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Variational Worn Stones

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Abstract

We introduce an evolution model à la Firey for a convex stone which tumbles on a beach and undertakes an erosion process depending on some variational energy, such as torsional rigidity, a principal Dirichlet Laplacian eigenvalue, or Newtonian capacity. Relying on the assumption of the existence of a solution to the corresponding parabolic flow, we prove that the stone tends to become asymptotically spherical. Indeed, we identify an ultimate shape of these flows with a smooth convex body whose ground state satisfies an additional boundary condition, and we prove symmetry results for the corresponding overdetermined elliptic problems. Moreover, we extend the analysis to arbitrary convex bodies: we introduce new notions of cone variational measures and we prove that, if such a measure is absolutely continuous with constant density, the underlying body is a ball.

1. Introduction

The aim of this paper is to propose a variational counterpart of Firey's seminal result stating that the fate of worn stones is that of becoming spherical [26]. Firey considered the evolution problem satisfied by the support functions $h(t,\cdot)$ of a family of convex bodies $C(t)$ in $\mathbb{R}^3$, obtained when an initial convex stone $C(0)$ tumbles on an abrasive plane and undertakes an erosion process, in which the rate of wear is proportional to the density of contact points with the plane per unit surface area, and also to the volume of the stone. The mathematical formulation reads as

$$\begin{cases} \frac{\partial h}{\partial t}(t,\xi) = -a|C(t)|G(t,\nu_t^{-1}(\xi)) & \text{on } (0,T)\times\mathbb{S}^{n-1}, \\ h(0,\xi) = h_0(\xi) & \text{on } \mathbb{S}^{n-1}, \\ h(t,\xi) \leqq b & \text{on } (0,T)\times\mathbb{S}^{n-1}, \end{cases} \tag{1}$$

where a and b are positive constants, ν_t^{-1} is the inverse Gauss map of $C(t)$, $G(t,\cdot)$ denotes the Gaussian curvature of $C(t)$, and $|C(t)|$ its volume. Under the assumption that $C(0)$ is smooth and centrally symmetric, and that, for some $T > 0$, the evolution problem (1) admits a smooth solution, Firey showed that actually $T = +\infty$, that convexity and symmetry are preserved along the flow, and that an ultimate shape must satisfy the following geometric condition: in every direction its support function h and Gaussian curvature G are proportional to each other. More precisely, denoting by ν its Gauss map and by c a positive constant, it satisfies

$$h(\xi) = c\, G(\nu^{-1}(\xi)) \qquad \forall \xi \in \mathbb{S}^2. \tag{2}$$

Firey then obtained the following symmetry result: if a smooth and centrally symmetric convex body in $\mathbb{R}^3$ satisfies the equality (2), it must be a ball. Moreover, he conjectured that, in dimension 3, the above results (often referred to as "convergence of solutions to a round point") should remain valid without the assumption of central symmetry; in higher dimensions, this is also called the generalized Firey conjecture.

Firey's vision of worn stones has inspired many deep developments up today, in both the fields of evolution equations, and of convex geometry.

On the side of evolution equations, the parabolic problem introduced by Firey, written in terms of a parametrization of the boundary and up to a renormalization, is nothing else than the Gaussian curvature flow. Such flow has been intensively studied in recent years, so that it is impossible to exhausdhely report the related literature. We refer to the recent monograph [4] for an account about the Gaussian curvature flow and more references on it. Let us just mention that nowadays a full resolution is available for both Firey's conjecture and its generalized version, the main steps of its history being the following: the existence of a solution to the Gauss curvature flow was proved in any space dimension by Chou [54]; the convergence to a round point was obtained in dimension $n = 3$ by Andrews [3], while in higher dimensions it follows from a convergence result by Guan–Ni [30], combined with a rigidity result by Brendle–Choi–Daskalopoulos [17].

On the side of convex geometry, the rigidity result by Firey is related to more general rigidity questions for convex bodies. In fact, condition (2) amounts to ask that a convex body K has the same *cone volume measure* V_K as a ball (see Sect. 2 for the definition). Accordingly, a generalized version of Firey rigidity problem reads as follows: for K belonging to the class $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$, and a ball B, investigate the validity of the implication

$$V_K = V_B \quad \Rightarrow \quad K = B. \tag{3}$$

In the above mentioned papers about the asymptotic behaviour of the Gaussian curvature flow, it has been proved that (3) holds under the assumption that K has its centroid at the origin [30, Proposition 3.3] or that K is smooth and strictly convex [17, Theorem 6].

Looking at (3), one is led in a natural way to the more general question whether the cone volume measure determines uniquely the associated convex body. In particular, denoting by $\mathcal{K}^n_*$ the class of centrally symmetric convex bodies in $\mathbb{R}^n$, the question is whether, given K and K_0 in $\mathcal{K}^n_*$, it holds that

$$V_K = V_{K_0} \quad \Rightarrow \quad K = K_0. \tag{4}$$

The relevance of this question lies in particular in its connection with one of the major open problems in convex geometry, namely the validity of the log-Brunn-Minkowski inequality

$$\left|(1-\lambda)\cdot K +_0 \lambda \cdot L\right| \geqq |K|^{1-\lambda}|L|^{\lambda} \qquad \forall K, L \in \mathcal{K}^n_*,\ \forall \lambda \in [0,1], \tag{5}$$

where

$$(1-\lambda)\cdot K +_0 \lambda \cdot L := \Big\{x \in \mathbb{R}^n \;\; x\cdot\xi \leqq h_K(\xi)^{1-\lambda} h_L(\xi)^{\lambda} \quad \forall \xi \in \mathbb{S}^{n-1}\Big\},$$

and h_K, h_L denote the support function of K and L respectively.

Inequality (5), which is a strengthening of the classical Brunn-Minkowsi inequality, has been proved in dimension $n = 2$ in [13], while it is currently open in higher dimensions (see the recent paper [10] for an extensive survey about the state of the art in this topic, and also our previous paper [23] for a related functional version). As shown in [13], proving the log-Brunn-Minkowski inequality is equivalent to proving that, for every fixed $K \in \mathcal{K}^n_*$, the minimization problem

$$\min\left\{\frac{1}{|K|}\int_{\mathbb{S}^{n-1}} \log(h_L)\,dV_K \;\; L \in \mathcal{K}^n_*\,,\; |K| = |L|\right\} \tag{6}$$

is solved by K itself. Moreover, it turns out that a solution K_0 to problem (6) exists and is a critical set, in the sense that it satisfies the equality of measures $V_{K_0} = V_K$. Hence the connection with the implication (4). For $n = 2$, Böröczky–Lutwak–Yang–Zhang [13] proved that such implication holds true, thus obtaining also the log-Brunn–Minkowski inequality.

To close the circle, the minimization problem (6) is not unrelated with the evolution of convex bodies by their Gaussian curvature. Indeed, in the particular case $K = B$, the integral in (6) is an entropy functional, firstly considered by Firey, which decreases along the Gaussian curvature flow (and actually a suitable generalization of this entropy monotonicity property is used by Guan-Ni to obtain (3) for bodies with centroid at the origin).

In this work we attack the new problem of studying the above topics when the volume functional is replaced by some variational energy, such as torsional rigidity, principal Dirichlet Laplacian eigenvalue, or Newtonian capacity. These functionals are object of study in many problems of classical and modern Calculus of Variations, such as isoperimetric type inequalities, concavity inequalities, inequalities involving polarity. In many cases, the behaviour of these functionals turn out to resemble that of the volume functional, in particular concerning the validity of inequalities of Brunn–Minkowski type (see [21]).

Thus, studying variational results à la Firey seems to be a very natural direction, which is, to the best of our knowledge, completely unexplored. To precise what we intend, let us focus our attention in particular on the case of torsional rigidity. In the physical case when $K \subset \mathbb{R}^2$ is the cross section of a cylindrical rod $K \times \mathbb{R}$ under torsion, its torsional rigidity $\mathcal{T}(K)$ is the torque required for unit angle of twist per unit length. From an analytical point of view, when K is more in general a n-dimensional convex body, its torsional rigidity (often abbreviated as torsion) is given by

$$\mathcal{T}(K) = \int_K u_K \, dx,$$

where u_K is the torsion function of K, i.e. the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in int } K \\ u = 0 & \text{on } \partial K \,. \end{cases}$$

We recall that, if we have a long cylindrical rod in the space $\mathbb{R}^3$ of cross-section K, the function u_K (also known as *warping function*) determines the state of the stress in the interior of the rod, as it is proportional to the out-of-plane displacement of the sections.

A classical result by Dahlberg [24] ensures that, for any $K \in \mathcal{K}^n$, the gradient of the torsion function is well-defined $\mathcal{H}^{n-1}$-a.e. on ∂K, and belongs to $L^2(\partial K)$. Moreover, it is also well-known that the Hadamard first variation of torsion can be expressed through an integral formula, which is the perfect analogue of the one valid for volume, replacing the surface area measure S_K by the *torsion first variation measure* μ_K, defined by

$$\mu_K := (\nu_K)_\sharp \big(|\nabla u_K|^2 \mathcal{H}^{n-1} \llcorner \partial K \big), \tag{7}$$

$(\nu_K)\sharp$ denotes the push forward through the Gauss map of K. By pursuing the analogy with the case of volume, we are led to introduce the *cone torsion measure* of K, as the positive measure on $\mathbb{S}^{n-1}$ defined by

$$\tau_K := h_K \, \mu_K,$$

where h_K denotes the support function of K.

