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Linear Instability Analysis on Compressible Navier–Stokes Equations with Strong Boundary Layer

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Abstract

A classical problem in fluid mechanics concerns the stability and instability of different hydrodynamic patterns in various physical settings, particularly in the high Reynolds number limit of laminar flows with boundary layers. Despite extensive studies when the fluid is governed by incompressible Navier-Stokes equations, there are very few mathematical results on the compressible fluid. This paper aims to introduce a new approach to studying the compressible Navier–Stokes equations in the subsonic and high Reynolds number regime, where a subtle quasi-compressible and Stokes iteration is developed. As a byproduct, we show the spectral instability of subsonic boundary layers.

Contents

1. Introduction

One of the most fundamental problems in fluid mechanics is understanding the physical mechanisms that lead to the stability or instability of hydrodynamic patterns. Most laminar flows are unstable at the high Reynolds number, and small perturbations will eventually cascade into turbulence. Under many circumstances, the early stage of such transition is the instability induced by viscous disturbance wave, now called Tollmien–Schlichting or T–S wave. For incompressible flow, the physical description of T–S waves can be found, for instance, in the pioneering work by Heisenberg, C.C. Lin, Tollmien and Schlichting, cf. [5,15,21,33], and Wasow [38] established a formal construction of them. Until recently, the most rigorous mathematical justification was given by Grenier–Guo–Nguyen [7].

From the physical point of view, it is important to study the compressible flow with boundary layers that arises from, for instance, the flow near the airfoil. The theoretical investigation can be traced back to Lees–Lin [19], in which Rayleigh's criterion for inviscid flow was extended to the compressible subsonic flow. Later on, the asymptotic expansion used in [19] near the critical layer was rigorously justified by Morawitz [29]. For more investigation from the physical perspective, we refer readers to [5,20,21,33] and the references therein. It is worth noting that the instability mechanisms studied in the literature are inviscid in nature, while the viscous transition mechanisms still need to be investigated. This paper aims to fill this gap by rigorously justifying the presence of T–S waves in the compressible boundary layer.

1.1. Problem and Main Result

Consider the 2D compressible Navier–Stokes equations for isentropic flow in half-space $\{(x, y) \mid x \in \mathbb{T}, y \in \mathbb{R}_+\}$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{U}) = 0, \\ \rho \partial_t \vec{U} + \rho \vec{U} \cdot \nabla \vec{U} + \nabla P(\rho) = \mu\varepsilon \Delta \vec{U} + \lambda\varepsilon \nabla(\nabla \cdot \vec{U}) + \rho \vec{F}, \\ \vec{U}|_{y=0} = \vec{0}. \end{cases} \tag{1.1}$$

In the above equations $\rho, \vec{U} = (u, v)$ and $P(\rho)$ stand for the density, velocity field and pressure of the fluid. The vector field $\vec{F}$ is a given external force. The constants $\mu > 0, \lambda \geqq 0$ are rescaled shear and bulk viscosity, respectively, while $0 < \varepsilon \ll 1$ is a small parameter which is proportional to the reciprocal of the Reynolds number. For simplicity and without loss of generality, the constant μ is set to 1 throughout paper.

A laminar boundary layer flow is defined by

$$(\rho_s, \vec{\mathbf{U}}_s) \stackrel{\text{def}}{=} (1, U_s(Y), 0), \ \ Y := \frac{y}{\sqrt{\varepsilon}}, \ \text{with } U_s(0) = 0, \lim_{Y \to +\infty} U_s(Y) = 1.$$

It is a steady solution to (1.1) with external force $\vec{F} = (-\partial_Y^2 U_s, 0)$.

In this work, to understand the (in)stability properties of the above boundary layer profile, we study the compressible Navier–Stokes system linearized around $(\rho_s, \vec{\mathbf{U}}_s)$. Denote the Mach number by $m := \frac{1}{\sqrt{P'(1)}}$. The linearization gives

$$\begin{cases} \partial_t \rho + U_s \partial_x \rho + \nabla \cdot \vec{u} = 0, \ t > 0, \ (x, y) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t \vec{u} + U_s \partial_x \vec{u} + m^{-2} \nabla \rho + v \partial_y U_s \vec{e}_1 - \varepsilon \Delta \vec{u} - \lambda \varepsilon \nabla(\nabla \cdot \vec{u}) - \rho \vec{F} = 0, \\ \vec{u}|_{y=0} = \vec{0}. \end{cases} \tag{1.2}$$

To study (1.2), we use the rescaled variable

$$\tau = \frac{t}{\sqrt{\varepsilon}}, \ X = \frac{x}{\sqrt{\varepsilon}}, \ Y = \frac{y}{\sqrt{\varepsilon}},$$

and then look for solution to the linearized compressible Navier–Stokes system in the following form:

$$(\tilde{\rho}, \tilde{u}, \tilde{v})(Y) e^{i\alpha(X - c\tau)}.$$

Plugging this ansatz into (1.2) yields to the system (we replace $(\tilde{\rho}, \tilde{u}, \tilde{v})$ by (ρ, u, v) for simplicity of notation)

$$\begin{cases} i\alpha(U_s - c)\rho + \mathrm{div}_\alpha(u, v) = 0, \\ \sqrt{\varepsilon}\Delta_\alpha u + \lambda i\alpha\sqrt{\varepsilon}\mathrm{div}_\alpha(u, v) - i\alpha(U_s - c)u - (i\alpha m^{-2} + \sqrt{\varepsilon}\partial_Y^2 U_s)\rho - v\partial_Y U_s = 0, \\ \sqrt{\varepsilon}\Delta_\alpha v + \lambda\sqrt{\varepsilon}\partial_Y \mathrm{div}_\alpha(u, v) - i\alpha(U_s - c)v - m^{-2}\partial_Y \rho = 0, \end{cases} \tag{1.3}$$

with no-slip boundary conditions

$$u|_{Y=0} = v|_{Y=0} = 0. \tag{1.4}$$

In (1.3), $\Delta_\alpha = (\partial_Y^2 - \alpha^2)$ and $\mathrm{div}_\alpha(u, v) = i\alpha u + \partial_Y v$ denote the Fourier modes of Laplacian and divergence operators respectively. For convenience, we denote by $\mathcal{L}(\rho, u, v)$ the linear operator (1.3). If for some $c \in \mathbb{C}$ with positive imaginary part $\mathrm{Im}c > 0$ and wave number $\alpha \in \mathbb{R}$, the boundary value problem (1.3) with (1.4) has a non-trivial solution, then the boundary layer profile $(\rho_s, \vec{\mathbf{U}}_s)$ is spectral unstable. Otherwise, thus is spectral stable.

In the analysis, we focus on a class of laminar boundary layer flows that satisfy the following assumptions:

- $U_s \in C^3(\overline{\mathbb{R}_+})$ and satisfies

$$U_s(0) = 0, \ U_s(Y) > 0, \ \lim_{Y \to +\infty} U_s(Y) = 1, \ \text{and } U_s'(0) = 1. \tag{1.5}$$

- There exist positive constants s_0, s_1 and s_2, such that

$$s_1 e^{-s_0 Y} \leqq \partial_Y U_s(Y) \leqq s_2 e^{-s_0 Y}, \ \forall Y \geqq 0. \tag{1.6}$$

- The boundary layer flow is assumed to be *uniformly subsonic*, that is $m \in (0, 1)$. Moreover, there exists a constant $\sigma_1 = \sigma_1(m) > 0$ such that for all $Y \geqq 0$, it holds

$$H(Y) \stackrel{\text{def}}{=} \frac{-\partial_Y^2 U_s(1 - m^2 U_s^2) - 2m^2 U_s |\partial_Y U_s|^2}{|\partial_Y^2 U_s| + |\partial_Y U_s|^2} \geqq \sigma_1. \tag{1.7}$$

- There exists a constant $\sigma_2 > 0$ such that for any $Y \geqq 0$, it holds

$$\left|\frac{\partial_Y^3 U_s}{\partial_Y^2 U_s}\right| + \frac{|\partial_Y^2 U_s|}{\partial_Y U_s} + \frac{1 - U_s}{\partial_Y U_s} \leqq \sigma_2. \tag{1.8}$$

Note that this class of profiles include the exponential profile $U_s(Y) = 1 - e^{-Y}$ with $\sigma_1(m) = \frac{1-m^2}{2}$ and $s_0 = s_1 = s_2 = \sigma_2 = 1$. Moreover, from (1.7), we have

$$-\partial_Y^2 U_s(Y) \geqq \frac{\sigma_1}{1 - m^2} |\partial_Y U_s(Y)|^2, \ \forall Y \geqq 0. \tag{1.9}$$

Hence, the profiles in this class are strictly concave.

The main result in the paper can be stated as follows:

Theorem 1.1. *Let the Mach number $m \in (0, \frac{1}{\sqrt{3}})$. There are positive constants $K_0 > 1$ and $\varepsilon_0 \in (0, 1)$, such that for any $\varepsilon \in (0, \varepsilon_0)$ and any wave number $\alpha = K\varepsilon^{\frac{1}{8}}$ with $K \geqq K_0$, there exists $c_\varepsilon \in \mathbb{C}$ with $\alpha Imc_\varepsilon \approx \varepsilon^{\frac{1}{4}}$, such that the linearized compressible Navier–Stokes system (1.2) admits a solution (ρ, u, v) in the form of*

$$(\rho, u, v)(t, x, y) = e^{\frac{i\alpha}{\sqrt{\varepsilon}}(x - c_\varepsilon t)} (\tilde{\rho}, \tilde{u}, \tilde{v})(Y), \ Y := \frac{y}{\sqrt{\varepsilon}}. \tag{1.10}$$

Here the profile $(\tilde{\rho}, \tilde{u}, \tilde{v}) \in H^1(\mathbb{R}_+) \times \left(H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)\right)^2$ and satisfies the eigenvalue problem (1.3).

Remark 1.2. In what follows, we present several remarks on Theorem 1.1.

(a) As shown in the proof, the bounds on the solution depend on some negative power of $1 - m$ that are uniform when m is in a compact set of $[0, \frac{1}{\sqrt{3}})$. Therefore, by taking the vanishing Mach number limit, we have the Tollmien–Schlichting wave solution for the incompressible flow that was analyzed in [7] by Grenier–Guo–Nguyen.

(b) The restriction of Mach number $m \in (0, \frac{1}{\sqrt{3}})$ should be technical and it is only used in obtaining a positive lower bound of $w_0 - U_s w_1$ in the proof of Lemma 3.4. However, the main purpose of this paper is to introduce a new approach to the stability and instability analysis on compressible fluid with strong boundary layers. Therefore, we will not pursue how the Mach number can be close to one in this work.

(c) The growing modes are supported in the frequency regime $n = \frac{\alpha}{\sqrt{\varepsilon}} = K\varepsilon^{-\frac{3}{8}}$ and grow like $\exp(K^{-\frac{2}{3}} n^{\frac{2}{3}} t)$ with $K \gg 1$. These parameters indicate a spectral instability in Gevrey space with index equal to $\frac{3}{2}$. How to justify the stability in Gevrey $\frac{3}{2}$ class for the linear problem (1.2) that shares the same index as in the incompressible case studied in [12] will be discussed in our future work.

(d) By constructing a suitable approximate growing mode, we can also obtain the same results as in Theorem 1.1 for the wave number $\alpha = C\varepsilon^{\beta}$, with $C > 0$ and $\beta \in (1/12, 1/8)$. Here we will not give the detailed analysis in this regime because we focus on the most unstable mode when $\beta = 1/8$. We refer to [22] on the incompressible MHD system for the related discussions and analysis.

1.2. Relevant Literature

Since this paper is motivated by the study of inviscid limit and Prandtl boundary layer expansion for incompressible Navier–Stokes equations, we briefly summarize some related works in this direction. Indeed, there are two main destabilizing mechanisms in the boundary layer that makes inviscid limit problem very challenging. The first one is induced by the inflexion points inside the boundary layer profile. In this case, the linearized Navier–Stokes system exhibits a strong ill-posedness below the analyticity regularity, cf. [6,8]. Therefore, the results of inviscid limit can be obtained only when the data is analytic at least near the boundary, cf. [18,25,30,32,34]. The second one is induced by the small disturbance around a monotone and concave boundary layer profile, called Tollmien–Schlichting instability that has been extensively studied in physical literatures and was justified rigorously in Grenier–Guo–Nguyen [7] by constructing a growing mode in Gevrey $3/2$ space to linearized incompressible Navier–Stokes equations. The main idea in [7] is to use the stream function and vorticity, that is, the Orr–Sommerfeld (OS) formulation, and then to solve it via an iteration based on Rayleigh and Airy equation that can be viewed as the inviscid and viscous approximation to the original OS equation. We also refer to Grenier-Nguyen [9] for a result of nonlinear instability for small data that depend on viscosity coefficient. On the other hand, this instability result was complemented by the work of Gérard-Varet-Maekawa-Masmoudi [12,13] that establishes the Gevrey stability of Euler plus Prandtl expansion with critical Gevrey index $3/2$; see also [2] for a result in L^{∞}-setting. Most recently, the formation of boundary layer is studied by Maekawa [26] using the Rayleigh profile. For the Sobolev data, the boundary layer expansion is only valid under certain symmetry assumptions or for steady flows, cf. [11,14,16,24,28] and the references therein. Finally, we refer to [3,4,10] for the instability analysis of boundary layer profile in different settings.

For compressible flow, even though there are many interesting results on the Navier–Stokes equations at high Reynolds number in different settings, cf. [1,23,31, 35–37,39,40] and the references therein, to our knowledge, the stability/instability properties of strong boundary layer for compressible Navier–Stokes equations have not yet been investigated. Compared to the incompressible Navier–Stokes equations, the major difficulty comes from the fact that the Orr–Sommerfeld formulation is no longer available for the compressible case. As a result, Rayleigh–Airy iteration approach used in [7] can not be applied, either the approaches used in [12,13]. Therefore, the novelty of this paper is to introduce a new iteration approach to study compressible flow in the subsonic and high Reynolds number regime. We believe that the analytic techniques developed in this work can be used in other related problems for subsonic flows.

In the next subsection, we present the strategy of the proof for better understanding of the detailed analysis in the follow sections.

1.3. Strategy of Proof

The instability analysis is based on several steps.

Step 1. Construction of the approximate growing mode. Similar as in the incompressible case [7], the T–S instability is driven by the interaction of inviscid and viscous perturbations. Set the approximate growing mode (its precise definition will be given in (2.32)) $\vec{\Xi}_{\text{app}} = (\rho_{\text{app}}, u_{\text{app}}, v_{\text{app}})$ as

$$(\rho_{\text{app}}, u_{\text{app}}, v_{\text{app}}) = (\rho^s_{\text{app}}, u^s_{\text{app}}, v^s_{\text{app}}) - \frac{v^s_{\text{app}}(0;c)}{v^f_{\text{app}}(0;c)}(\rho^f_{\text{app}}, u^f_{\text{app}}, v^f_{\text{app}}). \tag{1.11}$$

Here the slow mode $(\rho^s_{\text{app}}, u^s_{\text{app}}, v^s_{\text{app}})$ is an approximate solution to the inviscid system, the fast mode $(\rho^f_{\text{app}}, u^f_{\text{app}}, v^f_{\text{app}})$ is an approximate solution to the full system (1.3) which exhibits viscous boundary layer structure near $Y = 0$; see (2.19) and (2.30) for the precise definition of slow and fast modes respectively. Note that the approximate solutions defined in (1.11) have zero normal velocity at the boundary, that is $v_{\text{app}}(0;c) \equiv 0$. Then, to recover the no-slip boundary condition, inspired by [3,4], we analyze the zero point of $\mathcal{F}_{\text{app}}(c) \stackrel{\text{def}}{=} u^s_{\text{app}}(0;c) - \frac{v^s_{\text{app}}(0;c)}{v^f_{\text{app}}(0;c)} u^f_{\text{app}}(0;c)$ by applying Rouché's Theorem. Precisely, we study the equation $\mathcal{F}_{\text{app}}(c) = 0$ in a family of ε-dependent domains D_0 (see (2.34)). Then, by Rouché's Theorem, we can show that $\mathcal{F}_{\text{app}}(c)$ has the same number of zero points as a linear function $\mathcal{F}_{\text{ref}}(c)$ defined by (2.41). In addition, we prove that $|\mathcal{F}_{\text{app}}(c)|$ has a strictly positive lower bound on ∂D_0.

Step 2. Stability of the approximate growing mode. Since the approximate solution $\vec{\Xi}_{\text{app}}$ exhibits the instability already, this step is to show the existence of an exact solution near $\vec{\Xi}_{\text{app}}$. This is the most difficult and key step. For this, we study the resolvent problem

$$\begin{cases} i\alpha(U_s - c)\rho + \text{div}_\alpha(u, v) = 0, \\ \sqrt{\varepsilon}\Delta_\alpha u + \lambda i\alpha\sqrt{\varepsilon}\text{div}_\alpha(u, v) - i\alpha(U_s - c)u - v\partial_Y U_s - (i\alpha m^{-2} + \sqrt{\varepsilon}\partial_Y^2 U_s)\rho = f_u, \\ \sqrt{\varepsilon}\Delta_\alpha v + \lambda\sqrt{\varepsilon}\partial_Y \text{div}_\alpha(u, v) - i\alpha(U_s - c)v - m^{-2}\partial_Y\rho = f_v, \\ v|_{Y=0} = 0, \end{cases} \tag{1.12}$$

with a given inhomogeneous source term (f_u, f_v). Here, we emphasize that in (1.12) only normal velocity field v is prescribed on the boundary. Even though we relax the boundary constraint on u in (1.12), it is still difficult for existence because the presence of stretching term $\mathfrak{v}\partial_Y U_s$. This difficulty is overcome by the following three ingredients:

- Quasi-compressible approximation. When the inhomogeneous source $(f_u, f_v) \in H^1(\mathbb{R}_+)^2$, we introduce the following *quasi-compressible* system

$$\begin{cases} i\alpha(U_s - c)\varrho + i\alpha\mathfrak{u} + \partial_Y\mathfrak{v} = 0, \\ \sqrt{\varepsilon}\Delta_\alpha\left[\mathfrak{u} + (U_s - c)\varrho\right] - i\alpha(U_s - c)\mathfrak{u} - \mathfrak{v}\partial_Y U_s - i\alpha m^{-2}\varrho = f_u, \\ \sqrt{\varepsilon}\Delta_\alpha\mathfrak{v} - i\alpha(U_s - c)\mathfrak{v} - m^{-2}\partial_Y\varrho = f_v, \\ \mathfrak{v}|_{Y=0} = 0, \end{cases} \tag{1.13}$$

which will be denoted by $L_Q(\varrho, \mathfrak{u}, \mathfrak{v}) = (0, f_u, f_v)$. Note that the inviscid part of the original linear operator $\mathcal{L}$ is kept in L_Q, while the diffusion terms are modified to be divergence free. It turns out that for Mach number $m \in (0, 1)$, the system (1.13) exhibits a similar stream function-vorticity structure as the incompressible Navier–Stokes equations. In fact, if we introduce the "effective stream function" Ψ associated to the modified velocity variable $(\mathfrak{u} + (U_s - c)\varrho, \mathfrak{v})$ which satisfies

$$\partial_Y\Psi = \mathfrak{u} + (U_s - c)\varrho, \quad -i\alpha\Psi = \mathfrak{v}, \quad \Psi|_{Y=0} = 0,$$

then (1.13) can be reformulated in terms of Ψ as

$$\begin{aligned} \mathrm{OS}_{\mathrm{CNS}}(\Psi) &:= \frac{i}{n}\Lambda(\Delta_\alpha\Psi) + (U_s - c)\Lambda(\Psi) - \partial_Y(A^{-1}\partial_Y U_s)\Psi \\ &= f_v - \frac{1}{i\alpha}\partial_Y(A^{-1}f_u), \end{aligned} \tag{1.14}$$

where $n = \alpha/\sqrt{\varepsilon}$, $A(Y) = 1 - m^2(U_s - c)^2$ and $\Lambda(\Psi) = \partial_Y(A^{-1}\partial_Y\Psi) - \alpha^2\Psi$. Note that $A(Y)$ is invertible at least for $m \in (0, 1)$ and c near the origin. When the Mach number $m = 0$, we have $\Lambda = \Delta_\alpha$ and $A(Y) \equiv 1$. Thus $\mathrm{OS}_{\mathrm{CNS}}$ in this case reduces to the classical Orr–Sommerfeld operator for incompressible Navier–Stokes system. Therefore, $\mathrm{OS}_{\mathrm{CNS}}$ can be viewed as the compressible counterpart of the Orr–Sommerfeld equation, which to our best knowledge is for the first time derived in the literatures. This formulation motivates the notion "quasi-compressible" approximation.

We solve (1.14) with artificial boundary conditions $\Psi|_{Y=0} = \Lambda(\Psi)|_{Y=0} = 0$ that allows us to obtain the weighted estimates on $\Lambda(\Psi)$. One can see that when $m = 0$, these boundary conditions are simply the perfect-slip boundary conditions used in [2,11,13] for the study of incompressible Navier–Stokes equations. However, for the problem considered in this paper, the multiplier $w(Y) = -\partial_Y(A^{-1}\partial_Y U_s)$ is not real. Therefore both its leading and first order terms w_0, w_1 (see Lemma 3.3 for the precise definitions) play a role in the energy estimates. For the bound estimations, we essentially use $m \in (0, \frac{1}{\sqrt{3}})$ and the new structural condition (1.7) of the profile in order to show that the function $w_0 - U_s w_1$ has a positive lower bound, cf. (3.29) in the proof of Lemma 3.4.

