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Strong Well-Posedness of the Q-Tensor Model for Liquid Crystals: The Case of Arbitrary Ratio of Tumbling and Aligning Effects ξ

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Abstract

The Beris–Edwards model of nematic liquid crystals couples an equation for the molecular orientation described by the Q-tensor with a Navier–Stokes type equation with an additional non-Newtonian stress caused by the molecular orientation. Both equations contain a parameter $\xi \in \mathbb{R}$ measuring the ratio of tumbling and alignment effects. Previous well-posedness results largely vary on the space dimension n and the constraints of the parameter $\xi \in \mathbb{R}$. This work addresses strong well-posedness of this model, first locally and then globally for small initial data, both in the L^p-L^2-setting for $p > \frac{4}{4-n}$, in the general cases, i.e., for $n = 2, 3$ and without any restriction on ξ. The approach is based on methods from quasilinear equations and the fact that the associated linearized operator admits maximal L^p-L^2-regularity. The proof of the latter property relies on techniques from sectorial operators, Schur complements and $\mathcal{J}$-symmetry.

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1. Introduction

In physics there are various ways of describing order parameters in liquid crystals: the Doi–Onsager, the Landau–De Gennes and the Ericksen–Leslie theory. These lead to mathematical theories at various levels. The Ericksen-Leslie-model is a so-called vector model. Vector theories have the drawback that they do not

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respect head-to-tail symmetry (see [2]) which leads to difficulties when describing defect structures. Another type of model describing liquid crystal flow is the Q-tensor model, including the Landau-De Gennes theory. In contrast to vector models, it uses a symmetric, traceless 3×3-matrix Q to describe the alignment of molecules; they allow the description of the biaxiality of the alignment.

There exist several models which describe the dynamics of liquid crystals making use of the Q-tensor. We concentrate here on the Beris–Edwards system describing biaxial liquid crystals by the set of equations (1.1)–(1.6) given in [7] and below in (1.1). The evolution of Q is driven by the free energy of the molecules as well as by transport, distortion, tumbling and alignment effects caused by the flow. The flow field is forced by an additional non-Newtonian stress caused by the molecules orientation and expressed in terms of Q and ξ, where the parameter ξ measures the *ratio of tumbling and alignment effects*. For more information on various liquid crystal systems we refer to the monographs by Sonnet and Virga [29], and Virga [33] as well to the survey articles [16,31] and [35].

Previous analytical results on the Beris-Edwards model concentrated mainly on the case where the parameter ξ is zero or $n = 2$, see e.g. [8].

In what follows we analyze the Beris-Edwards model [7] in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary in dimensions $n = 2, 3$ and for arbitrary $\xi \in \mathbb{R}$. This Q-tensor model, going back to the work of de Gennes [12] describes, as written above, in contrast to the Ericksen–Leslie system, biaxial liquid crystals. The Beris–Edwards model is given by the following set of equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \operatorname{div}(\tau(Q,H) + \sigma(Q,H)), & \text{in } \Omega, t \in (0,T), \\ \operatorname{div} u = 0, & \text{in } \Omega, t \in (0,T), \\ \partial_t Q + (u \cdot \nabla)Q - S(\nabla u, Q) = \Gamma H, & \text{in } \Omega, t \in (0,T), \\ (u, \partial_\nu Q) = (0,0), & \text{on } \partial\Omega, t \in (0,T), \\ (u,Q)|_{t=0} = (v_0, Q_0), & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here

$$\begin{aligned} S(\nabla u, Q) :=& (\xi D(u) + W(u))(Q + \mathbb{I}/n) + (Q + \mathbb{I}/n)(\xi D(u) - W(u)) \\ &- 2\xi (Q + \mathbb{I}/n) \operatorname{tr}(Q \nabla u), \\ H(Q) :=& \lambda \Delta Q - aQ + b(Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/n) - c \operatorname{tr}(Q^2) Q, \\ \tau(Q,H) :=& -\lambda \nabla Q \odot \nabla Q - \xi (Q + \mathbb{I}/n) H - \xi H (Q + \mathbb{I}/n) \\ &+ 2\xi (Q + \mathbb{I}/n) \operatorname{tr}(QH), \\ \sigma(Q,H) :=& QH - HQ = \lambda(Q\Delta Q - \Delta Q Q) = \lambda \sigma(Q, \Delta Q), \end{aligned}$$

where u, p and Q describe the velocity, pressure and the molecular orientation of the liquid crystal, respectively. Moreover, Γ, λ, ν and a are positive constants and b and c are constants. For simplicity, we set $\nu = \Gamma = \lambda = a = b = c = 1$, which does not change our analysis concerning the existence and uniqueness of local solutions for arbitrary large data and global solutions for small data described in Theorems 3.1 and 3.2. This is, of course, different when we are considering as in Remark 3.3 the set of equilibria for equation (1.1).

More specifically, Q takes values in $\mathbb{S}_0^n$, the space of symmetric $n \times n$-matrices having trace zero. In three space dimensions, Q having one, two, or three different eigenvalues corresponds then to the case of isotropic, uniaxial and biaxial liquid crystals, respectively. The term S describes how the gradient of the velocity, ∇u, stretches and rotates the order parameter Q. The expression H relates to the variational derivative of the free energy functional which uses the one constant approximation for the Oseen-Frank energy of liquid crystals together with a Landau-de Gennes expression for the bulk energy, (cf. e.g. [3,7]). The terms τ and σ correspond to the symmetric and antisymmetric part of the stress tensor, respectively, and the parameter ξ describes the ratio of tumbling and alignment effects.

Moreover, $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ denote the symmetric and antisymmetric part of the gradient of u, respectively. Regarding matrix-related notation, in the above $\mathbb{I}$ stands for the $n \times n$-valued identity matrix (although we shall use it later also for other identities), tr describes the trace of a matrix, and the (i, j)-component of $\nabla Q \odot \nabla Q$ equals $\operatorname{tr}(\partial_i Q \partial_j Q)$.

Analytically, the Beris–Edwards model has been studied by many authors. The existing results on weak and strong solutions as well as their uniqueness properties depend, however, largely on the space dimension n and the constraints on the parameter ξ.

Concerning weak solutions, the first results were established by Paicu and Zarnescu in [22]. They considered the case $\Omega = \mathbb{R}^n$ and $\xi = 0$ and were able to prove global existence of weak solutions to Eq. (1.1) in dimension $n = 2$ and $n = 3$, as well as weak-strong uniqueness for $n = 2$. In [23] they expanded their result to the case where $\xi \neq 0$ is sufficiently small. The existence of weak solutions for the general case $\xi \in \mathbb{R}$ and $n = 2, 3$ was proven by Wilkinson [32]. He also established higher regularity results in two dimensions when $\xi = 0$ for a singular potential as bulk energy. Furthermore, Feireisl, Rocca, Schimperna and Zarnescu [11] proved existence of weak solutions for an isothermal variant with the above mentioned singular potential as bulk energy for periodic boundary conditions. Moreover, existence of global weak solutions with a more general energy functional was shown by Huang and Ding [17]. Abels, Dolzmann and Liu [3] showed local well-posedness and global existence for weak solutions.

Concerning strong solutions, Abels, Dolzmann, and Liu [4] established local, strong well-posedness of equations (1.1) with different boundary conditions for the case $\xi = 0$. Cavaterra, Rocca, Wu and Xu [8] showed global well-posedness of Eq. (1.1) in dimension $n = 2$ for general $\xi \in \mathbb{R}$. Furthermore, Xiao [34] showed local well-posedness and global existence for small data for the case $\xi = 0$ and with the additional assumption of $S = 0$. Global well-posedness and decay estimates for a modified stress tensor of equations (1.1) were shown by Schonbek and Shibata [28] in the case $\Omega = \mathbb{R}^n$. The above Q-tensor model for general values of $\xi \in \mathbb{R}$ but subject to inhomogeneous Dirichlet boundary conditions was investigated by Liu and Wang in [19] even in the setting of anisotropic elasticity. They proved the existence of a unique, local strong solution in the case of strong anchoring boundary conditions for Q for a non-empty subset of certain admissible initial data (u_0, Q_0)

belonging to $H^1_{0,\sigma} \times H^2$. Note that the size of this subset remains unclear while our approach does not require any such admissibility conditions.

It is the aim of this paper to give a rather complete understanding of the well-posedness of the Beris-Edwards model in dimension $n = 2, 3$ in the case of an arbitrary ratio of tumbling and aligning effects. Let us note that by well-posedness we mean well-posedness in the strong sense. We prove in Sect. 3 first that the Beris–Edwards model is locally, strongly well-posed for arbitrtay $\xi \in \mathbb{R}$ and $n = 2, 3$ and secondly, that the trivial equilibrium $v^* = 0$ is stable in the sense that for initial data close to this equilibrium, the solution exists globally and converges to some equilibrium point as $t \to \infty$.

Analytically, the main difficulty in the investigation of the Q-tensor model is given by the fact that the latter equations formulate a quasilinear, mixed order system where the diagonal parts are of second order operators, whereas the off-diagonal parts are of third and first order. Let us note that, in contrast to systems, whose principal parts give rise to well-posedness results, see e.g. [6,10], there is no general theory for mixed order systems. Introducing an anisotropic ground space, we obtain a system where every entry is of highest order. The analysis of this latter system is one of the main difficulties of our approach.