Relying on this new definition, it is natural to investigate the rigidity question analogue to the one of Firey, namely whether, for a given $K \in \mathcal{K}^n_*$,

$$\tau_K = \tau_B \quad \Rightarrow \quad K = B.$$

In the setting of smooth convex bodies, the equality $\tau_K = \tau_B$ amounts to ask that the torsion function u_K satisfies, for some positive constant c, the boundary condition

$$|\nabla u_K|^2 x \cdot \nu_K = c \, G_K \qquad \text{on } \partial K,$$

where G_K denotes the Gaussian curvature of ∂K. Hence, the corresponding rigidity problem consists in investigating symmetry of smooth convex bodies K such that the following overdetermined boundary value problem admits a solution:

$$\begin{cases} -\Delta u = 1 & \text{in int } K, \\ u = 0 & \text{on } \partial K, \\ |\nabla u|^2 x \cdot \nu_K = c \, G_K & \text{on } \partial K. \end{cases} \tag{8}$$

This problem is not covered by the very vast literature on Serrin-type problems [55], as the simultaneous presence of the support function and of curvature in the overdetermined boundary condition is completely new; we limit ourselves to quote the papers [15,27] about overdetermined problems in which curvatures are involved.

In Theorem 1 we establish symmetry for problem (8), under the weaker assumption that K has its centroid at the origin. At present, we do not know if the assumption on the centroid can be removed. The proof of Theorem 1 is obtained by an ad-hoc combination of different inequalities, such as the Saint–Venant inequality for torsion, and the isoperimetric inequality for the 2-affine surface area (or alternatively, an isoperimetric-type inequality proved in [19] and Blasckhe–Santaló inequality).

In the setting of arbitrary convex bodies, the equality $\tau_K = \tau_B$ cannot be formulated any longer as a pointwise equality for the gradient of the torsion function along the boundary; it just tells the measure τ_K is a constant multiple of the diffuse measure on the sphere, i.e.

$$\tau_K = c\,\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}. \tag{9}$$

In Theorem 4 we establish symmetry for convex bodies with centroid at the origin satisfying (9). We argue by adapting the arguments used in the regular case, but the proof is more delicate, as it involves the regularity properties of convex bodies having an absolutely continuous surface area measure; in this respect, we heavily rely on the results proved by Hug in the papers [34–36].

The symmetry results described so far are related to many further questions. In particular, in the spirit of Firey's work, it is natural to wonder whether the condition $\tau_K = \tau_B$ identifies K as the ultimate shape of some flow.

To answer this question, we imagine a new variational flow for worn stones which tumble on an abrasive plane and undertake an energy-based erosion process. Namely, we assume that the rate of wear is proportional to the density of contact points with the abrasive plane, no longer per unit surface area measure as in [26], but per unit torsion first variation measure; moreover, while in [26] the rate of wear is also taken to be proportional to the volume, we take it to be proportional to the torsional rigidity.

To write explicitly the problem, let $C(t)$ represent the evolution in time of an initial convex stone $C(0)$, and let σ be any small part of $\partial C(t)$. The measure of the set of directions for which the abrasive plane touches $C(t)$ at points in σ is given by $\mathcal{H}^{n-1}(\nu_t(\sigma))$, where ν_t is the Gauss map of $C(t)$. By a standard change of variables (see Proposition 1 (e)), we have

$$\mathcal{H}^{n-1}(\nu_t(\sigma)) = \int_\sigma G(t, y)\, d\mathcal{H}^{n-1}(y),$$

where $G(t, y)$ denotes the Gauss curvature of $\partial C(t)$ at the point y. Recalling the definitions of surface area measure and of torsion first variation measure for the convex body $C(t)$, the above integral can also be written as

$$\int_{\nu_t(\sigma)} G(t, \nu_t^{-1}(\xi))\, dS_{C(t)} \qquad \text{or} \qquad \int_{\nu_t(\sigma)} \frac{G(t, \nu_t^{-1}(\xi))}{|\nabla u_t(\nu_t^{-1}(\xi))|^2}\, d\mu_{C(t)},$$

where $u(t,\cdot)$ is the torsion function of $C(t)$. Thus we see that, while contact points have density $G(t,\nu_t^{-1}(\xi))$ with respect to the surface area measure $S_{C(t)}$, they have density $\frac{G(t,\nu_t^{-1}(\xi))}{|\nabla u(t,\nu_t^{-1}(\xi))|^2}$ with respect to the first variation measure $\mu_{C(t)}$ in (7). We conclude that the variational analogue of Firey's problem when replacing volume by torsion reads as follows:

$$\begin{cases} \frac{\partial h}{\partial t}(t,\xi) = -a\mathcal{T}(C(t))\frac{G(t,\nu_t^{-1}(\xi))}{|\nabla u(t,\nu_t^{-1}(\xi))|^2} & \text{on } (0,T)\times\mathbb{S}^{n-1}, \\ h(0,\xi) = h_0(\xi) & \text{on } \mathbb{S}^{n-1}, \\ h(t,\xi) \leqq b & \text{on } (0,T)\times\mathbb{S}^{n-1}. \end{cases} \tag{10}$$

A possible physical reason for considering such a model of abrasion process depending on the torsion first variation measure, can be the fact that, among the fundamental types of stresses undertaken by a body subject to abrasion, torsion appears to be the most relevant (in fact, typically the abrasion rate is less sensitive to traction, flexion, compression, or shear). On the other hand, the same model might also be used to describe other deterioration processes of erosion or corrosion. Indeed, the rate of deterioration of the body can be related to its mechanical properties, which in turn are affected by variations in the gradient of the warping function.

From a mathematical point of view, the main novelty of the evolutionary model (10) consists in its non-local nature, coming from the gradient of the warping function. Due to this non-local nature, getting global existence and regularity results seems to be a challenging problem (we refer to [20] for general nonlocal flows, and to the paper [33], appeared after the submission of this manuscript, for a logarithmic version of our flow). Proving the existence of a unique smooth solution to problem (10) is an interesting problem in parabolic flows, which is beyond the scopes of this paper, and will be object of further research.

As in Firey's paper, we assume that $C(0)$ is smooth and centrally symmetric, and that for some $T > 0$, the evolution problem (10) admits a smooth solution. Under this assumption, Theorem 7 establishes that $T = +\infty$, that convexity and symmetry are preserved along the flow, and that an ultimate shape has the same cone torsion measure as a ball. The proof relies basically on the Brunn-Minkowski inequality for torsion due to Borell [8].

To conclude, another question we wish to address is whether, by analogy with (4), for any pair of convex bodies $K, K_0 \in \mathcal{K}^n_*$, we have

$$\tau_K = \tau_{K_0} \quad \Rightarrow \quad K = K_0. \tag{11}$$

Such am implication is in turn related to the possibility of strengthening the Brunn-Minkowski inequality for torsion into a logarithmic Brunn-Minkowski inequality of the kind

$$\mathcal{T}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geqq \mathcal{T}(K)^{1-\lambda}\mathcal{T}(L)^{\lambda} \qquad \forall K, L \in \mathcal{K}^n_*,\ \forall \lambda \in [0,1]. \tag{12}$$

Indeed, by arguing as in the proof of Lemma 3.2 in [13], one can easily check that (12) is equivalent to proving that, for every fixed $K \in \mathcal{K}^n_*$, the minimization problem

$$\min\left\{\frac{1}{\mathcal{T}(K)}\int_{\mathbb{S}^{n-1}}\log(h_L)d\tau_K \ \ L\in\mathcal{K}^n_*\,,\ \mathcal{T}(K)=\mathcal{T}(L)\right\} \tag{13}$$

is solved by K itself; moreover, if the above problem admits solution K_0, it must satisfy the first order optimality condition $\tau_{K_0}=\tau_K$ (cf. the proof of Theorem 7.1 in [13]).

The inequality (12) (or the implication (11)) is likely very challenging, and it should be first proved (or disproved) in dimension $n=2$, see Sect. 6 for some related remarks.

To conclude, we point out that all the results discussed so far are valid also when torsional rigidity is replaced either by the principal frequency of the Dirichlet Laplacian, or (for $n\geqq 3$) by the Newtonian capacity. Also for this reason, we believe that the parabolic flows and logarithmic inequalities related to our variational worn stones are worth of further investigation.

Outline of the paper. In Sect. 2 we detail the definitions of cone variational measures and related representation formulas. In Sect. 3 we prove symmetry for smooth centred convex bodies K where overdetermined boundary value problems such as (8) admit a solution. In Sect. 4 we extend the rigidity results of the previous section to the general framework of arbitrary centred convex bodies. In Sect. 5 we relate the convex bodies studied in the previous sections to new variational flows. In Sect. 6 we give a few concluding remarks about logarithmic Brunn-Minkowski-type inequalities.

2. Cone Variational Measures and Related Representation Formulas

Let $\mathcal{K}^n$ be the class of convex bodies in $\mathbb{R}^n$ with the origin in their interior. In particular, we are going to consider convex bodies in $\mathcal{K}^n$ whose centroid

$$\mathrm{cen}(K)=\frac{1}{|K|}\int_K x\,dx$$

coincides with the origin. Here and below, we indicate by $|\cdot|$ the volume functional on $\mathcal{K}^n$. We shall also consider the subclass $\mathcal{K}^n_*$ of centrally symmetric convex bodies.