After we obtain Ψ which solves (1.14), the solution $(\varrho, \mathfrak{u}, \mathfrak{v})$ to (1.13) can be recovered in terms of Ψ, cf. (3.11) and (3.12). Here we would like to mention that (1.13) has a regularizing effect on density. That is, formally by applying div_α to the momentum equation in (1.13) and by noting that the diffusion term lies in the kernel of div_α, we have $\Delta_\alpha \varrho \in L^2(\mathbb{R}_+)$. This reveals an elliptic structure for the linearized compressible Navier–Stokes equations in the subsonic regime.

- Stokes approximation. Note that $(\varrho, \mathfrak{u}, \mathfrak{v})$ is not an exact solution to (1.12) and its error is

$$\begin{aligned} E_Q(\varrho, \mathfrak{u}, \mathfrak{v}) &\stackrel{\text{def}}{=} \mathcal{L}(\varrho, \mathfrak{u}, \mathfrak{v}) - L_Q(\varrho, \mathfrak{u}, \mathfrak{v}) \\ &= \Big(0, -\sqrt{\varepsilon}\Delta_\alpha\left[(U_s - c)\varrho\right] + \lambda\sqrt{\varepsilon} i\alpha \mathrm{div}_\alpha(\mathfrak{u}, \mathfrak{v}), \\ &\quad \sqrt{\varepsilon}\partial_Y \mathrm{div}_\alpha(\mathfrak{u}, \mathfrak{v})\Big). \end{aligned} \tag{1.15}$$

This error term involves a small factor of $\sqrt{\varepsilon}$ but lies only in $L^2(\mathbb{R}_+)$. This fact prevents us from using the standard fixed point argument to solve (1.12). To recover the regularity, we introduce another operator L_S that we call *Stokes approximation*. It is obtained from $\mathcal{L}$ by removing the stretching term, that is,

$$L_S(\xi, \phi, \psi) \stackrel{\text{def}}{=} \mathcal{L}(\xi, \phi, \psi) + (0, \psi\partial_Y U_s, 0).$$

To eliminate the error $E_Q(\varrho, \mathfrak{u}, \mathfrak{v})$, we then take (ξ, ϕ, ψ) as the solution to

$$L_S(\xi, \phi, \psi) = -E_Q(\varrho, \mathfrak{u}, \mathfrak{v}), \ \partial_Y \phi|_{Y=0} = \psi|_{Y=0} = 0.$$

By using the energy approach in the same spirit as Matsumura-Nishida [27] and Kawashima [17], we are able to show (ξ, ϕ, ψ) is in $H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$. Thus, the error term $E_S(\xi, \phi, \psi) := (0, \psi\partial_Y U_s, 0)$ is in the weighted space $H^2_w(\mathbb{R}_+)$ so that we can treat it as source term of L_Q. Therefore, we can iterate the above two approximations.

- Quasi-compressible-Stokes iteration. Recall that we have the following two decompositions of the linear operator $\mathcal{L}$:

$$\mathcal{L} = L_Q + E_Q = L_S + E_S.$$

The solvability to (1.12) can be justified via an iteration scheme that is illustrated as follows. Assume that at the N-th step we have an approximate solution in the form of $\sum_{j=0}^N \vec{\Xi}_j$ which satisfies

$$\mathcal{L}\left(\sum_{j=0}^N \vec{\Xi}_j\right) = (0, f_u, f_v) + \vec{\mathcal{E}}_N.$$

Here $\vec{\mathcal{E}}_N$ is an error term at this step. Provided that $\vec{\mathcal{E}}_N$ is smooth enough and has zero value at its first component, we can introduce a corrector

$$\vec{\Xi}_{N+1} = -L_Q^{-1}(\vec{\mathcal{E}}_N) + L_S^{-1} \circ E_Q \circ L_Q^{-1}(\vec{\mathcal{E}}_N),$$

where L_Q^{-1} and L_S^{-1} denote respectively the solution operators to quasi-compressible and Stokes approximate systems. The approximate solution at the $N+1$-step is therefore defined by $\sum_{j=0}^{N+1} \vec{\Xi}_j$. Then we have

$$\mathcal{L}\left(\sum_{j=1}^{N+1} \vec{\Xi}_j\right) = (0, f_u, f_v) + \vec{\mathcal{E}}_{N+1}$$
$$:= (0, f_u, f_v) + E_S \circ L_S^{-1} \circ E_Q \circ L_Q^{-1}(\vec{\mathcal{E}}_N).$$

A combination of the smallness of E_Q, the regularizing effect of L_S^{-1} and the strong decay property of E_S yields the contraction in $H_w^1(\mathbb{R}_+)$ of truncated error operator $E_S \circ L_S^{-1} \circ E_Q \circ L_Q^{-1}$ so that the convergence of series $\sum_{j=1}^{\infty} \vec{\Xi}_j$ in $H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$ follows, cf. the proof of Proposition 3.1.

Step. 3. Recovery of the no-slip boundary condition. We look for solutions to the original system (1.3) with $v|_{Y=0} = 0$ in the form of $(\rho, u, v) = (\rho_{\text{app}}, u_{\text{app}}, v_{\text{app}}) - (\rho_R, u_R, v_R)$. Here the remainder (ρ_R, u_R, v_R) solves $\mathcal{L}(\rho_R, u_R, v_R) = E$, $v_R|_{Y=0} = 0$, where E is the error due to the approximation $(\rho_{\text{app}}, u_{\text{app}}, v_{\text{app}})$. In this step, we need to decompose the error into regular and smallness parts as in (2.44). The remainder is divided accordingly into $\vec{\Xi}_R = \vec{\Xi}_{\text{re}} + \vec{\Xi}_{\text{sm}}$. The reason for such decomposition is that the regular part E_{re} coming from the rough approximation to Rayleigh equation has a worse bound than the smallness part so that we can compensate some extra order of ε by the favorable bounds of $\vec{\Xi}_{\text{re}}$ due to strong decay and H^1-regularity of E_{re}, cf. Proposition 3.1. Eventually, we can prove that $|u_R(0; c)| \leqq C\varepsilon^{\frac{1}{16}}$ on ∂D_0, which is smaller than $|u_{\text{app}}(0; c)|$. Then we conclude Theorem 1.1 by Rouché's Theorem.

The rest of the paper is organized as follows: in the next section, we will construct the approximate growing mode. In Sect. 3, we will show the solvability of the linearized system (1.3) with zero normal velocity condition in order to resolve the remainder due to the approximation. The proof is divided into several steps. Firstly, two approximate systems, that is, Quasi-compressible and Stokes approximations will be introduced in Sects. 3.1 and 3.2 respectively. Based on these two systems, the iteration scheme will be analyzed in Sect. 3.3. The proof of Theorem 1.1 will be given in the final section. In the Appendix, we will give the proof of the invertibility of operator Λ that is used in the construction of solution to the equation (3.16).

In the paper, for any $z \in \mathbb{C} \setminus \mathbb{R}_-$, we take the principle analytic branch of $\log z$ and z^k, $k \in (0, 1)$, that is

$$\log z \triangleq Log|z| + i\text{Arg}z, \;\; z^k \triangleq |z|^k e^{ik\text{Arg}z}, \;\; \text{Arg}z \in (-\pi, \pi].$$

Notations: Throughout the paper, C denotes a generic positive constant and C_a means that the generic constant depending on a. These constants may vary from line to line. $A \lesssim B$ and $A = O(1)B$ mean that there exists a generic constant C such that $A \leqq CB$. And $A \lesssim_a B$ implies that the constant C depends on a. Similar definitions hold for $A \gtrsim B$ and $A \gtrsim_a B$. Moreover, we use notation $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ denote the standard $L^2(\mathbb{R}_+)$ and $L^\infty(\mathbb{R}_+)$

norms respectively. For any $\eta > 0$, $L^\infty_\eta(\mathbb{R}_+)$ denotes the weighted Lebesgue space with the norm $\|f\|_{L^\infty_\eta} \triangleq \sup_{Y\in\mathbb{R}_+} \left|e^{\eta Y} f(Y)\right|$. And the weighted Sobolev space $W^{k,\infty}_\eta(\mathbb{R}_+)$ ($k \in \mathbb{N}$) has the norm $\|f\|_{W^{k,\infty}_\eta} = \sum_{j\leqq k} \|\partial_Y^j f\|_{L^\infty_\eta}$.

2. Approximate Growing Mode

In the following three subsections, we will construct the approximate growing mode that satisfies the no-slip boundary condition. Similar to the incompressible Navier–Stokes equations, this is based on the superposition of the slow mode and the fast mode that represent the interaction of the inviscid and viscous effects near the boundary.

2.1. Slow Mode

In this subsection, we will construct the slow mode to capture the inviscid behavior. For this, we consider the following system denoted by $\mathcal{I}(\rho, u, v) = \vec{0}$:

$$\begin{cases} i\alpha(U_s - c)\rho + \mathrm{div}_\alpha(u, v) = 0, \\ i\alpha(U_s - c)u + m^{-2} i\alpha\rho + v\partial_Y U_s = 0, \\ i\alpha(U_s - c)v + m^{-2}\partial_Y\rho = 0. \end{cases} \tag{2.1}$$

By introducing a new function $\Phi = \frac{i}{\alpha} v$, from $(2.1)_1$, we have

$$u = \partial_Y\Phi - (U_s - c)\rho. \tag{2.2}$$

Substituting this into $(2.1)_2$ yields

$$-m^{-2}A(Y)\rho = (U_s - c)\partial_Y\Phi - \Phi\partial_Y U_s,$$

where

$$A(Y) \stackrel{\text{def}}{=} 1 - m^2(U_s - c)^2. \tag{2.3}$$

Note that for the *uniformly subsonic* boundary layer, that is $m \in (0, 1)$, when $|c| \ll 1$, $A(Y)$ is invertible so that we can represent ρ in terms of Φ by

$$\rho = -m^2 A^{-1}(Y)\left[(U_s - c)\partial_Y\Phi - \Phi\partial_Y U_s\right]. \tag{2.4}$$

Plugging (2.4) into $(2.1)_3$, we derive the following equation for Φ, which can be viewed as an analogy of the classical Rayleigh equation in the compressible setting:

$$\mathrm{Ray}_{\mathrm{CNS}} \stackrel{\text{def}}{=} \partial_Y\left\{A^{-1}\left[(U_s - c)\partial_Y\Phi - \Phi\partial_Y U_s\right]\right\} - \alpha^2(U_s - c)\Phi = 0. \tag{2.5}$$

We remark that the equation (2.5) was firstly derived by Lees-Lin in [19] for the study of stability of shear flow in inviscid fluid. Thus (2.5) is sometimes referred to as Lees-Lin equation.

The slow mode will be constructed based on an approximate solution to (2.5). Since (2.5) has similar structure of the Rayleigh equation, the construction is similar as [7,22] for incompressible flow. In what follows we sketch the key steps to make the paper to be self-contained.

Starting from $\alpha = 0$, the equation (2.5) admits following two independent solutions

$$\begin{aligned}
&\varphi_+(Y) = (U_s - c),\\
&\varphi_-(Y) = (U_s - c)\int_1^Y \frac{1}{(U_s(X)-c)^2}\mathrm{d}X - m^2(U_s - c)Y, \ \text{for } \mathrm{Im}c > 0.
\end{aligned}$$

For $\alpha > 0$, to capture the decay property of the solution, we set

$$\begin{aligned}
\beta \overset{\text{def}}{=} \alpha A_\infty^{\frac{1}{2}}, \ \text{where } A_\infty = \lim_{Y\to+\infty} A(Y) &= 1 - m^2(1-c)^2\\
&= 1 - m^2 + O(1)|c|, \ \text{for } |c| \ll 1.
\end{aligned} \tag{2.6}$$

Then we define

$$\varphi_{+,\alpha}(Y) = e^{-\beta Y}\varphi_+(Y), \ \varphi_{-,\alpha}(Y) = e^{-\beta Y}\varphi_-(Y). \tag{2.7}$$

Direct computation yields the following error terms:

$$\mathrm{Ray}_{\mathrm{CNS}}\left(\varphi_{+,\alpha}\right) = -2\beta A^{-2}\partial_Y U_s\varphi_{+,\alpha} + A^{-1}(U_s - c)(\beta^2 - \alpha^2 A)\varphi_{+,\alpha}, \tag{2.8}$$

$$\mathrm{Ray}_{\mathrm{CNS}}\left(\varphi_{-,\alpha}\right) = -2\beta A^{-2}\partial_Y U_s\varphi_{-,\alpha} + A^{-1}(U_s - c)(\beta^2 - \alpha^2 A)\varphi_{-,\alpha} - 2\beta e^{-\beta Y}. \tag{2.9}$$

To have a better approximate solution for (2.5) up to $O(\alpha^2)$, the following approximate Green's function is needed:

$$G_\alpha(X,Y) \overset{\text{def}}{=} -(U_s(X)-c)^{-1}\begin{cases} e^{-\beta(Y-X)}\varphi_+(Y)\varphi_-(X), & X < Y,\\ e^{-\beta(Y-X)}\varphi_+(X)\varphi_-(Y), & X > Y.\end{cases}$$

Then we define a corrector

$$\varphi_{1,\alpha}(Y) \overset{\text{def}}{=} 2\int_0^\infty G(X,Y)A^{-2}(X)\partial_Y U_s(X)\varphi_{+,\alpha}(X)\mathrm{d}X, \tag{2.10}$$

and set

$$\Phi^s_{\mathrm{app}}(Y;c) \overset{\text{def}}{=} \varphi_{+,\alpha} + \beta\varphi_{1,\alpha}. \tag{2.11}$$

Hence, by (2.8) and (2.9), we have

$$\begin{aligned}
\mathrm{Ray}_{\mathrm{CNS}}\left(\Phi^s_{\mathrm{app}}\right) = &- 2\beta^2 A^{-2}U_s'\varphi_{1,\alpha} + 4\beta^2 e^{-\beta Y}\int_Y^\infty A^{-2}(X)U_s'(X)\varphi_+(X)\mathrm{d}X\\
&+ A^{-1}(U_s - c)(\beta^2 - \alpha^2 A)\Phi^s_{\mathrm{app}}\\
=&O(1)\alpha^2|\partial_Y U_s|.
\end{aligned} \tag{2.12}$$

In summary, Φ^s_{app} is the slow mode with properties given in the following lemma:

Lemma 2.1. *Let the Mach number $m \in (0, 1)$. Then for each $Y \geqq 0$, $\Phi^s_{app}(Y; c)$ is holomophic in the upper-half complex plane $\{c \in \mathbb{C} \mid Imc > 0\}$. Moreover, there exists $\gamma_1 \in (0, 1)$, such that if $Imc > 0$ and $|c| < \gamma_1$, the boundary values of Φ^s_{app} have the following expansions:*

$$\Phi^s_{app}(0; c) = -c + \frac{\alpha}{(1-m^2)^{\frac{1}{2}}} + O(1)\alpha|c \log Imc|, \tag{2.13}$$

$$\partial_Y \Phi^s_{app}(0; c) = 1 + O(1)\alpha|\log Imc|. \tag{2.14}$$

Proof. Since $\mathrm{Im}c > 0$, $U_s(Y) - c \neq 0$, $\forall Y \geqq 0$. The analyticity of Φ^s_{app} follows from the explicit formula (2.7), (2.10) and (2.11). Now we derive the boundary values $\Phi^s_{\mathrm{app}}(0; c)$ and $\partial_Y \Phi^s_{\mathrm{app}}(0; c)$.

Firstly, note that

$$\begin{aligned}\varphi_{1,\alpha}(Y; c) = &- 2e^{-\beta Y}\varphi_+(Y)\int_0^Y \varphi_-(X)A^{-2}(X)\partial_Y U_s(X)\mathrm{d}X\\ &- 2e^{-\beta Y}\varphi_-(Y)\int_Y^\infty \varphi_+(X)A^{-2}(X)\partial_Y U_s(X)\mathrm{d}X.\end{aligned} \tag{2.15}$$

Then it holds that

$$\begin{aligned}\varphi_{1,\alpha}(0; c) &= -2\varphi_-(0)\int_0^\infty A^{-2}(X)(U_s - c)\partial_Y U_s(X)\mathrm{d}X\\ &= -\frac{\varphi_-(0)}{m^2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}X}(A^{-1})\mathrm{d}X\\ &= -\frac{\varphi_-(0)}{m^2}\Big(A^{-1}(+\infty) - A^{-1}(0)\Big) = \frac{-\varphi_-(0)(1-2c)}{[1-m^2(1-c)^2][1-m^2c^2]}\\ &= -\varphi_-(0)\left(\frac{1}{1-m^2} + O(1)|c|\right), \text{ for } |c| \ll 1.\end{aligned} \tag{2.16}$$

Then by using (2.11) and the fact that $\beta = \alpha[(1-m^2)^{\frac{1}{2}} + O(1)|c|]$, one has

$$\Phi^s_{\mathrm{app}}(0; c) = \varphi_{+,\alpha}(0, c) + \beta\varphi_{1,\alpha}(0, c) = -c - \alpha\varphi_-(0)\left(\frac{1}{(1-m^2)^{\frac{1}{2}}} + O(1)|c|\right). \tag{2.17}$$

To estimate the boundary value $\partial_Y \Phi^s_{\mathrm{app}}(0; c)$, differentiating (2.15) yields that

$$\begin{aligned}\partial_Y \varphi_{1,\alpha}(Y; c) = &- \beta\varphi_{1,\alpha}(Y; c) + U_s'(Y)(U_s - c)^{-1}\varphi_{1,\alpha}(Y; c)\\ &- 2e^{-\beta Y}A(Y)(U_s - c)^{-1}\int_Y^\infty \varphi_+(X)A^{-2}(X)\partial_Y U_s(X)\mathrm{d}X.\end{aligned}$$

Similar to (2.16), by using $U_s'(0) = 1$, we obtain

$$\begin{aligned}\partial_Y \varphi_{1,\alpha}(0;c) &= -\beta\varphi_{1,\alpha}(0;c) - c^{-1}\varphi_{1,\alpha}(0;c)\\&\quad + 2c^{-1}A(0)\int_0^\infty \varphi_+(X)A^{-2}(X)\partial_Y U_s(X)\mathrm{d}X\\&= -\beta\varphi_{1,\alpha}(0;c) - c^{-1}\varphi_{1,\alpha}(0;c) + c^{-1}A(0)\left(\frac{1}{1-m^2} + O(1)|c|\right)\\&= \frac{1}{c(1-m^2)}(1+\varphi_-(0)) + O(1)(1+|\varphi_-(0)|).\end{aligned}$$

Here we have used (2.16) in the last identity. Consequently, it holds that

$$\begin{aligned}\partial_Y \Phi^s_{\mathrm{app}}(0;c) &= \partial_Y\varphi_{+,\alpha}(0;c) + \beta\partial_Y\varphi_{1,\alpha}(0;c)\\&= 1 + \frac{\alpha}{c(1-m^2)^{\frac{1}{2}}}(1+\varphi_-(0)) + O(1)|\alpha|(1+|\varphi_-(0)|).\end{aligned} \tag{2.18}$$

Finally, we have $\varphi_-(0) = -1 + O(1)|c\log \mathrm{Im} c|$, cf. Lemma 3.1 in [22]. Then by substituting this into (2.17) and (2.18), we obtain (2.13) and (2.14). The proof of Lemma 2.1 is completed. □

With Φ^s_{app}, we can define the slow mode of fluid quantities $\vec{\Xi}^s_{\mathrm{app}} \overset{\mathrm{def}}{=} (\rho^s_{\mathrm{app}}, u^s_{\mathrm{app}}, v^s_{\mathrm{app}})$ by using (2.2) and (2.4) as follows:

$$\begin{aligned}&v^s_{\mathrm{app}} = -i\alpha\Phi^s_{\mathrm{app}},\ \rho^s_{\mathrm{app}} = -m^2A^{-1}\left[(U_s-c)\partial_Y\Phi^s_{\mathrm{app}} - \Phi^s_{\mathrm{app}}\partial_Y U_s\right],\\&u^s_{\mathrm{app}} = \partial_Y\Phi^s_{\mathrm{app}} - (U_s-c)\rho^s_{\mathrm{app}}.\end{aligned} \tag{2.19}$$

One can directly check that $\vec{\Xi}^s_{\mathrm{app}}$ satisfies

$$\mathcal{G}(\vec{\Xi}^s_{\mathrm{app}}) = (0, 0, \mathrm{Ray}_{\mathrm{CNS}}(\Phi^s_{\mathrm{app}})), \tag{2.20}$$

where the error $\mathrm{Ray}_{\mathrm{CNS}}(\Phi^s_{\mathrm{app}})$ is given in (2.12). Therefore, $\vec{\Xi}^s_{\mathrm{app}}$ is an approximate solution to the inviscid equation (2.1) up to $O(\alpha^2)$.

