Homogenization of Griffith's Criterion for Brittle Laminates

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Communicated by K. BHATTACHARYA

Abstract

We consider a periodic, linear elastic laminate with a brittle crack, evolving along a prescribed path according to Griffith's criterion. We study the homogenized limit of this evolution, as the size of the layers vanishes. The limit evolution is governed again by Griffith's criterion, in terms of the energy release (of the homogenized elastic energy) and an effective toughness, which, in general, differs from the weak* limit of the periodic toughness. We provide a variational characterization of the effective toughness and, by the energy identity, we link the toughening effect (in the limit) to the micro-instabilities of the evolution (in the periodic laminate). Finally, we provide a couple of explicit calculations of the effective toughness in the anti-plane setting, showing, in particular, an example of toughening by elastic contrast.

1. Introduction

From the mathematical point of view, the homogenization of evolution equations is not a simple development of the (well established) homogenization theory for elliptic equations. A remarkable example was presented by Tartar in [23]. Consider a Cauchy problem of the form

$$\begin{cases} \dot{u}_\varepsilon(x,t) + a_\varepsilon(x)u_\varepsilon(x,t) = f(x,t) \\ u_\varepsilon(x,0) = u^0(x). \end{cases}$$

If $a_\varepsilon \overset{*}{\rightharpoonup} a_0$ (which is the case in homogenization), then, up to subsequences, $u_\varepsilon \rightharpoonup u_0$, where u_0 solves an evolution equation of the form

$$\dot{u}_0(x,t) + a_0(x)u_0(x,t) + \int_0^t K(x,t-\tau)u(\tau)\,\mathrm{d}\tau = f(x,t).$$

Therefore, a memory effect (in time) appears in the limit, as a consequence of the heterogeneity (in space) of the coefficients. This example was studied also in [15, Section 3.5.2] from a different perspective: setting the problem in spaces of measures, Mielke showed that different limit evolution may occur, as the system is endowed with different energy-dissipation structures. For instance, if $a_\varepsilon(x) = a(x/\varepsilon)$ and $f = 0$, then suitable choices of energy and dissipation lead to the following limit equations:

$$\dot{u}_0(x, t) + (\min a) \, u_0(x, t) = 0, \quad \dot{u}_0(x, t) + (\max a) \, u_0(x, t) = 0$$

(clearly, different subsequences are extracted from u_ε). In [15] this example is presented in the context of evolutionary Γ-convergence, which gives a notion of convergence for a class of doubly non-linear parabolic evolutions of the form

$$\begin{cases} \partial_v \mathcal{R}_\varepsilon(u_\varepsilon(x, t), \dot{u}_\varepsilon(x, t)) = -\nabla \mathcal{E}_\varepsilon(u_\varepsilon(x, t)), \\ u_\varepsilon(0, x) = u^\varepsilon(x), \end{cases}$$

usually when $\mathcal{E}_\varepsilon$ Γ-converge to $\mathcal{E}_0$ (for the general theory of Γ-convergence we refer to the book of Dal Maso [10]). When the dissipation is simply of the form $\mathcal{R}_\varepsilon(u, v) = \frac{1}{2}\|v\|^2$, the abstract results of Sandier and Serfaty [22] provide sufficient conditions which guarantee the (weak) convergence of u_ε to the solution u_0 of the parabolic problem

$$\begin{cases} \dot{u}_0(x, t) = -\nabla \mathcal{E}_0(u_0(x, t)), \\ u_0(0, x) = u^0(x). \end{cases}$$

For instance, under suitable regularity assumptions, convergence holds if the initial data is well-prepared, that is, if $u^\varepsilon \rightharpoonup u^0$ and $\mathcal{E}_\varepsilon(u^\varepsilon) \to \mathcal{E}_0(u^0)$, and if

$$\|\nabla \mathcal{E}_0\|(u) \leqq \Gamma\text{-}\liminf_{\varepsilon \to 0} \|\nabla \mathcal{E}_\varepsilon\|(u).$$

In the rate-independent setting, similar conditions are sufficient also for the convergence of balanced (or vanishing) viscosity evolutions; see for example [19]. The conditions in [22] are actually far more general, and include the case of ε-depending norms

$$\|\nabla \mathcal{E}_0\|_0(u) \leqq \Gamma\text{-}\liminf_{\varepsilon \to 0} \|\nabla \mathcal{E}_\varepsilon\|_\varepsilon(u), \tag{1}$$

which will be relevant also in our context. Finally, consider a family of minimizing movement (or implicit Euler) schemes of the form

$$u_\varepsilon((k + 1)\tau_\varepsilon) \in \operatorname{argmin}\left\{\mathcal{E}_\varepsilon(u) + \frac{1}{2}\tau_\varepsilon^{-1}\|u - u_\varepsilon(k\tau_\varepsilon)\|^2\right\},$$

where $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$. For when the energy $\mathcal{E}_\varepsilon$ is independent of ε, it is well known that (the interpolation of) the above time-discrete evolutions converge, as $\varepsilon \to 0$, to the gradient flow of $\mathcal{E}_\varepsilon$. However, by virtue of the examples provided in Braides [8], the limit evolution will, in general, depend on the relationship between ε (which parametrizes the family of energies) and the time step τ_ε.

From the mechanical point of view our work is motivated by experimental and numerical results on the effective toughness of brittle composites (see for instance [13,24] and the references therein). In general, when dealing with a periodically arranged composite, with a very fine microstructure, it is natural to introduce also an homogenized material reflecting "in average" the mechanical behavior of the composite. For instance, the equilibrium configuration (in terms of displacement, stress and energy) of a linear elastic composite can be approximated by the equilibrium configuration of a homogeneous linear elastic material, obtained with "suitable averages" of the stiffness tensors of the underlying constituents.

For quasi-static fracture propagation, the mechanical behavior is instead described by the crack set, at time t, together with the equilibrium configuration of the body, given the crack at time t; furthermore, the propagation is governed by a rate-independent evolution law for the crack (Griffith's criterion), together with momentum balance (an elliptic PDE). For the latter, (static) homogenization is suitable, and several mathematical approaches are available in the literature; see for example [5,9,17] and the references therein. For the former, that is, Griffith's criterion, there is a need instead to develop some kind of "evolutionary homogenization" or, better, an "evolutionary convergence" for rate-independent systems (specified hereafter). In practice, see for instance [13,24], the goal would be to find an effective "averaged" toughness in such a way that the quasi-static evolution of a crack in the brittle composite is approximated by the quasi-static evolution of a crack in a brittle homogeneous material characterized by the (static) homogenization of the stiffness tensors together with the effective toughness. In §2.8, we provide an example of a laminate with explicit computations of effective toughness and homogenized stiffness. This example highlights some peculiar features: the effective toughness depends on the volume fraction of the layers, on their toughness, and also on the coefficients of their stiffness matrices, furthermore, it can be larger than the toughness of the single brittle layers, and larger than the weak* limit of the periodic toughness. These properties are qualitatively consistent with the numerical and experimental results of Hossain et al. [13], where toughening (that is, the macroscopic increase in toughness) occurs as a result of the microscopic elastic and toughness heterogeneities. In mechanics, most of the interest is indeed driven by the toughening and the strengthening effects of composites [6]. In general, toughening is also due to several microscopic "geometrical features" of the crack, such as micro-cracking, branching, debonding and tortuosity; a mathematical result on toughening by the homogenization of highly oscillating crack paths is given in [4], where Barchiesi employs a modified Γ-convergence framework in order to take into account the irreversibility constraint. In the context of Coulomb friction, a characterization of the effective friction coefficient (depending on both elastic and friction coefficients of the layers) is provided in [3].

Let us turn to our results. We consider a brittle crack which propagates horizontally in a linear elastic laminate composed by n periodic layers; each layer is made of a layer of material A and B with volume ratio λ and $1 - \lambda$, respectively. We distinguish between vertical [13] and horizontal layers: in the former setting, the crack crosses the layers, while in the latter it lays in the interface between two layers. We do not consider a "surfing boundary condition" [13], but a Dirichlet boundary

condition $u = f(t)g$ for the displacement u on a portion $\partial_D \Omega$ of the boundary. For technical reasons, and making reference to the experimental and numerical results of [13], we confine a priori the geometry of the crack to a straight line, even if, in general, the crack may deflect at interfaces; see the well-known [12], which leads to (macroscopic) anisotropic toughness in layered materials [7].

We begin with the more delicate case, which is that of vertical layers. We denote the toughness by $G_n^c(l)$ and the energy release by $G_n(t, l)$, where t is time and l indicates the position of the crack tip. Note that G_n^c depends on l and that it is periodic. On the contrary, $G_n(t, \cdot)$ is not periodic and, to the best of our knowledge, it is not known if it is defined at the interfaces between the two materials; for this reason, we actually employ the right lower semi-continuous envelope, which is indeed the natural choice for Griffith's criterion (see Remark 2). By linearity, the dependence on time is much easier, since we can write $G_n(t, l) = f^2(t)\mathcal{G}_n(l)$, where $\mathcal{G}_n$ is computed with boundary condition $u = g$, that is, for $f(t) = 1$. Denoting by $\ell_n(t)$ the position of the tip at time t, the quasi-static propagation is governed by Griffith's criterion:

(i) $G_n(t, \ell_n(t)) \leqq G_n^c(\ell_n(t))$,
(ii) if $G_n(t, \ell_n(t)) < G_n^c(\ell_n(t))$ then $\dot{\ell}_n^+(t) = 0$ (that is, the right time derivative vanishes).

It is well known, see for example [18,21], that, in general, there are different solutions satisfying Griffith's criterion, and that these solutions may be discontinuous (in time). Discontinuities are basically due to the fact that Griffith's criterion is rate-independent (that is, invariant under time reparametrization). From the mathematical point of view, the behavior in the discontinuities characterizes the different notions of evolution; see [18]. Here, we denote by $J(\ell_n)$ the set times where ℓ_n jumps, and following [21], we require, besides (i) and (ii), that

(iii) if $t \in J(\ell_n)$ then $G_n(t, l) \geqq G_n^c(l)$ for every $l \in [\ell_n^-(t), \ell_n^+(t)]$.

This condition means that the transition between the equilibria $\ell_n^-(t)$ and $\ell_n^+(t)$ is unstable (or catastrophic). Notice that, for $l = \ell_n^-(t)$, the above conditions, put together, imply that $G_n(t, \ell_n^-(t)) = G_n^c(\ell_n^-(t))$, therefore both stable and unstable propagation occurs once the critical value is attained. Evolutions satisfying (i), (ii), and (iii) can be obtained also by a vanishing viscosity approach [14], taking the limit of continuous solutions of rate-dependent models. As observed in the numerical simulations [13], discontinuities often occur at those interfaces where the crack passes from the material with higher to that with lower toughness. Therefore, in our evolution ℓ_n, we should expect many of these "microscopic jumps" in the order n of the number of layers; these discontinuities, which "disappear" in the homogenized limit, will produce the toughening effect.

At this point, let us describe the homogenized limit. Denote by $G_{hom}(t, l)$ the energy release of the homogenized elastic energy. By Helly's theorem we know that $\ell_n \to \ell$ pointwise (upon extracting a subsequence). The goal is to define an effective toughness G_{eff}^c in such a way that the limit ℓ satisfies Griffith's criterion:

(i) $G_{hom}(t, \ell(t)) \leqq G_{eff}^c(\ell(t))$,
(ii) if $G_{hom}(t, \ell(t)) < G_{eff}^c(\ell(t))$ then $\dot{\ell}^+(t) = 0$ (that is, the right time derivative vanishes),

(iii) if $t \in J(\ell)$ then $G_{hom}(t, l) \geq G^c_{eff}(l)$ for every $l \in [\ell^-(t), \ell^+(t))$.

Notice that, in general, $G_{hom}(t, \cdot)$ is neither the pointwise nor the Γ-limit of $G_n(t, \cdot)$ (an explicit example is given in Section 2.8). However, a variational convergence in the spirit of (1) holds. Indeed, writing again that $G_{hom}(t, l) = f^2(t)\mathcal{G}_{hom}(l)$, the effective toughness is defined in such a way that

$$\frac{\mathcal{G}_{hom}(l)}{G^c_{eff}(l)} = \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n(l)}{G^c_n(l)} \,. \tag{2}$$

First of all, note that, in general, G^c_{eff} may depend on l and, through the energy release, on several other parameters of the problem, in particular, it does not depend only on the toughness of material A and B; in mechanical terms, it is an R-curve (see for example [2]). At the current stage, it is hard to guess if G^c_{eff} should always be a constant. In our examples it is so, on the contrary, in the experimental measures of [6], the effective toughness is surely not constant. What is clear is the fact that the effective toughness depends on the elastic contrast and it can be higher than the toughness of the underlying materials. Indeed, in the explicit calculations of Section 2.8, we show that (under suitable assumptions)

$$G^c_{eff} = \lambda \max\left\{G^c_A,\ G^c_B \frac{\mu_{B,1}}{\mu_{A,1}}\right\} + (1 - \lambda) \max\left\{G^c_A \frac{\mu_{A,1}}{\mu_{B,1}},\ G^c_B\right\},$$

where G^c_A and G^c_B are the toughness, while $\mu_{A,1}$ and $\mu_{B,1}$ are the components of the stiffness matrix in the anti-plane setting. Moreover, independently of this example, if $\mathcal{G}_n$ converge uniformly to $\mathcal{G}_{hom}$, then $G^c_{eff} = \max\{G^c_A, G^c_B\}$.

Finally, let us compare (2) with (1). In the context of fracture, the energy release $\mathcal{G}_n$ plays the role of the slope $\|\nabla \mathcal{E}_n\|$; the ratio $\mathcal{G}_n/G^c_n$ is instead some sort of energy release with respect to the dissipation metric induced by the toughness G^c_n, and plays the role of $\|\nabla \mathcal{E}_n\|_n$; in these terms, the effective toughness plays the role of the norm $\|\cdot\|_0$ in (1) or, more precisely, of the smallest norm which makes (1) true, and for this reason in the definition of G^c_{eff} an equality is needed.

In the case of horizontal layers with an interface crack the picture is much simpler. Indeed, the toughness G^c is constant and independent of n, and moreover the energy release $\mathcal{G}_n$ is well defined and converges locally uniformly to $\mathcal{G}_{hom}$; this is enough to prove that the evolutions ℓ_n converge to an evolution ℓ which satisfies Griffith's criterion with energy release G_{hom} and toughness $G^c_{eff} = G^c$.

2. Vertical Layers with Horizontal Crack

Let $\Omega = (0, L) \times (-H, H)$ be the (uncracked) reference configuration. For $l \in (0, L]$ let $K_l = [0, l] \times \{0\}$ be the crack; in the presence of a crack K_l the reference configuration is then $\Omega \setminus K_l$. We denote by $\partial_D \Omega$, independently of l, the union of the sets $\{0\} \times (-H, 0) \cup (0, H)$ and $\{L\} \times (-H, H)$. Let $g \in H^{1/2}(\partial_D \Omega)$. To fix the ideas,

$$g(x) = \begin{cases} -1 & x \in \{0\} \times (-H, 0), \\ 1 & x \in \{0\} \times (0, H), \\ 0 & x \in \{L\} \times (-H, H) \end{cases}$$
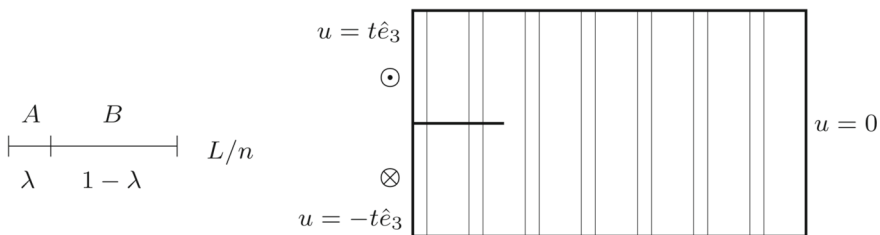
Fig. 1. An example of vertical layers and horizontal crack (along the mid line)

is an admissible function (see also Fig. 1). By abuse of notation we still denote by g a lifting of the boundary datum. Given $l \in (0, L]$, the spaces of admissible displacements and admissible variations are, respectively,

$$\mathcal{U}_l = \{u \in H^1(\Omega \backslash K_l) : u = g \text{ in } \partial_D \Omega\}, \quad \mathcal{V}_l = \{v \in H^1(\Omega \backslash K_l) : v = 0 \text{ in } \partial_D \Omega\}.$$

Note that $\mathcal{U}_l \neq \emptyset$, since $g \in H^{1/2}(\{L\} \times (-H, H))$. Clearly, if $l_1 < l_2$, then $\mathcal{U}_{l_1} \subset \mathcal{U}_{l_2}$, and thus $\mathcal{U}_l \subset \mathcal{U}_L \subset H^1(\Omega \backslash K_L)$ for every $l \in (0, L]$. All the spaces $\mathcal{V}_l$ and the sets $\mathcal{U}_l$ are endowed with the $H^1(\Omega \backslash K_L)$-norm. Splitting $\Omega \backslash K_L$ into the disjoint union of $\Omega^+ = (0, L) \times (0, H)$ and $\Omega^- = (0, L) \times (-H, 0)$, a uniform Poincaré inequality holds: there exists $C_P > 0$ such that, for every $l \in (0, L]$ and $v \in \mathcal{V}_l$,

$$\int_{\Omega \backslash K_l} |v|^2 \, \mathrm{d}x \leqq C_P \int_{\Omega \backslash K_l} |\nabla v|^2 \, \mathrm{d}x.$$

The conditions given above are independent of the periodic structure and hold throughout the paper, for both abstract results and explicit computations. Actually, some of the results hold under more general conditions, depending on the specific setting. For instance, the abstract results hold also in the plane-strain setting, see Section 2.6.1, however, in this context we do not have any explicit example; for this reason and for the sake of simplicity, we consider the antiplane setting presented above.

2.1. Energy and Energy Release

In this section we assume that Ω is a composite, made of $n \in \mathbb{N} \backslash \{0\}$ periodic vertical layers of thickness $l_n = L/n$; each layer itself is composed of a vertical layer of material A with thickness λl_n, for $\lambda \in (0, 1)$, and a vertical layer of material B with thickness $(1 - \lambda)l_n$. More precisely, let $l_{n,k} = k \, l_n$ for $k = 0, \ldots, n$ and $l_{n,k+\lambda} = l_{n,k} + \lambda l_n = (k + \lambda)l_n$ for $k = 0, \ldots, n - 1$; the layers of materials A are of the form $(l_{n,k}, l_{n,k+\lambda}) \times (-H, H)$ while those of material B are of the form $(l_{n,k+\lambda}, l_{n,k+1}) \times (-H, H)$. For later convenience, we also introduce the notation $\Lambda_n = \{l \in [0, L] : l = l_{n,k} \text{ or } l = l_{n,k+\lambda}\}$ so that the interfaces between material A and B (in the sound material) are of the form $\{l\} \times (-H, H)$ for $l \in \Lambda_n \backslash \{0, L\}$. We will also employ the notation $\Omega_{n,k} = (l_{n,k}, l_{n,k+1}) \times (-H, H)$ and

$$\Omega_{n,A} = \bigcup_{k=0}^{n-1} (l_{n,k}, l_{n,k+\lambda}) \times (-H, H), \quad \Omega_{n,B} = \bigcup_{k=0}^{n-1} (l_{n,k+\lambda}, l_{n,k+1}) \times (-H, H).$$

We define in a similar way the crack sets $K_{n,k} = (l_{n,k}, l_{n,k+1}) \times \{0\}$, and then $K_{n,A}$ and $K_{n,B}$.

We assume that A and B are elastic brittle materials. We denote by C_A the stiffness matrix of material A, of the form

$$C_A = \begin{pmatrix} \mu_{A,1} & 0 \\ 0 & \mu_{A,2} \end{pmatrix}, \quad \mu_{A,i} > 0 \text{ for } i = 1, 2,$$

and we denote by $G_A^c > 0$ the toughness of A; we employ a similar notation for material B. Let $C_n : \Omega \backslash (\Lambda_n \times (-H, H)) \to \{C_A, C_B\}$ be the periodic stiffness matrix and let $G_n^c : [0, L] \backslash \Lambda_n \to \{G_A^c, G_B^c\}$ be the periodic toughness.

The linear elastic energy $\mathcal{W}_{l,n} : \mathcal{U}_l \to \mathbb{R}$ is given by

$$\mathcal{W}_{l,n}(u) = \tfrac{1}{2} \int_{\Omega \backslash K_l} \nabla u \, C_n \nabla u^T \, dx.$$

Denote by $u_{l,n} \in \mathcal{U}_l$ the (unique) minimizer of the energy $\mathcal{W}_{l,n}$; clearly $u_{l,n}$ is also the (unique) solution of the variational problem

$$\int_{\Omega \backslash K_l} \nabla u \, C_n \nabla v^T \, dx = 0 \quad \text{for every } v \in \mathcal{V}_l.$$

Note that the above bi-linear form is coercive and continuous, uniformly with respect to $l \in (0, L]$ and $n \in \mathbb{N}$. In the sequel we will often employ the condensed (or reduced) elastic energy $\mathcal{E}_n : (0, L] \to \mathbb{R}$ given by

$$\mathcal{E}_n(l) = \mathcal{W}_{l,n}(u_{l,n}) = \min\{\mathcal{W}_{l,n}(u) : u \in \mathcal{U}_l\}.$$

The following lemma contains the fundamental properties of this energy (for n fixed):

Lemma 1. *The energy $\mathcal{E}_n$ is decreasing, continuous, and of class $C^1((0, L) \backslash \Lambda_n)$. In particular, if $l \in (l_{n,k}, l_{n,k+\lambda})$ the derivative takes the form*

$$\mathcal{E}_n'(l) = \int_{\Omega \backslash K_l} \nabla u_{l,n} C_n E \nabla u_{l,n}^T \phi' dx \quad where \quad E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

while $\phi \in W^{1,\infty}(0, L)$, $\mathrm{supp}(\phi) \subset (l_{n,k}, l_{n,k+\lambda})$, $\phi(l) = 1$ and, by abuse of notation, $\phi' = \partial_l \phi(x_1)$. A representation like (3) holds also for $l \in (l_{n,k+\lambda}, l_{n,k+1})$.

Remark 1. Note that $\mathcal{E}_n$ is not periodic, unless it is constant. Note also that the support of the auxiliary function ϕ shrinks when $n \to \infty$.

Proof. We provide a short proof, following [20]. For $l_1 < l_2$ we have $\mathcal{U}_{l_1} \subset \mathcal{U}_{l_2}$ and thus $\mathcal{E}_n(l_1) \geqq \mathcal{E}_n(l_2)$. If $l_m \to l$, then $\mathcal{E}_n(l_m) \to \mathcal{E}_n(l)$ and $u_{l_m,n} \to u_{l,n}$ strongly in $H^1(\Omega \backslash K_L)$.

