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The Feynman–Lagerstrom Criterion for Boundary Layers

THEODORE D. DRIVAS, SAMEER IYER & TRINH T. NGUYEN

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Abstract

We study the boundary layer theory for slightly viscous stationary flows forced by an imposed slip velocity at the boundary. According to the theory of Prandtl (in: International mathematical congress, Heidelberg, 1904; see Gesammelte Abhandlungen II, 1961) and Batchelor (J Fluid Mech 1:177–190, 1956), any Euler solution arising in this limit and consisting of a single "eddy" must have constant vorticity. Feynman and Lagerstrom (in: Proceedings of IX international congress on applied mechanics, 1956) gave a procedure to select the value of this vorticity by demanding a *necessary* condition for the existence of a periodic Prandtl boundary layer description. In the case of the disc, the choice—known to Batchelor (1956) and Wood (J Fluid Mech 2:77–87, 1957)—is explicit in terms of the slip forcing. For domains with non-constant curvature, Feynman and Lagerstrom give an approximate formula for the choice which is in fact only implicitly defined and must be determined together with the boundary layer profile. We show that this condition is also sufficient for the existence of a periodic boundary layer described by the Prandtl equations. Due to the quasilinear coupling between the solution and the selected vorticity, we devise a delicate iteration scheme coupled with a high-order energy method that captures and controls the implicit selection mechanism.

1. Introduction

Let $M \subset \mathbb{R}^2$ be a bounded, simply connected domain. Consider the Navier–Stokes equations

$$\partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p^\nu + \nu \Delta u^\nu \qquad \text{in } M, \tag{1}$$

$$\nabla \cdot u^\nu = 0 \qquad \text{in } M. \tag{2}$$

Motion is excited through the boundary, where stick boundary conditions are supplied

$$u^\nu \cdot \hat{n} = 0 \qquad \text{on } \partial M, \tag{3}$$

$$u^\nu \cdot \hat{\tau} = f \qquad \text{on } \partial M, \tag{4}$$

where $\hat{n}$ is the unit outer normal vector field on the boundary and $\hat{\tau} = \hat{n}^\perp$, the unit tangent field. In the above, there is a given autonomous slip velocity $f : \partial M \to \mathbb{R}$, which should be thought of as being generated by motion of the boundary (via so-called "stick" or "no slip" boundary conditions—the fluid velocity on the boundary matches its speed), and is responsible for the generation of complex fluid motions in the bulk. If the viscosity is large relative to the forcing, then it is easy to see that all solutions converge to a unique steady state as $t \to \infty$ [38]. However, as viscosity is decreased, one generally expects solutions to develop and retain non-trivial variation in time, perhaps even forever harboring turbulent behavior. There one general exception to this expectation in a special setting, proved in Sect. 3:

Theorem 1. (Absence of turbulence) *Let $M = \mathbb{D}$ be the disk of radius R and $f = \frac{1}{2}\omega_0 \mathsf{R}$ for any given $\omega_0 \in \mathbb{R}$ be a constant slip on the boundary. For any distributionally divergence-free $u_0 \in L^2$, the unique Leray-Hopf weak solution converges at long time to solid body rotation $u_{\mathsf{sb}} = \frac{1}{2}\omega_0 x^\perp$ having vorticity ω_0. In fact,*

$$\|u(t) - u_{\mathsf{sb}}\|_{L^2} \leq \|u_0 - u_{\mathsf{sb}}\|_{L^2} e^{-\lambda_1 \nu t}$$

where λ_1 is the first positive eigenvalue of $-\Delta$ with Dirichlet boundary conditions on $\mathbb{D}$.

Remark. The forcing (slip velocity) in Theorem 1 can be arbitrarily large and yet for any viscosity Navier–Stokes has a one-point attractor. This is the analogue of Marchioro's results on the absence of turbulence on the torus with 'gravest mode' body forcing [31,32].

Theorem 1 highlights a peculiarity of solid body rotation on the disk: if you center a circular basin on fluid on a record player, all motion will eventually be solid body. A question arises:

Question 1. What if the imposed slip is non-constant, or the domain is not a disk?

As mentioned above, one might expect that if either of these conditions is violated, time dependence generally survives. However, if the boundary forcing is special this need not be the case. For instance, consider the velocity field with constant (unit) vorticity on any M

$$u_* = K_M[1], \qquad K_M := \nabla^\perp \Delta_D^{-1}. \tag{5}$$

Any such velocity field satisfies both the Euler and Navier–Stokes equations in the bulk. As such, it is a stationary solution of Euler, and also of Navier–Stokes provided u_* is taken as initial data and it is forced consistently on the boundary:

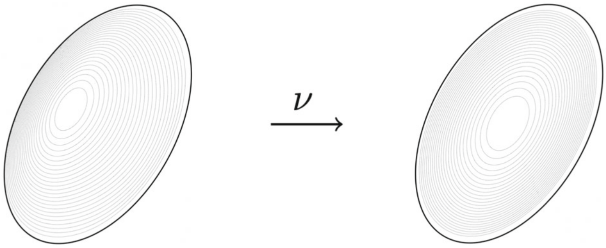

Fig. 1. Cartoon of streamlines of a steady Navier–Stokes solutions on the ellipse forced by an imposed slip $u^\nu \cdot \tau = f$ on the left. Streamlines of constant vorticity Euler solution $\psi_* = \frac{a^2}{2(1+a^2)}(\frac{x^2}{a^2} + y^2)$ shown on the right

$f_* = u_* \cdot \tau$. Thus, for any domain there is a family of non-trivial time-independent solutions uniformly in $\nu \geq 0$.

For force sufficiently close to that generated by a stationary Euler solution, asymptotic stability may occur but is a delicate issue. To begin to understanding these issues, we are interested in the question of the existence of sequence of *stationary* Navier–Stokes solutions approximating an Euler flow in this setting. The Prandtl–Batchelor theory [1,35] provides a restriction on the type of stationary Euler solutions that can arises as inviscid limits. Namely, it stipulates that they have constant vorticity within closed streamlines, so-called "eddies". See also Childress [4,5] and Kim [22,23]. The result, proved in Sect. 4, is:

Theorem 2. (Prandtl–Batchelor Theorem) *Let $M \subset \mathbb{R}^2$ be a simply connected domain with smooth boundary. Let $\psi : M \to \mathbb{R}$ be a $W^{4,1+}(M)$ streamfunction of a steady, non-penetrating solution of Euler u_e having a single stagnation point which is non-degenereate in a sense that the period of revolution of a particle is a differentiable function of the streamline. Suppose $\{u^\nu\}_{\nu>0}$ is a family satisfying* (1), (2) *together with*

$$\lim_{\nu\to 0} \|u^\nu - u_e\|_{H^{5/2+}(U)} \to 0, \tag{6}$$

for all interior open subsets $U \subset M$. Then $u_e = \omega_0 u_*$ for a constant $\omega_0 \in \mathbb{R}$ and u_* is (5).

See Fig. 1. for a cartoon of this convergence. In the above theorem, M can be thought of as a streamline of an Euler solution occupying some larger spatial domain. If the limiting Euler solution consists of multiple eddies, the above shows that, within each eddy, the vorticity tends to become constant. The vorticity of the resulting solution would be a staircase landscape separated, perhaps, by vortex sheets. Such a picture is consistent with the general expectation of the emergence of weak solutions in the inviscid limit on bounded domains [7,8,12]. We remark that similar selection principles to Theorem 2 appear also in two-dimensional passive scalar problems [33,36], and in steady heat distribution in three-dimensional integrable magnetic fields [10].

If the boundary data is a sufficiently small perturbation of the corresponding slip of an unit vorticity Euler flow u_* on whole vessel M,

$$f = u_* \cdot \tau + \varepsilon g. \tag{7}$$

then the inviscid limit of steady Navier–Stokes solutions might be expected to consist of just a single eddy having constant vorticity (5), that is, for some appropriate constant $\omega_0 \in \mathbb{R}$,

$$u^\nu \to \omega_0 u_* \qquad \text{as} \qquad \nu \to 0. \tag{8}$$

See Conjecture 1. This naturally leads to the following question:

Question 2. Given boundary data (3), (4), (7), how is the limiting vorticity ω_0 (8) selected?

This question was discussed by Batchelor (1956) [1] and Wood (1957) [40] for disk domains, and the resulting prediction is called the Batchelor–Wood formula. This analysis was done independently by Feynman–Lagerstrom (1956) [14] who also generalized this formula to domains with non-constant curvature. See also [29,30]. The idea is: Navier–Stokes with small viscosity should approximate Euler (constant vorticity) in the bulk of the domain, and interpolate to the given boundary conditions across a layer of width $\sqrt{\nu}$. In this layer, predicted by Prandtl [35], the leading order behavior of the fluid is captured by a simpler boundary layer equation which is supplied with data at infinity from Euler and at zero (the boundary), from Navier–Stokes. The complication is that the Euler solution is known only up to a constant multiple. The specific vorticity value ω_0 is then fixed by demanding the corresponding Prandtl equation for the boundary layer admits a periodic solution.

We note that the (Prandtl–Batchelor or Feynman–Lagerstrom) theory was developed to address the specific phenomenon which only arises for stationary flows on closed domains. For the much different setting of unsteady flows, the question of which Euler flow is achieved in the inviscid limit is essentially completely determined by the initial data. For stationary flows on non-closed domains, for example on $[0, L]\times\mathbb{R}_+$, there is an analogue of data prescribed on the sides $\{x = 0\}$, $\{x = L\}$ which similarly fixes the inviscid Euler flow (see for instance, results of [15], [21] for results in this direction). In contrast, for closed domains, the only prescribed data is the slip boundary data, $f(\theta)$. Therefore, the selection mechanism is less obvious to uncover, and historically motivated investigations of Prandtl–Batchelor or Feynman–Lagerstrom, and from a rigorous standpoint, the results in this paper.

On the disk $M = \mathbb{D}$ of radius R, this amounts to:

$$\overline{\omega_0}_n = \frac{⨍_{\mathbb{T}_L}(2gq_e^2 + \varepsilon g^2 q_e)\mathrm{d}s}{⨍_{\mathbb{T}_L} q_e^3 \mathrm{d}s} - \frac{1}{\omega_{0n-1}\varepsilon}\frac{\int_0^\infty y f(Q_{n-1}(y,s), s; \omega_{0n-1})\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3 \mathrm{d}s}. \tag{9}$$

This picture has been rigorously justified by Kim [24,25] for the boundary layer and recently by Fei, Gao, Lin and Tao [13] for Navier–Stokes. The latter constructs a sequence of steady Navier–Stokes solutions on the disk forced by (7) converging towards this predicted end state. In the case of a general domain, Feynman–Lagerstrom argued that selecting ω_0 to ensure a certain periodicity is a necessary

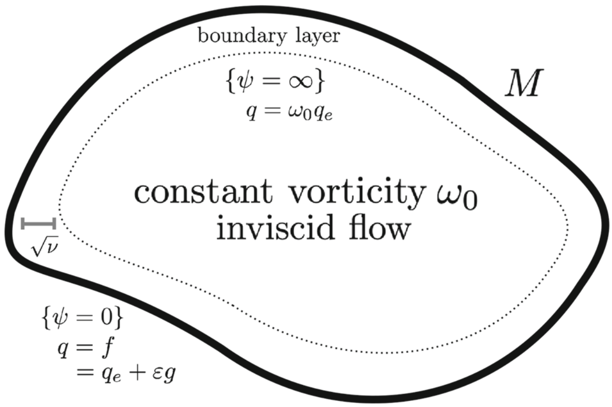

Fig. 2. Boundary layer geometry, depicted (abusing the required asymptotic $\nu \to 0$ for the sake of illustration) in von Mises coordinates. The level $\psi \in [0, \infty)$ denotes the distance from the boundary in a layer of size $\sqrt{\nu}$, upon rescaling so that this extends indefinetely. See Sect. 2. The unique vorticity value ω_0 so that such a boundary layer exists is selected nonlinearly via (12). It is given approximately by the Feynman-Lagerstrom formulae (13) or (14)

condition for the existence of such a layer and therefore for convergence, but did not speak to its sufficiency. We now review their theory.

Recall Prandtl's boundary layer equations written in von Mises coordinates $(s, \overline{\psi})$, where s is the periodic coordinate on the boundary and $\bar{\psi} := \psi/\sqrt{\nu}$ is the rescaled streamline coordinate. From hereon, we denote $\bar{\psi} = \psi$ and

$$q_e(s) := u_* \cdot \hat{\tau}(\gamma(s))$$

where $\gamma : [0, |\partial M|) \to \partial M$ is the arc-length parametrization of the boundary, is the tangential slip along the boundary of unit vorticity Euler solutions (5). The Prandtl equations—which determine an unknown function $q : [0, L) \times \mathbb{R}^+ \to \mathbb{R}$ which serves as an approximation of the tangential Navier–Stokes velocity $u^\nu \cdot \hat{\tau}$ in an $O(\sqrt{\nu})$ boundary layer—are (see [34]):

$$\partial_s Q - q\partial_\psi^2 Q = 0, \qquad Q = q^2 - \omega_0^2 q_e^2, \tag{10}$$

which is to be satisfied on $(\psi, s) \in [0, \infty) \times [0, L)$ where $L = |\partial M|$ is the length of the boundary. For completeness, we derive these equations in Sect. 2. The solution q must connect Navier–Stokes to Euler: at the boundary ($\psi = 0$), the solution q takes the Navier–Stokes data and away from the boundary ($\psi = \infty$), the solution q assumes the Eulerian behavior:

$$q(0, s) = f(s), \qquad q(\infty, s) = \omega_0 q_e(s)$$

which, for Q, translates to the data

$$Q(0, s) = f^2(s) - \omega_0^2 q_e^2(s), \qquad Q(\infty, s) = 0.$$

The solution q (or, equivalently, Q) of Eq. (10) must be periodic in the s variable (so that the boundary layer closes). See Fig. 2. Feynman–Lagerstrom noted that this leads to a self-consistency condition on ω_0. We will enforce this in the following way. First, we rewrite (10) as

$$\partial_s Q - \omega_0 q_e \partial_\psi^2 Q = \Big(1 - \frac{\omega_0 q_e}{\sqrt{\omega_0^2 q_e^2 + Q}}\Big)\partial_s Q.$$

Integrating the above equation over the boundary, we obtain

$$\partial_\psi^2 \int_0^L q_e(s) Q(\psi, s) \mathrm{d}s = \mathcal{N}^{\omega_0}[Q]$$

where the nonlinearity is explicitly

$$\mathcal{N}^{\omega_0}[Q](\psi) := -\frac{1}{\omega_0} \int_0^L \left(1 - \frac{\omega_0 q_e(s)}{\sqrt{\omega_0^2 q_e^2(s) + Q}}\right) \partial_s Q \mathrm{d}s. \tag{11}$$

Then, for some scalars A and B, we obtain the identity

$$\int_0^L q_e(s) Q(\psi, s) \mathrm{d}s = A + B\psi - \int_\psi^\infty (\psi - y) \mathcal{N}^{\omega_0}[Q](y) \mathrm{d}y.$$

From the boundary conditions, we have that $A = B = 0$. We thus obtain the nonlinear condition that at each $\psi \in \mathbb{R}_+$, we have

$$\int_0^L q_e(s) Q(\psi, s) \mathrm{d}s = \int_\psi^\infty (y - \psi) \mathcal{N}^{\omega_0}[Q](y) \mathrm{d}y.$$

Evaluating at $\psi = 0$, we find a nonlinear, nonlocal condition determining the constant ω_0:

$$\int_0^L q_e(s)(f^2(s) - \omega_0^2 q_e^2(s)) \mathrm{d}s = \int_0^\infty y \mathcal{N}^{\omega_0}[Q](y) \mathrm{d}y. \tag{12}$$

Remark. (Feynman–Lagerstrom formulae) Letting $1 - \omega_0^2 =: \varepsilon \overline{\omega}_0$ with $\overline{\omega}_0 = O(1)$, we anticipate $Q = O(\varepsilon)$. Since $\frac{1}{\sqrt{\omega_0^2 + x}} - \frac{1}{\sqrt{\omega_0^2}} = -\frac{x}{2\omega_0^3} + O(x^2)$, we see that $\mathcal{N}^{\omega_0}[Q](\psi) = O(Q^2) = O(\varepsilon^2)$. Moreover $\mathcal{N}^{\omega_0}[Q](y)$ is trivial in the case of the boundary having constant curvature $\kappa := \hat{\tau} \cdot \nabla \hat{n} \cdot \hat{\tau}$. Indeed, in this case of M being a disk, it is readily seen that the integrand in (11) is a total derivative in s and hence $\mathcal{N}^{\omega_0}[Q](\psi) \equiv 0$), see the next Remark. Thus, as pointed out by Feynman and Lagerstrom [14], to leading order, condition (12) is

$$\omega_0^2 = \frac{\int_0^L q_e(s) f^2(s) \mathrm{d}s}{\int_0^L q_e^3(s) \mathrm{d}s} + o(|\partial_s \kappa|). \tag{13}$$

This formula is exact (having $\partial_s \kappa = 0$) when M is the disk, and generally only for the disk.[1] Recalling $f(s) = q_e(s) + \varepsilon g(s)$ so that $f^2(s) - \omega_0^2 q_e^2(s) = (1 -$

[1] Among domains with smooth boundary. For Lipschitz domains, it holds also for regular polygons [39].

