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Regularity for Nonuniformly Elliptic Equations with p,q-Growth and Explicit x,u-Dependence

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Abstract

We are interested in the regularity of weak solutions u to the elliptic equation in divergence form as in (1.1), and more precisely in their local boundedness and their local Lipschitz continuity under *general growth conditions*, the so called p,q-*growth conditions*, as in (1.2) and (1.3) below. We found a unique set of assumptions to get all of these regularity properties at the same time; in the meantime we also found the way to treat a more general context, with explicit dependence on (x, u), in addition to the gradient variable $\xi = Du$. These aspects require particular attention, due to the p,q-context, with some differences and new difficulties compared to the standard case $p = q$.

Mathematics Subject Classification Primary 35D30, 35J15, 35J60; Secondary 35B45, 49N60

Dedicated to Enrico Giusti. *In submitting the tex version of the manuscript today, March 26, 2024, we learned with dismay of the passing of Enrico Giusti, a mathematician of great value, a man of cultural depth with multiple interests, our friend and colleague at the University of Florence. Among many other scientific activities, we mention here that his mathematical research largely concerns the regularity of solutions of elliptic partial differential equations and systems; in the context of this Journal we recall that he also had the role of Editor of the Archive of Rational Mechanics and Analysis. This work is dedicated to Enrico, with profound sadness, but also with great esteem and affection.*

Contents

1. Introduction

We consider elliptic equations of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i \left(x, u\left(x\right), Du\left(x\right)\right) = b\left(x, u\left(x\right), Du\left(x\right)\right), \qquad x \in \Omega, \tag{1.1}$$

where the vector field $a\left(x, u, \xi\right) = \left(a^i\left(x, u, \xi\right)\right)_{i=1,\ldots,n}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, and the right hand side $b\left(x, u, \xi\right)$ is a Carathéodory function in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Our assumptions are characterized by the fact that the second order elliptic equation in divergence form (1.1) explicitly depends on $(x, u) \in \Omega \times \mathbb{R}$, where Ω is an open set in $\mathbb{R}^n$, $n \geqq 2$, other than on the gradient variable $\xi \in \mathbb{R}^n$. Further characteristics are the ellipticity and growth condition in general form, the so called *p,q-conditions* as in (1.2) and (1.3) below.

We are interested in the regularity of the weak solutions u to (1.1), and more precisely in their local boundedness, their local Lipschitz continuity and higher differentiability under *general growth conditions*. The effect of these results, the gradient Du being *a-posteriori* locally bounded, is that the growth properties of the differential operator reduce to the so called *natural growth conditions* (a name which usually denotes the case $p = q$). Thus, having in force the local Lipschitz continuity considered in this manuscript, also *the $C^{1,\alpha}$ regularity can be deduced by the classical literature on regularity* under the same assumptions made in the context of natural growth, for instance as in [42, Chapter 4, Section 61] or in [40]; see the p,q- growth cases in [48, Section 7, Theorem D] and in [49, Corollary 2.2]. Regarding the higher regularity and Hölder continuity of the gradient, see also [26].

A few words about the well-known classical results on regularity for weak solutions to elliptic equations as in (1.1). A primary tool is the fundamental Hölder continuity result by De Giorgi [21], which has been extensively considered also in the book by Ladyzhenskaya-Ural'tseva [42, Chapter 4]. We also refer to the article by Evans [37], with explicit dependence of $\left(a^i\right)_{i=1,\ldots,n}$ in (1.1) only on the gradient variable and with right hand side $b=0$; the celebrated paper by DiBenedetto [29] on the $C^{1,\alpha}$-regularity for weak solutions of a class of degenerate elliptic equations; the famous $C^{1,\alpha}$-regularity result by Tolksdorf [59]; the article by Manfredi [46] on the $p-$Laplacian type integrals of the Calculus of Variations. Later, see also the well known articles by Lieberman [44] and Marcellini [49], the book by Giusti [40], the results by Duzaar-Mingione [32] and Cianchi-Maz'ya [9].

Our points of view are *the* p,q*-growth conditions* $(2 \leqq p \leqq q)$ with respect to the gradient variable $\xi := Du$, in this general context with (x,u) explicit dependence too. More precisely, the p*-ellipticity*

$$\sum_{i,j=1}^{n} \frac{\partial a^i}{\partial \xi_j} \lambda_i \lambda_j \geqq m \left(1+|\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \tag{1.2}$$

is valid for a positive constant m and for every $\lambda, \xi \in \mathbb{R}^n$ and $(x,u) \in \Omega \times \mathbb{R}$; and the q*-growth*

$$\left|\frac{\partial a^i}{\partial \xi_j}\right| \leqq M \left(\left(1+|\xi|^2\right)^{\frac{q-2}{2}} + |u|^{\alpha}\right), \tag{1.3}$$

for $M>0$, $0 \leqq \alpha < (q-2)\,\frac{p^*}{p}$ (if $q=2$ then $\alpha=0$) and for all $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $i,j \in \{1,2,\ldots,n\}$.

The interest in existence and regularity for weak solutions to elliptic equations in divergence form, under general growth conditions, has risen in the last decades, starting from the first related results in the 90's. Nowadays the literature on p,q-problems is large, and maybe there is no need to enter too much into details. However, we emphasize that a bound on the ratio $\frac{q}{p}$ of the type

$$1 \leqq \tfrac{q}{p} < 1 + O\left(\tfrac{1}{n}\right) \tag{1.4}$$

is necessary and, at the same time, another bound on the ratio $\frac{q}{p}$ of the same type is also sufficient for regularity. The first approach can be found in [48–51]. Not only in *the* $p,q-$*context*, but also for *general nonuniformly elliptic problems*, see [31,50–52].

We already said that a large literature exists on this subject. Recently Mingione gave a strong impulse with the introduction of the terminology (and not only terminology, but also fine results obtained with some colleagues of him, as detailed below) of *double phase integrals*, of the type

$$\int_{\Omega} \left\{ \tfrac{1}{p}\,|Du|^p + \tfrac{1}{q} a\left(x\right) |Du|^q \right\} \mathrm{d}x \tag{1.5}$$

and also their nondegenerate version

$$\int_{\Omega} \left\{ \tfrac{1}{p} \left(1+|Du|^2\right)^{\frac{p}{2}} + \tfrac{1}{q} a\left(x\right) \left(1+|Du|^2\right)^{\frac{q}{2}} \right\}, \tag{1.6}$$

whose Euler's first variation give rise to the differential equation in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \left(\left(1+|Du|^2\right)^{\frac{p-2}{2}} + a(x) \left(1+|Du|^2\right)^{\frac{q-2}{2}} \right) u_{x_i} \right\} = 0, \qquad x \in \Omega. \tag{1.7}$$

here the coefficient $a(x)$ is either Hölder continuous or local Lipschitz continuous in Ω. However, even more relevant, $a(x)$ is greater than or equal to zero in Ω, with the possibility to be equal to zero on a closed not empty subset of Ω. Therefore the energy integral in (1.5) or in (1.6), and the equation in (1.7) too, when $p < q$ behaves like a q-Laplacian in the subset of Ω where $a(x) > 0$ (and in this case the p-addendum plays the role of a *"lower order term"*), while it is a p-Laplacian in the subset of Ω where $a(x) = 0$.

The *double phase integrals in* (1.5) and in (1.6) are relevant examples of energy integrals with p,q-growth; many other examples exist: p-power times a logarithm, variable exponents $p(x)$, anisotropic integrands such as $\sum_{i=1}^{n} \left|u_{x_i}\right|^{p_i}$ with $p = \min_i \{p_i\}$ and $q = \max_i \{p_i\}$, and so on. Note, in particular, that the equation in (1.7) satisfies the p, q-growth conditions stated in (1.2), (1.3).

A special consideration about the recent interesting article by DeFilippis-Mingione [26], related to the regularity of local minimizers of a class of integrals of the calculus of variations which do not necessarily satisfy the Euler-Lagrange equation. In order to allow a comparison with the context considered here, since in [26, Section 2.2] the energy integral is expressed in a *splitting separated sum* with respect to the gradient dependence and on the u-dependence, we discuss a particular case of (1.1),

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ a^i(Du) \right\} = b(x, u); \tag{1.8}$$

here the vector field $\left\{a^i(\xi)\right\}_{i=1,2,\ldots,n}$ in the left hand side does not explicitly depend on x and u. In [26, Section 2.2] DeFilippis-Mingione allow a very general x-dependence of the right hand side $b(x, u)$; in fact they assume that the *primitive* $h(x, u) = \int_0^u b(x, t)\,\mathrm{d}t$ is a Carathéodory function, with $h(x, \cdot)$ Hölder continuous (and thus $b(x, \cdot)$ as *derivative* of $h(x, \cdot)$ may not exist) and $h(\cdot, u)$ in a suitable Lorentz class for every fixed u. The opposite situation here: the vector field $\left\{a^i(x, u, \xi)\right\}_{i=1,2,\ldots,n}$ depends on (x, u) too, with more strict assumptions on the x and u dependence in the right hand side $b(x, u, \xi)$, which here may also depend on the gradient variable ξ. Other differences are in force; for instance in the context of this paper we allow general elliptic equations without the symmetric assumption $\frac{\partial a^i}{\partial \xi_j} = \frac{\partial a^j}{\partial \xi_i}$, see (2.5) below.

Zhikov [62,63] first studied similar kinds of integrals with general growth in the context of *homogenization* and the *Lavrentiev's phenomenon* (in this context see also the recent article [23]). More recent contributions to regularity to minimizers of *double phase integrals* and for weak solutions to *double phase equations* are due to Baroni, Colombo and Mingione in [2,12]. *Double phase* and *variable exponents* are studied by Byun-Oh [6], Ragusa-Tachikawa [57], Fang-Rădulescu-Chao Zhang-Xia Zhang [38]. In the context of *variable exponents* and *Orlicz–Sobolev spaces* we mention Zhang-Rădulescu [61]. For *Orlicz–Sobolev spaces* see the Springer Lecture Notes by Diening-Harjulehto-Hästö-Ruzicka [30], the reference paper by Chlebicka [7]; see also Chlebicka-DeFilippis [8], Hästö-Ok [41]. About *quasi-convex integrals* of the calculus of variations under general growth conditions see in particular [4,13,20,39,47]; about partial regularity, after Schmidt [58] more recently DeFilippis [22] and DeFilippis-Stroffolini [28].

Recently many authors obtained new regularity results, mainly in local boundedness, higher summability, local Lipschitz continuity, $C^{1,\alpha}$. Most of the results deal with *interior* regularity, apart from Cianchi-Maz'ya [9,10], Bögelein-Duzaar-Marcellini-Scheven [5], DeFilippis-Piccinini [27], who proved Lipschitz continuity of weak solutions up to the boundary. About interior regularity for nonuniformly elliptic energy integrals with p,q-growth and general growth we quote for instance [3,15,16,19]; see also [1,34,35,60] and the references in the review articles [53,56].

What seems in the literature not often studied is the case when the differential equation is not the Euler's first variation of an energy integral. Not always the vector field $a(x,u,\xi) = \left(a^i(x,u,\xi)\right)_{i=1,\ldots,n}$ in the left hand side of the Equation (1.1) is the gradient, with respect to the variable $\xi \in \mathbb{R}^n$, of a function $f(x,u,\xi)$; on the contrary, in the literature on this subject, often the condition $a^i = \partial f/\partial \xi_i$, which simplifies the framework, is one of the main assumption. If f is of class C^2 in ξ then this variational assumption $a^i = \partial f/\partial \xi_i$ implies that

$$\frac{\partial a^i}{\partial \xi_j} = \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 f}{\partial \xi_j \partial \xi_i} = \frac{\partial a^j}{\partial \xi_i},$$

so that the $n \times n$ matrix $\left(\frac{\partial a^i}{\partial \xi_j}\right)$ is symmetric. In this manuscript *we do not assume that this matrix is symmetric*. Instead we assume a condition of the type (when u is bounded; see more precisely below in (2.5))

$$\left|\frac{\partial a^i}{\partial \xi_j} - \frac{\partial a^j}{\partial \xi_i}\right| \leqq M\left(1+|\xi|^2\right)^{\frac{p+q-4}{4}}, \tag{1.9}$$

which requires an *"intermediate growth"* with respect to $|\xi|$ (in fact the exponent $\frac{p+q-4}{2}$ is *"intermediate"*, that is it is the average between $p-2$ and $q-2$) of the antisymmetric terms of the matrix $\left(\frac{\partial a^i}{\partial \xi_j}\right)$. Condition (1.9) is automatically satisfied, for instance, either when the vector field $\left(a^i(x,u,\xi)\right)_{i=1,\ldots,n}$ has the variational structure $a^i = \partial f/\partial \xi_i$, or if $p = q$. This second possibility explains why in the literature the assumption (1.9) is not considered under the so called *natural growth conditions* (that is, when $p = q$); in fact if $p = q$ then (1.9) is an elementary consequence of the triangular inequality and of the growth assumption (1.3) when u is bounded.

An example can be easily constructed *by adding* to a p-Laplacian *a perturbation of the r-Laplacian* and one partial derivative for instance with respect to x_n. That is for $p \leqq r < (p+q)/2$, by adding an operator of the type

$$\frac{\partial}{\partial x_n}\left(\left(1+\left|u_{x_n}\right|^2\right)^{\frac{q-2}{2}} u_{x_n}\right)$$
$$+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{a_0(x)\left(\left(1+|Du|^2\right)^{\frac{r-2}{2}} u_{x_i}+g^i(Du)\right)\right\},$$

with a not symmetric and sufficiently small matrix $\left(\frac{\partial g^i}{\partial \xi_j}\right)$, in order to maintain the ellipticity of this operator. Then the vector field, obtained by *the sum of the two operators*, is p-elliptic, has q-growth, satisfies condition (1.9), but has not a variational structure, in the sense that it is not the Euler's first variation of an energy integral. Another even simpler example of elliptic operator satisfying the same p,q-growth conditions is

$$\frac{\partial}{\partial x_n}\left(\left(1+\left|u_{x_n}\right|^2\right)^{\frac{q-2}{2}} u_{x_n}\right)+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a^i(x,u(x),Du(x)),$$

the first addendum with q-exponent being as before, while the second addendum is *uniformly elliptic* with ellipticity and growth of order p. Finally, a third example can be of the form

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_{\xi_i}(u(x),Du(x))+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a^i(x,u(x),Du(x)),$$

with $f=f(u,\xi)$ being a convex function with respect to the gradient variable $\xi\in\mathbb{R}^n$, whose quadratic form of its second order derivatives is *semidefinite positive* and has a q-growth, that is $0\leqq\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}\lambda_i\lambda_j\leqq M(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2$; while as before the second addendum is *uniformly elliptic* of order $p<q$, not necessarily being the Euler's first variation of an energy functional.

The right hand side b in (1.1) may explicitly depend on all variables $(x,u,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$. Our growth assumption for b is

$$|b(x,u,\xi)|\leqq M\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}}+M|u|^{\delta-1}+b_0(x) \tag{1.10}$$

for almost every $x\in\Omega$, $1\leqq\delta<\frac{np}{n-p}=:p^*$ (when $p<n$) and $b_0\in L^{s_0}_{\mathrm{loc}}(\Omega)$ for some $s_0>n$. The general growth condition in the right hand side of (1.10) explicitly depends on (x,u,ξ) and does not allow us to assume the summability considered for instance by DeFilippis-Mingione [25] and by DeFilippis-Piccinini [27], with b_0 in the Lorentz space $L(n,1)(\Omega)$, but however with $M=0$ in (1.10), and with the functional inclusion $L^{n+\varepsilon}(\Omega)\subset L(n,1)(\Omega)\subset L^n(\Omega)$ for all $\varepsilon>0$. Note however that the *Lebesgue summability* $b_0\in L^{s_0}_{\mathrm{loc}}(\Omega)$ *with* $s_0>n$ is naturally assumed in the literature for regularity, see for instance Colombo-Figalli [11, Theorem 1.1] and [5, Theorem 1.1]. Moreover it is sharp: in fact the weaker condition $b_0\in L^n_{\mathrm{loc}}(\Omega)$ is not sufficient for the local Lipschitz continuity of the weak solution.

The principal part of the equation also depends on all variables $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and the growth assumptions for the variables (x, u) are (see the details in (2.3), (2.6))

$$\left|\frac{\partial a^i}{\partial u}\right| \leqq M\left(\left(1+|\xi|^2\right)^{\frac{p+q-4}{4}}+(1+|u|)^{\beta-1}\right),$$

$$\left|\frac{\partial a^i}{\partial x_s}\right| \leqq M(|u|)\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}}, \tag{1.11}$$

for $x \in \Omega$, $0 \leqq \beta < \frac{n(p-1)}{n-p} =: (p-1)\frac{p^*}{p}$ (when $p < n$) and for every $\xi \in \mathbb{R}^n$, $i, s = 1, 2, \ldots, n$. With respect to the x-dependence we could expect a more general assumption depending on a Sobolev summability; for instance of the type

$$\left|\frac{\partial a^i}{\partial x_s}\right| \leqq M(x, |u|)\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}},$$

where, for $|u| \leqq L$ with L fixed, $M \in L^r_{\mathrm{loc}}(\Omega)$ for some $r > n$. In this case we believe that the local Lipschitz continuity of the weak solution should be proved with a bound on the ratio q/p as in (1.4) depending on r too; precisely $\frac{q}{p} < 1+\frac{1}{n}-\frac{1}{r}$. This Lebesgue summability was introduced in a simpler context by Eleuteri-Marcellini-Mascolo [33] and later considered by DeFilippis-Mingione [24] too. Note that $(1.11)_2$ corresponds to $r = +\infty$.