Let us compute the derivative of the energy $\mathcal{E}_n(l)$ for $l \notin \Lambda_n$. For $h \ll 1$ let $\Psi_h(x) = (x_1 + h\phi(x_1), x_2)$ be a diffeomorphism of $\Omega \backslash K_l$ onto $\Omega \backslash K_{l+h}$. Note that

$\Psi_h(x) = x$ out of the layer $(l_{n,k}, l_{n,k+\lambda}) \times (-H, H)$. Using the change of variable $w = u \circ \Psi_h$ the energy $\mathcal{W}_{l+h,n} : \mathcal{U}_{l+h} \to \mathbb{R}$ reads as

$$\mathcal{W}_{l+h,n}(u) = \overline{\mathcal{W}}_{l,h,n}(w) = \tfrac{1}{2} \int_{\Omega \setminus K_l} \nabla w \, C_n (I + h M_h \phi') \nabla w^T \, dx,$$

where

$$M_h = \begin{pmatrix} -(1 + h\phi')^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

If $u_{l+h,n} \in \operatorname{argmin}\{\mathcal{W}_{l+h,n}(u) : u \in \mathcal{U}_{l+h}\}$, then $w_{l,h,n} = (u_{l+h,n} \circ \Psi_h) \in \operatorname{argmin}\{\overline{\mathcal{W}}_{l,h,n}(w) : w \in \mathcal{U}_l\}$ and $\mathcal{E}_n(l + h) = \overline{\mathcal{W}}_{l,h,n}(w_{l,h,n})$. Hence,

$$\mathcal{E}'_n(l) = \lim_{h \to 0} \frac{\mathcal{E}_n(l + h) - \mathcal{E}_n(l)}{h} = \lim_{h \to 0} \frac{\overline{\mathcal{W}}_{l,h,n}(w_{l,h,n}) - \mathcal{W}_{l,n}(u_{l,n})}{h}.$$

Writing the variational formulations for $w_{l,h,n}$ and $u_{l,n}$, we have

$$\int_{\Omega \setminus K_l} \nabla w_{l,h,n} \, C_n (I + h M_h \phi') \nabla v^T \, dx = \int_{\Omega \setminus K_l} \nabla u_{l,n} \, C_n \nabla v^T \, dx = 0 \quad \text{for every } v \in \mathcal{V}_l.$$

Then, being that $(w_{l,h,n} - u_{l,n}) \in \mathcal{V}_l$, we can re-write the energies as

$$\overline{\mathcal{W}}_{l,h,n}(w_{l,h,n}) = \tfrac{1}{2} \int_{\Omega \setminus K_l} \nabla w_{l,h,n} \, C_n (I + h M_h \phi') \nabla u_{l,n}^T \, dx,$$

$$\mathcal{W}_{l,n}(u_{l,n}) = \tfrac{1}{2} \int_{\Omega \setminus K_l} \nabla w_{l,h,n} \, C_n \nabla u_{l,n}^T \, dx.$$

Hence,

$$\frac{\overline{\mathcal{W}}_{l,h,n}(w_{l,h,n}) - \mathcal{W}_{l,n}(u_{l,n})}{h} = \tfrac{1}{2} \int_{\Omega \setminus K_l} \nabla w_{l,h,n} \, C_n M_h \nabla u_{l,n}^T \, \phi' \, dx.$$

As $h \to 0$, $M_h \to 2E$ in $L^\infty(\Omega; \mathbb{R}^{2 \times 2})$ and $w_{l,h,n} \rightharpoonup u_{l,n}$ in $H^1(\Omega \setminus K_l)$, therefore we obtain (3).

If $l_m \to l$ and $l_{n,k} < l < l_{n,k+\lambda}$ we can choose ϕ to be independent of m (sufficiently large). Moreover, $\nabla u_{l_m,n} \to \nabla u_{l,n}$ in $L^2(\Omega \setminus K_L, \mathbb{R}^2)$. Therefore, by the representation (3) we get $\mathcal{E}'_n(l_m) \to \mathcal{E}'_n(l)$. $\qquad\square$

By virtue of the above Proposition, for every $l \notin \Lambda_n$, we can define the energy release

$$\mathcal{G}_n(l) = -\partial_l^+ \mathcal{E}_n(l) = -\partial_l \mathcal{E}_n(l). \qquad (4)$$

In the sequel it will be convenient to extend the toughness and the energy release to Λ_n (even if in general the energy is not differentiable in Λ_n). We define

$$\mathcal{G}_n(l) = \liminf_{s \to l^+} \mathcal{G}_n(s) \text{ if } l \neq L \quad \text{and} \quad \mathcal{G}_n(L) = 0. \qquad (5)$$

With this definition the function $\mathcal{G}_n : [0, L] \to [0, +\infty]$ is right lower semi-continuous. In Remark 2 we will see that for $l \neq L$ this extension is actually the

only one compatible with Griffith's criterion; for $l = L$ it is instead suggested by technical convenience (strictly speaking, the right derivative of the reduced energies $\mathcal{E}_n$ do not make sense for $l = L$). Note that $\mathcal{G}_n$ may possibly take the value $+\infty$ in Λ_n. In a similar manner, we define

$$G_n^c(l) = \lim_{s \to l^+} G_n^c(s) \text{ if } l \neq L \quad \text{and} \quad G_n^c(L) = G_A^c. \tag{6}$$

Clearly, $G_n^c : [0, L] \to \{G_A^c, G_B^c\}$ is right continuous.

2.2. Quasi-Static Evolution by Griffith's Criterion

In this subsection we provide a notion of quasi-static propagation, following [14,18,21]. We consider a time depending boundary datum of the form $f(t)g$ where $f \in C^1([0, T])$ with $f(0) = 0$ and $f' > 0$ (non-monotone boundary conditions are considered in Section 4). Let $L_0 \in (0, L)$ be the initial crack. For $l \in [L_0, L]$ the space of admissible displacement is

$$U_{t,l} = \{u \in H^1(\Omega \backslash K_l) : u = f(t)g \text{ in } \partial_D \Omega\} = f(t)\mathcal{U}_l.$$

We denote by E_n and G_n the corresponding condensed elastic energy and energy release. For $t = 0$ we simply have that $E_n(0, \cdot) = G_n(0, \cdot) = 0$, since $u = 0$ on $\partial_D \Omega$. If $t > 0$, by linearity, we can write

$$E_n(t, l) = f^2(t)\,\mathcal{E}_n(l), \quad G_n(t, l) = f^2(t)\,\mathcal{G}_n(l).$$

The latter equality makes sense in generalized form also in the case $\mathcal{G}_n(l) = +\infty$. In particular, $E_n(\cdot, l)$ is of class $C^1([0, T])$ while, by Lemma 1, $E_n(t, \cdot)$ is decreasing, continuous and of class $C^1((0, L) \backslash \Lambda_n)$. The energy release rate $G_n(\cdot, l)$ is instead of class $C^1([0, T])$ if $\mathcal{G}_n(l) < +\infty$ (for example if $l \notin \Lambda_n$), otherwise $G_n(\cdot, l) = +\infty$ for $t > 0$ (in this case we will say that $G_n(\cdot, l)$ is continuous, with extended values). Finally, $G_n(t, \cdot)$ is right lower semi-continuous in $[0, L]$.

The following proposition provides our formulation of Griffith's criterion for the quasi-static propagation of a crack in a brittle material with periodic vertical layers:

Proposition 2. *There exists a unique non-decreasing, right continuous function $\ell_n : [0, T) \to [L_0, L]$ which satisfies the initial condition $\ell_n(0) = L_0$ and Griffith's criterion in the form*

(i) $G_n(t, \ell_n(t)) \leqq G_n^c(\ell_n(t))$ for every time $t \in [0, T)$;
(ii) if $G_n(t, \ell_n(t)) < G_n^c(\ell_n(t))$ then ℓ_n is right differentiable in t and $\dot{\ell}_n^+(t) = 0$;
(iii) if $t \in J(\ell_n)$ then $G_n(t, l) \geqq G_n^c(l)$ for every $l \in [\ell_n^-(t), \ell_n(t))$,

where $J(\ell_n)$ denotes the set of discontinuity points of ℓ_n.

Before proving Proposition 2 let us make a few comments. The uniqueness is true, thanks to the monotonicity of the boundary condition together with the right continuity of the evolution. The discontinuity points $t \in J(\ell_n)$ correspond to instantaneous non-equilibrium transitions of the system between the equilibria

$\ell_n^-(t)$ and $\ell_n(t) = \ell_n^+(t)$. Often discontinuities are tailored to the layers: roughly speaking, the crack jumps through the layer with smaller toughness, reaching the closest interface, where it "re-nucleates" before advancing in the layer with higher toughness. In this case the non-equilibrium condition $G_n(t, l) \geqq G_n^c(l)$ may not hold for $l = \ell_n^+(t) \in \Lambda_n$. For instance, it may happen that $G_n(t, l) > G_A^c$ for $l \in (l_{n,k}, l_{n,k+\lambda})$ while $G_n(t, l_{n,k+\lambda}) < G_B^c$. For this reason condition (ii) holds with the right derivative $\dot\ell_n^+(t)$; however, out of interfaces the full time derivative $\dot\ell_n(t)$ can be used (see Corollary 4). However, this is neither a general rule for the evolution neither an a-priori assumption in our analysis; in some cases the crack may as well cross multiple layers in a single discontinuity, depending on the parameters.

Strictly speaking, right continuity is not necessary for Griffith's criterion, however it is also not restrictive since ℓ_n turns out to be the (only) right continuous representative of any evolution λ_n which satisfies Griffith's criterion. More precisely, we have the following result, whose proof is postponed after Corollary 4:

Corollary 3. *If* $\lambda_n : [0, T] \to [L_0, L]$ *is non-decreasing, satisfies* $\lambda_n(0) = L_0$ *and Griffith's criterion, that is,*

(i) $G_n(t, \lambda_n(t)) \leqq G_n^c(\lambda_n(t))$ *for every time* $t \in [0, T]$;
(ii) *if* $G_n(t, \lambda_n(t)) < G_n^c(\lambda_n(t))$ *and* $t < T$ *then* λ_n *is right differentiable in* t *and* $\dot\lambda_n^+(t) = 0$;
(iii) *if* $t \in J(\lambda_n)$ *then* $G_n(t, l) \geqq G_n^c(l)$ *for every* $l \in [\lambda_n^-(t), \lambda_n^+(t))$,

then the right continuous representative of λ_n *coincides with* ℓ_n *in* $[0, T]$.

In other terms, $\{t \in [0, T] : \lambda_n(t) \neq \ell_n(t)\} \subset J(\ell_n) = J(\lambda_n)$. Roughly speaking, if $\lambda_n(t) \neq \ell_n(t)$, then t is a jump point, where $\lambda_n(t) = \ell_n^-(t)$ and $\lambda_n^+(t) = \ell_n(t)$.

Proof of Proposition 2. The proof is based on the following explicit representation of the evolution:

$$\ell_n(t) = \inf\{l \in [L_0, L] : G_n(t, l) < G_n^c(l)\}. \tag{7}$$

Note that the set $\{l \in [L_0, L] : G_n(t, l) < G_n^c(l)\}$ is not empty because, by definition, $G_n(t, L) = 0$. Thus ℓ_n is well defined and takes values in $[L_0, L]$. We will check that the function ℓ_n given by (7) is indeed the unique solution.

Clearly ℓ_n satisfies the initial condition because $G_n(0, L_0) = 0$.

Let us check monotonicity. Since the function f is increasing and G_n is non-negative, for every $0 \leqq t_1 < t_2 < T$, it holds that $G_n(t_1, l) = f^2(t_1)\mathcal{G}_n(l) \leqq f^2(t_2)\mathcal{G}_n(l) = G_n(t_2, l)$, and hence

$$\{l \in [L_0, L] : G_n(t_1, l) < G_n^c(l)\} \supset \{l \in [L_0, L] : G_n(t_2, l) < G_n^c(l)\}.$$

Taking the infimum yields that $\ell_n(t_1) \leqq \ell_n(t_2)$.

Next, we show that ℓ_n is right continuous. Since in general $G_n(\cdot, l)$ is continuous (with extended values) only in $(0, T)$, we distinguish between $t = 0$ and $t > 0$. In the former case, let $L_0 < l$ with $l \notin \Lambda_n$. Then as $\tau \to 0$ we have $G_n(\tau, l) \to 0 < G_n^c(l)$. Hence, (7) implies $\ell_n(\tau) \leqq l$ for $\tau > 0$ sufficiently small. By monotonicity

$\ell_n^+(0) \leq l$ and thus $\ell_n^+(0) \leq L_0$ by the arbitrariness of l. Let $t > 0$. By monotonicity $\ell_n^+(t) \leq \lim_{\tau \to t^+} \ell_n(\tau)$. Let us prove the opposite inequality. If $\lim_{\tau \to t^+} \ell_n(\tau) = L_0$ there is nothing to prove. Otherwise, let $L_0 \leq l^* < \lim_{\tau \to t^+} \ell_n(\tau)$ with $l^* \notin \Lambda_n$ (remember that Λ_n is a discrete set). By monotonicity, $l^* < \ell_n(\tau)$ for every $\tau > t$, and thus, applying the definition (7) to $\ell_n(\tau)$, we know that $G_n(\tau, l^*) \geq G_n^c(l^*)$ for every $\tau > t$. Passing to the limit as $\tau \to t^+$ by the continuity of $G_n(\cdot, l^*)$ it follows that $G_n(t, l^*) \geq G_n^c(l^*)$ for every $L_0 \leq l^* < \lim_{\tau \to t^+} \ell(\tau)$ with $l^* \notin \Lambda_n$. Hence $\ell_n(t) = \inf\{l \in [L_0, L] : G_n(t, l) < G_n^c(l)\} \geq l^*$. Taking the supremum in the right hand side for $L_0 \leq l^* < \lim_{\tau \to t^+} \ell(\tau)$ with $l^* \notin \Lambda_n$, we get $\ell_n(t) \geq \lim_{\tau \to t^+} \ell(\tau)$.

Let us check (i). If $\ell_n(t) < L$ then by (7) there exists a sequence $l_m \searrow \ell_n(t)$ such that $G_n(t, l_m) < G_n^c(l_m)$ and $l_m \notin \Lambda_n$. By the right lower semi-continuity of $G_n(t, \cdot)$ and the right continuity of G_n^c it follows that

$$G_n(t, \ell_n(t)) \leq \liminf_{m \to \infty} G_n(t, l_m) \leq \lim_{m \to \infty} G_n^c(l_m) = G_n^c(\ell_n(t)). \qquad (8)$$

If $\ell_n(t) = L$ there is nothing to prove since, by definition, $0 = G_n(t, L) < G_n^c(L)$.

Next we check (ii). Again, we consider separately $t = 0$ and $t > 0$. If $t = 0$ we have $G_n(0, L_0) < G_n^c(L_0)$. If $\ell_n(\tau) = L_0$ for some $\tau > 0$ then ℓ_n is constant up to τ and thus $\dot{\ell}_n^+(0) = 0$. On the contrary, assume that $\ell_n(\tau) > L_0$ for every $\tau > 0$. By (7) we know that $G_n(\tau, l) \geq G_n^c(l)$ for every $\tau > 0$ and every $L_0 \leq l < \ell_n(\tau)$. Hence $f^2(\tau)\mathcal{G}_n(l) \geq \min\{G_A^c, G_B^c\}$, and thus

$$f^2(\tau)\big(\mathcal{E}_n(L_0) - \mathcal{E}_n(\ell_n(\tau))\big) = f^2(\tau) \int_{L_0}^{\ell_n(\tau)} \mathcal{G}_n(l)\, \mathrm{d}l \geq (\ell_n(\tau) - L_0) \min\{G_A^c, G_B^c\}.$$

As a consequence

$$\lim_{\tau \to 0^+} \frac{\ell_n(\tau) - \ell_n(0)}{\tau} \leq \lim_{\tau \to 0^+} \frac{f^2(\tau) - f^2(0)}{\tau} \frac{\mathcal{E}_n(L_0) - \mathcal{E}_n(\ell_n(\tau))}{\min\{G_A^c, G_B^c\}} = 0,$$

where we used the continuity of the energy $\mathcal{E}_n$ and the fact that $f \in C^1([0, T])$ with $f(0) = 0$. Now, consider $t > 0$. If $G_n(t, \ell_n(t)) < G_n^c(\ell_n(t))$ then by continuity of $G_n(\cdot, l)$ we get $G_n(\tau, \ell_n(t)) < G_n^c(\ell_n(t))$ for τ in a (sufficiently small) right neighborhood of t. By (7) it follows that $\ell_n(\tau) \leq \ell_n(t)$; by monotonicity ℓ_n is constant in a right neighborhood of t.

Let us consider (iii). Let $t \in J(\ell_n)$. By right continuity $\ell_n(t) > L_0$. By definition $\ell_n(t) = \inf\{l \in [L_0, L] : G_n(t, l) < G_n^c(l)\}$, hence $G_n(t, l) \geq G_n^c(l)$ for every $L_0 \leq l < \ell_n(t)$ and in particular for $l \in [\ell_n^-(t), \ell_n(t))$.

To conclude, let us prove uniqueness. Assume, by contradiction, that there exists another right continuous evolution $\ell_* \neq \ell_n$ which satisfies Griffith's criterion. First, we claim that there exists a time $t \notin J(\ell_*)$ such that $\ell_*(t) \neq \ell_n(t)$; indeed, if $\{t \in [0, T] : \ell_*(t) \neq \ell_n(t)\} \subset J(\ell_*)$ then by right continuity of ℓ_* and ℓ_n we would get $\ell_* = \ell_n$ everywhere in $[0, T)$. So, let $t > 0$ be a continuity point of ℓ_* with $\ell_*(t) \neq \ell_n(t)$.

If $\ell_*(t) > \ell_n(t)$, then, by the definition (7) of ℓ_n, there exists $l_* \in (\ell_n(t), \ell_*(t))$ with $l_* \notin \Lambda_n$ such that $G_n(t, l_*) < G_n^c(l_*)$. Then, by continuity of the energy

release, there exists a neighborhood $[l', l'']$ of l_*, with $L_0 \leqq l'$ and $l'' < \ell_*(t)$, such that $G_n(t, l) < G_n^c(l)$ for every $l \in [l', l'']$. Let us see that this interval is a "barrier" for the evolution ℓ_*. First, note that, by monotonicity in time, $G_n(\tau, l) \leqq G_n(t, l) < G_n^c(l)$ for every $\tau \leqq t$ and every $l \in [l', l'']$. Moreover, ℓ_* takes all the values in the interval $[l', l'']$, indeed if $l \in [\ell_*^-(\tau), \ell_*(\tau))$ for some $\tau \leqq t$ and some $l \in [l', l'']$ then, by condition (iii) of Griffith's criterion, $G_n(\tau, l) \geqq G_n^c(l)$, which stands in contradiction with $G_n(\tau, l) < G_n^c(l)$. Therefore we can define $t_1 = \inf\{\tau : l' \leqq \ell_*(\tau)\}$ and $t_2 = \sup\{\tau : \ell_*(\tau) \leqq l''\}$. It follows that $G_n(\tau, \ell_*(\tau)) < G_n^c(l)$ for every $\tau \in [t_1, t_2)$ and thus, by condition (ii) of Griffith's criterion, ℓ_* is constantly equal to l' in $[t_1, t_2)$. If $t_2 = t$ then (being t a continuity point) we get $\ell_*(t) = l' < \ell_*(t)$. If $t_2 < t$ then $\ell_*^-(t_2) = l'$ while $\ell_*^+(t_2) \geqq l''$ and thus $t_2 \in J(\ell_*)$, this is again a contradiction since Griffith's criterion implies that $G_n(t_2, l) \geqq G_n^c(l)$ for every $l \in [l', l'']$.

Let $\ell_*(t) < \ell_n(t)$. By the definition of ℓ_n we know that $G_n(t, l) \geqq G_n^c(l) > 0$ for every $l \in [\ell_*(t), \ell_n(t))$. By monotonicity $G_n(\tau, l) > G_n^c(l)$ for every $\tau > t$ and every $l \in [\ell_*(t), \ell_n(t))$. If $\tau \to t^+$ then $\ell_*(\tau) \to \ell_*(t)$ and thus $\ell_*(\tau) \in [\ell_*(t), \ell_n(t))$ for every time τ sufficiently close to t; then we have $G_n(\tau, \ell_*(\tau)) > G_n^c(\ell_*(\tau))$, which is a contradiction with Griffith's criterion. □

In order to prove the energy identity we will need also the following corollary, which refines condition (ii) for $\ell_n(t) \notin \Lambda_n$:

Corollary 4. *If $\ell_n(t) \notin \Lambda_n$ and $G_n(t, \ell_n(t)) < G_n^c(\ell_n(t))$ then ℓ_n is constant in a (sufficiently small) neighborhood of t.*

Proof. Let $\ell_n(t) \in (l_{n,k}, l_{n,k+\lambda})$. The energy E_n is of class C^1 in $[0, T] \times (l_{n,k}, l_{n,k+\lambda})$ and thus the energy release G_n is continuous. Moreover, G_n^c is constant in $(l_{n,k}, l_{n,k+1})$. Hence, there exists $\delta > 0$ (sufficiently small) such that $G_n(\tau, l) < G_n^c(l)$ whenever $|\tau - t| < \delta$ and $|l - \ell_n(t)| < \delta$. By condition (iii) in Proposition 2 it follows that $t \notin J(\ell_n)$. By continuity of ℓ_n let us choose $\tau' < t$ such that $|\tau' - t| < \delta$ and $|\ell_n(\tau') - \ell_n(t)| < \delta$. Then, $G_n(\tau, \ell_n(\tau')) < G_n^c(\ell_n(\tau'))$ for every $\tau \in [\tau', t]$. By Proposition 2 it follows that $\ell_n(\tau) = \ell_n(t)$ is the unique solution for $\tau \in [t, t+\delta]$. The same argument applies if $\ell_n(t) \in (l_{n,k+\lambda}, l_{n,k+1})$. □

Proof of Corollary 3. By uniqueness it is enough to check that the right continuous representative of λ_n, here denoted by $\hat\lambda_n$, satisfies Griffith's criterion, in the sense of Proposition 2.

Arguing by contradiction it is easy to check that λ_n is right continuous in 0; indeed, if $\lim_{\tau \to 0^+} \lambda_n(\tau) > L_0$ then $0 \in J(\lambda_n)$ and thus $G_n(0, l) \geqq G_n^c(l) > 0$, however $G_n(0, l) = 0$. Hence $\hat\lambda_n(0) = L_0$.

Let us check condition (i). For $t = 0$ there is nothing to prove. For $t > 0$ we have $G_n(\tau, \lambda_n(\tau)) \leqq G_n^c(\lambda_n(\tau))$ for every $\tau > t$ and thus the regularity of energy release and toughness imply that

$$G_n(t, \hat\lambda_n(t)) \leqq \liminf_{\tau \to t^+} G_n(\tau, \lambda_n(\tau)) \leqq \lim_{\tau \to t^+} G_n^c(\lambda_n(\tau)) = G_n^c(\hat\lambda_n(t)).$$

Let us consider (iii). Clearly, $J(\lambda_n) = J(\hat\lambda_n)$ and $\hat\lambda_n^\pm(t) = \lambda_n^\pm(t)$. Hence the condition $G_n(t, l) \geqq G_n^c(l)$ for every $l \in [\hat\lambda_n^-(t), \hat\lambda_n^+(t))$ holds.

