}$$

4.3. Polynomial Bound for the Gradient of the Flow

In this section, we prove an algebraic upper bound for the $W^{1,1}$ norm of the Lagrangian flow $X(t,x)$. Despite the existence of a hyperbolic point, which causes exponential stretching/compression of particle trajectories, the result below asserts that, on average, the gradient growth is at most quadratic.

Proposition 4.6. *Denote by X the flow associated to b_{c}, then there is an absolute constant $C > 0$ such that*

$$\int_{\mathbb{T}^2} |D_x X(t,x)| \mathrm{d}x \leq C(1+t^2). \tag{4.12}$$

We expect the sharp estimate to be linear rather than quadratic, up to possible logarithmic corrections. Moreover, we believe that this property is true for more general Hamiltonians with non-degenerate hyperbolic points, although (4.12) can be saturated in degenerate cases. We will investigate these properties further in a subsequent work.

An immediate consequence of Proposition 4.6 and the interpolation inequality

$$\|f\|^2_{L^{4/3}} \lesssim \|\nabla f\|_{L^1} \|f\|_{\dot{H}^{-1}},$$

is that no datum can mix faster than $1/t^2$, even if its support contains the hyperbolic points.

Proof of Proposition 4.6. We reduce to the domain $(0,\pi)^2$. Recalling the definition of action-angle coordinates in Sect. 2.2 we have

$$X(t,x) = \Phi(\Psi(X(t,x)), H_{\mathsf{c}}(X(t,x))).$$

Since $H_{\mathsf{c}}(X(t,x)) = H_{\mathsf{c}}(x)$, we deduce by Lemma 2.4 that

$$X(t,x) = \Phi\left(\Psi(x) + \frac{t}{T(H_{\mathsf{c}}(x))}, H_{\mathsf{c}}(x)\right).$$

Differentiating with respect to x and recalling $b_{\mathsf{c}} = \nabla^{\perp} H_{\mathsf{c}}$ we obtain

$$|D_x X(t,x)| \leq |\partial_\theta \Phi| \left(|D\Psi| + \frac{t}{T^2(H_{\mathsf{c}}(x))} |T'(H_{\mathsf{c}}(x))||b(x)|\right) + |\partial_h \Phi||b(x)| \tag{4.13}$$

In order to estimate $|D\Psi|$, we repeat the computations in (2.8), taking into account the sharper estimates (4.11) for $|D\Phi|$, and using $T(h) \sim 1 + \ln(1/h)$: we have

$$|D\Psi| \leq |D\Phi^{-1}| \leq \frac{|D\Phi|}{|\det D\Phi|} \lesssim \frac{1}{h\sqrt{1-h}(1+\ln(1/h))}.$$

Plugging this estimate in (4.13), we obtain

$$|DX(t,x)| \lesssim (1+\ln(1/h))\sqrt{1-h}$$

$$\left(\frac{1}{h\sqrt{1-h}(1+\ln(1/h))}+\frac{t}{h(1+\ln(1/h))^2}|b(x)|\right)+\frac{1}{h\sqrt{1-h}}|b(x)|$$
$$=\frac{1}{h}+\frac{\sqrt{1-h}}{h(1+\ln(1/h))}|b(x)|t+\frac{1}{h\sqrt{1-h}}|b(x)|,$$

where $h = H_{\mathsf{c}}(x)$. We also have the trivial bound $|D_x X(t,x)| \le \mathrm{e}^{Lt}$, with $L = |Db_{\mathsf{c}}|_{L^\infty}$. Given $\bar{h} \in (0,1)$ and using the co-area formula we get

$$\begin{aligned}\int_{(0,\pi)^2}|D_xX(t,x)|\mathrm{d}x &= \int_{\{H_{\mathsf{c}}\le\bar h\}}|D_xX(t,x)|\mathrm{d}x+\int_{\bar h}^1\int_{\{H_{\mathsf{c}}=h\}}\frac{|D_xX(t,x)|}{|b(x)|}\mathrm{d}\mathscr{H}^1(x)\mathrm{d}h\\ &\lesssim \mathrm{e}^{Lt}|\{H_{\mathsf{c}}\le\bar h\}|+\int_{\bar h}^1\frac{1}{h}\int_{\{H_{\mathsf{c}}=h\}}\frac{1}{|b(x)|}\mathrm{d}\mathscr{H}^1(x)\mathrm{d}h\\ &\quad+\int_{\bar h}^1\left(\frac{\sqrt{1-h}}{h(1+\ln(1/h))}t+\frac{1}{h\sqrt{1-h}}\right)\mathscr{H}^1(\{H_{\mathsf{c}}=h\})\mathrm{d}h. \qquad (4.14)\end{aligned}$$

It is straightforward to check that

$$\left|\{H_{\mathsf{c}}\le\bar h\}\right|\lesssim\bar h\ln(1/\bar h),\qquad \mathscr{H}^1(\{H_{\mathsf{c}}=h\})\sim\sqrt{1-h}.$$

Moreover by Lemma 4.3 we have

$$\int_{\{H_{\mathsf{c}}=h\}}\frac{1}{|b(x)|}d\mathscr{H}^1(x)=T(h)\lesssim 1+\ln(1/h).$$

Plugging these estimates in (4.14), we get

$$\begin{aligned}\int_{(0,\pi)^2}|D_xX(t,x)|\mathrm{d}x &\lesssim \mathrm{e}^{Lt}\bar h(1+\ln(1/\bar h))+\int_{\bar h}^1\left(\frac{1+\ln(1/h)}{h}+\frac{t}{h\ln(1/h)}\right)\mathrm{d}h\\ &\lesssim \mathrm{e}^{Lt}\bar h(1+\ln(1/\bar h))+\ln(\bar h)^2+t\ln(\ln(1/\bar h)).\end{aligned}$$