Given $K\in\mathcal{K}^n$, we denote by h_K and ν_K respectively its support function and Gauss map. By definition, if m is a measure on ∂K, its push-forward $(\nu_K)_\sharp(m)$ through the Gauss map is the measure on $\mathbb{S}^{n-1}$ defined by

$$\int_{\mathbb{S}^{n-1}}\varphi\,d(\nu_K)_\sharp(m)=\int_{\partial K}\varphi\circ\nu_K\,dm\qquad\forall\varphi\in\mathcal{C}(\mathbb{S}^{n-1}).$$

We recall that the *surface area measure* S_K and the *cone volume measure* V_K are defined respectively by

$$S_K=(\nu_K)_\sharp(\mathcal{H}^{n-1}\llcorner\partial K)\,,\qquad V_K=h_K S_K;$$

moreover, denoting by $|V_K|$ the total variation of V_K, it holds that

$$\begin{aligned}&|K|=\tfrac{1}{n}\textstyle\int_{\mathbb{S}^{n-1}}h_K\,dS_K=\tfrac{1}{n}|V_K|,\\ &\tfrac{d}{dt}|K+tL|\Big|_{t=0^+}=\textstyle\int_{\mathbb{S}^{n-1}}h_L\,dS_K.\end{aligned}$$

The name cone volume measure is motivated by the fact that, when K is a polytope with facets F_i and unit outer normals ν_i, it holds that

$$V_K = \sum_i |\Delta(o, F_i)|\delta_{\nu_i}, \tag{14}$$

where δ_{ν_i} is a Dirac mass at ν_i and $\Delta(o, F_i)$ is the cone with apex o and basis F_i. In recent years, this notion of cone volume measure has been widely studied (see [5,11–14,29,40,41,44,46–48,53,57,58]).

Let now $F(K)$ be one of the following variational energies: torsional rigidity $\mathcal{T}(K)$, first Dirichlet Laplacian eigenvalue $\lambda_1(K)$, or Newtonian capacity $\operatorname{cap}(K)$ (the latter in dimension $n \geqq 3$). We denote by u_K the corresponding ground state, defined as the unique solution in $\Omega = \operatorname{int} K$ to the following elliptic boundary value problems (in the second case normalized so to have unit L^2-norm):

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus K \\ u = 1 & \text{on } \partial K \\ u(x) \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

It is well known that, by analogy with the case of volume, each of these functionals and its Hadamard derivative satisfy the following representation formulas (see respectively [22], [37] and [38])

$$\begin{aligned} &F(K) = \tfrac{1}{|\alpha|} \int_{\mathbb{S}^{n-1}} h_K \, d\mu_K \\ &\tfrac{d}{dt} F(K + tL)\Big|_{t=0^+} = \operatorname{sign}(\alpha) \int_{\mathbb{S}^{n-1}} h_L \, d\mu_K . \end{aligned}$$

Here α denotes the homogeneity degree of F under domain dilation (which equals equal to $n + 2$ for torsion, -2 for the eigenvalue, and $n - 2$ for capacity) and μ_K is the *first variation measure* of F, defined by

$$\mu_K := (\nu_K)_\sharp \big(|\nabla u_K|^2 \mathcal{H}^{n-1} \llcorner \partial K \big) \tag{15}$$

We set the following:

Definition 1. (cone variational measures) For any of the functionals above, with any $K \in \mathcal{K}^n$ we associate the positive measure on $\mathbb{S}^{n-1}$ defined by $h_K \, \mu_K$, being μ_K given by (15), and we denote it respectively by τ_K (*cone torsion measure*), by σ_K (*cone eigenvalue measure*) and by η_K (*cone capacitary measure*).

Let us point out that the terminology in Definition 1 is chosen to emphasize the analogy with the case of volume, though this is somehow an abuse; indeed, the analogue of the equality (14) is clearly false for our variational functionals, since they are not additive under the decomposition of a polytope in $\mathcal{K}^n$ into cones with apex o and bases at its facets.

Let us also notice that the above representation formulas can be rewritten as integrals over the boundary and in terms of the total variations of the corresponding cone variational measure, respectively as

$$\mathcal{T}(K) = \frac{1}{n+2}\int_{\partial K} |\nabla u_K|^2\, x\cdot\nu_K\, d\mathcal{H}^{n-1} = \frac{1}{n+2}|\tau_K|, \tag{16}$$

$$\lambda_1(K) = \frac{1}{2}\int_{\partial K} |\nabla u_K|^2\, x\cdot\nu_K\, d\mathcal{H}^{n-1} = \frac{1}{2}|\sigma_K|, \tag{17}$$

$$\mathrm{cap}(K) = \frac{1}{n-2}\int_{\partial K} |\nabla u_K|^2\, x\cdot\nu_K\, d\mathcal{H}^{n-1} = \frac{1}{n-2}|\eta_K| \qquad (n \geqq 3). \tag{18}$$

Remark 1. The cone variational measures introduced in Definition 1 are weakly* continuous with respect to the convergence of convex bodies in Hausdorff distance. Indeed, if K_n converge to K_∞ in Hausdorff distance, the support functions of K_n converge uniformly to the support function of K_∞, while the first variation measures of K_n converge weakly* to the first variation measure of K_∞ (cf. respectively [38, Thm. 3.1] for capacity, [37, Section 7] for the first eigenvalue, and [22, Thm. 6] for torsion).

3. Rigidity Results for Smooth Convex Bodies

In this section we deal with overdetermined boundary value problems on smooth convex bodies K, for which the Gauss map ν_K and the Gaussian curvature G_K can be classically defined. Whenever no confusion may arise, we omit the index K and we simply write ν and G. We denote by B a generic ball, by B_1 the unit ball, and by ω_n its Lebesgue measure.

Theorem 1. *Let $K \in \mathcal{K}^n$ have its centroid at the origin, and boundary of class $\mathcal{C}^2$. Assume that, for some constant $c > 0$, there exists a solution to the following overdetermined boundary value problem on $\Omega := \mathrm{int}K$:*

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ |\nabla u|^2 x\cdot\nu = c\,G & \text{on } \partial\Omega. \end{cases}$$

Then K is a ball.

The proof of Theorem 1 is based on the application of different inequalities for convex bodies, some involving torsional rigidity and some others being purely geometrical. We list them below:

(i) The Saint–Venant inequality for torsion [32,49]:

$$\frac{\mathcal{T}(K)}{|K|^{\frac{n+2}{n}}} \leqq \frac{\mathcal{T}(B)}{|B|^{\frac{n+2}{n}}},$$

with equality if and only if K is a ball. More explicitly, since $\mathcal{T}(B_1) = \frac{\omega_n}{n(n+2)}$,

$$\mathcal{T}(K) \leqq \frac{|K|^{\frac{n+2}{n}}}{n(n+2)\omega_n^{\frac{2}{n}}}, \tag{19}$$

with equality if and only if K is a ball.

(ii) The isoperimetric inequality of the p-affine surface area in the case $p = 2$:

$$\Theta_2(K) := \int_{\partial K} \frac{G^{\frac{2}{n+2}}}{(x\cdot\nu)^{\frac{n}{n+2}}}\, d\mathcal{H}^{n-1} \leqq n\,\omega_n^{\frac{4}{n+2}}\,|K|^{\frac{n-2}{n+2}}, \tag{20}$$

with equality if and only if K is an ellipsoid (see [43, Theorem 4.8], [56, Theorem 4.2] or [52, formula (10.49)]).

(iii) The isoperimetric-type inequality:

$$\frac{\mathcal{T}(K)^{1-\frac{1}{n+2}}}{\int_{\partial K}|\nabla u_K|^2} \leqq \frac{\mathcal{T}(B)^{1-\frac{1}{n+2}}}{\int_{\partial B}|\nabla u_B|^2} \tag{21}$$

see [19, Theorem 3.15].

(iv) The Blaschke–Santaló inequality [6,50] for convex bodies with centroid at the origin:

$$|K||K^o| \leqq \omega_n^2, \tag{22}$$

with equality if and only if K is an ellipsoid (see [42, Theorem 1′]).

Proof of Theorem 1. We are going to obtain an upper bound and a lower bound for the constant c appearing in the overdetermined boundary condition, starting from two different expressions of it. For the upper bound we use inequality (i), while the lower bound can be obtained by two different methods: either by using the inequality (ii), or by using the inequalities (iii)-(iv). We provide both methods because, as we shall see, the first one works to extend the result to the overdetermined boundary value problem associated with the Newtonian capacity, while the second one works for the first Laplacian Dirichlet eigenvalue. Since the upper bound and the lower bound turn out to match each other, we conclude that in particular Saint-Venant inequality (19) holds as an equality, and hence K must be a ball.

– *Upper bound for c*. We integrate over the boundary both sides of the pointwise overdetermined condition

$$|\nabla u|^2\, x\cdot\nu = c\, G \qquad \text{on } \partial K.$$

By using respectively the identity (16) and the change of variables formula (2.5.29) in [51], we obtain

$$\int_{\partial K} |\nabla u|^2\, x\cdot\nu\, d\mathcal{H}^{n-1} = (n+2)\mathcal{T}(K),$$

and

$$\int_{\partial K} G\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} 1\, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n. \tag{23}$$

Hence we get

$$c = \frac{n+2}{n\,\omega_n}\,\mathcal{T}(K)\,. \tag{24}$$

Then by using the Saint-Venant inequality (19) we obtain the upper bound

$$c \leqq \frac{1}{n^2}\left(\frac{|K|}{\omega_n}\right)^{\frac{n+2}{n}}, \tag{25}$$

with equality if and only if K is a ball.

– *Lower bound for c (first method).* We integrate on ∂K the pointwise overdetermined condition, after rewriting it as

$$|\nabla u| = c^{1/2}\left(\frac{G}{x\cdot\nu}\right)^{1/2} \qquad \text{on } \partial K.$$

Exploiting the fact that u solves the torsion problem, we obtain

$$|K| = c^{1/2}\int_{\partial K}\left(\frac{G}{x\cdot\nu}\right)^{1/2} d\mathcal{H}^{n-1},$$

i.e.

$$c = \frac{|K|^2}{\left[\int_{\partial K}\left(\frac{G}{x\cdot\nu}\right)^{1/2} d\mathcal{H}^{n-1}\right]^2}. \tag{26}$$

We now look at the integral appearing the denominator in the right-hand side of (26). Let us distinguish the cases $n=2$ and $n \geqq 3$.