$\omega_0^2)q_e^2(s) + 2\varepsilon g(s)q_e(s) + \varepsilon^2 g^2(s)$, to leading order in ε (the deviation of NS data from unit vorticity Euler slip) we have

$$\omega_0^2 = 1 + 2\varepsilon \frac{\int_0^L q_e^2(s)g(s)\mathrm{d}s}{\int_0^L q_e^3(s)\mathrm{d}s} + O(\varepsilon^2). \tag{14}$$

Remark. (Wood's formula when $M = \mathbb{D}$) On the disk of radius R, the constant vorticity solution (5) is solid body rotation $u_*(x) = \frac{1}{2}x^\perp$, so that $q_e = \frac{\mathsf{R}}{2}$ is a constant. In fact, by Serrin's theorem [37] constraining a domain admitting a solution of $\Delta\psi = 1$ with constant Neumann and Dirichlet data, the disk is the unique domain for which the solid body Euler solution has constant boundary slip velocity q_e. See also [26]. On the disk, (13) without an error term is exact and, with the circumference $L = 2\pi\mathsf{R}$, agrees with (9) of Wood [40].

In this paper, we rigorously establish this prediction by constructing a periodic boundary layer verifying the Feynman–Lagerstrom condition if the constant vorticity Euler solution u_* on M defined by (5) has no stagnation points on the boundary. We have

Theorem 3. (Existence of a periodic Prandtl boundary layer) *Let M be a simply connected domain with $L = |\partial M|$. Denote $\mathbb{T}_L = [0, L)$. Let $q_e : \mathbb{T}_L \to \mathbb{R}$ be a smooth, non-vanishing function. Let $f(s) := q_e(s) + \varepsilon g(s)$ with smooth $g : \mathbb{T}_L \to \mathbb{R}$. For all ε sufficiently small, depending only the data (q_e, g), there exists a unique constant $\omega_0 \in \mathbb{R}$ and function $q : \mathbb{T}_L \times \mathbb{R}^+ \to \mathbb{R}$ such that the pair (ω_0, q) solves the Prandtl equations on $\mathbb{T}_L \times \mathbb{R}^+$:*

$$\begin{aligned} \partial_s Q - q\partial_\psi^2 Q = 0, \qquad Q = q^2 - \omega_0^2 q_e^2, \\ Q(s,0) = f^2(s) - \omega_0^2 q_e^2(s), \\ Q(s,\infty) = 0. \end{aligned} \tag{15}$$

Moreover, the solution Q lies in the space $X_{2,50}$ defined by (25) and enjoys $\|Q\|_{X_{2,50}} \lesssim \varepsilon$. The selected vorticity ω_0 can be expressed as follows: there exists a constant $C > 0$ so that

$$\omega_0^2 = \frac{\int_0^L q_e(s) f^2(s)\mathrm{d}s}{\int_0^L q_e^3(s)\mathrm{d}s} + \omega_{\mathrm{Err}} \quad \textit{where} \quad |\omega_{\mathrm{Err}}| \le C\varepsilon^2. \tag{16}$$

The sign of ω_0 agrees with that of the background q_e which, in this case, is positive.

This theorem, proved in Sect. 5, provides the first rigorous confirmation of the Feynman–Lagerstrom formula, and justifies their claim that for $|f - q_e| \ll |f|$ (translating to $\varepsilon \ll 1$), the leading term in (13) serves as a good approximation for the selected vorticity.

The constant ω_0 satisfies (12), and has an explicit component, determined by $q_e(s)$ and $f(s)$, as well as an implicit component $\overline{\omega}_{\mathrm{Err}}$ which is smaller amplitude and for which we obtain bounds. We emphasize that the ω_0 appearing in (15) is *nonlinearity selected* as soon as the domain, M, is no longer a disk (for example,

M is an ellipse). This requires, at an analytical level, a delicate coupling between the choice of constant, ω_0, and the control of the solution Q in an appropriately chosen norm, which is the main innovation of our work.[2]

We anticipate the Prandtl system, which we analyze in this paper, to be stable in the inviscid limit for the full Navier–Stokes system. Indeed, this is what is proved in [13] when M is a disk and $u_e = x^{\perp}$. However, as discussed above, in that very special setting the constant ω_0 can be explicitly determined (9). In general, this is not the case and ω_0 is only implicitly determined by the condition (12) described above, making the inviscid limit more delicate. Nevertheless, we believe that the nonlinearly determined constant ω_0 will describe, to leading order in viscosity, the selection principle. That is, we believe that Navier–Stokes vorticity ω^{ν} should obey an asymptotic expansion

$$\omega^{\nu} = \omega_0 + O(\sqrt{\nu}) \tag{17}$$

in the interior of the domain. In fact, we issue the following

Conjecture 1. Let $M \subset \mathbb{R}^2$ be any simply connected domain such that the constant vorticity Euler solution u_* on M defined by (5) has a single eddy (streamfunction has a single, non-degenerate, critical point). Suppose that Navier–Stokes is forced by a slip of the form (7), e.g. $f = u_* \cdot \tau + \varepsilon g$ for some smooth function $g : \partial M \to \mathbb{R}$. Then, there exists an $\varepsilon_* := \varepsilon_*(M, g)$ such that for all $\varepsilon < \varepsilon_*$ we have weak convergence in $L^2(M)$

$$u^{\nu} \rightharpoonup \omega_0 u_* \qquad \text{as} \qquad \nu \to 0$$

along a sequence of steady Navier–Stokes solutions, where $\omega_0 := \omega_0(M, g)$ is (16) of Thm 3.

Of course, stronger convergence can be expected, along with a boundary layer description such as that established by [13] on the disk. The fact that moving boundaries can stabilize the inviscid limit is a well known phenomenon from the work of Guo and Nguyen [16] and Iyer [19,20]. Verifying the expansion (17) to prove the above conjecture will require substantially new ideas. In the context of elliptical domains M, this is work in preparation.

Finally we remark that the failure of a boundary layer to exist is indicative of the existence of multiple eddies: constant vorticity regions are separated by internal layers which can be thought of as free boundaries. This can happen either if the constant vorticity solution on that domain has multiple eddies, or if the given slip data is far from that of a constant vorticity slip (according to our Theorem 3). Kim [26] showed that if the Navier–Stokes boundary slip is only slightly negative in places, the Prandtl–Batchelor theory still applies to good approximation in the bulk. For the situation of being far from compatible slip data, see Kim and Childress [28] for an analytical investigation on a rectangle, Greengard and Kropinski [17]

[2] In this respect, the selection mechanism is similar to another arising in fluid dynamics: inviscid damping [2]. There, perturbations to certain stable shear flows return to equilibrium in a weak sense, but the which equilibrium they converge to must determined together with the entire time history of the solution.

for a numerical investigation on disk domains, and Henderson, Lopez and Stewart [18] for laboratory experiments.

2. Derivation of the Prandtl Boundary Layer Equation

In this section, we derive the Prandtl equations for any simply connected domain M. Assume that $s : [0, L] \to \partial M$ be the arc-length parametrization of the boundary ∂M. For $s \in \mathbb{T}_L$, let $\tau(s)$ and $n(s)$ be unit the tangential vector to the boundary ∂M. There exists $\delta > 0$ such that for any $x \in M$ such that $\mathrm{dist}(x, \partial M) < \delta$, there exists a unique $s \in \mathbb{T}_L$ and $x(s) \in \partial M$ such that

$$\mathrm{dist}(x, \partial M) = |x - x(s)|.$$

Moreover, one has the representation

$$x(s, z) = x(s) + zn(s),$$

where $x(s) \in \partial M$ and $z = \mathrm{dist}(x, \partial M)$, see [3]. The map

$$\begin{aligned} \{x \in M : \quad 0 < \mathrm{dist}(x, \partial M) < \delta\} &\to \mathbb{T}_L \times (0, \delta) \\ x &\to (z, s) \end{aligned}$$

is a diffeomorphism. We also define the following quantities for the domain M:

$$\gamma(s) = x_1''(s)x_2'(s) - x_1'(s)x_2''(s), \quad J(z, s) = 1 + z\gamma(s) > 0,$$

where γ represents the boundary curvature, and J is a Jacobian for a near-wall mapping used to derived the following form of Navier–Stokes, see [3] and Appendix 6.

Now for $x = x(s, z)$, we denote $\tau(s)$ and $n(s)$ to be the tangential and normal vector at $x(s) \in \partial M$ on the boundary. Consider the steady Navier–Stokes equations

$$\begin{aligned} u^\nu \cdot \nabla u^\nu + \nabla p^\nu &= \nu \Delta u^\nu, \\ \nabla \cdot u^\nu &= 0, \end{aligned}$$

written in the region $\mathrm{dist}(x, \partial M) < \delta$. We define

$$\begin{aligned} u_\tau(s, z) &= u^\nu(x) \cdot \tau(s) = u^\nu(x(s, z)) \cdot \tau(s), \\ u_n(s, z) &= u^\nu(x) \cdot n(s) = u^\nu(x(s, z)) \cdot n(s). \end{aligned}$$

By direct calculation, provided in Appendix 6, the Navier–Stokes equations become

$$\begin{aligned} &\frac{u_\tau}{J}\partial_s u_\tau + u_n \partial_z u_\tau - \frac{\gamma}{J} u_\tau u_n + \frac{1}{J}\partial_s p \\ &\quad = \nu \left\{ \frac{1}{J}\partial_z (J \partial_z u_\tau) + \frac{1}{J}\partial_s \left(\frac{1}{J}\partial_s u_\tau \right) - \frac{1}{J}\partial_s \left(\frac{\gamma u_n}{J} \right) - \frac{\gamma}{J} (\gamma u_\tau + \partial_s u_n) \right\}, \\ &\frac{u_\tau}{J}\partial_s u_n + u_n \partial_z u_n - \frac{\gamma}{J} u_\tau^2 + \partial_z p \\ &\quad = \nu \left\{ \frac{1}{J}\partial_z (J \partial_z u_n) + \frac{1}{J}\partial_s \left(\frac{1}{J}\partial_s u_n \right) - \frac{1}{J}\partial_s \left(\frac{\gamma u_\tau}{J} \right) - \frac{\gamma}{J} (\partial_s u_\tau - \gamma u_n) \right\}, \\ &\partial_z u_n + \frac{1}{J}\partial_s u_\tau - \frac{\gamma}{J} u_n = 0. \end{aligned}$$

Remark. On the disk with the usual polar coordinates $(\theta, r) \in \mathbb{T} \times [0, 1]$, we have

$$\gamma(\theta) = -1, \quad J(z, \theta) = \frac{1}{r}, \qquad u_\tau = u_\theta, \quad u_n = u_r.$$

Near the boundary, in a layer of width $\sqrt{\nu}$, we anticipate that Navier–Stokes velocity field (u_τ, u_n) will look like a small boundary layer correction (u_τ^P, v_n^P), to a constant vorticity $(\omega_0 q_e, 0)$ Euler flow, as discussed in the introduction. That is,

$$u_\tau(s, z) \sim \omega_0 q_e(s) + u_\tau^P(s, Z), \qquad u_n(s, z) \sim \sqrt{\nu} v_n^P(s, Z), \qquad Z = \frac{z}{\sqrt{\nu}},$$

where $\lim_{Z\to\infty} u_\tau^P(s, Z) = 0$. See discussion in Oleinik and Samokhin [34]. Plugging in this ansatz into the Navier–Stokes equations near the boundary, and using the approximation

$$\frac{1}{J} = \frac{1}{1+z\gamma(s)} = \frac{1}{1+\sqrt{\nu}Z\gamma(s)} \sim 1 - \sqrt{\nu}Z\gamma(s) + O(\nu),$$

we obtain the equations

$$\begin{aligned} &(\omega_0 q_e(s) + u_\tau^P)\partial_s(\omega_0 q_e(s) + u_\tau^P) + v_n^P \partial_Z(q_e(s) + u_\tau^P) + \partial_s p - \partial_Z^2 u_\tau^P = 0, \\ &\partial_Z p = 0 \end{aligned} \tag{18}$$

along with the divergence free condition

$$\partial_s(\omega_0 q_e(s) + u_\tau^P) + \partial_Z v_n^P = 0$$

Taking $Z \to \infty$ in the Eq. (18), we obtain

$$\partial_s p = -\omega_0^2 q_e(s) q_e'(s)$$

Replacing the pressure by the above into the equation (18), we obtain the *Prandtl equations*:

$$\begin{aligned} &(\omega_0 q_e(s) + u_\tau^P)\partial_s(\omega_0 q_e(s) + u_\tau^P) \\ &\quad + v_n^P \partial_Z(\omega_0 q_e(s) + u_\tau^P) - \omega_0^2 q_e(s) q_e'(s) - \partial_Z^2 u_\tau^P = 0 \\ &\partial_s(\omega_0 q_e(s) + u_\tau^P) + \partial_Z v_n^P = 0. \end{aligned}$$

Define the von Mises variables (s, ψ) such that

$$\partial_Z \psi(s, Z) = \omega_0 q_e(s) + u_\tau^P(s, Z), \qquad -\partial_s \psi(s, Z) = v_n^P(s, Z).$$

Let $q = q(s, \psi) = \omega_0 q_e(s) + u_\tau^P$, the Prandtl equation becomes

$$q(\partial_s q - v_n^P \partial_\psi q) + q v_n^P \partial_\psi q - \frac{1}{2}\partial_s\left(\omega_0^2 q_e(s)^2\right) - q\partial_\psi(q\partial_\psi q) = 0$$

which reduces to

$$\partial_s q^2 - \omega_0^2 \partial_s q_e^2 - q\partial_\psi^2 q^2 = 0.$$

Letting $Q = q^2 - \omega_0^2 q_e^2$, the above equation becomes (10), namely (See Fig. 3 for a visualization)

$$\partial_s Q - q\partial_\psi^2 Q = 0.$$

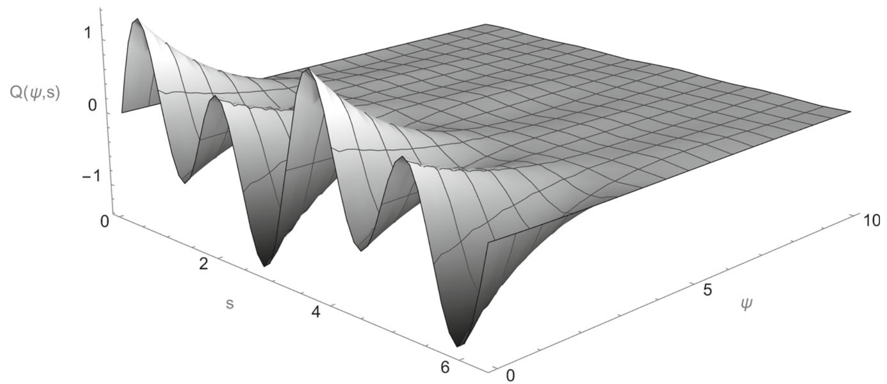

Fig. 3. Numerical solution Q of (10) for some representative data q_e and f. Thinking of s at “time", the equation is nonlinear heat sourced at the wall $\psi = 0$. Decay away from the boundary is rapid (exponential) in ψ

3. Proof of Theorem 1: Absence of Turbulence

Let $v^\nu = u^\nu - u_e$ be the difference of solutions of Euler and Navier–Stokes, where Navier–Stokes is forced by Euler's slip velocity. On general domains M, it satisfies

$$\begin{aligned}
&\partial_t v^\nu + (v^\nu + u_e)\cdot\nabla v^\nu + v^\nu\cdot\nabla u_e = -\nabla q + \nu\Delta v^\nu + \nu\Delta u_e && \text{in } M,\\
&\nabla\cdot v^\nu = 0 && \text{in } M,\\
&v^\nu\cdot\hat{n} = 0 && \text{on } \partial M,\\
&v^\nu\cdot\hat{\tau} = 0 && \text{on } \partial M.
\end{aligned}$$

Whence the error energy (which holds for Leray-Hopf solutions in dimension two) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^\nu\|_{L^2}^2 \le -\int_M v^\nu\cdot\nabla u_e\cdot v^\nu \mathrm{d}x - \nu\|\nabla v^\nu\|_{L^2}^2 + \nu\int v^\nu\cdot\Delta u_e.$$

In general, we may bound

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^\nu\|_{L^2}^2 \le \left(\|\nabla u_e\|_{L^\infty} - \frac{\nu\lambda_1}{2}\right)\|v^\nu\|_{L^2}^2 + \frac{2\nu}{\lambda_1}\|\Delta u_e\|_{L^2}^2,$$

since $v^\nu|_{\partial\mathbb{D}} = 0$ so we may apply the Poincaré inequality $\lambda_1\|v^\nu\|_{L^2} \le \|\nabla v^\nu\|_{L^2}^2$ where λ_1 is the first positive eigenvalue of $-\Delta_D$ on M. We remark, using the results of [38, Chapter 7] (which establish uniform bounds on the steady states), a similar energy identity can be used to prove global attraction of the unique steady state for Navier–Stokes forced by imposed slip on any domain, provided viscosity is large enough.

On the disk $M = \mathbb{D}$, if $u_e = u_{\mathsf{sb}} = \omega_0 x^\perp$ so that $\nabla u_{\mathsf{sb}} = \omega_0\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $\Delta u_e = 0$, we have

$$\int_{\mathbb{D}} v^\nu\cdot\nabla u_{\mathsf{sb}}\cdot v^\nu\mathrm{d}x = \int_{\mathbb{D}} v^\nu\cdot(v^\nu)^\perp\mathrm{d}x = 0.$$

On the disk of radius R, this is $\lambda_1 = (j_0/\mathsf{R})^2$ where j_0 is the first zero of J_0 the Bessel function of the first kind and order zero). We thus have the stated result.