Finally, let us check (ii). If $t \notin J(\lambda_n) = J(\hat{\lambda}_n)$ then $\lambda_n(t) = \hat{\lambda}_n(t)$ and $\dot{\lambda}_n^+(t) = 0$ coincides with the right derivative of $\hat{\lambda}_n(t)$, hence there is nothing to check. If $t \in J(\lambda_n) = J(\hat{\lambda}_n)$ we argue in the following way. We have $G_n(t, \hat{\lambda}_n(t)) < G_n^c(\hat{\lambda}_n(t))$, then for $\varepsilon > 0$ sufficiently small there exists a sequence $l_k \searrow \hat{\lambda}_n(t)$ such that $l_k \notin \Lambda_n$, $G_n^c(l_k) = G_n^c(\hat{\lambda}_n(t))$ is constant and

$$G_n(t, l_k) \leqq G_n^c(l_k) - \varepsilon \quad \text{for every } k \in \mathbb{N}.$$

Writing $G_n(t, l_k) = f^2(t)\mathcal{G}_n(l_k)$, by time continuity for every $0 < \varepsilon' < \varepsilon$ there exists $\delta > 0$ such that

$$G_n(\tau, l_k) < G_n^c(l_k) - \varepsilon' \quad \text{for every } k \in \mathbb{N} \text{ and every } \tau \in (t, t + \delta).$$

Since $l_k \notin \Lambda_n$ the energy release $\mathcal{G}_n$ is continuous, and thus for every $k \in \mathbb{N}$ there exists $l_k^\flat < l_k < l_k^\sharp < l_{k+1}$ such that

$$G_n(\tau, l) < G_n^c(l) \quad \text{for every } l \in (l_k^\flat, l_k^\sharp) \text{ and every } \tau \in (t, t + \delta).$$

As in proof of Proposition 2 the evolution λ_n cannot take value in the intervals $(l_k^\flat, l_k^\sharp)$ for $\tau \in (t, t + \delta)$. Indeed, there are no jumps, moreover, if $\lambda_n(\tau') \in (l_k^\flat, l_k^\sharp)$ then condition (ii) implies that the right derivative vanishes and thus λ_n is constant in $(\tau', t+\delta)$. Hence by monotonicity $\lambda_n(\tau) \leqq l_k^\sharp$ for every k and every $\tau \in (t, t+\delta)$; in conclusion $\hat{\lambda}_n(\tau) \leqq \inf l_k^\sharp = \hat{\lambda}_n(t)$ for every $\tau \in (t, t + \delta)$. $\qquad\square$

Remark 2. For $l \in \Lambda_n$, the extension (6) of the toughness seems the most natural, since

$$G_n^c(l) = \lim_{s \to l^+} G_n^c(s).$$

The extension (5), that is,

$$\mathcal{G}_n(l) = \liminf_{s \to l^+} \mathcal{G}_n(s),$$

turns out to be the only one compatible with Griffith's criterion. In principle, since the existence of $\partial_l^+ \mathcal{E}_n(l)$ is not known when $l \in \Lambda_n$ any value $\mathcal{G}_n(l)$ with

$$\liminf_{s \to l^+} \frac{\mathcal{E}_n(s) - \mathcal{E}_n(l)}{s - l} \leqq -\mathcal{G}_n(l) \leqq \limsup_{s \to l^+} \frac{\mathcal{E}_n(s) - \mathcal{E}_n(l)}{s - l} \tag{9}$$

could be a good candidate for the extension. To fix the ideas assume that $L_0 \in \Lambda_n$ and let $\tilde{\mathcal{G}}_n(L_0)$ be another extension of $\mathcal{G}_n$ in L_0. First of all, (9) implies that $\tilde{\mathcal{G}}_n(L_0) \geqq \mathcal{G}_n(L_0)$, indeed, by the mean value theorem we have

$$\mathcal{G}_n(L_0) = \liminf_{s \to L_0^+} \mathcal{G}_n(s) \leqq \liminf_{s \to L_0^+} -\frac{\mathcal{E}_n(s) - \mathcal{E}_n(L_0)}{s - L_0}$$

$$= -\limsup_{s \to L_0^+} \frac{\mathcal{E}_n(s) - \mathcal{E}_n(L_0)}{s - L_0} \leqq \tilde{\mathcal{G}}_n(L_0).$$

Let us consider

$$\tilde{\mathcal{G}}_n(L_0) > \mathcal{G}_n(L_0) = \liminf_{s \to L_0^+} \mathcal{G}_n(s).$$

Clearly, this condition makes sense only when $\liminf_{s \to L_0^+} \mathcal{G}_n(s) < +\infty$. Hence $0 < \tilde{\mathcal{G}}_n(L_0) \leqq +\infty$. Let

$$\tilde{G}_n(0, L_0) = 0, \quad \tilde{G}_n(t, L_0) = f^2(t)\,\tilde{\mathcal{G}}_n(L_0) \text{ for } t > 0.$$

Hence $\tilde{G}_n(t, L_0) > G_n(t, L_0)$ for $t > 0$. Assume that there exists a quasi-static evolution $\tilde{\ell}_n$ which satisfies Griffith's criterion with $\tilde{G}_n$ and G_n^c. We denote by t_* and t^* the points $t_* = \max\{t \geqq 0 : \tilde{G}_n(t, L_0) \leqq G_n^c(L_0)\}$ and $t^* = \max\{t \geqq 0 : G_n(t, L_0) \leqq G_n^c(L_0)\}$. We have $t_* < t^*$. Fix $t' \in (t_*, t^*)$. By the definition of $\mathcal{G}_n(L_0)$ there exists $l_m \searrow L_0$ such that $l_m \notin \Lambda_n$ and $\mathcal{G}_n(l_m) \to \mathcal{G}_n(L_0) < \tilde{\mathcal{G}}_n(L_0)$. Upon extracting a subsequence, not relabeled, we can assume that there exists $t_m < t'$ such that either $\tilde{\ell}_n(t_m) = l_m$ for every $m \in \mathbb{N}$ or $l_m \in [\tilde{\ell}_n^-(t_m), \ell_n(t_m))$ for every $m \in \mathbb{N}$.

In the former case, for every $t \in (t_m, t')$, we have

$$G_n(t, l_m) \leqq G_n(t', l_m) \to G_n(t', L_0) < G_n^c(L_0) = G_n^c(l_m).$$

Hence for $m \gg 1$ we have $G_n(t, l_m) < G^c(l_m)$ for every $t \in (t_m, t')$. As a consequence $\tilde{\ell}_n(t) = l_m$ in (t_m, t'). Repeating the argument for l_{m+1} leads to $\tilde{\ell}_n(t) = l_{m+1}$ in (t_{m+1}, t') which is a contradiction since $t_{m+1} < t_m$ and $l_{m+1} \neq l_m$.

In the other case, that is, when $l_m \in [\tilde{\ell}_n^-(t_m), \ell_n(t_m))$ for every $m \in \mathbb{N}$, we argue as follows. As above,

$$G_n(t_m, l_m) \leqq G_n(t', l_m) \to G_n(t', L_0) < G_n^c(L_0) = G_n^c(l_m)$$

and thus for $m \gg 1$ we have $G_n(t_m, l_m) < G^c(l_m)$ which contradicts condition (iii).

In summary, the right lower semi-continuity of the energy release $\mathcal{G}_n$ is necessary for the existence of a quasi-static evolution in the sense of Proposition 2.

Finally, note that the representation (7) is independent of the extension of $\mathcal{G}_n$ and G_n^c in $\Lambda_n \backslash \{L\}$; indeed, (7) can also be written as

$$\ell_n(t) = \inf\{l \in [L_0, L] \backslash \Lambda_n \text{ or } l = L : G_n(t, l) < G_n^c(l)\}.$$

2.3. Energy Identity

In the sequel the energy balance will be crucial to explain toughening. To this end, we introduce the potential energy $F : [0, T] \times [L_0, L] \to \mathbb{R}$ given by

$$F_n(t, l) = E_n(t, l) + D_n(l),$$

where $D_n : [L_0, L] \to \mathbb{R}$ is the dissipated energy given by

$$D_n(l) = \int_{L_0}^{l} G_n^c(s)\,\mathrm{d}s.$$

Note that the energy $F_n(\cdot, l)$ is differentiable in $[0, T]$ while $F_n(t, \cdot)$ is differentiable only in $[L_0, L] \backslash \Lambda_n$.

Corollary 5. *Let ℓ_n be the quasi-static evolution provided by Proposition [2]. For every $t \in [0, T)$ the following energy identity holds:*

$$F_n(t, \ell_n(t)) = \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \sum_{\tau \in J(\ell_n) \cap [0,t]} [\![F_n(\tau, \ell_n(\tau))]\!]. \qquad (10)$$

Where $[\![F_n(\tau, \ell_n(\tau))]\!] = F_n(\tau, \ell_n(\tau)) - F_n(\tau, \ell_n^-(\tau)) \leqq 0$.

We recall that $F_n(0, L_0) = 0$. The energy identity (10) can be written also as

$$E_n(t, \ell_n(t)) = \int_0^t P_n(t, \ell_n(t)) \, dt - D_n(\ell_n(t)) + \sum_{\tau \in J(\ell_n) \cap [0,t]} [\![F_n(\tau, \ell_n(\tau))]\!], \quad (11)$$

where $P_n(t, \ell_n(t)) = \partial_t F(t, l) = \partial_t E(t, l)$ is the power of "external forces". Therefore, part of the energy (supplied by the work of external forces) is stored in the elastic energy, part is dissipated by the crack, while part is dissipated in the discontinuity points. In this respect, we recall that the rate-independent evolutions ℓ_n can be obtained also from rate-dependent evolutions (see for example [14]) by vanishing viscosity; in this approach, the energy dissipated in the jumps turns out to be the limit of the energy dissipated in the rate-dependent processes.

Clearly, by right continuity the energy identity holds also in every time interval $[t_1, t_2]$, in the form

$$F_n(t_2, \ell_n(t_2)) = F_n(t_1, \ell_n(t_1)) + \int_{t_1}^{t_2} \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \sum_{\tau \in J(\ell_n) \cap (t_1, t_2]} [\![F_n(\tau, \ell_n(\tau))]\!].$$

Proof. In general the energy $F_n(t, \cdot)$ is not differentiable in Λ_n, thus the idea is to consider the sub-intervals in which the evolution takes values in $[L_0, L] \setminus \Lambda_n$, taking care of the possible discontinuities. In this way we also make reference to the stick-slip and re-nucleation effect at the interfaces, see [13].

Let $l_{n,k} = k l_n = kL/n$. Let k_0 and k_I such that

$$l_{n,k_0} \leqq L_0 < l_{n,k_1}, \quad l_{n,k_I-1} < \sup \ell_n \leqq l_{n,k_I}.$$

If $I = 0$ we have $\ell_n(t) \equiv L_0$, thus the energy identity (10) boils down to

$$F_n(t, \ell_n(t)) = \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau,$$

which is clearly true by the time regularity of the energy. If $I > 0$, consider the points $l_{n,k} \in \Lambda_n$ for $k = k_0, \ldots, k_I$ and define the times

$$t_{n,k_0} = \sup\{t : \ell_n(t) \leqq L_0\}, \quad t_{n,k_I}^\flat = \sup\{t : \ell_n(t) < l_{n,k_I}\}, \quad t_{n,k_I}^\sharp = T,$$

$$t_{n,k_i}^\flat = \sup\{t : \ell_n(t) < l_{n,k_i}\}, \quad t_{n,k_i}^\sharp = \inf\{t : \ell_n(t) > l_{n,k_i}\} \quad \text{for } 0 < i < I \text{ (if any).}$$

In this way we have a finite partition of $[0, T]$ given by

$$0 < t_{n,k_0} \leqq t_{n,k_1}^\flat \leqq t_{n,k_1}^\sharp \leqq \cdots \leqq t_{n,k_I}^\flat \leqq t_{n,k_I}^\sharp = T;$$

note that some of these points may coincide. We will prove (10) by induction, for $i = 0, \dots, I$.

In the interval $[0, t_{n,k_0})$ we have $\ell_n(t) = L_0$. Thus for every $t \in [0, t_{n,k_0})$ we can write

$$F_n(t, \ell_n(t)) = \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau.$$

Passing to the limit as $t \nearrow t_{n,k_0}$ we get

$$F_n(t_{n,k_0}, \ell_n^-(t_{n,k_0})) = \int_0^{t_{n,k_0}} \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau,$$

and then, by right continuity,

$$F_n(t_{n,k_0}, \ell_n(t_{n,k_0})) = \int_0^{t_{n,k_0}} \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + [\![F_n(t_{n,k_0}, \ell_n(t_{n,k_0}))]\!]. \quad (12)$$

If $t_{n,k_1}^\flat > t_{n,k_0}$ then $\ell_n(t) \in (L_0, l_{n,k_1}) \subset (l_{n,k_0}, l_{n,k_1})$ for $t \in (t_{n,k_0}, t_{n,k_1}^\flat)$. In the interval (L_0, l_{n,k_1}) the elastic energy $\mathcal{E}_n(t, \cdot)$ is of class C^1 while the dissipated energy D_n is affine, since G_n^c is constant. Thus in every subinterval $t_{n,k_0} < t' < t < t_{n,k_1}^\flat$ we can apply the chain rule in BV to get

$$F_n(t, \ell_n(t)) = F_n(t', \ell_n(t')) + \int_{t'}^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \int_{t'}^t \partial_l F_n(\tau, \ell_n(\tau)) \, d_{LC}\ell_n(\tau)$$

$$+ \sum_{\tau \in J(\ell_n) \cap (t',t]} [\![F_n(\tau, \ell_n(\tau))]\!],$$

where $d_{LC}\ell_n$ denotes the sum of the absolutely continuous and Cantor part of the measure $d\ell_n$. Note that $d_{LC}\ell_n$ is concentrated on the set of continuity points of ℓ_n. We claim that

$$\int_{t'}^t \partial_l F_n(\tau, \ell_n(\tau)) \, d_{LC}\ell_n(\tau) = 0.$$

Indeed, for $l \in (L_0, l_{n,k_1})$ we have $\partial_l F_n(\tau, l) = -G_n(\tau, l) + G_n^c(l)$. By Proposition 2 we known that $G_n(\tau, \ell_n(\tau)) \leqq G_n^c(\ell_n(\tau))$. If equality holds then $\partial_l F_n(\tau, \ell_n(\tau)) = 0$. If $G_n(\tau, \ell_n(\tau)) < G_n^c(\ell_n(\tau))$ then by Corollary 4 it follows that ℓ_n is constant in a neighborhood of τ, and thus $d_{LC}\ell_n(\tau) = 0$. The claim is proved. Now, passing to the limit as $t' \to t_{n,k_0}$ and using (12) we get

$$F_n(t, \ell_n(t)) = F_n(t_{n,k_0}, \ell(t_{n,k_0})) + \int_{t_{n,k_0}}^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau$$

$$+ \sum_{\tau \in J(\ell_n) \cap (t_{n,k_0}, t]} [\![F_n(\tau, \ell(\tau))]\!] \qquad (13)$$

$$= \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \sum_{\tau \in J(\ell_n) \cap [0,t]} [\![F_n(\tau, \ell(\tau))]\!].$$

Passing to the limit as $t \to t_{n,k_1}^{\flat}$ and taking into account the possible jump in $t_{n,k_1}^{\flat}$ yields

$$F_n(t_{n,k_1}^{\flat}, \ell_n(t_{n,k_1}^{\flat})) = \int_0^{t_{n,k_1}^{\flat}} \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \sum_{\tau \in J(\ell_n) \cap [0, t_{n,k_1}^{\flat}]} [\![F_n(\tau, \ell(\tau))]\!].$$

(14)

If $t_{n,k_1}^{\flat} = t_{n,k_0}$ there is nothing to prove since the previous identity boils down to (12).

If $t_{n,k_1}^{\flat} < t_{n,k_1}^{\sharp}$ then $\ell_n = l_{n,k_1}$ in the interval $(t_{n,k_1}^{\flat}, t_{n,k_1}^{\sharp})$, thus for every $t \in (t_{n,k_1}^{\flat}, t_{n,k_1}^{\sharp})$ we can write

$$F(t, \ell_n(t)) = \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \sum_{\tau \in J(\ell_n) \cap [0, t]} [\![F_n(\tau, \ell(\tau))]\!].$$

(15)

If $t_{n,k_1}^{\flat} = t_{n,k_1}^{\sharp}$ then (14) holds replacing $t_{n,k_1}^{\flat}$ with $t_{n,k_1}^{\sharp}$. We proceed in this way up to any time $t < T$. □

2.4. Compactness and Convergence of Evolutions

By Helly's Theorem it is well known that up to non-relabeled subsequences the evolutions ℓ_n converge to a certain limit ℓ. The goal of the next proposition is to provide a first characterization of the limit ℓ independently of the convergence of the energies.

Proposition 6. *Let ℓ_n be the sequence of quasi-static evolutions given by Proposition 2. There exists a subsequence (not relabeled) such that $\ell_n \to \ell$ pointwise in $[0, T)$. For $t \in [0, T)$ the right continuous representative of the limit ℓ is characterized by*

$$\ell^+(t) = \inf \left\{ l \in [L_0, L] : \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)} \right\},$$

(16)

where $1/f^2(0) = +\infty$.

Proof. We recall that

$$\Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) = \inf \left\{ \liminf_{n \to \infty} \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} : l_n \to l \right\}.$$

For the theory of Γ-convergence we refer the reader to [10].

Let $t > 0$. By Proposition 2 we know that $G_n(t, \ell_n(t)) \leqq G_n^c(\ell_n(t))$, hence $G_n(t, \ell_n(t))$ is finite. Writing $G_n(t, \ell_n(t)) = f^2(t) \mathcal{G}_n(\ell_n(t))$ we get

$$\frac{1}{f^2(t)} \geqq \frac{\mathcal{G}_n(\ell_n(t))}{G_n^c(\ell_n(t))}.$$

The above inequality holds also in generalized sense for $t = 0$. Since $\ell_n(t) \to \ell(t)$ pointwise for every $t \in [0, T)$ we have

$$\frac{1}{f^2(t)} \geq \liminf_{n \to \infty} \frac{\mathcal{G}_n(\ell_n(t))}{G_n^c(\ell_n(t))} \geq \inf \left\{ \liminf_{n \to \infty} \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} : l_n \to \ell(t) \right\}$$

$$= \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(\ell(t)).$$

Let $\tau > t \in [0, T)$. By the monotonicity of f we can write

$$\frac{1}{f^2(t)} > \frac{1}{f^2(\tau)} \geq \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(\ell(\tau)).$$

Hence

$$\ell(\tau) \in \left\{ l \in [L_0, L] : \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)} \right\} \qquad \text{for every } \tau > t$$

and then

$$\ell^+(t) \geq \inf \left\{ l \in [L_0, L] : \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)} \right\}.$$

Let us prove the opposite inequality. If $\ell^+(t) = L_0$ there is nothing to prove. Otherwise, let $L_0 \leq l < \ell^+(t) \leq \ell(\tau)$ for $\tau > t$. Let $\tau_n \searrow t^+$ such that $\ell_n(\tau_n) \to \ell^+(t)$ and let l_n be a "recovery sequence" for l, that is, $l_n \to l$ and

$$\liminf_{n \to \infty} \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} = \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l).$$

From the representation (7) of the evolution ℓ_n we know that

$$\ell_n(\tau_n) = \inf\{l \in [L_0, L] : G_n(\tau_n, l) < G_n^c(l)\}$$

$$= \inf \left\{ l \in [L_0, L] : \frac{1}{f^2(\tau_n)} > \frac{\mathcal{G}_n(l)}{G_n^c(l)} \right\}.$$

Being $l_n < \ell_n(\tau_n)$ for $n \gg 1$ (since $l < \ell^+(t)$) the above representation implies that

$$\frac{1}{f^2(\tau_n)} \leq \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)}.$$

Taking the liminf as $n \to \infty$ and remembering that l_n is a recovery sequence, we get

$$\frac{1}{f^2(t)} \leq \liminf_{n \to \infty} \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} = \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l).$$

Thus, every $l < \ell^+(t)$ does not belong to the set

$$\left\{ l \in [L_0, L] : \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)} \right\}$$

and then

$$\ell^+(t) \leqq \inf \left\{ l \in [L_0, L] : \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)} \right\},$$

which concludes the proof. □

Remark 3. The characterization (16) seems to be the natural choice combined with the pointwise convergence of a subsequence (extracted in Proposition 6) of the quasi-static evolutions ℓ_n. However, if there exists

$$\Gamma\text{-}\lim_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l), \tag{17}$$

then the whole sequence $\ell_n \overset{*}{\rightharpoonup} \ell$ in $BV(0, T)$. Indeed, each function ℓ_n is monotone and bounded in $[L_0, L]$, hence the sequence ℓ_n is weakly* pre-compact in $BV(0, T)$. Given any subsequence ℓ_{n_k} of ℓ_n there exists a further subsequence, denoted by ℓ_{n_i}, converging to some limit ℓ weakly in $BV(0, T)$ and pointwise in $(0, T)$. By Proposition 6 the limit ℓ (possibly depending on the subsequence) is characterized by

$$\ell^+(t) = \inf \left\{ l \in [L_0, L] : \Gamma\text{-}\liminf_{n \to \infty} \frac{\mathcal{G}_{n_i}}{G_{n_i}^c}(l) < \frac{1}{f^2(t)} \right\}$$

$$= \inf \left\{ l \in [L_0, L] : \Gamma\text{-}\lim_{n \to \infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)} \right\}.$$

Therefore, the evolution ℓ is actually independent of the subsequence ℓ_{n_k}. Clearly, this argument requires the existence of the Γ-limit in (17) while the Γ-liminf in (16) is always defined.

2.5. Homogenization of the Elastic Energy

In this section we will first recall a few results on the homogenization of the elastic energy. Let us introduce the homogenized stiffness matrix $\boldsymbol{C}_{hom}$ given by

$$\boldsymbol{C}_{hom} = \begin{pmatrix} \mu_{hom,1} & 0 \\ 0 & \mu_{hom,2} \end{pmatrix},$$

where $1/\mu_{hom,1} = \lambda/\mu_{A,1} + (1-\lambda)/\mu_{B,1}$ is the weak* limit of $1/\mu_{n,1}$ and $\mu_{hom,2} = \lambda\mu_{A,2} + (1-\lambda)\mu_{B,2}$ is the weak* limit of $\mu_{n,2}$. Note that the coefficients $\mu_{hom,i}$ are constant and independent of $l \in (0, L)$. We denote by $\mathcal{W}_{l,hom}$ the homogenized elastic energy and by $\mathcal{E}_{hom}$ its static condensation. By classical results on the homogenization of elliptic problems (see Appendix A) and on the energy release (see for example [20] and the references therein) we infer the following proposition.

Proposition 7. *If* $l_n \to l$ *then* $\mathcal{E}_n(l_n) \to \mathcal{E}_{hom}(l)$. *The energy* $\mathcal{E}_{hom}$ *is of class* $C^1(0, L)$.

Thanks to the regularity of $\mathcal{E}_{hom}$ we can define the energy release

$$\mathcal{G}_{hom}(l) = -\partial_l^+ \mathcal{E}_{hom}(l) = -\partial_l \mathcal{E}_{hom}(l) \quad \text{for } l \in (0, L), \quad \mathcal{G}_{hom}(L) = 0,$$

which is continuous in $(0, L)$. We remark that in general $\mathcal{G}_{hom}$ is neither the almost everywhere limit nor the Γ-liminf of $\mathcal{G}_n$ (an example is given in §2.8). However, thanks to the regularity and the convergence of the energies we have the following results, which will be useful in the sequel.

Lemma 8. *Let $\mathcal{G}_{n_i}$ be a subsequence of $\mathcal{G}_n$. For almost everywhere $l \in (0, L)$ it holds*

$$\liminf_{i \to \infty} \mathcal{G}_{n_i}(l) \leqq \mathcal{G}_{hom}(l)$$

(Note that the set where the above estimate holds may depend on the subsequence).