It is interesting to comment on previous approaches. Xiao [34] assumed that ξ as well as the coupling term S is zero. In this case, the system is of upper triangular form and solvability properties can be shown rather easily. Schonbek and Shibata [28] studied a modified stress tensor, which results in a system which is of perturbed lower triangular form. The perturbation is then controlled by an additional smallness assumption. Murata and Shibata [21] considered only the full space case with data close to zero. The linearisation then simplifies considerably to the case where the relevant and difficult terms vanish. Thus, it reduces to the case of constant coefficients. Let us emphasize that our approach includes the general case of arbitrary Q-tensors Q_0. Moreover, our strategy allows for dealing not only with $\mathbb{R}^n$ but with the physically relevant case of bounded domains. Abels, Dolzmann and Liu [4] assumed $\xi = 0$ and different boundary conditions. This results in additional symmetry properties of the system. They then show local strong well-posedness of a regularized system for which the different boundary conditions are necessary, and, secondly, they transfer this property to the limiting system by higher-order energy estimates. When performing these higher order energy estimates they exploit the above mentioned additional symmetry that arises only when $\xi = 0$. It is unclear whether their method can be generalized to the case of $\xi \neq 0$.

The rest of this article is structured as follows: in Sect. 2 we describe the Beris-Edwards model and its derivation in some detail. Section 3 is devoted to the presentation of our main results. Thereafter, we state well-posedness results for quasilinear evolution equations. Section 5 deals with the linearization of the quasilinear formulation of (1.1) and its maximal regularity properties. Here we make use of methods from Schur complements and $\mathcal{J}$-symmetry. We estimate the nonlinear terms in Sect. 6, before we prove the main theorems in Sect. 7.

2. Description of the Model

In this section we give a short explanation of the Q-tensor model described in (1.1). For a very thorough study of liquid crystal models we refer to the two monographs by Virga [33] and Sonnet and Virga [29].

Given a point $x \in \Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, let ϱ_x be the probability density function of the molecular orientation. These molecular orientations are elements of S^{n-1}, the unit sphere in $\mathbb{R}^n$. The Q-tensor is then defined as the traceless second-order moment of this probablity density function, i.e.,

$$Q(x) := \int_{S^{n-1}} \Big(\omega \otimes \omega - \frac{1}{n}\mathbb{I}\Big)\varrho_x(\omega) d\sigma^{n-1}(\omega), \quad x \in \Omega, \tag{2.1}$$

where σ^{n-1} denotes the surface measure on S^{n-1}. Since $\mathrm{tr}(\omega \otimes \omega) = |\omega|^2 = 1$, we see that $\mathrm{tr}\, Q = 0$. Thus $Q(x)$ belongs to $\mathbb{S}^n_{0,\mathbb{R}}$, the space of symmetric, traceless $n \times n$-matrices; that is,

$$Q(x) \in \mathbb{S}^n_{0,\mathbb{R}} := \{Q \in \mathbb{R}^{n\times n} : Q = Q^T, \mathrm{tr}\, Q = 0\}.$$

In dimension $n = 3$, $Q(x)$ has one, two or three different eigenvalues corresponding to the isotropic, uniaxial or biaxial liquid crystals, respectively. These models were investigated by Ball, Majumdar [1] and Ball and Zarnescu [2]. In the following, we summarize some of their ideas.

Since $Q(x)$ is symmetric, it has three (not necessarily different) eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. The representation (2.1) yields that $\lambda_1, \lambda_2, \lambda_3 \in [-\frac{1}{3}, \frac{2}{3}]$. Denoting by e_i the eigenvectors of λ_i, $i = 1, 2, 3$, of unit length, we see that $Q(x)$ can be represented as

$$Q(x) = (2\lambda_1 + \lambda_2)(e_1 \otimes e_1) + (\lambda_1 + 2\lambda_2)(e_2 \otimes e_2) - (\lambda_1 + \lambda_2)\mathbb{I}. \tag{2.2}$$

Now, if $Q(x)$ has only one distinct eigenvalue, this eigenvalue is zero and thus $Q(x) = 0$, which correponds to the isotropic case, i.e., the molecules are randomly distributed.

If $Q(x)$ has two different eigenvalues (without loss of generality $\lambda_2 = \lambda_3$, $\lambda_2 = -\frac{1}{2}\lambda_1$), then (2.2) implies that $Q(x)$ can be represented as

$$Q(x) = s\Big((e_1 \otimes e_1) - \frac{1}{3}\mathbb{I}\Big),$$

where $s := \frac{3}{2}\lambda_1 \in [-\frac{1}{2}, 1]$. The unit vector e_1 corresponds then to the director field in the Ericksen-Leslie model. For analytical results concerning this model we refer to [14–16]. This case corresponds to the uniaxial nematic state.

If $Q(x)$ has three different eigenvalues, then (2.2) yields that it can be respresented as

$$Q(x) = s_1\Big(e_1 \otimes e_1 - \frac{1}{3}\mathbb{I}\Big) + s_2\Big(e_2 \otimes e_2 - \frac{1}{3}\mathbb{I}\Big), \tag{2.3}$$

where $s_1 = 2\lambda_1 + \lambda_2$ and $s_2 = \lambda_1 + 2\lambda_2$ for $s_1, s_2 \in [-\frac{1}{2}, 1]$ and $s_1 \neq s_2$. This case corresponds to the biaxial nematic state, i.e., there are two axes of symmetry within the arrangement of the molecules. Here the direction of these two axes are e_1 and e_2, while s_1 and s_2 measure the respective degrees of the orientational ordering.

We now turn our attention to the free energy functional. For given constants $a, \lambda > 0, b, c \in \mathbb{R}$ and $Q : \Omega \to \mathbb{S}^n_{0,\mathbb{R}}$ its free energy functional is given by

$$\mathcal{F}(Q) = \int_\Omega \Big(\frac{\lambda}{2}|\nabla Q(x)|^2 + \frac{a}{2}\mathrm{tr}(Q(x)^2) - \frac{b}{3}\,\mathrm{tr}\,(Q(x)^3)\Big) + \frac{c}{4}\,\big(\mathrm{tr}\,(Q(x)^2)\big)^2 dx.$$

The first summand above amounts for the elastic energy, whereas the other terms stem from the Landau–de Gennes thermotropic energy (see [12]). For $Q \in \{Q \in H^2(\Omega, \mathbb{S}^n_{0.\mathbb{R}}) : \partial_\nu Q = 0 \text{ on } \partial\Omega\}$, the term

$$H := H(Q) := \lambda\Delta Q - aQ + b\Big(Q^2 - \mathrm{tr}(Q^2)\frac{\mathbb{I}}{n}\Big) - c\,\mathrm{tr}\,(Q^2)Q \tag{2.4}$$

relates to the above free energy by

$$\lim_{\varepsilon\to 0} \frac{\mathcal{F}(Q+\varepsilon\phi) - \mathcal{F}(Q)}{\varepsilon} = \int_\Omega \mathrm{tr}(-H(x)\phi(x))dx =< -H, \phi >_{L^2}$$

for any $\phi \in H^1(\Omega, \mathbb{S}^n_{0.\mathbb{R}})$.

The dynamic equation for the Q-tensor is then given by

$$\partial_t Q + (u\cdot\nabla)Q - S = \Gamma H \qquad \text{on} \quad (0,T)\times\Omega, \tag{2.5}$$

with a positive constant $\Gamma > 0$. It states that the change in time of Q is given by a convection term, a term S, which describes how the gradient of the velocity stretches and rotates the order parameter and a term H, which is derived from the above energy functional. The term S is given by

$$\begin{aligned} S(\nabla u, Q) := {}& (\xi D + W)(Q + \mathbb{I}/n) + (Q + \mathbb{I}/n)(\xi D - W) \\ & - 2\xi(Q + \mathbb{I}/n)\mathrm{tr}\,(Q\nabla u), \end{aligned}$$

where $D = D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and $W = W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ denote the symmetric and anti-symmetric part of the velocity gradient, respectively. Moreover, the parameter $\xi \in \mathbb{R}$ describes the ratio of tumbling and alignment effects.

The flow field is forced by an additional non-Newtonian stress caused by the molecules' orientation and expressed in terms of Q and ξ. It reads as

$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\,(\tau + \sigma) \qquad \text{on} \quad (0,T)\times\Omega, \tag{2.6}$$

where $\nu > 0$ is a constant and τ is the symmetric and σ the anti-symmetric part of the stress tensor given by

$$\begin{aligned} \tau(Q, H) :={}& -\lambda\nabla Q \odot \nabla Q - \xi(Q + \mathbb{I}/n)H - \xi H(Q + \mathbb{I}/n) \\ & + 2\xi(Q + \mathbb{I}/n)\,\mathrm{tr}(QH), \\ \sigma(Q) :={}& QH - HQ = \lambda(Q\Delta Q - \Delta Q Q) = \lambda\sigma(Q, \Delta Q), \end{aligned}$$

where the (i, j)-th component of $\nabla Q \odot \nabla Q$ equals $\mathrm{tr}(\partial_i Q \partial_j Q)$.

Lastly, there is the incompressibility condition

$$\operatorname{div} u = 0 \quad \text{on} \quad (0, T) \times \Omega. \tag{2.7}$$

Summing up, the Q-tensor model of Beris-Edwards is given by the momentum equation (2.6), the incompressibility condition (2.7), the dynamic equation for the order parameter (2.5) and Neumann boundary condutions for Q, Dirichlet boundary conditions for u as well as initial conditions u_0 and Q_0 for u and Q.

3. Main Results

Let $n \in \{2, 3\}$ and let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. For $p \in (1, \infty)$, $k \in \mathbb{N}$, we denote by $L^p(\Omega)$ and $W^k_p(\Omega)$ the Lebesgue space and Sobolev space of order k on the domain Ω equipped with the norms $\|\cdot\|_p$ and $\|\cdot\|_{W^k_p}$, respectively. We also denote $W^k_2(\Omega)$ by $H^k(\Omega)$. The space of of solenoidal functions $L^2_\sigma(\Omega)$ is given by $L^2_\sigma(\Omega) := \overline{\{u \in C^\infty(\overline{\Omega})^n : \operatorname{div} u = 0\}}^{\|\cdot\|_2}$.