If $n=2$, such integral is exactly the 2-affine surface area:

$$\int_{\partial K}\left(\frac{G}{x\cdot\nu}\right)^{1/2} d\mathcal{H}^{n-1} = \Theta_2(K).$$

If $n \geqq 3$, the same integral can be estimated in terms of the 2-affine surface area using Hölder's inequality. Specifically, using Hölder's inequality with the conjugate exponents $\beta = \frac{2n}{n+2}$ and $\beta' = \frac{2n}{n-2}$, it holds that

$$\begin{aligned}
\int_{\partial K}\left(\frac{G}{x\cdot\nu}\right)^{1/2} d\mathcal{H}^{n-1} &= \int_{\partial K}\frac{G^{1/n}}{(x\cdot\nu)^{1/2}}\cdot G^{\frac{2-n}{2n}}\, d\mathcal{H}^{n-1}\\
&\leqq \left[\int_{\partial K}\left(\frac{G^{1/n}}{(x\cdot\nu)^{1/2}}\right)^{\beta} d\mathcal{H}^{n-1}\right]^{1/\beta}\cdot\left[\int_{\partial K}\left(G^{\frac{2-n}{2n}}\right)^{\beta'} d\mathcal{H}^{n-1}\right]^{1/\beta'}\\
&= \left[\int_{\partial K}\frac{G^{\frac{2}{n+2}}}{(x\cdot\nu)^{\frac{n}{n+2}}}\, d\mathcal{H}^{n-1}\right]^{\frac{n+2}{2n}}\cdot\left[\int_{\partial K} G\, d\mathcal{H}^{n-1}\right]^{\frac{n-2}{2n}}\\
&= [\Theta_2(K)]^{\frac{n+2}{2n}}\cdot[n\omega_n]^{\frac{n-2}{2n}}\,.
\end{aligned}$$

Hence, for every $n \geqq 2$, we have that

$$\int_{\partial K} \left(\frac{G}{x \cdot \nu}\right)^{1/2} d\mathcal{H}^{n-1} \leqq \Theta_2(K)^{\frac{n+2}{2n}} (n\,\omega_n)^{\frac{n-2}{2n}} \leqq n\,\omega_n^{\frac{n+2}{2n}} |K|^{\frac{n-2}{2n}}, \tag{27}$$

where in the second inequality we have used the 2-affine isoperimetric inequality (20).

From (26) and (27), we get

$$c \geqq \frac{|K|^2}{n^2\,\omega_n^{\frac{n+2}{n}}\,|K|^{\frac{n-2}{n}}} = \frac{1}{n^2}\left(\frac{|K|}{\omega_n}\right)^{\frac{n+2}{n}}. \tag{28}$$

– *Lower bound for c (second method).* We integrate over ∂K the pointwise overdetermined condition, after rewriting it as

$$|\nabla u|^2 = c\,\frac{G}{x \cdot \nu} \qquad \text{on } \partial K.$$

Then, using also the expression of $\mathcal{T}(K)$ in (24), the isoperimetric inequality (21) reads:

$$\frac{c^{1-\frac{1}{n+2}}\left(\frac{n\omega_n}{n+2}\right)^{1-\frac{1}{n+2}}}{c\int_{\partial K}\frac{G}{x\cdot\nu}} \leqq \frac{\mathcal{T}(B)^{1-\frac{1}{n+2}}}{\int_{\partial B}|\nabla u_B|^2}.$$

Rising the above inequality to power $(n+2)$, and setting for brevity

$$\Lambda(B) := \frac{\mathcal{T}(B)^{1-\frac{1}{n+2}}}{\int_{\partial B}|\nabla u_B|^2},$$

we get

$$c \geqq \frac{\left(\frac{n\omega_n}{n+2}\right)^{n+1}}{\Lambda(B)^{n+2}\left(\int_{\partial K}\frac{G}{x\cdot\nu}\right)^{n+2}}. \tag{29}$$

We now look at the integral appearing the denominator in the right-hand side of (29). We transform it into an integral on $\mathbb{S}^{n-1}$ by the classical change of variables formula already quoted above. Then, by using Hölder's inequality with conjugate exponents n and $\frac{n}{n-1}$, the well-known representation formula for the volume of the dual body $|K^o| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K^{-n}$, and the Blaschke-Santaló inequality (22), we obtain

$$\begin{aligned} \int_{\partial K}\frac{G}{x\cdot\nu} = \int_{\mathbb{S}^{n-1}}\frac{1}{h_K} &\leqq \left(\int_{\mathbb{S}^{n-1}}\frac{1}{h_K^n}\right)^{\frac{1}{n}}(n\omega_n)^{\frac{n-1}{n}} \\ &= (n|K^o|)^{\frac{1}{n}}(n\omega_n)^{\frac{n-1}{n}} \\ &\leqq n\omega_n^{\frac{n+1}{n}}|K|^{-\frac{1}{n}}. \end{aligned} \tag{30}$$

We now combine (29) and (30). Taking also into account that $\mathcal{T}(B_1) = \frac{\omega_n}{n(n+2)}$, and $\int_{\partial B_1} |\nabla u_{B_1}|^2 = \frac{\omega_n}{n}$, so that

$$\Lambda(B)^{n+2} = \frac{n}{(n+2)^{n+1}\omega_n},$$

we arrive exactly at the inequality (28). □

– *Conclusion.* By comparing the upper bound (25) and the lower bound (28), we conclude that the equality sign holds in both inequalities. In particular, the equality sign in (25) implies that K is a ball. □

Theorem 2. *Let $K \in \mathcal{K}^n$ have its centroid at the origin, and boundary of class $\mathcal{C}^2$. Assume that, for some constant $c > 0$, there exists a solution to the following overdetermined boundary value problem on $\Omega := \operatorname{int} K$:*

$$\begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u|^2 x \cdot \nu = c\, G & \text{on } \partial\Omega. \end{cases}$$

Then K is a ball.

The proof of Theorem 2 is similar to the one of Theorem 1; it is based on the following inequalities:

(i) The Faber-Krahn inequality

$$\lambda_1(K)|K|^{2/n} \geqq \lambda_1(B)|B|^{2/n}, \tag{31}$$

with equality if and only if K is a ball, see e.g. [31, Section 3.2].

(ii) The isoperimetric-type inequality

$$\frac{\lambda_1(K)^{3/2}}{\int_{\partial K} |\nabla u_K|^2} \leqq \frac{\lambda_1(B)^{3/2}}{\int_{\partial B} |\nabla u_B|^2} \tag{32}$$

see [19, Theorem 3.15].

(iii) The Blaschke-Santaló inequality (22).

Proof of Theorem 2. Similarly as in the proof of Theorem 1, we are going to provide a matching upper and lower bound for the constant c appearing in the overdetermined boundary condition. A lower bound on c is obtained by arguing as done to obtain an upper bound in the proof of Theorem 1. First we integrate over ∂K the overdetermined condition written as

$$|\nabla u|^2\, x \cdot \nu = c\, G \qquad \text{on } \partial K.$$

Using the identity (17), and a change of variables as in (23), we deduce that

$$c = \frac{2}{n\,\omega_n}\,\lambda_1(K). \tag{33}$$

Then, by using the Faber-Krahn inequality (31), we obtain the lower bound

$$c \geqq \frac{2}{n} \omega_n^{\frac{2-n}{n}} \lambda_1(B)|K|^{-\frac{2}{n}}, \tag{34}$$

with equality if and only if K is a ball.

An upper bound on c is obtained by arguing as done to obtain a lower bound in the proof of Theorem 1, second method. Namely, we integrate on ∂K the overdetermined condition, after rewriting it as

$$|\nabla u|^2 = c \frac{G}{x \cdot \nu} \qquad \text{on } \partial K.$$

By using also the expression of $\lambda_1(K)$ in (33), the isoperimetric inequality (32) reads:

$$\frac{c^{\frac{3}{2}} \left(\frac{n\omega_n}{2}\right)^{\frac{3}{2}}}{c \int_{\partial K} \frac{G}{x \cdot \nu}} \leqq \frac{\lambda_1(B)^{\frac{3}{2}}}{\int_{\partial B} |\nabla u_B|^2}.$$

Rising the above inequality to power 2, and setting for brevity

$$\Lambda(B) := \frac{\lambda_1(B)^{\frac{3}{2}}}{\int_{\partial B} |\nabla u_B|^2},$$

we obtain

$$c \leqq \left(\frac{2}{n\omega_n}\right)^3 \Lambda(B)^2 \left(\int_{\partial K} \frac{G}{x \cdot \nu}\right)^2.$$

For the integral appearing at the right hand side of the above inequality, the estimate (30) found in the proof of Theorem 1 holds. Hence we get

$$c \leqq \frac{8}{n} \omega_n^{\frac{2-n}{n}} \Lambda(B)^2 |K|^{-\frac{2}{n}}. \tag{35}$$

To conclude the proof, it remains to show that the expressions at the right hand sides of the lower bound (34) and of the the upper bound (35) coincide. Indeed in this case the Faber-Krahn inequality must hold as an equality, yielding that K is a ball. The matching of our upper and lower bounds corresponds to the equality

$$2\lambda_1(B) = \int_{\partial B} |\nabla u_B|^2. \tag{36}$$

The validity of (36) is checked through some direct computations involving Bessel functions, that we enclose for the sake of completeness. We have (see for instance [39, Section 4])

$$\lambda_1(B_1) = (j_{\frac{n}{2}-1,1})^2, \tag{37}$$

where $j_{\frac{n}{2}-1,1}$ is the first zero of the Bessel function $J_{\frac{n}{2}-1}$, and

$$u_{B_1}(r) = C r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1} r).$$

The value of the constant C is determined by imposing that u_{B_1} has unit L^2-norm, yielding

$$C^2 = \left(n\omega_n \int_0^1 J^2(j_1 r)\, r\, dr\right)^{-1}$$

(here and in the sequel we have written for brevity $J := J_{\frac{n}{2}-1}$ and $j_1 := j_{\frac{n}{2}-1,1}$). Since, by known properties of Bessel functions (see [1, 11.4.5]) we have

$$\int_0^1 J^2(j_1 r) r\, dr = \frac{1}{2}(J'(j_1))^2\,,$$

we infer that

$$C^2 = \frac{2}{n\omega_n}\frac{1}{(J'(j_1))^2}.$$

Hence,

$$\int_{\partial B_1} |\nabla u_{B_1}|^2 = C^2 j_1^2 (J'(j_1))^2 = 2 j_1^2. \tag{38}$$

From (37) and (38), we see that (36) is satisfied and our proof is complete. □

Theorem 3. *Let $K \in \mathcal{K}^n$ have its centroid at the origin, and boundary of class $\mathcal{C}^2$. Assume that, for some constant $c > 0$, there exists a solution to the following overdetermined boundary value problem on the complement of K:*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus K \\ u = 1 & \text{on } \partial K \\ |\nabla u|^2 x \cdot \nu = c\, G & \text{on } \partial K \\ \lim\limits_{|x| \to +\infty} u(x) = 0. \end{cases}$$

Then K is a ball.