This research takes its origin from the authors' papers [17,55], where the local boundedness and the local Lipschitz continuity of weak solutions of the Equation (1.1) was studied, although with different assumptions in the two cases. The effort here was to find a unique set of assumptions to get all these regularity properties at the same time; in the meantime we also found the way to treat a more general context. The final regularity results, the local Lipschitz continuity and the higher differentiability, are stated in Section 2; the starting step for these regularity results, that is the local boundedness of the weak solutions, is stated and proved in Section 4.

In our opinion a relevant application of the regularity results proved in this manuscript relies in the authors' forthcoming paper [18], devoted to the *existence*—in the sense of Leray-Lions [43]—of weak solutions $W^{1,q}(\Omega)$ to the Dirichlet problem associated to the elliptic differential Equation (1.1): an aspect which requires particular attention due to the p,q-context when $q \neq p$, with some differences and new difficulties compared to the standard case $p = q$, and with a crucial application of the local regularity estimates obtained here.

2. Main Results

2.1. Assumptions

We study the elliptic Equation (1.1) under the following *general growth conditions* on the gradient variable $\xi = Du$, named *p, q-conditions*. In order to state the assumptions on vector field $a = a(x, u, \xi) = \left(a^i(x, u, \xi)\right)_{i=1,\dots,n}$, we start by the *ellipticity*, valid for some exponents p, q $(2 \leqq p \leqq q)$, a constant $m > 0$ and, for every $\lambda, \xi \in \mathbb{R}^n$, $(x, u) \in \Omega \times \mathbb{R}$,

$$\sum_{i,j=1}^{n} \frac{\partial a^i}{\partial \xi_j} \lambda_i \lambda_j \geqq m \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 . \tag{2.1}$$

For the same p, q, $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and for all $i, j = 1, 2, \dots, n$ we consider the *growth conditions*, for a constant $M > 0$,

$$\left|\frac{\partial a^i}{\partial \xi_j}\right| \leqq M(1 + |\xi|^2)^{\frac{q-2}{2}} + M\,|u|^{\alpha} , \tag{2.2}$$

$$\left|\frac{\partial a^i}{\partial u}\right| \leqq M(1 + |\xi|^2)^{\frac{p+q-4}{4}} + M\,(1 + |u|)^{\beta-1} , \tag{2.3}$$

$$0 \leqq \alpha < (q-2)\,\frac{p^*}{p} \quad \text{and} \quad 0 \leqq \beta < (p-1)\,\frac{p^*}{p} \tag{2.4}$$

(if $q = 2$ the first condition in (2.4) has to be read as $\alpha = 0$), where, as usual, $p^* := \frac{np}{n-p}$ if $p < n$. If $p \geqq n$ the conditions (2.4) simply reduces to $\alpha, \beta \geqq 0$, since in this case p^* is an arbitrary real number greater than p that we will assume satisfy (4.3); see Section 4 for more details.

Moreover, for every open set Ω', whose closure is contained in Ω, and for every $L > 0$, there exists a positive constant $M(L)$ (depending on Ω' and L) such that, for every $x \in \Omega'$, $\xi \in \mathbb{R}^n$ and for $|u| \leqq L$,

$$\left|\frac{\partial a^i}{\partial \xi_j} - \frac{\partial a^j}{\partial \xi_i}\right| \leqq M(L)\,(1 + |\xi|^2)^{\frac{p+q-4}{4}}, \tag{2.5}$$

$$\left|\frac{\partial a^i}{\partial x_s}\right| \leqq M(L)\,(1 + |\xi|^2)^{\frac{p+q-2}{4}}, \tag{2.6}$$

$i, j, s = 1, 2, \dots, n$. We also assume that

$$|a(x, 0, 0)| \in L^{\gamma}_{\mathrm{loc}}(\Omega), \qquad \forall\, i = 1, 2, \dots, n \tag{2.7}$$

for an exponent $\gamma > \max\left\{\frac{n}{p-1}; \frac{p}{p-1}\right\}$. In particular (2.7) is satisfied if $a(x, 0, 0)$ is a constant vector, or more generally if it is locally bounded in Ω.

The Carathéodory function $b(x, u, \xi)$ in the right hand side of (1.1), $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, for a nonnegative constant M satisfies the condition

$$|b(x, u, \xi)| \leqq M\left(1 + |\xi|^2\right)^{\frac{p+q-2}{4}} + M\,|u|^{\delta-1} + b_0(x) \tag{2.8}$$

for almost every $x \in \Omega$, for the same exponents p, q, for $1 \leqq \delta < p^*$ (see (4.4)) and for $b_0 \geq 0$, $b_0 \in L^{s_0}_{\mathrm{loc}}(\Omega)$ with $s_0 > n$ (see (5.18), (5.21)).

2.2. Discussion and Statement of a First Regularity Result

In the context of general growth conditions it is necessary to specify the functional class where to look for weak solutions. For instance, if we are studying a double phase equation as in (1.7), then this class can be defined by the functions in $W^{1,1}_{\mathrm{loc}}(\Omega)$ which make finite the energy integral (1.6), or in the degenerate case the integral in (1.5) which turns out to be an equivalent condition. More generally, if the differential equation is the Euler's first variation of an energy integral, then the natural class of functions where to look for a minimizer is the class which makes finite the energy integral. More difficult is the general case of a differential equation which is not the Euler's first variation of an integral, like in the context considered in this manuscript.

Under p,q-growth conditions with $p \leqq q$ the Sobolev class $W^{1,q}_{\mathrm{loc}}(\Omega)$ is the natural class where to look for solutions; see Definition 4.1 below, see also [55, Section 3.1] for a discussion about this aspect. At this stage we also use the summability $b_0 \in L^{q'}_{\mathrm{loc}}(\Omega)$, which is consequence of the assumption $b_0 \in L^{s_0}_{\mathrm{loc}}(\Omega)$ with $s_0 > n$; in fact, since $q \geqq 2$, $s_0 > n \geqq 2 \geqq \frac{q}{q-1} =: q'$. We note that in some cases, by using the a-priori regularity result in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, we can also show the existence of weak solutions of the associated Dirichlet problems in the Sobolev class $W^{1,p}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$; see [49, Section 4], [14,18,33]. When the differential equation is the Euler's first variation of an energy integral then we can look directly to minimizers. In this case we refer for instance to the higher integrability results for minimizers of energy integrals in [51, Theorem 2.1 and Remark 2.1] and in the well known article by Esposito-Leonetti-Mingione [36].

A main step in the proof of the following Theorem 2.1 is the local boundedness result for the weak solution obtained in Section 4. Therefore we adopt here the bound in (4.2)

$$\frac{q}{p} < 1 + \frac{1}{n}. \tag{2.9}$$

However, if the weak solution u to the PDE (1.1) is a-priori also locally bounded, that is if a-priori $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, then Theorem 2.1 also holds under the bound $\frac{q}{p} < 1 + \frac{2}{n}$ instead of (2.9); see [55, Theorem 3.3].

Addendum to the updated version of this manuscript: We thank the referee for her/his careful reading of the manuscript. She/he suggested to emphasize the way of choosing the Sobolev class where to look for solutions. We already described above in this section some details, in particular we mentioned the Sobolev classes of functions in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with "finite energy", and $W^{1,q}_{\mathrm{loc}}(\Omega)$. To make the distinction even clearer let us consider an integral of the calculus of variations to minimize, for instance of the simplest type

$$u \in W^{1,1}_{\mathrm{loc}}(\Omega) \;\to\; \int_{\Omega} f(Du)\,\mathrm{d}x, \tag{2.10}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given convex function. If f is bounded from below the integral in (2.10) is well defined, possibly with value $+\infty$. If f is *coercive* with power $p > 1$, that is $f(\xi) \geqq c\,|\xi|^p$ for some positive constant c and for some exponent $p > 1$, then for any *bounded* open set Ω the direct methods of the calculus of variations immediately show the existence of a minimizer in $W_0^{1,p}(\Omega) + u_0$, as soon as it has been fixed $u_0 \in W^{1,p}(\Omega)$ with *finite energy*, that is such that $f(Du_0) \in L^1(\Omega)$. Therefore, in this *simpler context* the *natural class* where to look for local minimizers is $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$, which by the coercivity assumption is equivalent to $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$. This was the approach adopted in [51, Theorem 2.1], one of the first regularity results—maybe the first after [48]—*for minimizers* under general growth conditions; there some more general growth conditions were considered, such as exponential growth too. In the special case of p,q-growth, it was proved [51, Remark 2.1] that if $q/p < 1 + 2/n$, then every local minimizer of (2.10) $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, with finite energy $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$, is locally Lipschitz continuous in Ω. See also the article [36] already cited above. Many relevant regularity results have been obtained when the variational problem is governed by an energy integral as in (2.10). An example is given by the *anisotropic case*, when the energy integral (2.10) assumes the form

$$\int_\Omega \left\{ |Du|^p + \sum_{i \in I} \left| u_{x_i} \right|^{p_i} \right\} \mathrm{d}x, \qquad I \subset \{1, 2, \ldots, n\},$$

where I can be any *proper* not empty subset of $\{1, 2, \ldots, n\}$, all $p_i \geqq 1$ and $q = \max\{p_i : i \in I\} \geqq p$. In this case the energy integrand $f(\xi) = |\xi|^p + \sum_{i \in I} |\xi_i|^{p_i}$ satisfies p,q-growth; the natural Sobolev space where to look for local minimizers is defined by the condition $f(Du) = |Du|^p + \sum_{i \in I} \left| u_{x_i} \right|^{p_i} \in L^1_{\mathrm{loc}}(\Omega)$ and explicitly gives the functional set $\left\{ u \in W^{1,p}_{\mathrm{loc}}(\Omega) : \ u_{x_i} \in L^{p_i} \ \ \forall\, i \in I \right\}$; a first regularity result for this anisotropic growth was obtained in [48], but of course [51, Theorem 2.1] applies too. Similarly, for functions $f = f(x, \xi)$ with x-dependence, for instance related to the so-called *double phase integrals* as in (1.5), (1.6), where the natural Sobolev class of functions u with finite energy is $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $f(x, Du) \simeq |Du|^p + a(x)\, |Du|^q \in L^1_{\mathrm{loc}}(\Omega)$; see the relevant papers [2,12] and more recently [25,28,33] under some more general conditions, noting that sometimes also a *relaxed energy functional* comes into play.

Different is the case considered in this manuscript, when u is a weak solution to a differential equation which does not admit a corresponding energy functional. Unless the differential equation has a special structure and/or is the Euler's first variation of a functional, in general under $p, q-$growth conditions an optimal class where to look for solutions is an open problem if $p \neq q$. Imagine for instance to start again by minimizing the energy integral (2.10) in the class $W_0^{1,p}(\Omega) + u_0$ by assuming that $c_1 |\xi|^p \leqq f(\xi) \leqq c_2 \left(1 + |\xi|^q\right)$ for some $c_1, c_2 > 0$, $p > 1$ and $u_0 \in W^{1,p}(\Omega)$, $f(Du_0) \in L^1(\Omega)$; let us denote by $u \in W_0^{1,p}(\Omega) + u_0$ a minimizer obtained by the direct methods of the calculus of variations. Now the question is: does u is a weak solution to the related Euler's equation? By the bound $f(\xi) \leqq c_2 \left(1 + |\xi|^q\right)$ and the Lebesgue dominated convergence theorem, if $q = p$ the answer is positive. But if $q > p$ the answer is: it depends, in general we do not know. The weak formulation in this general context could be meaningless, the pairings could be not well defined. A correct sufficient condition for u to be a weak solution of the Euler's equation is to belong to $W^{1,q}_{\mathrm{loc}}(\Omega)$. We hope, we are confident that $W^{1,q}_{\mathrm{loc}}(\Omega)$ is a sharp Sobolev class for a general situation. See the Definition 4.1 below for details, see also [49], [55, Section 3.1]. However existence in $W^{1,q}_{\mathrm{loc}}(\Omega)$ of weak solutions to the elliptic Equation (1.1) does not come for free; that is, if $q \neq p$ it is not a direct consequence of the classical existence theorems in functional analysis. However, if the vector field $\left(a^i(x,u,\xi)\right)_{i=1,\dots,n}$ in the Equation (1.1) does not explicitly depend on u, also if the differential equation is not the Euler's first variation of a functional, it is possible (see [49, Theorem 4.1], [14]) to give conditions for existence of weak solutions in the class $W^{1,q}_{\mathrm{loc}}(\Omega)$. With u-dependence the regularity results of Theorems 2.1, 2.2 below have been used by the authors in [18] as a main ingredient to existence of weak solutions in $W^{1,p}_0(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$, in the spirit of the celebrated Leray-Lions existence theorem [43], proved by Jean Leray and Jacques-Louis Lions in 1965 for the case $q = p$.

Here is one of our regularity results with u-dependence:

Theorem 2.1. *Under the ellipticity and growth conditions (2.1)–(2.8), if the exponents $q \geqq p \geqq 2$ satisfy the bound $\frac{q}{p} < 1 + \frac{1}{n}$, then every weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ to the differential Equation (1.1) is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$. Moreover, for every open set Ω' compactly contained in Ω, there exist constants $c, c', \alpha_0, \gamma > 0$ (depending on the $L^\infty\left(\Omega'\right)$ norm of u and on the data, but not on u) such that, for every ϱ and R with $0 < \varrho < R$ and $B_R(x_0) \subset \Omega'$,*

$$\|Du\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \leqq \left(\frac{c}{(R-\varrho)^n}\int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\,dx\right)^{\frac{\alpha_0}{p}}$$
$$\underset{\text{for } n>2}{=} \left(\frac{c}{(R-\varrho)^n}\left\|(1+|Du|^2)^{\frac{1}{2}}\right\|^p_{L^p(B_R)}\right)^{\frac{2}{(n+2)p-nq}}. \tag{2.11}$$

For the $n \times n$ matrix D^2u of the second derivatives of u the following estimates hold:

$$\int_{B_\varrho}\left|D^2u\right|^2 dx \leqq \frac{c}{(R-\varrho)^2}\int_{B_R}\left(1+|Du|^2\right)^{\frac{q}{2}} dx \tag{2.12}$$
$$\leqq \frac{c'}{(R-\varrho)^2}\left(\frac{1}{(R-\varrho)^{\gamma\vartheta(q-p)}}\int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\,dx\right)^{\frac{\alpha_0 q}{\vartheta p}}.$$

The explicit expression of α_0 in the gradient bound (2.11) is given by

$$\alpha_0 := \frac{\vartheta\frac{p}{q}}{1-\vartheta(1-\frac{p}{q})} \underset{\text{for } n>2}{=} \frac{2p}{(n+2)p-nq}, \tag{2.13}$$

where $\vartheta \geqq 1$ is the value $\vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2} \underset{\text{for} n>2}{=} \frac{2q}{np-(n-2)q}$ and $\gamma = \frac{n}{q}\vartheta$. Note that $\alpha_0 \geqq 1$ (in fact $\vartheta\frac{p}{q} \geqq 1-\vartheta(1-\frac{p}{q}) = 1-\vartheta+\vartheta\frac{p}{q}$ is equivalent to $\vartheta \geqq 1$) and, for the same reason, $\alpha_0 = 1$ if and only if $\vartheta = 1$, which is equivalent to $q = p$.

2.3. Statement of a Second Regularity Result

We may look for a $W^{1,\infty}_{\mathrm{loc}}(\Omega)-$ regularity result when in the above growth assumptions the average exponent $\frac{p+q}{2}$, middle point between p and q, is replaced by q; in this case—a priori—we consider less restrictive assumptions. More precisely, in view of the next Theorem 2.2, compared with Section 2.1 we now have the less restrictive growth conditions (obtained when in (2.3), (2.6), (2.8) $\frac{p+q}{2}$ is replaced by q)

$$\begin{cases} \left|\frac{\partial a^i}{\partial \xi_j}\right| \leqq M\left(1+|\xi|^2\right)^{\frac{q-2}{2}} + M\,|u|^{\alpha} \\ \left|\frac{\partial a^i}{\partial u}\right| \leqq M\left(1+|\xi|^2\right)^{\frac{q-2}{2}} + M\,(1+|u|)^{\beta-1} \\ \left|\frac{\partial a^i}{\partial x_s}\right| \leqq M\,(L)\left(1+|\xi|^2\right)^{\frac{q-1}{2}} \\ \left|a^i\,(x,0,0)\right| \in L^{\gamma}_{\mathrm{loc}}(\Omega) \\ |b\,(x,u,\xi)| \leqq M\left(1+|\xi|^2\right)^{\frac{q-1}{2}} + M\,|u|^{\delta-1} + b_0\,(x) \end{cases} \tag{2.14}$$

for every $\xi \in \mathbb{R}^n$, $i, j, s = 1, 2, \ldots, n$, u as before. The parameters satisfy the bounds

$$\begin{cases} 0 \leqq \alpha < (2q-p-2)\,\frac{p^*}{p} \\ 0 \leqq \beta < (p-1)\,\frac{p^*}{p} \\ \gamma > \max\left\{\frac{n}{p-1};\, \frac{p}{p-1}\right\} \\ 1 \leqq \delta < p^* \\ b_0 \in L^{s_0}_{\mathrm{loc}}(\Omega)\,, \quad s_0 > n. \end{cases} \tag{2.15}$$

When in (2.5) we replace $\frac{p+q}{2}$ by q we get $\left|\frac{\partial a^i}{\partial \xi_j} - \frac{\partial a^j}{\partial \xi_i}\right| \leqq M\,(L)\,(1+|\xi|^2)^{\frac{q-2}{2}}$; we do not need to state it among the other growth assumptions, since this time it is automatically satisfied as consequence of $(2.14)_1$ when $|u| \leqq L$.