Proof. Let $0 < l_1 < l_2 < L$. By Proposition 7 and by the regularity of the energies we can write

$$\int_{l_1}^{l_2} \mathcal{G}_{n_i}(s)\,\mathrm{d}s = \mathcal{E}_{n_i}(l_1) - \mathcal{E}_{n_i}(l_2) \quad \to \quad \mathcal{E}_{hom}(l_1) - \mathcal{E}_{hom}(l_2) = \int_{l_1}^{l_2} \mathcal{G}_{hom}(s)\,\mathrm{d}s.$$

Hence, by Fatou's Lemma we also have

$$\int_{l_1}^{l_2} \liminf_{i \to \infty} \mathcal{G}_{n_i}(s)\,\mathrm{d}s \leqq \liminf_{i \to \infty} \int_{l_1}^{l_2} \mathcal{G}_{n_i}(s)\,\mathrm{d}s = \int_{l_1}^{l_2} \mathcal{G}_{hom}(s)\,\mathrm{d}s,$$

which proves the lower inequality by the arbitrariness of l_1 and l_2. $\qquad\square$

Lemma 9. *For every $l \in (0, L)$ there exist a sequence $l_n^\sharp \to l^+$ such that $\mathcal{G}_{hom}(l) \geqq \limsup_{n \to \infty} \mathcal{G}_n(l_n^\sharp)$.*

Proof. By the continuity of $\mathcal{G}_{hom}$, for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\mathcal{G}_{hom}(s) < \mathcal{G}_{hom}(l) + \delta$ for every $s \in [l, l + \varepsilon)$.

We claim that there exists $m \in \mathbb{N}$ such that for every $n > m$ we can find $l_n \in (l, l + \varepsilon)$ with $\mathcal{G}_n(l_n) < \mathcal{G}_{hom}(l) + \delta$; from the claim the existence of $l_n^\sharp$ follows. By contradiction, assume that for every $m \in \mathbb{N}$ there exists $n_m > m$ such that $\mathcal{G}_{n_m}(s) \geqq \mathcal{G}_{hom}(l) + \delta$ for every $s \in (l, l + \varepsilon)$, then we can find a subsequence such that $\liminf_{m \to \infty} \mathcal{G}_{n_m}(s) \geqq \mathcal{G}_{hom}(l) + \delta > \mathcal{G}_{hom}(s)$ in $(l, l + \varepsilon)$, which is a contradiction with Lemma 8. $\qquad\square$

From the previous lemma we get the following corollary.

Corollary 10. *If $\mathcal{G}_{hom} \leqq \Gamma\text{-}\liminf_{n \to \infty} \mathcal{G}_n$ then $\mathcal{G}_{hom} = \Gamma\text{-}\lim_{n \to \infty} \mathcal{G}_n$.*

Proof. By Lemma 9 for every $l \in (0, L)$ we have

$$\Gamma\text{-}\limsup_{n \to \infty} \mathcal{G}_n(l) \leqq \limsup_{n \to \infty} \mathcal{G}_n(l_n^\sharp) \leqq \mathcal{G}_{hom}(l) \leqq \Gamma\text{-}\liminf_{n \to \infty} \mathcal{G}_n(l),$$

which concludes the proof. $\qquad\square$

As in the previous sections we consider the energy $E_{hom}(t, l) = f^2(t)\mathcal{E}_{hom}(l)$. Most important, we define $G_{hom}(t, l) = \partial_l E_{hom}(t, l) = f^2(t)\mathcal{G}_{hom}(l)$ for every $t \in [0, T]$ and $l \in (0, L)$; note that $\mathcal{G}_{hom}(l)$ is finite for $l \in (0, L)$. Finally, we define $G_{hom}(t, L) = 0$ for every $t \in [0, T]$.

2.6. Effective Toughness and Homogenization of Griffith's Criterion

The aim of this section is to provide an *effective toughness* G_{eff}^c in such a way that the limit evolution ℓ (given by Proposition 6) satisfies Griffith's criterion for G_{hom} and G_{eff}^c. Formally, regardless of the regularity of the energies involved, we may define G_{eff}^c as the function which makes the following identity true:

$$\frac{\mathcal{G}_{hom}}{G_{\text{eff}}^c}(l) = \Gamma\text{-}\liminf_{n\to\infty}\frac{\mathcal{G}_n}{G_n^c}(l). \tag{18}$$

More precisely, we define $G_{\text{eff}}^c : [0, L] \to [0, +\infty]$ as

$$G_{\text{eff}}^c(l) = \begin{cases} \dfrac{\mathcal{G}_{hom}(l)}{\Gamma\text{-}\liminf_{n\to\infty}\dfrac{\mathcal{G}_n}{G_n^c}(l)} & \text{if } \mathcal{G}_{hom}(l) \neq 0, \\[1em] +\infty & \text{if } \mathcal{G}_{hom}(l) = 0. \end{cases} \tag{19}$$

In the above definition we assume $G_{\text{eff}}^c(l) = +\infty$ when $\Gamma\text{-}\liminf_{n\to\infty}\dfrac{\mathcal{G}_n}{G_n^c}(l) = 0$; by the proof of Lemma 11 (see below) it follows that $\Gamma\text{-}\liminf_{n\to\infty}\dfrac{\mathcal{G}_n}{G_n^c}(l) < +\infty$. As far as Griffith's criterion is concerned, the definition of $G_{\text{eff}}^c(l)$ when $\mathcal{G}_{hom}(l) = 0$ is in some sense arbitrary, indeed, if the evolution takes, at time t, a value l where $\mathcal{G}_{hom}(l) = 0$ then the inequality $\mathcal{G}_{hom}(\tau, l) \leqq G_{\text{eff}}^c(l)$ is satisfied for every $\tau \geqq t$ and every non-negative value of G_{eff}^c, further, the evolution will be constantly equal to l for $\tau \geqq t$ (see Remark 5 for a different definition of G_{eff}^c when $\mathcal{G}_{hom}$ vanishes).

Note that G_{eff}^c is independent of time and of the evolution, however in principle it may depend on every parameter and every datum of the problem; in mechanical terms, G_{eff}^c is an R-curve (R standing for resistance, see for example [2]). In §2.8 we will give an example in which G_{eff}^c is actually a constant, depending on the volume fraction λ, the ratio between the shear moduli $\mu_{A,i}$ and $\mu_{B,i}$ and the toughness G_A^c and G_B^c. Before proceeding, we provide this lemma.

Lemma 11. *The effective toughness G_{eff}^c is upper semi-continuous (and thus Borel measurable). Moreover, $G_{\text{eff}}^c \geqq \min\{G_A^c, G_B^c\}$.*

Proof. It is enough to consider the set $\{\mathcal{G}_{hom} \neq 0\}$ since, by continuity, the set $\{\mathcal{G}_{hom} = 0\}$ is closed. The upper semi-continuity follows from the continuity of $\mathcal{G}_{hom}$ and the lower semi-continuity of the $\Gamma\text{-}\liminf$. Next, write

$$\Gamma\text{-}\liminf_{n\to\infty}\frac{\mathcal{G}_n}{G_n^c}(l) = \inf\left\{\liminf_{n\to\infty}\frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} : l_n \to l\right\}$$

$$\leqq \frac{1}{\min\{G_A^c, G_B^c\}}\inf\left\{\liminf_{n\to\infty}\mathcal{G}_n(l_n) : l_n \to l\right\}$$

$$\leqq \frac{\Gamma\text{-}\liminf_{n\to\infty}\mathcal{G}_n(l)}{\min\{G_A^c, G_B^c\}}.$$

By Lemma 8 we infer

$$\Gamma\text{-}\liminf_{n\to\infty}\mathcal{G}_n(l) \leqq \mathcal{G}_{hom}(l) \tag{20}$$

and thus

$$\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) \leqq \frac{\mathcal{G}_{hom}(l)}{\min\{G_A^c, G_B^c\}}. \tag{21}$$

Hence, $G_{eff}^c(l) \geqq \min\{G_A^c, G_B^c\}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A local upper bound for G_{eff}^c is given in Corollary 14, as a byproduct of the following theorem:

Theorem 12. *Let ℓ be the limit evolution given by Proposition 6. Then $\ell^+(t) = \ell_{hom}(t)$, where $\ell_{hom} : [0, T) \to [L_0, L]$ is the unique non-decreasing, right continuous function which satisfies the initial condition $\ell_{hom}(0) = L_0$ and Griffith's criterion:*

(i) $G_{hom}(t, \ell_{hom}(t)) \leqq G_{eff}^c(\ell_{hom}(t))$ for every time $t \in [0, T)$;
(ii) if $G_{hom}(t, \ell_{hom}(t)) < G_{eff}^c(\ell_{hom}(t))$ then ℓ_{hom} is right differentiable in t and $\dot\ell_{hom}^+(t) = 0$;
(iii) if $t \in J(\ell_{hom})$ then $G_{hom}(t, s) \geqq G_{eff}^c(s)$ for every $s \in [\ell_{hom}^-(t), \ell_{hom}(t))$.

Proof. First, we check that for every $t \in [0, T)$ we have

$$\{l \in [L_0, L] : G_{hom}(t, l) < G_{eff}^c(l)\} = \left\{l \in [L_0, L] : \Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) < \frac{1}{f^2(t)}\right\}. \tag{22}$$

If $t = 0$ then $G_{hom}(0, l) = 0$ and thus by Lemma 11 we have $G_{hom}(0, l) < \min\{G_A^c, G_B^c\} \leqq G_{eff}^c(l)$ for every $l \in [L_0, L]$. By (21) we can write

$$\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) < +\infty = \frac{1}{f^2(0)}$$

for every $l \in [L_0, L]$. If $t > 0$ we write $G_{hom}(t, l) = f^2(t)\mathcal{G}_{hom}(l)$. If $\mathcal{G}_{hom}(l) \neq 0$, using the definition of G_{eff}^c we get

$$G_{hom}(t, l) < G_{eff}^c(l) \quad \Leftrightarrow \quad f^2(t) < \frac{1}{\Gamma\text{-}\liminf\limits_{n\to\infty} \dfrac{\mathcal{G}_n}{G_n^c}(l)}.$$

If $\mathcal{G}_{hom}(l) = 0$ then $G_{hom}(t, l) = 0$ and thus $G_{hom}(t, l) < \min\{G_A^c, G_B^c\} \leqq G_{eff}^c(l)$. Moreover by (21) we have

$$\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) = 0 < \frac{1}{f^2(t)}.$$

By (22) we can now invoke Proposition 6 which gives the representation

$$\ell^+(t) = \inf\{l \in [L_0, L] : G_{hom}(t, l) < G_{eff}^c(l)\}. \tag{23}$$

At this point it is enough to follow the proof of Proposition 2, with a few minor differences, in order to show that the function ℓ^+ defined by (23) satisfies Griffith's criterion. The proof is concluded. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 4. In general, the limit evolution ℓ does not satisfy Griffith's criterion in terms of the energy release G_{hom}, of the homogenized energy E_{hom}, and the homogenized toughness $G_{hom}^c = \lambda G_A^c + (1 - \lambda) G_B^c$. Starting from the (16), which is independent of the limit energies, there are actually different ways to proceed. In our approach we define an effective toughness G_{eff}^c in such a way that the limit ℓ satisfies Griffith's criterion for G_{hom} and G_{eff}^c. This idea seems very natural in terms of convergence of the energies and is consistent with the approach followed in mechanics, see for example [13]. On the other hand it could be interesting also to fix G_{hom}^c and define an effective energy release $\mathcal{G}_{eff}$ taking the limit of $\mathcal{G}_n$. However, even if $\mathcal{G}_n$ has a "nice" integral representation (see Lemma 1) it seems difficult to study its limit, the main issue being the fact that the support of the auxiliary functions ϕ shrinks with n. In §3.4 we will consider a different setting in which the representation of $\mathcal{G}_n$ holds for a function ϕ independent of n (see Lemma 21); in that case the limit of $\mathcal{G}_n$ can be computed, and actually equals $\mathcal{G}_{hom}$.

2.6.1. Plane-Strain Setting   The definition (19) of effective toughness and the convergence result of Theorem 12 can be easily extended to the plane strain setting with minor changes in the proofs. On the contrary, the generalization of the example and of the results provided in §2.8 and §3 is not straightforward, indeed, the computations provided in that sections are tailored to the anti-plane setting, being based on the specific form of the homogenized energy and of the energy release.

We give a brief outline. Consider the same geometry for the reference domain, the cracks and the layers. Consider a boundary condition of the form $f(t)g$ where $g \in H^1(\partial_D \Omega; \mathbb{R}^2)$. The condensed elastic energy $\mathcal{E}_n$ is again of class $C^1((0, L) \backslash \Lambda_n)$, thus, as in Proposition 2, there exists a unique, right-continuous evolution ℓ_n which satisfies Griffith's criterion and which can be characterized by $\ell_n(t) = \inf\{l \in [L_0, L] : G_n(t, l) < G_n^c(l)\}$. The regularity of the energy $\mathcal{E}_n$ follows from an integral representation of the energy release, as that of Lemma 1, see for example [20].

Next, by classical results on H-convergence, see for example [11], the linear elastic energy E_n converge to a linear elastic energy E_{hom} with constant coefficients, in particular the condensed (or reduced) energy $\mathcal{E}_{hom}$ is of class $C^1(0, L)$.

By Helly's Theorem, upon extracting a subsequence, there exists a limit evolution ℓ. After defining the effective toughness, as in (19), and arguing as in Theorem 12 it follows that $\ell^+ = \ell_{hom}$, where ℓ_{hom} is again the unique right-continuous evolution which satisfies Griffith's criterion.

2.6.2. Evolutionary Γ-Convergence   In this section we give a slight generalization of Theorem 12, in order to provide an evolutionary Γ-convergence result for our rate independent system, in the spirit of [15]. To this end, let us introduce the rate independent dissipation functionals

$$\mathcal{R}_n(l, \dot{l}) = \begin{cases} G_n^c(l)\,\dot{l} & \dot{l} \geqq 0, \\ +\infty & \text{otherwise}, \end{cases} \qquad \mathcal{R}_{eff}(l, \dot{l}) = \begin{cases} G_{eff}^c(l)\,\dot{l} & \dot{l} \geqq 0, \\ +\infty & \text{otherwise}. \end{cases}$$

We refer the reader to Appendix 4 for the notion of balanced viscosity solution for the rate independent system $(E_n, \mathcal{R}_n)$, in the language of [16]; actually (see again

Appendix 4), these solutions coincide with the quasi-static evolutions of Corollary 3. Therefore the right continuous representative of any balanced viscosity solution of the rate independent system $(E_n, \mathcal{R}_n)$ coincides with ℓ_n, that is, the unique right continuous evolution provided by Proposition 2.

Corollary 13. *The rate independent systems $(E_n, \mathcal{R}_n)$ evolutionary Γ-converge to $(E_{hom}, \mathcal{R}_{eff})$.*

Proof. By [15, Definition 2.10] the above statement is equivalent to the following: let $L_n \in (0, L)$ such that $L_n \to L_0 \in (0, L)$ and let γ_n be a solution of the rate independent system $(E_n, \mathcal{R}_n)$ with intial condition L_n, then γ_n converge to γ_{hom} where γ_{hom} is a solution of the rate independent system $(E_{hom}, \mathcal{R}_{eff})$ with initial condition L_0.

By Helly's Theorem there exists a subsequence (not relabeled) such that $\gamma_n \to \gamma$ pointwise in $[0, T)$. In order to characterize the limit γ it is not restrictive to consider the convergence of the sequence ℓ_n of the right continuous representatives of γ_n. At this point, in order to apply Theorem 12 it is enough to "fix the initial condition". There are at least a couple of alternatives. The first is to follow, step by step, the proof of Theorem 12 employing the explicit representation

$$\ell_n(t) = \inf\{l \in [L_n, L] : G_n(t, l) < G_n^c(l)\}.$$

The second chance is instead to employ the change of variable $\psi_n : [0, L] \to [0, L]$ given by

$$\psi_n(l) = \begin{cases} lL_0/L_n & l < L_n, \\ (l - L_n)(L - L_0)/(L - L_n) + L_0 & l \geqq L_n, \end{cases}$$

and then consider the evolutions $\tilde{\ell}_n(t) = \psi_n \circ \ell_n(t)$, which satisfy $\tilde{\ell}_n(0) = L_0$. The (tedious) check of convergence is omitted. $\square$

2.6.3. An Upper Bound   Thanks to Griffith's criterion, we can provide also a simple on a posteriori estimate on the effective toughness.

Corollary 14. *Assume that $\sup\{\ell_{hom}(t) : t \in [0, T)\} > L_0$. Then, the effective toughness G_{eff}^c is locally bounded, more precisely,*

$$G_{eff}^c(l) \leqq f^2(T)\mathcal{G}_{hom}(l)\frac{\max\{G_A^c, G_B^c\}}{\min\{G_A^c, G_B^c\}}$$

for every $l \in \mathrm{co}\, \ell_{hom}([0, T))$, the convex envelope of the range $\ell_{hom}([0, T))$.

Proof. The convex envelope coincides with the set $\{l \in [L_0, L] : l \leqq \ell_{hom}(t)$ for some $t \in [0, T)\}$ and is used to "fill the gaps" in the discontinuity points.

If $l \in [\ell_{hom}^-(t), \ell_{hom}(t))$ for some $t \in J(\ell_{hom})$ then $G_{eff}^c(l) \leqq G_{hom}(t, l) \leqq f^2(T)\mathcal{G}_{hom}(l)$.

Otherwise, let $L_0 < l' < \ell_{hom}(t)$ for some $t \in (0, T)$. For any sequence $l_n' \to l'$, we argue as follows. Clearly, $l_n' < \ell_n(t)$ for $n \gg 1$. Remember that $\ell_n(t) = \inf\{l \in$

$[L_0, L] : G_n(t, l) < G_n^c(l)\}$, hence $G_n(t, l_n') \geqq G_n^c(l_n') \geqq \min\{G_A^c, G_B^c\}$ and thus $\mathcal{G}_n(l_n') \geqq \min\{G_A^c, G_B^c\}/f^2(t)$. As a consequence,

$$
\begin{aligned}
\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l') &= \inf\left\{ \liminf_{n\to\infty} \frac{\mathcal{G}_n(l_n')}{G_n^c(l_n')} : l_n' \to l' \right\} \\
&\geqq \frac{1}{\max\{G_A^c, G_B^c\}} \inf\left\{ \liminf_{n\to\infty} \mathcal{G}_n(l_n') : l_n' \to l' \right\} \\
&\geqq \frac{\min\{G_A^c, G_B^c\}}{f^2(t)\max\{G_A^c, G_B^c\}}.
\end{aligned}
$$

It follows that

$$
G_{\mathrm{eff}}^c(l') \leqq \frac{\max\{G_A^c, G_B^c\} f^2(t) \mathcal{G}_{\mathrm{hom}}(l')}{\min\{G_A^c, G_B^c\}} \leqq f^2(T) \mathcal{G}_{\mathrm{hom}}(l') \frac{\max\{G_A^c, G_B^c\}}{\min\{G_A^c, G_B^c\}}.
$$

We proved the inequality for any $l' \in (L_0, \sup\{\ell_{\mathrm{hom}}(t) : t \in [0, T)\})$. We conclude by the upper semi-continuity of G_{eff}^c.   □

In §2.8 we will see an example in which the effective toughness is strictly greater than $\max\{G_A^c, G_B^c\}$.

Remark 5. Joining the lower bound of Lemma 11 and the upper bound of Corollary 14 it is clear that the set $\mathrm{co}\,\ell_{\mathrm{hom}}([0, T))$ and the set $\{\mathcal{G}_{\mathrm{hom}} = 0\}$ are disjoint. Therefore, to the purpose of proving Theorem 12 it is enough to define G_{eff}^c in the set $\{\mathcal{G}_{\mathrm{hom}} \neq 0\}$. In other terms, we could provide alternative definitions of G_{eff}^c, which still makes Griffith's criterion true. For instance, let

$$
G_\infty^c(l) = \frac{\mathcal{G}_{\mathrm{hom}}(l)}{\Gamma\text{-}\liminf\limits_{n\to\infty} \dfrac{\mathcal{G}_n}{G_n^c}(l)}
$$

be defined for $l \in [L_0, L]$ such that $\mathcal{G}_{\mathrm{hom}}(l) \neq 0$. Then we could employ as effective toughness the function $\tilde{G}_{\mathrm{eff}}^c(l) = \inf\{G^c(l)\}$ where the infimum is taken over all the upper semi-continuous functions $G^c : [L_0, L] \to \mathbb{R}$ with $G^c \geqq G_\infty^c$ in the set $\{\mathcal{G}_{\mathrm{hom}} \neq 0\}$.

2.6.4. Effective Toughness Under Convergence of the Energy Release

In this subsection we consider a couple of cases in which the energy release $\mathcal{G}_n$ converge (in a suitable sense) to the energy release $\mathcal{G}_{\mathrm{hom}}$ and, as a consequence, the effective toughness G_{eff}^c is computed explicitly and equals the maximum between G_A^c and G_B^c.

Corollary 15. *Assume that $\mathcal{G}_n \to \mathcal{G}_{\mathrm{hom}}$ locally uniformly in $(0, L)$. If $\mathcal{G}_{\mathrm{hom}}(l) \neq 0$ then*

$$
G_{\mathrm{eff}}^c(l) = \max\{G_A^c, G_B^c\}.
$$

Proof. We recall that

$$G_{\text{eff}}^c(l) = \frac{\mathcal{G}_{\text{hom}}(l)}{\Gamma\text{-}\liminf\limits_{n\to\infty} \dfrac{\mathcal{G}_n}{G_n^c}(l)}.$$

By uniform convergence $\mathcal{G}_n(l_n) \to \mathcal{G}_{\text{hom}}(l)$ for every $l_n \to l$, hence

$$\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) = \inf\left\{ \liminf_{n\to\infty} \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} : l_n \to l \right\}$$

$$= \mathcal{G}_{\text{hom}}(l)\inf\left\{ \liminf_{n\to\infty} \frac{1}{G_n^c(l_n)} : l_n \to l \right\}$$

$$= \frac{\mathcal{G}_{\text{hom}}(l)}{\max\{G_A^c, G_B^c\}},$$

which concludes the proof.   □

The previous result can be slightly generalized as follows:

Corollary 16. *Let $\mathcal{G}_{\text{hom}}(l) \neq 0$. Assume that $\mathcal{G}_{\text{hom}}(l) \leqq \Gamma\text{-}\liminf_{n\to\infty} \mathcal{G}_n(l)$ and that there exists a "joint recovery sequence" for $\mathcal{G}_{\text{hom}}(l)$ and $G_{\text{eff}}^c(l)$, that is, $l_n \to l$ such that*

$$\limsup_{n\to\infty} G_n^c(l_n) = \max\{G_A^c, G_B^c\}, \quad \limsup_{n\to\infty} \mathcal{G}_n(l_n) = \mathcal{G}_{\text{hom}}(l).$$

Then $G_{\text{eff}}^c(l) = \max\{G_A^c, G_B^c\}$.