Remark. On the ellipse $u_{\mathsf{sb}} = \omega_0(-y, \alpha x)$ so that $\nabla u_{\mathsf{sb}} = \omega_0\begin{pmatrix}0 & -1\\ \alpha & 0\end{pmatrix}$ and $v\cdot\nabla u_{\mathsf{sb}}\cdot v = \omega_0(\alpha-1)v_1v_2$. It follows that provided

$$\nu > \nu_* := \omega_0\lambda_1^{-1}(1-\alpha),$$

then the solid body rotation solution is the global attractor. In particular, as the eccentricity of the elliptical domain goes to zero, $\alpha\to 1$ and the critical viscosity ν_* goes to zero. Curiously, all flows in this elliptical family are isochronal [41], meaning that the period of revolution of a particle does not depend on the particular streamline. As such, the form examples of cut points in group of area preserving diffeomorphisms of those domains, see discussion in [9,11]. The lack of differential rotation in the Euler solution may have important consequences for the asymptotic stability and realizability in the inviscid limit.

4. Proof of Theorem 2: Prandtl–Batchelor Theory

First, by [6, Lemma 5], under the stated assumptions we have that

$$\begin{aligned} \Delta\psi &= F(\psi) \quad \text{on } M, \\ \psi &= c_* \quad\quad \text{on } \partial M \end{aligned}$$

for some C^1 function $F : \mathbb{R} \to \mathbb{R}$ and constant $c_* \in \mathbb{R}$. Suppose without loss of generality that $\{\psi = 0\}$ is the unique critical point in M, so that $\text{rang}(\psi) = [0, c_*]$. By the assumption (6), we have the convergence $\psi^\nu \to \psi$ in $H^{7/2+}(U)$ and thus in $C^1(U)$ for all interior open subsets $U \subset M$. It follows that we have convergence of the streamlines (level sets of ψ^ν). Specifically, for any $c \in \text{rang}(\psi)$, the set $\{\psi^\nu = c\}$ is a closed streamline (at least for sufficiently small $\nu := \nu(c)$) converging to $\{\psi = c\}$. In what follows, for fixed c we assume ν is sufficiently small for the above to hold.

Integrating the Navier–Stokes vorticity balance in the sublevel set $\{\psi^\nu \leq c\}$

$$\begin{aligned} 0 &= \int_{\{\psi^\nu \leq c\}} \left[u^\nu \cdot \nabla\omega^\nu - \nu\Delta\omega^\nu \right] \mathrm{d}x \\ &= \int_{\{\psi^\nu = c\}} \left[(u^\nu \cdot \hat{n}^\nu)\omega^\nu - \nu\hat{n}^\nu \cdot \nabla\omega^\nu \right] \mathrm{d}\ell = -\nu \int_{\{\psi^\nu = c\}} \hat{n}^\nu \cdot \nabla\omega^\nu \mathrm{d}\ell, \end{aligned}$$

where $\hat{n}^\nu = \nabla\psi^\nu / |\nabla\psi^\nu|$ is the unit normal to streamlines $\{\psi^\nu = c\}$. Thus

$$\begin{aligned} \int_{\{\psi = c\}} \hat{n} \cdot \nabla\omega \mathrm{d}\ell &= \int_{\{\psi = c\}} \hat{n} \cdot \nabla\omega \mathrm{d}\ell - \int_{\{\psi^\nu = c\}} \hat{n}^\nu \cdot \nabla\omega^\nu \mathrm{d}\ell \\ &= \int_{\{\psi^\nu = c\}} \hat{n}^\nu \cdot \nabla(\omega - \omega^\nu) \mathrm{d}\ell - \int_\Omega \Delta F(\psi) \mathbf{1}_{\{\psi^\nu = c\}\Delta\{\psi = c\}} \mathrm{d}x \end{aligned}$$

where $A\Delta B$ denotes the symmetric difference between two sets. Under our assumptions, there exists an open set $O \subset M$ containing the streamline $\{\psi^\nu = c\}$ uniformly in ν. By the trace theorem,

$$\begin{aligned} \int_{\{\psi^\nu = c\}} n^\nu \cdot \nabla(\omega^\nu - \omega) \mathrm{d}\ell &\lesssim \|n^\nu\|_{H^{1/2+}(O)} \|\omega^\nu - \omega\|_{H^{3/2+}(O)} \\ &\lesssim \|\omega^\nu\|_{H^{3/2}(O)} \|\omega^\nu - \omega\|_{H^{3/2+}(O)}. \end{aligned}$$

Combined with the fact that $\omega = F(\psi)$, we find

$$F'(c) \int_{\{\psi = c\}} u \cdot \mathrm{d}\ell = \int_{\{\psi^\nu = c\}} \Delta(\omega^\nu - F(\psi)) \mathrm{d}x - \int \Delta F(\psi) \mathbf{1}_{\{\psi^\nu = c\}\Delta\{\psi = c\}} \mathrm{d}x.$$

Thus, for any $\delta > 0$, we have the bound

$$\begin{aligned} \left| F'(c) \int_{\Gamma(c)} u \cdot \mathrm{d}s \right| &\lesssim \|\omega^\nu\|_{H^{3/2}(O)} \|\omega^\nu - \omega\|_{H^{3/2+}(O)} \\ &\quad + \|\Delta F(\psi)\|_{L^{1+\delta}(M)} \left(\text{Area} \left(\{\psi^\nu = c\} \Delta \{\psi = c\} \right) \right)^{1/\delta}. \end{aligned}$$

Consequently, using (6) and taking the limit of the upper bound, we have

$$F'(c)\int_{\Gamma(c)} u\cdot \mathrm{d}s = 0.$$

By our hypotheses that ψ has a single stagnation point $\{\psi = 0\}$ in M, the circulation $\oint_{\{\psi=c\}} u\cdot \mathrm{d}\ell \neq 0$ for all $c \neq 0$. Thus, since F' is continuous, we must have that $F'(c) = 0$ for all $c \in \mathrm{rang}(\psi)$ so that $F = \omega_0$ for some $\omega_0 \in \mathbb{R}$.

5. Proof of Theorem 3

5.1. Iteration and Bootstraps

Here we produce a unique solution (Q, ω_0) of

$$\begin{aligned} &\partial_s Q - q\partial_\psi^2 Q = 0,\\ &Q := q^2 - \omega_0^2 q_e^2,\\ &Q(s,0) = \varepsilon\overline{\omega_0} q_e^2(s) + 2\varepsilon g(s) q_e(s) + \varepsilon^2 g^2(s),\\ &Q(s,\infty) = 0, \end{aligned} \tag{19a}$$

on $(s,\psi) \in \mathbb{T}\times\mathbb{R}^+$, for arbitrary $g : \mathbb{T} \to \mathbb{R}$ and sufficiently small $\varepsilon := \varepsilon(g; q_e)$. Here, ω_0 is to be determined together with Q, and we introduced $\overline{\omega_0} \in \mathbb{R}$ (anticipated to be an $O(1)$ quantity as it depends on ε) defined by

$$1 - \omega_0^2 = \varepsilon\overline{\omega_0}.$$

To prove this result, it is convenient to rewrite (19) as

$$\partial_s Q - \omega_0 q_e \partial_\psi^2 Q = \left(1 - \frac{\omega_0 q_e}{\sqrt{\omega_0^2 q_e^2 + Q}}\right)\partial_s Q. \tag{20}$$

We will study of following iteration scheme

$$\partial_s Q_n - \omega_{0n-1} q_e \partial_\psi^2 Q_n = \left(1 - \frac{\omega_{0n-1} q_e}{\sqrt{\omega_{0n-1}^2 q_e^2 + Q_{n-1}}}\right)\partial_s Q_{n-1}, \tag{21}$$

$$\begin{aligned} Q_n(s,0) &= (1-\omega_{0n}^2) q_e^2(s) + 2\varepsilon g(s) q_e(s) + \varepsilon^2 g^2(s)\\ &= \varepsilon\overline{\omega_0}_n q_e^2(s) + 2\varepsilon g(s) q_e(s) + \varepsilon^2 g^2(s),\\ Q_n(s,\infty) &= 0, \end{aligned}$$

with $Q_{-1} = 0, \omega_{0-1} = 1$. Schematically, we think that $(\omega_{0n-1}, Q_{n-1}) \mapsto \omega_{0n} \mapsto Q_n$, that is ω_{0n} is determined on the onset by a compatibility condition for the linear problem which depends on the prior iterate, and Q_n is subsequently solved for ω_{0n-1}. Let

$$f(Q, s; \omega_0) := \left(1 - \frac{\omega_0 q_e}{\sqrt{\omega_0^2 q_e^2 + Q}}\right)\partial_s Q = \left(1 - \sqrt{1 - \frac{Q}{\omega_0^2 q_e^2 + Q}}\right)\partial_s Q. \tag{22}$$

In this system ω_{0n} is chosen to enforce that

$$\overline{\omega_{0n}} = \frac{\fint_{\mathbb{T}_L}(2gq_e^2 + \varepsilon g^2 q_e)\mathrm{d}s}{\fint_{\mathbb{T}_L} q_e^3 \mathrm{d}s} - \frac{1}{\omega_{0n-1}\varepsilon}\frac{\int_0^\infty y f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}y}{\fint_{\mathbb{T}_L} q_e^3\mathrm{d}s}. \tag{23}$$

Conceptually, it is clearer to separate out the explicit component of $\overline{\omega_{0n}}$, which is $O(1)$ and independent of n, and the smaller amplitude implicit component of $\overline{\omega_{0n}}$ as follows

$$\begin{aligned} \overline{\omega_{0n}} &= \overline{\omega}_{0*} + \overline{\omega_{0\mathrm{Err},n}}, \\ \overline{\omega}_{0*} &:= \frac{\fint_{\mathbb{T}_L}(2gq_e^2 + \varepsilon g^2 q_e)\mathrm{d}s}{\fint_{\mathbb{T}_L} q_e^3\mathrm{d}s}, \\ \overline{\omega_{0\mathrm{Err},n}} &:= -\frac{1}{\omega_{0n-1}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y}{\fint_{\mathbb{T}_L} q_e^3\mathrm{d}s}. \end{aligned} \tag{24}$$

With this, we can solve the above equation for Q_n. For $\psi \geq 0$, we define

$$\langle\psi\rangle = 1 + \psi$$

For a function $f = f(s,\psi)$ defined on $\mathbb{T}_L \times \mathbb{R}_+$, we define

$$\begin{aligned} \|f\|_{X_{k,m}}^2 = &\sum_{k'=0}^{k}\sum_{m'=0}^{m}\left\{\|\langle\psi\rangle^{m'}\partial_s^{k'}f\|_{L^2(\mathbb{T}_L\times\mathbb{R}_+)}^2 + \|\langle\psi\rangle^{m'}\partial_s^{k'+1}f\|_{L^2(\mathbb{T}_L\times\mathbb{R}_+)}^2\right\} \\ &+ \sum_{k'=0}^{k}\sum_{m'=0}^{m}\left\{\|\langle\psi\rangle^{m'}\partial_\psi\partial_s^{k'}f\|_{L^2(\mathbb{T}_L\times\mathbb{R}_+)}^2 + \|\langle\psi\rangle^{m'}\partial_\psi^2\partial_s^{k'}f\|_{L^2(\mathbb{T}_L\times\mathbb{R}_+)}^2\right\}. \end{aligned} \tag{25}$$

We will construct the unique solution of the Eq. (20) in the space $X_{2,50}$. By the standard Sobolev embedding, we also have

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^2} + \|\partial_s f\|_{L^2} + \|\partial_\psi f\|_{L^2} + \|\partial_s\partial_\psi f\|_{L^2} \lesssim \|f\|_{X_{1,0}}.$$

Remark. (Exponential decay of Q for $\psi \gg 1$) In fact, one can prove existence in a space encoding exponential decay in ψ, as should be expect for a (nonlinear) heat equation with data at $\psi = 0$. For simplicity of presentation, we prove only algebraic decay but the requisite modifications involving exponential weights are standard.

Indeed, we have the following result that tells us this iteration is well-defined.

Lemma 4. *Let $n \geq 0$. Assume that $Q_{n-1} \in X_{2,50}$, and that* (27)–(28) *are valid until index $n-1$. Assume further that $\overline{\omega_{0n}}$ is defined according to* (23). *Then, there exists a unique solution, Q_n to the system* (21).

Proof. We first of all write the system (21) as follows

$$\begin{aligned} \partial_s Q_n - \omega_{0n-1} q_e \partial_\psi^2 Q_n &= F_{n-1}, \\ Q_n(s,0) &= b_{n-1}(s), \\ Q_n(s,\infty) &= 0, \end{aligned}$$

We introduce the variable

$$t = J(s), \qquad \frac{\mathrm{d}t}{\mathrm{d}s} = J'(s) = \omega_{0n-1} q_e(s).$$

We notice that $\frac{\mathrm{d}t}{\mathrm{d}s}$ is bounded above and below and hence determines an invertible transformation due to the fact that $\omega_{0n-1} q_e(s) > 0$. We also note that by writing $\omega_{0n-1} q_e(s) = \langle \omega_{0n-1} q_e \rangle + (\omega_{0n-1} q_e - \langle \omega_{0n-1} q_e \rangle)$, we have

$$t = \langle \omega_{0n-1} q_e \rangle s + \int_0^s \omega_{0n-1}(q_e - \langle q_e \rangle)\mathrm{d}s',$$

which maps $\mathbb{T}_L$ into $\mathbb{T}_{\omega_{0n-1}\langle q_e \rangle L}$. We next introduce

$$V_n(t,\psi) = V_n(J(s),\psi) = Q_n(s,\psi).$$

This object satisfies the system

$$\partial_t V_n - \partial_\psi^2 V_n = \frac{F_{n-1}}{\omega_{0n-1} q_e} =: G_{n-1}.$$

We expand the solution of V_n in a Fourier basis in the t variable as follows

$$ik\widehat{V}_n^k - \partial_\psi^2 \widehat{V}_n^k = \widehat{G}_{n-1}^k.$$

The zero mode equation is exactly the Feynman-Lagerstrom formula, (23). For the k'th mode, where $k \neq 0$, we write the explicit formula:

$$\hat{V}_n^k = e^{-\sqrt{ik}\psi} \hat{b}_{n-1}^k + \frac{1}{2\sqrt{ik}} \int_0^\infty \left(e^{-\sqrt{ik}(\psi+\psi')} - e^{-\sqrt{ik}|\psi-\psi'|} \right) \hat{G}_{n-1}^k d\psi',$$

where $\sqrt{ik}$ is the complex square root of ik with positive real part. We now observe that for $G_{n-1} \in X_{2,50}$, the above integrals converge to zero as $\psi \to \infty$ (when $k \neq 0$). This completes the proof.

We define the differences $\triangle\overline{\omega_0}_n$ and $\triangle Q_n$ to be

$$\begin{aligned} \triangle\overline{\omega_0}_n &= \overline{\omega_0}_{n-1} - \overline{\omega_0}_n, \\ &= \frac{1}{\omega_{0n-1}\varepsilon} \frac{\int_0^\infty y \int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3 \mathrm{d}s} \\ &\quad - \frac{1}{\omega_{0n-2}\varepsilon} \frac{\int_0^\infty y \int_0^L f(Q_{n-2}(y,s),s;\omega_{0n-2})\mathrm{d}s\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3 \mathrm{d}s}, \\ \triangle Q_n &= Q_n - Q_{n-1}. \end{aligned}$$

Differences in Q obey:

$$\begin{aligned}
&\partial_s(Q_n - Q_{n-1}) - \omega_{0n-1} q_e \partial_\psi^2 (Q_n - Q_{n-1}) \\
&\quad = \left(1 - \frac{\omega_{0n-1} q_e}{\sqrt{\omega_{0n-1}^2 q_e^2 + Q_{n-1}}}\right) \partial_s Q_{n-1} - \left(1 - \frac{\omega_{0n-2} q_e}{\sqrt{\omega_{0n-2}^2 q_e^2 + Q_{n-2}}}\right) \partial_s Q_{n-2}, \\
&\qquad + (\omega_{0n-1} - \omega_{0n-2}) \partial_\psi^2 Q_{n-1}, \\
&(Q_n - Q_{n-1})(s, 0) = \varepsilon(\overline{\omega_0}_n - \overline{\omega_0}_{n-1}) q_e^2(s), \\
&Q_n(s, \infty) = 0. \qquad (26)
\end{aligned}$$

Remark. (Obtaining the sharper bounds stated in Theorem 3) In what follows, we will bootstrap bounds of ε^{1-}, specifically $\varepsilon^{0.97}$ for $|\overline{\omega_0}_{\mathrm{Err},n}|$ and $\varepsilon^{0.99}$ for $\|Q_n\|_{X_{2,50}}$, although any power less than 1 would suffice by the same argument given below. This is not essential, it is to avoid keeping track of large constants for simplicity of the bootstrap argument. In fact, from these bounds one can deduce a posteriori sharper estimates of the form $|\overline{\omega_0}_{\mathrm{Err},n}| \le C_1\varepsilon$ and $\|Q_n\|_{X_{2,50}} \le C_2\varepsilon$ for some, possibly large, constants $C_1, C_2 > 0$ by taking the proved bounds on ω_0 and Q, returning to the equation, and performing the estimate again.