As we said above, the next theorem holds with the growth assumptions in (2.14), which are less restrictive than those of Theorem 2.1, however with a stronger a-priori summability condition on the weak solution u, a more restrictive bound for α in $(2.15)_1$ and for the quotient $\frac{q}{p}$.

Theorem 2.2. *Under the ellipticity* (2.1) *and the* p, q*-growth conditions* (2.14), (2.15)*, if* $q \geqq p \geqq 2$ *satisfy the bound* $\frac{q}{p} < 1 + \frac{1}{2n}$*, then every weak solution* $u \in W^{1,2q-p}_{\mathrm{loc}}(\Omega)$ *to the elliptic Equation* (1.1) *is of class* $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$ *and, for every open set* Ω' *whose closure is contained in* Ω*, there exist constants* $c, \alpha_1, R_0 > 0$ *(depending on the* $L^{\infty}\left(\Omega'\right)$ *norm of u and on the data, but not on u) such that, for every* ϱ *and* R *with* $0 < \varrho < R$ *and* $B_R(x_0) \subset \Omega'$,

$$\begin{aligned} \|Du\|_{L^{\infty}(B_\varrho;\mathbb{R}^n)} &\leqq \left(\frac{c}{(R-\varrho)^n}\int_{B_R}\left(1+|Du|^2\right)^{\frac{p}{2}}\,dx\right)^{\frac{\alpha_1}{p}} \\ &\underset{\text{for } n>2}{=} \left(\frac{c}{(R-\varrho)^n}\left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|^{p}_{L^p(B_R)}\right)^{\frac{1}{(n+1)p-nq}}. \end{aligned} \tag{2.16}$$

Moreover the L^2-local estimates (2.12) for the matrix D^2u of the second derivatives hold when we replace q by $2q - p$.

The exponent α_1 in (2.16), when $n > 2$, is given by

$$\alpha_1 := \frac{p}{(n+1)\,p - nq}. \tag{2.17}$$

Similarly to α_0 in (2.13), α_1 is also equal to 1 if and only if $q = p$; note that α_1 is well defined as a positive real number since $(n+1)\,p - nq > 0$, being this equivalent to the bound: $\frac{q}{p} < 1 + \frac{1}{n}$. We recall again that Theorem 2.2 holds for general differential Equation (1.1) without the symmetric assumption $\frac{\partial a^i}{\partial \xi_j} = \frac{\partial a^j}{\partial \xi_i}$.

Remark 2.3. If a-priori we know that the weak solution is locally bounded, then Theorem 2.1 also holds under the weaker bound $\frac{q}{p} < 1 + \frac{2}{n}$, while Theorem 2.2 also holds under the bound $\frac{q}{p} < 1 + \frac{1}{n}$. The reason relies on the fact that in this case it is not necessary to apply the local boundedness result of Theorem 4.2 and we can modify and adapt the method in [55] to the context considered here.

2.4. The Classical Case Under the So-called Natural Growth Conditions

Finally we observe that all the results of this paper hold in the particular case $q = p$, when the bound $\frac{q}{p} < 1 + \frac{1}{n}$ in (2.9) of course is satisfied. When $q = p$ the statements of the two Theorems 2.1 and 2.2 coincide each other; the ellipticity (2.1) and the growth conditions (2.14), (2.15) can be read with $q = p$ for the validity of the regularity in $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$ of the weak solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to the elliptic Equation (1.1). We already observed that in the gradient estimate (2.11) the exponent α_0, defined in (2.13), is greater than or equal to 1 and $\alpha_0 = 1$ if and only if $q = p$. Therefore in this case the L^∞-gradient local bound (2.11) for u takes the form

$$\|Du\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \leqq \left(\frac{c}{(R-\varrho)^n} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}}\, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{2.18}$$

while the $L^2(\Omega)$-local bound for the $n \times n$ matrix D^2u of the second derivatives of u is

$$\int_{B_\varrho} \left|D^2u\right|^2 \mathrm{d}x \leqq \frac{c}{(R-\varrho)^2} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}}\, \mathrm{d}x. \tag{2.19}$$

3. Linking Lemma

As before we use the notation $a(x,u,\xi) = \left(a^i(x,u,\xi)\right)_{i=1,\ldots,n}$; moreover, as usual, $(a(x,u,\xi), \xi)$ is the scalar product in $\mathbb{R}^n$ of $a(x,u,\xi)$ and ξ.

Lemma 3.1. *Under the ellipticity condition (2.1), the order-one growth conditions (2.2), (2.3), the bounds* $2 \leqq p \leqq q < p+2$, $0 \leqq \alpha < (q-2)\frac{p^*}{p}$ *(*$\alpha = 0$ *if* $q = 2$*) and* $0 \leqq \beta < (p-1)\frac{p^*}{p}$*, the following coercivity and zero-order growth conditions hold:*

(1) *For some positive constants* c_1, c_2 *and for all* $x \in \Omega$, $u \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^n$

$$(a(x,u,\xi), \xi) \geqq c_1 |\xi|^p - c_2 (1+|u|)^\theta - b_1(x), \tag{3.1}$$

with

$$\theta := \max\left\{\tfrac{2p}{p-q+2}, \beta \tfrac{p}{p-1}\right\} \tag{3.2}$$

and $b_1(x) := const \cdot \left\{1+|a(x,0,0)|^{\frac{p}{p-1}}\right\} \in L^{\gamma\frac{p-1}{p}}_{\text{loc}}$ *with* $\gamma > \max\left\{\frac{n}{p-1}; \frac{p}{p-1}\right\}$;

(2) *For some positive constants* c_3, c_4 *and for all* $x \in \Omega$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$

$$|a(x,u,\xi)| \leqq c_3 |\xi|^{q-1} + c_4 (1+|u|)^\lambda + b_2(x), \tag{3.3}$$

where λ*, when* $q > 2$*, is given by*

$$\lambda := \max\left\{2\tfrac{q-1}{q-p+2};\ \beta;\ \alpha\tfrac{q-1}{q-2}\right\}, \tag{3.4}$$

while if $q = 2$*, then* $\lambda := \max\{1;\ \beta\}$*; moreover* $b_2(x) := |a(x,0,0)| \in L^\gamma_{\text{loc}}$.

We recall that γ is defined in (2.7) and the summability condition here is the same as that one in the following Section, with $s_1 := \gamma\frac{p-1}{p} > 1$ and $L^{\gamma(p-1)/p}_{\text{loc}} = L^{s_1}_{\text{loc}}$, $s_1 > \frac{n}{p}$.

Proof. We start proving (1). We first observe that, for every $\varepsilon > 0$, by Young's inequality with conjugate exponents $\frac{p}{p-1}$ and p,

$$|(a(x,0,0), \xi)| \leqq |a(x,0,0)|\,|\xi| \leqq \tfrac{p-1}{p}\varepsilon^{-\frac{p}{p-1}} |a(x,0,0)|^{\frac{p}{p-1}} + \tfrac{1}{p}\varepsilon^p |\xi|^p. \tag{3.5}$$

Moreover, with the component notation $\xi = (\xi_i)_{i=1,\ldots,n}$, we have

$$\begin{aligned}
&(a(x,u,\xi), \xi) - (a(x,0,0), \xi) \\
&= \sum_{i=1}^n \left\{a^i(x,u,\xi) - a^i(x,0,0)\right\} \xi_i \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^n a^i(x,tu,t\xi)\,\xi_i \,\mathrm{d}t \\
&= \int_0^1 \left\{\sum_{i=1}^n a^i_u(x,tu,t\xi)\,\xi_i u + \sum_{i,j=1}^n a^i_{\xi_j}(x,tu,t\xi)\,\xi_i\xi_j\right\} \mathrm{d}t.
\end{aligned} \tag{3.6}$$

By the growth condition (2.3) and the ellipticity assumption (2.1) we get that

$$\begin{aligned}&(a(x,u,\xi),\xi)-(a(x,0,0),\xi)\\ &\geqq \int_0^1\left\{-nM\left(\left(1+|t\xi|^2\right)^{\frac{p+q-4}{4}}|\xi|\,|u|+|\xi|\,(1+|u|)^\beta\right)\right.\\ &\quad\left.+m\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}|\xi|^2\right\}\mathrm{d}t\\ &=\int_0^1\left\{-nM\left(\left(1+|t\xi|^2\right)^{\frac{p-2}{4}}|\xi|\left(1+|t\xi|^2\right)^{\frac{q-2}{4}}|u|+|\xi|\,(1+|u|)^\beta\right)\right.\\ &\quad\left.+\,m\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}|\xi|^2\right\}\mathrm{d}t\\ &\geqq\int_0^1\left\{-nM\left(\tfrac{\varepsilon^2}{2}\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}|\xi|^2+\tfrac{1}{2\varepsilon^2}\left(1+|t\xi|^2\right)^{\frac{q-2}{2}}|u|^2\right.\right.\\ &\quad\left.\left.+\,|\xi|\,(1+|u|)^\beta\right)+\,m\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}|\xi|^2\right\}\mathrm{d}t.\end{aligned}$$

For $\varepsilon>0$ sufficiently small we deduce that there exists a constant $c>0$ such that

$$\begin{aligned}&(a(x,u,\xi),\xi)-(a(x,0,0),\xi)\\ &\geqq\int_0^1\left\{-c\left(1+|t\xi|^2\right)^{\frac{q-2}{2}}|u|^2-nM\,|\xi|\,(1+|u|)^\beta\right.\\ &\quad\left.+\tfrac{m}{2}\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}|\xi|^2\right\}\mathrm{d}t.\end{aligned}\tag{3.7}$$

Let us first consider the case $q>2$. In the first addendum we take $t=1$ and we apply Young's inequality with conjugate exponents $\frac{p}{q-2}$ and $\frac{p}{p-q+2}$ (here we use the assumption $q<p+2$), while in the second addendum we consider the conjugate exponents p and $\frac{p}{p-1}$:

$$\begin{aligned}(a(x,u,\xi),\xi)-(a(x,0,0),\xi)\geqq&-c\frac{q-2}{p}\varepsilon^{\frac{p}{q-2}}\left(1+|\xi|^2\right)^{\frac{p}{2}}\\ &-c\frac{p-q+2}{p\varepsilon^{\frac{p}{p-q+2}}}(1+|u|)^{\frac{2p}{p-q+2}}\\ &-\frac{nM}{p}\varepsilon^p|\xi|^p-\frac{nM(p-1)}{p\,\varepsilon^{\frac{p}{p-1}}}(1+|u|)^{\beta\frac{p}{p-1}}\\ &+\tfrac{m}{2}|\xi|^2\int_0^1\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}\mathrm{d}t.\end{aligned}\tag{3.8}$$

If we "just forget 1+" we have

$$\tfrac{m}{2}|\xi|^2\int_0^1\left(1+|t\xi|^2\right)^{\frac{p-2}{2}}\mathrm{d}t\geqq\tfrac{m}{2}|\xi|^p\int_0^1 t^{p-2}\mathrm{d}t=\tfrac{m}{2(p-1)}|\xi|^p.\tag{3.9}$$

Moreover, for θ defined in (3.2) $\theta := \max\left\{\frac{2p}{p-q+2};\ \beta\frac{p}{p-1}\right\}$, we have $(1+|u|)^{\beta\frac{p}{p-1}} + (1+|u|)^{\frac{2p}{p-q+2}} \leqq 2\,(1+|u|)^{\theta}$. Therefore, for $\varepsilon > 0$ sufficiently small, taking into account (3.5), (3.8), (3.9) we obtain the statement in (1).

Finally, if $q = 2$ then $p = 2$. The inequality (3.7) implies

$$
\begin{aligned}
(a\,(x,u,\xi)\,,\xi) - (a\,(x,0,0)\,,\xi) \geqq& -c\,(1+|u|)^2 - \tfrac{nM}{2}\varepsilon^2\,|\xi|^2 \\
& - \tfrac{nM}{2\,\varepsilon^2}\,(1+|u|)^{2\beta} + \tfrac{m}{2}\,|\xi|^2\,,
\end{aligned}
$$

and we obtain the statement in (1) also in this case.

To prove (3.3) in (2) we fix $i \in \{1, 2, \ldots, n\}$ and we consider assumptions (2.2) and (2.3). We get

$$
\begin{aligned}
\left|a^i\,(x,u,\xi) - a^i\,(x,0,0)\right| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} a^i\,(x,tu,t\xi)\;\mathrm{d}t\right| \\
&= \left|\int_0^1 \left\{a_u^i\,(x,tu,t\xi)\,u + \sum_{j=1}^n a_{\xi_j}^i\,(x,tu,t\xi)\,\xi_i\right\}\mathrm{d}t\right| \\
&\leqq M\left((1+|\xi|^2)^{\frac{p+q-4}{4}}\,|u| + (1+|u|)^{\beta}\right) \\
&\quad + nM\left((1+|\xi|^2)^{\frac{q-2}{2}+\frac{1}{2}} + |\xi|\,|u|^{\alpha}\right). \qquad (3.10)
\end{aligned}
$$

Again, let us first consider the case $q > 2$. Similarly as before, we use Young's inequality with conjugate exponents $2\frac{q-1}{p+q-4}$ and $2\frac{q-1}{q-p+2}$ in the first addendum and with conjugate exponents $q-1$ and $\frac{q-1}{q-2}$ in the last addendum. We obtain

$$
\begin{aligned}
&\left|a^i\,(x,u,\xi) - a^i\,(x,0,0)\right| \\
&\leqq M\left(\tfrac{1}{2}\,\tfrac{p+q-4}{q-1}\left(1+|\xi|^2\right)^{\frac{q-1}{2}} + \tfrac{1}{2}\,\tfrac{q-p+2}{q-1}\,|u|^{2\frac{q-1}{q-p+2}} + (1+|u|)^{\beta}\right) \\
&\quad + nM\left(1+|\xi|^2\right)^{\frac{q-1}{2}} + nM\left(\tfrac{1}{q-1}\,|\xi|^{q-1} + \tfrac{q-2}{q-1}\,|u|^{\alpha\frac{q-1}{q-2}}\right) \\
&\leq c(M,n,p,q)\left\{\left(1+|\xi|^2\right)^{\frac{q-1}{2}} + (1+|u|)^{\lambda}\right\} \qquad (3.11)
\end{aligned}
$$

with λ in (3.4), that is $\lambda := \max\left\{2\frac{q-1}{q-p+2};\ \beta;\ \alpha\frac{q-1}{q-2}\right\}$, and (3.3) follows.

When $q = 2$ then $p = 2$, $\alpha = 0$. The inequality (3.10) says that

$$
\begin{aligned}
\left|a^i\,(x,u,\xi) - a^i\,(x,0,0)\right| &\leqq M\left(|u| + (1+|u|)^{\beta}\right) + nM\left((1+|\xi|^2)^{\frac{1}{2}} + |\xi|\right) \\
&\leqq 2nM(1+|\xi|^2)^{\frac{1}{2}} + 2M\,(1+|u|)^{\max\{1,\beta\}}\,.
\end{aligned}
$$

The final inequality (3.3) follows. □

4. Local Boundedness

In this section we prove a local boundedness result for weak solutions to the elliptic Equation (1.1). First we recall the definition of weak solution. In fact, in the context of p,q-growth conditions, $1 < p \leqq q$, it is necessary to use some care in choosing the Sobolev class where to look for solutions; for a discussion about this aspect see [55, Section 3.1], our previous Section 2.2 and Remark 4.3 below. In the following Definition 4.1 we look for solutions in the Sobolev class $W^{1,q}_{\mathrm{loc}}(\Omega)$.

Definition 4.1. A function $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1) if

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x, u, Du)\varphi_{x_i} + b(x, u, Du)\varphi \right\} \mathrm{d}x = 0 \tag{4.1}$$

for all $\varphi \in W^{1,q}(\Omega)$ with $\operatorname{supp}\varphi \Subset \Omega$.

Our aim in this section is to give conditions on the vector field $a(x, u, \xi) = \left(a^i(x, u, \xi)\right)_{i=1,\dots,n}$ and on the right hand side $b(x, u, \xi)$ which guarantee the local boundedness of all the weak solutions u to (4.1). Since a-priori $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, then u is also locally bounded in Ω when $q > n$, as a well known consequence of the Sobolev-Morrey embedding theorem. Therefore, for the local boundedness, we could limit ourselves in this section to consider the case $p \leqq q \leqq n$. However we adopt the more general framework $p, q \in \mathbb{R}$, $p \leqq q$ with the aim to get a final local L^∞-bound for u without upper restrictions on p, q, other than the natural bound on the ratio q/p in (4.2) below.