Proof. Clearly $G_n^c \leqq \max\{G_A^c, G_B^c\}$ and $\mathcal{G}_{\text{hom}}(l) = \Gamma\text{-}\lim_{n\to\infty} \mathcal{G}_n(l)$ by Corollary 10. Hence

$$\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) \geqq \frac{1}{\max\{G_A^c, G_B^c\}} \inf\left\{ \liminf_{n\to\infty} \mathcal{G}_n(l_n) : l_n \to l \right\} = \frac{\mathcal{G}_{\text{hom}}(l)}{\max\{G_A^c, G_B^c\}}.$$

By Γ-convergence we know that $\mathcal{G}_n(l_n) \to \mathcal{G}_{\text{hom}}(l)$. Hence

$$\Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) \leqq \liminf_{n\to\infty} \frac{\mathcal{G}_n(l_n)}{G_n^c(l_n)} = \frac{\mathcal{G}_{\text{hom}}(l)}{\max\{G_A^c, G_B^c\}}.$$

Hence,

$$\frac{\mathcal{G}_{\text{hom}}(l)}{\max\{G_A^c, G_B^c\}} = \Gamma\text{-}\liminf_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l),$$

which concludes the proof.   □

2.7. Energy Identity

Given (19) the effective dissipated energy is defined by

$$D_{eff}(l) = \int_{L_0}^{l} G_{eff}^c(s) \, \mathrm{d}s$$

and the potential energy by $F(t, l) = E_{hom}(t, l) + D_{eff}(l)$. By Corollary 14 it turns out that $D_{eff} \in AC(L_0, \ell_{hom}(T))$. Note that G_{eff}^c is Borel measurable since it is upper semi-continuous.

Corollary 17. *For every $t \in [0, T)$ the following energy identity holds:*

$$F(t, \ell_{hom}(t)) = \int_0^t \partial_t F(\tau, \ell_{hom}(\tau)) \, \mathrm{d}\tau + \sum_{\tau \in J_{hom} \cap [0,t]} [\![F(\tau, \ell_{hom}(\tau))]\!]. \tag{24}$$

There $J_{hom} = J(\ell_{hom})$ and $[\![F(\tau, \ell_{hom}(\tau))]\!] = F(\tau, \ell_{hom}(\tau)) - F(\tau, \ell_{hom}^-(\tau)) \leqq 0$.

Proof. In this case we cannot employ the proof of Corollary 5. Clearly, if $\ell_{hom}(t) \equiv L_0$ the proof is trivial by time regularity. As $\ell_{hom} \in BV(0, T)$ we can write

$$\mathrm{d}\ell_{hom} = \dot{\ell}_{hom} \mathrm{d}\tau + \mathrm{d}_C \ell_{hom} + \mathrm{d}_J \ell_{hom},$$

where $\dot{\ell}_{hom} \in L^1(0, T)$ denotes the density of the absolutely continuous part (with respect to the Lebesgue measure $\mathrm{d}\tau$), $\mathrm{d}_C \ell$ denotes the Cantor part, while

$$\mathrm{d}_J \ell_{hom} = \sum_{\tau \in J_{hom}} [\![\ell_{hom}(\tau)]\!] \delta_\tau.$$

For convenience, denote $\mathrm{d}_{LC} \ell_{hom} = \dot{\ell}_{hom} \mathrm{d}\tau + \mathrm{d}_C \ell$.

Being E_{hom} of class C^1 by the chain rule in BV, see for example [1], it follows that

$$E_{hom}(t, \ell_{hom}(t)) = E_{hom}(0, \ell_{hom}(0)) + \int_0^t \partial_t E_{hom}(\tau, \ell_{hom}(\tau)) \, \mathrm{d}\tau$$

$$+ \int_0^t \partial_l E_{hom}(\tau, \ell_{hom}(\tau)) \, \mathrm{d}_{LC} \ell_{hom} + \sum_{\tau \in J_{hom} \cap [0,t]} [\![E_{hom}(\tau, \ell_{hom}(\tau))]\!]. \tag{25}$$

By Griffith's criterion, it remains to prove that a similar representation holds also for the effective dissipated energy, i.e,

$$D_{eff}(\ell_{hom}(t)) = D_{eff}(\ell_{hom}(0)) + \int_0^t G_{eff}^c(\ell_{hom}(\tau)) \, \mathrm{d}_{LC} \ell_{hom} + \sum_{\tau \in J_{hom} \cap [0,t]} [\![D(\tau, \ell_{hom}(\tau))]\!]. \tag{26}$$

Remember that D_{eff} is not of class C^1, however, by definition we can write

$$D_{eff}(l_2) = D_{eff}(l_1) + \int_{l_1}^{l_2} G_{eff}^c(l) \, \mathrm{d}l, \tag{27}$$

for every $0 < l_1 < l_2 < L$. Hence

$$D_{\text{eff}}(\ell_{\text{hom}}(t)) = \int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l = \int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l_{|R} + \int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l_{|U},$$

where $R = \ell_{\text{hom}}([0, T))$ while $U = \text{co}\,\ell_{\text{hom}}([0, T))\backslash R$. We will check that

$$\int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l_{|U} = \sum_{\tau \in J_{\text{hom}} \cap [0,t]} [\![D(\ell_{\text{hom}}(\tau))]\!] \tag{28}$$

and that

$$\int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l_{|R} = \int_0^t G^c_{\text{eff}}(\ell_{\text{hom}}(\tau))\, \mathrm{d}_{LC}\ell_{\text{hom}}, \tag{29}$$

which, put together, imply (26).

By the right continuity of ℓ_{hom}

$$U = \bigcup_{\tau \in J_{\text{hom}}} U_\tau, \tag{30}$$

where U_τ denotes an interval of the form $(\ell_{\text{hom}}^-(\tau), \ell_{\text{hom}}(\tau))$ or $[\ell_{\text{hom}}^-(\tau), \ell_{\text{hom}}(\tau))$ (the fact that $\ell_{\text{hom}}(\tau)$ is excluded is due to the right continuity of ℓ_{hom}). Note that the sets U_τ are pairwise disjoint and remember that J_{hom} is a countable set. Then

$$\int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l_{|U} = \sum_{\tau \in J_{\text{hom}} \cap [0,t]} \int_{\ell_{\text{hom}}^-(\tau)}^{\ell_{\text{hom}}(\tau)} G^c_{\text{eff}}(l)\, \mathrm{d}l$$

$$= \sum_{\tau \in J_{\text{hom}} \cap [0,t]} D(\ell_{\text{hom}}(\tau)) - D(\ell_{\text{hom}}^-(\tau)).$$

In order to prove (29), we will first check that $\mathrm{d}l_{|R} = \mathrm{d}_{LC}^\sharp \ell_{\text{hom}}$, where $\mathrm{d}_{LC}^\sharp \ell_{\text{hom}}$ denotes the push forward of the measure $\mathrm{d}_{LC}\ell_{\text{hom}}$; then, by the change of variable formula for the push-forward, see for example [1], we get

$$\int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}l_{|R} = \int_{L_0}^{\ell_{\text{hom}}(t)} G^c_{\text{eff}}(l)\, \mathrm{d}_{LC}^\sharp \ell_{\text{hom}} = \int_0^t G^c_{\text{eff}}(\ell_{\text{hom}}(\tau))\, \mathrm{d}_{LC}\ell_{\text{hom}}.$$

Note that both $\mathrm{d}l_{|R}$ and $\mathrm{d}_{LC}^\sharp \ell_{\text{hom}}$ are Borel measure which do not contain atoms. Thus, it is enough to check that they coincide on closed intervals $[l_1, l_2] \subset \text{co}\,\ell_{\text{hom}}([0, T))$.

To evaluate the push-forward measure of $[l_1, l_2]$ we should compute the $(\mathrm{d}_{LC}\ell_{\text{hom}})$-measure of the set $(\ell_{\text{hom}})^{-1}([l_1, l_2])$. If $(\ell_{\text{hom}})^{-1}([l_1, l_2]) = \emptyset$ the measure vanishes. Otherwise, by monotonicity of ℓ_{hom} it turns out that $(\ell_{\text{hom}})^{-1}([l_1, l_2])$ is again an interval; more precisely, it is of the form $[t_1, t_2)$ or $[t_1, t_2]$, where $t_1 = \min\{(\ell_{\text{hom}})^{-1}([l_1, l_2])\}$ and $t_2 = \sup\{(\ell_{\text{hom}})^{-1}([l_1, l_2])\}$. Note that the point t_1 is always included by the right continuity of ℓ_{hom}, moreover, the $\mathrm{d}_{LC}\ell_{\text{hom}}$-measure

of $[t_1, t_2)$ or $[t_1, t_2]$ actually coincides with the $(d_{LC}\ell_{hom})$-measure of the open interval (t_1, t_2) since $d_{LC}\ell_{hom} = \dot{\ell}_{hom} d\tau + d_C \ell_{hom}$ does not contain atoms. Then

$$d_{LC}\ell_{hom}(t_1, t_2) = d\ell_{hom}(t_1, t_2) - d_J \ell_{hom}(t_1, t_2)$$
$$= \ell_{hom}^-(t_2) - \ell_{hom}(t_1) - \sum_{\tau \in J_{hom} \cap (t_1, t_2)} [\![\ell_{hom}(\tau)]\!]. \qquad (31)$$

Let us turn to the measure $dl_{|R}$. If $(\ell_{hom})^{-1}([l_1, l_2]) = \emptyset$ then $[l_1, l_2] \subset U$ and thus its $dl_{|R}$-measure vanishes. Otherwise, by (30) we can write

$$dl_{|R}[l_1, l_2] = (l_2 - l_1) - dl_{|U}[l_1, l_2] = (l_2 - l_1) - \sum_{\tau \in J_{hom}} |U_\tau \cap [l_1, l_2]|.$$

Let us study the measure $|U_\tau \cap [l_1, l_2]|$ as a function of $\tau \in J_{hom}$. Note that $\tau > 0$ since ℓ_{hom} is right continuous.

If $\tau < t_1$ then $\ell_{hom}(\tau) < l_1$ and thus $|U_\tau \cap [l_1, l_2]| = 0$. If $\tau > t_2$ then $\ell_{hom}^-(\tau) \geqq l_2$ (since $\ell_{hom}(t) > l_2$ for $t_2 < t < \tau$) and thus $|U_\tau \cap [l_1, l_2]| = 0$.

If $\tau = t_1$ then $l_1 \leqq \ell_{hom}(t_1) \leqq l_2$ while $\ell_{hom}^-(t_1) \leqq l_1$ (since $\ell_{hom}(t) < l_1$ for any $t < t_1$). It follows that $|U_{t_1} \cap [l_1, l_2]| = \ell_{hom}(t_1) - l_1$. Hence, we can write

$$-l_1 - |U_{t_1} \cap [l_1, l_2]| = -\ell_{hom}(t_1).$$

If $\tau \in (t_1, t_2)$ then $l_1 \leqq \ell_{hom}(t_1) \leqq l_2$ while $\ell_{hom}^-(\tau) \geqq l_1$ (since $\ell_{hom}(t) \geqq l_1$ for $t_1 \leqq t < \tau$).

Hence $[\ell_{hom}^-(\tau), \ell_{hom}(\tau)] \subset [l_1, l_2]$ and $|U_\tau \cap [l_1, l_2]| = [\![\ell_{hom}(\tau)]\!]$.

If $\tau = t_2$ and $t_2 \notin (\ell_{hom})^{-1}([l_1, l_2])$ then $\ell_{hom}(t_2) > l_2$ while $l_1 \leqq \ell_{hom}^-(t_2) \leqq l_2$ (since $l_1 \leqq \ell_{hom}^-(t) \leqq l_2$ for every $t \in [t_1, t_2)$). Then $|U_{t_2} \cap [l_1, l_2]| = l_2 - \ell_{hom}^-(t_2)$ and we can write

$$l_2 - |U_{t_2} \cap [l_1, l_2]| = \ell_{hom}^-(t_2).$$

If $\tau = t_2$ and $t_2 \in (\ell_{hom})^{-1}([l_1, l_2])$ then $\ell_{hom}(t_2) = l_2$; indeed, assume by contradiction that $\ell_{hom}(t_2) < l_2 \leqq \ell_{hom}(T)$ then $t_2 < T$ and (by right continuity) $\ell_{hom}(t) < l_2$ in a sufficiently small right neighborhood of t_2. Since $l_1 \leqq \ell_{hom}^-(t_2) \leqq l_2$ (as above) we get again

$$l_2 - |U_{t_2} \cap [l_1, l_2]| = \ell_{hom}^-(t_2).$$

In conclusion

$$dl_{|R}[l_1, l_2] = \ell_{hom}^-(t_2) - \ell_{hom}(t_1) - \sum_{\tau \in J_{hom} \cap (t_1, t_2)} [\![\ell_{hom}(\tau)]\!]. \qquad (32)$$

By (32) and (31) the measures $dl_{|R}$ and $d_{LC}^\sharp \ell_{hom}$ do coincide. $\square$

2.7.1. Toughening and Micro-instabilities   In this section we study the toughening effect resulting from our convergence result. As we will see toughening turns out to be the macroscopic effect of microscopic instabilities in the evolutions ℓ_n.

Let $G^c_{hom} = \lambda G^c_A + (1 - \lambda) G^c_B$ be the homogenized (or averaged) toughness, that is the weak* limit of G^c_n.

For simplicity, let us consider a time interval $[t_a, t_b]$ such that $\ell_{hom}(t_a) < \ell_{hom}(t_b)$ and $J(\ell_{hom}) \cap [t_a, t_b] = \emptyset$. We will show that

$$G^c_{hom} \leqq G^c_{eff}(l) \text{ for every } l \in [\ell_{hom}(t_a), \ell_{hom}(t_b)].$$

Let us rewrite the energy identity for ℓ_n in the form

$$E_n(t, \ell_n(t)) + \int_{L_0}^{\ell_n(t)} G_n(l) \, dl = \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + d_J F_n([0, t]), \quad (33)$$

where we used the shorthand notation

$$d_J F_n([0, t]) = \sum_{\tau \in J(\ell_n) \cap [0, t]} [\![F_n(\tau, \ell_n(\tau))]\!].$$

Thus, for every $t_a \leqq t_1 < t_2 \leqq t_b$ we can write

$$E_n(t_2, \ell_n(t_2)) - E_n(t_1, \ell_n(t_1)) - \int_{t_1}^{t_2} \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau$$

$$= - \int_{\ell_n(t_1)}^{\ell_n(t_2)} G^c_n(l) \, dl + d_J F_n((t_1, t_2]). \quad (34)$$

Since $\ell_n \to \ell_{hom}$ pointwise in $[0, T)$ by Proposition 7 we get

$$E_n(t_i, \ell_n(t_i)) \to E_{hom}(t_i, \ell_{hom}(t_i)) \quad \text{for } i = 1, 2.$$

Remember that $\partial_t F_n(\tau, l) = 2 f(\tau) \dot{f}(\tau) \mathcal{E}_n(l)$, hence $\partial_t F_n(\tau, \ell_n(\tau)) \to \partial_t F(\tau, \ell_{hom}(\tau))$ and by dominated convergence we get

$$\int_{t_1}^{t_2} \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau \to \int_{t_1}^{t_2} \partial_t F_{hom}(\tau, \ell_{hom}(\tau)) \, d\tau.$$

In summary, the left hand side of (34) converges to

$$E_{hom}(t_2, \ell_{hom}(t_2)) - E_{hom}(t_1, \ell_n(t_1)) - \int_{t_1}^{t_2} \partial_t F_{hom}(\tau, \ell_n(\tau)) \, d\tau$$

$$= - \int_{\ell_{hom}(t_1)}^{\ell_{hom}(t_2)} G^c_{eff}(l) \, dl,$$

where the identity follows from Corollary 17 and from the assumption $J(\ell_{hom}) \cap [t_a, t_b] = \emptyset$.

As a consequence the right hand side of (34) converges, that is,

$$\int_{\ell_n(t_1)}^{\ell_n(t_2)} G^c_n(l) \, dl + d_J F_n((t_1, t_2]) \to \int_{\ell_{hom}(t_1)}^{\ell_{hom}(t_2)} G^c_{eff}(l) \, dl.$$

As $G_n^c \overset{*}{\rightharpoonup} G_{hom}^c = \lambda G_A^c + (1-\lambda)G_B^c$,

$$\int_{\ell_n(t_1)}^{\ell_n(t_2)} G_n^c(l)\,dl \to \int_{\ell_{hom}(t_1)}^{\ell_{hom}(t_2)} G_{hom}^c(l)\,dl.$$

Hence

$$d_J F_n((t_1, t_2]) \to \int_{\ell_{hom}(t_1)}^{\ell_{hom}(t_2)} G_{hom}^c(l) - G_{eff}^c(l)\,dl.$$

Since $d_J F_n((t_1, t_2]) \leqq 0$ and it follows that

$$\int_{\ell_{hom}(t_1)}^{\ell_{hom}(t_2)} G_{hom}^c(l) - G_{eff}^c(l)\,dl \leqq 0.$$

By the arbitrariness of t_1 and t_2 it follows that $G_{hom}^c(l) \leqq G_{eff}^c(l)$ almost everywhere in $[\ell_{hom}(t_a), \ell_{hom}(t_a)]$. Since G_{hom}^c is constant and G_{eff}^c is upper semicontinuous the inequality actually holds everywhere in $[\ell_{hom}(t_a), \ell_{hom}(t_b)]$.

2.8. An Explicit Computation of the Effective Toughness

In this section we consider a specific case which highlights some peculiar features of the effective toughness, and in particular its dependence on the data. Besides the hypotheses of the previous section, we assume that $\mu_{B,1}\,\mu_{B,2} = \mu_{A,1}\,\mu_{A,2}$. No further assumptions are made on the toughness, thus it may occur $G_A^c \neq G_B^c$ or $G_A^c = G_B^c$. These assumptions include for instance the case in which material A is anisotropic and material B is obtained by a $\frac{\pi}{2}$ rotation of material A.

For convenience, let us recall the results of Proposition 7 and a few notations. The homogenized stiffness matrix is given by

$$C_{hom} = \begin{pmatrix} \mu_{hom,1} & 0 \\ 0 & \mu_{hom,2} \end{pmatrix},$$

where $1/\mu_{hom,1} = \lambda/\mu_{A,1} + (1-\lambda)/\mu_{B,1}$ and $\mu_{hom,2} = \lambda\mu_{A,2} + (1-\lambda)\mu_{B,2}$. The (condensed) elastic energy $\mathcal{E}_{hom}$ is of class $C^1(0, L)$ and the energy release is denoted by $\mathcal{G}_{hom}(l)$. We will prove that

$$G_{eff}^c = \lambda \max\left\{ G_A^c,\, G_B^c \frac{\mu_{B,1}}{\mu_{A,1}} \right\} + (1-\lambda) \max\left\{ G_A^c \frac{\mu_{A,1}}{\mu_{B,1}},\, G_B^c \right\}. \qquad (35)$$

In particular, the effective toughness is constant and depends also on the elastic contrast $\mu_{A,1}/\mu_{B,1}$. More precisely, we will prove that G_{eff}^c takes the above value where $\mathcal{G}_{hom} \neq 0$, which is enough for our purpose in the light of Remark 5.

Example 1. Assume that $\mu_{B,1} < \mu_{A,1}$ and $G_B^c \leqq G_A^c$. Then

$$G_{eff}^c = \lambda G_A^c + (1-\lambda)G_A^c \frac{\mu_{A,1}}{\mu_{B,1}} > \lambda G_A^c + (1-\lambda)G_A^c = G_A^c = \max\{G_A^c, G_B^c\}.$$

Next theorem states the main convergence result.

Theorem 18. *Let ℓ_n be the quasi-static evolutions given by Proposition 2. Then $\ell_n \overset{*}{\rightharpoonup} \ell_{hom}$ in $BV(0,T)$, where ℓ_{hom} is the unique non-decreasing, right continuous function which satisfies the initial condition $\ell_{hom}(0) = L_0$ and Griffith's criterion in the following form:*

(i) $G_{hom}(t, \ell_{hom}(t)) \leqq G_{eff}^c$ for every time $t \in [0,T)$;
(ii) if $G_{hom}(t, \ell_{hom}(t)) < G_{eff}^c$ then ℓ_{hom} is differentiable in t and $\dot{\ell}_{hom}(t) = 0$;
(iii) if $t \in J(\ell_{hom})$ then $G_{hom}(t,l) \geqq G_{eff}^c$ for every $l \in [\ell_{hom}^-(t), \ell_{hom}(t))$.

Note that in the above statement it is not necessary to extract any subsequence of ℓ_n. As the effective toughness is constant, the effective dissipated energy boils down to $D_{eff}(l) = G_{eff}^c(l - L_0)$; by Corollary 5 we get the energy identity for $F(t,u) = E_{hom}(t,u) + D_{eff}(l)$, stated hereafter.

Corollary 19. *For every $t \in [0,T)$, the following energy identity holds:*

$$F(t, \ell_{hom}(t)) = \int_0^t \partial_t F(\tau, \ell_{hom}(\tau)) \, d\tau + \sum_{\tau \in J_{hom} \cap [0,t]} [\![F(\tau, \ell_{hom}(\tau))]\!]. \qquad (36)$$

Remark 6. Note that the limit energy $F(t, \cdot)$ does not coincide with the Γ-limit of the energies $F_n(t, \cdot) = E_n(t, \cdot) + D_n(\cdot)$, which is given by

$$F_{hom}(t, l) = E_{hom}(t, l) + G_{hom}^c(l - L_0) \quad \text{where} \quad G_{hom}^c = \lambda G_A^c + (1 - \lambda) G_B^c.$$

Before proceeding we state, for the reader's convenience, the following elementary change of variable, which will be often used in the sequel:

Lemma 20. *Let O be an open set in $\mathbb{R}^2$ and $C \in \mathbb{R}^{2 \times 2}$. Let $\Phi : O \to \widehat{O}$ be a bi-Lipschitz map. For $z \in H^1(O)$ denote $\hat{z} = (z \circ \Phi^{-1}) \in H^1(\widehat{O})$, then*

$$\int_O \nabla z \, C \, \nabla z^T dx = \int_{\widehat{O}} \nabla \hat{z} \, \widehat{C} \nabla \hat{z}^T \, d\hat{x} \quad \text{where} \quad \widehat{C} = (D\Phi) C (D\Phi)^T (\det D\Phi)^{-1}.$$
(37)

In particular, if $\Phi(x_1, x_2) = (\alpha x_1, \beta x_2)$ and $C = \mu_1 \hat{e}_1 \otimes \hat{e}_1 + \mu_2 \hat{e}_2 \otimes \hat{e}_2$ then

$$\widehat{C} = \begin{pmatrix} \mu_1 \alpha/\beta & 0 \\ 0 & \mu_2 \beta/\alpha \end{pmatrix}.$$

Proof of Theorem 18. The Theorem follows from

$$\frac{\mathcal{G}_{hom}(l)}{\Gamma\text{-}\lim_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l)} = \lambda \max \left\{ G_A^c, \, G_B^c \frac{\mu_{B,1}}{\mu_{A,1}} \right\} + (1 - \lambda) \max \left\{ G_A^c \frac{\mu_{A,1}}{\mu_{B,1}}, \, G_B^c \right\}, \quad (38)$$

for $\mathcal{G}_{hom}(l) \neq 0$, after employing Remark 3 and Remark 5.