Also the proof of Theorem 3 follows the same strategy of Theorem 1. This is based on the following inequalities:

(i) The isoperimetric inequality for capacity [49]

$$\frac{\operatorname{cap}(K)}{|K|^{\frac{n-2}{n}}} \geqq \frac{\operatorname{cap}(B)}{|B|^{\frac{n-2}{n}}},$$

with equality if and only if K is a ball. More explicitly, since $\operatorname{cap}(B_1) = n(n-2)\omega_n$,

$$\operatorname{cap}(K) \geqq (n-2)^2 \omega_n^{\frac{2-n}{n}} |K|^{\frac{n-2}{n}}, \tag{39}$$

with equality if and only if K is a ball.

(ii) The isoperimetric inequality (20) for the p-affine surface area in the case $p = 2$.

Proof of Theorem 3. A lower bound for c is obtained by arguing as done to obtain an upper bound in the proof of Theorem 1. Specifically, by (18) and (23), we obtain

$$c = \frac{n-2}{n\,\omega_n}\,\mathrm{cap}(K). \tag{40}$$

Then, by using the isoperimetric inequality (39) for capacity, we get

$$c \geqq (n-2)^2 \omega_n^{\frac{2-n}{n}} |K|^{\frac{n-2}{n}}, \tag{41}$$

with equality if and only if K is a ball.

An upper bound for c is obtained by arguing as done to obtain a lower bound in the proof of Theorem 1, first method. We integrate over ∂K the overdetermined condition, after rewriting it as

$$|\nabla u| = c^{1/2} \left(\frac{G}{x\cdot\nu}\right)^{1/2}.$$

Using the equalities $\mathrm{cap}(K) = \int_{\partial K} |\nabla u|$ (see [28, p.27]) and (40), we obtain

$$\mathrm{cap}(K) = \frac{n\,\omega_n}{n-2} c = c^{1/2} \int_{\partial K} \left(\frac{G}{x\cdot\nu}\right)^{1/2} d\mathcal{H}^{n-1},$$

i.e.

$$c = \left(\frac{n-2}{n\omega_n}\right)^2 \left[\int_{\partial K} \left(\frac{G}{x\cdot\nu}\right)^{1/2} d\mathcal{H}^{n-1}\right]^2. \tag{42}$$

For the integral appearing at the right hand side of the above inequality, the estimate (27) found in the proof of Theorem 1 (through the use of Hölder inequality and the 2-affine isoperimetric inequality) holds. Thus we get

$$c \leqq \left(\frac{n-2}{n\omega_n}\right)^2 \left[n\,\omega_n^{\frac{n+2}{2n}} |K|^{\frac{n-2}{2n}}\right]^2 = (n-2)^2 \omega_n^{\frac{2-n}{n}} |K|^{\frac{n-2}{n}}. \tag{43}$$

Comparing (41) and (43), we see that our lower and upper bounds match each other, implying in particular that (41) must hold with equality sign, and hence that K is a ball. □

4. Rigidity Results for Arbitrary Convex Bodies

In this section we drop any smoothness assumption and we deal with arbitrary centred convex bodies having some cone variational measure equal to that of a ball.

Theorem 4. *Let $K \in \mathcal{K}^n$ have its centroid at the origin, and let τ_K be its cone torsion measure according to Definition 1. Assume that, for some positive constant c,*

$$\tau_K = c\,\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1} \text{ as measures on } \mathbb{S}^{n-1}. \tag{44}$$

Then K is a ball.

The proof of Theorem 4 requires some preliminaries about convex bodies which have absolutely continuous surface area measure (or equivalently which admit a curvature function); we gather them in Proposition 1, relying on some results proved in [34–36].

We recall that, if a convex body K has absolutely continuous surface area measure, by definition there exists a unique non-negative function $f_K \in L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1})$, called *curvature function* of K, such that its surface area measure satisfies $S_K = f_K \mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}$. Equivalently, we have

$$\begin{aligned} V_1(K, L) &:= \lim_{t \to 0^+} \frac{V(K + tL) - V(K)}{t} \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(\xi) f_K(\xi) d\mathcal{H}^{n-1}(\xi) \qquad \forall L \in \mathcal{K}^n. \end{aligned}$$

Below we follow the usual convention that $\nu \colon \partial K \to \mathbb{S}^{n-1}$ is a possibly multi-valued map associating to every $x \in \partial K$ the unit vectors of the normal cone to ∂K at x.

Moreover, using the notations in the cited papers of Hug, we set

- $\mathcal{M}(K)$:= the set of points $x \in \partial K$ such that ∂K is second order differentiable at x (so that $\nu(x)$ is a singleton and $G(x)$ is well-defined as the product of the principal curvatures);
- $(\partial K)_+$:= the set of points $x \in \partial K$ such that there exists an internal tangent ball touching ∂K at x (so that $G(x)$ is finite);
- $\exp^* K$:= the set of points $x \in \partial K$ such that there exists a closed ball B containing K with $x \in \partial B$ (so that $G(x) > 0$).

Proposition 1. *Assume that $K \in \mathcal{K}^n$ has absolutely continuous surface area measure, and let f_K be its curvature function. Then:*

(a) Denoting by $r_1(\xi), \dots, r_{n-1}(\xi)$ the generalized radii of curvature of ∂K at $\nu^{-1}(\xi)$, namely the eigenvalues of $D^2 h_K(\xi)|_{\xi^\perp}$, we have

$$f_K(\xi) = r(\xi) := \prod_{j=1}^{n-1} r_j(\xi) \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}. \tag{45}$$

(b) Setting

$$R(K) := \mathcal{M}(K) \cap (\partial K)_+ \cap \exp^* K,$$

we have that $\mathcal{H}^{n-1}(\partial K \setminus R(K)) = 0$, and ν is a bijection from $R(K)$ to $\nu(R(K))$.

If, in addition, $f_K(\xi) > 0$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$, then:

(c) $\nu(R(K))$ has full measure in $\mathbb{S}^{n-1}$, i.e. $\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus \nu(R(K))) = 0$.
(d) After possibly removing a $\mathcal{H}^{n-1}$-null set from $R(K)$, for every $\xi \in \nu(R(K))$, h_K is second order differentiable at ξ, $x := \nabla h_K(\xi) \in R(K)$ and $G(x) r(\xi) = 1$.

(e) For every nonnegative function $\psi \in L^1(\partial K, \mathcal{H}^{n-1} \llcorner \partial K)$ *it holds that*

$$\int_{\partial K} \psi \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \frac{\psi(\nu^{-1}(\xi))}{G(\nu^{-1}(\xi))} \, d\mathcal{H}^{n-1}(\xi).$$

Proof. The equality (45) is proved in [35, formula (2.8)].

Statement (b) is consequence of the facts that $\mathcal{H}^{n-1}(\partial K \setminus \mathcal{M}) = 0$, $\mathcal{H}^{n-1}(\partial K \setminus (\partial K)_+) = 0$ and $\mathcal{H}^{n-1}(\partial K \setminus \exp^* K) = 0$, following respectively from Alexandroff theorem [2], [45], and [36, Theorem 3.7(c)].

Assume now that $f_K > 0$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$. Recall that the curvature measure of K is the measure C_K, supported on ∂K, defined by

$$C_K(E) := \mathcal{H}^{n-1}(\nu(E)) \qquad \text{for every Borel set } E \subseteq \partial K.$$

By (45) and Theorem 2.3 in [36], it follows that C_K is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \partial K$. Since, by statement (b), the set $\partial K \setminus R(K)$ is $\mathcal{H}^{n-1}$-negligible, we conclude that also statement (c) holds, namely $\mathbb{S}^{n-1} \setminus \nu(R(K))$ is $\mathcal{H}^{n-1}$-negligible. To prove statement (d), we consider the subset of $\mathbb{S}^{n-1}$ given by

$$\Sigma := \left\{ \xi \in \nu(R(K)) \colon \ h_K \text{ is second order differentiable at } \xi \right\},$$

and we replace $R(K)$ with the possibly smaller set $\nu^{-1}(\Sigma)$. Then statement (d) follows from (c) and Alexandroff theorem.