In order to obtain a local boundedness result for the weak solutions of (1.1) we consider the assumptions of Section 2.1 and their consequences as stated in Lemma 3.1. However it could be useful to list explicitly here the hypotheses on the Carathéodory functions $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which we use for the local boundedness, assumptions which *sometimes are less restrictive* than those of Section 2.1 above. These *coercivity* and *growth assumptions* are stated in terms of some parameters $p, q, \lambda, \theta, \delta, s_0, s_1, s_2$. About p, q, we assume $p > 1$ and

$$\text{on the ratio } \tfrac{q}{p} \geqq 1: \qquad \tfrac{q}{p} < 1 + \min\left\{\tfrac{1}{n}, \tfrac{1}{p}\right\}. \tag{4.2}$$

Before writing the bounds of the other parameters, we remind that p^* denotes the Sobolev exponent appearing in the Sobolev embedding theorem for functions in $W^{1,p}(\Omega)$ with Ω bounded open set in $\mathbb{R}^n$; that is $p^* := \frac{np}{n-p}$ if $p < n$ and p^* equal any fixed real number greater than p if $p \geqq n$. In particular, if $p \geqq n$ we assume $s_0, s_1, s_2 > 1$ and, without loss of generality, we choose

$$p^* > \max\left\{\tfrac{p}{p-q+1};\ \tfrac{\lambda p}{q-1};\ \theta;\ \delta;\ \tfrac{ps_0}{s_0-1};\ \tfrac{ps_1}{s_1-1};\ \tfrac{ps_2}{s_2-1}\right\}. \tag{4.3}$$

If $p < n$ the conditions on the exponents $\lambda, \theta, \delta, s_0, s_1, s_2$ which we consider for the local boundedness of the weak solutions are

$$\text{on } \theta \geqq 0 \text{ and } \delta \geqq 1: \quad \theta, \delta < \tfrac{np}{n-p} =: p^*; \tag{4.4}$$

$$\text{on } \lambda \geqq 0: \quad \lambda < (q-1)\tfrac{p^*}{p}; \tag{4.5}$$

$$\text{on } s_0 \text{ and } s_1: \quad s_0, s_1 > \tfrac{n}{p}; \tag{4.6}$$

$$\text{on } s_2: \quad s_2 \geq \tfrac{q}{q-1} \text{ and } s_2 > \tfrac{n}{p}. \tag{4.7}$$

We note, in particular, that $s_0, s_1, s_2 > 1$.

In Section 5 the parameters θ and λ will be linked to p, q, α, β as described in (3.2) and (3.4); more precisely,

$$\theta := \max\left\{\tfrac{2p}{p-q+2}, \beta\tfrac{p}{p-1}\right\}, \tag{4.8}$$

and, when $q > 2$,

$$\lambda := \max\left\{2\tfrac{q-1}{q-p+2};\ \beta;\ \alpha\tfrac{q-1}{q-2}\right\}, \tag{4.9}$$

while, if $q = 2$, then $\lambda := \max\{1;\ \beta\}$.

4.1. Assumptions for the Local Boundedness

We start by (1), (2) in Lemma 3.1, in this slightly less restrictive form:

(i) There exist positive constants c_1, c_2, such that for almost every $x \in \Omega$ and every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$(a(x, u, \xi), \xi) \geqq c_1 |\xi|^p - c_2 (1 + |u|)^\theta - b_1(x), \tag{4.10}$$

with $b_1 \geqq 0$ and $b_1 \in L^{s_1}_{\mathrm{loc}}(\Omega)$ (here s_1 has the role stated in Lemma 3.1(1): $s_1 = \gamma\frac{p-1}{p}$);

(ii) There exist positive constants c_3, c_4, such that for almost every $x \in \Omega$, every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|a(x, u, \xi)| \leqq c_3 |\xi|^{q-1} + c_4 (1 + |u|)^\lambda + b_2(x) \tag{4.11}$$

with $b_2 \geqq 0$ and $b_2 \in L^{s_2}_{\mathrm{loc}}(\Omega)$ (here s_2 has the role stated in Lemma 3.1(2): $s_2 = \gamma$);

(iii) For the same exponents p, q, for $\delta \geqq 1$ and for a positive constant M

$$|b(x, u, \xi)| \leqq M\left(1 + |\xi|^2\right)^{\frac{p+q-2}{4}} + M|u|^{\delta-1} + b_0(x) \tag{4.12}$$

with $b_0 \geqq 0$ and $b_0 \in L^{s_0}_{\mathrm{loc}}(\Omega)$.

4.2. Statement of the Local Boundedness Result

The next local boundedness Theorem 4.2, valid for weak solutions to the differential Equation (1.1), will be used also in the proof of the local Lipschitz continuity Theorem 2.1.

Theorem 4.2. *Let $u \in W^{1,q}_{\text{loc}}(\Omega)$ be a weak solution to (1.1) under the assumptions (4.2)–(4.12). Consider $0 < R_0 \leqq 1$ with $B_{R_0}(x_0) \Subset \Omega$. Then there exists $\sigma > 0$, with $\frac{p^*-p}{\sigma} \geqq 1$, such that*

$$\|u\|_{L^\infty(B_{r/2}(x_0))} \leqq \frac{c}{r^{\frac{p}{\sigma(p-q+1)}}} \left(1 + \|u\|_{L^{p^*}(B_r(x_0))}\right)^{\frac{p^*-p}{\sigma}} \tag{4.13}$$

for every positive $r \leqq R_0$, where the constant c depends on the L^{s_0}-norm of b_0, the L^{s_1}-norm of b_1 in B_{R_0}, the L^{s_2}-norm of b_2 in B_{R_0} and it is independent of u.

The explicit expression for σ in (4.13) is

$$\sigma := p^* - \max\left\{\tfrac{p}{p-q+1};\ \tfrac{\lambda p}{q-1};\ \theta;\ \delta;\ \tfrac{p^*}{s_2}+1;\ \tfrac{p^*}{s_0}+1;\ \tfrac{p^*}{s_1}\right\}, \tag{4.14}$$

and we note that $\sigma > 0$; this is due to the bound (4.3) if $p \geqq n$, and to the bounds (4.2), (4.4), (4.5), (4.6) if $p < n$; in particular we notice that $\frac{p}{p-q+1} < p^*$ if and only if $\frac{q}{p} < 1 + \frac{1}{n}$, that is (4.2) holds, and, by (4.6) and (4.7), $s_i > \frac{n}{p} = \frac{p^*}{p^*-p} > \frac{p^*}{p^*-1}$ for every $i \in \{0, 1, 2\}$. Moreover, $\frac{p^*-p}{\sigma} \geq 1$, because $\frac{p}{p-q+1} \geq p$.

Remark 4.3. (about the summability condition on b) We discuss the summability of the integral form (4.1) of the elliptic Equation (1.1), in particular the correctness of the definition of the pairing

$$\int_\Omega b(x, u, Du)\,\varphi(x)\ \mathrm{d}x. \tag{4.15}$$

Since φ is a test function in $W^{1,q}(\Omega)$ with compact support in Ω, for the right hand side b, satisfying the bound in (4.12), for the x-dependence it is natural to assume the summability $b_0 \in L^{q'}_{\text{loc}}(\Omega)$, with as usual $\frac{1}{q} + \frac{1}{q'} = 1$. This summability $b_0 \in L^{q'}_{\text{loc}}(\Omega)$ is a consequence of the assumption in (2.8) $b_0 \in L^{s_0}_{\text{loc}}(\Omega)$ with $s_0 > n$; in fact, since $q \geqq 2$, $s_0 > n \geqq 2 \geqq \frac{q}{q-1} =: q'$. However the local boundedness result in Theorem 4.2 is obtained with the less restrictive bound in (4.12) $b_0 \in L^{s_0}_{\text{loc}}(\Omega)$ with $s_0 > \frac{n}{p}$. Therefore in this section still we need to show that the pairing (4.15) is well defined. This fact is a consequence of the imbedding of $W^{1,q}_0(\Omega)$ into $L^{q^*}(\Omega)$; in fact the test function $\varphi \in L^{q^*}(\Omega)$ and the integral (4.15) is also correctly defined if $b_0 \in L^{(q^*)'}_{\text{loc}}(\Omega)$. Let us show that $b_0 \in L^{s_0}_{\text{loc}}(\Omega) \subset L^{(q^*)'}_{\text{loc}}(\Omega)$ when $s_0 > \frac{n}{p}$ and $1 \leqq p < n$. Since $p \leqq q$ then $p^* \leqq q^*$ and

$$(q^*)' \leqq (p^*)' = \tfrac{p^*}{p^*-1} = \tfrac{np}{np-n+p};$$

this last quantity is less than or equal to $\frac{n}{p}$ if and only if $\frac{p}{np-n+p} \leqq \frac{1}{p}$, which is equivalent to $p^2-(n+1)\,p+n \leqq 0$; that is $1 \leqq p \leqq n$. Therefore $(q^*)' \leqq \frac{n}{p} < s_0$ and $L^{s_0}_{\mathrm{loc}}(\Omega) \subset L^{(q^*)'}_{\mathrm{loc}}(\Omega)$.

If $p = n$ the inclusion $L^{s_0}_{\mathrm{loc}}(\Omega) \subset L^{(q^*)'}_{\mathrm{loc}}(\Omega)$ is trivially satisfied by assuming q^* large enough, in dependence of $s_0 > 1$. If $p > n$ the test functions are bounded, therefore the pairing is well defined because b_0 is locally summable.

As far as the u-dependence it is concerned, we remark that the definition of the pairing is correct if $|u|^{\delta-1}\varphi \in L^1(\Omega)$. Since, by Sobolev embedding Theorem φ is in $L^{q^*}(\Omega)$ with compact support in Ω, then we need $u \in L^{(\delta-1)\frac{q^*}{q^*-1}}_{\mathrm{loc}}(\Omega)$. By Sobolev embedding Theorem $u \in L^{q^*}(\Omega)$, therefore the needed summability for u is satisfied if $(\delta-1)\frac{q^*}{q^*-1} \leq q^*$, or equivalently, $\delta \leq q^*$. This last condition holds, because $\delta < p^*$, see (4.4).

We conclude by considering the product $|Du|^{\frac{p+q-2}{2}}\varphi$ that we want to be summable in Ω. Reasoning as above, this happens if $\frac{p+q-2}{2}\frac{q^*}{q^*-1} \leq q$. If $q \geq n$ it is sufficient to choose q^* large enough, since $p < q+2$. If instead $q < n$ then we observe that $\frac{p+q-2}{2}\frac{q^*}{q^*-1} \leq q$ is consequence of $p \leq q$.

Remark 4.4. (about the summability condition on a) We discuss the well posedness of the pairing

$$\int_\Omega (a(x,u,Du), D\varphi(x))\,\mathrm{d}x. \tag{4.16}$$

Since φ is a test function in $W^{1,q}(\Omega)$ with compact support in Ω, we need to prove that $a(x,u,Du)$ is locally in $L^{q'}$, with as usual $\frac{1}{q}+\frac{1}{q'}=1$.

We use the inequality (4.11). By the Sobolev embedding theorem $u \in L^{q^*}_{\mathrm{loc}}$, with $q^* := \frac{nq}{n-q}$ if $q < n$; if instead $q \geqq n$ then q^* is a real number greater than q, that we can choose greater than $q\frac{p^*}{p}$. Therefore, to have the well posedness of (4.16), we need $|u|^\lambda \in L^{q'}_{\mathrm{loc}}$, that is $\lambda \leqq (q-1)\frac{q^*}{q}$. This condition is satisfied because $p \leqq q$ implies $\frac{p^*}{p} \leqq \frac{q^*}{q}$ and, by (4.5), $\lambda < (q-1)\frac{p^*}{p} \leqq (q-1)\frac{q^*}{q}$.

It remains to check if $b_2 \in L^{q'}_{\mathrm{loc}}$; that is, we need to check if $s_2 \geq q'$. This condition is satisfied because of (4.7).

In Section 5 we assume growth conditions on the derivatives of the vector field a, under the assumptions $2 \leqq p \leqq q$, that, as proved in Lemma 3.1(2), imply the existence of some positive constants c_3, c_4 such that for all $x \in \Omega$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$

$$|a(x,u,\xi)| \leqq c_3\,|\xi|^{q-1} + c_4\,(1+|u|)^\lambda + b_2(x), \tag{4.17}$$

with $b_2 = |a(\cdot,0,0)|$ and λ linked to the parameters in the assumptions on the derivatives of a by the relationship (4.9). We claim that $\lambda < (q-1)\frac{p^*}{p}$ holds. Indeed, $\lambda < (q-1)\frac{p^*}{p}$ is equivalent to assume the following inequalities if $q > 2$:

$$\frac{2p}{q-p+2} < p^*; \quad \beta < (q-1)\frac{p^*}{p}; \quad \alpha < (q-2)\frac{p^*}{p};$$

otherwise, if $q = 2$, they reduce to $p < p^*$, $\beta < \frac{p^*}{p}$. All these assumptions on α and β are satisfied due to (2.4), while the condition on q follows because $p \leqq q$ implies $\frac{2p}{q-p+2} \leqq \frac{p}{p-q+1}$ and this last term is smaller than p^*, see (4.3) and (4.2). We remark also that when we deal with the Lipschitz continuity we have $s_2 = \gamma$. Since $\gamma > \frac{n}{p-1}$ when $p < n$, then $\gamma \geqq q'$ too. Indeed, $\frac{n}{p-1} > \frac{p}{p-1} \geqq \frac{q}{q-1}$.

Remark 4.5. (about an upper bound of θ) In Section 5 the parameter θ is as in (4.8); that is $\theta := \max\left\{\frac{2p}{p-q+2}, \beta\frac{p}{p-1}\right\}$. We claim that $\theta < p^*$. Indeed, if $p \geq n$ this follows by choosing p^* large enough, see (4.3). If $p < n$, it is easy to check that if $\frac{q}{p} < 1 + \frac{2}{n}$, thus in particular if (4.2) holds, then $\frac{2p}{p-q+2} < p^*$. Therefore the conditions $\frac{q}{p} < 1 + \frac{2}{n}$ and $\beta < (p-1)\frac{p^*}{p}$ (see (2.4)) imply $\theta < p^*$.

4.3. The Caccioppoli's Inequality

We prove a Caccioppoli's inequality for the weak solutions of (1.1) under the assumptions stated in Section 4.1.

Proposition 4.6. *Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (1.1) under the assumptions in Section 4.1. Consider $B_{R_0}(x_0) \Subset \Omega$ and for every $k \in \mathbb{R}$, $k \geqq 0$, and every $R \leqq R_0$ consider the super-level sets $A_{k,R} := \{x \in B_R(x_0) : u(x) > k\}$. Then there exists c depending only on the data, but neither on u nor k, such that for every ρ, R such that $0 < \rho < R \leqq R_0 \leqq 1$,*

$$
\begin{aligned}
&\int_{A_{k,\rho}} |Du|^p \, dx \\
&\leqq \frac{c}{(R-\rho)^{\frac{p}{p-q+1}}} \int_{A_{k,R}} (u-k)^{\frac{p}{p-q+1}} \, dx \\
&\quad + \frac{c}{R-\rho} \|b_2 + 1\|_{L^{s_2}(B_{R_0})} \|u-k\|_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} \\
&\quad + c\|b_0 + 1\|_{L^{s_0}(B_{R_0})} \|u-k\|_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{1}{s_0}-\frac{1}{p^*}} \\
&\quad + c\|u-k\|^{\lambda\frac{p}{q-1}}_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\lambda\frac{p}{p^*(q-1)}} \\
&\quad + c\, \|u-k\|^{\theta}_{L^{p^*}(A_{k,R})} \, |A_{k,R}|^{1-\frac{\theta}{p^*}} \\
&\quad + c\, \|u-k\|^{\delta}_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{\delta}{p^*}} + c\,(1+k^{\tau})|A_{k,R}| \\
&\quad + c\|b_1\|_{L^{s_1}(B_{R_0})} |A_{k,R}|^{1-\frac{1}{s_1}},
\end{aligned} \tag{4.18}
$$

where

$$
\tau := \max\left\{\lambda\frac{p}{q-1}, \theta, \delta\right\}. \tag{4.19}
$$

Proof. We split the proof into steps.

Step 1 Consider $B_{R_0}(x_0) \Subset \Omega$, $0 < \rho < R \leqq R_0 \leqq 1$. Let $\eta \in C_0^\infty(B_R)$ be a cut-off function, satisfying the following assumptions:

$$0 \leqq \eta \leqq 1, \quad \eta \equiv 1 \text{ in } B_\rho(x_0), \quad |D\eta| \leqq \tfrac{2}{R-\rho}. \tag{4.20}$$

For every $k \geqq 1$ we define the test function φ_k as follows:

$$\varphi_k(x) := (u(x) - k)_+[\eta(x)]^\mu \quad \text{for almost every } x \in B_{R_0}(x_0),$$

where $(u(x) - k)_+ = \max\{u(x) - k, 0\}$ and

$$\mu := \tfrac{p}{p-q+1}. \tag{4.21}$$

Notice that μ is greater than 1 because $q > 1$ and that $\varphi_k \in W_0^{1,q}(B_{R_0}(x_0))$, $\operatorname{supp} \varphi_k \Subset B_R(x_0)$.

Step 2 Let us consider the super-level sets: $A_{k,R} := \{x \in B_R(x_0) : u(x) > k\}$. Using φ_k as a test function in (4.1) we get

$$\begin{aligned} I_1 := & \int_{A_{k,R}} (a(x, u, Du), Du)\, \eta^\mu \, dx \\ = & -\mu \int_{A_{k,R}} (a(x, u, Du), D\eta)\eta^{\mu-1}(u - k)\, dx \\ & - \int_{A_{k,R}} b(x, u, Du)(u - k)\, \eta^\mu \, dx =: I_2 + I_3. \end{aligned} \tag{4.22}$$

Now, we separately consider and estimate I_i, $i = 1, 2, 3$.