We consider the Edwards–Beris system as a quasilinear evolution equation in the groundspace

$$X_0 := X^u_0 \times X^Q_0 := L^2_\sigma(\Omega) \times H^1(\Omega, \mathbb{S}^n_0),$$

where

$$\mathbb{S}^n_0 = \{A \in \mathbb{C}^{n \times n} : A = A^T, \operatorname{tr}(A) = 0\}$$

denotes the space of symmetric, traceless $n \times n$-matrices, which is closed in $\mathbb{C}^{n\times n}$. The domain X_1 of the quasilinear Q-tensor operator is defined by

$$X_1 := X^u_1 \times X^Q_1 := D(A_2) \times D(\Delta^1_N)$$

where $A_2 := \mathbb{P}\Delta$ denotes the Stokes operator in $L^2_\sigma(\Omega)$ with domain

$$D(A_2) := X^u_1 := \{u \in H^2(\Omega)^n \cap L^2_\sigma(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

and $\Delta^1_N := -\Delta + \mathbb{I}$ the shifted Neumann-Laplacian on $H^1(\Omega, \mathbb{S}^n_0)$ with domain

$$D(\Delta^1_n) := X^Q_1 := \{Q \in H^3(\Omega, \mathbb{S}^n_0) : \partial_\nu Q = 0 \text{ on } \partial\Omega\}.$$

Let us note that the norm on $\mathbb{S}^n_0$ is given by $\|Q\|_{\mathbb{S}^n_0} = \operatorname{tr}(QQ^*)$, where Q^* denotes the congugate transpose of Q. For $p \in (1, \infty)$ the trace space X_γ is given as the real interpolation space

$$X_\gamma = (X_0, X_1)_{1-1/p,p} = (X^u_0, X^u_1)_{1-1/p,p} \times (X^Q_0, X^Q_1)_{1-1/p,p}.$$

We recall from [14] the following characterizations of the interpolation spaces involved:

$$(X^u_0, X^u_1)_{1-1/p,p} := \{u \in B^{2-2/p}_{2,p}(\Omega) \cap L^2_\sigma(\Omega), u = 0 \text{ on } \partial\Omega\},$$

$$(X_0^Q, X_1^Q)_{1-1/p,p} := \{Q \in B^{3-2/p}_{2,p}(\Omega), \partial_\nu Q = 0 \text{ on } \partial\Omega\}.$$

Note that

$$X_\gamma \hookrightarrow B^{2-2/p}_{2,p}(\Omega) \times B^{3-2/p}_{2,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \times C^1(\overline{\Omega}, \mathbb{S}^n_0),$$

provided that $1/p + n/4 < 1$. We are considering solutions in the space $\mathbb{E}$ of maximal L^p-regularity given by

$$\mathbb{E} := L^p(0, T; X_1) \cap H^{1,p}(0, T; X_0)$$

using vector valued Lebesgue and Sobolev spaces, cf. e.g. [25, Chapter 3]. The spaces $\mathbb{E}^u$ and $\mathbb{E}^Q$ are defined analogously. They denote function spaces related to the regularity of u and Q.

Given $p \in (1, \infty)$ and $(u_0, Q_0) \in X_\gamma$, we say that (u, Q) is a *local, strong solution* to equation (1.1), if (1.1) is satisfied almost everywhere on $(0, T)$ for some $T > 0$ and if $(u, Q) \in \mathbb{E}$. If the same assertion holds true for $T = \infty$, we call (u, Q) a *global, strong solution* to equation (1.1).

We are now in the position to state our main results, namely local well-posedness and global existence of (1.1) for arbitrary values of $\xi \in \mathbb{R}$ and $n \in \{2, 3\}$.

Theorem 3.1. (Local, strong well-posedness for arbitrary $\xi \in \mathbb{R}$). *Let* $n \in \{2, 3\}$ *and* $p > \frac{4}{4-n}$. *Let* $\xi \in \mathbb{R}$ *be arbitrary and assume that* $v_0 = (u_0, Q_0) \in \{u \in B^{2-2/p}_{2,p}(\Omega) \cap L^2_\sigma(\Omega), u = 0 \text{ on } \partial\Omega\} \times \{Q \in B^{3-2/p}_{2,p}(\Omega), \partial_\nu Q = 0 \text{ on } \partial\Omega\}$.

Then there exists $T = T(v_0) > 0$ *such that there exists a unique, strong solution* $v = (u, Q)$ *to equation (1.1) on* $(0, T)$ *lying in the regularity class*

$$v \in H^{1,p}\left(0, T; L^2_\sigma(\Omega) \times H^1(\Omega; \mathbb{S}^n_0)\right) \cap L^p\left(0, T; H^2(\Omega) \times H^3(\Omega; \mathbb{S}^n_0)\right).$$

Our second main result concerns global existence of strong solutions for initial data close to the trivial equlilibria $v^* = 0$ and reads as follows:

Theorem 3.2. (Global existence for small initial data). *Let* $p > \frac{4}{4-n}$ *and* $\xi \in \mathbb{R}$. *Then the equilibrium* $v^* = 0$ *of (1.1) is stable in* X_γ, *i.e., there exists* $\delta > 0$ *such that the strong solution* $v(t)$ *of (1.1) with initial value* $v_0 \in X_\gamma$ *and* $\|v_0\|_{X_\gamma} \leq \delta$ *exists globally and converges exponentially to* 0 *in* X_γ *as* $t \to \infty$.

Remark 3.3. One might wonder how the set of equilibria for equation (1.1) looks like. Let us note that in dimension $n = 2$ or $n = 3$, $v = (0, 0) \in \mathbb{R}^n \times \mathbb{S}^n_{0,\mathbb{R}}$ is the only spatially constant equilibrium for (1.1). This assertion depends, of course, on the chosen set of parameters $a = b = c = 1$. Here the set of spatially constant equilibria is given by

$$\{(u, Q) \in \mathbb{R}^n \times \mathbb{S}^n_{0,\mathbb{R}} : 0 = H(Q) = -Q + Q^2 - \text{tr}\,(Q^2)\mathbb{I}/n - \text{tr}\,(Q^2)Q\}.$$

In fact, for the case $n = 2$, $Q \in \mathbb{S}^2_{0,\mathbb{R}}$ can be represented as $\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, for some $\alpha, \beta \in \mathbb{R}$. The equality $Q^2 - \text{tr}\,(Q^2)\mathbb{I}/2 = 0$ yields $H(Q) = -Q - \text{tr}\,(Q^2)Q = -(1 + 2(\alpha^2 + \beta^2))Q$. Hence, $Q = 0$ is necessary for $H(Q) = 0$. This assertion remains true in the case of general coefficients $a > 0$ and $b, c \in \mathbb{R}$ provided $a + 2c(\alpha^2 + \beta^2) > 0$.

If $n = 3$, then by (2.3) Q can be represented as

$$Q(x) = s\Big(e_1 \otimes e_1) - \frac{1}{3}\mathbb{I}\Big) + r\Big(e_2 \otimes e_2) - \frac{1}{3}\mathbb{I}\Big),$$

where $s = 2\lambda_1 + \lambda$ and $r = \lambda_1 + 2\lambda_2$. Assuming $H(Q) = 0$ and substituting the above representation of Q into $H(Q)$ we obtain three equations for the coefficients s and r. One can show that these three equations do not have a real solution besides the trivial one given by $s = r = 0$. This implies $Q = 0$.

4. Background on Quasilinear Evolution Equations

In this section we briefly recall some results on quasilinear parabolic equations of the form

$$\dot{v} + A(v)v = F(v), \quad t > 0, \quad v(0) = v_0, \tag{4.1}$$

which are employed in the proofs of our main theorems.

We start by fixing some notation. For a Banach space X_0 let $A_0 : D(A_0) = X_1 \to X_0$ be a densely defined linear operator. For $p \in (1, \infty)$ and $0 < T \leq \infty$ we define the data space $\mathbb{F}(0, T)$ and the maximal regularity space $\mathbb{E}(0, T)$ by

$$\mathbb{F}(0, T) := L^p(0, T; X_0) \text{ and } \mathbb{E}(0, T) := W^{1,p}(0, T; X_0) \cap L^p(0, T; X_1).$$

Furthermore, for $p \in (1, \infty)$ we denote by $X_\gamma = (X_0, X_1)_{1-1/p}$ the time trace space. Let V be an open subset of X_γ.

The following assumptions are essential for showing existence and uniqueness of strong solutions to problem (4.1). (We denote the closed ball centered around 0 in $\mathbb{E}$ with radius $R > 0$ by $\overline{B_{\mathbb{E}}}(0, R)$.