2.2. Fast Mode

To capture the viscous effect of (1.3) in the boundary layer, we need to construct a boundary sublayer corresponding to the fast mode in the approximate solution. Let $z \overset{\mathrm{def}}{=} \delta^{-1}Y$. Here $0 < \delta \ll 1$ is the scale of boundary sublayer which will be determined later. Now we scale the density and velocity fields in the sublayer by setting

$$\mathfrak{p}(z) = \rho(Y),\ \ \mathcal{U}(z) = u(Y),\ \ \mathcal{V}(z) = (i\alpha\delta)^{-1}v(Y). \tag{2.21}$$

This leads to the following rescaled system associated to (1.3):

$$(U_s - c)\mathfrak{p} + \mathcal{U} + \partial_z \mathcal{V} = 0, \tag{2.22}$$

$$\partial_z^2 \mathcal{U} - in\delta^2 (U_s - c)\mathcal{U} - in\delta^3 U_s' \mathcal{V} - (m^{-2} in\delta^2 + \delta^2 \partial_Y^2 U_s)\mathfrak{p}$$
$$- \alpha^2\delta^2 \left[(1+\lambda)\mathcal{U} + \lambda \partial_z \mathcal{V}\right] = 0, \tag{2.23}$$

$$\partial_z^2 \mathcal{V} - in\delta^2 (U_s - c)\mathcal{V} - \alpha^2\delta^2 \mathcal{V} + \lambda\partial_z(\mathcal{U} + \partial_z \mathcal{V}) + im^{-2}\varepsilon^{-\frac{1}{2}}\alpha^{-1}\partial_z \mathfrak{p} = 0. \tag{2.24}$$

Here the constant is $n \stackrel{\text{def}}{=} \frac{\alpha}{\sqrt{\varepsilon}}$, which is the rescaled frequency. Recalling $U_s'(0) = 1$, we can rewrite $U_s(Y) - c$ as

$$U_s(Y) - c = U_s'(0)Y - c + [U_s(Y) - U_s'(0)Y] = \delta(z + z_0) + O(1)|\delta|^2|z|^2, \tag{2.25}$$

where $z_0 \stackrel{\text{def}}{=} -\delta^{-1}c$, and

$$U_s'(Y) = U_s'(0) + U_s'(Y) - U_s'(0) = 1 + O(1)|\delta||z|. \tag{2.26}$$

In view of (2.22)–(2.26), it is natural to set

$$\delta = e^{-\frac{1}{6}\pi i} n^{-\frac{1}{3}},$$

so that $in\delta^3 = 1$. Formally, we have the expansion

$$\mathfrak{p} = \mathfrak{p}_0 + \delta\mathfrak{p}_1 + \cdots, \quad \mathcal{U} = \mathcal{U}_0 + \delta\mathcal{U}_1 \cdots, \quad \mathcal{V} = \mathcal{V}_0 + \delta\mathcal{V}_1 \cdots.$$

Inserting this expansion into (2.22)–(2.24) and taking the leading order, we can derive the following system for $(\mathfrak{p}_0, \mathcal{U}_0, \mathcal{V}_0)$

$$\mathfrak{p}_0(z) = \mathcal{U}_0(z) + \partial_z \mathcal{V}_0(z) = 0, \tag{2.27}$$

$$\partial_z^2 \mathcal{U}_0(z) - (z + z_0)\mathcal{U}_0(z) - \mathcal{V}_0(z) = 0, \tag{2.28}$$

where the variable z lies in the segment $e^{\frac{1}{6}\pi i}\mathbb{R}_+$. From (2.27), we observe that the leading order terms of the density and divergence of velocity field vanish in the sublayer. We also require $(\mathcal{U}_0, \mathcal{V}_0)$ to concentrate near the boundary, that is,

$$\lim_{z\to\infty, z\in e^{\frac{1}{6}\pi i}\mathbb{R}_+} (\mathcal{U}_0, \mathcal{V}_0) = \vec{0}.$$

Differentiating (2.28), by (2.27), we obtain

$$\partial_z^4 \mathcal{V}_0 - (z + z_0)\partial_z^2 \mathcal{V}_0 = 0. \tag{2.29}$$

Therefore, from (2.27) and (2.29) we have

$$\mathcal{U}_0(z) = -\frac{\mathrm{Ai}(1, z + z_0)}{\mathrm{Ai}(2, z_0)}, \quad \mathcal{V}_0(z) = \frac{\mathrm{Ai}(2, z + z_0)}{\mathrm{Ai}(2, z_0)}.$$

Here $\mathrm{Ai}(1,z)$ and $\mathrm{Ai}(2,z)$ are respectively the first and second order primitives of the classical Airy function $\mathrm{Ai}(z)$ which is the solution to Airy equation

$$\partial_z^2 \mathrm{Ai} - z\mathrm{Ai} = 0.$$

$\mathrm{Ai}(2,z)$, $\mathrm{Ai}(1,z)$ and $\mathrm{Ai}(z)$ all vanish at infinity along $e^{\frac{1}{6}\pi i}\mathbb{R}_+$. They satisfy the relations $\partial_z \mathrm{Ai}(k,z) = \mathrm{Ai}(k-1,z)$, $k=1,2$, where $\mathrm{Ai}(0,z) \equiv \mathrm{Ai}(z)$. For the detailed construction of these profiles, we refer to [12].

Finally, by rescaling the leading order profile $(\mathfrak{p}_0, \mathcal{U}_0, \mathcal{V}_0)$ via (2.21), we define the fast mode as

$$\vec{\Xi}^f_{\mathrm{app}} = (\rho^f_{\mathrm{app}}, u^f_{\mathrm{app}}, v^f_{\mathrm{app}})(Y) \stackrel{\mathrm{def}}{=} (0, \mathcal{U}_0, i\alpha\delta\mathcal{V}_0)(\delta^{-1}Y). \tag{2.30}$$

Obviously,

$$u^f_{\mathrm{app}}(0;c) = -\frac{\mathrm{Ai}(1,z_0)}{\mathrm{Ai}(2,z_0)}, \ v^f_{\mathrm{app}}(0;c) = i\alpha\delta. \tag{2.31}$$

2.3. Approximate Growing Mode

Based on slow and fast modes constructed in the above two subsections, we are now ready to construct an approximate growing mode to (1.3) with boundary condition (1.4). Set

$$\begin{aligned}\vec{\Xi}_{\mathrm{app}}(Y;c) = (\rho_{\mathrm{app}}, u_{\mathrm{app}}, v_{\mathrm{app}})(Y;c) &\stackrel{\mathrm{def}}{=} \vec{\Xi}^s_{\mathrm{app}}(Y;c) - \frac{v^s_{\mathrm{app}}(0;c)}{v^f_{\mathrm{app}}(0;c)}\vec{\Xi}^f_{\mathrm{app}}(Y;c)\\ &= \vec{\Xi}^s_{\mathrm{app}}(Y;c) + \delta^{-1}\Phi^s_{\mathrm{app}}(0;c)\vec{\Xi}^f_{\mathrm{app}}(Y;c),\end{aligned} \tag{2.32}$$

where $\vec{\Xi}^s_{\mathrm{app}}$, $\vec{\Xi}^f_{\mathrm{app}}$ are defined in (2.19), (2.30) respectively, and the function $\Phi^s_{\mathrm{app}}(Y;c)$ is defined in (2.11) with boundary data satisfying (2.13) and (2.14). Thanks to (2.31), the normal velocity v_{app} satisfies the zero boundary condition, that is, $v_{\mathrm{app}}(0;c) \equiv 0$. Therefore, the approximate solution (2.32) satisfies the full no-slip boundary condition (1.4) if and only if the following function vanishes at some point c:

$$\mathcal{F}_{\mathrm{app}}(c) \stackrel{\mathrm{def}}{=} u_{\mathrm{app}}(0;c) = \partial_Y\Phi^s_{\mathrm{app}}(0;c) + c\rho^s_{\mathrm{app}}(0;c) - \delta^{-1}\Phi^s_{\mathrm{app}}(0;c)\frac{\mathrm{Ai}(1,z_0(c))}{\mathrm{Ai}(2,z_0(c))}.$$

To find the zero point of $\mathcal{F}_{\mathrm{app}}(c)$, we consider the Mach number $m \in (0,1)$ and the wave number $\alpha = K\varepsilon^{\frac{1}{8}}$ with $K \geqq 1$ being a large but fixed real number. Set

$$c_0 \stackrel{\mathrm{def}}{=} \left(\frac{K}{(1-m^2)^{\frac{1}{2}}} + K^{-1}(1-m^2)^{\frac{1}{4}}e^{\frac{1}{4}\pi i}\right)\varepsilon^{\frac{1}{8}} \tag{2.33}$$

and define a disk centered at c_0 by

$$D_0 \stackrel{\mathrm{def}}{=} \left\{c \in \mathbb{C} \mid |c - c_0| \leqq K^{-1-\theta}(1-m^2)^{\frac{1}{4}}\varepsilon^{\frac{1}{8}}\right\}, \tag{2.34}$$

with some constant $\theta \in (0,1)$. Clearly, for any $m \in (0,1)$, there exists a positive constant $\tau_0 > 0$ ($\tau_0 \to 0$ as $m \to 1$), such that for sufficiently large K, the following estimates hold for any $c \in D_0$:

$$\begin{aligned} &\mathrm{Im} c \geqq \tau_0 K^{-1} \varepsilon^{\frac{1}{8}},\ 0 < \arg c < \tau_0 K^{-2}, \\ &\text{and } \frac{K(1-\tau_0 K^{-2})}{(1-m^2)^{\frac{1}{2}}} \varepsilon^{\frac{1}{8}} \leqq |c| \leqq \frac{K(1+\tau_0 K^{-2})}{(1-m^2)^{\frac{1}{2}}} \varepsilon^{\frac{1}{8}}. \end{aligned} \tag{2.35}$$

With the above preparation, we will prove the following proposition about the existence of approximate growing mode:

Proposition 2.2. *Let $m \in (0,1)$. There exists a positive constant $K_0 > 1$, such that if $K \geqq K_0$, then there exists $\varepsilon_1 \in (0,1)$, such that for $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0$ with $\varepsilon \in (0, \varepsilon_1)$, the function $\mathcal{F}_{app}(c)$ has a unique zero point in D_0. Moreover, on the circle ∂D_0, it holds that*

$$|\mathcal{F}_{app}(c)| \geqq \frac{1}{2} K^{-\theta}. \tag{2.36}$$

Proof. The proof follows the approach used in Proposition 3.2 in the authors' paper [22] with Liu on the incompressible MHD system. For completeness, we sketch the main steps as follows. Firstly, we take K_0 sufficiently large so that (2.35) holds in the disk D_0. Then for $\alpha = K\varepsilon^{\frac{1}{8}}$ and any $c \in D_0$, by (2.13), (2.14) and (2.35), we have

$$\begin{aligned} \Phi^s_{\mathrm{app}}(0;c) &= -c + \frac{\alpha}{(1-m^2)^{\frac{1}{2}}} + O(1)\varepsilon^{\frac{1}{4}}|\log\varepsilon|, \\ \partial_Y \Phi^s_{\mathrm{app}}(0;c) &= 1 + O(1)\varepsilon^{\frac{1}{8}}|\log\varepsilon|. \end{aligned} \tag{2.37}$$

Thus from (2.4) the expression for ρ^s_{app}, one obtains

$$\begin{aligned} \rho^s_{\mathrm{app}}(0;c) &= m^2 A^{-1}(0) \left(c\partial_Y \Phi^s_{\mathrm{app}}(0;c) + U_s'(0)\Phi^s_{\mathrm{app}}(0;c) \right) \\ &= \frac{m^2}{1-m^2c^2} \left(\frac{\alpha}{(1-m^2)^{\frac{1}{2}}} + O(1)\varepsilon^{\frac{1}{4}}|\log\varepsilon| \right) \\ &= O(1)\varepsilon^{\frac{1}{8}}. \end{aligned} \tag{2.38}$$

Next, we consider the ratio $\frac{\mathrm{Ai}(1,z_0)}{\mathrm{Ai}(2,z_0)}$. Recall $z_0(c) = -\delta^{-1}c = -e^{\frac{1}{6}\pi i} K^{\frac{1}{3}} \varepsilon^{-\frac{1}{8}} c$. (2.35) implies that

$$|z_0| = \frac{K^{\frac{4}{3}}}{(1-m^2)^{\frac{1}{2}}}(1+\tau_0 K^{-2}), \text{ and } -\frac{5}{6}\pi < \arg z_0 < -\frac{5}{6}\pi + \tau_0 K^{-2}, \quad \forall c \in D_0. \tag{2.39}$$

Then using the asymptotic behavior of Airy profile (for example cf. [12]) and by (2.39), we obtain

$$\frac{\mathrm{Ai}(1,z_0)}{\mathrm{Ai}(2,z_0)} = -z_0^{\frac{1}{2}} + O(1)|z_0|^{-1} = -\frac{K^{\frac{2}{3}}}{(1-m^2)^{\frac{1}{4}}} e^{-\frac{5}{12}\pi i} + O(1)K^{-\frac{4}{3}}, \quad K \gg 1. \tag{2.40}$$

Now we set

$$\mathcal{F}_{\mathrm{ref}}(c) \stackrel{\mathrm{def}}{=} 1 + e^{-\frac{1}{4}\pi i} K (1-m^2)^{-\frac{1}{4}} \varepsilon^{-\frac{1}{8}} \left(-c + \frac{\alpha}{(1-m^2)^{\frac{1}{2}}} \right). \tag{2.41}$$

On one hand, there exists a unique zero point c_0 in (2.33) to the mapping $\mathcal{F}_{\mathrm{ref}}(c)$. On the boundary ∂D_0, it holds that

$$|\mathcal{F}_{\mathrm{ref}}(c)| = K^{-\theta}. \tag{2.42}$$

On the other hand, we can show that $\mathcal{F}_{\mathrm{ref}}(c)$ is the leading order of $\mathcal{F}_{\mathrm{app}}(c)$. In fact, by (2.37), (2.38) and (2.40), we have the following estimate on the difference:

$$\begin{aligned}
\left|\mathcal{F}_{\mathrm{app}}(c) - \mathcal{F}_{\mathrm{ref}}(c)\right| &\leqq \left| 1 + e^{\frac{1}{6}\pi i} K^{\frac{1}{3}} \varepsilon^{-\frac{1}{8}} \left(c - \frac{\alpha}{(1-m^2)^{\frac{1}{2}}} \right) \frac{\mathrm{Ai}(1,z_0)}{\mathrm{Ai}(2,z_0)} - \mathcal{F}_{\mathrm{ref}}(c) \right| \\
&\quad + C_{K,m} \varepsilon^{\frac{1}{8}} |\log \varepsilon| \\
&\leqq \left| 1 + e^{\frac{1}{6}\pi i} K^{\frac{1}{3}} \varepsilon^{-\frac{1}{8}} \left(c - \frac{\alpha}{(1-m^2)^{\frac{1}{2}}} \right) z_0^{\frac{1}{2}} - \mathcal{F}_{\mathrm{ref}}(c) \right| \\
&\quad + C_m K^{-2} + C_{K,m} \varepsilon^{\frac{1}{8}} |\log \varepsilon| \\
&\leqq C_m K^{-2} + C_{K,m} \varepsilon^{\frac{1}{8}} |\log \varepsilon|.
\end{aligned}$$

Here the positive constants C_m are independent of K and ε and $C_{K,m}$ depends on K and Mach number m, but not on ε. Now we take K_0 larger if needed and then take $\varepsilon_1 \in (0,1)$ suitably small such that for $\varepsilon \in (0, \varepsilon_1)$ and $K \geqq K_0$, it holds that

$$C_m K^{-2} + C_{K,m} \varepsilon^{\frac{1}{8}} |\log \varepsilon| < \frac{1}{2} K^{-\theta}.$$

Consequently, one obtains

$$|\mathcal{F}_{\mathrm{app}}(c) - \mathcal{F}_{\mathrm{ref}}(c)| \leqq \frac{1}{2} |\mathcal{F}_{\mathrm{ref}}(c)|, \ \forall c \in \partial D_0.$$

Combining this with (2.42) yields (2.36). Moreover, since $\mathrm{Ai}(1,z)$ and $\mathrm{Ai}(2,z)$ are analytic functions and $\mathrm{Ai}(2,z_0) \neq 0$ due to (2.40), both $\mathcal{F}_{\mathrm{app}}(c)$ and $\mathcal{F}_{\mathrm{ref}}(c)$ are analytic in D_0. Therefore, by Rouché's Theorem, $\mathcal{F}_{\mathrm{app}}(c)$ and $\mathcal{F}_{\mathrm{ref}}(c)$ have the same number of zeros in D_0. The proof of Proposition 2.2 is then completed. □

We now conclude this subsection by summarizing the relations between the parameters n (rescaled frequency), δ (scale of sublayer), α (wave number) and ε (viscosity) that will be used frequently later:

$$n = \frac{\alpha}{\sqrt{\varepsilon}}; \text{ and } \delta = e^{-\frac{1}{6}\pi i} n^{-\frac{1}{3}}.$$

If, in particular, $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0$, then

$$\alpha \approx |c| \approx \mathrm{Im} c \approx |\delta| \approx n^{-\frac{1}{3}} \approx \varepsilon^{\frac{1}{8}}, \tag{2.43}$$

where the relations may depend on K but not on ε.

2.4. Estimates on Error Terms

In this subsection, we will give the detailed estimate on the error of the approximate solution (2.32) by using a decomposition that takes the decay and regularity in Y into consideration.

<u>"*Regular + Smallness*" *decomposition*</u> : Precisely, the approximate solution $\vec{\Xi}_{\mathrm{app}}$ to (1.3) has the following error representation:

$$\mathcal{L}(\vec{\Xi}_{\mathrm{app}}) = (0, 0, E_{v,\mathrm{re}}) + (0, E_{u,\mathrm{sm}}, E_{v,\mathrm{sm}}). \tag{2.44}$$

Here the regular part

$$E_{v,\mathrm{re}} = \mathrm{Ray}_{\mathrm{CNS}}(\Phi^s_{\mathrm{app}}) \tag{2.45}$$

with $\mathrm{Ray}_{\mathrm{CNS}}(\Phi^s_{\mathrm{app}})$ defined in (2.12). Observe that $E_{v,\mathrm{re}}$ has strong decay in Y due to the background boundary layer profile, and the smallness part reads

$$\begin{aligned} E_{u,\mathrm{sm}} &= \sqrt{\varepsilon}\Delta_\alpha u^s_{\mathrm{app}} + \lambda i\alpha\sqrt{\varepsilon}\mathrm{div}_\alpha(u^s_{\mathrm{app}}, v^s_{\mathrm{app}}) - \sqrt{\varepsilon}\partial_Y^2 U_s \rho^s_{\mathrm{app}} + \eta\sqrt{\varepsilon}\alpha^2 u^f_{\mathrm{app}} \\ &\quad -i\alpha\eta(U_s(Y) - U_s'(0)Y)u^f_{\mathrm{app}} - \eta v^f_{\mathrm{app}}(\partial_Y U_s(Y) - \partial_Y U_s(0)), \\ E_{v,\mathrm{sm}} &= \sqrt{\varepsilon}\Delta_\alpha v^s_{\mathrm{app}} + \lambda\sqrt{\varepsilon}\partial_Y \mathrm{div}_\alpha(u^s_{\mathrm{app}}, v^s_{\mathrm{app}}) + \eta\sqrt{\varepsilon}\Delta_\alpha v^f_{\mathrm{app}} - i\alpha\eta(U_s - c)v^f_{\mathrm{app}}, \end{aligned} \tag{2.46}$$

where $\eta \overset{\mathrm{def}}{=} \delta^{-1}\Phi^s_{\mathrm{app}}(0; c)$. As we will see, the error terms $E_{u,\mathrm{sm}}$ and $E_{v,\mathrm{sm}}$ are of higher order in ε than $E_{v,\mathrm{re}}$.

The estimates on these error terms are summarized in the next proposition. Let us first define some weighted Sobolev spaces for later use:

$$\begin{aligned} L^2_w(\mathbb{R}_+) &\overset{\mathrm{def}}{=} \left\{ f \in L^2(\mathbb{R}_+) \,\middle|\, \|f\|_{L^2_w} \overset{\mathrm{def}}{=} \||\partial_Y^2 U_s|^{-\frac{1}{2}} f\|_{L^2} < \infty \right\}, \\ H^N_w(\mathbb{R}_+) &\overset{\mathrm{def}}{=} \left\{ f \in H^N(\mathbb{R}_+) \,\middle|\, \|f\|_{H^N_w} \overset{\mathrm{def}}{=} \sum_{j=0}^{N} \|\partial_Y^j f\|_{L^2_w} < \infty, \; N \text{ is a positive integer} \right\}. \end{aligned} \tag{2.47}$$

Recall $K_0 \geqq 1$ and $\varepsilon_1 \in (0, 1)$ are constants given in Proposition 2.2. For $K \geqq K_0$ and $\varepsilon \in (0, \varepsilon_1)$, the disk D_0 is defined in (2.34). The following proposition gives the precise error bound estimates:

Proposition 2.3. *Let the Mach number $m \in (0,1)$. There exists $\varepsilon_2 \in (0,\varepsilon_1)$, such that for $\varepsilon \in (0,\varepsilon_2)$, $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0$, the error terms $E_{v,re}$, $E_{u,sm}$ and $E_{v,sm}$ satisfy the estimates*

$$\|E_{v,re}(\cdot\,;c)\|_{H^1_w} \lesssim_K \varepsilon^{\frac{3}{16}}, \tag{2.48}$$

$$\|E_{u,sm}(\cdot\,;c)\|_{L^2} + \|E_{v,sm}(\cdot\,;c)\|_{L^2} \lesssim_K \varepsilon^{\frac{7}{16}}, \tag{2.49}$$

Here the constant K is uniform in ε.

The proof of Proposition 2.3 follows from a series of estimations on approximate solutions. First of all, we show some properties of corrector $\varphi_{1,\alpha}$ and the approximate solution Φ^s_{app} to Rayleigh operator that are defined in (2.10) and (2.11) respectively. Fix $m \in (0,1)$ and set $\beta_1 \stackrel{\text{def}}{=} \frac{1}{2}(1-m^2)^{\frac{1}{2}}\alpha$.