Choosing $\bar h = \mathrm{e}^{-Lt}$ we get the desired estimate. □

4.4. Global Mixing Estimate

Let $\rho_0 \in C^1(\mathbb{T}^2)$ be mean zero along the level sets of H_{c}, and consider $\rho : \mathbb{R}\times\mathbb{T}^2 \to \mathbb{R}$ the solution to (T) with $\rho(0,\cdot) = \rho_0$. To prove global mixing estimates, it is enough to study the flow in each invariant domain $(0,\pi)^2$, $(0,\pi)\times(-\pi,0)$, $(-\pi,0)^2$, $(-\pi,0)\times(0,\pi)$. More precisely, we split

$$\begin{aligned}\|\rho(t)\|_{H^{-1}(\mathbb{T}^2)} &= \sup_{\|\varphi\|_{H^1(\mathbb{T}^2)}\le1}\int_{\mathbb{T}^2}\rho(t)\varphi\mathrm{d}x\\ &\le \sup_{\|\varphi\|_{H^1(\mathbb{T}^2)}\le1}\int_{(0,\pi)^2}\rho(t)\varphi\mathrm{d}x+\sup_{\|\varphi\|_{H^1(\mathbb{T}^2)}\le1}\int_{(0,\pi)\times(-\pi,0)}\rho(t)\varphi\mathrm{d}x \qquad (4.15)\\ &\quad+\sup_{\|\varphi\|_{H^1(\mathbb{T}^2)}\le1}\int_{(-\pi,0)\times(0,\pi)}\rho(t)\varphi\mathrm{d}x+\sup_{\|\varphi\|_{H^1(\mathbb{T}^2)}\le1}\int_{(-\pi,0)^2}\rho(t)\varphi\mathrm{d}x\end{aligned}$$

and estimate each term separately using the special coordinates introduced in the previous section. We illustrate in details how to estimate the mixing rate in $(0,\pi)^2$, the analysis of the other domains being completely analogous.

We apply the stationary phase argument for the dynamics in the coordinates (θ, I) and the change of variables $\tilde{\Phi}$ given in (4.5). The function $f(t, \theta, I) := \rho(t, \tilde{\Phi}(\theta, I))$ solves the equation

$$\partial_t f + \frac{1}{\tilde{T}(I)} \partial_\theta f = 0 , \quad \text{for } (t, \theta, I) \in \mathbb{R} \times \mathbb{S}^1 \times \left[0, \frac{\pi}{2}\right].$$

By (2.7) and (4.6) we have

$$J_{\tilde{\Phi}}(\theta, I) = J_{\Phi}(\theta, I) \cos I = \tilde{T}(I) \cos I.$$

We set $g(I) := \tilde{T}(I) \cos I$, and work in the weighted Sobolev space H^1_g, defined by the norm

$$\|f\|^2_{H^1_g} := \int_{\mathbb{S}^1 \times (0, \frac{\pi}{2})} \left(|f(\theta, s)|^2 + |\partial_s f(\theta, s)|^2\right) g(s) \mathrm{d}s \mathrm{d}\theta.$$

Fix $\delta, \delta' \in (0, 1)$, by (A.4) we have

$$\sup_{\|\phi\|_{H^1_g} \leq 1} \int_\delta^{\frac{\pi}{2} - \delta'} \int_{\mathbb{S}^1} f(t, \theta, s) \phi(\theta, s) g(s) \mathrm{d}\theta \mathrm{d}s \leq \|f(0, \cdot)\|_{H^1_g} r(t)$$

with $r(t)$ given by (A.3), i.e.

$$r(t) = \frac{1}{t} \left\| g^{-\frac{1}{2}} \right\|^2_{L^2(\delta, \frac{\pi}{2} - \delta')} \left(\frac{\tilde{T}^2 g}{|\tilde{T}'|}(\delta) + \frac{\tilde{T}^2 g}{|\tilde{T}'|}(\frac{\pi}{2} - \delta') + \left\| \left(\frac{\tilde{T}^2 g}{\tilde{T}'} \right)' \right\|_{L^1(\delta, \frac{\pi}{2} - \delta')} \right)$$
$$+ \frac{1}{t} \left\| g^{-\frac{1}{2}} \right\|_{L^2(\delta, \frac{\pi}{2} - \delta')} \left\| \frac{\tilde{T}^2 g^{\frac{1}{2}}}{\tilde{T}'} \right\|_{L^2(\delta, \frac{\pi}{2} - \delta')} . \tag{4.16}$$

By Lemma 4.3 we have

$$\tilde{T}(I) \sim 1 + |\ln I|, \qquad |\tilde{T}'(I)| \sim I^{-1} \left(\frac{\pi}{2} - I \right), \qquad |\tilde{T}''(I)| \sim I^{-2} ,$$

hence each term in (4.16) can be estimated pointwise as

$$\left| \left(\frac{\tilde{T}^2 g}{\tilde{T}'} \right)' \right| = \left| \left(\frac{\tilde{T}^3 \cos I}{\tilde{T}'} \right)' \right| \lesssim \tilde{T}^2 + \frac{(1 + |\ln I|)^3 I^2}{\frac{\pi}{2} - I} + \frac{|\tilde{T}''| \tilde{T}^3 I^2}{\frac{\pi}{2} - I} \lesssim \frac{(1 + |\ln I|)^3}{\frac{\pi}{2} - I},$$
$$\left| \frac{\tilde{T}^2 g}{\tilde{T}'} \right| \lesssim (1 + |\ln I|)^3 I,$$
$$\left| \frac{\tilde{T}^2 g^{1/2}}{\tilde{T}'} \right| \lesssim \frac{(1 + |\ln I|)^{5/2} I}{\left(\frac{\pi}{2} - I\right)^{1/2}}.$$