Estimate of I_3. By (4.12) there exists $c(M, p, q)$ positive constant such that

$$|b\,(x, u, \xi)| \leqq c(M, p, q) \left\{ |\xi|^{\frac{p+q-2}{2}} + |u|^{\delta-1} + b_0(x) + 1 \right\};$$

therefore

$$\begin{aligned} I_3 \leqq c(M, p, q) & \int_{A_{k,R}} \eta^\mu \left\{ |Du|^{\frac{p+q-2}{2}} (u - k) \right. \\ & \left. + |u|^{\delta-1}(u - k) + (b_0 + 1)(u - k) \right\} dx. \end{aligned}$$

We estimate the right-hand side using the Young's inequality, with exponents $\frac{2p}{p+q-2}$ and $\frac{2p}{p-q+2}$. There exists $c > 0$, such that

$$\begin{aligned} & c(M, p, q)\, |Du|^{\frac{p+q-2}{2}} (u - k) \\ & \quad \leqq \frac{c_1}{4} |Du|^p + c(u - k)^{\frac{2p}{p-q+2}} \qquad \text{almost everywhere in } A_{k,R}, \end{aligned}$$

where c_1 is the constant in (4.10). Therefore,

$$
\begin{aligned}
I_3 \leqq \tfrac{c_1}{4} \int_{A_{k,R}} |Du|^p \eta^\mu \,\mathrm{d}x \\
&+ c \int_{A_{k,R}} \eta^\mu \left\{ (u-k)^{\frac{2p}{p-q+2}} + |u|^{\delta-1}(u-k) \right\} \mathrm{d}x \\
&+ c \int_{A_{k,R}} \eta^\mu (b_0+1)(u-k)\,\mathrm{d}x. \qquad (4.23)
\end{aligned}
$$

Collecting (4.22), (4.23) and using (4.10), we get

$$
\begin{aligned}
&\tfrac{3c_1}{4} \int_{A_{k,R}} |Du|^p \eta^\mu \,\mathrm{d}x \leqq I_2 \\
&+ c \int_{A_{k,R}} \left\{ |u|^\theta + (u-k)^{\frac{2p}{p-q+2}} + |u|^{\delta-1}(u-k) + (b_0+1)(u-k) + b_1 \right\} \mathrm{d}x. \qquad (4.24)
\end{aligned}
$$

Estimate of I_2. For almost every $x \in A_{k,R} \cap \{\eta \neq 0\}$ we have, by (4.11),

$$
|(a(x,u,Du), D\eta)| \leqq |D\eta| \left(c_3 |Du|^{q-1} + 2^\lambda c_4\, |u|^\lambda + b_2 + 2^\lambda c_4 \right),
$$

where c_3, c_4 are the constants in (4.11). For almost every $x \in A_{k,R} \cap \{\eta \neq 0\}$, by $q < p+1$, the Young's inequality with exponents $\frac{p}{q-1}$ and $\frac{p}{p-q+1}$, and noting that, by (4.21), $\mu - 1 = \mu \frac{q-1}{p}$, we get

$$
\mu c_3 |Du|^{q-1} |D\eta| (u-k) \eta^{\mu-1} \leqq \frac{c_1}{4} |Du|^p \eta^\mu + c\mu^{\frac{p}{p-q+1}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}}.
$$

Therefore,

$$
\begin{aligned}
I_2 \leqq \tfrac{c_1}{4} \int_{A_{k,R}} |Du|^p \eta^\mu \,\mathrm{d}x + c\mu^{\frac{p}{p-q+1}} \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \,\mathrm{d}x \\
&+ c \int_{A_{k,R}} |D\eta| \eta^{\mu-1} \left\{ |u|^\lambda + b_2 + 1 \right\} (u-k)\,\mathrm{d}x.
\end{aligned}
$$

By (4.24) and the inequality above, we get

$$
\begin{aligned}
\tfrac{c_1}{2} \int_{A_{k,R}} |Du|^p \eta^\mu \,\mathrm{d}x \leqq c \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \,\mathrm{d}x \\
&+ c \int_{A_{k,R}} |D\eta|\, |u|^\lambda (u-k)\,\mathrm{d}x + c \int_{A_{k,R}} |u|^{\delta-1}(u-k)\,\mathrm{d}x \\
&+ c \int_{A_{k,R}} \left\{ |u|^\theta + (u-k)^{\frac{2p}{p-q+2}} \right\} \mathrm{d}x + c \int_{A_{k,R}} |D\eta| (b_2+1)(u-k)\,\mathrm{d}x \\
&+ c \int_{A_{k,R}} (b_0+1)(u-k)\,\mathrm{d}x + c \int_{A_{k,R}} b_1\,\mathrm{d}x. \qquad (4.25)
\end{aligned}
$$

We have $|u|^\lambda (u-k)_+ \leqq c\left(|u-k|^{\lambda+1} + k^\lambda (u-k)_+\right)$ for some positive c depending only on p and q. By Young's inequality with exponents $\frac{p}{q-1}$ and $\frac{p}{p-q+1}$, we obtain

$$\int_{A_{k,R}} |D\eta|(u-k)^{\lambda+1}\,\mathrm{d}x \leqq c\int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}\,\mathrm{d}x + c\int_{A_{k,R}} (u-k)^{\lambda\frac{p}{q-1}}\,\mathrm{d}x.$$

Analogously,

$$\int_{A_{k,R}} |D\eta| k^\lambda (u-k)\,\mathrm{d}x \leqq c\int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}\,\mathrm{d}x + c\int_{A_{k,R}} k^{\lambda\frac{p}{q-1}}\,\mathrm{d}x.$$

Thus we get

$$\int_{A_{k,R}} |D\eta||u|^\lambda (u-k)\,\mathrm{d}x \leqq c\int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}\,\mathrm{d}x + c\int_{A_{k,R}} \left\{(u-k)^{\lambda\frac{p}{q-1}} + k^{\lambda\frac{p}{q-1}}\right\}\mathrm{d}x. \tag{4.26}$$

For almost every $x \in A_{k,R}$ $|u|^{\delta-1}(u-k)_+ \leqq c(u-k)^\delta + ck^\delta$ and we get

$$\int_{A_{k,R}} |u|^{\delta-1}(u-k)\,\mathrm{d}x \leqq c\int_{A_{k,R}} \left\{(u-k)^\delta + k^\delta\right\}\mathrm{d}x. \tag{4.27}$$

Analogously, for almost every $x \in A_{k,R}$ we have $|u|^\theta \leqq c(u-k)^\theta + ck^\theta$; therefore

$$\int_{A_{k,R}} |u|^\theta\,\mathrm{d}x \leqq c\int_{A_{k,R}} (u-k)^\theta\,\mathrm{d}x + ck^\theta |A_{k,R}|. \tag{4.28}$$

By Hölder's inequality, with exponents s_2 and $\frac{s_2}{s_2-1}$, we get

$$\int_{A_{k,R}} |D\eta|(b_2+1)(u-k)\,\mathrm{d}x \leqq c\left(\int_{A_{k,R}} |D\eta|^{\frac{s_2}{s_2-1}}(u-k)^{\frac{s_2}{s_2-1}}\,\mathrm{d}x\right)^{1-\frac{1}{s_2}} \|b_2+1\|_{L^{s_2}(B_{R_0})}.$$

Since $u \in L^{p^*}(B_{R_0})$ and $\frac{s_2}{s_2-1} < p^*$, by using the Hölder inequality with exponent $p^*\frac{s_2-1}{s_2}$ and (4.20),

$$\left(\int_{A_{k,R}} |D\eta|^{\frac{s_2}{s_2-1}}(u-k)^{\frac{s_2}{s_2-1}}\,\mathrm{d}x\right)^{1-\frac{1}{s_2}} \leqq \frac{c}{R-\rho}\|u-k\|_{L^{p^*}(A_{k,R})}|A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}}.$$

We conclude that

$$\int_{A_{k,R}} |D\eta|(b_2+1)(u-k)\,\mathrm{d}x \leqq \frac{c}{R-\rho}\|b_2+1\|_{L^{s_2}(B_{R_0})}\|u-k\|_{L^{p^*}(A_{k,R})}|A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}}. \tag{4.29}$$

Analogously, by the integrability assumption on b_0, $b_0 \in L^{s_0}(B_{R_0})$, and by Hölder's inequality,

$$\int_{A_{k,R}} (b_0+1)(u-k)\,\mathrm{d}x \leqq c\,\|b_0+1\|_{L^{s_0}(B_{R_0})}\,\|u-k\|_{L^{p^*}(A_{k,R})}\,|A_{k,R}|^{1-\frac{1}{s_0}-\frac{1}{p^*}}. \tag{4.30}$$

Of course,

$$\int_{A_{k,R}} b_1\,\mathrm{d}x \leqq \|b_1\|_{L^{s_1}(B_{R_0})}|A_{k,R}|^{1-\frac{1}{s_1}}. \tag{4.31}$$

Let us denote τ as in (4.19); that is $\tau := \max\left\{\lambda\frac{p}{q-1},\theta,\delta\right\}$; then $k^\theta + k^{\lambda\frac{p}{q-1}} + k^\delta \leqq 3(1+k^\tau)$. Collecting this last inequality, (4.25)–(4.31) and using (4.20) we obtain

$$\begin{aligned}\int_{A_{k,\rho}} |Du|^p\,\mathrm{d}x &\leqq \frac{c}{(R-\rho)^{\frac{p}{p-q+1}}}\int_{A_{k,R}} (u-k)^{\frac{p}{p-q+1}}\,\mathrm{d}x \\ &+ c\,\frac{1}{R-\rho}\|b_2+1\|_{L^{s_2}(B_{R_0})}\|u-k\|_{L^{p^*}(A_{k,R})}|A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} \\ &+ c\,\|b_0+1\|_{L^{s_0}(B_{R_0})}\|u-k\|_{L^{p^*}(A_{k,R})}|A_{k,R}|^{1-\frac{1}{s_0}-\frac{1}{p^*}} \\ &+ c\int_{A_{k,R}}\left((u-k)^{\lambda\frac{p}{q-1}} + (u-k)^\theta + (u-k)^\delta\right)\mathrm{d}x \\ &+ c\,(1+k^\tau)|A_{k,R}| + c\|b_1\|_{L^{s_1}(B_{R_0})}|A_{k,R}|^{1-\frac{1}{s_1}}. \end{aligned} \tag{4.32}$$

Step 3 Consider

$$J := c\int_{A_{k,R}}\left((u-k)^{\lambda\frac{p}{q-1}} + (u-k)^\theta + (u-k)^\delta\right)\mathrm{d}x,$$

that is the last integral at the right hand side of (4.32). By the definitions of p^* and of τ, given in (4.19), by the assumptions (4.3), (4.2), (4.4), (4.5) we conclude that $\tau < p^*$. Therefore, Hölder's inequality with exponent $p^*\frac{q-1}{p\lambda}$ implies that

$$\int_{A_{k,R}} (u-k)^{\lambda\frac{p}{q-1}}\,\mathrm{d}x \leqq \|u-k\|_{L^{p^*}(A_{k,R})}^{\lambda\frac{p}{q-1}}|A_{k,R}|^{1-\lambda\frac{p}{p^*(q-1)}}.$$

Hölder's inequality with exponent $\frac{p^*}{\theta}$ gives

$$\int_{A_{k,R}} (u-k)^\theta \, \mathrm{d}x \leqq \|u-k\|^\theta_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{\theta}{p^*}}$$

and, using Hölder's inequality with exponent $\frac{p^*}{\delta}$, we get

$$\int_{A_{k,R}} (u-k)^\delta \, \mathrm{d}x \leqq \|u-k\|^\delta_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{\delta}{p^*}}.$$

Therefore

$$J \leqq c\,\|u-k\|^{\lambda\frac{p}{q-1}}_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\lambda\frac{p}{p^*(q-1)}} + c\,\|u-k\|^\theta_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{\theta}{p^*}} + c\,\|u-k\|^\delta_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{\delta}{p^*}}.$$

By this estimate, together with (4.32), we conclude that Caccioppoli's inequality (4.18) holds. □

4.4. The Recursive Formula

Now we proceed towards the proof of our Theorem 4.2 by setting up the celebrated De Giorgi's iterative method. In what follows, we tacitly understand that all the assumptions and the notation introduced in the previous subsections do apply.

Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (1.1). Fix a point $x_0 \in \Omega$ and a real $R_0 \in (0, 1]$ in such a way that

$$B_{R_0}(x_0) \Subset \Omega. \tag{4.33}$$

For every fixed $R \in (0, R_0]$, we then define the following (decreasing) sequences:

$$\rho_h := \frac{R}{2}\left(1 + \frac{1}{2^h}\right) \quad \text{and} \quad \overline{\rho}_h := \frac{\rho_{h+1}+\rho_h}{2}, \qquad h \in \mathbb{N} \cup \{0\}. \tag{4.34}$$

Moreover, given any real number $d \geqq 1$, we consider the (increasing) sequence

$$k_h := d\left(1 - \frac{1}{2^{h+1}}\right), \quad h \in \mathbb{N} \cup \{0\}. \tag{4.35}$$

Finally, we define a sequence $(J_h)_{h\geqq 0}$ of non-negative numbers as follows:

$$J_h := \int_{A_{k_h,\rho_h}} (u-k_h)^{p^*} \, \mathrm{d}x. \tag{4.36}$$

Then the following result holds:

Proposition 4.7. *For every real number $d \geqq 1$,*

$$J_{h+1} \leqq c_* \left(\frac{1}{R}\right)^{\frac{p^*}{p-q+1}} \left(1 + \|u\|^{p^*}_{L^{p^*}(B_R)}\right)^{\frac{p^*}{p}\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}} \times \frac{1}{d^{\frac{p^*}{p}\sigma}} \left(2^{\frac{p^*}{p}} p^*\right)^h J_h^{\frac{p^*}{p}\left(1-\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}\right)}, \tag{4.37}$$

where σ is defined in (4.14)*; that is,*

$$\sigma := p^* - \max\left\{\frac{p}{p-q+1};\ \frac{\lambda p}{q-1};\ \theta;\ \delta;\ \frac{p^*}{s_2}+1;\ \frac{p^*}{s_0}+1;\ \frac{p^*}{s_1}\right\},$$

and c_* positive constant depending on the data, the L^{s_0}, L^{s_1} and the L^{s_2} norms in B_{R_0} of b_0, b_1, b_2, respectively, but it is independent of u and d.

We notice that, by assumptions (4.3), (4.6), $\frac{p^*}{p}\left(1 - \max\left\{\frac{1}{s_0}, \frac{1}{s_1}, \frac{1}{s_2}\right\}\right) > 1$.

Proof of Proposition 4.7. We explicitly observe that, since $(\rho_h)_h$ is decreasing and $(k_h)_h$ is increasing, the sequence $(J_h)_h$ is decreasing; in fact,

$$\begin{aligned} J_{h+1} &= \int_{A_{k_{h+1},\rho_{h+1}}} (u-k_{h+1})^{p^*}\,dx \leqq \int_{A_{k_{h+1},\rho_h}} (u-k_{h+1})^{p^*}\,dx \\ &\leqq \int_{A_{k_{h+1},\rho_h}} (u-k_h)^{p^*}\,dx \leqq \int_{A_{k_h,\rho_h}} (u-k_h)^{p^*}\,dx = J_h. \end{aligned} \tag{4.38}$$

Finally, by taking into account the definitions of J_h, k_h and ρ_h, we have

$$\begin{aligned} J_h &= \int_{A_{k_h,\rho_h}} (u-k_h)^{p^*}\,dx \geqq \int_{A_{k_{h+1},\rho_h}} (u-k_h)^{p^*}\,dx \\ &\geqq (k_{h+1}-k_h)^{p^*} \left|A_{k_{h+1},\rho_h}\right| = \left(\frac{d}{2^{h+2}}\right)^{p^*} \left|A_{k_{h+1},\rho_h}\right|. \end{aligned} \tag{4.39}$$

Let $(\eta_h)_{h\geqq 0}$ be a sequence in $C_c^\infty(\mathbb{R})$ such that *(i)* $0 \leqq \eta_h \leqq 1$ on $\mathbb{R}^n$; *(ii)* $\operatorname{supp}\eta_h \subseteq B(x_0,\overline{\rho}_h)$ and $\eta_h \equiv 1$ on $B(x_0,\rho_{h+1})$; *(iii)* $|D\eta_h| \leqq \frac{2^{h+4}}{R}$. In particular, $\eta_h \equiv 1$ on $A_{k_{h+1},\rho_{h+1}}$, so we have

$$\begin{aligned} J_{h+1}^{\frac{p}{p^*}} &= \left(\int_{A_{k_{h+1},\rho_{h+1}}} (u-k_{h+1})^{p^*}\,dx\right)^{\frac{p}{p^*}} \\ &\leqq \left(\int_{B(x_0,\overline{\rho}_h)} (\eta_h(u-k_{h+1})_+)^{p^*}\,dx\right)^{\frac{p}{p^*}} \\ &\leqq C_S^p \int_{B(x_0,\overline{\rho}_h)} \left|D(\eta_h(u-k_{h+1})_+)\right|^p\,dx, \end{aligned} \tag{4.40}$$

where c_S is the Sobolev constant. We estimate the last integral

$$\begin{aligned}\int_{B(x_0,\overline{\rho}_h)} \left|D(\eta_h(u-k_{h+1})_+)\right|^p \mathrm{d}x &\leqq \int_{B(x_0,\overline{\rho}_h)} \left\{|D\eta_h|\,(u-k_{h+1})_+ \; + \eta_h\,|Du|\,\chi_{A_{k_{h+1},\overline{\rho}_h}}\right\}^p \mathrm{d}x \\ &\leqq c\left\{\left(\frac{2^{h+4}}{R}\right)^p \int_{A_{k_{h+1},\overline{\rho}_h}} (u-k_{h+1})^p\,\mathrm{d}x + \int_{A_{k_{h+1},\overline{\rho}_h}} |Du|^p\,\mathrm{d}x\right\}. \end{aligned} \tag{4.41}$$

Collecting (4.40) and (4.41) we get

$$J_{h+1}^{\frac{p}{p^*}} \leqq c\left\{\left(\frac{2^{h+4}}{R}\right)^p \int_{A_{k_{h+1},\overline{\rho}_h}} (u-k_{h+1})^p\,\mathrm{d}x + \int_{A_{k_{h+1},\overline{\rho}_h}} |Du|^p\,\mathrm{d}x\right\}. \tag{4.42}$$