(A1) The operators $A : X_\gamma \to \mathcal{L}(X_1, X_0)$ are a family of closed linear operators, and for every $R > 0$ there exists $L(R) > 0$ such that, for all $v, \overline{v}, w \in \overline{B_{\mathbb{E}}}(0, R)$, it holds that

$$\|A(v(\cdot))w(\cdot) - A(\overline{v}(\cdot))w(\cdot)\|_{\mathbb{E}} \leq L(R)\|v - \overline{v}\|_{\mathbb{E}}\|w\|_{\mathbb{E}}.$$

(A2) The map $F : [0, T] \times X_\gamma \to X_0$ satisfies $F(\cdot, v(\cdot)) \in \mathbb{F}$ for all $v \in \mathbb{E}$, and for some $k \in \mathbb{N}_0$ there exists $C > 0$ such that for all $v, \overline{v} \in \mathbb{E}$ one has

$$\|F(\cdot, v(\cdot)) - F(\cdot, \overline{v}(\cdot))\|_{\mathbb{E}} \leq C(\|v\|_{L^\infty(0,T;X_\gamma)} + \|\bar{v}\|_{L^\infty(0,T;X_\gamma)} + 1)^k (\|v\|_{\mathbb{E}} + \|\bar{v}\|_{\mathbb{E}})\|v - \bar{v}\|_{\mathbb{E}},$$

(A3) The operator $-A_0 := -A(0)$ admits maximal L^p-regularity on X_0.

Proposition 4.1. *Let $p \in (1,\infty)$, $v_0 \in V$ be given and suppose that A and F satisfy the assumptions (A1),(A2),(A3).*

Then there is $a = a(v_0) > 0$ and $r = r(v_0) > 0$ with $\overline{B_{X_\gamma}}(v_o, r) \subset V$ such that (4.1) admits a unique, strong solution

$$v = v(\cdot, v_1) \in \mathbb{E}(0,a) \cap C([0,a]; V)$$

on $[0,a]$ for any initial value $v_1 \in \overline{B_{X_\gamma}}(v_o, r)$.

The next result provides information about the continuation of local solutions. Let

$$\mathcal{E} := \{v \in X_1 : A(v)v = F(v)\}$$

denote the set of equilibria to (4.1). For $v^* \in \mathcal{E}$ we set $u = v - v^*$. The equation for u then reads as

$$\dot{u}(t) + A_0 u(t) = G(u(t)), \quad t > 0, \quad u(0) = u_0, \tag{4.2}$$

where $u_0 = v_0 - v^*$ and $A_0 = A(v^*) + B_1 - B_2$ with $B_j \in \mathcal{L}(X_\gamma, X_0)$ and $G(u) = G_1(u) + G_2(u)$, where $G_1(u) = [F(v^* + u) - F(u) - B_2 u] - [(A(v^* + u) - A(v^*))v^* - B_1 u$ for $u \in X_\gamma$ and $G_2(u, w) = (A(v^*) - A(v^* + u))w$ for $u \in X_\gamma$ and $w \in X_1$. The operators B_1 and B_2 resemble the derivatives of A and F at v^*. It is hence natural to assume that there exist constants $L, r_0, \delta > 0$ such that for $\varepsilon > 0$ small enough it follows that

(S1) $\|A(v^* + v)v^* - A(v^*)v^* - B_1 v\|_{X_0} \le \varepsilon \|v\|_{X_\gamma}, \quad \|v\|_{X_\gamma} \le r_0$,
(S2) $\|h_\delta[F(v^* + v) - F(v^*) - B_2 v]\|_{\mathbb{F}(T)} \le \varepsilon \|h_\delta v\|_{\mathbb{E}(T)}$, for $T > 0$ and all $v \in \mathbb{E}(T)$ with $\|v\|_{L^\infty(0,T;X_\gamma)} \le r_0$, where $h_\delta(t) = e^{\delta t}$,
(S3) $\|A(v^* + v) - A(v^*)\|_{\mathcal{L}(X_1, X_0)} \le L\|v\|_{X_\gamma}, \quad \|v\|_{X_\gamma} < r_0$.

If $A(v)$ and $F(v)$ are Fréchet differentiable at $v^* \in X_\gamma$, then (S1),(S2),(S3) hold with $B_1 = A'(v^*)v^*$ and $B_2 = F'(v^*)$ and $\varepsilon \to 0$ as $r \to 0$.

Proposition 4.2. *Suppose $p \in (1,\infty)$, $v^* \in \mathcal{E}$ and let B_1, B_2 be such that assumptions (S1)-(S3) are satisfied. Suppose that $A(v^*)$ has maximal L^p-regularity and that the spectrum $\sigma(A_0)$ of $A_0 = A(v^*) + B_1 - B_2$ is contained in the open right half plane.*

Then there exists $\rho \in (0, r_0]$ such that for each $v_0 \in B_\rho(v^*) \subset X_\gamma$ there exists a unique, global solution $v \in H^{1,p}_{loc}(\mathbb{R}_+; X_0) \cap L^p_{loc}(\mathbb{R}_+; X_1)$ satisfying

$$e^{\delta t}(v - v^*) \in H^{1,p}(\mathbb{R}_+; X_0) \cap L^p(\mathbb{R}_+; X_1) \cap C_0(\mathbb{R}_+; X_\gamma).$$

In particular, the equilibrium v^* of the quasilinear problem (4.1) is exponentially stable in the space X_γ.

The above results are only minor modifications of results due to Clément and Li [9], Prüss [24], Prüss, Simonett and Zacher [27] and Prüss, Simonett and Wilke [26]. A convenient reference for these type of results are the monographs by Amann [5,6] and Prüss and Simonett [25, Chapter 5].

5. Reformulation as Quasilinear Evolution Equation

In this section we rewrite the system (1.1) as a quasilinear evolution equation of the form

$$\dot{v} + A(v)v = F(v), \quad t > 0, \quad v(0) = v_0,$$

in the space $X_0 = L^2_\sigma(\Omega) \times H^1(\Omega, \mathbb{S}^n_0)$, where $\mathbb{S}^n_0$ is defined as in Sect. 3 as the space of all symmetric, traceless $n \times n$-matrices over $\mathbb{C}$. We then show that the mapping $A_\xi(\hat{Q}) : X_1 \to X_0$ (defined precisely below in (5.4)) is sectorial with spectral angle $\phi_{A_\xi(\hat{Q})} < \frac{\pi}{2}$. In particular, $A_\xi(\hat{Q})$ has maximal L^p-L^2-regularity. Our idea to show sectoriality of the underlying linear operator $A_\xi(\hat{v})$ is to exploit a classical result for unbounded operators in Hilbert spaces (cf. Kato [18], Section V.3.2). It implies that for operators with non-empty resolvent set, firstly, the spectrum is contained in the numerical range and, secondly, that the resolvent scales inversely to the distance of the numerical range. Consequently, such operators are sectorial (cf. e.g. [10]) if their numerical range is contained in a sector. We hence aim to show that $A_\xi(\hat{Q})$ is invertible as well as that its numerical range lies a a certain sector. In order to verify these two properties we take adavantage of certain symmetry properties, in particular $\mathcal{J}$-symmetry, and show that its first Schur complement is closed and invertible.

In what follows, we will use the notation $[A, B] = AB - BA$ as well as the notation $\{A, B\} = AB + BA$ for the commutator and the anticommutator, respectively, of two matrices $A, B \in \mathbb{C}^{n\times n}$. Moreover, we split the terms $\tau = \tau_h + \tau_l$ and $H = H_h + H_l$ into higher and lower order terms and obtain then

$$\begin{aligned}
S(\nabla u, Q) &= -[Q, W] + \xi\big(2/nD + \{Q, D\} - 2(Q + \mathbb{I}/n)\operatorname{tr}(Q\nabla u)\big),\\
H_h(Q) &= \Delta Q - Q,\\
H_l(Q) &= (Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/n) - \operatorname{tr}(Q^2)Q,\\
\tau_h(Q) &= \xi\big(2/nH_h + \{Q, H_h\} - 2(Q + \mathbb{I}/n)\operatorname{tr}(QH_h)\big),\\
\tau_l(Q) &= 2\xi(Q + \mathbb{I}/n)(\operatorname{tr}(Q^3) - \operatorname{tr}(Q^2)^2) - \nabla Q \odot \nabla Q - 2\xi(Q + \mathbb{I}/n)H_l,\\
\sigma(Q) &= -[Q, (-\Delta + \mathbb{I})Q].
\end{aligned}$$

We now define two linear mappings on $\mathbb{M}^n_0 := \{Q \in \mathbb{C}^{n\times n} : \operatorname{tr} Q = 0\}$ and recall that $\mathbb{S}^n_{0,\mathbb{C}} := \{A \in \mathbb{C}^{n\times n} : A = A^T, \operatorname{tr} A = 0\}$ in order to rewrite the terms $\sigma + \tau_h$ and S in a way which indicates better derivatives. More precisely, for $\xi \in \mathbb{R}$ and $Q \in \mathbb{S}^n_0$ we define the mappings $S_\xi(Q) \in \mathcal{L}(\mathbb{S}^n_0, \mathbb{M}^n_0)$ and $\tilde{S}_\xi(Q) \in \mathcal{L}(\mathbb{M}^n_0, \mathbb{S}^n_0)$ as

$$\begin{aligned}
S_\xi(Q)A &:= [Q, A] - \tfrac{2\xi}{n}A - \xi\{Q, A\} + 2\xi(Q + \mathbb{I}/n)\operatorname{tr}(QA),\\
\tilde{S}_\xi(Q)B &:= [\overline{Q}, B^W] - \tfrac{2\xi}{n}B^D - \xi\{\overline{Q}, B^D\} + 2\xi(\overline{Q} + \mathbb{I}/n)\operatorname{tr}(\overline{Q}B^D),
\end{aligned}$$

where B^D and B^W are defined as $B^D := \frac{1}{2}(B + B^T)$ and $B^W := B - B^D$ and where $\overline{Q}$ denotes the complex conjugate of Q. In particular, for real-valued u and Q this gives

$$\tau_h(Q) + \sigma(Q) = S_\xi(Q)(-\Delta + \mathbb{I})Q, \quad \text{and} \quad -S(\nabla u, Q) = \tilde{S}_\xi(Q)\nabla u.$$