Lemma 2.4. *Let γ_1 be the constant given in Lemma 2.1. There exists $\gamma_2 \in (0,\gamma_1)$, such that for any c lies in the half disk $\{c \in \mathbb{C} \mid Imc > 0,\ |c| < \gamma_2\}$ and $\alpha \in (0,1)$, the corrector $\varphi_{1,\alpha}$ satisfies*

$$\|\varphi_{1,\alpha}\|_{L^\infty_{\beta_1}} + \alpha^{\frac{1}{2}}\|\varphi_{1,\alpha}\|_{L^2} \lesssim 1, \tag{2.50}$$

$$\|\partial_Y\varphi_{1,\alpha}\|_{L^\infty_{\beta_1}} \lesssim 1 + |\log Imc|,\ \ \|\partial_Y\varphi_{1,\alpha}\|_{L^2} \lesssim 1, \tag{2.51}$$

$$\|\partial_Y^2\varphi_{1,\alpha}\|_{L^\infty_{\beta_1}} + (Imc)^{-\frac{1}{2}}\|\partial_Y^2\varphi_{1,\alpha}\|_{L^2} \lesssim (Imc)^{-1}, \tag{2.52}$$

$$\|\partial_Y^3\varphi_{1,\alpha}\|_{L^\infty_{\beta_1}} + (Imc)^{-\frac{1}{2}}\|\partial_Y^3\varphi_{1,\alpha}\|_{L^2} \lesssim (Imc)^{-2}. \tag{2.53}$$

Moreover, if, in addition, $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0$, we have the following uniform bounds:

$$\alpha^{\frac{1}{2}}\|\Phi^s_{app}\|_{L^2} + \|\partial_Y\Phi^s_{app}\|_{H^1} + \|\Phi^s_{app}\|_{W^{2,\infty}_{\beta_1}} + (Imc)^{\frac{1}{2}}\|\partial_Y^3\Phi^s_{app}\|_{L^2} + Imc\|\partial_Y^3\Phi^s_{app}\|_{L^\infty_{\beta_1}} \lesssim 1. \tag{2.54}$$

Proof. Recall that $\beta = \alpha\left[(1-m^2)^{\frac{1}{2}} + O(1)|c|\right]$ from (2.6) and $A(Y) = 1 - m^2U_s^2 + O(1)|c|$ from (2.3). Taking $\gamma_2 \in (0,\gamma_1)$ suitably small, we have $\mathrm{Re}\beta > \beta_1$ and $|A^{-1}| \leqq \frac{1}{2(1-m^2)} \lesssim 1$ for $|c| < \gamma_2$. Then the proof of (2.50)–(2.53) follows from an argument exactly as in Lemma 3.6 in [22] by using the explicit expression (2.15). Hence, we omit it for brevity.

For (2.54), we recall (2.11) and observe that

$$\alpha^{\frac{1}{2}}\|\varphi_{+,\alpha}\|_{L^2} + \|\partial_Y\varphi_{+,\alpha}\|_{H^2} + \|\varphi_{+,\alpha}\|_{W^{3,\infty}_{\beta_1}} \lesssim 1. \tag{2.55}$$

By (2.43), we have $\alpha/\mathrm{Im}c \lesssim 1$. Thus putting (2.50)–(2.53) and (2.55) together yields the desired estimate (2.54). The proof of the lemma is then completed. □

By Lemma 2.4, we can immediately obtain the following estimates on the slow mode $\vec{\Xi}^s_{\text{app}}$ given in (2.19):

Corollary 2.5. *If $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0 \cap \{c \in \mathbb{C} \mid |c| < \gamma_2\}$, the slow mode $\vec{\Xi}^s_{app}$ satisfies the following estimates:*

$$m^{-2}\|\rho^s_{app}\|_{H^1} + \|u^s_{app}\|_{H^1} + \|v^s_{app}\|_{H^2} \lesssim 1, \tag{2.56}$$

$$m^{-2}\|\partial_Y^2\rho^s_{app}\|_{L^2} + \|\partial_Y^2 u^s_{app}\|_{L^2} \lesssim (Imc)^{-\frac{1}{2}}. \tag{2.57}$$

Proof. The estimate on $v^s_{\rm app}$ follows from (2.54) directly. For $\rho^s_{\rm app}$, using (2.54) gives

$$m^{-2}\|\rho^s_{\rm app}\|_{L^2} \lesssim \|\partial_Y\Phi^s_{\rm app}\|_{L^2} + \|\partial_Y U_s\|_{L^2}\|\Phi^s_{\rm app}\|_{L^\infty} \lesssim 1.$$

Differentiating (2.4) with respect to Y yields the formulas

$$\begin{aligned} m^{-2}\partial_Y\rho^s_{\rm app} &= -m^{-2}\rho^s_{\rm app}A^{-1}A' \\ &\quad -A^{-1}(U_s-c)\partial_Y^2\Phi^s_{\rm app} + \Phi^s_{\rm app}A^{-1}\partial_Y^2 U_s, \end{aligned}$$

and

$$\begin{aligned} m^{-2}\partial_Y^2\rho^s_{\rm app} &= -2m^{-2}A^{-1}\partial_Y A\partial_Y\rho^s_{\rm app} - m^{-2}A^{-1}\partial_Y^2 A\rho^s_{\rm app} \\ &\quad -A^{-1}\partial_Y\left[(U_s-c)\partial_Y^2\Phi^s_{\rm app} - \Phi^s_{\rm app}\partial_Y^2 U_s\right]. \end{aligned}$$

Taking L^2-norm and by (2.54), we obtain

$$m^{-2}\|\partial_Y\rho^s_{\rm app}\|_{L^2} \lesssim m^{-2}\|\rho^s_{\rm app}\|_{L^2} + \|\partial_Y^2\Phi^s_{\rm app}\|_{L^2} + \|\Phi^s_{\rm app}\|_{L^\infty}\|\partial_Y^2 U_s\|_{L^2} \lesssim 1,$$

and

$$\begin{aligned} m^{-2}\|\partial_Y^2\rho^s_{\rm app}\|_{L^2} &\lesssim m^{-2}\|(\rho^s_{\rm app}, \partial_Y\rho^s_{\rm app})\|_{L^2} + \|\partial_Y\Phi^s_{\rm app}\|_{H^2} + \|\partial_Y^3 U_s\|_{L^2}\|\Phi^s_{\rm app}\|_{L^\infty} \\ &\lesssim ({\rm Im}c)^{-\frac{1}{2}}. \end{aligned}$$

The velocity field $u^s_{\rm app}$ can be estimated in the same way so that we omit the details. And this completes the proof of the corollary. □

The next lemma gives some pointwise estimates on the fast mode $(u^f_{\rm app}, v^f_{\rm app})$ defined in (2.30). The proof follows from Lemma 3.9 in [22] by using the pointwise estimate of Airy profiles. Thus, we omit the details, for brevity.

Lemma 2.6. *The fast mode (u^f_{app}, v^f_{app}) has the pointwise estimates*

$$\left|\partial_Y^k u^f_{app}(Y;c)\right| \lesssim n^{\frac{k}{3}}e^{-\tau_1 n^{\frac{1}{3}}Y},\ k=0,1,2 \tag{2.58}$$

$$\left|\partial_Y^k v^f_{app}(Y;c)\right| \lesssim n^{\frac{k-2}{3}}e^{-\tau_1 n^{\frac{1}{3}}Y},\ k=0,1,2, \tag{2.59}$$

for some constant $\tau_1 > 0$ which does not depend on n.

With the above estimates, we are now ready to prove Proposition 2.3 as follows:

Proof of Proposition 2.3. We start with proving (2.48) for $E_{v,\mathrm{re}}$. Recall the definition (2.45) and explicit formula (2.12). By taking $\varepsilon_2 > 0$ suitably small, such that $D_0 \subset \{\mathrm{Im} c > 0, |c| \leqq \gamma_2\}$, we have $\mathrm{Re}\beta > \beta_1$ and $|A| \gtrsim 1$. Then thanks to (1.8), (2.6), the bounds (2.50)–(2.54) and the fact that

$$|\beta^2 - \alpha^2 A| \lesssim \alpha^2 |A(+\infty) - A(Y)| \lesssim \alpha^2 |1 - U_s(Y)|,$$

we have

$$\begin{aligned}
|E_{v,\mathrm{re}}(Y)| &\lesssim \alpha^2 \partial_Y U_s(Y) e^{-\beta_1 Y} \|\varphi_{1,\alpha}\|_{L^\infty_{\beta_1}} + \alpha^2 e^{-\beta_1 Y}(1 - U_s(Y)) \\
&\quad + |\beta^2 - \alpha^2 A| e^{-\beta_1 Y} \|\Phi^s_{\mathrm{app}}\|_{L^\infty_{\beta_1}} \\
&\lesssim \alpha^2 \partial_Y U_s(Y) e^{-\beta_1 Y} \|\varphi_{1,\alpha}\|_{L^\infty_{\beta_1}} + \alpha^2 e^{-\beta_1 Y}(1 - U_s(Y))(1 + \|\Phi^s_{\mathrm{app}}\|_{L^\infty_{\beta_1}}) \\
&\lesssim \alpha^2 e^{-\beta_1 Y} \partial_Y U_s(Y).
\end{aligned}$$

This, combined with the concavity (1.9), gives the following weighted estimate:

$$\|E_{v,\mathrm{re}}\|_{L^2_w} \lesssim \alpha^2 \| |\partial_Y^2 U_s|^{-\frac{1}{2}} \partial_Y U_s \|_{L^\infty} \|e^{-\beta_1 Y}\|_{L^2} \lesssim \alpha^{\frac{3}{2}} \lesssim \varepsilon^{\frac{3}{16}}. \tag{2.60}$$

Moreover, differentiating (2.12) yields

$$\begin{aligned}
|\partial_Y E_{v,\mathrm{re}}| \ &\lesssim \alpha^2 \left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s| \right) |\varphi_{1,\alpha}| + \alpha^2 |\partial_Y U_s| \Big(e^{-\beta_1 Y} + |\partial_Y \varphi_{1,\alpha}| \\
&\quad + |\Phi^s_{\mathrm{app}}| + |\partial_Y \Phi^s_{\mathrm{app}}| \Big).
\end{aligned}$$

With this, by the bounds (2.50)–(2.54) and the concavity (1.9), we obtain

$$\begin{aligned}
\|\partial_Y E_{v,\mathrm{re}}\|_{L^2_w} &\lesssim \alpha^2 \|e^{-\beta_1 Y}\|_{L^2} \left(1 + \| |\partial_Y^2 U_s|^{-\frac{1}{2}} \partial_Y U_s \|_{L^\infty} \right) \\
&\qquad \times \left(1 + \|\varphi_{1,\alpha}\|_{W^{1,\infty}_{\beta_1}} + \|\Phi^s_{\mathrm{app}}\|_{W^{1,\infty}_{\beta_1}} \right) \\
&\lesssim \alpha^{\frac{3}{2}} \lesssim \varepsilon^{\frac{3}{16}}.
\end{aligned} \tag{2.61}$$

Putting (2.60) and (2.61) together yields the estimate (2.48) on part of the error with decay.

Now we turn to estimate the part of error with smallness $(E_{u,\mathrm{sm}}, E_{v,\mathrm{sm}})$ which is defined in (2.46). Keeping in mind the bounds of parameters (2.43) and

$$|\eta| = |\delta|^{-1} |\Phi^s_{\mathrm{app}}(0; c)| \lesssim_K 1, \quad \forall c \in D_0,$$

because of (2.13). Note that also $|U_s(Y) - U_s'(0)Y| \lesssim Y^2$, $|\partial_Y U_s(Y) - \partial_Y U_s(0)| \lesssim Y$ and

$$\|\mathrm{div}_\alpha (u^s_{\mathrm{app}}, v^s_{\mathrm{app}})\|_{H^1} \leqq C \|u^s_{\mathrm{app}}\|_{H^1} + C \|v^s_{\mathrm{app}}\|_{H^2}.$$

By the bounds on $(u^s_{\mathrm{app}}, v^s_{\mathrm{app}})$ and $(u^f_{\mathrm{app}}, v^f_{\mathrm{app}})$ given in Corollary 2.5 and Lemma 2.6, we have

$$\begin{aligned}\|E_{u,\mathrm{sm}}\|_{L^2} &\lesssim \sqrt{\varepsilon}(\|u^s_{\mathrm{app}}\|_{H^2} + \|v^s_{\mathrm{app}}\|_{H^2} + \|\rho^s_{\mathrm{app}}\|_{L^2}) + \sqrt{\varepsilon}\alpha^2\|u^f_{\mathrm{app}}\|_{L^2} \\ &\quad + \alpha\|Y^2 u^f_{\mathrm{app}}\|_{L^2} + \|Y v^f_{\mathrm{app}}\|_{L^2} \\ &\lesssim \sqrt{\varepsilon}(1 + |\mathrm{Im} c|^{-\frac{1}{2}}) + \sqrt{\varepsilon}\alpha^2\|e^{-\tau_1 n^{\frac{1}{3}} Y}\|_{L^2} + \alpha\|Y^2 e^{-\tau_1 n^{\frac{1}{3}} Y}\|_{L^2} \\ &\quad + n^{-\frac{2}{3}}\|Y e^{-\tau_1 n^{\frac{1}{3}} Y}\|_{L^2} \\ &\lesssim \sqrt{\varepsilon}(1 + |\mathrm{Im} c|^{-\frac{1}{2}}) + \sqrt{\varepsilon}\alpha^2 n^{-\frac{1}{6}} + \alpha n^{-\frac{5}{6}} + n^{-\frac{7}{6}} \lesssim_K \varepsilon^{\frac{7}{16}}, \end{aligned} \tag{2.62}$$

and

$$\begin{aligned}\|E_{v,\mathrm{sm}}\|_{L^2} &\lesssim \sqrt{\varepsilon}(\|v^s_{\mathrm{app}}\|_{H^2} + \|u^s_{\mathrm{app}}\|_{H^1}) + \sqrt{\varepsilon}\|v^f_{\mathrm{app}}\|_{H^2} \\ &\quad + \alpha\left(|c|\|v^f_{\mathrm{app}}\|_{L^2} + \|Y v^f_{\mathrm{app}}\|_{L^2}\|Y^{-1}U_s\|_{L^\infty}\right) \\ &\lesssim \sqrt{\varepsilon} + \sqrt{\varepsilon}\|e^{-\tau_1 n^{\frac{1}{3}} Y}\|_{L^2} + \alpha n^{-\frac{2}{3}}\left(|c|\|e^{-\tau_1 n^{\frac{1}{3}} Y}\|_{L^2} + \|Y e^{-\tau_1 n^{\frac{1}{3}} Y}\|_{L^2}\right) \\ &\lesssim \sqrt{\varepsilon}(1 + n^{-\frac{1}{6}}) + \alpha|c|n^{-\frac{5}{6}} + \alpha n^{-\frac{7}{6}} \lesssim_K \varepsilon^{\frac{1}{2}}. \end{aligned} \tag{2.63}$$

Estimates (2.62) and (2.63) give the bound (2.49) for $(E_{u,\mathrm{sm}}, E_{v,\mathrm{sm}})$. Then the proof of the proposition is completed. □

3. Solvability of Remainder System

In this section, we will construct a solution to the resolvent problem

$$\begin{cases} i\alpha(U_s - c)\rho + \mathrm{div}_\alpha(u, v) = 0, \\ \sqrt{\varepsilon}\Delta_\alpha u + \lambda i\alpha\sqrt{\varepsilon}\mathrm{div}_\alpha(u, v) - i\alpha(U_s - c)u - v\partial_Y U_s - (m^{-2} i\alpha + \sqrt{\varepsilon}\partial_Y^2 U_s)\rho = f_u, \\ \sqrt{\varepsilon}\Delta_\alpha v + \lambda\sqrt{\varepsilon}\partial_Y \mathrm{div}_\alpha(u, v) - i\alpha(U_s - c)v - m^{-2}\partial_Y\rho = f_v, \\ v|_{Y=0} = 0, \end{cases} \tag{3.1}$$

where (f_u, f_v) is a given inhomogeneous source term. If $(f_u, f_v) \in H^1(\mathbb{R}_+)^2$, we define the operator

$$\Omega(f_u, f_v) \overset{\mathrm{def}}{=} f_v - \frac{1}{i\alpha}\partial_Y(A^{-1} f_u). \tag{3.2}$$

Recall (2.47) the weighted function space $L^2_w(\mathbb{R}_+)$. The following is the main result in this section:

Proposition 3.1. (Solvability of resolvent problem) *Let the Mach number $m \in (0, \frac{1}{\sqrt{3}})$. There exists $\varepsilon_3 \in (0, 1)$, such that for any $\varepsilon \in (0, \varepsilon_3)$, $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0$, the following two statements hold.*

(1) If $(f_u, f_v) \in L^2(\mathbb{R}_+)^2$, then there exists a solution $\vec{\Xi} = (\rho, u, v) \in H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$ to (3.1) which satisfies the following estimates:

$$\|(m^{-1}\rho, u, v)\|_{L^2} \lesssim \frac{1}{\alpha(Imc)^2}\|(f_u, f_v)\|_{L^2}, \tag{3.3}$$

$$\|m^{-2}\partial_Y\rho\|_{L^2} + \|div_\alpha(u, v)\|_{H^1} \lesssim \frac{1}{\alpha(Imc)^2}\|(f_u, f_v)\|_{L^2}, \tag{3.4}$$

$$\|(\partial_Y u, \partial_Y v)\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{\alpha(Imc)^{\frac{1}{2}}}\|(f_u, f_v)\|_{L^2}, \tag{3.5}$$

$$\|(\partial_Y^2 u, \partial_Y^2 v)\|_{L^2} \lesssim \frac{n}{\alpha Imc}\|(f_u, f_v)\|_{L^2}. \tag{3.6}$$

(2) If in addition we have $(f_u, f_v) \in H^1(\mathbb{R}_+)^2$ with $\|\Omega(f_u, f_v)\|_{L^2_w} < \infty$, then there exists a solution $\vec{\Xi} = (\rho, u, v) \in H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$ to (3.1) which satisfies the following improved estimates

$$\|(m^{-1}\rho, u, v)\|_{H^1} \lesssim \frac{1}{Imc}\|\Omega(f_u, f_v)\|_{L^2_w} + \frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|div_\alpha(f_u, f_v)\|_{L^2}, \tag{3.7}$$

$$\|(\partial_Y^2 u, \partial_Y^2 v)\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{(Imc)^{\frac{1}{2}}}\|\Omega(f_u, f_v)\|_{L^2_w} + \frac{1}{Imc}\left(\frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|div_\alpha(f_u, f_v)\|_{L^2}\right). \tag{3.8}$$

Moreover, if $(f_u, f_v)(\cdot\,; c)$ is analytic in c with values in L^2, then the solution $\vec{\Xi}(\cdot\,; c)$ is analytic with values in $H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$.

Remark 3.2. (a) By Sobolev embedding $H^1(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$, the mapping $u(0; c) : D_0 \mapsto \mathbb{C}$ is analytic.
(b) The solutions to (3.1) are in general not unique because we do not prescribe the boundary data at $Y = 0$ for u.
(c) The constants in estimates (3.3)–(3.8) are uniform for $m \in (0, m_0]$ with any $m_0 \in (0, \frac{1}{\sqrt{3}})$.
(d) As one can see from the proof, the argument also works for a wider regime of parameters:

$$|\alpha| \lesssim 1, \ |c| \ll 1, \ \frac{|c|^2}{\mathrm{Im}c} \ll 1, \ \frac{1}{n(\mathrm{Im}c)^2} \ll 1. \tag{3.9}$$

In fact, the boundedness of wave number α is essentially used in the proof. In addition, we require c to satisfy (3.103) so that $c \in \Sigma_Q \cap \Sigma_S$ where Σ_Q and Σ_S are resolvent sets of L_Q and L_S respectively. Moreover, in view of (3.114), we require smallness of $\frac{1}{n(\mathrm{Im}c)^2}$ in order to establish the convergence of iteration. These requirements can be fulfilled by the smallness in (3.9).

As mentioned in the Introduction, the proof of Proposition 3.1 is based on the following two newly introduced decompositions, that is, quasi-compressibile approximation and the Stokes approximation.

3.1. Quasi-Compressible Approximation

Following the strategy described in the Introduction, we first consider the approximate problem

$$\begin{cases} i\alpha(U_s-c)\varrho+\mathrm{div}_\alpha(\mathfrak{u},\mathfrak{v})=0,\\ \sqrt{\varepsilon}\Delta_\alpha\left(\mathfrak{u}+(U_s-c)\varrho\right)-i\alpha(U_s-c)\mathfrak{u}-\mathfrak{v}\partial_Y U_s-i\alpha m^{-2}\varrho=s_1,\\ \sqrt{\varepsilon}\Delta_\alpha\mathfrak{v}-i\alpha(U_s-c)\mathfrak{v}-m^{-2}\partial_Y\varrho=s_2,\\ \mathfrak{v}|_{Y=0}=0,\end{cases} \tag{3.10}$$

with a given inhomogeneous source term (s_1, s_2).