I. Let $\alpha = \mu_{B,1}/\mu_{A,1}$. For $S = \lambda L \alpha + (1 - \lambda) L$ let $\widehat{\Omega} = (0, S) \times (-H, H)$. We consider a bi-Lipschitz piecewise affine map $\Phi_n : \Omega \to \widehat{\Omega}$ of the form $\Phi_n(x_1, x_2) = (\phi_n(x_1), x_2)$ where

$$\phi_n(x_1) = \int_0^{x_1} \alpha_n \, ds \quad \text{and} \quad \alpha_n = \begin{cases} \alpha & \text{in } K_{n,A} \\ 1 & \text{in } K_{n,B}. \end{cases}$$

We employ the notation $\widehat{\Omega}_{n,A}$ to denote the set $\Phi_n(\Omega_{n,A})$ and similarly for all the sets introduced in §2.1. We denote $\hat{g}_n = g \circ \Phi_n^{-1}$ and consider the spaces

$$\widehat{\mathcal{U}}_s = \{\hat{u} \in H^1(\widehat{\Omega} \setminus K_s) : \hat{u} = \hat{g}_n \text{ in } \partial_D \widehat{\Omega}\} \quad \text{for } s \in (0, S].$$

Let $s = \phi_n(l)$. For $u \in \mathcal{U}_l$ and $\hat{u} = u \circ \Phi_n^{-1} \in \widehat{\mathcal{U}}_s$ we have and

$$\mathcal{W}_{l,n}(u) = \widehat{\mathcal{W}}_{s,n}(\hat{u}) = \tfrac{1}{2} \int_{\widehat{\Omega} \setminus \widehat{K}_s} \nabla \hat{u} \, \widehat{C}_n \nabla \hat{u}^T \, dy.$$

Under the assumption $\mu_{B,1} \mu_{B,2} = \mu_{A,1} \mu_{A,2}$ and with the above choice of $\alpha = \mu_{B,1}/\mu_{A,1}$, it turns out that $\widehat{C}_n$ is constant in $\widehat{\Omega}$; indeed we have

$$\widehat{C}_n = \begin{pmatrix} \alpha \mu_{A,1} & 0 \\ 0 & \mu_{A,2}/\alpha \end{pmatrix} = \begin{pmatrix} \mu_{B,1} & 0 \\ 0 & \mu_{B,2} \end{pmatrix} \text{ in } \widehat{\Omega}_{A,n}, \quad \widehat{C}_n = \begin{pmatrix} \mu_{B,1} & 0 \\ 0 & \mu_{B,2} \end{pmatrix} \text{ in } \widehat{\Omega}_{B,n}.$$

Note also that $\widehat{C}_n$ is actually independent of n; therefore in the sequel we will skip the dependence on n in the notation. The reduced energy $\widehat{\mathcal{E}}$ is then of class $C^1(0, S)$; in particular the energy release $\widehat{\mathcal{G}}$ is continuous in $(0, S)$. Clearly, by the change of variable,

$$\widehat{\mathcal{E}}(\phi_n(l)) = \mathcal{E}_n(l) \quad \text{and thus} \quad \widehat{\mathcal{G}}(\phi_n(l)) \, \phi_n'(l) = \mathcal{G}_n(l) \text{ for } l \in (0, L) \setminus \Lambda_n. \quad (39)$$

II. Let $\alpha_{hom} = \alpha \lambda + (1 - \lambda)$. Let $\Phi : \Omega \to \widehat{\Omega}$ be the linear map $\Phi(x_1, x_2) = (\phi(x_1), x_2)$ for $\phi(x_1) = \alpha_{hom} x_1$. With this change of variable, for $s = \phi(l) = \alpha_{hom} l$, we write

$$\mathcal{W}_{l,hom}(u) = \widehat{\mathcal{W}}_{s,hom}(\hat{u}) = \tfrac{1}{2} \int_{\widehat{\Omega} \setminus \widehat{K}_s} \nabla \hat{u} \, \widehat{C}_{hom} \nabla \hat{u}^T \, dy,$$

where

$$\widehat{C}_{hom} = \begin{pmatrix} \alpha_{hom} \, \mu_{hom,1} & 0 \\ 0 & \mu_{hom,2}/\alpha_{hom} \end{pmatrix}.$$

As expected, $\widehat{C}_{hom} = \widehat{C}$; indeed,

$$\frac{1}{\mu_{hom,1}} = \frac{\lambda}{\mu_{A,1}} + \frac{1 - \lambda}{\mu_{B,1}} = \frac{\lambda \mu_{B,1} + (1 - \lambda)\mu_{A,1}}{\mu_{A,1} \mu_{B,1}}, \quad \alpha_{hom} = \frac{\mu_{B,1}}{\mu_{A,1}} \lambda + (1 - \lambda)$$
$$= \frac{\lambda \mu_{B,1} + (1 - \lambda)\mu_{A,1}}{\mu_{A,1}},$$

and thus $\alpha_{hom} \, \mu_{hom,1} = \mu_{B,1}$, moreover, recalling that $\mu_{B,1} \mu_{B,2} = \mu_{A,1} \mu_{A,2}$,

$$\mu_{hom,2} = \lambda \mu_{A,2} + (1 - \lambda)\mu_{B,2}, \quad \alpha_{hom} = \frac{\mu_{A,2}}{\mu_{B,2}} \lambda + (1 - \lambda) = \frac{\lambda \mu_{A,2} + (1 - \lambda)\mu_{B,2}}{\mu_{B,2}},$$

and then $\mu_{hom,2}/\alpha_{hom} = \mu_{B,2}$. In conclusion,

$$\widehat{\mathcal{E}}(\phi(l)) = \mathcal{E}_{hom}(l) \quad \text{and thus} \quad \widehat{\mathcal{G}}(\phi(l))\alpha_{hom} = \mathcal{G}_{hom}(l) \text{ for } l \in (0, L). \quad (40)$$

III. By (39) we can write

$$\frac{\mathcal{G}_n(l)}{G_n^c(l)} = \frac{\widehat{\mathcal{G}}(\phi_n(l))\,\phi_n'(l)}{G_n^c(l)}.$$

Note that $\phi_n \to \phi$ uniformly in $[0, L]$. Then, being $\widehat{\mathcal{G}}$ continuous, $\widehat{\mathcal{G}} \circ \phi_n \to \widehat{\mathcal{G}} \circ \phi$ locally uniformly in $(0, L)$. As a consequence

$$\Gamma\text{-}\lim_{n\to\infty} \frac{\mathcal{G}_n}{G_n^c}(l) = \widehat{\mathcal{G}}(\phi(l))\, \Gamma\text{-}\lim_{n\to\infty} \frac{\phi_n'}{G_n^c}(l). \tag{41}$$

Writing, explicitly,

$$\frac{\phi_n'(l)}{G_n^c(l)} = \begin{cases} \alpha/G_A^c & \text{in } K_{nA}, \\ 1/G_B^c & \text{in } K_{nB}, \end{cases}$$

we get $\Gamma\text{-}\lim_{n\to\infty} \dfrac{\phi_n'}{G_n^c}(l) = \min\{\alpha/G_A^c, 1/G_B^c\}$. If $\mathcal{G}_{hom}(l) \neq 0$, then $\widehat{\mathcal{G}}(\phi(l)) \neq 0$, and by (40) and (41),

$$\begin{aligned}
G_{eff}^c(l) &= \frac{\mathcal{G}_{hom}(l)}{\Gamma\text{-}\lim\limits_{n\to\infty} \dfrac{\mathcal{G}_n}{G_n^c}(l)} = \frac{\alpha_{hom}\widehat{\mathcal{G}}(\phi(l))}{\widehat{\mathcal{G}}(\phi(l))\min\{\alpha/G_A^c, 1/G_B^c\}} \\
&= \left(\alpha\lambda + (1-\lambda)\right)\max\{G_A^c/\alpha, G_B^c\} \\
&= \lambda\max\left\{G_A^c,\, G_B^c\frac{\mu_{B,1}}{\mu_{A,1}}\right\} + (1-\lambda)\max\left\{G_A^c\frac{\mu_{A,1}}{\mu_{B,1}},\, G_B^c\right\}.
\end{aligned}$$

The proof of (38) is concluded. □

Remark 7. In general $\mathcal{G}_{hom}$ is neither the pointwise nor the Γ-limit of $\mathcal{G}_n$. Indeed, by (39) and (40) we have

$$\widehat{\mathcal{G}}(\phi_n(l))\,\phi_n'(l) = \mathcal{G}_n(l) \ \text{ and } \ \widehat{\mathcal{G}}(\phi(l))\alpha_{hom} = \mathcal{G}_{hom}(l).$$

We know that $\widehat{\mathcal{G}} \circ \phi_n \to \widehat{\mathcal{G}} \circ \phi$ locally uniformly but the pointwise limit of ϕ_n' does not exists. Moreover, $\Gamma\text{-}\lim_{n\to\infty}\phi_n'(l) = \min\{\alpha, 1\}$, while $\alpha_{hom} = \lambda\alpha + (1-\lambda) = \min\{\alpha, 1\}$ if and only if $\alpha = 1$; the latter case is trivial, since $\mu_{B,1} = \mu_{A,1}$ and $\mu_{B,1}\,\mu_{B,2} = \mu_{A,1}\,\mu_{A,2}$ imply that the stiffness matrix $\boldsymbol{C}_n$ is constant.

3. Horizontal Layers with Horizontal Crack

3.1. Energy and Energy Release

As in the previous section we assume that the uncracked reference configuration is the set $\Omega = (0, L) \times (-H, H)$ and that the crack set is of the form $K_l = [0, l] \times \{0\}$ for $l \in (0, L)$. On the contrary, here we assume that Ω is made periodic horizontal layers of thickness $h_n = H/n$ for $n \in 2\mathbb{N}$ (the fact that n is even implies that K_l
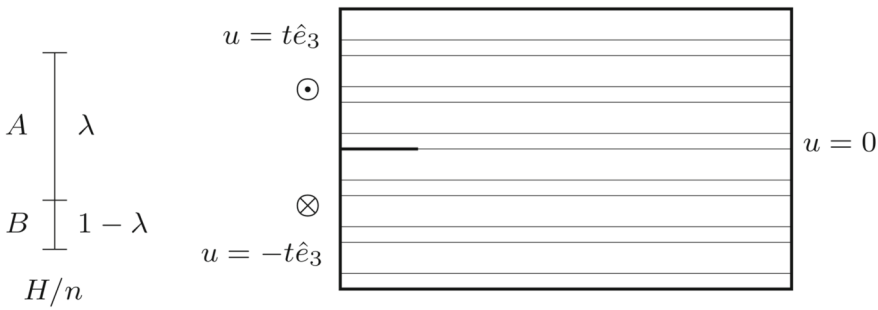
Fig. 2. An example of geometry in the case of an interfacial crack with horizontal layers

is contained in the interface between two layers). For $\lambda \in (0, 1)$ each layer is itself made of a horizontal layer of material A, with thickness λh_n, and an horizontal layer of material B, with thickness $(1 - \lambda)h_n$ (see Fig. 2). In this geometry crack may actually propagate along the interface or penetrate into the layer [12]. The convergence result presented in the sequel holds only in the former case, since the latter is still out of reach in our analysis.

For functional spaces, boundary condition, energies and energy release we employ the same assumptions and notation of the previous section, apart from toughness which here is simply denoted by G^c, since it is independent of n.

Following the proof of Lemma 1 it not difficult to show

Lemma 21. $\mathcal{E}_n : [0, L] \to \mathbb{R}$ *is decreasing and continuous. Moreover, it is of class* $C^1(0, L)$ *and*

$$\mathcal{E}'_n(l) = \int_{\Omega \setminus K_l} \nabla u_{l,n} \boldsymbol{C}_n \boldsymbol{E} \nabla u^T_{l,n} \, \phi' \mathrm{d}x \quad \text{for} \quad \boldsymbol{E} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (42)$$

where $\phi \in W^{1,\infty}(0, L)$ *with* $\mathrm{supp}(\phi) \subset (0, L)$, $0 \leqq \phi \leqq 1$ *and* $\phi(l) = 1$.

Remark 8. Comparing with Lemma 1 note that here ϕ is independent of n, in particular its support does not shrink with n. Finally, note that $\mathcal{E}_n$ is of class C^1 in the whole $(0, L)$.

3.2. Quasi-Static Evolution by Griffith's Criterion

Following the proofs of Proposition 2 and Corollary 5 we obtain the following results:

Proposition 22. *There exists a unique non-decreasing, right continuous function* $\ell_n : [0, T) \to [L_0, L]$ *which satisfies the initial condition* $\ell_n(0) = L_0$ *and Griffith's criterion, in the following form:*

  (i) $G_n(t, \ell_n(t)) \leqq G^c$ *for every time* $t \in [0, T)$;
  (ii) *if* $G_n(t, \ell_n(t)) < G^c$ *then* ℓ_n *is differentiable in* t *and* $\dot{\ell}_n(t) = 0$;
  (iii) *if* $t \in J(\ell_n)$ *then* $G_n(t, l) \geqq G^c$ *for every* $l \in [\ell_n^-(t), \ell_n(t))$.

Corollary 23. *If $G_n(t, \ell_n(t)) < G^c$ then ℓ_n is constant in a neighborhood of t.*

Corollary 24. *For every $t \in [0, T)$ the following energy identity holds:*

$$F_n(t, \ell_n(t)) = \int_0^t \partial_t F_n(\tau, \ell_n(\tau)) \, d\tau + \sum_{\tau \in J(\ell_n) \cap [0,t]} [\![F_n(\tau, \ell_n(\tau))]\!]. \qquad (43)$$

3.3. Homogenization of Griffith's Criterion

Here the homogenized stiffness matrix is

$$C_{hom} = \begin{pmatrix} \mu_{hom,1} & 0 \\ 0 & \mu_{hom,2} \end{pmatrix},$$

where $\mu_{hom,1} = \lambda \mu_{A,1} + (1 - \lambda)\mu_{B,1}$ is the weak* limit of $\mu_{n,1}$ while $1/\mu_{hom,2} = \lambda/\mu_{A,2} + (1 - \lambda)/\mu_{B,2}$ is the weak* limit of $1/\mu_{n,2}$.

Accordingly, we introduce the homogenized elastic energy $\mathcal{W}_{l,hom}$ and the reduced homogenized energy $\mathcal{E}_{hom}$. Arguing as in Lemma 1, it turns out that the energy $\mathcal{E}_{hom}$ is of class $C^1(0, L)$, and hence the energy release $\mathcal{G}_{hom}(l) = -\partial_l \mathcal{E}_{hom}(l)$ is well defined in $(0, L)$. We set $\mathcal{G}_{hom}(L) = 0$.

In this setting it turns out that $G^c_{eff} = G^c$; indeed, we have the following result:

Theorem 25. *Let ℓ_n be the quasi-static evolutions given by Proposition 22. There exists a subsequence (not relabeled) such that $\ell_n \to \ell$ pointwise in $[0, T)$. Then $\ell^+ = \ell_{hom}$, where ℓ_{hom} is the unique non-decreasing, right continuous function which satisfies the initial condition $\ell_{hom}(0) = L_0$ and Griffith's criterion:*

(i) $G_{hom}(t, \ell_{hom}(t)) \leqq G^c$ for every time $t \in [0, T)$;
(ii) if $G_{hom}(t, \ell_{hom}(t)) < G^c$ then $\dot{\ell}_{hom}(t) = 0$;
(iii) if $t \in J(\ell_{hom})$ then $G_{hom}(t, l) \geqq G^c$ for every $l \in [\ell^-_{hom}(t), \ell_{hom}(t))$.

Corollary 26. *For every $t \in [0, T)$ the following energy identity holds:*

$$F_{hom}(t, \ell_{hom}(t)) = \int_0^t \partial_t F_{hom}(\tau, \ell_{hom}(\tau)) \, d\tau + \sum_{\tau \in J(\ell_{hom}) \cap [0,t]} [\![F_{hom}(\tau, \ell_{hom}(\tau))]\!].$$

$$(44)$$

The proof of Theorem 25 is a consequence of Corollary 15 and Remark 5, together with the following result on the convergence of the energy release:

Theorem 27. $\mathcal{G}_n$ *converge to* $\mathcal{G}_{hom}$ *locally uniformly in* $(0, L)$.

The proof of this theorem is contained in the next subsection.

3.4. Convergence of the Energy Release

Lemma 28. $\mathcal{G}_n$ is uniformly bounded in $C^{0,1/2}([L', L''])$ for every $0 < L' < L'' < L$.

Proof. By minimality $\mathcal{W}_{l,n}(u_{l,n}) \leqq \mathcal{W}_{l,n}(g) \leqq C$ where C is independent of n. Hence, by uniform coercivity, $\|\nabla u_{l,n}\|_{L^2} \leqq C$ for every $l \in [0, L]$.

Now, following [20] we prove that there exists $C > 0$ such that, for every $0 \leqq l_1 < l_2 \leqq L$ and every $n \in 2\mathbb{N}$, it holds that

$$\|\nabla u_{l_1,n} - \nabla u_{l_2,n}\|_{L^2} \leqq C|\mathcal{E}_n(l_1) - \mathcal{E}_n(l_2)|^{1/2}. \tag{45}$$

By the variational formulation we have

$$\int_{\Omega \setminus K_{l_1}} \nabla u_{l_1,n} \, \boldsymbol{C}_n \nabla v_1^T \, \mathrm{d}y = 0 \quad \text{for every } v_1 \in \mathcal{V}_{l_1},$$

$$\int_{\Omega \setminus K_{l_2}} \nabla u_{l_2,n} \, \boldsymbol{C}_n \nabla v_2^T \, \mathrm{d}y = 0 \quad \text{for every } v_2 \in \mathcal{V}_{l_2}.$$

Since $(u_{l_1,n} - u_{l_2,n}) \in \mathcal{V}_{l_2}$, we get

$$\int_{\Omega \setminus K_{l_2}} \nabla u_{l_2,n} \, \boldsymbol{C}_n \nabla (u_{l_1,n} - u_{l_2,n})^T \, \mathrm{d}y = 0.$$

Then, by monotonicity and uniform coercivity of the energy, we get

$$|\mathcal{E}_n(l_1) - \mathcal{E}_n(l_2)| = \mathcal{E}_n(l_1) - \mathcal{E}_n(l_2)$$

$$= \tfrac{1}{2} \int_\Omega \nabla u_{l_1,n} \, \boldsymbol{C}_n \nabla u_{l_1,n}^T - \nabla u_{l_2,n} \, \boldsymbol{C}_n \nabla u_{l_2,n}^T \, \mathrm{d}y$$

$$= \tfrac{1}{2} \int_\Omega \nabla (u_{l_1,n} + u_{l_2,n}) \, \boldsymbol{C}_n \nabla (u_{l_1,n} - u_{l_2,n})^T \, \mathrm{d}y$$

$$= \tfrac{1}{2} \int_\Omega \nabla u_{l_1,n} \, \boldsymbol{C}_n \nabla (u_{l_1,n} - u_{l_2,n})^T \, \mathrm{d}y$$

$$= \tfrac{1}{2} \int_\Omega \nabla (u_{l_1,n} - u_{l_2,n}) \, \boldsymbol{C}_n \nabla (u_{l_1,n} - u_{l_2,n})^T \, \mathrm{d}y$$

$$\geqq C \|\nabla u_{l_1,n} - \nabla u_{l_2,n}\|_{L^2}^2.$$

Next, we show that given $0 < L' < L'' < L$ there exists $C > 0$ s.t.

$$\mathcal{G}_n(l) \leqq C \quad \text{for every } l \in [L', L''] \text{ and every } n \in 2\mathbb{N}.$$

Let $\phi \in C_c^\infty(0, L)$ with $0 \leqq \phi \leqq 1$ and $\phi = 1$ in $[L', L'']$ such that

$$\mathcal{E}_n'(l) = \int_\Omega \nabla u_{l,n} \, \boldsymbol{C}_n \boldsymbol{E} \, \nabla u_{l,n}^T \, \phi' \, \mathrm{d}y \quad \text{for} \quad \boldsymbol{E} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\mathcal{G}_n(l) \leqq C$ (note that C depends on ϕ' and thus on L' and L''). It follows that the energies $\mathcal{E}_n$ are uniformly Lipischitz continuous in $[L', L'']$ and thus, by (45),

$$\|\nabla u_{l_1,n} - \nabla u_{l_2,n}\|_{L^2} \leqq C|l_1 - l_2|^{1/2}.$$

As a consequence, using the integral representation of $\mathcal{G}_n = -\mathcal{E}'_n$ we get

$$|\mathcal{G}_n(l_1) - \mathcal{G}_n(l_2)| \leqq C|l_1 - l_2|^{1/2},$$

which concludes the proof.   □

By Ascoli-Arzelà Theorem, with the aid of a diagonal argument, Lemma 28 implies

Corollary 29. *There exists a subsequence (non relabeled) and a limit $\mathcal{G}$ such that $\mathcal{G}_n \to \mathcal{G}$ locally uniformly in $(0, L)$.*

We claim that $\mathcal{G} = \mathcal{G}_{\text{hom}}$. To this end, we will need a couple of lemmas on the local convergence of the energy.