Applying the Helmholtz projection $\mathbb{P}$ in $L^2(\Omega)$ to the equation $(1.1)_1$ yields along with equation $(1.1)_3$ the following system:

$$\partial_t u - \mathbb{P}\Delta u - \mathbb{P}\operatorname{div}(\sigma + \tau_h) = \mathbb{P}\left(\operatorname{div}\tau_l - (u\cdot\nabla)u\right), \tag{5.1}$$

$$\partial_t Q - S - H_h = H_l - (u\cdot\nabla)Q. \tag{5.2}$$

Setting $v = (u, Q)$, the system (1.1) can be now rewritten equivalently as a quasilinear evolution equation on X_0 of the form

$$v'(t) + A_\xi(v(t))v(t) = F(v(t)),\ t \in (0, T), \quad v(0) = v_0, \tag{5.3}$$

with $v_0 = (u_0, Q_0)$ and where A_ξ is given by

$$\begin{aligned} A_\xi(\hat{v}) &:= A_\xi(\hat{Q}) \\ &:= \begin{pmatrix} \mathcal{A} & \mathbb{P}\operatorname{div} S_\xi(\hat{Q})(-\Delta + \mathbb{I}) \\ \tilde{S}_\xi(\hat{Q})\nabla & \Delta_N^1 \end{pmatrix}, \quad \hat{v} = (\hat{u}, \hat{Q}) \in X_\gamma. \end{aligned} \tag{5.4}$$

Here $\mathcal{A}$ denotes the Stokes operator in $L^2_\sigma(\Omega)$ given by

$$\begin{aligned} \mathcal{A} &:= -\mathbb{P}\Delta, \\ D(\mathcal{A}) &:= \{u \in H^2(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

and Δ_N^1 denotes the Neumann Laplacian on $H^1(\Omega, \mathbb{S}_0^{\mathrm{n}})$ shifted by the identity, i.e.,

$$\begin{aligned} \Delta_N^1 &:= -\Delta + \mathbb{I}, \\ D(\Delta_N^1) &:= \{Q \in H^3(\Omega, \mathbb{S}_0^{\mathrm{n}}) : \partial_\nu Q = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Moreover, the nonlinear term $F(v) = (\mathbb{P}f_1(v), f_2(v))$ is given by

$$\begin{cases} f_1(v) := & -(u\cdot\nabla)u - \operatorname{div}(\nabla Q \odot \nabla Q) + 2\xi \operatorname{div}\left((Q + \mathbb{I}/n)(\operatorname{tr}(Q^3) - \operatorname{tr}(Q^2)^2)\right) \\ & -2\xi \operatorname{div}\left((Q + \mathbb{I}/n)(Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/n - \operatorname{tr}(Q^2)Q)\right), \\ f_2(v) := & (Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/n) - \operatorname{tr}(Q^2)Q - (u\cdot\nabla)Q. \end{cases} \tag{5.5}$$

Our aim is now to show that the operator $A_\xi(\hat{Q})$ has the property of maximal L^p-L^2-regularity provided $p > \frac{4}{4-n}$, $\hat{Q} \in X_\gamma^Q$ and $\xi \in \mathbb{R}$. Let us start by showing that the numerical range of $A_\xi(\hat{Q})$ lies in a certain sector in the right half plane.

5.1. Numerical Range

We start by showing that S_ξ and $\tilde{S}_\xi$ are adjoint to each other on the spaces of tracelesss and spaces of symmetric, traceless matrices.

Lemma 5.1. *Let $\xi \in \mathbb{R}$ and $Q \in \mathbb{S}_0^n$. Then $(S_\xi(Q))^* = \tilde{S}_\xi(Q)$, i.e., if $A \in \mathbb{S}_0^n$ and $B \in \mathbb{M}_0^n$, then*

$$\langle S_\xi(Q)A, B\rangle_{\mathbb{C}^{n\times n}} = \langle A, \tilde{S}_\xi(Q)B\rangle_{\mathbb{C}^{n\times n}}.$$

Proof. Let us note that $\operatorname{tr}(ab) = 0$ for symmetric $a \in \mathbb{C}^{n\times n}$ and skewsymmetric $b \in \mathbb{C}^{n\times n}$ and let us recall the notation $B^D = \frac{1}{2}B + \frac{1}{2}B^T$ and $B^W = B - B^D$. We verify the assertion for the four terms of S_ξ seperately.
Term 1: Commutator. Note that $[a, \tilde{a}]$ is skewsymmetric and $[a, b]$ is symmetric for symmetric $a, \tilde{a} \in \mathbb{C}^{n\times n}$ and skewsymmetric $b \in \mathbb{C}^{n\times n}$. It follows that

$$\begin{aligned}\operatorname{tr}\left([Q, A]\overline{B^T}\right) &= \operatorname{tr}\left([Q, A]\overline{(-B^W)}\right) = \operatorname{tr}\left(QA\overline{(-B^W)}\right) - \operatorname{tr}\left(AQ\overline{(-B^W)}\right)\\ &= \operatorname{tr}\left(A\overline{(-B^W)}Q\right) - \operatorname{tr}\left(AQ\overline{(-B^W)}\right) = \operatorname{tr}\left(A\overline{\left[\overline{Q}, B^W\right]}\right)\\ &= \operatorname{tr}\left(A\overline{\left[\overline{Q}, B^W\right]^T}\right).\end{aligned}$$

Term 2: Anticommutator. Note that $\{a, \tilde{a}\}$ is symmetric for symmetric $a, \tilde{a} \in \mathbb{C}^{n\times n}$. Hence,

$$\begin{aligned}\operatorname{tr}\left(\{Q, A\}\overline{B^T}\right) &= \operatorname{tr}\left(\{Q, A\}\overline{B^D}\right) = \operatorname{tr}\left(QA\overline{B^D}\right) + \operatorname{tr}\left(AQ\overline{B^D}\right)\\ &= \operatorname{tr}\left(A\overline{B^D}Q\right) + \operatorname{tr}\left(AQ\overline{B^D}\right) = \operatorname{tr}\left(A\overline{\left\{\overline{Q}, B^D\right\}}\right)\\ &= \operatorname{tr}\left(A\overline{\left\{\overline{Q}, B^D\right\}^T}\right).\end{aligned}$$

Term 3: The constant term. We have

$$\operatorname{tr}(A\overline{B^T}) = \operatorname{tr}(A\overline{B^D}) = \operatorname{tr}(A\overline{(B^D)^T}).$$

Term 4: The trace term. Since A and B are traceless we get

$$\begin{aligned}\operatorname{tr}\left((Q + \mathbb{I}/n)\operatorname{tr}(QA)\overline{B^T}\right) &= \operatorname{tr}(AQ)\operatorname{tr}(Q\overline{B^D}) = \operatorname{tr}\left(A\overline{(\overline{Q} + \mathbb{I}/n)\operatorname{tr}(\overline{Q}B^D)}\right)\\ &= \operatorname{tr}\left(A\overline{(\overline{Q} + \mathbb{I}/n)^T\operatorname{tr}(\overline{Q}B^D)}\right).\end{aligned}$$

Hence, in total, we get

$$\begin{aligned}\langle S_\xi(Q)A, B\rangle_{\mathbb{C}^{n\times n}} &= \operatorname{tr}\left(([Q, A] - \tfrac{2\xi}{n}A - \xi\{Q, A\} + 2\xi(Q + \mathbb{I}/n)\operatorname{tr}(QA))\overline{B^T}\right)\\ &= \operatorname{tr}\left(A\overline{([\overline{Q}, B^W] - \tfrac{2\xi}{n}B^D - \xi\{\overline{Q}, B^D\} + 2\xi(\overline{Q} + \mathbb{I}/n)\operatorname{tr}(\overline{Q}B^D))^T}\right)\\ &= \langle A, \tilde{S}_\xi(Q)B\rangle_{\mathbb{C}^{n\times n}}.\end{aligned}$$

□

The above lemma allows us to compute $\langle A_\xi(\hat{Q})v, v\rangle_{X_0}$ precisely, as follows:

Lemma 5.2. *Let* $p > \frac{4}{6-n}$, $\hat{Q} \in X^Q_\gamma$, $\xi \in \mathbb{R}$ *and* $v = (u, Q) \in X_1$. *Then*

$$\langle A_\xi(\hat{Q})v, v\rangle_{X_0} = \|\nabla u\|^2_{L^2} + \|Q\|^2_{H^2} + \|\nabla Q\|^2_{L^2} + i\operatorname{Im}(2\langle\nabla u, S_\xi(\hat{Q})(-\Delta + \mathbb{I})Q\rangle_{L^2}).$$

Proof. By the boundary conditions for u and Q, $\mathbb{P}u = u$ and Lemma 5.1 we obtain

$$\begin{aligned}\langle A_\xi(\hat{Q})v, v\rangle_{X_0} &= \langle \mathcal{A}u, u\rangle_{L^2} + \langle \Delta_N^1 Q, Q\rangle_{H^1} + \langle \mathbb{P}\operatorname{div} S_\xi(\hat{Q})(-\Delta+\mathbb{I})Q, u\rangle_{L^2}\\ &\quad + \langle \tilde{S}_\xi(\hat{Q})\nabla u, Q\rangle_{H^1}\\ &= \|\nabla u\|_{L^2}^2 + \|Q\|_{H^2}^2 + \|\nabla Q\|_{L^2}^2 - \langle S_\xi(\hat{Q})(-\Delta+\mathbb{I})Q, \nabla u\rangle_{L^2}\\ &\quad + \langle \nabla u, S_\xi(\hat{Q})(-\Delta+\mathbb{I})Q\rangle_{L^2}\\ &= \|\nabla u\|_{L^2}^2 + \|Q\|_{H^2}^2 + \|\nabla Q\|_{L^2}^2\\ &\quad + i\operatorname{Im}(2\langle \nabla u, S_\xi(\hat{Q})(-\Delta+\mathbb{I})Q\rangle_{L^2}).\end{aligned}$$