By the continuity equation $(3.10)_1$, we can define an "effective stream function" Ψ satisfying that

$$\partial_Y\Psi=\mathfrak{u}+(U_s-c)\varrho,\ \ -i\alpha\Psi=\mathfrak{v},\ \ \Psi|_{Y=0}=0. \tag{3.11}$$

Then, by $(3.10)_2$, we can express the density ρ in terms of Ψ as

$$m^{-2}\varrho(Y)=-A^{-1}(Y)\left[\frac{i}{n}\Delta_\alpha\partial_Y\Psi+(U_s-c)\partial_Y\Psi-\Psi\partial_Y U_s+(i\alpha)^{-1}s_1\right]. \tag{3.12}$$

Substituting (3.12) into $(3.10)_3$, we derive the following equation for Ψ which can be viewed as the Orr–Sommerfeld equation in the compressible setting:

$$\mathrm{OS}_{\mathrm{CNS}}(\Psi)\overset{\mathrm{def}}{=}\frac{i}{n}\Lambda(\Delta_\alpha\Psi)+(U_s-c)\Lambda(\Psi)-\partial_Y(A^{-1}\partial_Y U_s)\Psi=\Omega(s_1,s_2),\ \ Y>0. \tag{3.13}$$

Here the modified vorticity operator Λ is given by

$$\begin{aligned}&\Lambda: H^2(\mathbb{R}_+)\cap H_0^1(\mathbb{R}_+)\to L^2(\mathbb{R}_+),\\ &\Lambda(\Psi)\overset{\mathrm{def}}{=}\partial_Y(A^{-1}\partial_Y\Psi)-\alpha^2\Psi,\end{aligned} \tag{3.14}$$

and Ω is given in (3.2).

In order to solve (3.13), we consider the following boundary condition:

$$\Psi|_{Y=0}=\Lambda(\Psi)|_{Y=0}=0. \tag{3.15}$$

If Ψ solves problem (3.13) with boundary conditions (3.15), it is straightforward to check that (ρ, u, v) defined by (3.11) and (3.12) is a solution to (3.10).

Thus, in what follows, we consider the boundary value problem

$$\mathrm{OS}_{\mathrm{CNS}}(\Psi)=h,\ \ Y>0,\ \ \Psi|_{Y=0}=\Lambda(\Psi)|_{Y=0}=0, \tag{3.16}$$

with a given inhomogeneous source $h\in L_w^2(\mathbb{R}_+)$. Let us first introduce the multiplier

$$w(Y)\overset{\mathrm{def}}{=}-\left(\partial_Y\left(A^{-1}\partial_Y U_s\right)\right)^{-1}. \tag{3.17}$$

A straightforward computation yields the properties of w stated in the following lemma:

Lemma 3.3. *Let $m \in (0,1)$ and U_s satisfy (1.5)–(1.8). There exists $\gamma_3 > 0$, such that if $|c| < \gamma_3$, $w(Y)$ has the expansion*

$$w = w_0 + c w_1 + O(1)|c|^2 |\partial_Y^2 U_s|^{-1}. \tag{3.18}$$

Here w_0 and w_1 are given by

$$\begin{aligned} w_0 = \frac{(1-m^2U_s^2)^2}{H\left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s|\right)},\ w_1 &= \frac{4m^2 U_s(1-m^2U_s^2)}{H\left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s|\right)} \\ &- \frac{2m^2(1-m^2U_s^2)^2\left(|\partial_Y U_s|^2 - U_s\partial_Y^2 U_s\right)}{H^2\left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s|\right)^2}, \end{aligned} \tag{3.19}$$

where the function $H(Y)$ is defined in (1.7). Moreover, it holds that

$$w_0(Y) \approx |w(Y)| \approx |\partial_Y^2 U_s(Y)|^{-1}. \tag{3.20}$$

Set the function space

$$\mathbb{X} \overset{\text{def}}{=} \Big\{ \Psi \in H^3(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+),\ \Lambda(\Psi)|_{Y=0} = 0 \\ \Big|\ \|\partial_Y \Psi, \alpha\Psi\|_{L^2} + \|\Lambda(\Psi)\|_{L^2_w} + \|\partial_Y \Lambda(\Psi)\|_{L^2_w} < \infty \Big\}.$$

For the problem (3.16), we have

Lemma 3.4. (A priori estimates) *Let $m \in (0, \frac{1}{\sqrt{3}})$ and $\Psi \in \mathbb{X}$ be a solution to (3.16). There exists $\gamma_4 \in (0,\gamma_3)$, such that for $\alpha \in (0,1)$ and c lies in*

$$\Sigma_Q \overset{def}{=} \{c \in \mathbb{C} \mid Imc > \max\{\gamma_4^{-1}|c|^2,\ \gamma_4^{-1}n^{-1}\},\ |c| < \gamma_4\}, \tag{3.21}$$

then Ψ satisfies the following estimates

$$\|(\partial_Y\Psi, \alpha\Psi)\|_{L^2} + \|\Lambda(\Psi)\|_{L^2_w} \leqq \frac{C}{Imc}\|h\|_{L^2_w}, \tag{3.22}$$

$$\|\left(\partial_Y\Lambda(\Psi), \alpha\Lambda(\Psi)\right)\|_{L^2_w} \leqq \frac{Cn^{\frac{1}{2}}}{(Imc)^{\frac{1}{2}}}\|h\|_{L^2_w}. \tag{3.23}$$

Proof. Taking inner product of (3.16) with the multiplier $-w\overline{\Lambda(\Psi)}$ leads to

$$\underbrace{-\frac{i}{n}\int_0^\infty w\Lambda(\Delta_\alpha\Psi)\overline{\Lambda(\Psi)}\mathrm{d}Y}_{J_1} + \underbrace{\int_0^\infty -(U_s - c)w|\Lambda(\Psi)|^2\mathrm{d}Y}_{J_2} + \underbrace{\int_0^\infty -\Psi\overline{\Lambda(\Psi)}\mathrm{d}Y}_{J_3} \\ + \underbrace{\int_0^\infty hw\overline{\Lambda(\Psi)}\mathrm{d}Y}_{J_4} = 0. \tag{3.24}$$

Now we estimate $J_1 - J_4$ separately. Let us consider J_3 first. By integrating by parts and using the boundary condition $\Psi|_{Y=0} = 0$, we obtain

$$J_3 = \int_0^\infty \bar{A}^{-1}|\partial_Y \Psi|^2 + \alpha^2|\Psi|^2 \mathrm{d}Y. \tag{3.25}$$

Recalling (2.3) about the definition of $A(Y)$, we have

$$\bar{A}^{-1} = (1 - m^2U_s^2)^{-1} - 2m^2U_s(1 - m^2U_s^2)^{-2}\bar{c} + O(1)|c|^2.$$

With this identity, the assumption $m \in (0, 1)$, and (1.5) for the positivity of U_s, we can deduce from (3.25) that

$$\mathrm{Re}J_3 \gtrsim \alpha^2\|\Psi\|_{L^2}^2 + (1 - O(1)|c|)\|\partial_Y\Psi\|_{L^2}^2, \tag{3.26}$$

and

$$\mathrm{Im}J_3 \gtrsim \mathrm{Im}c\|m|U_s|^{\frac{1}{2}}\partial_Y\Psi\|_{L^2}^2 - O(1)|c|^2\|\partial_Y\Psi\|_{L^2}^2, \tag{3.27}$$

where the constants may depend on m but not on either ε or c.

For J_2, we obtain from the expansion (3.18) and bound (3.20) that

$$-(U_s - c)w = -U_s w_0 + (w_0 - U_s w_1)\,c + O(1)|c|^2|\partial_Y^2 U_s|^{-1}. \tag{3.28}$$

Using the explicit formula (3.19) of w_0 and w_1 gives

$$\begin{aligned} w_0 - U_s w_1 &= \frac{(1 - m^2U_s^2)}{H^2\left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s|\right)}\Big\{(1 - 5m^2U_s^2)H \\ &\quad + \frac{2m^2(1 - m^2U_s^2)(U_s|\partial_Y U_s|^2 - U_s^2\partial_Y^2 U_s)}{|\partial_Y U_s|^2 + |\partial_Y^2 U_s|}\Big\} \\ &= \frac{(1 - m^2U_s^2)}{H^2\left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s|\right)^2}\Big\{(1 - 5m^2U_s^2)\Big[(1 - m^2U_s^2)|\partial_Y^2 U_s| \\ &\quad - 2m^2U_s|\partial_Y U_s|^2\Big] + 2m^2(1 - m^2U_s^2)(U_s|\partial_Y U_s|^2 + U_s^2|\partial_Y^2 U_s|)\Big\} \\ &= \frac{(1 - m^2U_s^2)}{H^2\left(|\partial_Y U_s|^2 + |\partial_Y^2 U_s|\right)^2}\Big\{(1 - 3m^2U_s^2)(1 - m^2U_s^2)|\partial_Y^2 U_s| \\ &\quad + 8m^4U_s^3|\partial_Y U_s|^2\Big\}. \end{aligned} \tag{3.29}$$

Since $m \in (0, \frac{1}{\sqrt{3}})$, we have $(1 - 3m^2U_s^2) \geqq (1 - 3m^2) > 0$. Thus, by (1.9) and (3.29), it holds that

$$C_1|\partial_Y^2 U_s|^{-1} \leqq w_0 - U_s w_1 \leqq C_2|\partial_Y^2 U_s|^{-1}, \tag{3.30}$$

where the positive constants C_1 and C_2 are uniform in ε and c. Therefore, taking real and imaginary part of J_2 respectively and using the bounds (3.20), (3.28) and (3.30) yield

$$|\mathrm{Re}J_2| \lesssim C\|\Lambda(\Psi)\|_{L_w^2}^2, \tag{3.31}$$

and

$$\mathrm{Im} J_2 \gtrsim \left(\mathrm{Im} c - O(1)|c|^2\right) \|\Lambda(\Psi)\|^2_{L^2_w}. \tag{3.32}$$

For J_1, we rewrite

$$J_1 = \frac{-i}{n}\int_0^\infty w\Delta_\alpha \Lambda(\Psi)\overline{\Lambda(\Psi)}\mathrm{d}Y + \frac{i}{n}\int_0^\infty w[\Delta_\alpha, \Lambda](\Psi)\overline{\Lambda(\Psi)}\mathrm{d}Y := J_{11} + J_{12}, \tag{3.33}$$

where $[\Delta_\alpha, \Lambda](\Psi)$ is the commutator $\Delta_\alpha[\Lambda(\Psi)] - \Lambda[\Delta_\alpha(\Psi)]$. By integrating by parts and using the boundary condition $\Lambda(\Psi)|_{Y=0} = 0$, we obtain

$$J_{11} = \frac{i}{n}\int_0^\infty w\left(|\partial_Y \Lambda(\Psi)|^2 + \alpha^2|\Lambda(\Psi)|^2\right)\mathrm{d}Y + \frac{i}{n}\int_0^\infty \partial_Y w \partial_Y \Lambda(\Psi)\overline{\Lambda(\Psi)}\mathrm{d}Y. \tag{3.34}$$

Then by (1.8) and (3.20), we have

$$\begin{aligned}\left\|\partial_Y w \partial_Y^2 U_s\right\|_{L^\infty} &= \|w^2 \partial_Y^2(A^{-1}\partial_Y U_s)\partial_Y^2 U_s\|_{L^\infty} \\ &\leqq \|w\partial_Y^2 U_s\|^2_{L^\infty}\left(\|\partial_Y U_s\|_{L^\infty} + \||\partial_Y^2 U_s|^{-1}|\partial_Y U_s|^2\|_{L^\infty}\right. \\ &\quad \left. + \||\partial_Y^2 U_s|^{-1}|\partial_Y^3 U_s\|_{L^\infty}\right) \leqq C.\end{aligned}$$

Thus the last integral on the right hand side of (3.34) is bounded by

$$\begin{aligned}\left|\frac{i}{n}\int_0^\infty \partial_Y w \partial_Y \Lambda(\Psi)\overline{\Lambda(\Psi)}\mathrm{d}Y\right| &\lesssim \frac{1}{n}\|\partial_Y w \partial_Y^2 U_s\|_{L^\infty}\|\partial_Y \Lambda(\Psi)\|_{L^2_w}\|\Lambda(\Psi)\|_{L^2_w} \\ &\lesssim \frac{1}{n}\|\partial_Y \Lambda(\Psi)\|_{L^2_w}\|\Lambda(\Psi)\|_{L^2_w}.\end{aligned} \tag{3.35}$$

By taking real and imaginary parts of J_{11} respectively and using (3.35), we deduce that

$$\begin{aligned}|\mathrm{Re} J_{11}| &\lesssim \frac{1}{n}\int_0^\infty |\mathrm{Im} w|\left(|\partial_Y \Lambda(\Psi)|^2 + \alpha^2|\Lambda(\Psi)|^2\right)\mathrm{d}Y \\ &\quad + \frac{1}{n}\|\partial_Y \Lambda(\Psi)\|_{L^2_w}\|\Lambda(\Psi)\|_{L^2_w}, \\ &\lesssim \frac{|c|}{n}\|(\partial_Y \Lambda(\Psi), \alpha\Lambda(\Psi))\|^2_{L^2_w} + \frac{1}{n}\|\partial_Y \Lambda(\Psi)\|_{L^2_w}\|\Lambda(\Psi)\|_{L^2_w},\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\mathrm{Im} J_{11} &\gtrsim \frac{1}{n}\int_0^\infty \mathrm{Re} w\left(|\partial_Y \Lambda(\Psi)|^2 + \alpha^2|\Lambda(\Psi)|^2\right)\mathrm{d}Y - \frac{1}{n}\|\partial_Y \Lambda(\Psi)\|_{L^2_w}\|\Lambda(\Psi)\|_{L^2_w} \\ &\gtrsim \frac{1}{n}\|(\partial_Y \Lambda(\Psi), \alpha\Lambda(\Psi))\|^2_{L^2_w} - \frac{1}{n}\|\partial_Y \Lambda(\Psi)\|_{L^2_w}\|\Lambda(\Psi)\|_{L^2_w},\end{aligned} \tag{3.37}$$

where we have used the fact that

$$|\mathrm{Im} w| \lesssim |\mathrm{Im} c| w_1 + O(1)|c|^2|\partial_Y^2 U_s|^{-1} \lesssim |c||\partial_Y^2 U_s|^{-1}$$

and

$$\mathrm{Re}w \gtrsim w_0 - |c||w_1| - O(1)|c|^2||\partial_Y^2 U_s|^{-1} \gtrsim |\partial_Y^2 U_s|^{-1}$$

by the expansion (3.18).

Next we estimate J_{12}. Recall (3.14) the definition of Λ. We have

$$\begin{aligned}[\Delta_\alpha, \Lambda](\Psi) &= \Delta_\alpha[\Lambda(\Psi)] - \Lambda[\Delta_\alpha(\Psi)] = \partial_Y^3(A^{-1}\partial_Y\Psi) - \partial_Y(A^{-1}\partial_Y^3\Psi)\\ &= 2\partial_Y(A^{-1})\partial_Y^3\Psi + 3\partial_Y^2(A^{-1})\partial_Y^2\Psi + \partial_Y^3(A^{-1})\partial_Y\Psi. \end{aligned} \tag{3.38}$$

We rewrite $\partial_Y^2\Psi$ and $\partial_Y^3\Psi$ as

$$\begin{aligned}\partial_Y^2\Psi &= A\Lambda(\Psi) + A^{-1}\partial_Y A\partial_Y\Psi + \alpha^2 A\Psi,\\ \partial_Y^3\Psi &= A\partial_Y\Lambda(\Psi) + 2\Lambda(\Psi)\partial_Y A + \partial_Y\Psi\left(A^{-1}\partial_Y^2 A + \alpha^2 A\right) + 2\alpha^2\Psi\partial_Y A.\end{aligned}$$

By $\alpha \in (0, 1)$, it holds that

$$|\partial_Y^2\Psi| \lesssim |\Lambda(\Psi)| + |\partial_Y\Psi| + \alpha|\Psi|,\ |\partial_Y^3\Psi| \lesssim |\partial_Y\Lambda(\Psi)| + |\Lambda(\Psi)| + |\partial_Y\Psi| + \alpha|\Psi|. \tag{3.39}$$

By (1.8), we have

$$\begin{aligned}&\left|\partial_Y(A^{-1})\right| \leqq C|\partial_Y U_s|,\ \left|\partial_Y^2(A^{-1})\right| \leqq C|\partial_Y^2 U_s| + C|\partial_Y U_s|^2 \leqq C|\partial_Y U_s|,\\ &\left|\partial_Y^3(A^{-1})\right| \leqq C|\partial_Y^3 U_s| + C|\partial_Y^2 U_s\partial_Y U_s| + C|\partial_Y U_s|^3 \leqq C|\partial_Y U_s|.\end{aligned} \tag{3.40}$$

Then applying the bounds (3.39) and (3.40) to (3.38) gives

$$|[\Delta_\alpha, \Lambda](\Psi)| \lesssim |\partial_Y U_s|\left(|\partial_Y\Lambda(\Psi)| + |\Lambda(\Psi)| + |\partial_Y\Psi| + \alpha|\Psi|\right),$$

which by (1.9) implies

$$\begin{aligned}\|[\Delta_\alpha, \Lambda](\Psi)\|_{L^2_w} &\lesssim \||\partial_Y^2 U_s|^{-\frac{1}{2}}\partial_Y U_s\|_{L^\infty}\left(\|\partial_Y\Lambda(\Psi)\|_{L^2} + \|\Lambda(\Psi)\|_{L^2} + \|\partial_Y\Psi\|_{L^2} + \alpha\|\Psi\|_{L^2}\right)\\ &\lesssim \|\partial_Y\Lambda(\Psi)\|_{L^2} + \|\Lambda(\Psi)\|_{L^2} + \|\partial_Y\Psi\|_{L^2} + \alpha\|\Psi\|_{L^2}.\end{aligned} \tag{3.41}$$

Substituting (3.41) into J_{12} and using Cauchy-Schwarz inequality yield

$$\begin{aligned}|J_{12}| &\lesssim \frac{1}{n}\|w\partial_Y^2 U_s\|_{L^\infty}\|\Lambda(\Psi)\|_{L^2_w}\|[\Delta_\alpha, \Lambda](\Psi)\|_{L^2_w}\\ &\lesssim \frac{1}{n}\|\Lambda(\Psi)\|_{L^2_w}\left(\|\partial_Y\Lambda(\Psi)\|_{L^2_w} + \|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y\Psi, \alpha\Psi)\|_{L^2}\right),\end{aligned} \tag{3.42}$$

where we have used (3.20). By (3.36), (3.37) and (3.42), we can deduce from real and imaginary parts of (3.33) that

$$\begin{aligned}|\mathrm{Re}J_1| \lesssim &\frac{|c|}{n}\|(\partial_Y\Lambda(\Psi), \alpha\Lambda(\Psi))\|_{L^2_w}^2\\ &+ \frac{1}{n}\|\Lambda(\Psi)\|_{L^2_w}\left(\|\partial_Y\Lambda(\Psi)\|_{L^2_w} + \|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y\Psi, \alpha\Psi)\|_{L^2}\right),\end{aligned} \tag{3.43}$$

and

$$\operatorname{Im} J_1 \gtrsim \frac{1}{n} \|(\partial_Y \Lambda(\Psi), \alpha \Lambda(\Psi))\|_{L^2_w}^2 - \frac{1}{n} \|\Lambda(\Psi)\|_{L^2_w} \Big(\|\partial_Y \Lambda(\Psi)\|_{L^2_w} + \|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2} \Big). \tag{3.44}$$

Finally, for J_4, we have by Cauchy-Schwarz inequality that

$$J_4 \lesssim \|w \partial_Y^2 U_s\|_{L^\infty} \|h\|_{L^2_w} \|\Lambda(\Psi)\|_{L^2_w} \lesssim \|h\|_{L^2_w} \|\Lambda(\Psi)\|_{L^2_w}. \tag{3.45}$$

Thus, we have completed the estimation on $J_1 - J_4$.