To estimate the last integral in (4.42), we use Caccioppoli's estimate (4.18) with $k = k_{h+1}$, $\rho = \overline{\rho}_h$, $R = \rho_h$, thus obtaining

$$\begin{aligned}\int_{A_{k_{h+1},\overline{\rho}_h}} |Du|^p\,\mathrm{d}x \leqq{}& c\left(\frac{2^{h+3}}{R}\right)^{\frac{p}{p-q+1}} \int_{A_{k_{h+1},\rho_h}} (u-k_{h+1})^{\frac{p}{p-q+1}}\,\mathrm{d}x \\ &+ c\,\frac{2^{h+3}}{R}\,\|b_2+1\|_{L^{s_2}(B_{R_0})} J_h^{\frac{1}{p^*}} |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} \\ &+ c\,\|b_0+1\|_{L^{s_0}(B_{R_0})} J_h^{\frac{1}{p^*}} |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s_0}-\frac{1}{p^*}} \\ &+ c\,J_h^{\frac{\lambda p}{(q-1)p^*}} |A_{k_{h+1},\rho_h}|^{1-\frac{\lambda p}{(q-1)p^*}} + c\,J_h^{\frac{\theta}{p^*}} |A_{k_{h+1},\rho_h}|^{1-\frac{\theta}{p^*}} \\ &+ c\,J_h^{\frac{\delta}{p^*}} |A_{k_{h+1},\rho_h}|^{1-\frac{\delta}{p^*}} + c\,(1+k_{h+1}^{\tau})|A_{k_{h+1},\rho_h}| \\ &+ c\,\|b_1\|_{L^{s_1}(B_{R_0})} |A_{k_{h+1},\rho_h}|^{1-\frac{1}{s_1}}. \end{aligned} \tag{4.43}$$

By (4.39) we get

$$\left|A_{k_{h+1},\rho_{h+1}}\right| \leqq \left|A_{k_{h+1},\rho_h}\right| \leqq \left(\frac{2^{h+2}}{d}\right)^{p^*} J_h = 4^{p^*}\left(\frac{2^h}{d}\right)^{p^*} J_h. \tag{4.44}$$

Collecting (4.43), (4.44), and using that $k_{h+1} \leqq d$, we get

$$\begin{aligned}\int_{A_{k_{h+1},\overline{\rho}_h}} |Du|^p\,\mathrm{d}x \leqq{}& c\left(\frac{2^{h+3}}{R}\right)^{\frac{p}{p-q+1}} \int_{A_{k_{h+1},\rho_h}} (u-k_{h+1})^{\frac{p}{p-q+1}}\,\mathrm{d}x \\ &+ c\,\frac{2^h}{R}\,\|b_2+1\|_{L^{s_2}(B_{R_0})} \left(4\,\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_2}-1} J_h^{1-\frac{1}{s_2}} \\ &+ c\,\|b_0+1\|_{L^{s_0}(B_{R_0})} \left(4\,\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_0}-1} J_h^{1-\frac{1}{s_0}} \\ &+ c\left\{\left(4\,\frac{2^h}{d}\right)^{p^*-\frac{\lambda p}{q-1}} + \left(4\,\frac{2^h}{d}\right)^{p^*-\theta} + \left(4\,\frac{2^h}{d}\right)^{p^*-\delta}\right\} J_h \\ &+ c\,(1+d^{\tau})\left(4\,\frac{2^h}{d}\right)^{p^*} J_h + c\,\|b_1\|_{L^{s_1}(B_{R_0})} \left(4\,\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_1}} J_h^{1-\frac{1}{s_1}}. \end{aligned} \tag{4.45}$$

We now estimate the integral at the right hand side:

$$\begin{aligned}&\int_{A_{k_{h+1},\rho_h}} (u-k_{h+1})^{\frac{p}{p-q+1}}\,\mathrm{d}x\\ &\quad\leqq \left(\int_{A_{k_{h+1},\rho_h}} (u-k_{h+1})^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*(p-q+1)}} |A_{k_{h+1},\rho_h}|^{1-\frac{p}{p^*(p-q+1)}}.\end{aligned}$$

This, by (4.44), implies

$$\int_{A_{k_{h+1},\rho_h}} (u-k_{h+1})^{\frac{p}{p-q+1}}\,\mathrm{d}x \leqq c\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{p}{p^*(p-q+1)}\right)} J_h. \tag{4.46}$$

Collecting (4.45), (4.46) and using $k_{h+1}\leqq d$, we get

$$\begin{aligned}&\int_{A_{k_{h+1},\bar{\rho}_h}} |Du|^p\,\mathrm{d}x\\ &\leqq c\left(\frac{1}{R}\right)^{\frac{p}{p-q+1}}\left(\frac{1}{d}\right)^{p^*\left(1-\frac{p}{p^*(p-q+1)}\right)} 2^{p^*h} J_h\\ &\quad+c\,\|b_2+1\|_{L^{s_2}(B_{R_0})}\frac{1}{R}\left(\frac{1}{d}\right)^{p^*-\frac{p^*}{s_2}-1} 2^{p^*\left(1-\frac{1}{s_2}\right)h} J_h^{1-\frac{1}{s_2}}\\ &\quad+c\,\|b_0+1\|_{L^{s_0}(B_{R_0})}\left(\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_0}-1} J_h^{1-\frac{1}{s_0}}\\ &\quad+c\left\{\left(\frac{2^h}{d}\right)^{p^*-\frac{\lambda p}{q-1}}+\left(\frac{2^h}{d}\right)^{p^*-\theta}+\left(\frac{2^h}{d}\right)^{p^*-\delta}+(1+d^{\tau})\left(\frac{2^h}{d}\right)^{p^*}\right\} J_h\\ &\quad+c\|b_1\|_{L^{s_1}(B_{R_0})}\left(\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_1}} J_h^{1-\frac{1}{s_1}},\end{aligned} \tag{4.47}$$

with a constant c depending on $n,\ p,\ q,\ \lambda,\ \theta,\ \delta,\ s_0,\ s_1,\ s_2$ and the embedding Sobolev constant c_S, but depending neither on d, h nor u. We now put together (4.42) and (4.47); taking into account that Hölder's inequality and (4.44) imply

$$\int_{A_{k_{h+1},\bar{\rho}_h}} (u-k_{h+1})^p\,\mathrm{d}x \leqq c\left(\frac{2^h}{d}\right)^{p^*-p} J_h,$$

we obtain

$$
\begin{aligned}
J_{h+1}^{\frac{p}{p^*}} \leqq c & \left(\left(\frac{1}{R}\right)^p \left(\frac{1}{d}\right)^{p^*\left(1-\frac{p}{p^*}\right)} + \left(\frac{1}{R}\right)^{\frac{p}{p-q+1}} \left(\frac{1}{d}\right)^{p^*\left(1-\frac{p}{p^*(p-q+1)}\right)} \right) 2^{p^* h} J_h \\
& + c\, \|b_2+1\|_{L^{s_2}(B_{R_0})} \frac{1}{R} \left(\frac{1}{d}\right)^{p^*-\frac{p^*}{s_2}-1} 2^{p^*\left(1-\frac{1}{s_2}\right)h} J_h^{1-\frac{1}{s_2}} \\
& + c\, \|b_0+1\|_{L^{s_0}(B_{R_0})} \left(\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_0}-1} J_h^{1-\frac{1}{s_0}} \\
& + c \left\{ \left(\frac{2^h}{d}\right)^{p^*-\frac{\lambda p}{q-1}} + \left(\frac{2^h}{d}\right)^{p^*-\theta} + \left(\frac{2^h}{d}\right)^{p^*-\delta} + (1+d^\tau)\left(\frac{2^h}{d}\right)^{p^*} \right\} J_h \\
& + \|b_1\|_{L^{s_1}(B_{R_0})} \left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{1}{s_1}\right)} J_h^{1-\frac{1}{s_1}}.
\end{aligned}
$$

Notice that $J_h \leqq \|u\|^{p^*}_{L^{p^*}(B_R)}$ for every $h \in \mathbb{N}$, so that

$$
\begin{aligned}
& \max\{J_h;\ J_h^{1-\frac{1}{s_0}};\ J_h^{1-\frac{1}{s_1}};\ J_h^{1-\frac{1}{s_2}}\} \\
& \quad \leqq (1+\|u\|^{p^*}_{L^{p^*}(B_R)})^{\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}} J_h^{1-\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_{h+1}^{\frac{p}{p^*}} \leqq & c\,(1+\|u\|^{p^*}_{L^{p^*}(B_R)})^{\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}} \Big(1+\|b_2+1\|_{L^{s_2}(B_{R_0})} \\
& + \|b_0+1\|_{L^{s_0}(B_{R_0})} + \|b_1\|_{L^{s_1}(B_{R_0})}\Big) \times \\
& \times \left\{ \left(\left(\frac{1}{R}\right)^p \left(\frac{1}{d}\right)^{p^*-p} + \left(\frac{1}{R}\right)^{\frac{p}{p-q+1}} \left(\frac{1}{d}\right)^{p^*-\frac{p}{p-q+1}} \right) 2^{p^* h} \right. \\
& + \frac{1}{R}\left(\frac{1}{d}\right)^{p^*-\frac{p^*}{s_2}-1} 2^{p^*\left(1-\frac{1}{s_2}\right)h} + \left(\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_0}-1} \\
& + \left(\frac{2^h}{d}\right)^{p^*-\frac{\lambda p}{q-1}} + \left(\frac{2^h}{d}\right)^{p^*-\theta} + \left(\frac{2^h}{d}\right)^{p^*-\delta} + (1+d^\tau)\left(\frac{2^h}{d}\right)^{p^*} \\
& \left. + \left(\frac{2^h}{d}\right)^{p^*-\frac{p^*}{s_1}} \right\} J_h^{1-\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}}. \qquad (4.48)
\end{aligned}
$$

We now majorize the right hand side. Since $R_0 \leqq 1$ and $q \geqq 1$,

$$
\max\left\{1;\ \frac{1}{R};\ \left(\frac{1}{R}\right)^{\frac{p}{p-q+1}}\right\} \leqq \left(\frac{1}{R}\right)^{\frac{p}{p-q+1}} \qquad \text{for every } R \in (0, R_0].
$$

Note that $p \leq \frac{p}{p-q+1}$. Taking into account that $d \geqq 1$ and denoting σ as in (4.14), that is

$$\sigma := p^* - \max\left\{\frac{p}{p-q+1};\ \frac{\lambda p}{q-1};\ \theta;\ \delta;\ \frac{p^*}{s_2}+1;\ \frac{p^*}{s_0}+1;\ \frac{p^*}{s_1}\right\},$$

by inequality (4.48) we obtain

$$J_{h+1}^{\frac{p}{p^*}} \leqq c_0\left(\tfrac{1}{R}\right)^{\frac{p}{p-q+1}}\left(1+\|b_2+1\|_{L^{s_2}(B_{R_0})}+\|b_0+1\|_{L^{s_0}(B_{R_0})}+\|b_1\|_{L^{s_1}(B_{R_0})}\right)$$
$$\times \tfrac{1}{d^\sigma}\left(2^{p^*}\right)^h\left(1+\|u\|^{p^*}_{L^{p^*}(B_R)}\right)^{\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}} J_h^{1-\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}}.$$

Raising at the power $\frac{p^*}{p}$, we get (4.37), with

$$c_* := \max\left\{1, c_0^{\frac{p^*}{p}}\right\}\Big(1+\|b_1+1\|_{L^{s_1}(B_{R_0})}+\|b_0+1\|_{L^{s_0}(B_{R_0})}$$
$$+\|b_1\|_{L^{s_1}(B_{R_0})}\Big)^{\frac{p^*}{p}}. \tag{4.49}$$

□

4.5. Proof of the Local Boundedness Result

With Proposition 4.7 at hand, we are ready to provide the proof of one of our main result, namely Theorem 4.2. Before doing this, we remind the following very classical lemma of Real Analysis (see, for example, [40, Lemma 7.1]):

Lemma 4.8. *Let $(z_h)_{h\geqq 0}$ be a sequence of positive real numbers satisfying the recursive relation*

$$z_{h+1} \leqq L\,\zeta^h z_h^{1+\alpha} \qquad (h \in \mathbb{N}\cup\{0\}), \tag{4.50}$$

where $L, \alpha > 0$ and $\zeta > 1$. If $z_0 \leqq L^{-\frac{1}{\alpha}}\zeta^{-\frac{1}{\alpha^2}}$, then $z_h \leqq \zeta^{-\frac{h}{\alpha}} z_0$ for every $h \geqq 0$. In particular, $z_h \to 0$ as $h \to \infty$.

Proof of Theorem 4.2. Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (1.1) under the assumptions in Section 4.1. Consider $B_{R_0}(x_0) \Subset \Omega$ with $0 < R_0 \leqq 1$. Moreover, let $d \geqq 1$ (to be chosen later on) and let $(J_h)_{h\geqq 0}$ be the sequence defined in (4.36). Owing to Proposition 4.7, for every $R \leqq R_0$ we have the estimate

$$J_{h+1} \leqq L\left(2^{\frac{p^*}{p}p^*}\right)^h J_h^{1+\alpha} \qquad (h \in \mathbb{N}\cup\{0\}), \tag{4.51}$$

where α is

$$\alpha := \tfrac{p^*}{p}\left(1-\max\left\{\tfrac{1}{s_0},\tfrac{1}{s_1},\tfrac{1}{s_2}\right\}\right)-1, \tag{4.52}$$

and the constant L is given by

$$L := c_* \left(1 + \|u\|^{p^*}_{L^{p^*}(B_R)}\right)^{\frac{p^*}{p}\max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}} \frac{1}{d^{\frac{p^*}{p}\sigma}} \left(\frac{1}{R}\right)^{\frac{p^*}{p-q+1}},$$

where σ is defined in (4.14) and c_*, independent of d, is defined as in (4.49). We notice that $\alpha > 0$, because s_0, s_1, s_2 satisfy (4.3), (4.6), (4.7).

We now claim that it is possible to choose $d \geqq 1$ in such a way that

$$J_0 := \int_{A_{\frac{d}{2},R}} \left(u - \tfrac{d}{2}\right)^{p^*} \mathrm{d}x \leqq L^{-1/\alpha} \left(2^{\frac{p^*}{p}p^*}\right)^{-1/\alpha^2}. \tag{4.53}$$

In fact, by definition of J_0 and since $u \in L^{p^*}_{\mathrm{loc}}(\Omega)$, we have

$$J_0 \leqq \int_{B_R} |u|^{p^*}\,\mathrm{d}x < \infty;$$

thus, reminding that $R_0 \leqq 1$, condition (4.53) is clearly fulfilled if we choose

$$d := \frac{(c_*)^{\frac{p}{p^*\sigma}}}{R^{\frac{p}{\sigma(p-q+1)}}} 2^{\frac{p^*}{\alpha\sigma}} \left(1 + \|u\|^{p^*}_{L^{p^*}(B(x_0,R))}\right)^{\frac{p\alpha}{p^*\sigma} + \max\left\{\frac{1}{s_0},\frac{1}{s_1},\frac{1}{s_2}\right\}\frac{1}{\sigma}},$$

that is, taking into account (4.52),

$$d := \frac{(c_*)^{\frac{p}{p^*\sigma}}}{R^{\frac{p}{\sigma(p-q+1)}}} 2^{\frac{p^*}{\alpha\sigma}} \left(1 + \|u\|^{p^*}_{L^{p^*}(B(x_0,R))}\right)^{\frac{p^*-p}{p^*\sigma}}. \tag{4.54}$$

Notice that $d \geqq 1$, because $c_* \geqq 1$ and $R_0 \leqq 1$. With (4.53) at hand and d as in (4.54), we are entitled to apply Lemma 4.8. As a consequence, we obtain

$$\lim_{h\to\infty} J_h = \lim_{h\to\infty} \int_{A_{k_h,\rho_h}} (u - k_h)^{p^*}\,\mathrm{d}x = \int_{A_{d,R/2}} (u-d)^{p^*}\,\mathrm{d}x = 0. \tag{4.55}$$

Since, by definition, $u - d > 0$ on $A_{d,R/2}$, from (4.55) we then conclude that

$$\left|A_{d,R/2}\right| = 0, \qquad \text{whence} \quad u \leqq d \text{ for almost every } x \in B_{R/2}(x_0).$$

To prove that u is locally bounded from below, we can reason analogously, using the sub-level sets of u. Therefore we obtain that there exists c'_* such that $-u \leqq d'$ almost everywhere in $B_{\frac{R}{2}}$, with

$$d' = \frac{(c'_*)^{\frac{p}{p^*\sigma}}}{R^{\frac{p}{\sigma(p-q+1)}}} 2^{\frac{p^*}{\alpha\sigma}} \left(1 + \|u\|^{p^*}_{L^{p^*}(B(x_0,R))}\right)^{\frac{p^*-p}{p^*\sigma}}.$$

We have thus proven that $u \in L^\infty(B_{R/2}(x_0))$. Due to the arbitrariness of x_0 and R_0, we get $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and (4.13) follows. □

5. Local Lipschitz Continuity and Higher Differentiability

We start with the assumptions of Section 2.1 in order to prove Theorem 2.1. The other local Lipschitz continuity Theorem 2.2 will need some specific considerations which we will propose below. Thus again, as in Definition 4.1, we consider a weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ of the differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i (x, u(x), Du(x)) = b(x, u(x), Du(x)), \qquad x \in \Omega, \tag{5.1}$$

under the ellipticity condition

$$\sum_{i,j=1}^{n} \frac{\partial a^i}{\partial \xi_j} \lambda_i \lambda_j \geqq m \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2, \tag{5.2}$$

and the order-one growth conditions (2.2), (2.3). The weak solution u is locally bounded in Ω. In fact, since $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, this property is satisfied when $q > n$, as application of the Sobolev-Morrey embedding theorem. In the remaining case $p \leqq q \leqq n$ we apply Theorem 4.2; in fact, by the Linking Lemma 3.1 the assumptions of Section 4 hold, more precisely, (4.10) and (4.11) hold and, by Theorem 4.2, the weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is locally bounded in Ω. We observe that we use the result of Theorem 4.2 only when $p \leqq n$; in this case the assumption (4.2), that is $\frac{q}{p} < 1 + \min\left\{\frac{1}{n}, \frac{1}{p}\right\}$, simply reduces to $\frac{q}{p} < 1 + \frac{1}{n}$. In this section we use the boundedness results of Theorem 4.2 with only these constraints on p, q. In particular $q < p + \frac{p}{n} \leqq p + 1$.