Plugging these estimates in (A.3), we get

$$r(t) \lesssim \left|\ln \delta'\right|^2 \frac{1}{t} . \tag{4.17}$$

Let $\Omega \subset (0,\pi)^2$ be an open set such that $\mathrm{spt}(\rho(t)) \cap (0,\pi)^2 \subset \Omega$ for every $t \in \mathbb{R}$. Denoting by $\mathcal{L}^2$ the Lebesgue measure on $(0,\pi)^2$, we estimate

$$\begin{aligned}
&\sup_{\|\varphi\|_{H^1}\le 1} \int_{(0,\pi)^2} \rho(t)\varphi \mathrm{d}x \\
&= \sup_{\|\varphi\|_{H^1}\le 1} \left(\int_{\tilde\Phi(\mathbb{S}^1\times(0,\delta))} \rho(t)\varphi \mathrm{d}x + \int_{\tilde\Phi(\mathbb{S}^1\times(\delta,\frac{\pi}{2}-\delta'))} \rho(t)\varphi \mathrm{d}x + \int_{\tilde\Phi(\mathbb{S}^1\times(\frac{\pi}{2}-\delta',\frac{\pi}{2}))} \rho(t)\varphi \mathrm{d}x \right) \\
&\lesssim C(\varepsilon)\|\rho_0\|_{L^\infty} \left[\mathcal{L}^2 \left(\left(\tilde\Phi(\mathbb{S}^1\times(0,\delta)) \cup \tilde\Phi\left(\mathbb{S}^1 \times \left(\frac{\pi}{2}-\delta', \frac{\pi}{2}\right)\right)\right) \cap \Omega \right) \right]^{1-\varepsilon} \\
&\quad + \sup_{\|\varphi\|_{H^1}\le 1} \int_{\tilde\Phi(\mathbb{S}^1\times(\delta,\frac{\pi}{2}-\delta'))} \rho(t)\varphi \mathrm{d}x,
\end{aligned}$$

where we used the a priori estimate $\|\rho(t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}$ and the two-dimensional Sobolev embedding $\|\varphi\|_{L^p} \le C(p)\|\varphi\|_{H^1} \le C(p)$ for any $p < \infty$. Performing the computations in Section A.1, we deduce from (4.17) and Lemma 4.5 that

$$\sup_{\|\varphi\|_{H^1}\le 1} \int_{\tilde\Phi(\mathbb{S}^1\times(\delta,\frac{\pi}{2}-\delta'))} \rho(t)\varphi \mathrm{d}x \lesssim \frac{\|\rho_0\|_{H^1}}{t}(1+\mathrm{Lip}\tilde\Phi)^2 \left|\ln \delta'\right|^2 \lesssim \frac{\|\rho_0\|_{H^1}}{t}\delta^{-2} \left|\ln \delta'\right|^2 .$$

Let $\kappa < 1/5$ be a small parameter. We claim that for every $\varepsilon > 0$ we have the following:

(1) If $\mathrm{spt}(\rho_0) \cap (0,\pi)^2 \subset \mathbb{S}^1 \times (\kappa, \pi-\kappa)$, then

$$\sup_{\|\varphi\|_{H^1}\le 1} \int_{(0,\pi)^2} \rho(t)\varphi \mathrm{d}x \le C(\varepsilon,\kappa,\|\rho_0\|_{H^1},\|\rho_0\|_{L^\infty}) \frac{1}{t^{1-\varepsilon}} . \tag{4.18}$$

(2) If $\mathrm{spt}(\rho_0) \cap (0,\pi)^2 \subset \mathbb{S}^1 \times (0,\pi)^2 \setminus B_\kappa((\pi/2,\pi/2))$, then

$$\sup_{\|\varphi\|_{H^1}\le 1} \int_{(0,\pi)^2} \rho(t)\varphi \mathrm{d}x \le C(\varepsilon,\kappa,\|\rho_0\|_{H^1},\|\rho_0\|_{L^\infty}) \frac{1}{t^{\frac{1}{3}-\varepsilon}} . \tag{4.19}$$

To prove (1), we set $\Omega := \mathbb{S}^1 \times (\kappa', \pi-\kappa')$ for some $0 < \kappa' < \kappa$ in such a way that it contains $\mathrm{spt}(\rho(t)) \cap (0,\pi)^2$ for any $t \in \mathbb{R}$. Hence, there exists $\delta(\kappa) < 1/5$, depending only on κ, such that $\tilde\Phi\left(\mathbb{S}^1 \times (0,\delta(\kappa))\right) \cap \Omega = \emptyset$. The analysis above with $\delta = \delta(\kappa)$ gives

$$\sup_{\|\varphi\|_{H^1}\le 1} \int_{(0,\pi)^2} \rho(t)\varphi \mathrm{d}x \lesssim C(\varepsilon)\|\rho_0\|_{L^\infty}(\delta')^{2-2\varepsilon} + C(\kappa)\|\rho_0\|_{H^1}|\ln\delta'|^2\frac{1}{t} ,$$

where we used that

$$\mathcal{L}^2\left(\tilde\Phi\left(\mathbb{S}^1 \times \left(\frac{\pi}{2}-\delta', \frac{\pi}{2}\right)\right)\right) \lesssim (\delta')^2 .$$

The sought conclusion follows by optimizing δ'.

To prove (2) we argue analogously. The right domain to consider is

$$\Omega := (0,\pi)^2 \subset (0,\pi)^2 \setminus B_{2\kappa}((\pi/2,\pi/2)),$$

and we use the volume estimate

$$\mathcal{L}^2\left(\tilde\Phi(\mathbb{S}^1\times(0,\delta))\right) \lesssim \delta(1+|\ln\delta|) .$$

Proof of Theorem 3. Let $\rho_0 \in C^1(\mathbb{T}^2)$ be mean zero along the level sets of H_{c}, and consider $\rho : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$ the solution to (T) with $\rho(0, \cdot) = \rho_0$. The lower bound on the mixing rate in (1.4) and the upper bound on the enhanced dissipation rate in (1.5) are a consequence of Theorem 2. We then prove the upper bound on the mixing rate in (1.4).