Lemma 30. *For $0 < a < b < L$, let $\Omega_{(a,b)} = (a, b) \times (-H, H)$. Then*

$$\int_{\Omega_{(a,b)} \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \, \mathrm{d}x \; \to \; \int_{\Omega_{(a,b)} \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \nabla u_{l,\text{hom}}^T \, \mathrm{d}x.$$

Proof. For $\varepsilon > 0$ let $\theta_\varepsilon \in C_c^\infty(0, L)$ with $0 \leqq \theta_\varepsilon \leqq 1$, $\theta_\varepsilon = 1$ in (a, b), and such that

$$\int_{\Omega \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \nabla u_{l,\text{hom}}^T \theta_\varepsilon \, \mathrm{d}x \leqq \varepsilon + \int_{\Omega_{(a,b)} \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \nabla u_{l,\text{hom}}^T \, \mathrm{d}x.$$

Then, by the properties of θ_ε and since $u_{l,n}\theta_\varepsilon \in \mathcal{V}_l$ by the variational formulation we can write

$$\int_{\Omega_{(a,b)} \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \, \mathrm{d}x \leqq \int_{\Omega \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \theta_\varepsilon \, \mathrm{d}x$$

$$= \int_{\Omega \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \left(\nabla (u_{l,n}\theta_\varepsilon) - u_{l,n}\nabla\theta_\varepsilon \right)^T \mathrm{d}x$$

$$= -\int_{\Omega \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla\theta_\varepsilon^T u_{l,n} \, \mathrm{d}x.$$

As $\boldsymbol{C}_n \nabla u_{l,n}^T \rightharpoonup \boldsymbol{C}_{\text{hom}} \nabla u_{l,\text{hom}}^T$ in $L^2(\Omega \setminus K_l; \mathbb{R}^2)$ and since $u_{l,n} \to u_{l,\text{hom}}$ in $L^2(\Omega \setminus K_l)$ the last term converge to

$$-\int_{\Omega \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \nabla\theta_\varepsilon^T u_{l,\text{hom}} \, \mathrm{d}x =$$

$$= -\int_{\Omega \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \left(\nabla (u_{l,\text{hom}}\theta_\varepsilon) - \nabla u_{l,\text{hom}}\theta_\varepsilon \right)^T \mathrm{d}x$$

$$= \int_{\Omega \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \nabla u_{l,\text{hom}}^T \theta_\varepsilon \, \mathrm{d}x$$

$$\leqq \varepsilon + \int_{\Omega_{(a,b)} \setminus K_l} \nabla u_{l,\text{hom}} \, \boldsymbol{C}_{\text{hom}} \nabla u_{l,\text{hom}}^T \, \mathrm{d}x.$$

Hence

$$\limsup_{n\to\infty} \int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,n} C_n \nabla u_{l,n}^T \, \mathrm{d}x \leqq \varepsilon + \int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,\mathrm{hom}} C_{\mathrm{hom}} \nabla u_{l,\mathrm{hom}}^T \, \mathrm{d}x.$$

For $\varepsilon > 0$ let $\psi_\varepsilon \in C_c^\infty(0,L)$ with $0 \leqq \psi_\varepsilon \leqq 1$, $\psi_\varepsilon = 0$ in $(0,L)\setminus(a,b)$, and such that

$$\int_{\Omega\setminus K_l} \nabla u_{l,\mathrm{hom}} C_{\mathrm{hom}} \nabla u_{l,\mathrm{hom}}^T \psi_\varepsilon \, \mathrm{d}x \geqq -\varepsilon + \int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,\mathrm{hom}} C_{\mathrm{hom}} \nabla u_{l,\mathrm{hom}}^T \, \mathrm{d}x.$$

Then, arguing as above,

$$\int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,n} C_n \nabla u_{l,n}^T \, \mathrm{d}x \geqq \int_{\Omega\setminus K_l} \nabla u_{l,n} C_n \nabla u_{l,n}^T \psi_\varepsilon \, \mathrm{d}x,$$

where the last term converges to

$$\int_{\Omega\setminus K_l} \nabla u_{l,\mathrm{hom}} C_{\mathrm{hom}} \nabla u_{l,\mathrm{hom}}^T \psi_\varepsilon \, \mathrm{d}x \geqq -\varepsilon + \int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,\mathrm{hom}} C_{\mathrm{hom}} \nabla u_{l,\mathrm{hom}}^T \, \mathrm{d}x.$$

Hence

$$\liminf_{n\to\infty} \int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,n} C_n \nabla u_{l,n}^T \, \mathrm{d}x \geqq -\varepsilon + \int_{\Omega_{(a,b)}\setminus K_l} \nabla u_{l,\mathrm{hom}} C_{\mathrm{hom}} \nabla u_{l,\mathrm{hom}}^T \, \mathrm{d}x.$$

We conclude by the arbitrariness of $\varepsilon > 0$. $\qquad\square$

Lemma 31. *For $0 < a < b < L$, let $\Omega_{(a,b)} = (a,b) \times (-H,H)$. Then*

$$\int_{\Omega_{(a,b)}\setminus K_l} \mu_{n,1} |\partial_{x_1} u_{l,n}|^2 \, \mathrm{d}x \;\to\; \int_{\Omega_{(a,b)}\setminus K_l} \mu_{\mathrm{hom},1} |\partial_{x_1} u_{l,\mathrm{hom}}|^2 \, \mathrm{d}x. \qquad (46)$$

Proof. We employ a change of variable. Let $\widehat{\Omega} = (0,L) \times (-S,S)$ where $S = H\left(\lambda/\mu_{B,1} + (1-\lambda)/\mu_{A,1}\right)$. Denote by $\Pi_{n,A}$ the projection of $\Omega_{n,A}$ on the verical axis, that is,

$$\Pi_{n,A} = \{x_2 \in (-H,H) : (x_1,x_2) \in \Omega_{n,A} \text{ for } x_1 \in (0,L)\},$$

and similarly for $\Pi_{n,B}$. Let $\Phi_n : \Omega \to \widehat{\Omega}$ be the bi-Lipschitz piecewise affine map $\Phi_n(x_1,x_2) = (x_1, \phi_n(x_2))$ where

$$\phi_n(x_2) = \int_0^{x_2} \beta_n \, \mathrm{d}s \quad \text{and} \quad \beta_n = \begin{cases} 1/\mu_{B,1} & \text{in } \Pi_{n,A}, \\ 1/\mu_{A,1} & \text{in } \Pi_{n,B}. \end{cases}$$

Denote $\hat{g}_n = g \circ \Phi_n^{-1}$ and

$$\widehat{\mathcal{U}}_{l,n} = \{\hat{u} \in H^1(\widehat{\Omega}\setminus K_l) : \hat{u} = \hat{g}_n \text{ in } \partial_D\widehat{\Omega}\} \quad \widehat{\mathcal{V}}_l = \{\hat{v} \in H^1(\widehat{\Omega}\setminus K_l) : \hat{v} = 0 \text{ in } \partial_D\widehat{\Omega}\}.$$

For $u \in \mathcal{U}_l$ and $\hat{u} = (u \circ \Phi_n^{-1}) \in \widehat{\mathcal{U}}_{l,n}$ we have

$$\mathcal{W}_{l,n}(u) = \widehat{\mathcal{W}}_{l,n}(\hat{u}) = \frac{1}{2} \int_{\widehat{\Omega}\setminus K_l} \nabla\hat{u} \, \widehat{C}_n \nabla\hat{u}^T \, \mathrm{d}y$$

where (see Lemma 20)

$$\widehat{C}_n = \begin{pmatrix} \hat{\mu}_{n,1} & 0 \\ 0 & \hat{\mu}_{n,2} \end{pmatrix} \quad \text{with} \quad \hat{\mu}_{n,1} = \mu_{A,1}\,\mu_{B,1} \quad \text{and} \quad \hat{\mu}_{n,2} = \begin{cases} \mu_{A,2}/\mu_{B,1} & \text{in } \widehat{\Omega}_{n,A}, \\ \mu_{B,2}/\mu_{A,1} & \text{in } \widehat{\Omega}_{n,B}. \end{cases}$$

Note that $\hat{\mu}_{n,1}$ is constant and independent of $n \in 2\mathbb{N}$. We denote $\hat{\mu}_{A,2} = \mu_{A,2}/\mu_{B,1}$ and $\hat{\mu}_{B,2} = \mu_{B,2}/\mu_{A,1}$. Note that the relative size of the layers $\widehat{\Omega}_{n,A}$ and $\widehat{\Omega}_{n,B}$ is, respectively,

$$\hat{\lambda} = \frac{\lambda/\mu_{B,1}}{\lambda/\mu_{B,1} + (1-\lambda)/\mu_{A,1}} = \frac{\lambda\mu_{A,1}}{\lambda\mu_{A,1} + (1-\lambda)\mu_{B,1}}, \quad (1-\hat{\lambda}) = \frac{(1-\lambda)\mu_{B,1}}{\lambda\mu_{A,1} + (1-\lambda)\mu_{B,1}}.$$

Therefore, by Proposition 32, the homogenized stiffness matrix in the rescaled domain $\widehat{\Omega}$ is given by

$$\widehat{C}_{hom} = \begin{pmatrix} \hat{\mu}_{hom,1} & 0 \\ 0 & \hat{\mu}_{hom,2} \end{pmatrix},$$

where $\hat{\mu}_{hom,1} = \mu_{A,1}\,\mu_{B,1}$, while

$$\frac{1}{\hat{\mu}_{hom,2}} = \frac{\hat{\lambda}}{\hat{\mu}_{A,2}} + \frac{1-\hat{\lambda}}{\hat{\mu}_{B,2}} = \frac{\mu_{A,1}\,\mu_{B,1}}{\lambda\mu_{A,1} + (1-\lambda)\mu_{B,1}} \left(\frac{\lambda}{\mu_{A,2}} + \frac{(1-\lambda)}{\mu_{B,2}} \right).$$

Next, let $\Phi : \Omega \to \widehat{\Omega}$ be the bi-Lipschitz affine map $\Phi(x_1, x_2) = (x_1, \phi(x_2))$, where

$$\phi(x_2) = \beta x_2 \quad \text{for} \quad \beta = \lambda/\mu_{B,1} + (1-\lambda)/\mu_{A,1} = \frac{\lambda\mu_{A,1} + (1-\lambda)\mu_{B,1}}{\mu_{A,1}\mu_{B,1}}.$$

Note that $\beta_n \overset{*}{\rightharpoonup} \beta$. With this change of variable the stiffness matrix C_{hom} becomes $\widehat{C}_{hom}$, indeed (see Lemma 20),

$$\mu_{hom,1} = \lambda\mu_{A,1} + (1-\lambda)\mu_{B,1} \quad \text{and} \quad \frac{1}{\mu_{hom,2}} = \frac{\lambda}{\mu_{A,2}} + \frac{(1-\lambda)}{\mu_{B,2}},$$

and thus $\mu_{hom,1}/\beta = \mu_{A,1}\,\mu_{B,1} = \hat{\mu}_{hom,1}$ and

$$\frac{1}{\beta\mu_{hom,2}} = \frac{\mu_{A,1}\,\mu_{B,1}}{\lambda\mu_{A,1} + (1-\lambda)\mu_{B,1}} \left(\frac{\lambda}{\mu_{A,2}} + \frac{(1-\lambda)}{\mu_{B,2}} \right) = \frac{1}{\hat{\mu}_{hom,2}}.$$

Since β_{hom} is the weak* limit of β_n, $\phi_n \to \phi$ uniformly. Let $\hat{g} = g \circ \Phi^{-1}$. Note that $\hat{g}_n$ is a bounded sequence in $H^1(\widehat{\Omega} \backslash K_L)$, moreover, $\hat{g}_n \to \hat{g}$ in $L^2(\partial_D \widehat{\Omega})$.

Let $\hat{u}_{l,n} \in \text{argmin}\{\widehat{\mathcal{W}}_{l,n}(\hat{u}) : \hat{u} \in \widehat{\mathcal{U}}_{l,n}\}$ and $u_{l,hom} \in \text{argmin}\{\mathcal{W}_{l,hom}(u) : u \in \mathcal{U}_l\}$. We know that $\hat{u}_{l,n} \rightharpoonup \hat{u}_{l,hom}$ in $H^1(\widehat{\Omega} \backslash K_l)$. Since $\hat{\mu}_{n,1} = \hat{\mu}_{hom,1} = \mu_{A,1}\,\mu_{B,1}$ we get

$$\int_{\widehat{\Omega}_{(a,b)} \backslash K_l} \hat{\mu}_{hom,1}\, |\partial_{y_1} \hat{u}_{l,hom}|^2 \, \mathrm{d}y \leqq \liminf_{n \to \infty} \int_{\widehat{\Omega}_{(a,b)} \backslash K_l} \hat{\mu}_{n,1}\, |\partial_{y_1} \hat{u}_{l,n}|^2 \, \mathrm{d}y,$$

where $\widehat{\Omega}_{(a,b)} = (a,b) \times (-S, S)$. Clearly, $\hat{u}_{l,n} = u_{l,n} \circ \Phi_n^{-1}$, where $u_{l,n} \in$ argmin$\{\mathcal{W}_{l,n}(u) : u \in \mathcal{U}_l\}$ and $\hat{u}_{l,\text{hom}} = u_{l,\text{hom}} \circ \Phi^{-1}$ where $u_{l,\text{hom}} \in$ argmin$\{\mathcal{W}_{l,\text{hom}}(u) :$ $u \in \mathcal{U}_l\}$. Therefore, applying the changes of variable we get

$$\int_{\Omega_{(a,b)} \setminus K_l} \mu_{\text{hom},1} |\partial_{x_1} u_{l,\text{hom}}|^2 \, \mathrm{d}x \leqq \liminf_{n \to \infty} \int_{\Omega_{(a,b)} \setminus K_l} \mu_{n,1} |\partial_{x_1} u_{l,n}|^2 \, \mathrm{d}x.$$

Using a different change of variable we can apply the above argument in such a way that

$$\int_{\Omega_{(a,b)} \setminus K_l} \mu_{\text{hom},2} |\partial_{x_2} u_{l,\text{hom}}|^2 \, \mathrm{d}x \leqq \liminf_{n \to \infty} \int_{\Omega_{(a,b)} \setminus K_l} \mu_{n,2} |\partial_{x_2} u_{l,n}|^2 \, \mathrm{d}x.$$

By Lemma 30 we have

$$\int_{\Omega_{(a,b)} \setminus K_l} \mu_{n,1} |\partial_{x_1} u_{l,n}|^2 + \mu_{n,2} |\partial_{x_2} u_{l,n}|^2 \, \mathrm{d}x$$

which converges to

$$\int_{\Omega_{(a,b)} \setminus K_l} \mu_{\text{hom},1} |\partial_{x_1} u_{l,\text{hom}}|^2 + \mu_{\text{hom},2} |\partial_{x_2} u_{l,\text{hom}}|^2 \, \mathrm{d}x,$$

and thus

$$\limsup_{n \to \infty} \int_{\Omega_{(a,b)} \setminus K_l} \mu_{n,1} |\partial_{x_1} u_{l,n}|^2 \, \mathrm{d}x$$

$$= \int_{\Omega_{(a,b)} \setminus K_l} \mu_{\text{hom},1} |\partial_{x_1} u_{l,\text{hom}}|^2 + \mu_{\text{hom},2} |\partial_{x_2} u_{l,\text{hom}}|^2 \, \mathrm{d}x$$

$$- \liminf_{n \to \infty} \int_{\Omega_{(a,b)} \setminus K_l} \mu_{n,2} |\partial_{x_2} u_{l,n}|^2 \, \mathrm{d}x$$

$$\leqq \int_{\Omega_{(a,b)} \setminus K_l} \mu_{\text{hom},1} |\partial_{x_1} u_{l,\text{hom}}|^2 \, \mathrm{d}x.$$

The proof is concluded. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 27. It is enough to show that $\mathcal{G}_n \to \mathcal{G}_{\text{hom}}$ pointwise in $(0, L)$.

Let $r > 0$ such that $0 < 2r < l < L - 2r$. Let $\phi_r : [0, L] \to [0, 1]$ be the Lipschitz map defined by $\phi_r = 0$ in $(0, r)$ and in $(L - r, L)$, $\phi_r = 1$ in $(2r, L - 2r)$, ϕ_r is affine in $(r, 2r)$ and in $(L - 2r, L - r)$. By Lemma 21 we can write

$$\mathcal{E}'_n(l) = \int_{\Omega \setminus K_l} \nabla u_{l,n} \boldsymbol{C}_n \boldsymbol{E} \nabla u_{l,n}^T \phi_r' \mathrm{d}x \quad \text{for} \quad \boldsymbol{E} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\boldsymbol{I}$ denotes the identity matrix; a similar representation holds also for $\mathcal{G}_{\text{hom}}$. Then, splitting $\boldsymbol{E}$ as above,

$$\mathcal{G}_n(l) = -\mathcal{E}'_n(l) = -\int_{\Omega \setminus K_l} \nabla \hat{u}_{l,n} \, \boldsymbol{C}_n \nabla \hat{u}_{l,n}^T \, \phi_r' \, \mathrm{d}x + 2 \int_{\Omega \setminus K_l} \mu_{n,1} |\partial_{x_1} u_{l,n}|^2 \phi_r' \, \mathrm{d}x.$$

By the definition of ϕ_r, we can write

$$\int_{\Omega \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \, \phi_r' \, dx = r^{-1} \int_{\Omega_{(r,2r)} \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \, dx$$
$$- r^{-1} \int_{\Omega_{(L-2r,L-r)} \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \, dx.$$

Hence by Lemma 30 we get

$$\int_{\Omega \setminus K_l} \nabla u_{l,n} \, \boldsymbol{C}_n \nabla u_{l,n}^T \, \phi_r' \, dx \;\to\; \int_{\Omega \setminus K_l} \nabla u_{l,hom} \, \boldsymbol{C}_{hom} \nabla u_{l,hom}^T \, \phi_r' \, dx.$$

In the same way we get

$$\int_{\Omega \setminus K_l} \mu_{n,1} |\partial_{x_1} u_{l,n}|^2 \phi_r' \, dx \;\to\; \int_{\Omega \setminus K_l} \mu_{hom,1} |\partial_{x_1} u_{l,hom}|^2 \phi_r' \, dx.$$

Hence $\mathcal{G}_n(l) \to \mathcal{G}_{hom}(l)$. $\square$

4. Non-monotone Boundary Condition

Consider a Dirichlet boundary condition of the type $f(t)g$ where $f \in C^1([0, T])$ with $f(0) = 0$ (f is not necessarily monotone). We will provide an existence and convergence result.

Let $\hat{\ell}_n$ and $\hat{\ell}_{hom}$ be the unique solutions obtained with the monotone boundary condition $u = sg$ and $s \in [0, S]$, where $S = \max\{|f(t)| : t \in [0, T]\}$. By Theorem 12 we know that $\hat{\ell}_n \to \hat{\ell}$ pointwise in $[0, S)$ and that $\hat{\ell}^+ = \hat{\ell}_{hom}$.

Following [18, Section 12] we introduce the non-decreasing function $\bar{f}(t) = \max\{|f(\tau)| : \tau \in [0, t]\}$ and we define $\ell_n = \hat{\ell}_n \circ \bar{f}$. We will prove that

(1) ℓ_n is a quasi-static evolution in the sense of Proposition 2, that is, $\ell_n : [0, T) \to [L_0, L]$ is non-decreasing, right continuous, satisfies the initial condition $\ell_n(0) = L_0$ and Griffith's criterion, in the following form:
   (i) $G_n(t, \ell_n(t)) \leqq G_n^c(\ell_n(t))$ for every time $t \in [0, T)$;
   (ii) if $G_n(t, \ell_n(t)) < G_n^c(\ell_n(t))$ then ℓ_n is right differentiable in t and $\dot{\ell}_n^+(t) = 0$;
   (iii) if $t \in J(\ell_n)$ then $G_n(t, l) \geqq G_n^c(l)$ for every $l \in [\ell_n^-(t), \ell_n(t))$;
2) $\ell_n = \hat{\ell}_n \circ \bar{f}$ converge to $\ell = \hat{\ell} \circ \bar{f}$ pointwise in $[0, T)$ (up to subsequences); the evolution ℓ is non-decreasing, satisfies the initial condition and Griffith's criterion, in the following form:
   (i) $G_{hom}(t, \ell(t)) \leqq G_{eff}^c(\ell(t))$ for every time $t \in [0, T)$;
   (ii) if $G_{hom}(t, \ell(t)) < G_{eff}^c(\ell(t))$ then ℓ is right differentiable in t and $\dot{\ell}^+(t) = 0$;
   (iii) if $t \in J(\ell)$ then $G_{hom}(t, l) \geqq G_{eff}^c(l)$ for every $l \in [\ell^-(t), \ell^+(t))$.

Let us prove (1). Clearly ℓ_n is non-decreasing, since both $\hat{\ell}_n$ and $\bar{f}$ are non-decreasing, and $\ell_n(0) = \hat{\ell}_n \circ f(0) = \hat{\ell}_n(0) = L_0$. The right continuity of $\hat{\ell}_n$ together with the monotonicity and continuity of $\bar{f}$ imply the right continuity of ℓ_n. It remains to check Griffith's criterion, but in this case the representation formula (7) does not hold. Before proceeding, note that for $s = \bar{f}(t)$ we have

$$\hat{\ell}_n^-(s) \leqq \ell_n^-(t) \leqq \ell_n^+(t) \leqq \hat{\ell}_n(s). \tag{47}$$

(i) We know that $G_n(s, \hat{\ell}_n(s)) \leqq G_n^c(\hat{\ell}_n(s))$ for every $s \in [0, S)$; writing $s = \bar{f}(t)$ and $\ell_n = \hat{\ell} \circ \bar{f}$ yields

$$\bar{f}^2(t)\mathcal{G}_n(\ell_n(t)) = \bar{f}^2(t)\mathcal{G}_n(\hat{\ell}_n \circ \bar{f}(t)) = G_n(\bar{f}(t), \hat{\ell}_n \circ \bar{f}(t)) \leqq G_n^c(\hat{\ell}_n \circ \bar{f}(t))$$
$$= G_n^c(\ell_n(t)).$$

Since $f^2 \leqq \bar{f}^2$ we get

$$G_n(t, \ell_n(t)) = f^2(t)\mathcal{G}_n(\ell_n(t)) \leqq \bar{f}^2(t)\mathcal{G}_n(\ell_n(t)) \leqq G_n^c(\ell_n(t)).$$

(ii) If $|f(t)| = \bar{f}(t)$ then $G_n(t, \ell_n(t)) < G_n^c(\ell_n(t))$ reads as

$$\bar{f}^2(t)\mathcal{G}_n(\hat{\ell}_n \circ \bar{f}(t)) = f^2(t)\mathcal{G}_n(\hat{\ell}_n \circ \bar{f}(t)) < G_n^c(\hat{\ell}_n \circ \bar{f}(t))$$

which, for $s = \bar{f}(t)$, gives $G_n(s, \hat{\ell}_n(s)) < G_n^c(\hat{\ell}_n(s))$. Since $\bar{f}$ is continuous and monotone non-decreasing, it turns out that

$$0 \leqq \dot{\ell}_n^+(t) \leqq \lim_{h \to 0^+} \frac{\hat{\ell}_n(s+h) - \hat{\ell}_n(s)}{h} = 0.$$

If $|f(t)| < \bar{f}(t)$ then $\bar{f}(t) = f(t^*)$ for some $t^* < t$. Moreover, by the continuity of f there exists $\delta > 0$ such that $\bar{f}(t') = f(t^*)$ for every $|t' - t| < \delta$. Hence $\ell_n(t') = \hat{\ell}_n \circ \bar{f}(t') = \ell_n \circ f(t^*)$ for every $|t' - t| < \delta$; hence ℓ_n is constant in a neighborhood of t.

(iii) Let $t \in J(\ell_n)$. First, note that $\bar{f}(t) = |f(t)|$; indeed, if $\bar{f}(t) > |f(t)|$ then, repeating the argument above, ℓ_n would be constant in a neighborhood of t and thus $t \notin J(\ell_n)$. By (47) it turns out that $s \in J(\hat{\ell}_n)$. Hence, $G_n(s, l) \geqq G_n^c(l)$ for every $l \in [\hat{\ell}_n^-(s), \hat{\ell}_n^+(s))$; the substitution $s = \bar{f}(t)$ and (47) lead to

$$G_n(t, l) = f^2(t)\mathcal{G}_n(l) = \bar{f}^2(t)\mathcal{G}_n(l) = G_n(s, l) \geqq G_n^c(l)$$

for every $l \in [\ell_n^-(t), \ell_n^+(t))$.

Let us prove 2). Before proceeding, note that in general $\ell^+ = (\hat{\ell} \circ \bar{f})^+ \neq \hat{\ell}^+ \circ \bar{f} = \hat{\ell}_{hom} \circ \bar{f}$, therefore we cannot employ directly the properties of $\hat{\ell}_{hom}$. Instead, we first check Griffith's criterion for $\hat{\ell}$. We know that $\hat{\ell}^+ = \hat{\ell}_{hom}$ and thus $J(\hat{\ell}) = J(\hat{\ell}_{hom})$, moreover $\hat{\ell}^-(s) = \hat{\ell}_{hom}^-(s)$ and $\hat{\ell}^+(s) = \hat{\ell}_{hom}(s)$ for every $s \in J(\hat{\ell})$, while $\hat{\ell}(s) = \hat{\ell}_{hom}(s)$ for every $s \in [0, S) \setminus J(\hat{\ell})$.

(iii) By Griffith's criterion for $\hat{\ell}_{hom}$, if $s \in J(\hat{\ell}_{hom}) = J(\hat{\ell})$ then $G_{hom}(s, l) \geqq G_{eff}^c(l)$ for every $l \in [\hat{\ell}_{hom}^-(s), \hat{\ell}_{hom}(s)) = [\hat{\ell}^-(s), \hat{\ell}^+(s))$.

(ii) If $G_{hom}(s, \hat{\ell}(s)) < G^c_{eff}(\hat{\ell}(s))$ then either $s \notin J(\hat{\ell})$ or $s \in J(\hat{\ell})$ and $\hat{\ell}(s) = \hat{\ell}_{hom}(s)$ (otherwise the opposite inequality would hold). In both the cases $\hat{\ell}(s) = \hat{\ell}_{hom}(s)$ and their right derivatives coincide. Therefore, $\hat{\ell}$ is right differentiable in s and its right derivative vanishes.

(i) We employ the definition of G^c_{eff}. If $\mathcal{G}_{hom}(\hat{\ell}(s)) = 0$ or $G^c_{eff}(\hat{\ell}(s)) = +\infty$ there is nothing to prove. Otherwise, by pointwise convergence

$$\frac{G_{hom}(s, \hat{\ell}(s))}{G^c_{eff}(\hat{\ell}(s))} \leqq \liminf_{n \to \infty} \frac{G_n(s, \hat{\ell}_n(s))}{G^c_n(\hat{\ell}_n(s))} \leqq 1,$$

which implies that $G_{hom}(s, \hat{\ell}(s)) \leqq G^c_{eff}(\hat{\ell}(s))$ for every $s \in [0, S)$.