Therefore the weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is locally bounded in Ω. We read again assumptions (2.2), (2.3) by taking into account the local boundedness of u: for every open set Ω', whose closure is contained in Ω, there exists a constant $L > 0$ such that $\|u\|_{L^\infty(\Omega')} \leqq L$; thus, by (2.2), (2.3), there exist a positive constant $M(L)$ (depending on Ω' and L; precisely $M(L) = 2M \max\{1; L^\alpha; (1+L)^{\beta-1}\}$) such that, for every $x \in \Omega'$, $\lambda, \xi \in \mathbb{R}^n$ and, for $|u| \leqq L$

$$\left|\frac{\partial a^i}{\partial \xi_j}\right| \leqq M(L) \left(1 + |\xi|^2\right)^{\frac{q-2}{2}}, \qquad \left|\frac{\partial a^i}{\partial u}\right| \leqq M(L) \left(1 + |\xi|^2\right)^{\frac{p+q-4}{4}}. \tag{5.3}$$

Conditions (5.3) correspond to the assumptions (3.7) in Marcellini [55]. Together with the ellipticity condition (5.1) and the growth conditions (2.5), (2.6) they give the full landscape in order to state that the vector field $a(x, u, \xi) = \left(a^i(x, u, \xi)\right)_{i=1,\ldots,n}$ satisfies all the assumptions taken under consideration in [55] to study the local Lipschitz continuity of the weak solution u. To this aim, it remains only to analyze the assumption on the right hand side $b(x, u, \xi)$ in the differential Equation (5.1).

In this paper the growth assumption on $b(x, u, \xi)$ in (2.8) is (here with $M(L) = 2M \max\{1; L^{\delta-1}\}$)

$$|b(x, u, \xi)| \leqq M(L) \left(1 + |\xi|^2\right)^{\frac{p+q-2}{4}} + b_0(x), \tag{5.4}$$

for almost every $x \in \Omega'$, and for all $\lambda, \xi \in \mathbb{R}^n$ and $|u| \leqq L$. The difference with the growth assumption in [55] is the addendum $b_0(x)$ in the right hand side of (5.4), with $b_0 \in L^{s_0}_{\mathrm{loc}}(\Omega)$ for some $s_0 > n$, which is posed equal to zero in [55]. Therefore in the following we analyze in which way it is possible to modify the argument of [55] in order to handle this term.

5.1. Proof of Theorems 2.1 and 2.2

As already said, we start with the proof of Theorem 2.1. Under the notation of Section 5 in [55] we consider the Equation (5.1) and a test function $\varphi \in W_0^{1,q}(\Omega')$ of the form

$$\varphi = \Delta_{-h}\left(\eta^2 g(\Delta_h u)\right),$$

where $\Delta_{-h}\psi$ denote as usual the difference quotient of a function $\psi(x)$ in the s-direction and g is a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$, with $0 < g'(t) \leqq L$ for all $t \in \mathbb{R}$, and $\eta \in C_0^1(\Omega')$, $\eta \geqq 0$, $\Omega' \subset\subset \Omega$. We can see that $\varphi \in W_0^{1,q}(\Omega')$. More precisely, following Section 5.5.2 in [55], for generic $k \in \mathbb{N}$ and $\beta \geqq 0$ we choose $g(t) = t\left(1+t^2\right)^{\beta/2}$ when $t \in [-k, k]$ and $g(t)$ affine out of the interval $[-k, k]$ in such a way that globally $g \in C^1(\mathbb{R})$. *For reader's convenience we adopt here the symbols in* [55]; *we note that* β *here is different and independent of the same symbol previously used in this manuscript in Section* 4. For $t \in [-k, k]$ the derivative of g holds: $g'(t) = \left(1+t^2\right)^{\beta/2-1}\left(1+(\beta+1)t^2\right)$, then

$$0 < g'(t) \leqq (\beta+1)\left(1+t^2\right)^{\beta/2} \tag{5.5}$$

for all $t \in [-k, k]$, and also for all $t \in \mathbb{R}$, since $g'(t)$ is constant out of the interval $[-k, k]$; precisely $g'(t) = g'(k) = g'(-k)$ when $t \notin [-k, k]$, and thus for such t-values $g'(t) = g'(k) \leqq (\beta+1)\left(1+k^2\right)^{\beta/2} \leqq (\beta+1)\left(1+t^2\right)^{\beta/2}$.

We insert $\varphi = \Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)$ in the weak form of the differential Equation (5.1) and we obtain

$$\int_\Omega \sum_{i=1}^n a^i(x, u, Du(x))\left(\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right)_{x_i} \mathrm{d}x + \int_\Omega b(x, u, Du)\left(\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right) \mathrm{d}x = 0. \tag{5.6}$$

The integral with the vector field $a(x, u, \xi) = \left(a^i(x, u, \xi)\right)_{i=1,\ldots,n}$ can be estimated exactly as done in Section 5 of [55]. In fact, by proceeding as in subsection

5.2 of [55], we get

$$\begin{aligned}
&\frac{1}{c}\int_0^1 \mathrm{d}t \int_\Omega \eta^2 g'(\Delta_h u) \cdot \\
&\quad \cdot \left(1 + |(1-t)\,Du(x) + t\,Du(x+the_s)|^2\right)^{\frac{p-2}{2}} |\Delta_h Du|^2 \,\mathrm{d}x \\
&\leqq \int_0^1 \mathrm{d}t \int_\Omega \eta^2 g'(\Delta_h u)\left(1 + |Du(x)|^2 + |Du(x+the_s)|^2\right)^{\frac{q}{2}} \mathrm{d}x \\
&\quad + \int_0^1 \mathrm{d}t \int_\Omega 2\eta\,|D\eta| \cdot |g(\Delta_h u)| \left(1 + |Du(x)|^2 + |Du(x+the_s)|^2\right)^{\frac{q-1}{2}} \mathrm{d}x \\
&\quad + \int_0^1 \mathrm{d}t \int_\Omega |D\eta|^2 \cdot \frac{g^2(\Delta_h u)}{g'(\Delta_h u)} \left(1 + |Du(x)|^2 + |Du(x+the_s)|^2\right)^{\frac{q-2}{2}} \mathrm{d}x \\
&\quad + \int_\Omega b(x,u,Du)\left(\Delta_h(\eta^2 g(\Delta_h u)\right) \mathrm{d}x. \qquad (5.7)
\end{aligned}$$

5.2. Estimate of the Right Hand Side b

For the integral related to the term with $b(x,u,\xi)$ we use the growth assumption (5.4): $|b(x,u,\xi)| \leqq M(L)\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}} + b_0(x)$ and we get

$$\begin{aligned}
&\left|\int_\Omega b(x,u,Du)\left(\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right)\mathrm{d}x\right| \\
&\quad \leqq M(L)\left|\int_\Omega \left(1+|Du|^2\right)^{\frac{p+q-2}{4}} \cdot \left(\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right)\mathrm{d}x\right| \\
&\qquad + \left|\int_\Omega |b_0(x)| \cdot \left|\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right| \mathrm{d}x\right|.
\end{aligned}$$

Reasoning as in [55] (see in particular Section 5.3.7 of [55]), we represent $\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)$ in this way

$$\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right) = \int_0^1 \left(2\eta\eta_{x_s}\, g(\Delta_h u) + \eta^2 g'(\Delta_h u)\,\Delta_h u_{x_s}\right)\mathrm{d}t,$$

where the arguments in the last integrands are $x - the_s$. Therefore

$$\begin{aligned}
&\left|\int_\Omega b(x,u,Du)\left(\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right)\mathrm{d}x\right| \qquad (5.8)\\
&\leqq M(L)\int_\Omega \left(1+|Du|^2\right)^{\frac{p+q-2}{4}} \\
&\quad \cdot \left(\int_0^1 \left|2\eta\eta_{x_s}\, g(\Delta_h u) + \eta^2 g'(\Delta_h u)\,\Delta_h u_{x_s}\right| \mathrm{d}t\right)\mathrm{d}x \qquad (5.9)\\
&\quad + \int_0^1 \left|\int_\Omega |b_0(x)| \cdot \left|2\eta\eta_{x_s}\, g(\Delta_h u) + \eta^2 g'(\Delta_h u)\,\Delta_h u_{x_s}\right| \mathrm{d}x\right| \mathrm{d}t.
\end{aligned}$$

(5.10)

The above term in (5.9) is identical to the correspondent term in [55] (see in particular Section 5.3.7 of [55]) and it can be estimated as in (5.15) of [55]. Thus we limit here to estimate the addendum in (5.10). Recalling that $g'(t) > 0$ for all $t \in \mathbb{R}$, we start by using the inequalities

$$
\begin{aligned}
2\eta \left|\eta_{x_s}\right| \left|b_0(x)\right| \left|g(\Delta_h u)\right| &= 2\eta \left|\eta_{x_s}\right| \left|b_0(x)\right| \frac{\left|g(\Delta_h u)\right|}{\left(g'(\Delta_h u)\right)^{1/2}} \left(g'(\Delta_h u)\right)^{1/2} \\
&\leqq |D\eta|^2 \frac{g^2(\Delta_h u)}{g'(\Delta_h u)} + \eta^2 b_0^2(x)\, g'(\Delta_h u); \\
\left|b_0(x)\right| \cdot \left|\Delta_h u_{x_s}\right| &\leqq \varepsilon \left|\Delta_h u_{x_s}\right|^2 + \frac{1}{4\varepsilon} b_0^2(x).
\end{aligned}
$$

As before we denote by $\Omega' = \operatorname{supp}\eta$, which is a compact set contained in Ω. Moreover, since $b_0 \in L^{s_0}_{\mathrm{loc}}(\Omega)$ for $s_0 > n$, and in particular $b_0 \in L^{s_0}\left(\Omega'\right)$, we can also use Hölder's inequality with exponents $\frac{s_0}{2}$ and $\frac{s_0}{s_0-2}$. From (5.10) we obtain

$$
\begin{aligned}
&\left|\int_\Omega \left|b_0(x)\right| \cdot \left|\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right| \mathrm{d}x\right| \\
&= \int_0^1 \left|\int_\Omega \left|b_0(x)\right| \cdot \left|2\eta\eta_{x_s}\, g(\Delta_h u) + \eta^2 g'(\Delta_h u)\, \Delta_h u_{x_s}\right| \mathrm{d}x\right| \mathrm{d}t \\
&\leqq \int_0^1 \mathrm{d}t \int_\Omega |D\eta|^2 \frac{g^2(\Delta_h u)}{g'(\Delta_h u)} \mathrm{d}x + \int_0^1 \mathrm{d}t \int_\Omega \eta^2 b_0^2(x)\, g'(\Delta_h u)\, \mathrm{d}x \\
&\quad + \varepsilon \int_\Omega \eta^2 g'(\Delta_h u) \left|\Delta_h u_{x_s}\right|^2 \mathrm{d}x + \frac{1}{4\varepsilon} \int_0^1 \mathrm{d}t \int_\Omega \eta^2 b_0^2(x)\, g'(\Delta_h u)\, \mathrm{d}x \\
&\leqq \int_0^1 \mathrm{d}t \int_\Omega |D\eta|^2 \frac{g^2(\Delta_h u)}{g'(\Delta_h u)} \mathrm{d}x + \varepsilon \int_\Omega \eta^2 g'(\Delta_h u) \left|\Delta_h u_{x_s}\right|^2 \mathrm{d}x \\
&\quad + \left(1 + \frac{1}{4\varepsilon}\right) \left(\int_{\Omega'} \left|b_0(x)\right|^{s_0} \mathrm{d}x\right)^{\frac{2}{s_0}} \cdot \int_0^1 \mathrm{d}t \left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}} \mathrm{d}x\right)^{\frac{s_0-2}{s_0}}.
\end{aligned}
\tag{5.11}
$$

With the aim to estimate the last addendum in (5.11) we observe that the exponent $\frac{s_0}{s_0-2} > 1$ is strictly less than $\frac{n}{n-2}$, since $s_0 > n$ (for simplicity, we limit here to consider the details for the case $n > 2$; for $n = 2$ we can proceed similarly with small modifications). We represent $\frac{s_0}{s_0-2}$ as *convex combination* of 1 and $\frac{n}{n-2}$

$$
\frac{s_0}{s_0-2} = t + \frac{n}{n-2}(1-t), \qquad \text{with} \quad t = \frac{s_0-n}{s_0-2} \quad \text{and} \quad 1-t = \frac{n-2}{s_0-2}.
$$

Let λ be a positive real parameter that we will fix later; a computation shows that $\lambda^{-\frac{n}{s_0-n}t + \frac{n}{n-2}(1-t)} = 1$. By Hölder's inequality with exponents $\frac{1}{t}$ and $\frac{1}{1-t}$

$$
\begin{aligned}
&\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}}\,\mathrm{d}x\\
&\quad= \int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{t+\frac{n}{n-2}(1-t)}\,\mathrm{d}x\\
&\quad= \int_\Omega \left(\lambda^{-\frac{n}{s_0-n}}\eta^2 g'(\Delta_h u)\right)^{t}\left(\lambda\eta^2 g'(\Delta_h u)\right)^{\frac{n}{n-2}(1-t)}\,\mathrm{d}x\\
&\quad\leqq \left(\lambda^{-\frac{n}{s_0-n}}\int_\Omega \eta^2 g'(\Delta_h u)\,\mathrm{d}x\right)^{t}\left(\lambda^{\frac{n}{n-2}}\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{1-t}.
\end{aligned}
$$

We first recall that $t = \frac{s_0-n}{s_0-2}$ and $1-t = \frac{n-2}{s_0-2}$. Then, we use Young's inequality with exponents $\frac{s_0}{s_0-n}$ and $\frac{s_0}{n}$

$$
\begin{aligned}
&\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}}\,\mathrm{d}x\right)^{\frac{s_0-2}{s_0}}\\
&\leqq \left(\lambda^{-\frac{n}{s_0-n}}\int_\Omega \eta^2 g'(\Delta_h u)\,\mathrm{d}x\right)^{t\frac{s_0-2}{s_0}}\left(\lambda^{\frac{n}{n-2}}\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{(1-t)\frac{s_0-2}{s_0}}\\
&= \left(\lambda^{-\frac{n}{s_0-n}}\int_\Omega \eta^2 g'(\Delta_h u)\,\mathrm{d}x\right)^{\frac{s_0-n}{s_0}}\left(\lambda^{\frac{n}{n-2}}\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{s_0}}\\
&\leqq \frac{s_0-n}{s_0}\lambda^{-\frac{n}{s_0-n}}\int_\Omega \eta^2 g'(\Delta_h u)\,\mathrm{d}x + \frac{n}{s_0}\lambda\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}}.
\end{aligned}
\tag{5.12}
$$

The function η has compact support in Ω and we can apply the Sobolev inequality with exponents 2 and $2^* := \frac{2n}{n-2}$ and with the Sobolev constant $c = c(n)$ depending only on the dimension n

$$
\begin{aligned}
\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} &= \left(\int_\Omega \left(\eta\left(g'(\Delta_h u)\right)^{\frac{1}{2}}\right)^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\\
&\leqq c(n)\int_\Omega \left|D\left(\eta\left(g'(\Delta_h u)\right)^{\frac{1}{2}}\right)\right|^2\,\mathrm{d}x.
\end{aligned}
\tag{5.13}
$$

Collecting (5.12), (5.13) we get

$$
\begin{aligned}
\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}}\,\mathrm{d}x\right)^{\frac{s_0-2}{s_0}} &\leqq \frac{s_0-n}{s_0}\lambda^{-\frac{n}{s_0-n}}\int_\Omega \eta^2 g'(\Delta_h u)\,\mathrm{d}x\\
&\quad+ \frac{n}{s_0}\lambda c(n)\int_\Omega \left|D\left(\eta\left(g'(\Delta_h u)\right)^{\frac{1}{2}}\right)\right|^2\,\mathrm{d}x.
\end{aligned}
$$

A simple computation gives

$$
D\left(\eta\left(g'(\Delta_h u)\right)^{\frac{1}{2}}\right) = D\eta\left(g'(\Delta_h u)\right)^{\frac{1}{2}} + \frac{1}{2}\eta\left(g'(\Delta_h u)\right)^{-\frac{1}{2}} g''(\Delta_h u)\,\Delta_h Du,
$$

and we continue the previous estimate with

$$\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}} \, dx\right)^{\frac{s_0-2}{s_0}}$$
$$\leqq \int_\Omega \left(\tfrac{s_0-n}{s_0}\lambda^{-\frac{n}{s_0-n}}\eta^2 + 2\tfrac{n}{s_0}\lambda c(n)\,|D\eta|^2\right) g'(\Delta_h u) \, dx$$
$$+\tfrac{1}{2}\tfrac{n}{s_0}\lambda c(n)\int_\Omega \eta^2 \frac{\left(g''(\Delta_h u)\right)^2}{g'(\Delta_h u)}\,|\Delta_h Du|^2 \, dx. \qquad (5.14)$$