Let χ be a smooth function, constant along the levels of H_{c} such that $\chi = 1$ in $B_{\frac{1}{100}}((\pi/2, \pi/2))$ and $\chi = 0$ in the complement of $B_{\frac{1}{10}}((\pi/2, \pi/2))$. It turns out that $\chi\rho(t)$ and $(1-\chi)\rho(t)$ both solve (T). Hence, we can apply (4.18) and (4.19) to get

$$
\begin{aligned}
\sup_{\|\varphi\|_{H^1}\leq 1} \int_{(0,\pi)^2} \rho(t)\varphi \mathrm{d}x &\leq \sup_{\|\varphi\|_{H^1}\leq 1} \int_{(0,\pi)^2} \chi\rho(t)\varphi \mathrm{d}x + \sup_{\|\varphi\|_{H^1}\leq 1} \int_{(0,\pi)^2} (1-\chi)\rho(t)\varphi \mathrm{d}x \\
&\leq C(\varepsilon, \kappa, \|\chi\rho_0\|_{H^1}, \|\chi\rho_0\|_{L^\infty}) \frac{1}{t^{1-\varepsilon}} \\
&\quad + C(\varepsilon, \kappa, \|(1-\chi)\rho_0\|_{H^1}, \|(1-\chi)\rho_0\|_{L^\infty}) \frac{1}{t^{\frac{1}{3}-\varepsilon}} \\
&\leq C(\varepsilon, \kappa, \|\rho_0\|_{H^1}, \|\rho_0\|_{L^\infty}) \frac{1}{t^{\frac{1}{3}-\varepsilon}}\,.
\end{aligned}
$$

To estimate $\|\rho(t)\|_{H^{-1}(\mathbb{T}^2)}$ we use the decomposition (4.15), and sum all the contributions. This concludes the proof. □

4.5. Towards Enhanced Dissipation

Given the upper bound on the mixing rate (1.4), we can obtain a partial enhanced dissipation result for the standard cellular flow (Theorem 4), by adapting the strategy of [13, Theorem 2.1]. Define the average on the streamlines of a function $g \in L^2(\mathbb{T}^2)$ as

$$
P_0 g(x) = \fint_{\Gamma(x)} g \,\mathrm{d}\mathscr{H}^1,
$$

where $\Gamma(x)$ is the connected component of $\{H_{\mathsf{c}} = H_{\mathsf{c}}(x)\}$ containing x, and $P_\perp = 1 - P_0$. Clearly, P_0 and $P_\perp$ are orthogonal projections in L^2. Moreover, $P_0 b \cdot \nabla = 0$ and $P_\perp$ commutes with $b \cdot \nabla$. However, they are not in general well-behaved with the Laplace operator, so that [13, Theorem 2.1] cannot be applied directly. The mixing estimate (1.4) can be phrased as

$$
\|P_\perp \rho(t)\|_{H^{-1}} \leq \frac{a}{\langle t\rangle^p} \|\rho_0\|_{H^1}, \qquad \forall t \geq 0, \quad \forall \rho_0 \in H^1, \quad P_0\rho_0 = 0,
$$

for some constant $a \geq 1$ and $p = 1/3^-$. We will restrict ourselves to initial data supported in one of the cells. We need the following preliminary result.

Lemma 4.7. *Let $\epsilon \in (0, 1)$, and consider $\eta_\epsilon = \eta_\epsilon(h) : [0, 1] \to [0, 1]$ a smooth, increasing cut-off such that $\eta_\epsilon \equiv 1$ on $\{H_{\mathsf{c}} \geq \epsilon\}$ and $\eta_\epsilon \equiv 0$ on $\{H_{\mathsf{c}} \leq \epsilon/2\}$. Then there exists a constant $C = C(H) > 0$ such that*

$$
\|P_0 g(\eta_\epsilon)\|^2_{H^1} + \|P_\perp(g\eta_\epsilon)\|^2_{H^1} \leq C\epsilon^{-1}\|(g\eta_\epsilon)\|^2_{H^1} \leq C\epsilon^{-3}\|g\|^2_{H^1}, \tag{4.20}
$$

for any $g \in H^1$.

Proof. It is enough to prove the estimate for $P_0(g\eta_\epsilon)$. By the coarea formula it is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}h}\int_{\{H_\mathsf{c}=h\}} g\,\mathrm{d}\mathscr{H}^1 = \int_{\{H_\mathsf{c}=h\}} \nabla\cdot\left(\frac{\nabla H_\mathsf{c}}{|\nabla H_\mathsf{c}|}g\right)|\nabla H_\mathsf{c}|\,\mathrm{d}\mathscr{H}^1\,,$$

for every smooth function g supported away from the critical points of H_c. Fix $h\in\mathbb{R}$. It immediately follows that

$$\begin{aligned}|\nabla P_0 g(x)| &\lesssim |\nabla H_\mathsf{c}(x)|\left(P_0\left(\frac{|g|}{|\nabla H_\mathsf{c}|}\right)(x) + P_0\left(\frac{1}{|\nabla H_\mathsf{c}|}\right)(x)P_0(|g|)(x)\right) + P_0(|\nabla g|)(x)\\ &\lesssim C(h)P_0(|g|+|\nabla g|)(x)\end{aligned}$$

for every $x\in\{H_\mathsf{c}=h\}$, where

$$C(h) := \frac{\sup_{H_\mathsf{c}=h}|\nabla H|}{\inf_{H_\mathsf{c}=h}|\nabla H|}\,.$$

Notice that $C(h)\lesssim\epsilon^{-1/2}$ for every $x\in\{H_\mathsf{c}\geq\epsilon\}$. In particular, it holds

$$\|\nabla P_0(g\eta_\epsilon)\|_{L^2} \lesssim \epsilon^{-1/2}(\|g\eta_\epsilon\|_{L^2} + \|\nabla(g\eta_\epsilon)\|_{L^2}) \lesssim \epsilon^{-1/2}\|\nabla(g\eta_\epsilon)\|_{L^2}\,.$$

The second inequality in (4.20) follows by standard properties of cut-off functions, and hence the proof is over. □

In order to prove Theorem 4, we need to estimate the difference between solutions of (A-D) and (T) with the same initial datum. We consider $\tau_0>0$ and ρ,ρ^ν be the solutions of

$$\partial_t\rho + b_\mathsf{c}\cdot\nabla\rho = 0, \qquad \partial_t\rho^\nu + b_\mathsf{c}\cdot\nabla\rho^\nu = \nu\Delta\rho^\nu$$

with $\rho(\tau_0)=\rho^\nu(\tau_0)=\rho^{\tau_0}\in H^1$.