We will establish the following bounds

$$|\overline{\omega_0}_{\mathrm{Err},n}| \le \varepsilon^{0.97}, \qquad (27)$$

$$\|Q_n\|_{X_{2,50}} \le \varepsilon^{0.99}, \qquad (28)$$

$$|\Delta\overline{\omega_0}_n| \le \varepsilon^{1.97} |\Delta\overline{\omega_0}_{n-1}| + \varepsilon^{-0.02} \|\Delta Q_{n-1}\|_{X_{1,4}}, \qquad (29)$$

$$\|\Delta Q_n\|_{X_{2,50}} \le \varepsilon^{\frac{3}{4}} \|\Delta Q_{n-1}\|_{X_{2,50}} + \varepsilon^{\frac{1}{2}} |\Delta\overline{\omega_0}_n| + \varepsilon^{\frac{3}{2}} |\Delta\overline{\omega_0}_{n-1}|, \qquad (30)$$

which immediately imply the main result. The bounds (27)–(28) show that $(\overline{\omega_0}_{\mathrm{Err},n}, Q_n) \in B_{\varepsilon^{1.99},\varepsilon^{0.99}} \subset \mathbb{R} \times X_{2,50}$, whereas the bounds (29)–(30) show that iteration converges to a unique fixed point. A standard fixed point result imply that these bootstrap bounds give the main theorem:

Proof. We insert the bound (29) into the second term on the right-hand side of (30) in order to get the following

$$\begin{aligned}
|\Delta\overline{\omega_0}_n| &\le \varepsilon^{1.97} |\Delta\overline{\omega_0}_{n-1}| + \varepsilon^{-0.02} \|\Delta Q_{n-1}\|_{X_{2,50}}, \\
\|\Delta Q_n\|_{X_{2,50}} &\le \varepsilon^{\frac{1}{4}} \|\Delta Q_{n-1}\|_{X_{2,50}} + \varepsilon^{\frac{3}{2}} |\Delta\overline{\omega_0}_{n-1}|.
\end{aligned}$$

Define $\boldsymbol{Y}_n := (\Delta Q_n, \varepsilon\Delta\overline{\omega_0}_n) \in X_{2,50} \times \mathbb{R}$, endowed with the product norm. Then

$$\|\boldsymbol{Y}_n\|_{X_{2,50}\times\mathbb{R}} \le \varepsilon^{\frac{1}{5}} \|\boldsymbol{Y}_{n-1}\|_{X_{2,50}\times\mathbb{R}}.$$

It is therefore clear that

$$\begin{aligned}
\|\boldsymbol{Y}_n\|_{X_{2,50}\times\mathbb{R}} &\le \varepsilon^{\frac{n}{5}} \|\boldsymbol{Y}_0\|_{X_{2,50}\times\mathbb{R}} \\
&\le \varepsilon^{\frac{n}{5}} (\|Q_{-1}\|_{X_{2,50}} + \|Q_0\|_{X_{2,50}} + \varepsilon|\overline{\omega_0}_{-1}| + \varepsilon|\overline{\omega_0}_0|)
\end{aligned}$$

$$\leq C\varepsilon^{\frac{n}{5}}.$$

This then implies that $(Q_n, \overline{\omega_0}_n)$ is a Cauchy sequence in $X_{2,50} \times \mathbb{R}$, and hence converges to a limit, $(Q_\infty, \overline{\omega_0}_\infty)$. We can therefore pass to the limit in Eq. (21) as well as in (23) to conclude that $(Q_\infty, \overline{\omega_0}_\infty)$ satisfy the system (19).

We now prove uniqueness. We assume that $(Q_1, \omega_{0,1})$ and $(Q_2, \omega_{0,2})$ are two solutions to (19) in the space $X_{2,50} \times \mathbb{R}$. We may therefore write an analogous Eq. (26) on $Q_1 - Q_2$ (without the iteration), which reads:

$$\begin{aligned}
&\partial_s(Q_1 - Q_2) - \omega_{0,1} q_e \partial_\psi^2 (Q_1 - Q_2) \\
&\quad = \left(1 - \frac{\omega_{0,1} q_e}{\sqrt{\omega_{0,1}^2 q_e^2 + Q_1}}\right)\partial_s Q_1 - \left(1 - \frac{\omega_{0,2} q_e}{\sqrt{\omega_{0,2}^2 q_e^2 + Q_2}}\right)\partial_s Q_2, \\
&\qquad + (\omega_{0,1} - \omega_{0,2})\partial_\psi^2 Q_2, \\
&(Q_1 - Q_2)(s, 0) = \varepsilon(\overline{\omega_0}_{,1} - \overline{\omega_0}_{,2}) q_e^2(s), \\
&(Q_1 - Q_2)(s, \infty) = 0,
\end{aligned}$$

as well as the analogue of expression (26) (again without the iteration)

$$\begin{aligned}
\overline{\omega_0}_1 - \overline{\omega_0}_2 = &\frac{1}{\omega_{0,2}\varepsilon} \frac{\int_0^\infty y \int_0^L f(Q_2(y,s), s; \omega_{0,2}) \mathrm{d}s\mathrm{d}y}{\fint_{\mathbb{T}_L} q_e^3 \mathrm{d}s} \\
&- \frac{1}{\omega_{0,1}\varepsilon} \frac{\int_0^\infty y \int_0^L f(Q_1(y,s), s; \omega_{0,1}) \mathrm{d}s\mathrm{d}y}{\fint_{\mathbb{T}_L} q_e^3 \mathrm{d}s}.
\end{aligned}$$

Re-applying the *a-priori* estimates on these systems results in the following bounds:

$$\begin{aligned}
|\overline{\omega_0}_{,1} - \overline{\omega_0}_{,2}| &\leq \epsilon^{1.97} |\overline{\omega_0}_{,1} - \overline{\omega_0}_{,2}| + \epsilon^{-0.02} \|Q_1 - Q_2\|_{X_{2,50}}, \\
\|Q_1 - Q_2\|_{X_{2,50}} &\leq \varepsilon^{\frac{3}{4}} \|Q_1 - Q_2\|_{X_{2,50}} + \varepsilon^{\frac{1}{2}} |\overline{\omega_0}_{,1} - \overline{\omega_0}_{,2}|,
\end{aligned}$$

which are the analogues of (29)–(30). The two bounds above clearly imply that $\overline{\omega_0}_{,1} = \overline{\omega_0}_{,2}$ and $Q_1 = Q_2$. This proves uniqueness.

5.2. $\overline{\omega_0}_{\mathrm{Err},n}$ *Estimates*

Here, we will establish the bootstrap bound (27). Indeed,

Lemma 5. *Assume* (27)–(30) *are valid until the index* $n-1$. *Then* $\overline{\omega_0}_{\mathrm{Err},n}$ *satisfies:*

$$|\overline{\omega_0}_{\mathrm{Err},n}| \leq \varepsilon^{0.97}.$$

Proof. Recall the expression (24), after which we estimate as follows

$$\begin{aligned}
|\overline{\omega_0}_{\mathrm{Err},n}| &\leq \frac{1}{\varepsilon} \left| \int_0^\infty y \int_0^L f(Q_{n-1}(y,s), s; \omega_{0n-1}) \mathrm{d}s\mathrm{d}y \right| \\
&\lesssim \frac{1}{\varepsilon} \left\| \left(1 - \frac{\omega_{0n-1} q_e}{\sqrt{\omega_{0n-1}^2 q_e^2 + Q_{n-1}}}\right) \partial_s Q_{n-1} \langle\psi\rangle^4 \right\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{\varepsilon}\left\|\left(1-\frac{\omega_{0n-1}q_e}{\sqrt{\omega_{0n-1}^2q_e^2+Q_{n-1}}}\right)\|_{L^\infty}\|\partial_s Q_{n-1}\langle\psi\rangle^4\right\|_{L^2}\\
&=\frac{1}{\varepsilon}\left\|1-\sqrt{1-\frac{Q_{n-1}}{\omega_{0n-1}^2q_e^2+Q_{n-1}}}\right\|_{L^\infty}\left\|\partial_s Q_{n-1}\langle\psi\rangle^4\right\|_{L^2}\\
&\lesssim \frac{1}{\varepsilon}\left\|\frac{Q_{n-1}}{\omega_{0n-1}^2q_e^2+Q_{n-1}}\right\|_{L^\infty}\left\|\partial_s Q_{n-1}\langle\psi\rangle^4\right\|_{L^2}\lesssim \frac{1}{\varepsilon}\varepsilon^{0.99}\varepsilon^{0.99}<\varepsilon^{0.97},
\end{aligned}
$$

where we have invoked the bootstrap bound (28) in the final step, as well as the L^∞ estimate

$$
\begin{aligned}
\left\|\frac{Q_{n-1}}{\omega_{0n-1}^2q_e^2+Q_{n-1}}\right\|_{L^\infty} &\lesssim \frac{1}{\omega_{0n-1}^2q_e^2-\|Q_{n-1}\|_{L^\infty}}\|Q_{n-1}\|_{L^\infty}\\
&\lesssim \frac{1}{\omega_{0n-1}^2q_e^2-\|Q_{n-1}\|_{X_{1,0}}}\|Q_{n-1}\|_{X_{1,0}}\\
&\lesssim \frac{1}{\omega_{0n-1}^2q_e^2-C\varepsilon^{0.99}}\varepsilon^{0.99}\lesssim \varepsilon^{0.99}.
\end{aligned}
$$

Above, we have used the following Sobolev inequality on $\mathbb{T}\times\mathbb{R}_+$, which reads $\|f\|_{L^\infty}\lesssim\|f\|_{X_{1,0}}$. This Sobolev embedding will be used repeatedly to estimate nonlinear terms.

5.3. $\Delta\overline{\omega_0}_n$ Estimates

Here we prove the following lemma.

Lemma 6. *Assume* (27)–(30) *are valid until the index $n-1$. Assume* (27) *and* (28) *are valid until index n. Then the following bound hol*ds

$$
|\Delta\overline{\omega_0}_n|\leq \epsilon^{1.97}|\Delta\overline{\omega_0}_{n-1}|+\varepsilon^{-0.02}\|\Delta Q_{n-1}\|_{X_{1,4}}.
$$

Proof. We use the expression

$$
\begin{aligned}
\Delta(\overline{\omega_0}_n) &= \overline{\omega_0}_{n-1}-\overline{\omega_0}_n\\
&:= \frac{1}{\omega_{0n-1}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3\mathrm{d}s}\\
&\quad -\frac{1}{\omega_{0n-2}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-2}(y,s),s;\omega_{0n-2})\mathrm{d}s\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3\mathrm{d}s}\\
&= \frac{1}{\omega_{0n-1}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3\mathrm{d}s}\\
&\quad -\frac{1}{\omega_{0n-2}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y}{⨍_{\mathbb{T}_L} q_e^3\mathrm{d}s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\omega_{0n-2}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y}{\fint_{\mathbb{T}_L} q_e^3\mathrm{d}s}\\
&-\frac{1}{\omega_{0n-2}\varepsilon}\frac{\int_0^\infty y\int_0^L f(Q_{n-2}(y,s),s;\omega_{0n-2})\mathrm{d}s\mathrm{d}y}{\fint_{\mathbb{T}_L} q_e^3\mathrm{d}s}\\
&= I_1 + I_2.
\end{aligned}
$$

To estimate I_1, we have

$$
\begin{aligned}
|I_1| &= \frac{|\omega_{0n-2}-\omega_{0n-1}|}{\varepsilon\omega_{0n-1}\omega_{0n-2}}\frac{|\int_0^\infty y\int_0^L f(Q_{n-1}(y,s),s;\omega_{0n-1})\mathrm{d}s\mathrm{d}y|}{\fint_{\mathbb{T}_L} q_e^3\mathrm{d}s}\\
&\lesssim \frac{1}{\epsilon}|\omega_{0n-2}-\omega_{0n-1}|\|\langle\psi\rangle^4 f(Q_{n-1},s;\omega_{0n-1})\|_{L^2}\\
&\lesssim \frac{1}{\epsilon}|\omega_{0n-2}-\omega_{0n-1}|\epsilon^{0.99}\epsilon^{0.99}\\
&= \frac{1}{\epsilon}|\omega_{0n-2}^2-\omega_{0n-1}^2||\omega_{0n-2}+\omega_{0n-1}|^{-1}\epsilon^{0.99}\epsilon^{0.99}\\
&\lesssim \frac{1}{\epsilon}\epsilon|\overline{\omega_0}_{n-2}-\overline{\omega_0}_{n-1}|\epsilon^{0.99}\epsilon^{0.99}\\
&\leq \frac{1}{2}\epsilon^{1.97}|\Delta\overline{\omega_0}_{n-1}|.
\end{aligned}
$$

To estimate I_2, we need to use the identity (22) to estimate

$$
\begin{aligned}
f(Q,\omega_0)-f(\overline{Q},\overline{\omega_0}) &= \left(1-\sqrt{1-\frac{Q}{\omega_0^2q_e^2+Q}}\right)\partial_s Q\\
&\quad-\left(1-\sqrt{1-\frac{\overline{Q}}{\overline{\omega_0}^2q_e^2+\overline{Q}}}\right)\partial_s \overline{Q}\\
&= \left(1-\sqrt{1-\frac{Q}{\omega_0^2q_e^2+Q}}\right)\partial_s Q\\
&\quad-\left(1-\sqrt{1-\frac{Q}{\omega_0^2q_e^2+Q}}\right)\partial_s \overline{Q}\\
&\quad+\left(1-\sqrt{1-\frac{Q}{\omega_0^2q_e^2+Q}}\right)\partial_s \overline{Q}\\
&\quad-\left(1-\sqrt{1-\frac{\overline{Q}}{\overline{\omega_0}^2q_e^2+\overline{Q}}}\right)\partial_s \overline{Q}\\
&= BD_1 + BD_2.
\end{aligned}
$$

Clearly, using the inequality $|1-\sqrt{1-x}|\leq |x|$ for $x\leq 1$, we have

$$
\|BD_1\langle\psi\rangle^4\|_{L^2}\lesssim \left\|1-\sqrt{1-\frac{Q}{\omega_0^2q_e^2+Q}}\right\|_{L^\infty}\|\partial_s(Q-\overline{Q})\langle\psi\rangle^4\|_{L^2}
$$

$$\lesssim \|Q\|_{X_{1,0}}\|Q-\overline{Q}\|_{X_{0,4}}.$$

and, using the inequality $|\sqrt{1-x}-\sqrt{1-y}| \lesssim |x-y|$ for $x, y \ll 1$, we have

$$\begin{aligned}
\|BD_2\langle\psi\rangle^4\|_{L^2} &\lesssim \|\partial_s\overline{Q}\langle\psi\rangle^4\|_{L^2}\left\|\frac{Q}{\omega_0^2q_e^2+Q}-\frac{\overline{Q}}{\overline{\omega_0}^2q_e^2+\overline{Q}}\right\|_{L^\infty}\\
&\lesssim \|\partial_s\overline{Q}\langle\psi\rangle^4\|_{L^2}\left\|\frac{\overline{\omega_0}^2Q-\omega_0^2\overline{Q}}{(\omega_0^2q_e^2+Q)(\overline{\omega_0}^2q_e^2+\overline{Q})}\right\|_{L^\infty}\\
&\lesssim \|\partial_s\overline{Q}\langle\psi\rangle^4\|_{L^2}\left\|Q-\overline{Q}\right\|_{L^\infty}+\|\partial_s\overline{Q}\langle\psi\rangle^4\|_{L^2}\|Q\|_{L^\infty}|\omega_0^2-\overline{\omega_0}^2|\\
&\lesssim \|\overline{Q}\|_{X_{0,4}}\|Q-\overline{Q}\|_{X_{1,0}}+\|\overline{Q}\|_{X_{0,4}}\|Q\|_{X_{1,0}}|\omega_0^2-\overline{\omega_0}^2|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|I_2| &\lesssim \frac{1}{\epsilon}\|Q_{n-1}\|_{X_{1,0}}\|Q_{n-1}-Q_{n-2}\|_{X_{0,4}}+\frac{1}{\epsilon}\|Q_{n-2}\|_{X_{0,4}}\|Q_{n-1}-Q_{n-2}\|_{X_{1,0}}\\
&\quad+\frac{1}{\epsilon}\|Q_{n-2}\|_{X_{0,4}}\|Q_{n-1}\|_{X_{1,0}}|{\omega_0}_{n-1}^2-{\omega_0}_{n-2}^2|\\
&\lesssim \frac{1}{\varepsilon}\epsilon^{0.99}\|\Delta Q_{n-1}\|_{X_{1,4}}+\frac{1}{\varepsilon}\epsilon^{0.99}\epsilon^{0.99}\epsilon|\overline{\omega_0}_{n-1}-\overline{\omega_0}_{n-2}|\\
&\leq \varepsilon^{-0.02}\|\Delta Q_{n-1}\|_{X_{1,4}}+\frac{\varepsilon^{1.97}}{2}|\overline{\omega_0}_{n-1}-\overline{\omega_0}_{n-2}|.
\end{aligned}$$

Pairing these bounds together, we get the desired result.

5.4. Abstract Q Estimates

For future use, it turns out we will have a need to develop our estimates on a slightly more abstract system. Therefore, we consider

$$\begin{aligned}
\partial_s Q-\omega_0q_e\partial_\psi^2Q &= F+\partial_\psi^2G, \qquad (31)\\
Q(s,0) &= b(s),\\
Q(s,\infty) &= 0.
\end{aligned}$$

We develop a high-order energy method to treat equation (31). We commute ∂_s^k to obtain

$$\begin{aligned}
\partial_s Q^{(k)}-\omega_0q_e\partial_\psi^2Q^{(k)} &= F^{(k)}+\partial_\psi^2G^{(k)}+A_{\mathrm{comm},k}, \qquad (32)\\
Q^{(k)}(s,0) &= b^{(k)}(s),\\
Q^{(k)}(s,\infty) &= 0,
\end{aligned}$$

where the commutator term

$$A_{\mathrm{comm},k} := \sum_{k'=0}^{k-1}\binom{k}{k'}\partial_s^{k-k'}q_e\partial_\psi^2Q^{(k')},$$

and where we adopt the short-hand $f^{(k)}(s,\psi) := \partial_s^k f(s,\psi)$ for an abstract function $f(s,\psi)$.