□

The regularity of the map $S_\xi(\hat{Q})$ can be described as follows:

Lemma 5.3. *a) Let $p > \frac{4}{6-n}$, $\hat{Q} \in X_\gamma^Q$ and $\xi \in \mathbb{R}$. Then*

$$S_\xi(\hat{Q}) \in C(\overline{\Omega}, \mathcal{L}(\mathbb{S}_{0,\mathbb{C}}^n, \mathbb{C}^{n\times n})) \quad \textit{and} \quad \tilde{S}_\xi(\hat{Q}) \in C(\overline{\Omega}, \mathcal{L}(\mathbb{M}_{0,\mathbb{C}}^n, \mathbb{C}^{n\times n})).$$

b) If in addition $p > \frac{4}{4-n}$, then

$$S_\xi(\hat{Q}) \in C^1(\overline{\Omega}, \mathcal{L}(\mathbb{S}_{0,\mathbb{C}}^n, \mathbb{C}^{n\times n})) \quad \textit{and} \quad \tilde{S}_\xi(\hat{Q}) \in C^1(\overline{\Omega}, \mathcal{L}(\mathbb{M}_{0,\mathbb{C}}^n, \mathbb{C}^{n\times n})).$$

Proof. The first assertion follows by the embedding $X_\gamma^Q \hookrightarrow C(\overline{\Omega}, \mathbb{S}_{0,\mathbb{C}}^n)$, the fact that the Frobenius norm is submultiplicative and the estimate $\|S_\xi(\hat{Q})A\|_{L^\infty} \leq C(\|\hat{Q}\|_{X_\gamma^Q} + 1)^2\|A\|$. The second assertion follows in the same way by noticing that the assumption implies $X_\gamma^Q \hookrightarrow C^1(\overline{\Omega}, \mathbb{S}_{0,\mathbb{C}}^n)$. □

We proceed by showing that the numerical range of $A_\xi(\hat{Q})$ lies in a certain sector of the right half plane. Here, given $\varphi \in [0, \pi)$, Σ_φ denotes an open sector in the complex plane of angle φ.

Proposition 5.4. (Numerical range of $A_\xi(\hat{Q})$) *Let $p > \frac{4}{6-n}$, $q = 2$, $\hat{Q} \in X_\gamma^Q$, and $\xi \in \mathbb{R}$. Then the numerical range $\mathcal{W}(A_\xi(\hat{Q}))$ of $A_\xi(\hat{Q})$ lies in a sector of the right half plane, i.e., there exists an angle $\varphi \in [0, \pi/2)$ such that*

$$\mathcal{W}(A_\xi(\hat{Q})) \subset \Sigma_\varphi,$$

where φ depends on ξ, p, n, $\|\hat{Q}\|_{X_\gamma^Q}$, and Ω. In particular,

$$\mathcal{W}(-A_\xi(\hat{Q})) \subset (\Sigma_{\pi-\varphi})^c.$$

Proof. Let $v = (u, Q) \in X_1$ such that $\|v\|_{X_0} = 1$. By Lemma 5.3, Cauchy-Schwarz and Young's inequality we obtain

$$|2\langle \nabla u, S_\xi(\hat{Q})(-\Delta+\mathbb{I})Q\rangle_{L^2}| \leq \|S_\xi(\hat{Q})\|_{L^\infty(\Omega, \mathcal{L}(\mathbb{S}_0^n, \mathbb{C}^{n\times n}))}(\|\nabla u\|_{L^2}^2 + \|Q\|_{H^2}^2).$$

Thus, $z := \langle A_\xi(\hat{Q})v, v\rangle_{X_0}$ satisfies by Lemma 5.2 the properties

$$\mathrm{Re}(z) > 0, \quad \text{and} \quad |\mathrm{Im}(z)| \le \|S_\xi(\hat{Q})\|_{L^\infty(\Omega,\mathcal{L}(\mathbb{S}_0^n,\mathbb{C}^{n\times n}))}|\mathrm{Re}(z)|.$$

Hence,

$$|\arg(z)| = |\arctan(\mathrm{Im}(z)/\mathrm{Re}(z))| \le \arctan\left(\|S_\xi(\hat{Q})\|_{L^\infty(\Omega,\mathcal{L}(\mathbb{S}_0^n,\mathbb{C}^{n\times n}))}\right) =: \varphi. \quad \square$$

Our next aim is to show that $A_\xi(\hat{Q})$ is invertible.

5.2. Invertibility

Our next aim is to show that the mapping $A_\xi(\hat{Q})$ defined on X_0 and having domain X_1 is invertible for all $\hat{Q} \in X_\gamma^Q$ and all $\xi \in \mathbb{R}$. It is well known that the invertibility of this map is closely related to the one of its Schur complement at $\lambda = 0$. For $\lambda \in \varrho(\Delta_N^1)$, the first Schur complement is given by

$$\begin{aligned} S_1(\lambda) &:= \mathcal{A} - \mathbb{P}\mathrm{div} S_\xi(\hat{Q})(-\Delta + \mathbb{I})(\Delta_N^1 - \lambda)^{-1}\tilde{S}_\xi(\hat{Q})\nabla, \\ D(S_1(\lambda)) &:= H^2(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega). \end{aligned}$$

At $\lambda = 0$, the first Schur complement $S_1(0)$ is given by a Stokes operator with variable coefficients of the form

$$S_1(0) = -\mathbb{P}\Delta - \mathbb{P}\,\mathrm{div}\, S_\xi(\hat{Q})(\tilde{S}_\xi(\hat{Q})^*\nabla$$

with domain $D(S_1(0)) = H^2(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega)$. We show first that the Schur complement $S_1(0)$ is invertible.

Lemma 5.5. (Invertibility of the Schur complement) *Let $p > \frac{4}{4-n}$, $\hat{Q} \in X_\gamma^Q$, and $\xi \in \mathbb{R}$. Then*

$$\begin{aligned} S_1(\lambda) &:= \mathcal{A} - \mathbb{P}div S_\xi(\hat{Q})(-\Delta + \mathbb{I})(\Delta_N^1 - \lambda)^{-1}\tilde{S}_\xi(\hat{Q})\nabla, \\ D(S_1(\lambda)) &:= H^2(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega). \end{aligned}$$

is invertible on $L_\sigma^2(\Omega)$.

Proof. Setting $W := H_0^1(\Omega) \cap L_\sigma^2(\Omega)$, we note that the mapping $a : W \times W \to \mathbb{R}$ given by

$$a(u, v) = \langle(\mathbb{I} + S_\xi(\hat{Q})(S_\xi(\hat{Q}))^*)\nabla u, \nabla v\rangle_{L^2}$$

defines a positive, closed, symmetric sesquilinear form on $L_\sigma^2(\Omega)$ satisfying $a(u, u) \ge C_\Omega\|u\|_2^2$ for some $C_\Omega > 0$. There exists thus a unique operator A_a on $L_\sigma^2(\Omega)$ with domain $D(A_a)$ and with $\sigma(A_a) \subset [C_\Omega, \infty)$ such that for every $u \in D(A_a)$ and $v \in V$ we have $a(u, v) = \langle A_a u, v\rangle$. In particular, A_a is invertible.

Let now $u \in D(S_1(0))$. Then $S_1(0)u \in L_\sigma^2(\Omega)$ and thus by integration by parts we get for all $v \in H_0^1(\Omega) \cap L_\sigma^2(\Omega)$ that

$$\langle S_1(0)u, v\rangle_{L^2} = \langle(\mathbb{I} + S_\xi(\hat{Q})(S_\xi(\hat{Q}))^*)\nabla u, \nabla v\rangle_{L^2} = \langle A_a u, v\rangle_{L^2}.$$

Therefore, $u \in D(A_a)$ and $A_a u = S_1(0)u$.

Conversely, let $u \in D(A_a)$. Then $A_a u \in L^2_\sigma(\Omega)$ and

$$\langle (\mathbb{I} + S_\xi(\hat{Q})(S_\xi(\hat{Q}))^*)\nabla u, \nabla v\rangle_{L^2} = \langle A_a u, v\rangle_{L^2}$$

holds for all $v \in H^1_0(\Omega) \cap L^2_\sigma(\Omega)$. Thus, a classical result due to Giaquinta and Modica [13, Theorem 1.3] implies that $u \in H^2(\Omega)$ and hence $u \in \mathcal{D}(S_1(0))$. □

Using the theory of Schur complements we next show that the operator $A_\xi(\hat{Q})$ is invertible.