We have the following energy inequalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^\nu\|_{L^2}^2 + 2\nu\|\nabla\rho^\nu\|_{L^2}^2 = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho^\nu\|_{L^2}^2 + 2\nu\|\Delta\rho^\nu\|_{L^2}^2 \leq 2\|\nabla\rho^\nu\|_{L^2}^2. \tag{4.21}$$

Integrating (4.21) in time we obtain a bound on the integral of the H^2 norm as

$$2\nu\int_{\tau_0}^{\tau_0+t}\|\Delta\rho^\nu(s)\|_{L^2}^2\mathrm{d}s \leq 2\int_{\tau_0}^{\tau_0+t}\|\nabla\rho^\nu(s)\|_{L^2}^2\mathrm{d}s + \|\nabla\rho^\nu(\tau_0)\|_{L^2}^2. \tag{4.22}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^\nu-\rho\|_{L^2}^2 = 2\nu\langle\Delta\rho^\nu,\rho^\nu-\rho\rangle_{L^2} \leq 2\nu\|\Delta\rho^\nu\|_{L^2}\|\rho^\nu-\rho\|_{L^2},$$

we deduce $\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^\nu - \rho\|_{L^2} \leq \nu\|\Delta\rho^\nu\|_{L^2}$: integrating in time and recalling $\rho(\tau_0) = \rho^\nu(\tau_0)$ and (4.22) we get

$$\begin{aligned}\|(\rho^\nu - \rho)(\tau_0 + t)\|_{L^2}^2 &\leq \left(\nu \int_{\tau_0}^{\tau_0+t} \|\Delta\rho^\nu(s)\|_{L^2}\mathrm{d}s\right)^2 \\ &\leq \nu t \int_{\tau_0}^{\tau_0+t} \nu\|\Delta\rho^\nu(s)\|_{L^2}^2\mathrm{d}s \\ &\leq \nu t \left(\int_{\tau_0}^{\tau_0+t} \|\nabla\rho^\nu(s)\|_{L^2}^2\mathrm{d}s + \frac{\|\nabla\rho^\nu(\tau_0)\|_{L^2}^2}{2}\right). \end{aligned} \tag{4.23}$$

We are now in position to prove Theorem 4.

Proof of Theorem 4. For $q \in \left(\frac{24}{25}, 1\right)$, we have to show that for all $\nu < \nu_0$ to be determined, we have the inequality

$$2\nu \int_0^{\nu^{-q}} \|\rho^\nu(t)\|_{H^1}^2\mathrm{d}t \geq \delta\|\rho_0\|_{L^2}^2, \qquad \text{with } \delta = \frac{1}{8}.$$

Without loss of generality, we can assume by linearity that $\|\rho_0\|_{L^2} = 1$. Towards a contradiction, we assume that

$$2\nu \int_0^{\nu^{-q}} \|\rho^\nu(t)\|_{H^1}^2\mathrm{d}t < \delta. \tag{4.24}$$

Following the proof in [13, Theorem 2.1], there exists a $\tau_1 \in [0, \nu^{-q} - \nu^{-q/2}]$ so that

$$\nu \int_{\tau_1}^{\tau_1+\nu^{-q/2}} \|\rho^\nu(s)\|_{H^1}^2\mathrm{d}s < \delta\nu^{q/2}. \tag{4.25}$$

We can also find $\tau_0 \in [\tau_1, \tau_1 + \nu^{-q/2}/2]$ such that

$$\nu\|\rho^\nu(\tau_0)\|_{H^1}^2 < 2\delta\nu^q. \tag{4.26}$$

Moreover,

$$\nu \int_{\tau_0}^{\tau_0+\nu^{-q/2}/2} \|\rho^\nu(s)\|_{H^1}^2\mathrm{d}s < \delta\nu^{q/2}, \tag{4.27}$$

by (4.25). Now we take $\rho^{\tau_0} = \rho^\nu(\tau_0)$ as the initial datum for the inviscid problem (T) with initial time τ_0 and denote the solution by $\rho(t + \tau_0)$ with $t \geq 0$. Using that $\|\rho^\nu(\tau_0)\|_{L^2} \leq 1$, estimate (4.23), the properties of τ_0 in (4.26), (4.27), we find

$$\begin{aligned}\|\rho^\nu(\tau_0 + t) - \rho(\tau_0 + t)\|_{L^2}^2 &\leq \nu t \left(\int_{\tau_0}^{\tau_0+t} \|\nabla\rho^\nu(s)\|_{L^2}^2\mathrm{d}s + \frac{\|\nabla\rho^\nu(\tau_0)\|_{L^2}^2}{2}\right) \\ &\leq \frac{\delta}{2}\left(1 + \nu^{q/2}\right), \end{aligned} \tag{4.28}$$

for all $t \in [0, \frac{1}{2}\nu^{-q/2}]$.