By taking imaginary part of (3.24) and using previous bounds (3.27), (3.32), (3.44) and (3.45) for $J_1 - J_4$, we have

$$\begin{aligned} &\frac{1}{n} \|(\partial_Y \Lambda(\Psi), \alpha \Lambda(\Psi))\|_{L^2_w}^2 + \operatorname{Im} c \left(\|\Lambda(\Psi)\|_{L^2_w}^2 + \|m|U_s|^{\frac{1}{2}} \partial_Y \Psi\|_{L^2}^2 \right) \\ &\quad \lesssim \frac{1}{n} \|\Lambda(\Psi)\|_{L^2_w} \left(\|\partial_Y \Lambda(\Psi)\|_{L^2_w} + \|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2} \right) \\ &\qquad + |c|^2 \left(\|\Lambda(\Psi)\|_{L^2_w}^2 + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2}^2 \right) + \|h\|_{L^2_w} \|\Lambda(\Psi)\|_{L^2_w}. \end{aligned} \tag{3.46}$$

Similarly, taking real part of (3.24) and using (3.26), (3.31), (3.43) and (3.45) give

$$\begin{aligned} \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2}^2 \lesssim & \frac{1}{n} \|\Lambda(\Psi)\|_{L^2_w} \left(\|\partial_Y \Lambda(\Psi)\|_{L^2_w} + \|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2} \right) \\ & + \|\Lambda(\Psi)\|_{L^2_w}^2 + \frac{|c|}{n} \|(\partial_Y \Lambda(\Psi), \alpha \Lambda(\Psi))\|_{L^2_w}^2 + \|h\|_{L^2_w} \|\Lambda(\Psi)\|_{L^2_w}. \end{aligned} \tag{3.47}$$

Multiplying estimate (3.47) by $\operatorname{Im} c$, suitably combining it with (3.46) and using Young's inequality, we can obtain that

$$\begin{aligned} &\frac{1}{n} \|(\partial_Y \Lambda(\Psi), \alpha \Lambda(\Psi))\|_{L^2_w}^2 + \operatorname{Im} c \left(\|\Lambda(\Psi)\|_{L^2_w}^2 + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2}^2 \right) \\ &\quad \lesssim \frac{1}{n} \|\Lambda(\Psi)\|_{L^2_w} \left(\|\partial_Y \Lambda(\Psi)\|_{L^2_w} + \|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2} \right) \\ &\qquad + |c|^2 \left(\|\Lambda(\Psi)\|_{L^2_w}^2 + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2}^2 \right) + \frac{|c|}{n} \|(\partial_Y \Lambda(\Psi), \alpha \Lambda(\Psi))\|_{L^2_w}^2 \\ &\qquad + \|h\|_{L^2_w} \|\Lambda(\Psi)\|_{L^2_w} \\ &\quad \leqq \left(\frac{1}{2n} + \frac{C|c|}{n} \right) \|(\partial_Y \Lambda(\Psi), \alpha \Lambda(\Psi)\|_{L^2_w}^2 + \operatorname{Im} c \left(\frac{1}{2} + \frac{C}{n \operatorname{Im} c} + \frac{C|c|^2}{\operatorname{Im} c} \right) \\ &\qquad \times \left(\|\Lambda(\Psi)\|_{L^2_w}^2 + \|(\partial_Y \Psi, \alpha \Psi)\|_{L^2}^2 \right) + \frac{C}{\operatorname{Im} c} \|h\|_{L^2_w}^2. \end{aligned} \tag{3.48}$$

By taking $\gamma_4 \in (0, \gamma_3)$ suitably small such that

$$C|c| \leqq C\gamma_4 \leqq \frac{1}{4}, \text{ and } \frac{C}{n \operatorname{Im} c} + \frac{C|c|^2}{\operatorname{Im} c} \leqq 2C\gamma_4 \leqq \frac{1}{4}, \ \forall c \in \Sigma_Q,$$

we can absorb the first and second terms on the right hand side of (3.48) by the left hand side. Thus,

$$\|\Lambda(\Psi)\|_{L^2_w} + \|(\partial_Y \Psi, \alpha\Psi)\|_{L^2} \leqq \frac{C}{\mathrm{Im}c}\|h\|_{L^2_w}, \text{ and } \|(\partial_Y \Lambda(\Psi), \alpha\Lambda(\Psi))\|_{L^2_w} \leqq \frac{Cn^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|h\|_{L^2_w}.$$

The above two inequalities immediately imply the estimates (3.22) and (3.23). The proof of the lemma is completed. □

With the a priori estimates in Lemma 3.4, we can prove the existence, uniqueness and analytic dependence on c of the solution to the compressible Orr–Sommerfeld equation (3.16) in the following lemma:

Lemma 3.5. (Construction of the solution) *Let* $m \in (0, \frac{1}{\sqrt{3}})$, $\alpha \in (0, 1)$ *and* $c \in \Sigma_Q$. *If* $\|h\|_{L^2_w} < \infty$, *there exists a unique solution* $\Psi \in \mathbb{X}$ *to* (3.16) *which satisfies estimates* (3.22) *and* (3.23). *Moreover, if* $h(\cdot\,;c)$ *is analytic in* c *in* $L^2_w(\mathbb{R}_+)$, *then* $\Psi(\cdot\,;c)$ *is analytic in* $\mathbb{X}$.

Remark 3.6. By elliptic regularity, the solution Ψ is in $H^4(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+)$.

Proof. The proof is based on a cascade of approximate process and a continuity argument. First of all, we set $W \overset{\text{def}}{=} \Lambda(\Psi)$ and reformulate (3.16) as

$$\begin{cases} \frac{i}{n}\Lambda(\Delta_\alpha \Lambda^{-1}W) + (U_s - c)W + w^{-1}\Lambda^{-1}W = h, \ Y > 0, \\ W|_{Y=0} = 0. \end{cases} \tag{3.49}$$

Here the inverse operator $\Lambda^{-1} : L^2(\mathbb{R}_+) \to H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+)$ is constructed in Lemma 5.1. If one can show the solvability of (3.49) in $H^1_w(\mathbb{R}_+)$, then by Lemma 5.1, $\Psi \overset{\text{def}}{=} \Lambda^{-1}(W) \in \mathbb{X}$ and it solves the equation (3.16). Now we elaborate the construction of solution to (3.49) in the following three steps.

Step 1. Fix any parameter $l > 0$. We start from an auxiliary problem

$$T_l(W) \overset{\text{def}}{=} \frac{i}{n}\Delta_\alpha W + (U_s - c - il)W + w^{-1}\Lambda^{-1}W = h, \ W|_{Y=0} = 0. \tag{3.50}$$

We claim that there exists $l_0 > 0$, such that if $c \in \Sigma_Q$ and $\|h\|_{L^2_w} < \infty$, then (3.50) admits a unique solution $W \in H^2_w(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+)$ and the solution operator $T^{-1}_{l_0} : L^2_w(\mathbb{R}_+) \to H^2_w(\mathbb{R}_+)$ is analytic in c. To prove this claim, we define a sequence of approximate solutions $\{W_k\}_{k=0}^\infty$ by the following equations

$$\left[\mathrm{Airy} - (c + il)\right](W_{k+1}) = h - w^{-1}\Lambda^{-1}W_k, \ W_{k+1}\big|_{Y=0} = 0, \ W_0 \equiv 0, \tag{3.51}$$

where $\mathrm{Airy} \overset{\text{def}}{=} \frac{i}{n}\Delta_\alpha + U_s : H^2_w(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \to L^2_w(\mathbb{R}_+)$ is the Airy operator. For any $c \in \Sigma_Q$ and $l > 0$, by direct energy method, it is straightforward to check that $c+il$ lies in the resolvent set of Airy operator. Thus by an inductive argument, we can solve W_k and establish its analytic dependence on c from (3.51). In order to take the limit $k\to\infty$, we need some uniform estimates. Applying the multiplier $(\partial_Y^2 U_s)^{-1}\bar{W}_{k+1}$ to (3.51), we have

$$\begin{aligned}\frac{i}{n}\int_0^\infty (\partial_Y^2 U_s)^{-1}\bar{W}_{k+1}\Delta_\alpha W_{k+1}\mathrm{d}Y+\int_0^\infty (\partial_Y^2 U_s)^{-1}(U_s-c-il)|W_{k+1}|^2\mathrm{d}Y\\ =-\int_0^\infty (\partial_Y^2 U_s)^{-1}w^{-1}\Lambda^{-1}(W_k)\bar{W}_{k+1}\mathrm{d}Y+\int_0^\infty (\partial_Y^2 U_s)^{-1}h\bar{W}_{k+1}\mathrm{d}Y.\end{aligned}\tag{3.52}$$

By Cauchy-Schwarz inequality, we deduce that

$$\left|\int_0^\infty (\partial_Y^2 U_s)^{-1}h\bar{W}_{k+1}\mathrm{d}Y\right|\leqq C\|h\|_{L^2_w}\|W_{k+1}\|_{L^2_w}.\tag{3.53}$$

By using (3.20) and the bound in (5.1) for Λ^{-1}, we have

$$\begin{aligned}\left|\int_0^\infty (\partial_Y^2 U_s)^{-1}w^{-1}\Lambda^{-1}(W_k)\bar{W}_{k+1}\mathrm{d}Y\right| &\leqq C\||w|^{-1}|\partial_Y^2 U_s|^{-\frac12}\|_{L^\infty}\|W_{k+1}\|_{L^2_w}\|\Lambda^{-1}(W_k)\|_{L^2}\\ &\leqq C\|W_{k+1}\|_{L^2_w}\|(1+Y)W_k\|_{L^2}\\ &\leqq C\|W_{k+1}\|_{L^2_w}\|W_k\|_{L^2_w}.\end{aligned}\tag{3.54}$$

Integration by parts yields

$$\begin{aligned}\frac{i}{n}\int_0^\infty (\partial_Y^2 U_s)^{-1}\bar{W}_{k+1}\Delta_\alpha W_{k+1}\mathrm{d}Y =&\frac{i}{n}\left(\|\partial_Y W_{k+1}\|_{L^2_w}^2+\alpha^2\|W_{k+1}\|_{L^2_w}^2\right)\\ &+\frac{i}{n}\int_0^\infty \frac{\partial_Y^3 U_s}{(\partial_Y^2 U_s)^2}\partial_Y W_{k+1}\bar{W}_{k+1}\mathrm{d}Y.\end{aligned}\tag{3.55}$$

By (1.8), the last integral in the above equality is bounded by

$$\begin{aligned}\left|\frac{i}{n}\int_0^\infty \frac{\partial_Y^3 U_s}{(\partial_Y^2 U_s)^2}\partial_Y W_{k+1}\bar{W}_{k+1}\mathrm{d}Y\right| &\leqq \frac{1}{n}\left\|\frac{\partial_Y^3 U_s}{\partial_Y^2 U_s}\right\|_{L^\infty}\|\partial_Y W_{k+1}\|_{L^2_w}\|W_{k+1}\|_{L^2_w}\\ &\leqq \frac{C}{n}\|\partial_Y W_{k+1}\|_{L^2_w}\|W_{k+1}\|_{L^2_w}.\end{aligned}\tag{3.56}$$

By taking the imaginary part of (3.52), and using the bounds obtained in (3.53)–(3.56) with Young's inequality, we have

$$\begin{aligned}&\frac{1}{n}\left(\|\partial_Y W_{k+1}\|_{L^2_w}^2+\alpha^2\|W_{k+1}\|_{L^2_w}^2\right)+(\mathrm{Im}c+l)\|W_{k+1}\|_{L^2_w}^2\\ &\quad\leqq C\|W_{k+1}\|_{L^2_w}\left(\|h\|_{L^2_w}+\|W_k\|_{L^2_w}+\frac{1}{n}\|\partial_Y W_{k+1}\|_{L^2_w}\right)\\ &\quad\leqq \frac{1}{2n}\|\partial_Y W_{k+1}\|_{L^2_w}^2+\frac{\mathrm{Im}c+l}{2}\left(1+\frac{C}{n(\mathrm{Im}c+l)}\right)\|W_{k+1}\|_{L^2_w}^2\\ &\qquad+\frac{C}{\mathrm{Im}c+l}\left(\|h\|_{L^2_w}^2+\|W_k\|_{L^2_w}^2\right).\end{aligned}\tag{3.57}$$

We choose $\gamma_4 > 0$ smaller if needed so that $\frac{C}{n\mathrm{Im}c} < \frac{1}{2}$, for any $c \in \Sigma_Q$. Then (3.57) gives

$$\|W_{k+1}\|_{L^2_w} \leqq \frac{C}{\mathrm{Im}c + l}\left(\|W_k\|_{L^2_w} + \|h\|_{L^2_w}\right),$$
$$\|(\partial_Y W_{k+1}, \alpha W_{k+1})\|_{L^2_w} \leqq \frac{Cn^{\frac{1}{2}}}{(\mathrm{Im}c + l)^{\frac{1}{2}}}\left(\|W_k\|_{L^2_w} + \|h\|_{L^2_w}\right).$$

Now we take the difference $W_{k+1} - W_k$. A Similar argument gives

$$\|W_{k+1} - W_k\|_{L^2_w} \leqq \frac{C}{\mathrm{Im}c + l}\|W_k - W_{k-1}\|_{L^2_w},$$
$$\|\partial_Y W_{k+1} - \partial_Y W_k\|_{L^2_w} \leqq \frac{Cn^{\frac{1}{2}}}{(\mathrm{Im}c + l)^{\frac{1}{2}}}\|W_k - W_{k-1}\|_{L^2_w}.$$

By taking l suitably large, such that $\frac{C}{\mathrm{Im}c+l} \leqq \frac{C}{l} \leqq \frac{1}{2}$, $\{W_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $H^1_w(\mathbb{R}_+)$. This implies the existence of a limit function $W = \lim_{k\to\infty} W_k$ in $H^1_w(\mathbb{R}_+)$ that is the solution to (3.50). By the elliptic regularity, W_k converges to W in $H^2_w(\mathbb{R}_+)$. Moreover, by induction, each W_k is analytic in c, so is W by uniform convergence. This justifies the claim and step 1 is completed.

Step 2. (Bootstrap from $T_{l_0}^{-1}$ to T_0^{-1}). Consider the equation (3.50) for any fix $l \in [0, l_0]$ with W^l as its solution. Applying the multiplier $-w\bar{W}^l$ and using the same argument as in Lemma 3.4, we can show that W^l satisfies

$$\|W^l\|_{L^2_w} \leqq \frac{C}{\mathrm{Im}c}\|h\|_{L^2_w}, \ \|(\partial_Y W^l, \alpha W^l)\|_{L^2_w} \leqq \frac{Cn^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|h\|_{L^2_w}, \ \forall c \in \Sigma_Q, \tag{3.58}$$

where the constant C is uniform in $l \in [0, l_0]$. Now we take $l_1 = l_0 - \lambda$ for some fixed constant $0 < \lambda < 2C^{-1}\gamma_4^{-1}n^{-1}$ and construct the solution $W^{l_1} = T_{l_1}^{-1}(h)$ through the following iteration

$$W_{k+1}^{l_1} = T_{l_0}^{-1}\left(-i\lambda W_k^{l_1} + h\right), \ W_0^{l_1} \equiv 0.$$

Applying the a priori estimate (3.58) to $W_{k+1}^{l_1} - W_k^{l_1}$ yields that

$$\begin{aligned}\|W_{k+1}^{l_1} - W_k^{l_1}\|_{L^2_w} &\leqq \frac{C\lambda}{\mathrm{Im}c}\|W_k^{l_1} - W_{k-1}^{l_1}\|_{L^2_w} \\ &\leqq \frac{1}{2\gamma_4 n\mathrm{Im}c}\|W_k^{l_1} - W_{k-1}^{l_1}\|_{L^2_w} \leqq \frac{1}{2}\|W_k^{l_1} - W_{k-1}^{l_1}\|_{L^2_w}, \\ \|\partial_Y W_{k+1}^{l_1} - \partial_Y W_k^{l_1}\|_{L^2_w} &\leqq \frac{C\lambda n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|W_k^{l_1} - W_{k-1}^{l_1}\|_{L^2_w} \\ &\leqq C\|W_k^{l_1} - W_{k-1}^{l_1}\|_{L^2_w}, \ \forall c \in \Sigma_Q.\end{aligned}$$

Hence, $\{W_k^{l_1}\}_{k=0}^{\infty}$ is a Cauchy sequence in H_w^1 and it has a limit $W^{l_1} = \lim_{k\to\infty} W_k^{l_1}$. It is straightforward to check that W^{l_1} is in $H_w^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$ and satisfies (3.50) with $l = l_1$. Moreover, from the previous step we have already shown that each $W_k^{l_1}$ is analytic in c. Thus analyticity of W^{l_1} follows from the uniform convergence. Thus we have completed the construction solution operator $T_{l_1}^{-1}$. Noting that W^{l_1} satisfies the a priori estimate (3.58), we can take $l_2 = l_1 - \lambda$ and construct the solution operator $T_{l_2}^{-1}$ in the same way. Repeating the same procedure, we can eventually establish the existence and analytic dependence on c of the solution operator T_0^{-1}.

Step 3. We now solve the original system (3.49) by using the following iteration:

$$T_0(W_{k+1}) = h + \frac{i}{n}[\Delta_\alpha, \Lambda](\Lambda^{-1}(W_k)), \ Y > 0, \ W_{k+1}|_{Y=0} = 0, \ W_0(Y) \equiv 0.$$

By using the bounds in (3.41) and (5.1) on the commutator $[\Delta_\alpha, \Lambda]$ and Λ^{-1} respectively, we have

$$\begin{aligned}\left\|[\Delta_\alpha, \Lambda](\Lambda^{-1}(W_k))\right\|_{L_w^2} &\leqq C\left(\|\partial_Y W_k\|_{L^2} + \|W_k\|_{L^2} + \|(\partial_Y\Lambda^{-1}W_k, \alpha\Lambda^{-1}W_k)\|_{L^2}\right)\\ &\leqq C\left(\|\partial_Y W_k\|_{L^2} + \|W_k(1+Y)\|_{L^2}\right) \leqq C\|W_k\|_{H_w^1}.\end{aligned}$$

Then applying the a priori bound (3.58) to $W_{k+1} - W_k$, gives

$$\begin{aligned}\|W_{k+1} - W_k\|_{L_w^2} &\leqq \frac{C}{n\mathrm{Im}c}\|W_k - W_{k-1}\|_{H_w^1},\\ \|\partial_Y W_{k+1} - \partial_Y W_k\|_{L_w^2} &\leqq \frac{C}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{1}{2}}}\|W_k - W_{k-1}\|_{H_w^1}.\end{aligned}$$

By taking $\gamma_4 > 0$ smaller if needed such that $\frac{C}{n\mathrm{Im}c} + \frac{C}{n^{\frac{1}{2}}\mathrm{Im}c^{\frac{1}{2}}} \leqq C\gamma_4^{\frac{1}{2}}(1+\gamma_4^{\frac{1}{2}}) < \frac{1}{2}$ for $c \in \Sigma_Q$, we show that $\{W_k\}_{k=0}^{\infty}$ is a Cauchy sequence in $H_w^1(\mathbb{R}_+)$. Let $W := \lim_{k\to\infty} W_k$. By the elliptic regularity and H_w^1-convergence, it is straightforward to check that W_k converges to W in $H_w^2(\mathbb{R}_+)$ and W is a solution to (3.49). Moreover, since each W_k is analytic in c and the convergence is uniform in $c \in \Sigma_Q$, we conclude that W is analytic in c. The uniqueness of solution follows from the a priori estimates obtained in Lemma 3.4. Then the proof of the lemma is completed. □

Now let Ψ be the solution to (3.16) with $h = \Omega(s_1, s_2)$ and Ω defined in (3.2). In terms of the fluid variables $(\varrho, \mathfrak{u}, \mathfrak{v})$ given in (3.11) and (3.12), we have the following proposition for the solvability of the quasi-compressible approximation system (3.10):

Proposition 3.7. (Solvability of quasi-compressible system) *Under the same assumption on parameters m, α and c as in Lemma 3.5, if $\vec{s} = (s_1, s_2) \in H^1(\mathbb{R}_+)^2$*

and $\|\Omega(s_1, s_2)\|_{L^2_w} < \infty$, *there exists a solution* $(\varrho, \mathfrak{u}, \mathfrak{v}) \in H^2(\mathbb{R}_+)^3$ *to the quasi-compressible approximation system* (3.10). *Moreover,* $(\varrho, \mathfrak{u}, \mathfrak{v})$ *satisfies the estimates*

$$\|\mathfrak{u}\|_{H^1} + \|(m^{-2}\varrho, \mathfrak{v})\|_{H^2} + \alpha^{-1}\|div_\alpha(\mathfrak{u}, \mathfrak{v})\|_{H^1} \lesssim \frac{1}{Imc}\|\Omega(s_1, s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} + \|s_2\|_{L^2} + \|div_\alpha(s_1, s_2)\|_{L^2}, \quad (3.59)$$

and

$$\|\partial_Y^2 \mathfrak{u}\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{(Imc)^{\frac{1}{2}}}\|\Omega(s_1, s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} + \|s_2\|_{L^2} + \|div_\alpha(s_1, s_2)\|_{L^2}. \quad (3.60)$$

Furthermore, if both $\vec{s}(\cdot\,; c)$ *and* $\Omega(\cdot\,; c)$ *are analytic in* c *in* $H^1(\mathbb{R}_+)$ *and* $L^2_w(\mathbb{R}_+)$ *respectively, then* $(\varrho, \mathfrak{u}, \mathfrak{v})(\cdot\,; c)$ *is analytic in* c *in* $H^2(\mathbb{R}_+)$.

Remark 3.8. If $\operatorname{div}_\alpha(s_1, s_2) = 0$, then by (3.11), (3.67) and regularity of Ψ it is easy to deduce that $(\varrho, \mathfrak{u}, \mathfrak{v}) \in H^3(\mathbb{R}_+)^3$. This reveals the elliptic structure for linearized compressible Navier–Stokes equations around the subsonic boundary layer profile.