Finally, let us prove (e). Following [34], for every $r > 0$ let $(\partial K)_r$ be the set of all points of $x \in \partial K$ such that there exists an internal tangent ball of radius r touching ∂K at x. From Lemma 2.3 in [34], the map $\nu \llcorner (\partial K)_r$ is Lipschitz continuous, and its approximate $(n-1)$-dimensional Jacobian is

$$\text{ap } J_{n-1} \nu \llcorner (\partial K)_r(x) = G(x), \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in (\partial K)_r.$$

Recalling that $G(x) > 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial K$, for every positive $n \in \mathbb{N}$ we can apply Federer's coarea formula (see [25, Theorem 3.2.22]) to the non-negative function

$$h_n(x) := \frac{\psi(x)}{G(x)} \chi_{(\partial K)_{1/n}}(x),$$

obtaining

$$\int_{\partial K} h_n \, G \, d\mathcal{H}^{n-1} = \int_{(\partial K)_{1/n}} h_n \, G \, d\mathcal{H}^{n-1} = \int_{\nu((\partial K)_{1/n})} \frac{\psi \circ \nu^{-1}}{G \circ \nu^{-1}} \, d\mathcal{H}^{n-1}.$$

Since $(\partial K)_+ = \bigcup_n (\partial K)_{1/n}$, the change of variable formula (e) follows by using (c), and applying Lebesgue monotone convergence theorem. □

We are now in a position to give the

Proof of Theorem 4. The idea is to follow the same proof line of Theorem 1. However, this cannot be done directly, since we do not have any longer a pointwise identity holding along the boundary. So, we have to prove first of all that the constant c appearing in (44) can be still expressed by the two different formulas (24) and

(26). To obtain formula (24), it is enough to observe that the two measures in the overdetermined conditions must have the same total variation: using (16), we obtain

$$|\tau_K| = (n+2)\mathcal{T}(K),$$

and hence

$$c = \frac{|\tau_K|}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} = \frac{n+2}{n\,\omega_n}\,\mathcal{T}(K).$$

To obtain also formula (26), we need first to prove the following

Claim: the equality (44) implies that K has absolutely continuous surface area measure, and in addition its curvature function is strictly positive.

This amount to show that, if the equality (44) holds, for a Borel set $E \subset \mathbb{S}^{n-1}$, the following implications hold:

$$\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}(E) = 0 \quad \Rightarrow \; S_K(E) = 0; \tag{46}$$

$$\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}(E) > 0 \quad \Rightarrow \; S_K(E) > 0. \tag{47}$$

Indeed, (46) implies that K admits a curvature function f_K; then, since $S_K = f_K\, \mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}$, (47) implies that f_K is strictly positive $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$.

Let $E \subset \mathbb{S}^{n-1}$ be a Borel set with $\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}(E) = 0$. By (44), we have $\tau_K(E) = 0$. Recalling that, by definition, we have

$$\tau_K(E) = \int_{\nu^{-1}(E)} |\nabla u|^2 x \cdot \nu \, d\mathcal{H}^{n-1}, \tag{48}$$

and

$$S_K(E) = \mathcal{H}^{n-1}(\partial K \cap \nu^{-1}(E)),$$

the implication (46) follows from the fact that

$$|\nabla u|^2 x \cdot \nu > 0 \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial K.$$

Indeed, the term $x \cdot \nu$ remains strictly positive since by assumption K contains the origin in its interior, while the term $|\nabla u|^2$ remains strictly positive by Hopf boundary point lemma (since K admits an inner touching ball at $\mathcal{H}^{n-1}$-a.e. $x \in \partial K$, see e.g. [45]).

Let $E \subset \mathbb{S}^{n-1}$ be a Borel set with $\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}(E) > 0$. From (44), it follows that $\tau_K(E) = c\,\mathcal{H}^{n-1}(E) > 0$. In turn, by (48), this implies that $\mathcal{H}^{n-1}(\nu^{-1}(E)) > 0$, or equivalently that $S_K(E) > 0$, proving (47).

Since we have just proved that K admits a positive curvature function f_K, recalling also the definition of cone torsion measure τ_K, the equality (44) can be reformulated as

$$h_K\, \nu_\sharp(|\nabla u|^2 \mathcal{H}^{n-1} \llcorner \partial K) = \frac{c}{f_K}\, \nu_\sharp(\mathcal{H}^{n-1} \llcorner \partial K).$$

Namely, for every Borel set $E \subset \mathbb{S}^{n-1}$, it holds that

$$\int_{\nu^{-1}(E)} x \cdot \nu |\nabla u|^2 \, d\mathcal{H}^{n-1} = c \int_{\nu^{-1}(E)} \frac{1}{f_K \circ \nu} \, d\mathcal{H}^{n-1}. \tag{49}$$

We now invoke Proposition 1. By statements (a) and (d), we can rewrite the right hand side of the above equality as

$$\int_{\nu^{-1}(E)} \frac{1}{f_K \circ \nu} \, d\mathcal{H}^{n-1} = \int_{\nu^{-1}(E)} G \, d\mathcal{H}^{n-1},$$

so that (49) turns into

$$\int_{\nu^{-1}(E)} x \cdot \nu |\nabla u|^2 \, d\mathcal{H}^{n-1} = c \int_{\nu^{-1}(E)} G \, d\mathcal{H}^{n-1}. \tag{50}$$

By Proposition 1, statements (b) and (c), we know that $\nu \llcorner R(K)$ is a bijection from $R(K)$ to $\nu(R(K))$, with $\mathcal{H}^{n-1}(\partial K \setminus R(K)) = 0$ and $\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus \nu(R(K))) = 0$. In view of this fact, the integral equality (50) yields the pointwise equality

$$|\nabla u| = c^{1/2} \left(\frac{G(x)}{x \cdot \nu(x)} \right)^{1/2} \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial K.$$

Integrating over ∂K, we see that (26) holds.

Having established the validity of (24) and (26), the remaining of the proof can proceed in the analogous way as in Theorem 1. Indeed, the inequalities used in the proof of Theorem 1 do not need any smoothness assumption. In particular, for the

2-affine isoperimetric inequality for convex bodies which admit a curvature function, we refer to [40, Theorem 3] (see also [34, Remark p. 296]). □

As well as the inequalities used in the proof of Theorem 1, also the inequalities used in the proof of Theorems 2 and 3 do not need any smoothness assumption. Thus we obtain the following results for convex bodies with the same cone eigenvalue measure or the same cone capacity measure as a ball. We omit their proofs since they are analogous to the one of Theorem 4 detailed above.

Theorem 5. *Let $K \in \mathcal{K}^n$ have its centroid at the origin, and let σ_K be its cone eigenvalue measure according to Definition 1. Assume that, for some positive constant c,*

$$\sigma_K = c\, \mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1} \text{ as measures on } \mathbb{S}^{n-1}.$$

Then K is a ball.

Theorem 6. *Let $K \in \mathcal{K}^n$, $n \geqq 3$, have its centroid at the origin, and let η_K be its cone capacity measure according to Definition 1. Assume that, for some positive constant c,*

$$\eta_K = c\, \mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1} \text{ as measures on } \mathbb{S}^{n-1}.$$

Then K is a ball.

5. Ultimate Shapes of Variational Flows

In this section we show that, in the smooth centrally symmetric setting, the convex bodies analysed in Sect. 3 can be identified with the ultimate shapes of new variational flows, under the basic assumption that they admit a solution. We present a unified treatment, by considering the evolution problem

$$\begin{cases} \frac{\partial h}{\partial t}(t,\xi) = -aF(C(t))\frac{G(t,\nu_t^{-1}(\xi))}{|\nabla u(t,\nu_t^{-1}(\xi))|^2} & \text{on } (0,T)\times\mathbb{S}^{n-1}, \\ h(0,\xi) = h_0(\xi) & \text{on } \mathbb{S}^{n-1}, \\ h(t,\xi) \leqq b & \text{on } (0,T)\times\mathbb{S}^{n-1}. \end{cases} \tag{51}$$

where F may denote either the torsional rigidity, or the principal Dirichlet Laplacian eigenvalue, or the Newtonian capacity (the latter for $n \geqq 3$), and u is the corresponding ground state.

Let us remark that, in terms of the parametrization of $\partial C(t)$ by its inverse Gauss map $X(t,\cdot) : \mathbb{S}^{n-1} \ni \xi \to \nu_t^{-1}(\xi) \in \partial C(t)$, the evolution equation in (51) can be written as

$$\frac{\partial X}{\partial t}(t,\xi) = -\frac{G(t,X)}{|\nabla u|(t,X)^2}F(C(t))\,\xi \qquad \text{on } (0,T)\times\mathbb{S}^{n-1}$$

(indeed, we have $h(t,\xi) = \nu_t^{-1}(\xi)\cdot\xi$, so that $\frac{\partial h}{\partial t}(t,\xi) = \frac{\partial X}{\partial t}(t,\nu_t^{-1}(\xi))\cdot\xi$).

Thus, up to the factor $F(C(t))$ (which is just a rescaling allowing to have global existence in time), the unique crucial difference between (51) and the classical Gaussian curvature flow is the presence of the squared modulus of the ground state gradient.

We denote by α the homogeneity degree of F under domain dilations (see Sect. 4). The following statement deals with the cases of torsion and first eigenvalue, see however Remark 2 for the case of capacity.