We now estimate $P_0\rho^\nu$: since $P_0\rho_0 = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P_0\rho^\nu\|_{L^2}^2 = 2\nu\langle P_0\rho^\nu, P_0\Delta\rho^\nu\rangle \leq 2\nu\left|\langle\rho^\nu, \Delta\rho^\nu\rangle\right| \leq 2\nu\|\rho^\nu\|_{H^1}^2,$$

after an integration in time we deduce by (4.24) that

$$\|P_0\rho^\nu(t)\|_{L^2}^2 \leq \delta \qquad \forall t \in \left[0, \nu^{-q}\right].$$

Consider a cutoff η_ϵ as in Lemma 4.7 and set $\phi_\epsilon^2 = 1 - \eta_\epsilon^2$, where $\epsilon > 0$ will be chosen later. Notice that, by construction, P_0, $P_\perp$ commute with the multiplication by $\eta_\epsilon, \phi_\epsilon$. For $R > 0$, denote by $P_{\leq R}$ the projection onto the span of the Fourier modes $\{\mathrm{e}^{in\cdot x}\}$ with $|n|^2 \leq R$. Since $P_\perp$ and the multiplication by η_ϵ commute with the transport operator, the inviscid mixing estimate (4.29), implies that

$$\|P_{\leq R}P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2 \leq \frac{a^2R}{\langle t\rangle^{2p}}\|P_\perp\eta_\epsilon\rho^\nu(\tau_0)\|_{H^1}^2, \qquad \forall t \geq 0. \tag{4.29}$$

In particular, by (4.26) and Lemma 4.7, we deduce that for every $t \in \left[\frac{\nu^{-q/2}}{4}, \frac{\nu^{-q/2}}{2}\right]$, it holds

$$\|P_{\leq R}P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2 \leq 2C\frac{a^2R}{\epsilon^3\langle t\rangle^{2p}}\delta\nu^{q-1} \leq 2\cdot 4^{2p}Ca^2R\epsilon^{-3}\delta\nu^{q(p+1)-1}.$$

It follows that

$$\begin{aligned}
&\|P_{>R}P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2\\
&\quad= \|P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2 - \|P_{\leq R}P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2\\
&\quad= \|\eta_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2 - \|P_0\eta_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2 - \|P_{\leq R}P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2\\
&\quad\geq \|\eta_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2 - \delta - 2\cdot 4^{2p}Ca^2R\epsilon^{-3}\delta\nu^{q(p+1)-1}
\end{aligned}$$

By (4.28), and recalling $\|\rho^\nu(\tau_0)\|_{L^2} \leq 1$, we obtain the following estimate for ρ^ν:

$$\begin{aligned}
\|P_{>R}P_\perp\eta_\epsilon\rho^\nu(\tau_0+t)\|_{L^2}^2 &\geq \|P_{>R}P_\perp\eta_\epsilon\rho(\tau_0+t)\|_{L^2}^2 - 2\|\rho^\nu(\tau_0+t) - \rho(\tau_0+t)\|_{L^2}\\
&\geq \|\eta_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2 - \delta - \sqrt{2\delta\left(1+\nu^{q/2}\right)} - 2\cdot 4^{2p}Ca^2R
\end{aligned} \tag{4.30}$$

We now estimate from below $\|\eta_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2$: given $\bar{p} \in (2, \infty)$, by Hölder's inequality, the embedding of H^1 in $L^{\bar{p}}$, and since $\mathrm{supp}(\phi_\epsilon) \subset \{H_{\mathsf{c}} \leq \epsilon\}$, there is $C_{\bar{p}} > 0$ such that

$$\|\phi_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2 \leq |\{H_{\mathsf{c}} \leq \epsilon\}|^{1-\frac{2}{\bar{p}}}\|\rho^\nu(\tau_0)\|_{L^{\bar{p}}}^2 \leq C_{\bar{p}}|\epsilon\log\epsilon|^{1-\frac{2}{\bar{p}}}\|\rho^\nu(\tau_0)\|_{H^1}^2,$$

Therefore, assuming $\bar{p} > 3$,

$$\begin{aligned}
\|\eta_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2 &\geq \|\rho^\nu(\tau_0)\|_{L^2}^2 - \|\phi_\epsilon\rho^\nu(\tau_0)\|_{L^2}^2\\
&\geq 1 - \delta - C_{\bar{p}}|\epsilon\log\epsilon|^{1-\frac{2}{\bar{p}}}\|\rho^\nu(\tau_0)\|_{H^1}^2\\
&\geq 1 - \delta - 2C_{\bar{p}}|\epsilon\log\epsilon|^{1-\frac{2}{\bar{p}}}\delta\nu^{q-1}\\
&\geq 1 - \delta - \tilde{C}_{\bar{p}}\epsilon^{1-\frac{3}{\bar{p}}}\delta\nu^{q-1}
\end{aligned}$$

for some $\tilde{C}_{\bar{p}} > 0$. We choose $\epsilon = \frac{\nu^{(1-q)\left(1-\frac{3}{\bar{p}}\right)}}{\tilde{C}_{\bar{p}}^{1-\frac{3}{\bar{p}}}}$ to ensure $\|\eta_\epsilon \rho^\nu(\tau_0)\|_{L^2}^2 \geq 1 - 2\delta$. Therefore, by (4.30),

$$\|P_{>R} P_\perp \eta_\epsilon \rho^\nu(\tau_0 + t)\|_{L^2}^2 \geq 1 - 3\delta - \sqrt{2\delta\left(1 + \nu^{q/2}\right)}$$
$$-2 \cdot 4^{2p} C a^2 R \epsilon^{-3} \delta \nu^{q(p+1)-1},$$

so that

$$\|P_\perp \eta_\epsilon \rho^\nu(\tau_0 + t)\|_{H^1}^2 \geq R\left(1 - 3\delta - \sqrt{2\delta\left(1 + \nu^{q/2}\right)}\right)$$
$$-R^2\left(2 \cdot 4^{2p} C a^2 \epsilon^{-3} \delta \nu^{q(p+1)-1}\right). \tag{4.31}$$