Proof. It is straightforward to check that $(\varrho, \mathfrak{u}, \mathfrak{v})$ satisfies (3.10). The analyticity directly follows from Lemma 3.4. It remains to show the estimates (3.59) and (3.60). Firstly, by using bounds given in (3.22), (3.23) with $h = \Omega(s_1, s_2)$ and (3.39), we obtain that

$$\|\partial_Y^2\Psi\|_{L^2} + \|(\partial_Y\Psi, \alpha\Psi)\|_{L^2} \lesssim \|\Lambda(\Psi)\|_{L^2} + \|(\partial_Y\Psi, \alpha\Psi)\|_{L^2} \lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1, s_2)\|_{L^2_w}, \quad (3.61)$$

and

$$\|\partial_Y^3\Psi\|_{L^2} \lesssim \|\partial_Y\Lambda(\Psi)\|_{L^2} + \|\Lambda(\Psi)\|_{L^2} + \|(\partial_Y\Psi, \alpha\Psi)\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\left(1 + \frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{1}{2}}}\right)\|\Omega(s_1, s_2)\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|\Omega(s_1, s_2)\|_{L^2_w}, \quad (3.62)$$

where we have used $n\mathrm{Im}c \gtrsim 1$ for $c \in \Sigma_Q$. Then by $\mathfrak{v} = -i\alpha\Psi$, (3.61) and $\alpha \in (0, 1)$, it holds that

$$\|\mathfrak{v}\|_{H^2} \lesssim \|\partial_Y^2\Psi\|_{L^2} + \|(\partial_Y\Psi, \alpha\Psi)\|_{L^2} \lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1, s_2)\|_{L^2_w}. \quad (3.63)$$

Next we estimate ϱ. Recall (3.12) for its representation. Since $\Psi|_{Y=0} = 0$, we can use Hardy inequality $\|Y^{-1}\Psi\|_{L^2} \leqq 2\|\partial_Y\Psi\|_{L^2}$, and the bounds given in (3.61), (3.62) to obtain

$$
\begin{aligned}
m^{-2}\|\varrho\|_{L^2} &\lesssim \frac{1}{n}\|\partial_Y^3\Psi\|_{L^2} + (1+\frac{\alpha^2}{n})\|\partial_Y\Psi\|_{L^2} + \|Y^{-1}\Psi\|_{L^2}\|Y\partial_Y U_s\|_{L^\infty} \\
&\quad + \frac{1}{\alpha}\|s_1\|_{L^2} \\
&\lesssim \frac{1}{n}\|\partial_Y^3\Psi\|_{L^2} + \|\partial_Y\Psi\|_{L^2} + \frac{1}{\alpha}\|s_1\|_{L^2} \\
&\lesssim \left(\frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{1}{2}}} + \frac{1}{\mathrm{Im}c}\right)\|\Omega(s_1,s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} \\
&\lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1,s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2}.
\end{aligned} \tag{3.64}
$$

For $\partial_Y\varrho$, differentiating (3.12) yields that

$$
\begin{aligned}
-m^{-2}\partial_Y\varrho &= \mathrm{OS}_{\mathrm{CNS}}(\Psi) + \alpha^2\left(\frac{i}{n}\Delta_\alpha\Psi + (U_s-c)\Psi\right) - \frac{i}{\alpha}\partial_Y(A^{-1}s_1) \\
&= \Omega(s_1,s_2) + \alpha^2\left(\frac{i}{n}\Delta_\alpha\Psi + (U_s-c)\Psi\right) - \frac{i}{\alpha}\partial_Y(A^{-1}s_1) \\
&= s_2 + \alpha^2\left(\frac{i}{n}\Delta_\alpha\Psi + (U_s-c)\Psi\right),
\end{aligned} \tag{3.65}
$$

where we have used the equation (3.13) in second identity. Taking L^2-norm in (3.65) and using bound (3.61), we can further deduce that

$$
\begin{aligned}
m^{-2}\|\partial_Y\varrho\|_{L^2} &\lesssim \|s_2\|_{L^2} + \frac{\alpha^2}{n}\|\partial_Y^2\Psi\|_{L^2} + \alpha(1+\frac{\alpha^2}{n})\|\alpha\Psi\|_{L^2} \\
&\lesssim \|s_2\|_{L^2} + \|\partial_Y^2\Psi\|_{L^2} + \alpha\|\Psi\|_{L^2} \\
&\lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1,s_2)\|_{L^2_w} + \|s_2\|_{L^2}.
\end{aligned} \tag{3.66}
$$

Now we estimate $\partial_Y^2\varrho$. By using (3.12) and (3.65), we have

$$
-m^{-2}\Delta_\alpha\varrho = \mathrm{div}_\alpha(s_1,s_2) + \alpha^2(U_s-c)^2\varrho + 2\alpha^2\Psi\partial_Y U_s. \tag{3.67}
$$

Then taking L^2 norm leads to

$$
\begin{aligned}
m^{-2}\|\partial_Y^2\varrho\|_{L^2} &\lesssim \|\mathrm{div}_\alpha(s_1,s_2)\|_{L^2} + \alpha^2(1+m^{-2})\|\varrho\|_{L^2} + \alpha^2\|\Psi\|_{L^2} \\
&\lesssim \|\mathrm{div}_\alpha(s_1,s_2)\|_{L^2} + m^{-2}\|\varrho\|_{L^2} + \alpha\|\Psi\|_{L^2} \\
&\lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1,s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} + \|\mathrm{div}_\alpha(s_1,s_2)\|_{L^2}.
\end{aligned} \tag{3.68}
$$

Here we have used (3.61) and (3.64) in the last inequality. Therefore, H^2-estimate of ϱ follows from (3.64), (3.66) and (3.68). Since $\mathrm{div}_\alpha(\mathfrak{u},\mathfrak{v}) = -i\alpha(U_s-c)\varrho$, by using (3.64) and (3.66) we have

$$
\alpha^{-1}\|\mathrm{div}_\alpha(\mathfrak{u},\mathfrak{v})\|_{H^1} \lesssim \|\varrho\|_{H^1} \lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1,s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} + \|s_2\|_{L^2}. \tag{3.69}
$$

Finally, for $\mathfrak{u}$, by using (3.11), (3.61), (3.62), (3.64), (3.66) and (3.68), we obtain that

$$\|\mathfrak{u}\|_{H^1} \lesssim \|\partial_Y \Psi\|_{H^1} + \|\varrho\|_{H^1} \lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_1, s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} + \|s_2\|_{L^2}, \tag{3.70}$$

$$\|\partial_Y^2 \mathfrak{u}\|_{L^2} \lesssim \|\partial_Y^3 \Psi\|_{L^2} + \|\varrho\|_{H^2} \lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|\Omega(s_1, s_2)\|_{L^2_w} + \frac{1}{\alpha}\|s_1\|_{L^2} + \|s_2\|_{L^2} + \|\mathrm{div}_\alpha(s_1, s_2)\|_{L^2}. \tag{3.71}$$

Putting the estimates in (3.63), (3.64), (3.66), (3.68)–(3.70) together yields the estimate (3.59). Note that (3.60) directly follows from (3.71). Then the proof of proposition is completed. □

3.2. Stokes Approximation

In this section, we study the Stokes system with advection

$$\begin{cases} i\alpha(U_s - c)\xi + \mathrm{div}_\alpha(\phi, \psi) = q_0, \\ \sqrt{\varepsilon}\Delta_\alpha \phi + \lambda i\alpha\sqrt{\varepsilon}\mathrm{div}_\alpha(\phi, \psi) - i\alpha(U_s - c)\phi - (i\alpha m^{-2} + \sqrt{\varepsilon}\partial_Y^2 U_s)\xi = q_1, \\ \sqrt{\varepsilon}\Delta_\alpha \psi + \lambda\sqrt{\varepsilon}\partial_Y \mathrm{div}_\alpha(\phi, \psi) - i\alpha(U_s - c)\psi - m^{-2}\partial_Y \xi = q_2, \\ \partial_Y \phi|_{Y=0} = \psi|_{Y=0} = 0, \end{cases} \tag{3.72}$$

with a given inhomogeneous source term $\vec{q} = (q_0, q_1, q_2) \in H^1(\mathbb{R}_+) \times L^2(\mathbb{R}_+)^2$. Compared with original system (3.1), in (3.72) we remove the stretching term $-\psi\partial_Y U_s$ in the momentum equation. We impose the Neumann boundary condition $\partial_Y \phi|_{Y=0} = 0$ on the tangential velocity for obtaining estimates on the higher order derivatives. The following proposition gives the solvability of (3.72):

Proposition 3.9. *Let* $m \in (0, 1)$. *Assume that* $\alpha \in (0, 1)$ *and* $\frac{1}{n} = \frac{\sqrt{\varepsilon}}{\alpha} \ll 1$. *There exists* $\gamma_5 \in (0, 1)$, *such that for any* c *lies in*

$$\Sigma_S \overset{def}{=} \{c \in \mathbb{C} \mid Imc > \gamma_5^{-1} n^{-1},\ |c| < \gamma_5\}, \tag{3.73}$$

the system (3.72) *admits a unique solution* $(\xi, \phi, \psi) \in H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$. *Moreover,* (ξ, ϕ, ψ) *satisfies the following estimates:*

$$\|(m^{-1}\xi, \phi, \psi)\|_{L^2} \leqq \frac{C}{\alpha Imc}\|(m^{-1}q_0, q_1, q_2)\|_{L^2}, \tag{3.74}$$

$$\|(\partial_Y\phi, \alpha\phi)\|_{L^2} + \|\partial_Y\psi, \alpha\psi\|_{L^2} \leqq \frac{Cn^{\frac{1}{2}}}{\alpha(Imc)^{\frac{1}{2}}}\|(m^{-1}q_0, q_1, q_2)\|_{L^2}, \tag{3.75}$$

$$\|div_\alpha(\phi, \psi)\|_{H^1} + m^{-2}\|\partial_Y\xi\|_{L^2} \leqq \frac{C}{Imc}\|(m^{-1}q_0, q_1, q_2)\|_{L^2} + C\|q_0\|_{H^1}, \tag{3.76}$$

$$\|\Delta_\alpha\phi, \Delta_\alpha\psi)\|_{L^2} \leqq \frac{Cn}{\alpha Imc}\|(m^{-1}q_0, q_1, q_2)\|_{L^2} + C\|q_0\|_{H^1}. \tag{3.77}$$

Here the positive constant C *does not depend on either* α *or* ε. *Furthermore, if* $\vec{q}(\cdot\,; c)$ *is analytic in* c *in* $H^1(\mathbb{R}_+) \times L^2(\mathbb{R}_+)^2$, *then* $(\xi, \phi, \psi)(\cdot\,; c)$ *is analytic in* c *in* $H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$.

Remark 3.10. We will use the bounds given in (3.74)–(3.77) only when $q_0 = 0$ in the proof of convergence of iteration.

Remark 3.11. In view of (3.75)–(3.77) with $q_0 = 0$, the divergence part $\mathrm{div}_\alpha(\phi, \psi)$ and the density ξ of the solution have better estimates than other components because there is no strong sublayer related to these two fluid components. This stronger estimate is crucial in the proof of convergence of the iteration later.

Proof. We first focus on the a priori estimates (3.74)–(3.77). By taking inner product of $(3.72)_2$ and $(3.72)_3$ with $-\bar{\phi}$ and $-\bar{\psi}$ respectively then integrating by parts, we obtain

$$\begin{aligned}
&\sqrt{\varepsilon}\left(\|(\partial_Y\phi, \alpha\phi)\|_{L^2}^2 + \|(\partial_Y\psi, \alpha\psi)\|_{L^2}^2\right) + \lambda\sqrt{\varepsilon}\|\mathrm{div}_\alpha(\phi, \psi)\|_{L^2}^2 \\
&\quad + i\alpha\int_0^\infty (U_s - c)\left(|\phi|^2 + |\psi|^2\right)\mathrm{d}Y \\
&\quad - m^{-2}\int_0^\infty \xi\overline{\mathrm{div}_\alpha(\phi, \psi)}\mathrm{d}Y = \int_0^\infty -(q_1 + \sqrt{\varepsilon}\partial_Y^2 U_s\xi)\bar{\phi} - q_2\bar{\psi}\mathrm{d}Y.
\end{aligned} \tag{3.78}$$

By Cauchy-Schwarz and Young's inequalities, it holds that

$$\begin{aligned}
\left|\int_0^\infty (q_1 + \sqrt{\varepsilon}\partial_Y^2 U_s\xi)\bar{\phi} + q_2\bar{\psi}\mathrm{d}Y\right| &\lesssim \|(q_1, q_2)\|_{L^2}\|(\phi, \psi)\|_{L^2} + \sqrt{\varepsilon}\|\xi\|_{L^2}\|\phi\|_{L^2} \\
&\lesssim \|(q_1, q_2)\|_{L^2}\|(\phi, \psi)\|_{L^2} + C\sqrt{\varepsilon}(\|\xi\|_{L^2}^2 + \|\phi\|_{L^2}^2).
\end{aligned} \tag{3.79}$$

By using the continuity equation, $\overline{\mathrm{div}_\alpha(\phi,\psi)} = i\alpha(U_s-\bar{c})\bar{\xi}+\bar{q}_0$, and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathrm{Re}\left(-m^{-2}\int_0^\infty \xi\overline{\mathrm{div}_\alpha(\phi,\psi)}\mathrm{d}Y\right) &= \mathrm{Re}\left(-i\alpha m^{-2}\int_0^\infty (U_s-\bar{c})|\xi|^2\mathrm{d}Y\right)\\ &\quad - m^{-2}\mathrm{Re}\left(\int_0^\infty \xi\bar{q}_0\mathrm{d}Y\right)\\ &\geqq \alpha\mathrm{Im}c\|m^{-1}\xi\|_{L^2}^2 - m^{-2}\|\xi\|_{L^2}\|q_0\|_{L^2}.\end{aligned} \tag{3.80}$$

By (3.79) and (3.80), the real part of (3.78) gives that

$$\begin{aligned}&\sqrt{\varepsilon}\left(\|(\partial_Y\phi,\alpha\phi)\|_{L^2}^2+\|(\partial_Y\psi,\alpha\psi)\|_{L^2}^2\right)+\lambda\sqrt{\varepsilon}\|\mathrm{div}_\alpha(\phi,\psi)\|_{L^2}^2\\ &\quad+\alpha\mathrm{Im}c\|(m^{-1}\xi,\phi,\psi)\|_{L^2}^2\\ &\quad\leqq C\sqrt{\varepsilon}(\|\xi\|_{L^2}^2+\|\phi\|_{L^2}^2)+C\|(m^{-1}\xi,\phi,\psi)\|_{L^2}\|(m^{-1}q_0,q_1,q_2)\|_{L^2}.\end{aligned} \tag{3.81}$$

By taking $\gamma_5\in(0,1)$ sufficiently small so that $\frac{C\sqrt{\varepsilon}}{\alpha\mathrm{Im}c}\leqq\frac{C}{n\mathrm{Im}c}\leqq C\gamma_5\leqq\frac{1}{4}$, $\forall c\in\Sigma_S$, we can absorb the first term on the right hand side of (3.81) by the left hand side. Thus we get

$$\|(m^{-1}\xi,\phi,\psi)\|_{L^2}\leqq\frac{C}{\alpha\mathrm{Im}c}\|(m^{-1}q_0,q_1,q_2)\|_{L^2},$$

and

$$\begin{aligned}\|(\partial_Y\phi,\alpha\phi)\|_{L^2}+\|(\partial_Y\psi,\alpha\psi)\|_{L^2} &\leqq C\varepsilon^{-\frac14}\|(m^{-1}\xi,\phi,\psi)\|_{L^2}^{\frac12}\|(m^{-1}q_0,q_1,q_2)\|_{L^2}^{\frac12}\\ &\leqq\frac{C}{\varepsilon^{\frac14}\alpha^{\frac12}(\mathrm{Im}c)^{\frac12}}\|(m^{-1}q_0,q_1,q_2)\|_{L^2}\\ &\leqq\frac{Cn^{\frac12}}{\alpha(\mathrm{Im}c)^{\frac12}}|(m^{-1}q_0,q_1,q_2)\|_{L^2}.\end{aligned} \tag{3.82}$$

This completes the proof of (3.74) and (3.75).

Next we estimate $\|\partial_Y\xi\|_{L^2}$ and $\|\mathrm{div}_\alpha(\phi,\psi)\|_{H^1}$. Define $\omega\overset{\mathrm{def}}{=}\partial_Y\phi-i\alpha\psi$ and denote $\mathcal{D}:=\mathrm{div}_\alpha(\phi,\psi)$. Then

$$\Delta_\alpha\phi=\partial_Y\omega+i\alpha\mathcal{D},\ \ \Delta_\alpha\psi=-i\alpha\omega+\partial_Y\mathcal{D}, \tag{3.83}$$

and $\omega|_{Y=0}=0$ because of the boundary conditions in (3.72). Thus, we can rewrite $(3.72)_2$ and $(3.72)_3$ as

$$i\alpha m^{-2}\xi=\sqrt{\varepsilon}\partial_Y\omega+\sqrt{\varepsilon}(1+\lambda)i\alpha\mathcal{D}-i\alpha(U_s-c)\phi-q_1-\sqrt{\varepsilon}\partial_Y^2U_s\xi, \tag{3.84}$$

$$m^{-2}\partial_Y\xi=-\sqrt{\varepsilon}i\alpha\omega+\sqrt{\varepsilon}(1+\lambda)\partial_Y\mathcal{D}-i\alpha(U_s-c)\psi-q_2. \tag{3.85}$$

By taking inner product of (3.84) and (3.85) with $-i\alpha\bar{\xi}$ and $\partial_Y\bar{\xi}$ respectively, we deduce that

$$m^{-2}\|\partial_Y\xi,\alpha\xi\|_{L^2}^2=\underbrace{\sqrt{\varepsilon}\int_0^\infty-\partial_Y\omega i\alpha\bar{\xi}-i\alpha\omega\partial_Y\bar{\xi}\mathrm{d}Y}_{J_5}+\underbrace{\sqrt{\varepsilon}(1+\lambda)\int_0^\infty\alpha^2\mathcal{D}\bar{\xi}+\partial_Y\mathcal{D}\partial_Y\bar{\xi}\mathrm{d}Y}_{J_6}$$
$$+\underbrace{\int_0^\infty(q_1+\sqrt{\varepsilon}\partial_Y^2U_s\xi)i\alpha\bar{\xi}-q_2\partial_Y\bar{\xi}\mathrm{d}Y}_{J_7}+\underbrace{\int_0^\infty i\alpha(U_s-c)(i\alpha\bar{\xi}\phi-\psi\partial_Y\bar{\xi})\mathrm{d}Y}_{J_8}. \tag{3.86}$$

Integrating by parts and using boundary condition $\omega|_{Y=0}=0$ yield that

$$J_5=i\alpha\sqrt{\varepsilon}\bar{\xi}\omega|_{Y=0}=0. \tag{3.87}$$

For J_6, by using the continuity equation $(3.72)_1$, we have

$$\mathcal{D}=-i\alpha(U_s-c)\xi+q_0,\ \partial_Y\mathcal{D}=-i\alpha(U_s-c)\partial_Y\xi-i\alpha\partial_YU_s\xi+\partial_Yq_0, \tag{3.88}$$

which implies that

$$\begin{aligned}J_6=&-\sqrt{\varepsilon}(1+\lambda)\int_0^\infty i\alpha(U_s-c)(|\partial_Y\xi|^2+\alpha^2|\xi|^2)\mathrm{d}Y\\&-i\alpha\sqrt{\varepsilon}(1+\lambda)\int_0^\infty\partial_YU_s\xi\partial_Y\bar{\xi}\mathrm{d}Y\\&+\sqrt{\varepsilon}(1+\lambda)\int_0^\infty\partial_Yq_0\partial_Y\bar{\xi}+\alpha^2q_0\bar{\xi}\mathrm{d}Y.\end{aligned}$$

For last two terms on the right hand side, we obtain by Cauchy-Schwarz and Young's inequalities that

$$\begin{aligned}\left|i\alpha\sqrt{\varepsilon}(1+\lambda)\int_0^\infty\partial_YU_s\xi\partial_Y\bar{\xi}\mathrm{d}Y\right|&\leqq C\sqrt{\varepsilon}\|\partial_Y\xi\|_{L^2}\|\alpha\xi\|_{L^2}\\&\leqq C\sqrt{\varepsilon}(\|\partial_Y\xi\|_{L^2}^2+\alpha^2\|\xi\|_{L^2}^2),\end{aligned}$$

and

$$\begin{aligned}\sqrt{\varepsilon}(1+\lambda)\left|\int_0^\infty\partial_Yq_0\partial_Y\bar{\xi}+\alpha^2q_0\bar{\xi}\mathrm{d}Y\right|&\leqq C\sqrt{\varepsilon}\|(\partial_Y\xi,\alpha\xi)\|_{L^2}\|(\partial_Yq_0,\alpha q_0)\|_{L^2}\\&\leqq\frac{m^{-2}}{8}\|(\partial_Y\xi,\alpha\xi)\|_{L^2}^2+Cm^2\varepsilon\|(\partial_Yq_0,\alpha q_0)\|_{L^2}^2.\end{aligned}$$

Thus, taking real part of J_6 gives

$$\begin{aligned}\mathrm{Re}J_6\leqq&-\alpha\mathrm{Im}c\sqrt{\varepsilon}(1+\lambda)\|(\partial_Y\xi,\alpha\xi)\|_{L^2}^2+\left(C\sqrt{\varepsilon}+\frac{m^{-2}}{8}\right)\|(\partial_Y\xi,\alpha\xi)\|_{L^2}^2\\&+Cm^2\varepsilon\|(\partial_Yq_0,\alpha q_0)\|_{L^2}^2\\\leqq&\frac{m^{-2}}{4}\|(\partial_Y\xi,\alpha\xi)\|_{L^2}^2+Cm^2\varepsilon\|(\partial_Yq_0,\alpha q_0)\|_{L^2}^2,\end{aligned} \tag{3.89}$$

for $0 < \varepsilon \ll 1$ being sufficiently small. Again, by Young's inequality, we get

$$
\begin{aligned}
|J_7| + |J_8| &\leqq C \left(\|(q_1, q_2)\|_{L^2} + \alpha \|(\phi, \psi)\|_{L^2} \right) \|(\partial_Y \xi, \alpha \xi)\|_{L^2} + \frac{C\sqrt{\varepsilon}}{\alpha} \|\alpha \xi\|_{L^2}^2 \\
&\leqq \frac{m^{-2}}{4} \|(\partial_Y \xi, \alpha \xi)\|_{L^2}^2 + C m^2 (\|(q_1, q_2)\|_{L^2}^2 + \alpha^2 \|(\phi, \psi)\|_{L^2}^2),
\end{aligned} \tag{3.90}
$$

where we have used $\frac{1}{n} = \frac{\sqrt{\varepsilon}}{\alpha} \ll 1$. By (3.87), (3.89) and (3.90), the real part of (3.86) yields

$$
m^{-2} \|(\partial_Y \xi, \alpha \xi)\|_{L^2} \leqq C \|(q_1, q_2)\|_{L^2} + C\sqrt{\varepsilon} \|(\partial_Y q_0, \alpha q_0)\|_{L^2} + C\alpha \|(\phi, \psi)\|_{L^2}. \tag{3.91}
$$

Moreover, by (3.88) and (3.91), we obtain

$$
\begin{aligned}
\|\mathrm{div}_\alpha (\phi, \psi)\|_{H^1} &\leqq C \|(\partial_Y \xi, \alpha \xi)\|_{L^2} + C \|q_0\|_{H^1} \\
&\leqq C \|(q_1, q_2)\|_{L^2} + C \|q_0\|_{H^1} + C\alpha \|(\phi, \psi)\|_{L^2}.
\end{aligned} \tag{3.92}
$$

Putting the bound (3.74) on $\|(\phi, \psi)\|_{L^2}$ into (3.91) and (3.92) yields the estimate

$$
\begin{aligned}
\|\mathrm{div}_\alpha (\phi, \psi)\|_{H^1} + m^{-2} \|(\partial_Y \xi, \alpha \xi)\|_{L^2} &\leqq C(1 + \frac{1}{\mathrm{Im} c}) \|(m^{-1} q_0, q_1, q_2)\|_{L^2} \\
&\quad + C(1 + \sqrt{\varepsilon}) \|q_0\|_{H^1} \\
&\leqq \frac{C}{\mathrm{Im} c} \|(m^{-1} q_0, q_1, q_2)\|_{L^2} + \|q_0\|_{H^1}.
\end{aligned}
$$

Hence, (3.76) holds.