Proposition 7. *Assume that the boundary condition $b(s)$ and the source term $F + \partial_\psi^2 G$ satisfy the Feynman–Lagerstrom compatibilty condition* (23). *Then the solution Q to* (31) *obeys the following inequality:*

$$\|Q\|_{X_{k,m}} \lesssim \sum_{k'=0}^{k} \|\partial_s^{k'} F\langle\psi\rangle^{m+4}\|_{L^2} + \sum_{k'=0}^{k}\sum_{j=0}^{2} \|\partial_s^{k'}\partial_\psi^j G\langle\psi\rangle^m\|_{L^2} + \|b\|_{H_s^{k+1}}. \tag{33}$$

The first task is we lift the boundary condition $b(s)$ by considering the lift function

$$L(s,\psi) := L[b](s,\psi) := e^{-\psi} b(s)$$

and consequently

$$\mathring{Q} := Q - L[b],$$

which satisfies the following system

$$\begin{aligned} \partial_s \mathring{Q} - \omega_0 q_e \partial_\psi^2 \mathring{Q} &= F + \partial_\psi^2 G + G_{\text{Lift}}, \\ \mathring{Q}(s,0) &= 0, \\ \mathring{Q}(s,\infty) &= 0 \end{aligned} \tag{34}$$

where

$$G_{\text{Lift}} := e^{-\psi}(\omega_0 q_e b(s) - b'(s)). \tag{35}$$

We will need to work in higher order norms. Therefore, we present the equations upon commuting ∂_s^k to (34), which yield

$$\begin{aligned} \partial_s \mathring{Q}^{(k)} - \omega_0 q_e \partial_\psi^2 \mathring{Q}^{(k)} &= F^{(k)} + \partial_\psi^2 G^{(k)} + G_{\text{Lift}}^{(k)} + C_{\text{comm},k}, \\ \mathring{Q}^{(k)}(s,0) &= 0, \\ \mathring{Q}^{(k)}(s,\infty) &= 0. \end{aligned} \tag{36}$$

Above, we define the commutator term as follows:

$$C_{\text{comm},k} := \omega_0 \mathbf{1}_{k\geq 1} \sum_{k'=0}^{k-1} \binom{k}{k'} \partial_s^{k-k'} q_e \partial_\psi^2 \mathring{Q}^{(k')}. \tag{37}$$

Lemma 8. *For any $\delta > 0$ the following bounds hold (where the constant $C_\delta \uparrow \infty$ as $\delta \downarrow 0$):*

$$\begin{aligned} \|\partial_\psi Q^{(k)}\langle\psi\rangle^m\|_{L^2}^2 \leq{}& C_\delta \|F^{(k)}\langle\psi\rangle^m\|_{L^2}^2 + C_\delta \|\partial_\psi^2 G^{(k)}\langle\psi\rangle^m\|_{L^2}^2 + C_\delta \|b\|_{H_s^{k+1}}^2 \\ &+ \delta \|Q^{(k)}\langle\psi\rangle^m\|_{L^2}^2 + C_\delta \mathbf{1}_{k\geq 1} \sum_{k'=0}^{k-1} \|\partial_\psi Q^{(k')}\langle\psi\rangle^m\|_{L^2}^2 \\ &+ \mathbf{1}_{m\geq 1} \|Q^{(k)}\langle\psi\rangle^{m-1}\|_{L^2}^2. \end{aligned} \tag{38}$$

Proof. We multiply (36) by $\mathring{Q}^{(k)}\langle\psi\rangle^{2m}$ and integrate by parts to get the identity

$$\begin{aligned}
&\frac{\partial_s}{2}\int_{\mathbb{R}_+}|\mathring{Q}^{(k)}|^2\langle\psi\rangle^{2m}\mathrm{d}\psi+\omega_0 q_e(s)\int_{\mathbb{R}_+}|\partial_\psi\mathring{Q}^{(k)}|^2\langle\psi\rangle^{2m}\mathrm{d}\psi\\
&=\int(F^{(k)}+\partial_\psi^2G^{(k)})\mathring{Q}^{(k)}\langle\psi\rangle^{2m}\mathrm{d}\psi+\int G^{(k)}_{\mathrm{Lift}}\mathring{Q}^{(k)}\langle\psi\rangle^{2m}\mathrm{d}\psi\\
&\quad+\frac{2m(2m-1)}{2}\omega_0q_e(s)\int|\mathring{Q}^{(k)}|^2\langle\psi\rangle^{2m-2}\mathrm{d}\psi\\
&\quad-\int\omega_0\mathbf{1}_{k\geq1}\sum_{k'=0}^{k-1}\binom{k}{k'}\partial_s^{k-k'}q_e\partial_\psi\mathring{Q}^{(k')}\partial_\psi\mathring{Q}^{(k)}\langle\psi\rangle^{2m}\mathrm{d}\psi\\
&\quad-\int\omega_0\mathbf{1}_{k\geq1}\sum_{k'=0}^{k-1}\binom{k}{k'}\partial_s^{k-k'}q_e\partial_\psi\mathring{Q}^{(k')}\mathring{Q}^{(k)}2m\langle\psi\rangle^{2m-1}.
\end{aligned}$$

We now integrate in $s\in\mathbb{T}$, and the ∂_s term drops out due to periodicity. This implies

$$\begin{aligned}
\left\|\partial_\psi\mathring{Q}^{(k)}\langle\psi\rangle^m\right\|_{L^2}^2\lesssim&\left|\int\int(F^{(k)}+\partial_\psi^2G^{(k)})\mathring{Q}^{(k)}\langle\psi\rangle^{2m}\mathrm{d}\psi\mathrm{d}s\right|\\
&+\left|\int\int G^{(k)}_{\mathrm{Lift}}\mathring{Q}^{(k)}\langle\psi\rangle^{2m}\mathrm{d}\psi\mathrm{d}s\right|\\
&+\left\|\mathring{Q}^{(k)}\langle\psi\rangle^{m-1}\right\|_{L^2}^2\\
&+\left|\int\int\omega_0\mathbf{1}_{k\geq1}\sum_{k'=0}^{k-1}\binom{k}{k'}\partial_s^{k-k'}q_e\partial_\psi\mathring{Q}^{(k')}\partial_\psi\mathring{Q}^{(k)}\langle\psi\rangle^{2m}\mathrm{d}\psi\mathrm{d}s\right|\\
&+\left|\int\int\omega_0\mathbf{1}_{k\geq1}\sum_{k'=0}^{k-1}\binom{k}{k'}\partial_s^{k-k'}q_e\partial_\psi\mathring{Q}^{(k')}\mathring{Q}^{(k)}\langle\psi\rangle^{2m-1}\mathrm{d}\psi\mathrm{d}s\right|.
\end{aligned}$$

This implies

$$\begin{aligned}
\left\|\partial_\psi\mathring{Q}^{(k)}\langle\psi\rangle^m\right\|_{L^2}^2\leq&\,\delta\left\|\mathring{Q}^{(k)}\langle\psi\rangle^m\right\|_{L^2}^2+C_\delta\left\|F^{(k)}\langle\psi\rangle^m\right\|_{L^2}^2+C_\delta\left\|\partial_\psi^2G^{(k)}\langle\psi\rangle^m\right\|_{L^2}^2\\
&+C_\delta\|b\|_{H_s^{k+1}}+C_0\mathbf{1}_{\{m\geq1\}}\left\|\mathring{Q}^{(k)}\langle\psi\rangle^{m-1}\right\|_{L^2}^2\\
&+C_\delta\mathbf{1}_{k\geq1}\sum_{k'=0}^{k-1}\left\|\partial_\psi\mathring{Q}^{(k')}\langle\psi\rangle^m\right\|_{L^2}^2+\delta\left\|\partial_\psi\mathring{Q}^{(k)}\langle\psi\rangle^m\right\|_{L^2}^2,
\end{aligned}$$

where $\delta>0$ is small and $C_\delta\sim\delta^{-1}$. The result follows immediately, using the fact that

$$\mathring{Q}^{(k)}=Q^{(k)}-e^{-\psi}b^{(k)}(s).$$

This concludes the proof of the lemma.

Lemma 9. *Let $k \geq 0, m \geq 0$. The solution $Q^{(k)}$ to (32) satisfies the following estimate:*

$$\|\partial_s Q^{(k)} \langle \psi \rangle^m \|_{L^2}^2 \lesssim \|F^{(k)} \langle \psi \rangle^m \|_{L^2}^2 + \|\partial_\psi^2 G^{(k)} \langle \psi \rangle^m \|_{L^2}^2 + \|b\|_{H_s^{k+1}}^2 + \mathbf{1}_{m \geq 1} \|\partial_\psi Q^{(k)} \langle \psi \rangle^{m-1} \|_{L^2}^2 + \mathbf{1}_{k \geq 1} \sum_{k'=0}^{k-1} \|\partial_\psi^2 Q^{(k')} \langle \psi \rangle^m \|_{L^2}^2. \tag{39}$$

Proof. We multiply (36) by $\frac{1}{q_e(s)} \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^{2m}$ and integrate by parts to produce

$$\begin{aligned}
&\int \frac{1}{q_e(s)} (F^{(k)} + \partial_\psi^2 G^{(k)}) \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^{2m} \mathrm{d}\psi + \int \frac{1}{q_e(s)} G_{\mathrm{Lift}}^{(k)} \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^{2m} \mathrm{d}\psi \\
&\quad + \int \frac{1}{q_e(s)} C_{\mathrm{comm},k} \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^{2m} \mathrm{d}\psi \\
&= \int \frac{1}{q_e(s)} |\partial_s \mathring{Q}^{(k)}|^2 \langle \psi \rangle^{2m} \mathrm{d}\psi + \omega_0 \int \partial_\psi^2 \mathring{Q}^{(k)} \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^{2m} \mathrm{d}\psi \\
&= \int \frac{1}{q_e(s)} |\partial_s \mathring{Q}|^2 \langle \psi \rangle^{2m} \mathrm{d}\psi - \omega_0 \frac{\partial_s}{2} \int |\partial_\psi \mathring{Q}|^2 \langle \psi \rangle^{2m} \mathrm{d}\psi \\
&\quad - \omega_0 2m \int \partial_\psi \mathring{Q}^{(k)} \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^{2m-1} \mathrm{d}\psi.
\end{aligned}$$

Above, we have used the homogeneous boundary condition for $\mathring{Q}$ to integrate by parts. We now integrate over $s \in \mathbb{T}_L$ and use periodicity to eliminate the second term on the right-hand side above, which results in

$$\begin{aligned}
&\int \int \frac{1}{q_e(s)} \left| \partial_s \mathring{Q}^{(k)} \right|^2 \langle \psi \rangle^{2m} \mathrm{d}\psi \mathrm{d}s \\
&\leq C_\delta \left\| F^{(k)} \langle \psi \rangle^m \right\|_{L^2}^2 + C_\delta \left\| \partial_\psi^2 G^{(k)} \langle \psi \rangle^m \right\|_{L^2}^2 \\
&\quad + C_\delta \left\| G_{\mathrm{Lift}}^{(k)} \langle \psi \rangle^m \right\|_{L^2}^2 + C_\delta \left\| C_{\mathrm{comm},k} \langle \psi \rangle^m \right\|_{L^2}^2 \\
&\quad + \delta \left\| \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^m \right\|_{L^2}^2 + C_\delta \left\| \partial_\psi \mathring{Q}^{(k)} \langle \psi \rangle^{m-1} \right\|_{L^2}^2.
\end{aligned}$$

Recalling (37), (35) and absorbing the last term on the right-hand side to the left, we get

$$\begin{aligned}
\left\| \partial_s \mathring{Q}^{(k)} \langle \psi \rangle^m \right\|_{L^2} &\leq C_\delta \left(\left\| F^{(k)} \langle \psi \rangle^m \right\|_{L^2}^2 + \left\| \partial_\psi^2 G^{(k)} \langle \psi \rangle^m \right\|_{L^2}^2 + \|b\|_{H_s^{k+1}}^2 \right) \\
&\quad + C_\delta \mathbf{1}_{k \geq 1} \sum_{k'=0}^{k-1} \left\| \partial_\psi^2 \mathring{Q}^{(k')} \langle \psi \rangle^m \right\|_{L^2}^2.
\end{aligned}$$

We conclude the proof of the lemma, upon using the fact that $\mathring{Q}^{(k)} = Q^{(k)} - e^{-\psi} b^{(k)}(s)$.

We now need to estimate the zero mode of $Q^{(k)}$. Clearly, this is nontrivial only for $k = 0$ (when $k \geq 1$ there is no zero mode).

Lemma 10. *The zero mode, $Q^{(=0)}$, to the solution of (31), satisfies the following bound:*

$$\|Q^{(=0)}\langle\psi\rangle^m\|_{L^2_\psi}^2 \lesssim \|Q^{(\neq 0)}\langle\psi\rangle^m\|_{L^2_\psi}^2 + \|F\langle\psi\rangle^{m+4}\|_{L^2}^2 + \|G\langle\psi\rangle^m\|_{L^2}^2. \tag{40}$$

Proof. We integrate Eq. (31) to generate the identity for each $\psi \in \mathbb{R}_+$:

$$\partial_\psi^2 \int_0^L q_e(s) Q(s,\psi)\mathrm{d}s = -\frac{1}{\omega_0}\int_0^L (F + \partial_\psi^2 G)\mathrm{d}s,$$

after which we integrate twice from ∞ to get

$$\begin{aligned}\int_0^L q_e(s)Q(s,\psi)\mathrm{d}s &= -\frac{1}{\omega_0}\int_\psi^\infty\int_{\psi'}^\infty\int_0^L (F+\partial_\psi^2 G)\mathrm{d}s\mathrm{d}\psi''\psi' \\ &= -\frac{1}{\omega_0}\int_\psi^\infty\int_{\psi'}^\infty\int_0^L F\mathrm{d}s\mathrm{d}\psi''\psi' - \frac{1}{\omega_0}\int_0^L G\mathrm{d}s.\end{aligned}$$

We now separate out the left-hand side

$$\begin{aligned}\int_0^L q_e(s)Q(s,\psi)\mathrm{d}s &= \langle q_e\rangle Q^{(=0)}(\psi) + \int_0^L (q_e(s) - \langle q_e\rangle)Q(s,\psi)\mathrm{d}s \\ &= \langle q_e\rangle Q^{(=0)}(\psi) + \int_0^L (q_e - \langle q_e\rangle)\Big(Q^{(=0)}(\psi) + Q^{(\neq 0)}(s,\psi)\Big)\mathrm{d}s \\ &= \langle q_e\rangle Q^{(=0)}(\psi) + \int_0^L q_e^{(\neq 0)}(s)Q^{(\neq 0)}(s,\psi)\mathrm{d}s.\end{aligned}$$

This implies

$$\begin{aligned}Q^{(=0)}(\psi) = &-\frac{1}{\omega_0\langle q_e\rangle}\int_\psi^\infty\int_{\psi'}^\infty\int_0^L F\mathrm{d}s\mathrm{d}\psi''\mathrm{d}\psi' - \frac{1}{\omega_0\langle q_e\rangle}\int_0^L G\mathrm{d}s \\ &- \frac{1}{\langle q_e\rangle}\int_0^L q_e^{(\neq 0)}(s)Q^{(\neq 0)}(s,\psi)\mathrm{d}s.\end{aligned}$$

We therefore obtain

$$\begin{aligned}\|Q^{(=0)}\langle\psi\rangle^m\|_{L^2_\psi}^2 \lesssim &\int_{\mathbb{R}_+}\Big(\int_0^L q_e^{(\neq 0)}(s)Q^{(\neq 0)}(s,\psi)\mathrm{d}s\Big)^2\langle\psi\rangle^{2m}\mathrm{d}\psi \\ &+ \int_{\mathbb{R}_+}\langle\psi\rangle^{2m}\Big(\int_\psi^\infty\int_{\psi'}^\infty\int_0^L F\Big)^2\mathrm{d}\psi + \|G^{(=0)}\langle\psi\rangle^m\|_{L^2_\psi}^2 \\ =: &\,\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.\end{aligned}$$

Clearly, $\mathcal{I}_3$ is majorized by the last term on the right-hand side of (40). We will estimate the first term above, which we call $\mathcal{I}_1$. Using Hölder's inequality, we get

$$\mathcal{I}_1 := \int_{\mathbb{R}_+}\langle\psi\rangle^{2m}\Big(\int_0^L q_e^{(\neq 0)}(s)Q^{(\neq 0)}(s,\psi)\mathrm{d}s\Big)^2\mathrm{d}\psi$$

$$\lesssim \|Q^{(\neq 0)}\langle\psi\rangle^m\|_{L^2}^2.$$

To estimate $\mathcal{I}_2$, we need to pay weights as follows using Cauchy-Schwartz:

$$\Big|\int_{\psi'}^{\infty}\int_0^L F\Big| \lesssim \|F\langle\psi\rangle^{m+4}\|_{L^2}\langle\psi'\rangle^{-m-\frac{7}{2}},$$

which therefore implies that $|\mathcal{I}_2| \lesssim \|F\langle\psi\rangle^{m+4}\|_{L^2}$. This completes the proof.

Lemma 11. *Let $k \geq 0, m \geq 0$. The solution $Q^{(k)}$ to (32) satisfies the following estimate:*

$$\|\partial_\psi^2 Q^{(k)}\langle\psi\rangle^m\|_{L^2}^2 \lesssim \|\partial_s Q^{(k)}\langle\psi\rangle^m\|_{L^2}^2 + \|F^{(k)}\langle\psi\rangle^m\|_{L^2}^2 + \|\partial_\psi^2 G^{(k)}\langle\psi\rangle^m\|_{L^2}^2 + \mathbf{1}_{k\geq 1}\sum_{k'=0}^{k-1}\|\partial_\psi^2 Q^{(k')}\langle\psi\rangle^m\|_{L^2}^2. \tag{41}$$

Proof. We simply rearrange Eq. (32) and apply L^2 norm to both sides.