The choice $\delta = \frac{1}{8}$ ensures that $\left(1 - 3\delta - \sqrt{2\delta\left(1 + \nu^{q/2}\right)}\right) > \frac{1}{16}$ for sufficiently small ν, therefore choosing the optimal $R = \left(1 - 3\delta - \sqrt{2\delta\left(1 + \nu^{q/2}\right)}\right) \Big/ 4^{2p+1} C a^2 \epsilon^{-3} \delta \nu^{q(p+1)-1}$, recalling the choice of ϵ, and using again Lemma 4.7, we get

$$\|\rho^\nu(\tau_0 + t)\|_{H^1}^2 \geq \frac{\epsilon^3}{C} \|P_\perp \eta_\epsilon \rho^\nu(\tau_0 + t)\|_{H^1}^2 \geq \frac{\nu^{7-q(7+p)-6(1-q)\frac{3}{\bar{p}}}}{4^{2p+4} \tilde{C}_{\bar{p}}^{6\left(1-\frac{3}{\bar{p}}\right)} C^2 a^2}. \tag{4.32}$$

Integrating this estimate for $t \in \left[\frac{\nu^{-q/2}}{4}, \frac{\nu^{-q/2}}{2}\right]$ and recalling (4.27), we get

$$\frac{1}{8} = \delta > \nu^{1-\frac{q}{2}} \int_{\tau_0}^{\tau_0 + \nu^{-q/2}/2} \|\rho^\nu(s)\|_{H^1}^2 \mathrm{d}s > \frac{\nu^{8-q(8+p)-6(1-q)\frac{3}{\bar{p}}}}{2 \cdot 4^{2p+4} \tilde{C}_{\bar{p}}^{6\left(1-\frac{3}{\bar{p}}\right)} C^2 a^2}, \tag{4.33}$$

which gives a contradiction for ν sufficiently small provided $8 - q(8 + p) - 6(1 - q)\frac{3}{\bar{p}} < 0$. Since we have shown the mixing estimate for the cellular flow for $p \in (0, \frac{1}{3})$ and $\bar{p} \geq 3$ is arbitrarily large, we deduce that the statement holds for every $q > \frac{8}{8+\frac{1}{3}} = \frac{24}{25}$. □

Remark 4.8. [Data supported away from the critical point] If in Theorem 4 we consider initial data supported away from the critical point, we can improve the enhanced dissipation time-scale significantly. In the same spirit as above, *we now fix* $\epsilon > 0$ so that the support of the datum is contained in the set $\{H_{\mathsf{c}} \geq \epsilon\}$. The main observation is that such data are mixed by (T) at the faster rate $1/t^p$ with $p \in (0, 1)$.

In this way, the analogue of (4.31) still holds (with ϵ small enough independent of ν) and thus (4.32) now reads

$$\|\rho^\nu(\tau_0 + t)\|_{H^1}^2 \geq \frac{1}{C} \nu^{1-q(p+1)},$$

for some constant $C \geq 1$. Arguing as in (4.33), the constraint simplifies to $2 - q(p+2) < 0$, in analogy with [13, Theorem 2.1]. In turn, this gives $q > 2/3$ for the standard cellular flow.

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Appendix A. Analytic mixing and stationary phase in action-angle variables

Let $\Omega = \mathbb{S}^1 \times (0, \pi/2)$ and $g \in C^1(0, \pi/2)$. We define the weighted Sobolev space H^1_g through the norm

$$\|f\|^2_{H^1_g} = \int_\Omega \left(|f(\theta, s)|^2 + |\partial_s f(\theta, s)|^2\right) g(s)\mathrm{d}s\mathrm{d}\theta.$$

Via a stationary phase argument, the next theorem gives an explicit bound on the solution of a transport equation that is helpful in estimating the decay of correlation.

Theorem 5. *Fix $\delta, \delta' \in (0, 1/4)$. Let $g, T \in C^1(0, \pi/2)$ be positive functions, and let $f : \mathbb{R} \times \Omega \to \mathbb{R}$ solve*

$$\partial_t f + \frac{1}{T(s)} \partial_\theta f = 0, \quad \textit{for } (t, \theta, I) \in \mathbb{R} \times \Omega. \tag{A.1}$$

Assume further that $\int_{\mathbb{S}^1} f(0,\theta,s)\mathrm{d}\theta = 0$ *for any* $s \in (0,1)$. *Then, for any* $\varphi \in C^1(\Omega)$ *it holds*

$$\int_{\mathbb{S}^1\times(\delta,1-\delta')} f(t,\theta,s)\varphi(\theta,s)g(s)\,\mathrm{d}s\,\mathrm{d}\theta \le \|f(0,\cdot)\|_{H^1_g}\|\varphi\|_{H^1_g}\, r(t) \tag{A.2}$$

where

$$\begin{aligned} r(t) = \frac{1}{t}\left\|g^{-\frac{1}{2}}\right\|^2_{L^2(\delta,\pi/2-\delta')}&\left(\frac{T^2 g}{|T'|}(\delta) + \frac{T^2 g}{|T'|}(\pi/2-\delta')\right.\\ &\left.+\left\|\left(\frac{T^2 g}{T'}\right)'\right\|_{L^1(\delta,\pi/2-\delta')} + \left\|\frac{T^2 g^{\frac{1}{2}}}{T'}\right\|_{L^2(\delta,\pi/2-\delta')}\right)\\ &+\frac{1}{t}\left\|g^{-\frac{1}{2}}\right\|_{L^2(\delta,\pi/2-\delta')}\left\|\frac{T^2 g^{\frac{1}{2}}}{T'}\right\|_{L^2(\delta,\pi/2-\delta')}. \end{aligned} \tag{A.3}$$

Proof. Expanding in Fourier series in the variable θ we obtain

$$f(t,\theta,s) = \sum_{k\in 4\mathbb{Z}} f_k(t,s)\mathrm{e}^{ik\theta}, \qquad f_k(t,s) = f_k(0,s)\mathrm{e}^{-\frac{ikt}{T(s)}}.$$