Theorem 7. *Let $F = \mathcal{T}$ or $F = \lambda_1$. Assume that, for some maximal time $T > 0$, there exists a unique family of convex bodies $C(t)$, defined for $t \in [0,T)$, which are smooth (at least of class $\mathcal{C}^3$) and such that their support function $\xi \mapsto h(t,\xi)$ satisfies (51). Then, setting $\gamma = \frac{an\omega_n}{|\alpha|}$,*

(i) it holds that $F(t) = F_0\exp(-\gamma t)$, where F_0 is the energy at $t = 0$; moreover, the family $C(t)$ is strictly decreasing by inclusion;

(ii) the flow is defined for every $t \geqq 0$ (i.e., $T = +\infty$), and the sets $C(t)$ are strictly convex bodies;

(iii) the following entropy is decreasing along the flow:

$$\mathcal{E}(t) := \int_{\mathbb{S}^{n-1}} \log h(t,\xi)\, d\mathcal{H}^{n-1}(\xi) + \gamma n\omega_n t;$$

(iv) assuming in addition that $C(0)$ is centrally symmetric, we have that $C(t)$ is centrally symmetric for every $t \geqq 0$; moreover, if $\{t_n\} \to +\infty$, up to subsequences $\exp(\gamma t_n)C(t_n)$ *converge in Hausdorff distance to a centrally symmetric convex body, called an* ultimate shape *for problem (10), whose cone torsion measure or cone eigenvalue measure is absolutely continuous with respect to $\mathcal{H}^{n-1}\llcorner\mathbb{S}^{n-1}$, with constant density.*

Remark 2. As it can be seen by direct inspection of the proof below, for $F = \text{cap}$, thanks to the Brunn-Minkowski inequality proved in [7], statement (i) continues to hold, but in principle statement (ii) may fail, because the positivity of the capacity does not imply that the corresponding convex body is nondegenerate. However, if it occurs that the flow is defined for all times, then also statements (iii)–(iv) hold true.

Proof. We follow the proof line of [26, Thm. 1, Thm. 2].

(i) Set

$$F_1(K, L) := \frac{1}{\alpha} \lim_{t\to 0^+} \frac{F(K+tL) - F(K)}{t} = \frac{1}{|\alpha|} \int_{\mathbb{S}^{n-1}} h_L \, d\mu_K ,$$

where μ_K is the first variation measure given by (15); recall that, by the Brunn-Minkowski inequality satisfied by F (see [9] for torsion and [16] for the first eigenvalue), it holds that

$$F_1(K, L) \geqq F(K)^{1-\frac{1}{\alpha}} F(L)^{\frac{1}{\alpha}} . \tag{52}$$

Rewrite the equation as

$$\frac{|\nabla u(t, \nu_t^{-1}(\xi))|^2}{G(t, \nu_t^{-1}(\xi))} \lim_{s\to 0} \frac{1}{s}\big[h(t+s, \xi) - h(t, \xi)\big] = -aF(t). \tag{53}$$

Let us denote by $(0, T')$, with $T' \leqq T$, the maximal interval of existence of a smooth solution to this equation.

Integrating over $\mathbb{S}^{n-1}$ with respect to ξ, we obtain

$$|\alpha| \lim_{s\to 0} \frac{F_1(t, t+s) - F(t)}{s} = -an\omega_n F(t),$$

where we used for shortness the notation $F_1(t, t') := F_1(C(t), C(t'))$, $F(t) := F(C(t))$.

Then, using (52), we arrive at

$$-\frac{an\omega_n}{|\alpha|} F(t) \geqq F(t)^{1-\frac{1}{\alpha}} \frac{d}{dt}\Big(F^{\frac{1}{\alpha}}(t)\Big) = \frac{1}{\alpha}\frac{d}{dt} F(t). \tag{54}$$

Consider now the convex body $\widetilde{C}(t)$ with support function $\widetilde{h}(t, \xi) := h(t, \xi) \exp(\gamma t)$. Denoting by $\widetilde{u}$, $\widetilde{\nu}_t$, $\widetilde{G}$, $\widetilde{F}$ respectively its torsion function, Gauss map, Gaussian curvature, and torsional rigidity, it holds that

$$\begin{aligned} &\widetilde{G}(t, \widetilde{\nu}_t^{-1}(\xi)) = \exp\big(-(n-1)\gamma t\big) G(t, \nu_t^{-1}(\xi)), \\ &|\nabla \widetilde{u}(t, \widetilde{\nu}_t^{-1}(\xi))|^2 = \exp\big((\alpha - n)\gamma t\big) |\nabla u(t, \nu_t^{-1}(\xi))|^2, \\ &\widetilde{F}(t) = \exp(\alpha\gamma t) F(t) . \end{aligned}$$

Taking $\gamma = \frac{an\omega_n}{|\alpha|}$, by (54), we have

$$(\operatorname{sign} \alpha) \frac{d}{dt} \widetilde{F}(t) \leqq 0. \tag{55}$$

The equation satisfied by $\widetilde{h}$ is

$$\frac{|\nabla\widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}\Big[\frac{\partial\widetilde{h}}{\partial t}(t,\xi)-\gamma\widetilde{h}(t,\xi)\Big]=-a\widetilde{F}(t). \tag{56}$$

Integrating this equation on $\mathbb{S}^{n-1}$, by the choice of γ and the representation formula for $\widetilde{F}(t)$, we infer that

$$\int_{\mathbb{S}^{n-1}}\frac{|\nabla\widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}\frac{\partial\widetilde{h}}{\partial t}(t,\xi)\,d\mathcal{H}^{n-1}(\xi)=0.$$

Consequently, differentiating under the sign of integral (thanks to the smoothness assumption made on $C(t)$ and standard elliptic boundary regularity),

$$\frac{d}{dt}\widetilde{F}(t)=\frac{1}{|\alpha|}\int_{\mathbb{S}^{n-1}}\frac{\partial}{\partial t}\Big[\frac{|\nabla\widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}\Big]\widetilde{h}(t,\xi)\,d\mathcal{H}^{n-1}(\xi),$$

that we may rewrite as

$$\frac{d}{dt}\widetilde{F}(t)=\lim_{s\to 0}\frac{\widetilde{F}_1(t+s,t)-\widetilde{F}(t)}{s},$$

where we have set $\widetilde{F}_1(t+s,t):=F_1(\widetilde{C}(t+s),\widetilde{C}(t))$. Then, using (52) in a similar way as above, we obtain

$$\frac{d}{dt}\widetilde{F}(t)\geqq\Big[\widetilde{F}(t)^{\frac{1}{\alpha}}\frac{d}{dt}\Big(\widetilde{F}^{1-\frac{1}{\alpha}}(t)\Big)\Big]=\Big(1-\frac{1}{\alpha}\Big)\frac{d}{dt}\widetilde{F}(t),$$

which shows that

$$(\operatorname{sign}\alpha)\frac{d}{dt}\widetilde{F}(t)\geqq 0. \tag{57}$$

By combining (55) and (57), we see that $\widetilde{F}(t)$ is equal to a constant, precisely to $F_0:=\widetilde{F}(0)=F(0)$, and hence

$$F(t)=F_0\exp(-\alpha\gamma t).$$

Since $F(t)>0$ for every $t\in[0,T')$, from the equation we see in particular that the family of convex sets $C(t)$ is strictly decreasing with respect to t.

(ii) We recall that, for a convex body K, we have the inequalities

$$\mathcal{T}(K)\leqq\frac{|K|^{\frac{n+2}{n}}}{n(n+2)\omega_n^{\frac{2}{n}}}\qquad\text{and}\qquad\lambda_1(K)\geqq\frac{\pi^2}{(\text{minimal width}(K))^2} \tag{58}$$

(the first one is the Saint-Venant inequality (19), while for the second one we refer e.g. to [18, Proposition 11]). It follows that $C(t)$ is a nondegenerate convex body for every $t\in(0,T')$. Moreover, the strict convexity of $C(t)$ follows from the fact that, in equation (53), the right hand side is strictly negative. We infer that $T'=+\infty$, so that $T=+\infty$, and $C(t)$ is strictly convex for every $t\geqq 0$.

(iii) Dividing equation (56) by $\frac{|\nabla \widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}\widetilde{h}(t,\xi)$ and recalling that $\widetilde{F}(t) \equiv F_0$, we get

$$\frac{d}{dt}\log \widetilde{h}(t,\xi) - \gamma = -aF_0 \frac{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}{|\nabla \widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}\frac{1}{\widetilde{h}(t,\xi)}.$$

By applying Hölder's inequality

$$\left(\int_{\mathbb{S}^{n-1}} f\right)\left(\int_{\mathbb{S}^{n-1}} \frac{1}{f}\right) \geqq n^2\omega_n^2,$$

with

$$f = \frac{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}{|\nabla \widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}\frac{1}{\widetilde{h}(t,\xi)},$$

we get

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \tfrac{d}{dt}\log \widetilde{h}(t,\xi)\, d\mathcal{H}^{n-1}(\xi) &\leqq \gamma n\omega_n - aF_0 n^2\omega_n^2 \Big[\int_{\mathbb{S}^{n-1}} \tfrac{|\nabla \widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}\widetilde{h}(t,\xi)\Big]^{-1} \\ &= \tfrac{an^2\omega_n^2}{|\alpha|} - aF_0 n^2\omega_n^2\big(|\alpha|F_0\big)^{-1} = 0.\end{aligned}$$

By interchanging derivative and integral on $\mathbb{S}^{n-1}$, we obtain that the entropy $\mathcal{E}(t)$ decreases along the flow.