Proposition 5.6. *Suppose that* $p > \frac{4}{4-n}$, $\hat{Q} \in X^Q_\gamma$, *and* $\xi \in \mathbb{R}$. *Then the operator* $A_\xi(\hat{Q})$ *is invertible.*

Proof. Recall from (5.4) that the operator $A_\xi(\hat{Q})$ is given by

$$A_\xi(\hat{Q}) = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix},$$

where

$$\mathfrak{a} := \mathcal{A}, \quad \mathfrak{b} := \mathbb{P}\operatorname{div} S_\xi(\hat{Q})(-\Delta + \mathbb{I}), \quad \mathfrak{c} := \tilde{S}_\xi(\hat{Q})\nabla, \quad \mathfrak{d} := \Delta^1_N. \tag{5.6}$$

Since $S_1(0) = \mathfrak{a} - \mathfrak{b}\mathfrak{d}^{-1}\mathfrak{c}$ is invertible by Lemma 5.5, we find that

$$\begin{pmatrix} (S_1(0))^{-1} & -(S_1(0))^{-1}\mathfrak{b}\mathfrak{d}^{-1} \\ -\mathfrak{d}^{-1}\mathfrak{c}(S_1(0))^{-1} & \mathfrak{d}^{-1} + \mathfrak{d}^{-1}\mathfrak{c}(S_1(0))^{-1}\mathfrak{b}\mathfrak{d}^{-1} \end{pmatrix} \in \mathcal{L}(X_0, X_1).$$

One can easily check that the above is the inverse of $A_\xi(\hat{Q}) \in \mathcal{L}(X_1, X_0)$. □

We are now in the position to prove that $A_\xi(\hat{Q})$ is sectorial with spectral angle $\phi_{A_\xi(\hat{Q})} < \frac{\pi}{2}$.

Proposition 5.7. *Let* $p > \frac{4}{4-n}$, $\hat{Q} \in X^Q_\gamma$ *and* $\xi \in \mathbb{R}$. *Then the operator* $A_\xi(\hat{Q}) : X_1 \to X_0$ *is sectorial with spectral angle* $\phi_{A_\xi(\hat{Q})} < \frac{\pi}{2}$. *In particular,* $A_\xi(\hat{Q})$ *has maximal* L^p-L^2*-regularity.*

Proof. Proposition 5.6 yields that $A_\xi(\hat{Q})$ is invertible. Therefore (cf. e.g. [18, Theorem V.3.2]), we find that $\sigma(-A_\xi(Q)) \subset \mathcal{W}(-A_\xi(Q))$ and

$$\|(\lambda + A_\xi(Q))^{-1}\|_{\mathcal{L}(X_0)} \leq \frac{1}{\operatorname{dist}(\lambda, \mathcal{W}(-A_\xi(Q)))}, \quad \lambda \in \mathcal{W}(-A_\xi(Q))^c.$$

By Proposition 5.4, there exists an angle $\varphi \in [0, \pi/2)$, such that $\mathcal{W}(-A_\xi(Q)) \subset (\Sigma_{\pi-\varphi})^c$. Let $\varphi_2 \in (\varphi, \pi/2]$. This yields

$$\|(\lambda + A_\xi(Q))^{-1}\|_{\mathcal{L}(X_0)} \leq \frac{1}{\operatorname{dist}(\lambda, (\Sigma_{\pi-\varphi})^c)} \leq \frac{1}{\sin(\varphi - \varphi_2)|\lambda|}.$$

Hence $A_\xi(\hat{Q})$ is sectorial with spectral angle $\phi_{A_\xi(\hat{Q})} \leq \varphi \in [0, \pi/2)$. Since $A_\xi(\hat{Q})$ is a sectorial on the Hilbert space X_0 with spectral angle less than $\pi/2$, it is also $\mathcal{R}$-sectorial and admits hence maximal L^p-L^2-regularity; see e.g. [10] for details. □

6. Nonlinear Estimates

We start this section by showing that the mapping $A_\xi : X_\gamma \to \mathcal{L}(X_1, X_0)$ is locally Lipschitz continuous.

Lemma 6.1. *Let $p > \frac{4}{4-n}$. Then there exists a constant $C > 0$ such that for all $\hat{Q}, \hat{Q}_1, \hat{Q}_2 \in X_\gamma^Q$ and $M \in H^1(\Omega, \mathbb{C}^{n\times n})$ one has*

$$\|[\hat{Q}, M]\|_{H^1} + \|\{\hat{Q}, M\}\|_{H^1} \le C\|\hat{Q}\|_{X_\gamma^Q}\|M\|_{H^1},$$
$$\|(\hat{Q}_1 + \mathbb{I}/n)tr\,(\hat{Q}_2 M)\|_{H^1} \le C(\|\hat{Q}_1\|_{X_\gamma^Q} + 1)\|\hat{Q}_2\|_{X_\gamma^Q}\|M\|_{H^1}.$$

The above assumption on p allows, due to Sobolev embeddings, to prove the assertion of Lemma 6.1 easily by means of Hölder's inequality.

Lemma 6.2. *Let $p > \frac{4}{4-n}$, $\hat{Q} \in X_\gamma^Q$, and $\xi \in \mathbb{R}$. Then $A_\xi(\hat{Q}) \in \mathcal{L}(X_1, X_0)$.*

Proof. Applying Lemma 6.1 yields that there exists a constant $C > 0$ such that for all $u \in X_1^u$ and all $Q \in X_1^Q$ we have

$$\|\mathfrak{c}(\hat{Q})u\|_{X_0^Q} \le C(\|\overline{\hat{Q}}\|_{X_\gamma^Q} + 1)^2\|u\|_{H^2} \qquad \text{as well as}$$
$$\|\mathfrak{b}Q\|_{X_0^u} \le C(\|\hat{Q}\|_{X_\gamma^Q} + 1)^2\|\mathfrak{d}Q\|_{H^1},$$

where $\mathfrak{d}Q$ is defined as above by $\mathfrak{d}Q = (-\Delta + \mathbb{I})Q$. Hence,

$$\|\mathfrak{c}\hat{Q}u\|_{X_0^Q} \le C\|u\|_{X_1^u} \quad \text{and} \quad \|\mathfrak{b}\hat{Q}Q\|_{x_0^u} \le C\|\mathfrak{d}Q\|_{H^1} \le C\|Q\|_{X_1^Q},$$

where C depends on $\xi, p, n, \hat{Q}$ and Ω. It hence follows that $A_\xi(\hat{Q}) \in \mathcal{L}(X_1, X_0)$. □

We are now in the position to show that the mapping $A_\xi : X_\gamma \to \mathcal{L}(X_1, X_0)$ is locally Lipschitz continuous.

Lemma 6.3. (Local Lipschitz continuity of A_ξ) *Let $p > \frac{4}{4-n}$ and $\xi \in \mathbb{R}$. Then $A_\xi : X_\gamma \to \mathcal{L}(X_1, X_0)$ is locally Lipschitz continuous, i.e. for all $R > 0$ there exists a constant $K(R) > 0$ such that*

$$\|A_\xi(\hat{Q}_1) - A_\xi(\hat{Q}_2)\|_{\mathcal{L}(X_1,X_0)} \le K(R)\|\hat{Q}_1 - \hat{Q}_2\|_{X_\gamma^Q}$$

for all $\hat{Q}_1, \hat{Q}_2 \in X_\gamma^Q$ satisfying $\|\hat{Q}_1\|_{X_\gamma^Q}, \|\hat{Q}_1\|_{X_\gamma^Q} \le R$.

Proof. Lemma 6.1 yields

$$\begin{aligned}\|\mathfrak{b}(\hat{Q}_1)Q - \mathfrak{b}(\hat{Q}_2)Q\|_{X_0^u} \le\ & \|[\hat{Q}_1 - \hat{Q}_2, \mathfrak{d}Q]\|_{H^1} + |\xi|\|\{\hat{Q}_1 - \hat{Q}_2, \mathfrak{d}Q\}\|_{H^1}\\ &+ 2|\xi|\|(\hat{Q}_1 + \mathbb{I}/n)\mathrm{tr}(\hat{Q}^1 - \hat{Q}^2)\mathfrak{d}Q\|_{H^1}\\ &+ 2|\xi|\|\hat{Q}_1 - \hat{Q}_2)\mathrm{tr}(\hat{Q}^2)\mathfrak{d}Q)\|_{H^1}\end{aligned}$$

$$\leq C(\|\hat{Q}_1\|_{X^Q_\gamma} + \|\hat{Q}_2\|_{X^Q_\gamma} + 1)\|\hat{Q}_1 - \hat{Q}_2\|_{X^Q_\gamma}\|Q\|_{X^Q_1}.$$

Similarly, for $u \in X_1^u$ we show that

$$\|\mathfrak{c}(\hat{Q}_1)u - \mathfrak{c}(\hat{Q}_2)u\|_{X^Q_0} \leq C(\|\hat{Q}_1\|_{X^Q_\gamma} + \|\hat{Q}_2\|_{X^Q_\gamma} + 1)\|\hat{Q}_1 - \hat{Q}_2\|_{X^Q_\gamma}\|u\|_{X^u_1}.$$

Combining these estimates we obtain, for $v = (u, Q) \in X_1$, that

$$\|A_\xi(\hat{Q}_1)v - A_\xi(\hat{Q}_2)v\|_{X_0} \leq C(\|\hat{Q}_1\|_{X^Q_\gamma} + \|\hat{Q}_2\|_{X^Q_\gamma} + 1)\|\hat{Q}_1 - \hat{Q}_2\|_{X^Q_\gamma}\|v\|_{X_1}.$$

This finishes the proof with $K(R) = C(2R + 1)$. □

We now turn our attention to the nonlinear terms $F(v)$. Before starting, we state an auxiliary lemma, which allows us to keep track of the time dependence in embedding theorems.

Lemma 6.4. *Let $s \geq s' \geq 0$ and $p \in [1, \infty)$. Then*

$$H^{s,p}(0, T) \overset{s-s'}{\hookrightarrow} H^{s',p}(0, T),$$

where $\overset{\eta}{\hookrightarrow}$ stands for an embedding with embedding constant CT^η, $C > 0$ independent of T.