Notice that $f_0(0,s) = 0$ for every $s \in (0,1)$. We compute

$$\int_\delta^{\pi/2-\delta'}\int_{\mathbb{S}^1} f(t,\theta,s)\varphi(\theta,s)g(s)\mathrm{d}\theta\mathrm{d}s = \sum_k \int_\delta^{\pi/2-\delta'} f_k(t,s)\overline{\varphi_k(s)}g(s)\mathrm{d}s$$

Integrating by parts, for every $k \in 4\mathbb{Z}$ we have

$$\begin{aligned} &\int_\delta^{\pi/2-\delta'} f_k(t,s)\overline{\varphi_k(s)}g(s)\mathrm{d}s\\ &\quad= \int_\delta^{\pi/2-\delta'} f_k(0,s)\frac{1}{ikt}\frac{T^2(s)}{T'(s)}\frac{d}{\mathrm{d}s}\left(\mathrm{e}^{-\frac{ikt}{T(s)}}\right)\overline{\varphi_k(s)}g(s)\mathrm{d}s\\ &= f_k(0,s)\frac{1}{ikt}\frac{T^2(s)}{T'(s)}\mathrm{e}^{-\frac{ikt}{T(s)}}\overline{\varphi_k(s)}g(s)\Big|_{s=\delta}^{\pi/2-\delta'}\\ &\quad-\frac{1}{ikt}\int_\delta^{\pi/2-\delta'}\partial_s f_k(0,s)\frac{T^2(s)}{T'(s)}\mathrm{e}^{-\frac{ikt}{T(s)}}\overline{\varphi_k(s)}g(s)\mathrm{d}s\\ &\quad-\frac{1}{ikt}\int_\delta^{\pi/2-\delta'} f_k(0,s)\left(\frac{T^2 g}{T'}\right)'(s)\mathrm{e}^{-\frac{ikt}{T(s)}}\overline{\varphi_k(s)}\mathrm{d}s\\ &\quad-\frac{1}{ikt}\int_\delta^{\pi/2-\delta'} f_k(0,s)\frac{T^2(s)}{T'(s)}\mathrm{e}^{-\frac{ikt}{T(s)}}\overline{\varphi_k'(s)}g(s)\mathrm{d}s\,. \end{aligned}$$

We can estimate $\|f_k(0,\cdot)\|_{L^\infty} \le \|f_k(0,\cdot)\|_{L^1} + \|\partial_s f_k(0,\cdot)\|_{L^1} \le \|f_k(0,\cdot)\|_{H^1_g}$ $\left\|g^{-\frac{1}{2}}\right\|_{L^2}$ and $\|\varphi_k\|_{L^\infty} \le \|\varphi_k\|_{H^1_g}\left\|g^{-\frac{1}{2}}\right\|_{L^2}$ to get

$$\left|\int_\delta^{1\pi/2-\delta'} f_k(t,s)\overline{\varphi_k(s)}g(s)\mathrm{d}s\right| \le \frac{r(t)}{|k|}\|f_k(0,\cdot)\|_{H^1_g}\|\varphi_k\|_{H^1_g},$$

with

$$r(t) = \frac{1}{t}\left\|g^{-\frac{1}{2}}\right\|_{L^2}^2\left(\frac{T^2 g}{|T'|}(\delta)+\frac{T^2 g}{|T'|}(1-\delta')+\left\|\left(\frac{T^2 g}{T'}\right)'\right\|_{L^1}\right) + \frac{1}{t}\left\|g^{-\frac{1}{2}}\right\|_{L^2}\left\|\frac{T^2 g^{\frac{1}{2}}}{T'}\right\|_{L^2}.$$

Summing over k we get

$$\left|\int_{\mathbb{S}^1\times(\delta,\pi/2-\delta')} f(t,\theta,s)\varphi(\theta,s)g(s)\,\mathrm{d}s\,\mathrm{d}\theta\right| \le \|f(0,\cdot)\|_{H^1_g}\|\varphi\|_{H^1_g} r(t). \quad \text{(A.4)}$$

This concludes the proof. □

A.1 Change of Variables

The introduction of action-angle variables simplify the structure of a general transport equation to an equation of the form (A.1). The estimate (A.2) then provides a correlation estimate which can be understood in a negative Sobolev space with respect to the action-angle coordinates. Therefore, an estimate on the change of coordinates may be needed to understand mixing in the usual H^{-1} sense.

Lemma A.1. *Let $\rho = \rho(t,x)$ and $f(t,\theta,s) = \rho(t,\Phi(\theta,s))$, for some change of coordinate Φ. Assume that the Jacobian $J_\Phi(\theta,s) = g(s)$ for some smooth positive function g. If*

$$\sup_{\|\phi\|_{H^1_g}\le 1}\int f(t,\theta,s)\phi(\theta,s)g(s)\mathrm{d}s \le \|f(0,\cdot)\|_{H^1_g} r(t)$$

for some $r(t)$, then

$$\|\rho(t)\|_{H^{-1}} \lesssim \|\rho(0)\|_{H^1}(1+\mathrm{Lip}(\Phi))^2 r(t).$$

Proof. The proof is a direct computation

$$\begin{aligned}\|\rho(t)\|_{H^{-1}} &= \sup_{\|\psi\|_{H^1}\le 1}\int \rho(t,x)\psi(x)\mathrm{d}x\\ &= \sup_{\|\psi\|_{H^1}\le 1}\int f(t,\theta,s)\psi(\Phi(\theta,s))g(s)\mathrm{d}\theta\mathrm{d}s\\ &\le \sup_{\|\phi\|_{H^1_g}\le 1+\mathrm{Lip}(\Phi)}\int f(t,\theta,s)\phi(\theta,s)g(s)\mathrm{d}\theta ds\\ &\le (1+\mathrm{Lip}(\Phi))\|f(0,\cdot)\|_{H^1_g} r(t)\\ &\le (1+\mathrm{Lip}(\Phi))^2\|\rho(0,\cdot)\|_{H^1} r(t),\end{aligned}$$

as needed. □

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