(iv) The central symmetry of $C(t)$ corresponds to the condition $h(t,\xi) = h(t,-\xi)$, which is satisfied since the map $h(t,\xi)$ is a solution to problem (51), and by assumption such problems admits a unique solution. Now, integrating on $[0,t]$ the inequality $\mathcal{E}' \leqq 0$, we obtain

$$\mathcal{E}(t) = \int_{\mathbb{S}^{n-1}} \log \widetilde{h}(t,\xi)\, d\mathcal{H}^{n-1}(\xi) \leqq \int_{\mathbb{S}^{n-1}} \log \widetilde{h}_0(\xi)\, d\mathcal{H}^{n-1}(\xi) = \mathcal{E}(0).$$

By exploiting the central symmetry of $\widetilde{C}(t)$ this implies, by arguing as in [26, proof of Thm. 2], that the convex bodies $\widetilde{C}(t)$ lie into some fixed ball independent of t; moreover, applying the inequalities (58) it follows that, for every $t \geqq 0$, $\widetilde{C}(t)$ contains a fixed ball centered at the origin. So there exist $R > r > 0$ independent of t such that

$$B_r(0) \subseteq \widetilde{C}(t) \subseteq B_R(0) \qquad \forall t \in [0,+\infty). \tag{59}$$

In particular this implies that the monotone decreasing map $\mathcal{E}(t)$ is bounded from below, and so it converges to a finite limit as $t \to +\infty$; thus we have

$$\lim_{t\to+\infty}\frac{d}{dt}\mathcal{E}(t) = \lim_{t\to+\infty}\frac{d}{dt}\int_{\mathbb{S}^{n-1}} \log \widetilde{h}(t,\xi)\, d\mathcal{H}^{n-1}(\xi) = 0. \tag{60}$$

Set

$$g(t,\xi) := 1 - \frac{1}{\gamma}\frac{\partial}{\partial t}\log \widetilde{h}(t,\xi).$$

We have

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} g(t,\xi)\,d\mathcal{H}^{n-1}(\xi) = n\omega_n - \tfrac{1}{\gamma}\int_{\mathbb{S}^{n-1}} \tfrac{\partial}{\partial t}\log \widetilde{h}(t,\xi)\,d\mathcal{H}^{n-1}(\xi)\\ &\int_{\mathbb{S}^{n-1}} \tfrac{1}{g(t,\xi)}\,d\mathcal{H}^{n-1}(\xi) = \tfrac{\gamma}{aF_0}\int_{\mathbb{S}^{n-1}} \widetilde{h}(t,\xi)\frac{|\nabla\widetilde{u}(t,\widetilde{\nu}_t^{-1}(\xi))|^2}{\widetilde{G}(t,\widetilde{\nu}_t^{-1}(\xi))}\,d\mathcal{H}^{n-1} = n\omega_n,\end{aligned}$$

where to compute the integral of g^{-1} we have used (56). It follows that

$$\int_{\mathbb{S}^{n-1}} \left(\sqrt{g} - \frac{1}{\sqrt{g}}\right)^2 d\mathcal{H}^{n-1}(\xi) = -\frac{1}{\gamma}\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}\log \widetilde{h}(t,\xi)\,d\mathcal{H}^{n-1}(\xi)$$

and hence by (60) we have that $g(t,\xi) \to 1$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$ as $t \to +\infty$. By (59), the support functions $\widetilde{h}(t,\cdot)$ are bounded from above and from below on $\mathbb{S}^{n-1}$ by positive constants independent of t. It follows that

$$\lim_{t\to+\infty} \frac{\partial \widetilde{h}}{\partial t}(t,\xi) = 0 \qquad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{S}^{n-1}. \tag{61}$$

Take now a sequence $\{t_n\} \to +\infty$. By (59) and by the compactness of the Hausdorff distance, $\widetilde{C}(t_n)$ converges to a nondegenerate convex body $\widetilde{C}_\infty$. Then the cone variational measures of $\widetilde{C}(t_n)$ converge weakly* to the cone variational measure of $\widetilde{C}_\infty$ (see Remark 1). Thus, if we set

$$\psi_n(\xi) := \frac{|\nabla\widetilde{u}(t_n,\widetilde{\nu}_{t_n}^{-1}(\xi))|^2}{\widetilde{G}(t_n,\widetilde{\nu}_{t_n}^{-1}(\xi))}\widetilde{h}(t_n,\xi)$$

we have that ψ_n converge weakly in $L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1}\llcorner\mathbb{S}^{n-1})$ to ψ_∞, being ψ_∞ the density of the cone variational measure of $\widetilde{C}_\infty$.

On the other hand, by passing to the limit as $n \to +\infty$ in the equation (56) written at $t = t_n$, recalling that $F(t_n) = F_0$, and exploiting (61), we obtain the following pointwise convergence

$$\lim_n \psi_n(\xi) = \frac{a}{\gamma}F_0 \qquad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{S}^{n-1}. \tag{62}$$

Moreover, we have

$$\lim_n \int_{\mathbb{S}^{n-1}} \psi_n(\xi)\,d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \psi_\infty(\xi)\,d\mathcal{H}^{n-1},$$

(see respectively [22, eq. (23)] for torsion, [38, eq. (3.15)] for capacity and [37, Section 7] for principal eigenvalue), so that ψ_n converge to ψ_∞ strongly in $L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1}\llcorner\mathbb{S}^{n-1})$.

In particular, from (62) we conclude that $\psi_\infty = \frac{a}{\gamma}F_0$, namely the cone variational measure of $\widetilde{C}_\infty$ is a constant multiple of $\mathcal{H}^{n-1}\llcorner\mathbb{S}^{n-1}$. □

6. Concluding Remarks

Remark 3. In dimension $n = 2$ and for K smooth, the existence of a solution to the logarithmic Minkowski problem (13) easily follows from the results proved in [13, Section 6]. Indeed, it is not restrictive to assume $\mathcal{T}(K) = 1$ and to minimize over convex bodies with unit torsion. Let $\{L_j\}$ be a minimizing sequence. By arguing as in the proof of Lemma 6.2 in [13], it is possible to find a sequence of origin-symmetric parallelograms $\{P_j\}$, with $\mathcal{T}(P_j) = 1$ and orthogonal diagonals, such that $P_j \subseteq L_j \subseteq 2P_j$. If $\{L_j\}$ is not bounded, we have that also $\{P_j\}$ is not bounded; moreover, by the assumption $\mathcal{T}(L_j) = 1$, the monotonicity of torsion by inclusions, and Saint-Venant inequality (19), the volume of P_j is bounded from below by a positive constant independent of j. Then we can apply Lemma 6.1 in [13] to infer that the sequence $\int_{\mathbb{S}^1} \log h_{P_j}\, dV_K$ is not bounded from above. Since by Hopf's boundary lemma $|\nabla u_K|^2$ has a strictly positive minimum on ∂K, this implies that also the sequence $\int_{\mathbb{S}^1} \log h_{L_j}\, d\tau_K$ is not bounded from above, contradicting the fact that $\{L_j\}$ is a minimizing sequence. Hence, $\{L_j\}$ remains bounded, and by Blaschke's selection theorem we may select a converging subsequence; its limit turns out to be an origin symmetric convex body with unit torsion which solves problem (13).

Remark 4. In any space dimension, for $K = B$ the logarithmic Minkowski problem (13) is solved uniquely by the ball. Namely, using for brevity the notation $\overline{\tau}_B$ for the normalized cone torsion measure, i.e. $\overline{\tau}_B := \tau_B/\mathcal{T}(B)$, we have

$$\int_{\mathbb{S}^{n-1}} \log\Big(\frac{h_L}{h_B}\Big) d\overline{\tau}_B \geqq \frac{1}{n+2} \log\Big(\frac{\mathcal{T}(L)}{\mathcal{T}(B)}\Big) \qquad \forall L \in \mathcal{K}^n_*.$$

Indeed, it was proved by Guan–Ni [30, Proposition 1.1] that

$$\int_{\mathbb{S}^{n-1}} \log\Big(\frac{h_L}{h_B}\Big) d\overline{V}_B \geqq \frac{1}{n} \log\Big(\frac{|L|}{|B|}\Big) \qquad \forall L \in \mathcal{K}^n_*, \tag{63}$$

with equality if and only if $L = B$, where $\overline{V}_B := V_B/|B|$ denotes the normalized cone volume measure of B. Since B is a ball, we have $\overline{V}_B = \overline{\tau}_B$. Then, by using (63) and the Saint-Venant inequality, we obtain

$$\int_{\mathbb{S}^{n-1}} \log\Big(\frac{h_L}{h_B}\Big) d\overline{\tau}_B \geqq \frac{1}{n} \log\Big(\frac{|L|}{|B|}\Big) \geqq \frac{1}{n+2} \log\Big(\frac{\mathcal{T}(L)}{\mathcal{T}(B)}\Big) \qquad \forall L \in \mathcal{K}^n_*.$$

Remark 5. In dimension $n = 2$, the logarithmic Brunn-Minkowski inequality for the first eigenvalue and for the torsion can be easily tested on the class of rectangles. For any $x > 0$, set $R_x = (0, x) \times (0, 1)$. Given $\ell_1, \ell_2 > 0$, and $\lambda \in (0, 1)$, we have

$$(1-\lambda) \cdot R_{\ell_1} +_0 \lambda \cdot R_{\ell_2} = R_\ell \qquad \text{with } \ell = \ell_1^{1-\lambda} \ell_2^{\lambda}.$$

Then, by using the explicit formulas

$$\begin{aligned} \lambda_1(R_\ell) &= \pi^2\big(1 + \tfrac{1}{\ell^2}\big) \\ \mathcal{T}(R_\ell) &= \tfrac{\ell^3}{12} - \tfrac{16\ell^4}{\pi^5} \textstyle\sum_{k \geq 0} \tfrac{e^{(2k+1)\pi/\ell}-1}{e^{(2k+1)\pi/\ell}+1} \tfrac{1}{(2k+1)^5}\,, \end{aligned}$$

the inequalities

$$\lambda_1(R_\ell) \leqq \lambda_1(R_{\ell_1})^{1-\lambda}\lambda_1(R_{\ell_2})^{\lambda} \quad \text{and} \quad \mathcal{T}(R_\ell) \geqq \mathcal{T}(R_{\ell_1})^{1-\lambda}\mathcal{T}(R_{\ell_2})^{\lambda}$$

are confirmed, respectively, via explicit straightforward computations and via computations done by Mathematica.

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Declarations

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