We recall that, for generic $k \in \mathbb{N}$ and $\beta \geqq 0$, $g(t) = t\left(1+t^2\right)^{\beta/2}$ when $t \in [-k, k]$ and $g(t)$ affine out of the interval $[-k, k]$ in such a way that globally $g \in C^1(\mathbb{R})$. Its derivative $g'(t)$ is continuous in $\mathbb{R}$, and in fact it is Lipschitz continuous in $\mathbb{R}$. For $t \in [-k, k]$ we have

$$\begin{cases} g'(t) = \left(1+t^2\right)^{\beta/2-1}\left(1+(\beta+1)\,t^2\right) \\ g''(t) = \beta t\left(1+t^2\right)^{\beta/2-2}\left(3+(\beta+1)\,t^2\right); \end{cases}$$

while when $t \notin [-k, k]$ then $g'(t)$ is constant and $g''(t) = 0$. We obtain

$$\frac{\left|g''(t)\right|}{g'(t)} \leqq \frac{\beta\,|t|\left(1+t^2\right)^{\beta/2-2}\left(3+(\beta+1)\,t^2\right)}{\left(1+t^2\right)\left(1+t^2\right)^{\beta/2-2}\left(1+(\beta+1)\,t^2\right)} \leqq 3\beta\frac{|t|}{1+t^2} \leqq \tfrac{3}{2}\beta$$

and then, taking the square of both sides, $\frac{\left(g''(\Delta_h u)\right)^2}{g'(\Delta_h u)} \leqq \frac{9}{4}\beta^2 g'(\Delta_h u)$. Going back to (5.14), we obtain

$$\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}} \, dx\right)^{\frac{s_0-2}{s_0}}$$
$$\leqq \int_\Omega \left(\tfrac{s_0-n}{s_0}\lambda^{-\frac{n}{s_0-n}}\eta^2 + 2\tfrac{n}{s_0}\lambda c(n)\,|D\eta|^2\right) g'(\Delta_h u) \, dx$$
$$+\tfrac{9}{8}\tfrac{n}{s_0}\lambda\beta^2 c(n)\int_\Omega \eta^2 g'(\Delta_h u)\,|\Delta_h Du|^2 \, dx. \qquad (5.15)$$

Here we change notation, in principle by posing $\mu = \lambda\beta^2$ for every $\beta > 0$; for $\beta = 0$ there is not necessity of this change. More precisely, with the aim to avoid the denominator β^2, which is not uniformly far from zero, we pose $\mu = \lambda\left(\beta^2+1\right)$ and we increase the last addendum in (5.15) by changing β^2 with β^2+1; then of course $\lambda = \frac{\mu}{\beta^2+1}$ and $\lambda \leqq \mu$ for all $\beta \geqq 0$. Thus

$$\left(\int_\Omega \left(\eta^2 g'(\Delta_h u)\right)^{\frac{s_0}{s_0-2}} \, dx\right)^{\frac{s_0-2}{s_0}}$$
$$\leqq \int_\Omega \left(\tfrac{s_0-n}{s_0}\left(\tfrac{\beta^2+1}{\mu}\right)^{\frac{n}{s_0-n}}\eta^2 + 2\tfrac{n}{s_0}\mu c(n)\,|D\eta|^2\right) g'(\Delta_h u) \, dx$$
$$+\tfrac{9}{8}\tfrac{n}{s_0}\mu c(n)\int_\Omega \eta^2 g'(\Delta_h u)\,|\Delta_h Du|^2 \, dx. \qquad (5.16)$$

By (5.11) and (5.16) we get the final estimate

$$\left|\int_{\Omega}|b_0(x)|\cdot\left|\Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right|\,\mathrm{d}x\right| \tag{5.17}$$

$$\leqq \int_0^1 \mathrm{d}t\int_{\Omega}|D\eta|^2\frac{g^2(\Delta_h u)}{g'(\Delta_h u)}\,\mathrm{d}x+\varepsilon\int_{\Omega}\eta^2 g'(\Delta_h u)\left|\Delta_h u_{x_s}\right|^2\,\mathrm{d}x \tag{5.18}$$

$$+\left(1+\tfrac{1}{4\varepsilon}\right)\|b_0\|^2_{L^{s_0}(\Omega')}\left\{c(n,s_0,\beta,\mu)\int_0^1\mathrm{d}t\int_{\Omega}\left(\eta^2+|D\eta|^2\right)g'(\Delta_h u)\,\mathrm{d}x\right. \tag{5.19}$$

$$\left.+\tfrac{9}{8}\tfrac{n}{s_0}\mu c(n)\int_0^1\mathrm{d}t\int_{\Omega}\eta^2 g'(\Delta_h u)\,|\Delta_h Du|^2\,\mathrm{d}x\right\}, \tag{5.20}$$

where $c(n,s_0,\beta,\mu)=\max\left\{\frac{s_0-n}{s_0}\left(\frac{\beta^2+1}{\mu}\right)^{\frac{n}{s_0-n}};\ \frac{2n}{s_0}\mu c(n)\right\}$ depends only on n, s_0, β, μ; in particular it depends on powers of the parameter β. We observe that this constant diverges to $+\infty$ as $\mu\to 0^+$, but this fact will not be a problem, since in the next section we will fix a (sufficiently small) value of $\mu>0$.

5.3. Conclusion

As explained below, all the addenda in (5.18), (5.19), (5.20) can be reabsorbed in (5.7), and we obtain (cfr. with Section 5.4 in [55])

$$\begin{aligned}
&\frac{1}{c}\int_0^1\mathrm{d}t\int_{\Omega}\eta^2 g'(\Delta_h u)\left(1+|(1-t)\,Du(x)+t\,Du(x+the_s)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du|^2\,\mathrm{d}x\\
&\quad\leqq\int_0^1\mathrm{d}t\int_{\Omega}\left(\eta^2+|D\eta|^2\right)g'(\Delta_h u)\left(1+|Du(x)|^2+|Du(x+the_s)|^2\right)^{\frac{q}{2}}\,\mathrm{d}x\\
&\qquad+\int_0^1\mathrm{d}t\int_{\Omega}2\eta\,|D\eta|\cdot|g(\Delta_h u)|\left(1+|Du(x)|^2+|Du(x+the_s)|^2\right)^{\frac{q-1}{2}}\,\mathrm{d}x\\
&\qquad+\int_0^1\mathrm{d}t\int_{\Omega}|D\eta|^2\cdot\frac{g^2(\Delta_h u)}{g'(\Delta_h u)}\left(1+|Du(x)|^2+|Du(x+the_s)|^2\right)^{\frac{q-2}{2}}\,\mathrm{d}x.
\end{aligned} \tag{5.21}$$

In particular, the ε-addendum in (5.18) can be reabsorbed in the left side of (5.21) if ε is sufficiently small; then the addendum in (5.20) above, although with the large factor $\left(1+\frac{1}{4\varepsilon}\right)$ in front (however now with ε fixed), can be reabsorbed in the left side of (5.21) by considering μ sufficiently small. The dependence of the right hand side of the estimate (5.21) on powers of β (precisely, the dependence of the constant c on powers of β) is allowed.

It remains only to follow the argument of [55] (see also details in [54, Section 4]) to conclude that the gradient Du of the weak solution is locally bounded in Ω, as in (2.11). In fact, by Theorem 3.3 in [55] we can say that there exist constants $c,\alpha_0,\gamma,R_0>0$ (depending on the $L^{\infty}\left(\Omega'\right)$ norm of u and on the data, but not on

u) such that, for every ϱ and R such that $0 < \varrho < R < R_0$,

$$\|Du\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \leqq \left(\frac{c}{(R-\varrho)^{\frac{\gamma q}{\vartheta p}}}\left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_R)}\right)^{\alpha_0} \underset{\text{for } n>2}{=} \left(\frac{c}{(R-\varrho)^{\frac{\gamma q}{\vartheta p}}}\left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_R)}\right)^{\frac{2p}{(n+2)p-nq}}. \tag{5.22}$$

The explicit expression of the exponent α_0 above (5.22) is given in (2.13), with $\vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2} \underset{\text{for } n>2}{=} \frac{2q}{np-(n-2)q}$ and $\gamma = \frac{n}{q}\vartheta$. Therefore $\frac{\gamma q}{\vartheta p} = \frac{n}{p}$, and also, from (5.22),

$$\|Du\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \leqq \left(\frac{c}{(R-\varrho)^n}\int_{B_R}\left(1+|Du|^2\right)^{\frac{p}{2}}\,\mathrm{d}x\right)^{\frac{\alpha_0}{p}}, \tag{5.23}$$

which corresponds to the stated estimate (2.11).

The $W^{2,2}_{\mathrm{loc}}(\Omega)$-bound stated in (2.12) can be similarly obtained in this way: we first use the bound (5.21) with $g(t)$ as above: $g(t) = t\left(1+t^2\right)^{\beta/2}$, $\beta \geqq 0$.

In the special case $\beta = 0$ we have $g(t) = t$, $g'(t) = 1$ and

$$\begin{aligned}
&\frac{1}{c}\int_0^1 \mathrm{d}t \int_\Omega \eta^2\left(1+|(1-t)\,Du(x)+t\,Du(x+the_s)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du|^2\,\mathrm{d}x\\
&\leqq \int_0^1 \mathrm{d}t\int_\Omega\left(\eta^2+|D\eta|^2\right)\left(1+|Du(x)|^2+|Du(x+the_s)|^2\right)^{\frac{q}{2}}\mathrm{d}x\\
&\quad+\int_0^1 \mathrm{d}t\int_\Omega 2\eta\,|D\eta|\cdot|\Delta_h u|\left(1+|Du(x)|^2+|Du(x+the_s)|^2\right)^{\frac{q-1}{2}}\mathrm{d}x\\
&\quad+\int_0^1 \mathrm{d}t\int_\Omega |D\eta|^2\cdot(\Delta_h u)^2\left(1+|Du(x)|^2+|Du(x+the_s)|^2\right)^{\frac{q-2}{2}}\mathrm{d}x.
\end{aligned}$$

Similarly to [55] we can go to the limit as $h \to 0$. In the left hand side we go to the limit by lower semicontinuity and in the limit $\left|D^2u\right|^2$ appears. In the limit as $h \to 0$ all the three integrands in the right hand side can be estimated by the q-power of the gradient Du. More precisely, in the limit as $h \to 0$ we obtain (cfr. with (5.18) in [55, Remark 5.1])

$$\frac{1}{c}\int_0^1 \mathrm{d}t\int_\Omega \eta^2\left(1+|Du|^2\right)^{\frac{p-2}{2}}\left|D^2u\right|^2\mathrm{d}x \leqq \int_0^1 \mathrm{d}t\int_\Omega\left(\eta^2+|D\eta|^2\right)\left(1+|Du|^2\right)^{\frac{q}{2}}\mathrm{d}x. \tag{5.24}$$

The integral with respect to $t \in [0,1]$ is not more necessary. We fix η as usual. Precisely we consider concentric balls B_R and B_ρ compactly contained in Ω, with $\varrho < R < R_0 = R_0(\varepsilon, n, s_0)$; then we consider a test function $\eta \in C^1_0(B_R)$,

$0 \leqq \eta \leqq 1$ in B_R, $\eta = 1$ in B_ϱ and $|D\eta| \leqq 2/(R-\varrho)$. We obtain the simplified version of (5.24)

$$\int_{B_\varrho} \left|D^2 u\right|^2 \, dx \leqq c\left(1 + \tfrac{4}{(R-\varrho)^2}\right) \int_{B_R} \left(1 + |Du|^2\right)^{\frac{q}{2}} \, dx. \tag{5.25}$$

Since $0 < \varrho < R < R_0$, then $\frac{4}{(R-\varrho)^2} \geqq \frac{4}{R_0^2}$ and thus $1 \leqq \frac{R_0^2}{4}\frac{4}{(R-\varrho)^2}$. Therefore, with a different constant which we continue to denote by c, we also have

$$\int_{B_\varrho} \left|D^2 u\right|^2 \, dx \leqq \tfrac{c}{(R-\varrho)^2} \int_{B_R} \left(1 + |Du|^2\right)^{\frac{q}{2}} \, dx. \tag{5.26}$$

Therefore the first $W^{2,2}_{\mathrm{loc}}(\Omega)$-bound stated in (2.12) is obtained. We now make use of the interpolation formula in [55, Remark 6.1] (see also [49, Theorem 3.1, formula (3.4)])

$$\left\|\left(1 + |Du|^2\right)^{\frac{1}{2}}\right\|_{L^q(B_\varrho)} \leqq \left(\frac{c}{(R-\varrho)^{\gamma\left(\frac{q}{p}-1\right)}} \left\|\left(1 + |Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_R)}^{\frac{1}{\vartheta}}\right)^{\alpha_0}, \tag{5.27}$$

where α_0, ϑ are expressed in (2.13) and $\gamma = \frac{n}{q}\vartheta$. We iterate (5.26), (5.27) in B_ρ, $B_{(R+\varrho)/2}$, B_R; more precisely we consider (5.26) with the balls B_ϱ, $B_{(R+\varrho)/2}$ and (5.27) with $B_{(R+\varrho)/2}$, B_R. With different constants c we obtain

$$\begin{aligned}
\int_{B_\varrho} \left|D^2 u\right|^2 \, dx &\leqq \frac{c}{(R-\varrho)^2} \left\|\left(1 + |Du|^2\right)^{\frac{1}{2}}\right\|_{L^q(B_{(R+\varrho)/2})}^q \\
&\leqq \frac{c}{(R-\varrho)^2} \left(\frac{1}{(R-\varrho)^{\gamma\left(\frac{q}{p}-1\right)}} \left\|\left(1 + |Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_R)}^{\frac{1}{\vartheta}}\right)^{\alpha_0 q} \\
&= \frac{c}{(R-\varrho)^2} \left(\frac{1}{(R-\varrho)^{\gamma\vartheta(q-p)}} \int_{B_R} \left(1 + |Du|^2\right)^{\frac{p}{2}} \, dx\right)^{\frac{\alpha_0 q}{\vartheta p}} \\
&= \frac{c}{(R-\varrho)^{2+\alpha_0\gamma q\left(\frac{q}{p}-1\right)}} \left\|\left(1 + |Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_R)}^{\frac{\alpha_0 q}{\vartheta}}.
\end{aligned} \tag{5.28}$$

This is the conclusion of the $W^{2,2}_{\mathrm{loc}}(\Omega)$-estimate, as stated in (2.12). Note that in the special case $q = p$ all the parameters in this estimate simplify, $\alpha_0 = \vartheta = 1$, and the bounds (2.19), (5.26) are reproduced.

The proof of Theorem 2.1 is now complete.

The computations in this Section 5 are useful for proving Theorem 2.2 too. More precisely we are now under the ellipticity (2.1) and the p,q-growth conditions

(2.14), (2.15). In particular, other than the ellipticity assumption (2.1), with the growth conditions

$$\begin{cases} \left|\frac{\partial a^i}{\partial \xi_j}\right| \leqq M(1+|\xi|^2)^{\frac{q-2}{2}} + M\,|u|^{\alpha} \\ \left|\frac{\partial a^i}{\partial u}\right| \leqq M(1+|\xi|^2)^{\frac{q-2}{2}} + M\,|u|^{\beta-1} \\ \left|\frac{\partial a^i}{\partial x_s}\right| \leqq M\,(L)\,(1+|\xi|^2)^{\frac{q-1}{2}} \\ \left|a^i\,(x,0,0)\right| \in L^{\gamma}_{\text{loc}} \\ |b\,(x,u,\xi)| \leqq M(1+|\xi|^2)^{\frac{q-1}{2}} + M\,|u|^{\delta-1} + b_0\,(x) \end{cases} \tag{5.29}$$

Conditions $(5.29)_2$, $(5.29)_3$, $(5.29)_5$ respectively correspond to (2.3), (2.6), (2.8) when we replace $\frac{p+q}{2}$ by q. For a better understanding, let us denote by $r := \frac{p+q}{2}$; then in accordance $q = 2r - p$. This means that, if for instance $(5.29)_2$ corresponds to (2.3) when we replace $\frac{p+q}{2}$ by q, likewise in $(5.29)_1$ we should replace q with $2q - p$. In fact, since $2q - p \geqq q$, then if $(5.29)_1$ holds, then all the more so

$$\left|\frac{\partial a^i}{\partial \xi_j}\right| \leqq M(1+|\xi|^2)^{\frac{2q-p-2}{2}} + M\,|u|^{\alpha}$$

when we limit α (recall that, in Theorem 2.1, $0 \leqq \alpha < (q-2)\frac{p^*}{p}$) with the corresponding bound

$$0 \leqq \alpha < (2q - p - 2)\frac{p^*}{p},$$

as stated in $(2.15)_1$. Note that, when $q = p$, the two constraints for α coincide each other. About the condition $\frac{q}{p} < 1 + \frac{1}{n}$, when we replace q by $2q - p$ we obtain $\frac{2q-p}{p} < 1 + \frac{1}{n}$; that is, $\frac{q}{p} < 1 + \frac{1}{2n}$. Finally, the exponent in the $W^{1,\infty}$-local estimate (2.16), which in Theorem 2.1 when $n \geqq 3$ was equal to $\frac{2p}{(n+2)p-nq}$, now becomes $\frac{p}{(n+1)p-nq}$. The proof of Theorem 2.2 is complete too.

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