Proof of Proposition 7. Consolidating the bounds (38), (39), (40), (41), we proved (33).

5.5. Q_n Estimates

Lemma 12. *Assume (27) is valid up to index n and (28) is valid up to index $n-1$. Then*

$$\|Q_n\|_{X_{2,50}} \leq \epsilon^{0.99}.$$

Proof. For this bound, motivated by Eq. (21), we set

$$F := \left(1 - \frac{\omega_{0_{n-1}} q_e}{\sqrt{\omega_{0_{n-1}}^2 q_e^2 + Q_{n-1}}}\right)\partial_s Q_{n-1},$$
$$G := 0,$$
$$b := \varepsilon\overline{\omega_0}_n q_e^2(s) + 2\varepsilon g(s) q_e(s) + \varepsilon^2 g^2(s).$$

According to (33), we fix $k = 2, m = 50$, which results in

$$\|Q_n\|_{X_{2,50}} \lesssim \sum_{k'=0}^{2}\|\partial_s^{k'} F\langle\psi\rangle^{54}\|_{L^2} + \|b\|_{H_s^3}. \tag{42}$$

We therefore estimate the two quantities appearing on the right-hand side above. To make notation simpler, we define

$$U_{n-1} := \frac{Q_{n-1}}{\omega_{0_{n-1}}^2 q_e^2 + Q_{n-1}}, \tag{43}$$

then

$$F = \left(1 - \sqrt{1 - U_{n-1}}\right) \partial_s Q_{n-1}.$$

By a direct calculation, we have the following identities

$$\begin{aligned}
\partial_s U_{n-1} &= \frac{\partial_s Q_{n-1}}{\omega 0_{n-1}^2 q_e^2 + Q_{n-1}} - \frac{Q_{n-1}}{(\omega 0_{n-1}^2 q_e^2 + Q_{n-1})^2} (\omega 0_{n-1}^2 \partial_s q_e^2 + \partial_s Q_{n-1}), \\
\partial_s^2 U_{n-1} &= \frac{\partial_s^2 Q_{n-1}}{\omega 0_{n-1}^2 q_e^2 + Q_{n-1}} - 2 \frac{\partial_s Q_{n-1}}{(\omega 0_{n-1}^2 q_e^2 + Q_{n-1})^2} (\omega 0_{n-1}^2 \partial_s q_e^2 + \partial_s Q_{n-1}) \\
&\quad + 2 \frac{Q_{n-1}}{(\omega 0_{n-1}^2 q_e^2 + Q_{n-1})^3} (\omega 0_{n-1}^2 \partial_s q_e^2 + \partial_s Q_{n-1})^2 \\
&\quad - \frac{Q_{n-1}}{(\omega 0_{n-1}^2 q_e^2 + Q_{n-1})^2} (\omega 0_{n-1}^2 \partial_s^2 q_e^2 + \partial_s^2 Q_{n-1}).
\end{aligned}$$

First we estimate $\|F \langle \psi \rangle^{54}\|_{L^2}$. We have

$$\begin{aligned}
\|F \langle \psi \rangle^{54}\|_{L^2} &\lesssim \|U_{n-1} \langle \psi \rangle^4\|_{L^\infty} \|\partial_s Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \frac{1}{1 - \|Q_{n-1}\|_{L^\infty}} \|Q_{n-1} \langle \psi \rangle^4\|_{L^\infty} \|\partial_s Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \frac{1}{1 - \varepsilon^{0.99}} \|Q_{n-1} \langle \psi \rangle^4\|_{X_{1,4}} \|\partial_s Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \|Q_{n-1}\|_{X_{1,4}} \|Q_{n-1}\|_{X_{0,50}} \\
&\lesssim \varepsilon^{0.99} \varepsilon^{0.99}.
\end{aligned}$$

We now show that $\left\|\partial_s F \langle \psi \rangle^{54}\right\|_{L^2} \lesssim \epsilon^{0.99} \epsilon^{0.99}$. We have

$$\begin{aligned}
\partial_s F &= (1 - \sqrt{1 - U_{n-1}}) \partial_s^2 Q_{n-1} + \frac{1}{2} (1 - U_{n-1})^{-\frac{1}{2}} \partial_s U_{n-1} \partial_s Q_{n-1} \\
&=: A_1 + A_2.
\end{aligned}$$

We first bound A_1. We have

$$\begin{aligned}
\|A_1 \langle \psi \rangle^{54}\|_{L^2} &\lesssim \|U_{n-1} \langle \psi \rangle^4\|_{L^\infty} \|\partial_s^2 Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \frac{1}{1 - \|Q_{n-1}\|_{L^\infty}} \|Q_{n-1} \langle \psi \rangle^4\|_{L^\infty} \|\partial_s^2 Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \frac{1}{1 - \varepsilon^{0.99}} \|Q_{n-1}\|_{X_{1,4}} \|\partial_s^2 Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \|Q_{n-1}\|_{X_{1,4}} \|Q_{n-1}\|_{X_{1,50}} \\
&\lesssim \varepsilon^{0.99} \varepsilon^{0.99}.
\end{aligned}$$

We now estimate A_2. We have

$$\begin{aligned}
\|A_2 \langle \psi \rangle^{54}\|_{L^2} &\lesssim \|\partial_s U_{n-1} \langle \psi \rangle^4\|_{L^\infty} \|\partial_s Q_{n-1} \langle \psi \rangle^{50}\|_{L^2} \\
&\lesssim \|Q_{n-1}\|_{X_{2,4}} \|Q_{n-1}\|_{X_{0,50}}
\end{aligned}$$

$$\lesssim \varepsilon^{0.99}\varepsilon^{0.99}.$$

We now show that

$$\left\|\partial_s^2 F\langle\psi\rangle^{54}\right\|_{L^2} \lesssim \epsilon^{0.99}\epsilon^{0.99}.$$

By a direct calculation, we get

$$\begin{aligned}\partial_s^2 F &= (1-\sqrt{1-U_{n-1}})\partial_s^3 Q_{n-1} + (1-U_{n-1})^{-\frac{1}{2}}\partial_s U_{n-1}\partial_s^2 Q_{n-1}\\ &\quad + (1-U_{n-1})^{-\frac{1}{2}}\partial_s^2 U_{n-1}\partial_s Q_{n-1} + (1-U_{n-1})^{-\frac{3}{2}}|\partial_s U_{n-1}|^2\partial_s^2 Q_{n-1}\\ &=: B_1 + B_2 + B_3 + B_4.\end{aligned}$$

We first establish the following bounds on the auxiliary quantities U_{n-1}. We have

$$\begin{aligned}\|\partial_s U_{n-1}\langle\psi\rangle^m\|_{L^\infty} &\lesssim \|\partial_s Q_{n-1}\langle\psi\rangle^m\|_{L^\infty} + \|Q_{n-1}\langle\psi\rangle^m\|_{L^\infty}(1+\|\partial_s Q_{n-1}\|_{L^\infty})\\ &\lesssim \|Q_{n-1}\|_{X_{2,m}} + \|Q_{n-1}\|_{X_{2,m}}(1+\|Q_{n-1}\|_{X_{2,0}})\\ &\lesssim \|Q_{n-1}\|_{X_{2,m}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\partial_s^2 U_{n-1}\langle\psi\rangle^m\|_{L^2} &\lesssim \|\partial_s^2 Q_{n-1}\langle\psi\rangle^m\|_{L^2} + \|\partial_s Q_{n-1}\langle\psi\rangle^m\|_{L^2}(1+\|\partial_s Q_{n-1}\|_{L^\infty})\\ &\quad + \|Q_{n-1}\langle\psi\rangle^m\|_{L^2}(1+\|\partial_s Q_{n-1}\|_{L^\infty})^2\\ &\quad + \|Q_{n-1}\langle\psi\rangle^m\|_{L^\infty}(1+\|\partial_s^2 Q_{n-1}\|_{L^2})\\ &\lesssim \|Q_{n-1}\|_{X_{2,m}}.\end{aligned}$$

We can now estimate of $\left\|\partial_s^2 F\langle\psi\rangle^{54}\right\|_{L^2}$. We first bound B_1. We have

$$\begin{aligned}\|B_1\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|U_{n-1}\langle\psi\rangle^4\|_{L^\infty}\|\partial_s^3 Q_{n-1}\langle\psi\rangle^{50}\|_{L^2}\\ &\lesssim \frac{1}{1-\|Q_{n-1}\|_{L^\infty}}\|Q_{n-1}\langle\psi\rangle^4\|_{L^\infty}\|\partial_s^3 Q_{n-1}\langle\psi\rangle^{50}\|_{L^2}\\ &\lesssim \frac{1}{1-\varepsilon^{0.99}}\|Q_{n-1}\|_{X_{1,4}}\|Q_{n-1}\|_{X_{2,50}}\\ &\lesssim \|Q_{n-1}\|_{X_{1,4}}\|Q_{n-1}\|_{X_{2,50}}\\ &\lesssim \varepsilon^{0.99}\varepsilon^{0.99}.\end{aligned}$$

We next move to B_2, for which we have

$$\begin{aligned}\|B_2\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s U_{n-1}\langle\psi\rangle^4\|_{L^\infty}\|\partial_s^2 Q_{n-1}\langle\psi\rangle^{50}\|_{L^2}\\ &\lesssim \|Q_{n-1}\|_{X_{2,4}}\|Q_{n-1}\|_{X_{1,50}}\\ &\lesssim \varepsilon^{0.99}\varepsilon^{0.99}.\end{aligned}$$

As for B_3, we have

$$\begin{aligned}\|B_3\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s^2 U_{n-1}\langle\psi\rangle^{50}\|_{L^2}\|\partial_s Q_{n-1}\langle\psi\rangle^4\|_{L^\infty}\\ &\lesssim \|\partial_s^2 U_{n-1}\langle\psi\rangle^{50}\|_{L^2}\|\partial_s Q_{n-1}\langle\psi\rangle^4\|_{H_s^1 H_\psi^1}\end{aligned}$$

$$\lesssim \|Q_{n-1}\|_{X_{2,50}} \|Q_{n-1}\|_{X_{2,4}}$$
$$\lesssim \varepsilon^{0.99} \varepsilon^{0.99}.$$

We finally conclude with an estimate on B_4, for which we have

$$\begin{aligned}\|B_4\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s U_{n-1}\langle\psi\rangle^2\|_{L^\infty}^2 \|\partial_s^2 Q_{n-1}\langle\psi\rangle^{50}\|_{L^2}\\ &\lesssim \|Q_{n-1}\|_{X_{2,2}}^2 \|Q_{n-1}\|_{X_{1,50}}\\ &\lesssim \varepsilon^{0.99}\varepsilon^{0.99}\varepsilon^{0.99}.\end{aligned}$$

To conclude the proof of lemma, we need to estimate the H_s^3 norm of b,

$$\|b\|_{H_s^3} \lesssim \varepsilon|\omega_{0n}|\|q_e\|_{H_s^3} + \varepsilon\|g\|_{H_s^3}\|q_e\|_{H_s^3} + \varepsilon^2\|g\|_{H_s^3}^2 \lesssim \varepsilon.$$

Therefore, according to (60), the lemma is proven.

5.6. ΔQ_n Estimates

Our main objective in this section is to close the final bootstrap bound, (30). We begin with a lemma which allows us to control our auxiliary quantity, U_{n-1}, introduced in (43).

Lemma 13. *Let $0 \le m \le 50$. Assume (27) - (30) are valid until the index $n-1$. Assume (27) - (29) are valid until the index n. The quantities U_{n-1}, U_{n-2} satisfy:*

$$\sum_{j=0}^{1} \|(\partial_s^j U_{n-1} - \partial_s^j U_{n-2})\langle\psi\rangle^m\|_{L^\infty} \lesssim \|\Delta Q_{n-1}\|_{X_{2,m}} + \epsilon^{1.99}|\Delta\overline{\omega_0}_{n-1}|, \quad (44)$$

$$\sum_{j=0}^{2} \|(\partial_s^j U_{n-1} - \partial_s^j U_{n-2})\langle\psi\rangle^m\|_{L^2} \lesssim \|\Delta Q_{n-1}\|_{X_{2,m}} + \epsilon^{1.99}|\Delta\overline{\omega_0}_{n-1}|. \quad (45)$$

Proof. Recalling (43), we have

$$\begin{aligned}U_{n-1} - U_{n-2} &= \Big(\frac{Q_{n-1}}{\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1}} - \frac{Q_{n-2}}{\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1}}\Big)\\
&\quad + Q_{n-2}\Big(\frac{1}{\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1}} - \frac{1}{\omega_{0}{}_{n-2}^2 q_e^2 + Q_{n-2}}\Big)\\
&= \frac{\Delta Q_{n-1}}{\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1}} + \frac{Q_{n-2}}{(\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1})(\omega_{0}{}_{n-2}^2 q_e^2 + Q_{n-2})}\Delta Q_{n-1}\\
&\quad + \frac{Q_{n-2}}{(\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1})(\omega_{0}{}_{n-2}^2 q_e^2 + Q_{n-2})} q_e^2(\omega_{0}{}_{n-2}^2 - \omega_{0}{}_{n-1}^2)\\
&= \alpha\Delta Q_{n-1} + \gamma\varepsilon|\Delta\overline{\omega_0}_{n-1}|, \qquad (46)\end{aligned}$$

where the coefficients are defined by

$$\alpha := \frac{1}{\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1}} + \frac{Q_{n-2}}{(\omega_{0}{}_{n-1}^2 q_e^2 + Q_{n-1})(\omega_{0}{}_{n-2}^2 q_e^2 + Q_{n-2})},$$

$$\gamma := \frac{Q_{n-2}}{(\omega 0_{n-1}^2 q_e^2 + Q_{n-1})(\omega 0_{n-2}^2 q_e^2 + Q_{n-2})} q_e^2.$$

According to our bootstraps, we claim the following bounds. There exists a decomposition of $\partial_s^2 \alpha = \alpha_A + \alpha_B$ such that

$$\|\alpha\|_{L^\infty} + \|\partial_s \alpha\|_{L^\infty} \lesssim 1, \tag{47}$$

$$\|\alpha_A\|_{L^\infty} + \|\alpha_B\|_{L^2} \lesssim 1, \tag{48}$$

$$\|\gamma \langle\psi\rangle^m\|_{L^\infty} + \|\partial_s \gamma \langle\psi\rangle^m\|_{L^\infty} \lesssim \varepsilon^{0.99}, \tag{49}$$

$$\|\partial_s^2 \gamma \langle\psi\rangle^m\|_{L^2} \lesssim \varepsilon^{0.99}, \tag{50}$$

$$\|\gamma \langle\psi\rangle^m\|_{L^2} + \|\partial_s \gamma \langle\psi\rangle^m\|_{L^2} \lesssim \varepsilon^{0.99}. \tag{51}$$

We will prove these bounds as follows. First, we define

$$\begin{aligned}
\alpha^{(1)} &:= \frac{1}{\omega 0_{n-1}^2 q_e^2 + Q_{n-1}} = \frac{1}{D_{n-1}}, \\
\alpha^{(2)} &:= \frac{Q_{n-2}}{(\omega 0_{n-1}^2 q_e^2 + Q_{n-1})(\omega 0_{n-2}^2 q_e^2 + Q_{n-2})} = \frac{Q_{n-2}}{D_{n-1} D_{n-2}}, \\
D_n &:= \omega 0_n^2 q_e^2 + Q_n.
\end{aligned}$$

after which the following identities are valid:

$$\alpha = \alpha^{(1)} + \alpha^{(2)}, \qquad \gamma = q_e^2 \alpha^{(2)}. \tag{52}$$

We will henceforth prove the following bounds. We claim there exists a decomposition of $\partial_s^2 \alpha^{(1)} = \alpha_A^{(1)} + \alpha_B^{(1)}$, where

$$\|\alpha^{(1)}\|_{L^\infty} + \|\partial_s \alpha^{(1)}\|_{L^\infty} \lesssim 1, \tag{53}$$

$$\|\alpha_A^{(1)}\|_{L^\infty} + \|\alpha_B^{(1)}\|_{L^2} \lesssim 1, \tag{54}$$

$$\|\alpha^{(2)} \langle\psi\rangle^m\|_{L^\infty} + \|\partial_s \alpha^{(2)} \langle\psi\rangle^m\|_{L^\infty} \lesssim \varepsilon^{0.99}, \tag{55}$$

$$\|\partial_s^2 \alpha^{(2)} \langle\psi\rangle^m\|_{L^2} \lesssim \varepsilon^{0.99}, \tag{56}$$

$$\|\alpha^{(2)} \langle\psi\rangle^m\|_{L^2} + \|\partial_s \alpha^{(2)} \langle\psi\rangle^m\|_{L^2} \lesssim \varepsilon^{0.99}, \tag{57}$$

upon which using (52), we obtain (47), (48), (49), (50), and (51).

Proof of (53). Clearly, we have

$$\|\alpha^{(1)}\|_{L^\infty} \lesssim \frac{1}{\inf |D_{n-1}|} \lesssim \frac{1}{\omega 0_{n-1}^2 q_e^2 - \|Q_{n-1}\|_{L^\infty}} \lesssim \frac{1}{1 - \varepsilon^{0.99}} \lesssim 1.$$

Next, we have the identity $\partial_s \alpha^{(1)} = \frac{\partial_s D_{n-1}}{D_{n-1}^2}$. Since we have already established a lower bound on D_{n-1}, it suffices to estimate $\partial_s D_{n-1}$:

$$\|\partial_s \alpha^{(1)}\|_{L^\infty} \lesssim \|\partial_s D_{n-1}\|_{L^\infty} \lesssim \|\omega 0_{n-1}^2 \partial_s \{q_e^2\}\|_{L^\infty} + \|\partial_s Q_{n-1}\|_{L^\infty} \lesssim 1 + \varepsilon^{0.99} \lesssim 1,$$

where we have invoked the bootstraps (27) and (28). This proves the bound (53).