At this point, employing the change of variable $s = \bar{f}(t)$ and arguing as in point (1) it follows that Griffith's criterion holds for $\ell = \hat{\ell} \circ \bar{f}$.

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Appendix A: Balanced Viscosity Solutions and Griffith's Criterion

For sake of completeness, we provide the notion of balanced viscosity solutions for the rate independent system $(\mathcal{E}_n, \mathcal{R}_n)$, making reference to [16] with a few differences due to irreversibility and lack of regularity in Λ_n. Moreover we briefly show that the evolution ℓ_n (provided by Proposition 2) is a balanced viscosity solution and, vice versa, that each right continuous balanced viscosity solution satisfies Griffith's criterion. A similar reasoning applies to ℓ_{hom}.

Following [16] a right continuous evolution γ_n is a balance viscosity evolution for the rate independnt system $(\mathcal{E}_n, \mathcal{R}_n)$ if it satisfies the following conditions for every time $t \in [0, T]$:

$$\partial_{l'} \mathcal{R}_n(\gamma_n(t), 0) + \partial_l E_n(t, \gamma_n(t)) \ni 0 \tag{48}$$

$$E_n(t, \gamma_n(t)) = E_n(0, L_0) + \int_0^t P_n(t, \gamma_n(t)) \, dt - \text{Var}_{f_n}(\gamma_n; [0, t]). \tag{49}$$

We recall that the state dependent dissipation $\mathcal{R}_n : [L_0, L] \times \mathbb{R} \to [0, +\infty]$ is given by

$$\mathcal{R}_n(l, l') = \begin{cases} G_n^c(l) \, l' & l' \geqq 0, \\ +\infty & l' < 0. \end{cases}$$

We denote by $\partial_{l'} \mathcal{R}_n$ the subdifferential of the dissipation $\mathcal{R}_n(l, \cdot)$ and by $\partial_l E_n(t, l)$ the partial derivative of the reduced elastic energy. Then $\partial_{l'} \mathcal{R}_n(l, 0) = (-\infty, G_n^c(l)]$ while $\partial_l E_n(t, l) = -G_n(t, l)$, if $l \notin \Lambda_n$. If $l \in \Lambda_n$ the energy $\mathcal{E}_n$ is not differentiable, therefore following (5) we set $\partial_l E_n(t, l) = -G_n(t, l)$.

Next, let us introduce the dissipation "distance" $\mathsf{dist}_n : [L_0, L] \times [L_0, L] \to [0, +\infty]$ given by

$$\mathsf{dist}_n(l_0, l_1) = \inf \left\{ \int_0^1 \mathcal{R}_n(l(r), l'(r)) \, dr : l \in AC(0, 1), \ l(0) = l_0, \ l(1) = l_1 \right\}$$

and the Finsler dissipation cost $\Delta_{\mathfrak{f}_n} : [0, T] \times [L_0, L] \times [L_0, L] \to [0, +\infty]$ given by

$$\Delta_{\mathfrak{f}_n}(t, l_0, l_1) = \inf \left\{ \int_0^1 \mathfrak{f}_n(t, l(r), l'(r)) \, dr : l \in AC(0, 1), \ l(0) = l_0, \ l(1) = l_1 \right\},$$

where

$$\mathfrak{f}_n(t, l(r), l'(r)) = \mathcal{R}_n(l(r), l'(r)) + \mathrm{dist}\big(-\partial_l E(t, l(r)), \partial_{l'} \mathcal{R}_n(l(r), 0)\big) |l'(r)|.$$

Finally, let us introduce the notation

$$\mathrm{Var}_{\mathfrak{f}_n}(\gamma_n; [0, t]) = \mathrm{Var}_{\mathsf{dist}_n}(\gamma_n; [0, t]) - \mathrm{Jmp}_{\mathsf{dist}_n}(\gamma_n; [0, t]) + \mathrm{Jmp}_{\mathfrak{f}_n}(\gamma_n; [0, t]),$$

where

$$\mathrm{Var}_{\mathsf{dist}_n}(\gamma_n; [0, t]) = \sup \left\{ \sum_{i=0}^{m-1} \mathsf{dist}_n(\gamma_n(t_{i+1}), \gamma_n(t_i)) : 0 = t_0 \leqq \cdots \leqq t_m = t \right\},$$

$$\mathrm{Jmp}_{\mathsf{dist}_n}(\gamma_n; [0, t]) = \sum_{\tau \in J(\gamma_n) \cap [0, t]} \mathsf{dist}_n(\gamma_n(\tau), \gamma_n^-(\tau)),$$

$$\mathrm{Jmp}_{\mathfrak{f}_n}(\gamma_n; [0, t]) = \sum_{\tau \in J(\gamma_n) \cap [0, t]} \Delta_n(\tau, \gamma_n(t), \gamma_n^-(t)),$$

(remember that here γ_n is right continuous).

First, let us check that the evolution ℓ_n provided by Proposition 2 is a balanced viscosity solution. The inequality $G_n(t, \ell_n(t)) \leqq G_n^c(\ell_n(t))$ reads as

$$-\partial E_n(t, \ell_n(t)) \in (-\infty, G_n^c(\ell_n(t))] = \partial_{l'} \mathcal{R}_n(\ell_n(t), 0),$$

which is (48). Next, let us check that the energy identity (11) implies (49), showing that

$$\mathrm{Var}_{\mathfrak{f}_n}(\ell_n; [0, t]) = \mathrm{Var}_{\mathsf{dist}_n}(\ell_n; [0, t]) - \mathrm{Jmp}_{\mathsf{dist}_n}(\ell_n; [0, t]) + \mathrm{Jmp}_{\mathfrak{f}_n}(\ell_n; [0, t]),$$

$$= D_n(\ell_n(t)) - \sum_{\tau \in J(\ell_n) \cap [0, t]} [\![F_n(\tau, \ell_n(\tau))]\!].$$

First of all, note that in our irreversible setting it is not restrictive to consider $l' \geqq 0$ in the definition of both dist_n and $\Delta_{\mathfrak{f}_n}$. By explicit computations, it is easy to check that

$$\mathsf{dist}_n(l_0, l_1) = \begin{cases} D_n(l_1) - D_n(l_0) & \text{if } l_0 \leqq l_1, \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence, by monotonicity and right continuity of ℓ_n,

$$\text{Var}_{\text{dist}_n}(\ell_n; [0, t]) = D_n(\ell_n(t)),$$

$$\text{Jmp}_{\text{dist}_n}(\ell_n; [0, t]) = \sum_{\tau \in J(\ell_n) \cap [0,t]} \text{dist}_n(\ell_n(t), \ell_n^-(t)) = \sum_{\tau \in J(\ell_n) \cap [0,t]} [\![D_n(\ell_n(\tau))]\!].$$

Next, note that

$$\text{dist}\big(-\partial_l E(t, l), \partial_{l'} \mathcal{R}_n(l, 0)\big) = [G(t, l) - G_n^c(l)]_+$$

By condition (iii) of Griffith's criterion for $\tau \in J(\ell_n)$ we have $G_n(\tau, l) - G_n^c(l) \geqq 0$ for every $l \in [\ell_n^-(\tau), \ell_n(\tau))$, and thus

$$\mathfrak{f}_n(\tau, l(r), l'(r)) = G_n(\tau, l(r)) \, l'(r)$$

for every monotone $l \in AC(0, 1)$ with $l(0) = \ell_n^-(\tau)$ and $l(1) = \ell_n(\tau)$. It follows easily that

$$\Delta_n(\tau, \ell_n(\tau), \ell_n^-(\tau)) = -[\![E_n(\tau, \ell_n(\tau))]\!],$$

and thus

$$\text{Jmp}_{\mathfrak{f}_n}(\ell_n; [0, t]) = \sum_{\tau \in J(\ell_n) \cap [0,t]} -[\![E_n(\tau, \ell_n(\tau))]\!].$$

Conversely, let γ_n be a (right continuous) balanced viscosity solution. If γ_n satisfies (48) then

$$(-\infty, G_n^c(\gamma_n(t))] - G_n(t, \gamma_n(t)) \ni 0,$$

that is $G_n(t, \gamma_n(t)) \leqq G_n^c(\gamma_n(t))$. Let us check that $G_n(\tau, l) \geqq G_n^c(l)$ for $\tau \in J(\gamma_n)$ and every $l \in [\gamma_n^-(\tau), \gamma_n(\tau)]$. In the interval $[\tau - \delta, \tau + \delta]$ (for some $\delta > 0$) the energy identity (49) reads as

$$E_n(\tau + \delta, \gamma_n(\tau + \delta)) - E_n(\tau - \delta, \gamma_n(\tau - \delta)) = \int_{\tau-\delta}^{\tau+\delta} P_n(t, \gamma_n(t)) \, dt$$

$$-\text{Var}_{\mathfrak{f}_n}(\gamma_n; [\tau - \delta, \tau + \delta]).$$

Hence, taking the limit as $\delta \to 0^+$, we get

$$E_n(\tau, \gamma_n^-(\tau)) - E_n(\tau, \gamma_n(\tau)) = \int_{\gamma_n^-(\tau)}^{\gamma_n(\tau)} G_n(\tau, s) \, ds = \lim_{\delta \to 0^+} \text{Var}_{\mathfrak{f}_n}(\gamma_n; [\tau - \delta, \tau + \delta]). \tag{50}$$

It is easy to check that

$$\lim_{\delta \to 0^+} \text{Var}_{\mathfrak{f}_n}(\gamma_n; [\tau - \delta, \tau + \delta]) = \Delta(\tau, \gamma_n^-(\tau), \gamma_n(\tau))$$

$$= \int_{\gamma_n^-(\tau)}^{\gamma_n(\tau)} G_n^c(s) + [G_n(\tau, s) - G^c(s)]_+ ds.$$

Hence, by (50),

$$\int_{\gamma_n^-(\tau)}^{\gamma_n(\tau)} G_n(\tau, s) - G_n^c(s) \, ds = \int_{\gamma_n^-(\tau)}^{\gamma_n(\tau)} [G_n(\tau, s) - G^c(s)]_+ ds,$$

which in turn implies $G_n(\tau, s) - G_n^c(s) \geqq 0$ for almost everywhere $s \in (\gamma_n^-(\tau), \gamma_n(\tau))$. By the regularity of energy release and toughness we conclude that $G_n(\tau, s) \geqq G_n^c(s)$ holds for every $s \in [\gamma_n^-(\tau), \gamma_n(\tau))$. It remains to show that $G_n(\tau, \gamma_n(\tau)) < G_n^c(\gamma_n(\tau))$ implies $\dot{\gamma}_n^+(\tau) = 0$. Let us first assume that $\ell_n(\tau) \notin \Lambda_n$. Then, by continuity of G_n and G_n^c it follows that $G_n(t, l) < G_n^c(l)$ for every (t, l) such that $|\tau - t| < \delta$ and $|l - \ell_n(\tau)| < \delta$. By right continuity we can further assume that $|\gamma_n(t) - \gamma_n(\tau)| < \delta$ for $t \in (\tau, \tau + \delta)$. The fact that $G_n(t, l) < G_n^c(l)$ implies that there are no jumps in $t \in (\tau, \tau + \delta)$, otherwise (iii) (which we already proved) would be violated. Hence for $t \in (\tau, \tau + \delta)$ the energy identity reads as

$$E_n(t, \gamma_n(t)) = E_n(\tau, \gamma_n(\tau)) + \int_\tau^t P_n(t, \gamma_n(t))\, \mathrm{d}t - \mathrm{Var}_{\mathrm{dist}_n}(\gamma_n; [\tau, t])$$

Writing

$$E_n(t, \gamma_n(t)) - E_n(\tau, \gamma_n(\tau)) = \int_\tau^t \partial_t E_n(t, \gamma_n(t))\, \mathrm{d}t + \int_\tau^t \partial_l E_n(t, \gamma_n(t))\, d\gamma_n(t)$$

$$\int_\tau^t P_n(t, \gamma_n(t))\, \mathrm{d}t = \int_\tau^t \partial_t E_n(t, \gamma_n(t))\, \mathrm{d}t$$

$$\mathrm{Var}_{\mathrm{dist}_n}(\gamma_n; [\tau, t]) = \int_\tau^t G_n^c(\gamma_n(t))\, d\gamma_n(t),$$

it follows that

$$\int_\tau^t G_n(t, \gamma_n(t))\, d\gamma_n(t) = \int_\tau^t G_n^c(\gamma_n(t))\, d\gamma_n(t).$$

This is a contradiction with $G_n(t, \gamma_n(t)) < G_n^c(\gamma_n(t))$ unless $d\gamma_n = 0$, which implies the thesis. It $\gamma_n(\tau) \in \Lambda_n$ then there exists a sequence $l_m \searrow \gamma_n(\tau)$ such that $l_m \notin \Lambda_n$ and $G_n(\tau, l_m) \to G_n(\tau, \gamma_n(\tau)) < G_n^c(\gamma_n(\tau)) = G_n^c(l_m)$. Since $l_m \notin \Lambda_n$, arguing as above it follows that $\gamma_n(t) < l_m$ in a right neighborhood of τ, uniformly with respect to $m \in \mathbb{N}$. Hence, taking the infimum with respect to $m \in \mathbb{N}$, we get $\gamma_n(t) \leqq \gamma_n(\tau)$; by monotonicity, γ_n is constant in a right neighborhood of τ.

A. Homogenization of Displacement, Stress, and Energy

This section contains classic results on the homogenization of Dirichlet energies (see for example [17] or [9] and the references therein). We provide a short proof, adapted to the case of horizontal layers. We recall that

$$\mathbf{C}_{hom} = \begin{pmatrix} \mu_{hom,1} & 0 \\ 0 & \mu_{hom,2} \end{pmatrix},$$

where $1/\mu_{hom,1} = \lambda/\mu_{A,1} + (1-\lambda)/\mu_{B,1}$ and $\mu_{hom,2} = \lambda\mu_{A,2} + (1-\lambda)\mu_{B,2}$. For $l \in [0, L]$ we denote

$$u_{l,n} \in \mathrm{argmin}\,\{\mathcal{W}_{l,n}(u) : u \in \mathcal{U}_l\}, \quad u_{l,hom} \in \mathrm{argmin}\,\{\mathcal{W}_{l,hom}(u) : u \in \mathcal{U}_l\}.$$

Proposition 32. *Let $l \in [0, L]$. Then $u_{l,n} \rightharpoonup u_{l,hom}$ in $H^1(\Omega \backslash K_l)$ and $\mathbf{C}_n \nabla u_{l,n}^T \rightharpoonup \mathbf{C}_{hom} \nabla u_{l,hom}^T$ in $L^2(\Omega \backslash K_l; \mathbb{R}^2)$.*

Proof. By uniform coercivity $u_{l,n}$ is a bounded sequence in $H^1(\Omega\backslash K_l)$ and thus $C_n\nabla u_{l,n}^T$ is a bounded sequence in $L^2(\Omega\backslash K_l; \mathbb{R}^2)$. Then, up to non-relabeled subsequences, $u_{l,n} \rightharpoonup u_{l,\infty}$ in $H^1(\Omega\backslash K_l)$ while $C_n\nabla u_{l,n}^T \rightharpoonup \sigma_{l,\infty}$ in $L^2(\Omega\backslash K_l; \mathbb{R}^2)$. We will prove that

$$\sigma_{l,\infty} = C_{hom}\nabla u_{l,\infty}^T. \tag{51}$$

From this identity it follows that $u_{l,\infty} = u_{l,hom}$; indeed, $u_{l,\infty} \in \mathcal{U}_l$ and, passing to the limit in the variational formulation

$$\int_{\Omega\backslash K_l} \nabla v\, C_n\nabla u_{l,n}^T \,\mathrm{d}x = 0 \quad \text{for every } v \in \mathcal{V}_l,$$

we easily get that $u_{l,\infty}$ is the unique solution of the variational problem

$$\int_{\Omega\backslash K_l} \nabla v\, C_{hom}\nabla u_{l,\infty}^T \,\mathrm{d}x = 0 \quad \text{for every } v \in \mathcal{V}_l.$$

The idea to prove (51) is to show that $(\sigma_{l,\infty})_i = \mu_{hom,i}\,\partial_{x_i} u_{l,\infty}$ for $i = 1, 2$, by passing to the limit in the constitutive relation, that is, $(\sigma_{l,n})_i = \mu_{n,i}\,\partial_{x_i} u_{l,n}$ for $i = 1, 2$.
I. Let $i = 1$. For every $\varphi \in C_c^\infty(\Omega\backslash K_l)$ we have

$$\int_{\Omega\backslash K_l} (\sigma_{l,n})_1\varphi/\mu_{n,1} \,\mathrm{d}x = \int_{\Omega\backslash K_l} (\partial_{x_1} u_{l,n})\varphi \,\mathrm{d}x.$$

By weak convergence we can pass to the limit in the right hand side, getting

$$\int_{\Omega\backslash K_l} (\partial_{x_1} u_{l,n})\varphi \,\mathrm{d}x \to \int_{\Omega\backslash K_l} (\partial_{x_1} u_{l,\infty})\varphi \,\mathrm{d}x.$$

Let us prove that

$$\int_{\Omega\backslash K_l} (\sigma_{l,n})_1\varphi/\mu_{n,1} \,\mathrm{d}x \to \int_{\Omega\backslash K_l} (\sigma_{l,\infty})_1\varphi/\mu_{hom,1} \,\mathrm{d}x. \tag{52}$$

Let $I = (-H, 0) \cup (0, H)$ and write

$$\int_{\Omega\backslash K_l} (\sigma_{l,n})_1\varphi/\mu_{n,1} \,\mathrm{d}x = \int_{(0,L)} \left(\int_I (\sigma_{l,n})_1(x_1, x_2)\varphi(x_1, x_2) \,\mathrm{d}x_2 \right) \Big/ \mu_{n,1}(x_1) \,\mathrm{d}x_1$$

$$= \int_{(0,L)} s_n(x_1)/\mu_{n,1}(x_1) \,\mathrm{d}x_1, \tag{53}$$

where we have introduced the auxiliary function s_n given by

$$s_n(x_1) = \int_I (\sigma_{l,n})_1(x_1, x_2)\varphi(x_1, x_2) \,\mathrm{d}x_2, \quad \text{for almost everywhere } x_1 \in (0, L).$$

We claim that $s_n \to s_\infty$ strongly in $L^1(0, L)$, where

$$s_\infty(x_1) = \int_I (\sigma_{l,\infty})_1(x_1, x_2)\,\varphi(x_1, x_2) \,\mathrm{d}x_2$$

from which (52) follows from (53) because $1/\mu_{hom,1}$ is the weak* limit of $1/\mu_{n,1}$ in $L^\infty(0, L)$.

Denote by σ_n the (vector valued) function $\sigma_n(x_1) = (\sigma_{l,n})_1(x_1, \cdot)$; since $\boldsymbol{\sigma}_n$ is bounded in $L^2(\Omega \backslash K_l)$, it turns out that σ_n is bounded in $L^2(0, L; L^2(I))$. We claim that σ_n' is bounded in $L^2(0, L; H^{-1}(I))$. Let us start writing

$$\|\sigma_n'\|_{L^2(0,L;H^{-1}(I))} = \sup \int_{(0,L)} \langle \sigma_n', z \rangle \, dx_1,$$

where the supremum is taken over all the (vector valued) functions z with $\|z\|_{L^2(0,L;H_0^1(I))} \leq 1$ and $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}(I)$ and $H_0^1(I)$. The set of functions of the form $z = \phi\psi$ for $\phi \in C_c^\infty(0, L)$ and $\psi \in C_c^\infty(I)$ is dense in $L^2(0, L; H_0^1(I))$. For any such z we can write

$$\int_{(0,L)} \langle \sigma_n', z \rangle \, dx_1 = \int_{(0,L)} \langle \sigma_n', \psi \rangle \phi \, dx_1 = - \int_{(0,L)} \langle \sigma_n, \psi \rangle \phi' \, dx_1$$

$$= - \int_{(0,L)} \int_I \sigma_n(x_1, x_2) \psi(x_2) \partial_{x_1} \phi(x_1) \, dx_2 \, dx_1$$

$$= - \int_{\Omega \backslash K_l} \boldsymbol{\sigma}_{l,n} \partial_{x_1} z \, dx \leq \|\boldsymbol{\sigma}_{l,n}\|_{L^2(\Omega \backslash K_l)}.$$

Then, by Schauder's theorem and Aubin-Lions lemma there exists a subsequence (not relabeled) such that $\sigma_n \to \sigma_\infty$ (strongly) in $L^2(0, L; H^{-1}(I))$. As the stresses are L^2 functions, writing

$$s_n(x_1) =_{H^{-1}(I)} \langle \sigma_n(x_1), \varphi(x_1, \cdot) \rangle_{H_0^1(I)} \quad \text{and} \quad s_\infty(x_1) =_{H^{-1}(I)} \langle \sigma_\infty(x_1), \varphi(x_1, \cdot) \rangle_{H_0^1(I)}$$

it follows that $s_n \to s$ in $L^1(0, L)$.

II. Let $i = 2$. Since $\mu_{n,2}$ is independent of x_2, for every $\varphi \in C_c^\infty(\Omega \backslash K_l)$ we have

$$\int_{\Omega \backslash K_l} (\boldsymbol{\sigma}_{l,n})_2 \, \varphi \, dx = \int_{\Omega \backslash K_l} (\partial_{x_2}(\mu_{n,2} u_{l,n})) \varphi \, dx = - \int_{\Omega \backslash K_l} \mu_{n,2} u_{l,n} \partial_{x_2} \varphi \, dx.$$

We have $\mu_{n,2} \partial_{x_2} \varphi \overset{*}{\rightharpoonup} \mu_{hom,2} \partial_{x_2} \varphi$ in $L^\infty(\Omega \backslash K_l)$ and $u_{l,n} \to u_{l,\infty}$ in $L^1(\Omega \backslash K_l)$ (by compact embedding), therefore

$$- \int_{\Omega \backslash K_l} \mu_{n,2} u_{l,n} \partial_{x_2} \varphi \, dx \to - \int_{\Omega \backslash K_l} \mu_{hom,2} u_{l,hom} \partial_{x_2} \varphi \, dx$$

$$= \int_{\Omega \backslash K_l} \mu_{hom,2}(\partial_{x_2} u_{l,hom}) \varphi \, dx$$

$$= \int_{\Omega \backslash K_l} (\boldsymbol{\sigma}_{l,hom})_2 \varphi \, dx.$$

The proof is concluded. $\qquad\square$

By Proposition 32 and standard arguments it follows the convergence of the reduced energies, that is, $\mathcal{E}_n \to \mathcal{E}_{hom}$ pointwise in $[0, L]$. Further, it is not difficult to also prove the following result:

Corollary 33. *Let $l_n \in [0, L]$ such that $l_n \to l$. Then $\nabla u_{l_n,n} \rightharpoonup \nabla u_{l,hom}$ and $C_n \nabla u_{l_n,n}^T \rightharpoonup C_{hom} \nabla u_{l,hom}^T$ in $L^2(\Omega; \mathbb{R}^2)$. Moreover $\mathcal{E}_n(l_n) \to \mathcal{E}_{hom}(l)$.*

As by-product of the previous Corollary the energy $\mathcal{E}_n$ converge uniformly to $\mathcal{E}_{hom}$ in $[0, L]$.

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