Proof. Set $m := \lfloor p(s - s')\rfloor$ and $1/r := m + 1/p + s' - s \in [1/p, 1)$. Sobolev embeddings and Hölder's inequality yield

$$\begin{aligned} &H^{s,p} \hookrightarrow H^{m+s',r} \overset{1-\frac{1}{r}}{\hookrightarrow} H^{m+s',1} \hookrightarrow H^{m-1+s',\infty} \overset{1}{\hookrightarrow} H^{m-1+s',1} \\ &\hookrightarrow \ldots \hookrightarrow H^{s',\infty} \overset{\frac{1}{p}}{\hookrightarrow} H^{s',p}. \end{aligned}$$

□

The following lemma allows us to keep track of the time dependence for typical product terms of the form $v_1 \partial v_2$ where v_1, v_2 belong to the maximal regularity space:

Lemma 6.5. *Let $p, q \in (1, \infty)$ be such that $2/3p + n/3q \leq 1$. Then for all $v_1, v_2 \in Y_1 := L^p(0, T; H^{2,q}) \cap H^{1,p}(0, T; L^q)$, $\partial \in \{\partial_x, \partial_y, \partial_z\}$ and $\eta \in [0, \frac{3}{2}\left(1 - \frac{2}{3p} - \frac{n}{3q}\right)]$ it follows that*

$$\|v_1 \partial v_2\|_{L^p(L^q)} \leq CT^\eta \|v_1\|_{Y_1}\|v_2\|_{Y_1}.$$

Proof. Let $\theta_1 = \frac{2\eta}{3} + \frac{2}{3p}$ and $\theta_2 = \frac{1}{2}\theta_1$. Then, by an application of the Mixed Derivative Theorem [25], Lemma 6.4 and Sobolev embeddings, we get

$$\begin{aligned} &Y_1 \hookrightarrow H^{\theta_1,p}(0, T; H^{2-2\theta_1,q}) \overset{2\eta/3}{\hookrightarrow} H^{2/3p,p}(0, T; H^{2-2\theta_1,q}) \hookrightarrow L^{3p}(0, T; L^{3q}), \\ &Y_1 \hookrightarrow H^{\theta_2,p}(0, T; H^{2-2\theta_2,q}) \overset{\eta/3}{\hookrightarrow} H^{1/3p,p}(0, T; H^{2-2\theta_2,q}) \hookrightarrow L^{3p/2}(0, T; H^{1,3q/2}), \end{aligned}$$

where $\overset{\eta}{\hookrightarrow}$ stands for an embedding with embedding constant CT^η and $C > 0$ independent of T. Thus, Hölder's inequality implies that

$$\|v_1 \partial v_2\|_{L^p(0,T;L^q)} \leq \|v_1\|_{L^{3p}(0,T;L^{3q})}\|v_2\|_{L^{3p/2}(0,T;H^{1,3q/2})} \leq CT^\eta \|v_1\|_{Y_1}\|v_2\|_{Y_1}.$$

□

After these preparations we are now able to estimate the nonlinear term $F(v)$.

Proposition 6.6. (Nonlinear Estimates). *Let $\xi \in \mathbb{R}$ and $p \geq 2$. Then there exists $k \in \mathbb{N}_0$ such that for all $v, \bar{v} \in \mathbb{E}$*

$$\begin{aligned}&\|F(v) - F(\bar{v})\|_{\mathbb{F}} \\ &\leq C(\|v\|_{L^\infty(0,T;X_\gamma)} + \|\bar{v}\|_{L^\infty(0,T;X_\gamma)} + 1)^k(\|v\|_{\mathbb{E}} + \|\bar{v}\|_{\mathbb{E}})\|v - \bar{v}\|_{\mathbb{E}}.\end{aligned}$$

Moreover, there exists $\delta_0 > 0$ such that for all $v \in \mathbb{E}$, $\|v\|_{\mathbb{E}} \leq r$, $\delta \in [0, \delta_0]$, and for $h_\delta(t) = e^{\delta t}$ it follows that

$$\|h_\delta[F(v) - F(0)]\|_{\mathbb{F}} \leq \varepsilon\|h_\delta v\|_{\mathbb{E}}.$$

Proof. Let us note that the conditions of Lemma 6.5 are satisfied if $p \geq 2$ and $q = 2$. In this case we have

$$X_\gamma \hookrightarrow L^q_\sigma(\Omega) \times C(\overline{\Omega}; \mathbb{S}^n_0).$$

We estimate the four terms of $f_1(v)$ by means of Lemma 6.5 as follows: For the first term we obtain

$$\|(u \cdot \nabla)u - (\bar{u} \cdot \nabla)\bar{u}\|_{L^p(L^2)} \leq CT^\eta\|u - \bar{u}\|_{\mathbb{E}^u}(\|u\|_{\mathbb{E}^u} + \|\bar{u}\|_{\mathbb{E}^u}),$$

and for the second one

$$\|\operatorname{div}(\nabla Q \odot \nabla Q) - \operatorname{div}(\nabla\bar{Q} \odot \nabla\bar{Q})\|_{L^p(L^2)} \leq CT^\eta(\|Q\|_{\mathbb{E}^Q} + \|\bar{Q}\|_{\mathbb{E}^Q})\|Q - \bar{Q}\|_{\mathbb{E}^Q},$$

where we used the short hand notation $L^p(L^2) = L^p(0, T; L^2(\Omega))$. Lemma 6.5 implies furthermore

$$\begin{aligned}&\|\operatorname{div}((Q + \mathbb{I}/n)(\operatorname{tr}(Q^3) - \operatorname{tr}(Q^2)^2)) - \operatorname{div}((\bar{Q} + \mathbb{I}/n)(\operatorname{tr}(\bar{Q}^3 - \bar{Q}^2)^2))\|_{L^p(L^2)} \\ &\quad\leq CT^\eta\left(\|Q\|_{L^\infty(X_\gamma)} + \|\bar{Q}\|_{L^\infty(X_\gamma)} + 1\right)^3(\|Q\|_{\mathbb{E}^Q} + \|\bar{Q}\|_{\mathbb{E}^Q})\|Q - \bar{Q}\|_{\mathbb{E}^Q},\end{aligned}$$

as well as

$$\begin{aligned}&\|\operatorname{div}((Q + \mathbb{I}/n)(Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/n - \operatorname{tr}(Q^2)Q)) \\ &\qquad - \operatorname{div}((\bar{Q} + \mathbb{I}/n)(\bar{Q}^2 - \operatorname{tr}(\bar{Q}^2)\mathbb{I}/n - \operatorname{tr}(\bar{Q}^2)\bar{Q}))\|_{L^p(L^2)} \\ &\quad\leq CT^\eta\left(\|Q\|_{L^\infty(X_\gamma)} + \|\bar{Q}\|_{L^\infty(X_\gamma)} + 1\right)^2(\|Q\|_{\mathbb{E}^Q} + \|\bar{Q}\|_{\mathbb{E}^Q})\|Q - \bar{Q}\|_{\mathbb{E}^Q}.\end{aligned}$$

Concerning the terms of $f_2(v)$ we estimate

$$\begin{aligned}\|(u \cdot \nabla)Q - (\bar{u} \cdot \nabla)\bar{Q}\|_{L^p(H^1)} \leq CT^\eta\big(&\|u\|_{\mathbb{E}^u}\|(Q - \bar{Q}, \nabla(Q - \bar{Q})\|_{\mathbb{E}^Q} \\ &+\|u - \bar{u}\|_{\mathbb{E}^u}\|(\bar{Q}, \nabla\bar{Q})\|_{\mathbb{E}^Q}\big)\end{aligned}$$

and

$$\begin{aligned}&\|(Q^2 - \operatorname{tr}(Q^2)\mathbb{I}/n) - (\bar{Q}^2 - \operatorname{tr}(\bar{Q}^2)\mathbb{I}/n) - \operatorname{tr}(Q^2)Q + \operatorname{tr}(\bar{Q}^2)\bar{Q}\|_{L^p(H^1)} \\ &\quad\leq CT^\eta\left(\|Q\|_{L^\infty(X_\gamma)} + \|\bar{Q}\|_{L^\infty(X_\gamma)} + 1\right)(\|Q\|_{\mathbb{E}^Q} + \|\bar{Q}\|_{\mathbb{E}^Q})\|Q - \bar{Q}\|_{\mathbb{E}^Q}.\end{aligned}$$

Finally, the second assertion can be proven similarly. We therefore omit the details. □

7. Proof of the Main Theorems

The previous sections are organized in such a way that we may conclude the proof of the two main results, Theorem 3.1 and Theorem 3.2, by the results given in Sect. 4. Indeed, concerning local existence of a unique, strong solution to system (1.1) we verify the assumptions of Proposition 4.1 as follows.

Proof. By Lemma 6.3, the operator-valued function $A_\xi : X_\gamma \to \mathcal{L}(X_1, X_0)$ is locally Lipschitz continuous. Proposition 5.7 implies that $A_\xi(v_0)$ has maximal L^p-regularity. By Proposition 6.6 the nonlinear term F fulfils the assumption (A2) of Proposition 4.1. Hence, the assertion of Theorem 3.1 follows from Proposition 4.1.

Concerning global existence, we note first that $v^* = 0$ is a trivial equilibrium and that we may verify conditions (S1)-(S3) for $B_1 = B_2 = 0$. The Lipschitz condition (S1) for A_ξ holds true by Lemma 6.3. Moreover, by Proposition 6.6, also condition (S2) holds true. Since in our case $B_1 = B_2 = 0$, we see that $A_0 = A(0)$ has maximal regularity and that by Propostion 5.6 its spectrum is contained in the open right half plane. Thus the assertion of Theorem 3.2 follows from Proposition 4.2. □

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Declarations

Conflict of interest On behalf of all the authors, the corresponding author states that there is no Conflict of interest.

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