Proof of (54). For this bound, we differentiate once more to find the identity

$$
\begin{aligned}
\partial_s^2 \alpha^{(1)} &= \frac{\partial_s^2 D_{n-1}}{D_{n-1}^2} - 2\frac{|\partial_s D_{n-1}|^2}{D_{n-1}^3} \\
&= [\frac{1}{D_{n-1}^2} {\omega_0}_{n-1}^2 \partial_s^2 \{q_e^2\} + \frac{\partial_s D_{n-1}}{D_{n-1}^3} {\omega_0}_{n-1}^2 \partial_s \{q_e^2\}] \\
&\quad + [\frac{1}{D_{n-1}^2} \partial_s^2 Q_{n-1} + \frac{\partial_s D_{n-1}}{D_{n-1}^3} \partial_s Q_{n-1}] \\
&=: \alpha_A^{(1)} + \alpha_B^{(1)}.
\end{aligned}
$$

We estimate

$$
\|\alpha_A^{(1)}\|_{L^\infty} \lesssim |\omega_{0n-1}|^2 + \|\partial_s D_{n-1}\|_{L^\infty} |\omega_{0n-1}|^2 \lesssim 1,
$$

and

$$
\|\alpha_B^{(1)}\|_{L^2} \lesssim \|\partial_s^2 Q_{n-1}\|_{L^2} + \|\partial_s D_{n-1}\|_{L^\infty} \|\partial_s Q_{n-1}\|_{L^2} \lesssim \varepsilon^{0.99}.
$$

This proves the bound (54).

Proof of (55). We turn now to the definition of $\alpha^{(2)}$. We will use freely the bounds $|D_{n-1}| + |D_{n-2}| \gtrsim 1$ and $\|\partial_s D_{n-1}\|_{L^\infty} + \|\partial_s D_{n-2}\|_{L^\infty} \lesssim 1$, which have already been established. First, we have

$$
\|\alpha^{(2)} \langle \psi \rangle^m \|_{L^\infty} \lesssim \|Q_{n-2} \langle \psi \rangle^m\|_{L^\infty} \lesssim \|Q_{n-2}\|_{X_{2,m}} \lesssim \varepsilon^{0.99}.
$$

Next, we have the identity

$$
\partial_s \alpha^{(2)} = \frac{\partial_s Q_{n-2}}{D_{n-1} D_{n-2}} - \frac{Q_{n-2} \partial_s \{D_{n-1} D_{n-2}\}}{D_{n-1}^2 D_{n-2}^2}, \tag{58}
$$

from which we obtain

$$
\begin{aligned}
\|\partial_s \alpha^{(2)} \langle \psi \rangle^m \|_{L^\infty} &\lesssim \|\partial_s Q_{n-2} \langle \psi \rangle^m \|_{L^\infty} + \|Q_{n-2} \langle \psi \rangle^m\|_{L^\infty} (\|D_{n-1}\|_{L^\infty} \|\partial_s D_{n-2}\|_{L^\infty} \\
&\quad + \|D_{n-2}\|_{L^\infty} \|\partial_s D_{n-1}\|_{L^\infty}) \\
&\lesssim \| \ Q_{n-2}\|_{X_{2,m}} + \|Q_{n-2}\|_{X_{1,m}} (\|D_{n-1}\|_{L^\infty} \|\partial_s D_{n-2}\|_{L^\infty} \\
&\quad + \|D_{n-2}\|_{L^\infty} \|\partial_s D_{n-1}\|_{L^\infty}) \\
&\lesssim \varepsilon^{0.99}.
\end{aligned}
$$

This proves the bound (55).

Proof of (56). We differentiate (58) again to obtain the identity

$$
\partial_s^2 \alpha^{(2)} = \frac{\partial_s^2 Q_{n-2}}{D_{n-1} D_{n-2}} - 2\frac{\partial_s Q_{n-2} \partial_s \{D_{n-1} D_{n-2}\}}{D_{n-1}^2 D_{n-2}^2} + 2\frac{Q_{n-2} |\partial_s \{D_{n-1} D_{n-2}\}|^2}{D_{n-1}^3 D_{n-2}^3},
$$

after which we obtain the bound

$$
\|\partial_s^2 \alpha^{(2)} \langle \psi \rangle^m \|_{L^2} \lesssim \|\partial_s^2 Q_{n-2} \langle \psi \rangle^m \|_{L^2} + \|\partial_s Q_{n-2} \langle \psi \rangle^m \|_{L^2} \|\partial_s \{D_{n-1} D_{n-2}\}\|_{L^\infty}
$$

$$
\begin{aligned}
&+ \|Q_{n-2}\langle\psi\rangle^m\|_{L^2}\|\partial_s\{D_{n-1}D_{n-2}\}\|_{L^\infty}^2\\
&\lesssim \|Q_{n-2}\|_{X_{2,m}}\\
&\lesssim \varepsilon^{0.99}.
\end{aligned}
$$

This proves the bound (56).

Proof of (57). We have

$$
\|\alpha^{(2)}\langle\psi\rangle^m\|_{L^2} \lesssim \|Q_{n-2}\langle\psi\rangle^m\|_{L^2} \lesssim \|Q_{n-2}\|_{X_{2,m}} \lesssim \varepsilon^{0.99},
$$

and upon using (58), we have

$$
\begin{aligned}
\|\partial_s\alpha^{(2)}\langle\psi\rangle^m\|_{L^2} \lesssim\ & \|\partial_s Q_{n-2}\langle\psi\rangle^m\|_{L^2} + \|Q_{n-2}\langle\psi\rangle^m\|_{L^2}(\|D_{n-1}\|_{L^\infty}\|\partial_s D_{n-2}\|_{L^\infty}\\
& + \|D_{n-2}\|_{L^\infty}\|\partial_s D_{n-1}\|_{L^\infty})\\
\lesssim\ & \|Q_{n-2}\|_{X_{2,m}} + \|Q_{n-2}\|_{X_{1,m}}(\|D_{n-1}\|_{L^\infty}\|\partial_s D_{n-2}\|_{L^\infty}\\
& + \|D_{n-2}\|_{L^\infty}\|\partial_s D_{n-1}\|_{L^\infty})\\
\lesssim\ & \varepsilon^{0.99}.
\end{aligned}
$$

We have therefore established (53)–(57) and hence (47)–(51). From here, the desired estimates, (44)–(45), follow from an application of the product rule applied to the identity (46). Indeed, we have:

$$
\begin{aligned}
\|(U_{n-1}-U_{n-2})\langle\psi\rangle^m\|_{L^\infty} &\lesssim \|\alpha\|_{L^\infty}\|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^\infty} + \|\gamma\langle\psi\rangle^m\|_{L^\infty}\varepsilon|\Delta\overline{\omega_0}_{n-1}|\\
&\lesssim \|\Delta Q_{n-1}\|_{X_{2,m}} + \varepsilon^{1.99}|\Delta\overline{\omega_0}_{n-1}|,
\end{aligned}
$$

where we have used the bounds (47) and (49). In L^2, we similarly have

$$
\begin{aligned}
\|(U_{n-1}-U_{n-2})\langle\psi\rangle^m\|_{L^2} &\lesssim \|\alpha\|_{L^\infty}\|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2} + \|\gamma\langle\psi\rangle^m\|_{L^2}\varepsilon|\Delta\overline{\omega_0}_{n-1}|\\
&\lesssim \|\Delta Q_{n-1}\|_{X_{2,m}} + \varepsilon^{1.99}|\Delta\overline{\omega_0}_{n-1}|,
\end{aligned}
$$

where we have used the bounds (47) and (51).

Next, we have upon differentiating (46), the identity

$$
\partial_s\{U_{n-1}-U_{n-2}\} = \alpha\partial_s\Delta Q_{n-1} + \Delta Q_{n-1}\partial_s\alpha + \varepsilon|\Delta\omega_{0n-1}|\partial_s\gamma,
$$

after which we have the following L^∞ bound:

$$
\begin{aligned}
\|\partial_s\{U_{n-1}-U_{n-2}\}\langle\psi\rangle^m\|_{L^\infty} \lesssim\ & \|\alpha\|_{L^\infty}\|\partial_s\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^\infty}\\
& + \|\partial_s\alpha\|_{L^\infty}\|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^\infty}\\
& + \varepsilon\|\partial_s\gamma\langle\psi\rangle^m\|_{L^\infty}|\Delta\omega_{0n-1}|\\
\lesssim\ & \|\Delta Q_{n-1}\|_{X_{2,m}} + \varepsilon^{1.99}|\Delta\omega_{0n-1}|,
\end{aligned}
$$

where we have invoked (47) and (49). In L^2, we similarly have

$$
\begin{aligned}
\|\partial_s\{U_{n-1}-U_{n-2}\}\langle\psi\rangle^m\|_{L^2} \lesssim\ & \|\alpha\|_{L^\infty}\|\partial_s\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2}\\
& + \|\partial_s\alpha\|_{L^\infty}\|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2}\\
& + \varepsilon\|\partial_s\gamma\langle\psi\rangle^m\|_{L^2}|\Delta\omega_{0n-1}|
\end{aligned}
$$

$$\lesssim \|\Delta Q_{n-1}\|_{X_{2,m}} + \varepsilon^{1.99}|\Delta\omega_{0n-1}|,$$

where we have used the bounds (47) and (51).

Differentiating (46) twice in s, we obtain the identity

$$\begin{aligned}\partial_s^2\{U_{n-1} - U_{n-2}\} &= \alpha\partial_s^2\Delta Q_{n-1} + \Delta Q_{n-1}\partial_s^2\alpha + 2\partial_s\Delta Q_{n-1}\partial_s\alpha + \varepsilon|\Delta\omega_{0n-1}|\partial_s^2\gamma\\ &= \alpha\partial_s^2\Delta Q_{n-1} + \Delta Q_{n-1}\alpha_A + \Delta Q_{n-1}\alpha_B + 2\partial_s\Delta Q_{n-1}\partial_s\alpha\\ &\quad + \varepsilon|\Delta\omega_{0n-1}|\partial_s^2\gamma,\end{aligned}$$

where we use the decomposition $\partial_s^2\alpha = \alpha_A + \alpha_B$. We now estimate the L^2 norm as follows:

$$\begin{aligned}&\|\partial_s^2\{U_{n-1} - U_{n-2}\}\langle\psi\rangle^m\|_{L^2}\\ &\quad\lesssim \|\alpha\|_{L^\infty}\|\partial_s^2\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2} + \|\alpha_A\|_{L^\infty}\|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2}\\ &\qquad + \|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^\infty}\|\alpha_B\|_{L^2} + \|\partial_s\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2}\|\partial_s\alpha\|_{L^\infty}\\ &\qquad + \varepsilon\|\partial_s^2\gamma\langle\psi\rangle^m\|_{L^2}|\Delta\omega_{0n-1}|\\ &\quad\lesssim \|\partial_s^2\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2} + \|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2} + \|\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^\infty}\\ &\qquad + \|\partial_s\Delta Q_{n-1}\langle\psi\rangle^m\|_{L^2} + \varepsilon^{1.99}|\Delta\omega_{0n-1}|\\ &\quad\lesssim \|\Delta Q_{n-1}\|_{X_{2,m}} + \varepsilon^{1.99}|\Delta\omega_{0n-1}|,\end{aligned}$$

where we have used the bounds (47)–(50). We have therefore established the bounds (44)–(45), and this concludes the proof of the lemma.

Lemma 14. *Assume* (27)–(30) *are valid until the index* $n-1$. *Assume* (27)–(29) *are valid until the index* n. *Then*

$$\|\Delta Q_n\|_{X_{2,50}} \leq \varepsilon^{\frac{3}{4}}\|\Delta Q_{n-1}\|_{X_{2,50}} + \varepsilon^{\frac{1}{2}}|\Delta\overline{\omega_0}_n| + \varepsilon^{\frac{3}{2}}|\Delta\overline{\omega_0}_{n-1}|. \tag{59}$$

Proof. For this estimate, motivated by (26), we set

$$\begin{aligned}F &:= \left(1 - \frac{\omega_{0n-1}q_e}{\sqrt{\omega_{0n-1}^2q_e^2 + Q_{n-1}}}\right)\partial_s Q_{n-1} - \left(1 - \frac{\omega_{0n-2}q_e}{\sqrt{\omega_{0n-2}^2q_e^2 + Q_{n-2}}}\right)\partial_s Q_{n-2},\\ G &:= (\omega_{0n-1} - \omega_{0n-2})Q_{n-1},\\ b &:= \varepsilon(\overline{\omega_0}_n - \overline{\omega_0}_{n-1})q_e^2(s).\end{aligned}$$

According to (33), we fix $k = 2$, $m = 50$, which results in

$$\|\Delta Q_n\|_{X_{2,50}} \lesssim \sum_{k'=0}^{2}\|\partial_s^{k'}F\langle\psi\rangle^{54}\|_{L^2} + \sum_{k'=0}^{2}\sum_{j=0}^{2}\|\partial_s^{k'}\partial_\psi^j G\langle\psi\rangle^{50}\|_{L^2} + \|b\|_{H_s^3}. \tag{60}$$

We therefore estimate the two quantities appearing on the right-hand side above. We first address the term F, which we rewrite as follows

$$F = (1 - \sqrt{1 - U_{n-1}})\partial_s Q_{n-1} - (1 - \sqrt{1 - U_{n-2}})\partial_s Q_{n-2}$$

$$= (1-\sqrt{1-U_{n-1}})\partial_s \Delta Q_{n-1} + \partial_s Q_{n-2}(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}}) := F_1 + F_2.$$

An identical calculation to the estimate of the forcing, F, in Lemma 12 results in the bound

$$\|F_1 \langle\psi\rangle^{54}\|_{H_s^2 L_\psi^2} \lesssim \varepsilon^{0.99} \|\Delta Q_{n-1}\|_{X_{2,50}}.$$

We develop the following identities

$$\begin{aligned}
\partial_s F_2 &= \partial_s^2 Q_{n-2}\left(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}}\right) \\
&\quad - \frac{1}{2}\partial_s Q_{n-2}\Big(\frac{\partial_s U_{n-1}}{\sqrt{1-U_{n-1}}} - \frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-2}}}\Big) \\
&= \partial_s^2 Q_{n-2}\left(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}}\right) \\
&\quad - \frac{1}{2}\partial_s Q_{n-2}\frac{\partial_s U_{n-1} - \partial_s U_{n-2}}{\sqrt{1-U_{n-1}}} \\
&\quad - \frac{1}{2}\partial_s Q_{n-2}\Big(\frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-1}}} - \frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-2}}}\Big) \\
&= C_1 + C_2 + C_3.
\end{aligned}$$

We estimate $\partial_s F_2$ as follows. First,

$$\begin{aligned}
\|C_1\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}})\langle\psi\rangle^4\|_{L^\infty}\|\partial_s^2 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2} \\
&\lesssim \|(U_{n-1}-U_{n-2})\langle\psi\rangle^4\|_{L^\infty}\|\partial_s^2 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2} \\
&\lesssim \|(U_{n-1}-U_{n-2})\langle\psi\rangle^4\|_{L^\infty}\|Q_{n-2}\|_{X_{1,50}} \\
&\lesssim \varepsilon^{0.99}\|(U_{n-1}-U_{n-2})\langle\psi\rangle^4\|_{L^\infty}.
\end{aligned}$$

Next, to estimate C_2, we have

$$\begin{aligned}
\|C_2\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s Q_{n-2}\langle\psi\rangle^{50}\|_{L^\infty}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^2} \\
&\lesssim \|\partial_s Q_{n-2}\|_{X_{2,50}}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^2} \\
&\lesssim \varepsilon^{0.99}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^2}.
\end{aligned}$$

Finally, to estimate C_3, we have

$$\begin{aligned}
\|C_3\langle\psi\rangle^{54}\|_{L^2} &\lesssim \left\|\partial_s Q_{n-2}\langle\psi\rangle^{50}\right\|_{L^2}\|\partial_s U_{n-2}\langle\psi\rangle^4\|_{L^\infty}\|U_{n-1}-U_{n-2}\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{0,50}}\|Q_{n-2}\|_{X_{2,4}}\|U_{n-1}-U_{n-2}\|_{L^\infty} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99}\|U_{n-1}-U_{n-2}\|_{L^\infty}.
\end{aligned}$$

We now move to the second derivative, $\partial_s^2 F_2$, which we will treat as follows:

$$\partial_s^2 F_2 = \partial_s C_1 + \partial_s C_2 + \partial_s C_3.$$

We have

$$\partial_s C_1 = \partial_s^3 Q_{n-2}(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}})$$

$$
\begin{aligned}
&- \frac{1}{2}\partial_s^2 Q_{n-2}\Big(\frac{\partial_s U_{n-1}}{\sqrt{1-U_{n-1}}} - \frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-2}}}\Big) \\
&= \partial_s^3 Q_{n-2}(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}}) - \frac{1}{2}\partial_s^2 Q_{n-2}\Big(\frac{\partial_s U_{n-1} - \partial_s U_{n-2}}{\sqrt{1-U_{n-1}}}\Big) \\
&- \frac{1}{2}\partial_s^2 Q_{n-2}\Big(\frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-1}}} - \frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-2}}}\Big) \\
&=: C_{1,1} + C_{1,2} + C_{1,3}.
\end{aligned}
$$

First, we estimate

$$
\begin{aligned}
\|C_{1,1}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \Big\|(\sqrt{1-U_{n-1}} - \sqrt{1-U_{n-2}})\langle\psi\rangle^4\Big\|_{L^\infty} \Big\|\partial_s^3 Q_{n-2}\langle\psi\rangle^{50}\Big\|_{L^2} \\
&\lesssim \Big\|(U_{n-1} - U_{n-2})\langle\psi\rangle^4\Big\|_{L^\infty} \|\partial_s^3 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2} \\
&\lesssim \Big\|(U_{n-1} - U_{n-2})\langle\psi\rangle^4\Big\|_{L^\infty} \|Q_{n-2}\|_{X_{2,50}} \\
&\lesssim \varepsilon^{0.99}\|(U_{n-1} - U_{n-2})\langle\psi\rangle^4\|_{L^\infty}.
\end{aligned}
$$

Next, to estimate $C_{1,2}$, we have

$$
\begin{aligned}
\|C_{1,2}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s^2 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{1,50}}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^\infty} \\
&\lesssim \varepsilon^{0.99}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^\infty}.
\end{aligned}
$$

Next, to estimate $C_{1,3}$, we have

$$
\begin{aligned}
\|C_{1,3}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s^2 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2}\|\partial_s U_{n-2}\langle\psi\rangle^4\|_{L^\infty}\|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{1,50}}\|Q_{n-2}\|_{X_{2,4}}\|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99}\|U_{n-1} - U_{n-2}\|_{L^\infty}.
\end{aligned}
$$

We next move to the $\partial_s C_2$ contributions, for which we record the identity

$$
\begin{aligned}
\partial_s C_2 &= -\frac{1}{2}\partial_s^2 Q_{n-2}\frac{\partial_s U_{n-1} - \partial_s U_{n-2}}{\sqrt{1-U_{n-1}}} - \frac{1}{2}\partial_s Q_{n-2}\frac{\partial_s^2 U_{n-1} - \partial_s^2 U_{n-2}}{\sqrt{1-U_{n-1}}} \\
&+ \frac{1}{4}\partial_s Q_{n-2}\partial_s U_{n-1}(1-U_{n-1})^{-\frac{3}{2}}(\partial_s U_{n-1} - \partial_s U_{n-2}) \\
&=: C_{2,1} + C_{2,2} + C_{2,3}.
\end{aligned}
$$

We first estimate $C_{2,1}$ for which we have

$$
\begin{aligned}
\|C_{2,1}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s^2 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{1,50}}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^\infty} \\
&\lesssim \varepsilon^{0.99}\|(\partial_s U_{n-1} - \partial_s U_{n-2})\langle\psi\rangle^4\|_{L^\infty}.
\end{aligned}
$$

Next, we have

$$
\|C_{2,2}\langle\psi\rangle^{54}\|_{L^2} \lesssim \|\partial_s Q_{n-2}\langle\psi\rangle^{50}\|_{L^\infty}\|(\partial_s^2 U_{n-1} - \partial_s^2 U_{n-2})\langle\psi\rangle^4\|_{L^2}
$$

$$\lesssim \|\partial_s Q_{n-2}\|_{X_{2,50}} \|(\partial_s^2 U_{n-1} - \partial_s^2 U_{n-2})\langle\psi\rangle^4\|_{L^2}$$
$$\lesssim \varepsilon^{0.99} \|(\partial_s^2 U_{n-1} - \partial_s^2 U_{n-2})\langle\psi\rangle^4\|_{L^2}.$$

Finally, we have the $C_{2,3}$ contribution for which we estimate

$$\begin{aligned}
\|C_{2,3}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s Q_{n-2}\langle\psi\rangle^{50}\|_{L^\infty} \|\partial_s U_{n-1}\langle\psi\rangle^4\|_{L^\infty} \|(\partial_s U_{n-1} - \partial_s U_{n-2})\|_{L^2} \\
&\lesssim \|Q_{n-2}\|_{X_{2,50}} \|Q_{n-1}\|_{X_{2,4}} \|(\partial_s U_{n-1} - \partial_s U_{n-2})\|_{L^2} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99} \|(\partial_s U_{n-1} - \partial_s U_{n-2})\|_{L^2}.
\end{aligned}$$

We next compute $\partial_s C_3$, which results in the following identity,

$$\begin{aligned}
\partial_s C_3 &= \frac{1}{2}\partial_s^2 Q_{n-2}\Big(\frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-1}}} - \frac{\partial_s U_{n-2}}{\sqrt{1-U_{n-2}}}\Big) \\
&\quad + \frac{1}{2}\partial_s Q_{n-2}\Big(\frac{\partial_s^2 U_{n-2}}{\sqrt{1-U_{n-1}}} - \frac{\partial_s^2 U_{n-2}}{\sqrt{1-U_{n-2}}}\Big) \\
&\quad - \frac{1}{4}\partial_s Q_{n-2}\partial_s U_{n-2}\frac{1}{(1-U_{n-1})^{\frac{3}{2}}}(\partial_s U_{n-1} - \partial_s U_{n-2}) \\
&\quad - \frac{1}{4}\partial_s Q_{n-2}|\partial_s U_{n-2}|^2\left(\frac{1}{(1-U_{n-1})^{\frac{3}{2}}} - \frac{1}{(1-U_{n-2})^{\frac{3}{2}}}\right) =: \sum_{i=1}^{4} C_{3,i}.
\end{aligned}$$

We estimate first $C_{3,1}$ as follows

$$\begin{aligned}
\|C_{3,1}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s^2 Q_{n-2}\langle\psi\rangle^{50}\|_{L^2} \|\partial_s U_{n-2}\langle\psi\rangle^4\|_{L^\infty} \|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{1,50}} \|Q_{n-2}\|_{X_{2,4}} \|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99} \|U_{n-1} - U_{n-2}\|_{L^\infty}.
\end{aligned}$$

Next, we estimate $C_{3,2}$ as follows

$$\begin{aligned}
\|C_{3,2}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s Q_{n-2}\langle\psi\rangle^{50}\|_{L^\infty} \|\partial_s^2 U_{n-2}\langle\psi\rangle^4\|_{L^2} \|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{2,50}} \|\partial_s^2 U_{n-2}\langle\psi\rangle^4\|_{L^2} \|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \|Q_{n-2}\|_{X_{2,50}} \|Q_{n-2}\|_{X_{2,4}} \|U_{n-1} - U_{n-2}\|_{L^\infty} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99} \|U_{n-1} - U_{n-2}\|_{L^\infty}.
\end{aligned}$$

Next, we estimate $C_{3,3}$ as follows

$$\begin{aligned}
\|C_{3,3}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s Q_{n-2}\langle\psi\rangle^{50}\|_{L^\infty} \|\partial_s U_{n-2}\langle\psi\rangle^4\|_{L^\infty} \|\partial_s U_{n-1} - \partial_s U_{n-2}\|_{L^2} \\
&\lesssim \|Q_{n-2}\|_{X_{2,50}} \|Q_{n-2}\|_{X_{2,4}} \|\partial_s U_{n-1} - \partial_s U_{n-2}\|_{L^2} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99} \|\partial_s U_{n-1} - \partial_s U_{n-2}\|_{L^2}.
\end{aligned}$$

Finally, we estimate $C_{3,4}$ as follows

$$\begin{aligned}
\|C_{3,4}\langle\psi\rangle^{54}\|_{L^2} &\lesssim \|\partial_s Q_{n-2}\langle\psi\rangle^{50}\|_{L^\infty} \|\partial_s U_{n-2}\langle\psi\rangle^2\|_{L^\infty}^2 \|U_{n-1} - U_{n-2}\|_{L^2} \\
&\lesssim \|Q_{n-2}\|_{X_{2,50}} \|Q_{n-2}\|_{X_{2,2}}^2 \|U_{n-1} - U_{n-2}\|_{L^2} \\
&\lesssim \varepsilon^{0.99}\varepsilon^{0.99}\varepsilon^{0.99} \|U_{n-1} - U_{n-2}\|_{L^2}.
\end{aligned}$$

Now, upon invoking (44)–(45), the above estimates give

$$
\begin{aligned}
&\|F_2\langle\psi\rangle^{54}\|_{H^2_s L^2_\psi} \\
&\lesssim \varepsilon^{0.99}\Big(\sum_{j=0}^{1}\|(\partial_s^j U_{n-1}-\partial_s^j U_{n-2})\langle\psi\rangle^4\|_{L^\infty} \\
&\quad+\sum_{j=0}^{2}\|(\partial_s^j U_{n-1}-\partial_s^j U_{n-2})\langle\psi\rangle^4\|_{L^2}\Big) \\
&\lesssim \varepsilon^{0.99}\left(\|\Delta Q_{n-1}\|_{X_{2,4}}+\epsilon^{1.99}|\Delta\overline{c}_{n-1}|\right).
\end{aligned}
$$

Next, we clearly have

$$
\begin{aligned}
&\sum_{k'=0}^{2}\sum_{j=0}^{2}\|\partial_s^{k'}\partial_\psi^j G\langle\psi\rangle^{50}\|_{L^2} \le |\omega_{0n-1}-\omega_{0n-2}|\sum_{k'=0}^{2}\sum_{j=0}^{2}\left\|\partial_s^{k'}\partial_\psi^j Q_{n-1}\langle\psi\rangle^{50}\right\|_{L^2} \\
&\lesssim \frac{1}{\left|\omega_{0n-1}+\omega_{0n-2}\right|}|\omega_{0n-1}^2-\omega_{0n-2}^2|\|Q_{n-1}\|_{X_{2,50}} \\
&\lesssim \varepsilon\varepsilon^{0.99}|\overline{\omega_0}_{n-1}-\overline{\omega_0}_{n-2}|.
\end{aligned}
$$

Finally, we have the boundary condition

$$
\|b\|_{H^3_s} \lesssim \varepsilon\|q_e\|_{H^3}^3|\Delta\overline{\omega_0}_n|.
$$

Consolidating all the above bounds with estimate (59) concludes the proof of the lemma.

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Declarations

Conflict of interest The authors declare the are no conflict of interest.

Appendix A. Derivation of Near-Boundary Navier–Stokes Equations

In this section, we give the detailed calculations for the Navier–Stokes equations claimed in Sect. 2. We recall the standard identities, which will be used in the next lemmas:

$$n'(s) = \gamma(s)\tau(s), \qquad \tau'(s) = \gamma(s)n(s).$$

We recall that the map

$$\begin{aligned} \{x \in M : \quad 0 < \operatorname{dist}(x, \partial M) < \delta\} &\to \mathbb{T}_L \times (0, \delta) \\ x &\to (s, z) = (s(x_1, x_2), z(x_1, x_2)) \end{aligned}$$

is a diffeomorphism. In this transformation, we have

$$\nabla_x s = \frac{\tau}{J}, \qquad \nabla_x z = n(s).$$

For a vector field $u : M \to \mathbb{R}^2$, we also have

$$\begin{aligned} u_1 &= n_2 u_\tau - \tau_2 u_n = \tau_1 u_\tau - \tau_2 u_n, \\ u_2 &= -n_1 u_\tau + \tau_1 u_n = \tau_2 u_\tau + \tau_1 u_n. \end{aligned}$$

Lemma 15. *For any vector field $u : M \to \mathbb{R}^2$ and scalar function $f : M \to \mathbb{R}$ supported near the boundary ∂M there holds*

$$u \cdot \nabla f = \frac{u_\tau}{J} \partial_s f + u_n \partial_z f.$$

In particular, by choosing $u = \tau$ and $u = n$ respectively, there hold

$$\tau \cdot \nabla f = \frac{1}{J} \partial_s f, \qquad n \cdot \nabla f = \partial_z f.$$

Proof. This follows by direct calculation. We have

$$\begin{aligned} u \cdot \nabla f = u \cdot \nabla_x f(s, z) &= u_1 \left(\partial_s f \partial_{x_1} s + \partial_z f \partial_{x_1} z\right) + u_2 \left(\partial_s f \partial_{x_2} s + \partial_z f \partial_{x_2} z\right) \\ &= u_1 \left(\partial_s f \frac{\tau_1}{J} + \partial_z f n_1\right) + u_2 (\partial_s f \cdot \frac{\tau_2}{J} + \partial_z f n_2) \\ &= \frac{u \cdot \tau}{J} \partial_s f + (u \cdot n) \partial_z f. \end{aligned}$$

Lemma 16. *The following identities holds for any given vector field $u : M \to \mathbb{R}^2$:*

$$\begin{aligned} (u \cdot \nabla u) \cdot \tau &= \left(\frac{u_\tau}{J}, u_n\right) \cdot \nabla_{s,z} u_\tau - \frac{\gamma(s)}{J} u_\tau u_n, \\ (u \cdot \nabla u) \cdot n &= \left(\frac{u_\tau}{J}, u_n\right) \cdot \nabla_{s,z} u_n - \frac{\gamma(s)}{J} u_\tau^2, \\ \Delta u \cdot \tau &= \frac{1}{J} \partial_z (J \partial_z u_\tau) + \frac{1}{J} \partial_s \left(\frac{1}{J} \partial_s u_\tau\right) - \frac{1}{J} \partial_s \left(\frac{\gamma u_n}{J}\right) - \frac{\gamma}{J} \left(\gamma u_\tau + \partial_s u_n\right), \\ \Delta u \cdot n &= \frac{1}{J} \partial_z (J \partial_z u_n) + \frac{1}{J} \partial_s \left(\frac{1}{J} \partial_s u_n\right) - \frac{1}{J} \partial_s \left(\frac{\gamma u_\tau}{J}\right) - \frac{\gamma}{J} \left(\partial_s u_\tau - \gamma u_n\right), \\ \nabla \cdot u &= \frac{1}{J} \left(\partial_s u_\tau - \gamma(s) u_n\right) + \partial_z u_n, \\ \nabla^\perp \cdot u &= \partial_1 u_2 - \partial_2 u_1 = \frac{\gamma}{J} u_\tau - \partial_z u_\tau + \partial_s u_n. \end{aligned}$$

Proof. We check the first, the third and the fifth identities only, and the proofs for other identities are similar. We have

$$\begin{aligned}(u\cdot\nabla u)\cdot\tau &= (u\cdot\nabla_x u_1)\tau_1 + (u\cdot\nabla_x u_2)\tau_2\\ &= \left(\frac{u_\tau}{J}, u_n\right)\cdot\nabla_{s,z}u_1\tau_1 + \left(\frac{u_\tau}{J}, u_n\right)\cdot\nabla_{s,z}u_2\tau_2\\ &= \left(\frac{u_\tau}{J}, u_n\right)\cdot\nabla_{s,z}u_\tau - \frac{u_\tau}{J}\left(\tau_1'(s)u_1 + \tau_2'(s)u_2\right).\end{aligned}$$

We note that

$$\begin{aligned}\tau_1'u_1 + \tau_2'u_2 &= \gamma n_1(\tau_1 u_\tau - \tau_2 u_n) + \gamma n_2(\tau_2 u_\tau + \tau_1 u_n)\\ &= \gamma(-\tau_2)(\tau_1 u_\tau - \tau_2 u_n) + \gamma\tau_1(\tau_2 u_\tau + \tau_1 u_n)\\ &= -\gamma\tau_1\tau_2 u_\tau + \gamma\tau_2^2 u_n + \gamma\tau_1\tau_2 u_\tau + \gamma\tau_1^2 u_n = \gamma u_n.\end{aligned}$$

Combining the above with the previous calculation, we obtain

$$(u\cdot\nabla u)\cdot\tau = \left(\frac{u_\tau}{J}, u_n\right)\cdot\nabla_{s,z}u_\tau - \frac{\gamma(s)}{J}u_\tau u_n.$$

Now we show the third identity. We have

$$\begin{aligned}\Delta u\cdot\tau = \Delta u_1\cdot\tau_1 + \Delta u_2\cdot\tau_2 &= \sum_{i=1}^{2}\frac{1}{J}\left(\partial_z(J\partial_z u_i) + \partial_s\left(\frac{1}{J}\partial_s u_i\right)\right)\tau_i\\ &= \frac{1}{J}\partial_z(J\partial_z u_\tau) + \frac{1}{J}\partial_\theta\left(\frac{1}{J}\partial_\theta(u\cdot\tau)\right) - \frac{1}{J}\partial_s\left(\frac{1}{J}u\cdot\partial_s\tau\right) - \sum_i\frac{1}{J}\partial_s u_i\tau_i'\\ &= \frac{1}{J}\partial_z(J\partial_z u_\tau) + \frac{1}{J}\partial_\theta\left(\frac{1}{J}\partial_s u_\tau\right) - \frac{1}{J}\partial_s\left(\frac{\gamma u_n}{J}\right) - \frac{\gamma}{J}\left(\gamma u_\tau + \partial_s u_n\right).\end{aligned}$$

For incompressibility, we find

$$\begin{aligned}\nabla\cdot u = \partial_{x_1}u_1 + \partial_{x_2}u_2 &= \sum_i \partial_{x_i}s\partial_s u_i + \partial_{x_i}z\partial_z u_i = \sum_i\frac{\tau_i}{J}\partial_s u_i + n_i\partial_z u_i\\ &= \frac{1}{J}\left(\partial_s u_\tau - \gamma(s)u_n\right) + \partial_z u_n.\end{aligned}$$

The proof is complete.

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T. D. Drivas
Department of Mathematics,
Stony Brook University,
Stony Brook
NY
11794 USA.
e-mail: tdrivas@math.stonybrook.edu

and

S. Iyer
Department of Mathematics,
University of California,
Davis
CA
95616 USA.
e-mail: sameer@math.ucdavis.edu

and

T. T. Nguyen
Department of Mathematics,
University of Wisconsin-Madison,
Madison
WI
53706 USA.
e-mail: tnguyen67@wisc.edu

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