Finally, we derive the estimate on $\|(\partial_Y \omega, \alpha \omega)\|_{L^2}$. By taking inner products of (3.84) and (3.85) with $\partial_Y \bar{\omega}$ and $i\alpha \bar{\omega}$ respectively then using the fact that

$$
\int_0^\infty (i\alpha \mathcal{D} \partial_Y \bar{\omega} + i\alpha \bar{\omega} \partial_Y \mathcal{D}) \mathrm{d}Y = \int_0^\infty (i\alpha \xi \partial_Y \bar{\omega} + \partial_Y \xi i\alpha \bar{\omega}) \mathrm{d}Y = 0,
$$

we obtain

$$
\begin{aligned}
\sqrt{\varepsilon} \|(\partial_Y \omega, \alpha \omega)\|_{L^2}^2 &= \int_0^\infty (q_1 + \sqrt{\varepsilon} \partial_Y^2 U_s \xi) \partial_Y \bar{\omega} + q_2 i\alpha \bar{\omega} \mathrm{d}Y \\
&\quad + \int_0^\infty i\alpha (U_s - c) \left(\phi \partial_Y \omega + \psi i\alpha \bar{\omega} \right) \mathrm{d}Y \\
&\leqq C \left(\|(q_1, q_2)\|_{L^2} + \sqrt{\varepsilon} \|\xi\|_{L^2} + \alpha \|(\phi, \psi)\|_{L^2} \right) \|(\partial_Y \omega, \alpha \omega)\|_{L^2},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|(\partial_Y \omega, \alpha \omega)\|_{L^2} &\leqq \frac{C}{\sqrt{\varepsilon}} \|(q_1, q_2)\|_{L^2} + Cn \|(\xi, \phi, \psi)\|_{L^2} \\
&\leqq \frac{Cn}{\alpha \mathrm{Im} c} (1 + \mathrm{Im} c) \|(m^{-1} q_0, q_1, q_2)\|_{L^2} \\
&\leqq \frac{Cn}{\alpha \mathrm{Im} c} \|(m^{-1} q_0, q_1, q_2)\|_{L^2}.
\end{aligned} \tag{3.93}
$$

By combining this with the bound (3.76) on $\|\mathrm{div}_\alpha(\phi, \psi)\|_{H^1}$ and recalling (3.83), we have

$$\begin{aligned}\|(\Delta_\alpha\phi, \Delta_\alpha\psi)\| &\leqq C\|(\partial_Y\omega, \alpha\omega)\|_{L^2} + C\|\mathrm{div}_\alpha(\phi, \psi)\|_{H^1}\\ &\leqq \frac{Cn}{\alpha \mathrm{Im} c}\left(1+\sqrt{\varepsilon}\right)\|(m^{-1}q_0, q_1, q_2)\|_{L^2} + C\|q_0\|_{H^1}\\ &\leqq \frac{Cn}{\alpha \mathrm{Im} c}\|(m^{-1}q_0, q_1, q_2)\|_{L^2} + C\|q_0\|_{H^1},\end{aligned}$$

which is (3.77). The uniqueness of solution follows from the a priori bounds (3.74)–(3.77).

As for the construction of solution, we introduce a parameter $\eta \in [0, 1]$ and study a sequence of auxiliary problems $L_{S,\eta}(\xi^\eta, \phi^\eta, \psi^\eta) = (q_0, q_1, q_2)$ as follows:

$$\begin{cases} i\alpha\eta(U_s - c)\xi^\eta + \mathrm{div}_\alpha(\phi^\eta, \psi^\eta) = q_0,\\ \sqrt{\varepsilon}\Delta_\alpha\phi^\eta + \eta\lambda i\alpha\sqrt{\varepsilon}\mathrm{div}_\alpha(\phi^\eta, \psi^\eta) - i\alpha\eta(U_s - c)\phi^\eta - (i\alpha m^{-2} + \eta\sqrt{\varepsilon}\partial_Y^2 U_s)\xi^\eta = q_1,\\ \sqrt{\varepsilon}\Delta_\alpha\psi^\eta + \eta\lambda\sqrt{\varepsilon}\partial_Y\mathrm{div}_\alpha(\phi^\eta, \psi^\eta) - i\alpha\eta(U_s - c)\psi^\eta - m^{-2}\partial_Y\xi^\eta = q_2,\\ \partial_Y\phi^\eta|_{Y=0} = \psi^\eta|_{Y=0} = 0. \end{cases} \tag{3.94}$$

When $\eta = 0$, (3.94) reduces to the classical Stokes system for incompressible flow:

$$\begin{aligned}&\mathrm{div}_\alpha(\phi^0, \psi^0) = q_0,\ \sqrt{\varepsilon}\Delta_\alpha\phi^0 - i\alpha m^{-2}\xi^0 = q_1,\ \sqrt{\varepsilon}\Delta_\alpha\psi^0 - m^{-2}\partial_Y\xi^0 = q_2,\\ &\partial_Y\phi^0|_{Y=0} = \psi^0|_{Y=0} = 0.\end{aligned}$$

It is standard to show the existence and uniqueness of solution $(\xi^0, \phi^0, \psi^0) \in H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$ for any $(q_0, q_1, q_2) \in H^1(\mathbb{R}_+) \times L^2(\mathbb{R}_+)^2$. Moreover, by repeating previous energy estimates to (3.94) and slightly modifying the proof of bounds (3.82), (3.91), (3.92) and (3.93), one can deduce the following estimates on $(\xi^\eta, \phi^\eta, \psi^\eta)$

$$\begin{aligned}\|(\partial_Y\phi^\eta, \alpha\phi^\eta)\|_{L^2} + \|(\partial_Y\psi^\eta, \alpha\psi^\eta)\|_{L^2} &\leqq C\varepsilon^{-\frac14}\|(m^{-1}\xi^\eta, \phi^\eta, \psi^\eta)\|_{L^2}^{\frac12}\\ &\quad \|(m^{-1}q_0, q_1, q_2)\|_{L^2}^{\frac12},\\ m^{-2}\|(\partial_Y\xi^\eta, \alpha\xi^\eta)\|_{L^2} + \|\mathrm{div}_\alpha(\phi^\eta, \psi^\eta)\|_{H^1} &\leqq C\|(q_1, q_2)\|_{L^2}\\ &\quad + C\|q_0\|_{H^1} + C\alpha\|(\phi^\eta, \psi^\eta)\|_{L^2},\\ \|(\partial_Y\omega^\eta, \alpha\omega^\eta)\|_{L^2} &\leqq \frac{C}{\sqrt{\varepsilon}}\|(q_1, q_2)\|_{L^2}\\ &\quad + Cn\|(m^{-1}\xi^\eta, \phi^\eta, \psi^\eta)\|_{L^2},\end{aligned}$$

where $\omega^\eta = \partial_Y\phi^\eta - i\alpha\psi^\eta$ and the constant $C > 0$ does not depend on η. Putting above inequalities together yields the following uniform-in-η estimate

$$\|\xi^\eta\|_{H^1} + \|(\phi^\eta, \psi^\eta)\|_{H^2} \leqq C(\varepsilon, \alpha)\|(m^{-1}q_0, q_1, q_2)\|_{L^2} + C(\varepsilon, \alpha)\|q_0\|_{H^1},$$

where the constant $C(\varepsilon, \alpha)$ may depend on ε and α, but not on $\eta \in [0, 1]$. Thus the existence of solution to (3.72) as well as its analytic dependence on c can be established by the same bootstrap argument as in Lemma 3.5. By uniqueness, the solution obtained satisfies the bounds (3.74)–(3.77). And this completes the proof of the proposition. $\square$

3.3. Quasi-Compressible-Stokes Iteration

In this subsection, we will construct a solution $\vec{\Xi} = (\varrho, \mathfrak{u}, \mathfrak{v})$ to the linearized system (3.1) via an iteration scheme based on the solutions to quasi-compressible and Stokes approximations given in Propositions 3.7 and 3.9.

We first consider the case when source term $(f_u, f_v) \in L^2(\mathbb{R}_+)^2$. At zeroth step, we define $\vec{\Xi}_0 = (\xi_0, \phi_0, \psi_0)$ as the solution to Stokes approximate system

$$L_S(\xi_0, \phi_0, \psi_0) = (0, f_u, f_v), \tag{3.95}$$

which yields an error

$$\vec{\mathscr{E}}_0 \stackrel{\text{def}}{=} \mathscr{L}(\xi_0, \phi_0, \psi_0) - L_Q(\xi_0, \phi_0, \psi_0) = (0, -\psi_0 \partial_Y U_s, 0)\,.$$

Because of the regularizing effect of solution operator to Stokes approximation L_S, this error has higher regularity and fast decay so that $\vec{\mathscr{E}}_0 \in H^2_w(\mathbb{R}_+)$. We can then eliminate it by considering $(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1)$ as the solution to quasi-compressible approximation

$$L_Q(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1) = -\vec{\mathscr{E}}_0. \tag{3.96}$$

Then we have

$$\mathscr{L}(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1) - L_Q(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1) = E_Q(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1), \tag{3.97}$$

where the error operator E_Q is defined in (1.15). According to Proposition 3.7, the solution $(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1)$ is in $H^2(\mathbb{R}_+)^3$. Thus the error term $E_Q(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1)$ is in $L^2(\mathbb{R}_+)$. This allows us to correct this error by using the solution (ξ_1, ϕ_1, ψ_1) to the Stokes approximate system again:

$$L_S(\xi_1, \phi_1, \psi_1) = -E_Q(\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1) \tag{3.98}$$

Now we set $\vec{\Xi}_1 = (\varrho_1, \mathfrak{u}_1, \mathfrak{v}_1) + (\xi_1, \phi_1, \psi_1)$ as the approximate solution as the first step, which together with $\vec{\Xi}_0$ generates an error term

$$\vec{\mathscr{E}}_1 \stackrel{\text{def}}{=} \mathscr{L}(\vec{\Xi}_0 + \vec{\Xi}_1) - (0, f_u, f_v) = (0, -\psi_1 \partial_Y U_s, 0)\,.$$

Now we can iterate the above process. Given the approximate solution $\vec{\Xi}_j$ as well as the error

$$\vec{\mathscr{E}}_j = \big(0, -\psi_j \partial_Y U_s, 0\big)$$

in the j-th ($j \geqq 1$) step, we define the $j+1$-order approximate solution $\vec{\Xi}_{j+1}$ as

$$\vec{\Xi}_{j+1} = (\varrho_{j+1}, \mathfrak{u}_{j+1}, \mathfrak{v}_{j+1}) + (\xi_{j+1}, \phi_{j+1}, \psi_{j+1}),$$

where $(\varrho_{j+1}, \mathfrak{u}_{j+1}, \mathfrak{v}_{j+1})$ is the solution to quasi-compressible system

$$L_Q(\varrho_{j+1}, \mathfrak{u}_{j+1}, \mathfrak{v}_{j+1}) = -\vec{\mathscr{E}}_j, \tag{3.99}$$

and $(\xi_{j+1}, \phi_{j+1}, \psi_{j+1})$ solves the Stokes approximate system

$$L_S(\xi_{j+1}, \phi_{j+1}, \psi_{j+1}) = -E_Q(\varrho_{j+1}, \mathfrak{u}_{j+1}, \mathfrak{v}_{j+1}). \tag{3.100}$$

Observe that for each positive integer $N \geqq 0$, it holds that

$$\mathcal{L}\left(\sum_{j=0}^{N} \vec{\Xi}_j\right) = (0, f_u, f_v) + \vec{\mathscr{E}}_N,$$

where the error term in N-th step is $\vec{\mathscr{E}}_N = (0, -\psi_N \partial_Y U_s, 0)$. Therefore, at this point, formally the series $\vec{\Xi} = \sum_{j=0}^{\infty} \vec{\Xi}_j$ gives a solution to the original system (3.1).

If in addition $f_u, f_v \in H^1(\mathbb{R}_+)$ and $\|\Omega(f_u, f_v)\|_{L^2_w} < \infty$ where operator Ω is defined in (3.2), then we introduce $(\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0) \in H^2(\mathbb{R}_+)^3$ as the solution to the quasi-compressible system

$$L_Q(\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0) = (0, f_u, f_v), \tag{3.101}$$

which yields an error term

$$\begin{aligned}
\vec{\mathscr{E}}_{-1} &\stackrel{\text{def}}{=} \mathcal{L}(\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0) - L_Q(\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0)\\
&= \Big(0, -\sqrt{\varepsilon}\Delta_\alpha\big[(U_s - c)\varrho_0\big] + \lambda i\alpha\sqrt{\varepsilon}\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\\
&\quad - \sqrt{\varepsilon}\partial_Y^2 U_s \varrho_0, \lambda\sqrt{\varepsilon}\partial_Y \mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\Big)\\
&= E_Q(\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0).
\end{aligned} \tag{3.102}$$

The new error term $\mathscr{E}_{-1}$ is in $L^2(\mathbb{R}_+)^3$. So we can take $\vec{\Upsilon} = (\tilde{\rho}, \tilde{u}, \tilde{v})$ as the solution to original linear system (3.1) with inhomogeneous source term $-\mathscr{E}_{-1}$, that is $\mathcal{L}(\vec{\Upsilon}) = -\vec{\mathscr{E}}_{-1}$. Then it is clear that $\vec{\Xi} \stackrel{\text{def}}{=} (\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0) + \vec{\Upsilon}$ defines a solution to (3.1).

The above iteration can be rigorously justified by proving the convergence of iteration that is given in Proposition 3.1.

Proof of Proposition 3.1. Recall the bounds on the parameters $|c|$ and n in (2.43). We can take $0 < \varepsilon \ll 1$ suitably small such that the following bounds hold for any $c \in \overline{D_0}$:

$$|c| < \min\{\gamma_4, \gamma_5\},\ n\mathrm{Im}c \gtrsim \varepsilon^{-\frac{1}{4}} \geqq \max\{2\gamma_4^{-1}, 2\gamma_5^{-1}\},\ |c|^{-2}\mathrm{Im}c \gtrsim \varepsilon^{-\frac{1}{8}} \gtrsim 2\gamma_4^{-1}. \tag{3.103}$$

Here the constants γ_4 and γ_5 are given in Proposition 3.7 and 3.9 respectively. Thus, we have $\overline{D_0} \subsetneq \Sigma_Q \cap \Sigma_S$, where Σ_Q and Σ_S are resolvent sets of L_Q and

L_S, which are defined in (3.21) and (3.73) respectively. From (3.99), we know that $(\varrho_{j+1}, \mathfrak{u}_{j+1}, \mathfrak{v}_{j+1})$ is the solution to quasi-compressible approximation (3.10) with inhomogeneous source term $s_{1,j+1} = \psi_j \partial_Y U_s$, $s_{2,j+1} = 0$. Then we have

$$\Omega(s_{1,j+1}, s_{2,j+1}) = \frac{i}{\alpha}\partial_Y\left(A^{-1}\psi_j\partial_Y U_s\right) = -\frac{i}{\alpha}\varpi^{-1}\psi_j + \frac{i}{\alpha}A^{-1}\partial_Y U_s \partial_Y \psi_j.$$

To eliminate the singular factor α^{-1}, we use the fact that $\partial_Y\psi_j = \mathrm{div}_\alpha(\phi_j, \psi_j) - i\alpha\phi_j$, the two bounds given in (1.9), (3.20) and Hardy inequality to obtain

$$\begin{aligned}\|\Omega(s_{1,j+1}, s_{2,j+1})\|_{L^2_w} &\lesssim \frac{1}{\alpha}\||\partial_Y^2 U_s|^{-\frac{1}{2}}|\varpi|^{-1}Y\|_{L^\infty}\|Y^{-1}\psi_j\|_{L^2}\\ &\quad + \frac{1}{\alpha}\||\partial_Y^2 U_s|^{-\frac{1}{2}}\partial_Y U_s\|_{L^\infty}\|\partial_Y\psi_j\|_{L^2}\\ &\lesssim \frac{1}{\alpha}\|\partial_Y\psi_j\|_{L^2} \lesssim \frac{1}{\alpha}\|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}.\end{aligned} \tag{3.104}$$

Similarly, we get

$$\begin{aligned}\|s_{1,j+1}\|_{L^2} &\lesssim \|Y\partial_Y U_s\|_{L^\infty}\|Y^{-1}\psi_j\|_{L^2} \lesssim \|\partial_Y\psi_j\|_{L^2}\\ &\lesssim \|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \alpha\|\phi_j\|_{L^2}.\end{aligned} \tag{3.105}$$

$$\|\mathrm{div}_\alpha(s_{1,j+1}, s_{2,j+1})\|_{L^2} \lesssim \alpha\|s_{1,j+1}\|_{L^2} \lesssim \alpha\|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \alpha^2\|\phi_j\|_{L^2}. \tag{3.106}$$

Thus, by applying bounds given in (3.59) and (3.60) in Proposition 3.7 to $(\varrho_{j+1}, \mathfrak{u}_{j+1}, \mathfrak{v}_{j+1})$ and using (3.104)–(3.106), we obtain

$$\begin{aligned}&\|\mathfrak{u}_{j+1}\|_{H^1} + \|(m^{-2}\varrho_{j+1}, \mathfrak{v}_{j+1})\|_{H^2} + \alpha^{-1}\|\mathrm{div}_\alpha(\mathfrak{u}_{j+1}, \mathfrak{v}_{j+1})\|_{H^1}\\ &\quad\lesssim \frac{1}{\mathrm{Im}c}\|\Omega(s_{1,j+1}, s_{2,j+1})\|_{L^2_w} + \alpha^{-1}(\|s_{1,j+1}\|_{L^2} + \alpha\|s_{2,j+1}\|_{L^2})\\ &\qquad + \|\mathrm{div}_\alpha(s_{1,j+1}, s_{2,j+1})\|_{L^2}\\ &\quad\lesssim \left(1 + \alpha^2 + \frac{1}{\mathrm{Im}c}\right)\left(\alpha^{-1}\|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right)\\ &\quad\lesssim \frac{1}{\mathrm{Im}c}\left(\alpha^{-1}\|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right),\end{aligned} \tag{3.107}$$

and

$$\begin{aligned}\|\partial_Y^2 u_{j+1}\|_{L^2} &\lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|\Omega(s_{1,j+1}, s_{2,j+1})\|_{L^2_w} + \alpha^{-1}(\|s_{1,j+1}\|_{L^2} + \alpha\|s_{2,j+1}\|_{L^2})\\ &\quad + \|\mathrm{div}_\alpha(s_{1,j+1}, s_{2,j+1})\|_{L^2}\\ &\lesssim \left(1 + \alpha^2 + \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\right)\left(\alpha^{-1}\|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right)\\ &\lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\left(\alpha^{-1}\|\mathrm{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right).\end{aligned} \tag{3.108}$$

Here we have also used $\alpha \in (0, 1)$.

Next according to (3.100), we solve $(\xi_{j+1}, \phi_{j+1}, \psi_{j+1})$ from the Stokes approximation (3.72) with inhomogeneous source term $q_{0,j+1} = 0$,

$$q_{1,j+1} = \sqrt{\varepsilon}\Delta_\alpha\big[(U_s - c)\varrho_{j+1}\big] - \lambda\sqrt{\varepsilon}i\alpha\text{div}_\alpha(\mathfrak{u}_{j+1}, \mathfrak{v}_{j+1}) + \sqrt{\varepsilon}\partial_Y^2 U_s \varrho_{j+1},$$
$$q_{2,j+1} = -\lambda\sqrt{\varepsilon}\partial_Y \text{div}_\alpha(\mathfrak{u}_{j+1}, \mathfrak{v}_{j+1}).$$

By (3.107), we have

$$\begin{aligned}\|q_{1,j+1}, q_{2,j+1}\|_{L^2} &\lesssim \sqrt{\varepsilon}\left(\|\varrho_{j+1}\|_{H^2} + \|\text{div}_\alpha(\mathfrak{u}_{j+1}, \mathfrak{v}_{j+1})\|_{H^1}\right)\\ &\lesssim \sqrt{\varepsilon}\left(\|m^{-2}\varrho_{j+1}\|_{H^2} + \alpha^{-1}\|\text{div}_\alpha(\mathfrak{u}_{j+1}, \mathfrak{v}_{j+1})\|_{H^1}\right)\\ &\lesssim \frac{\sqrt{\varepsilon}}{\text{Im}c}\left(\alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right). \end{aligned} \tag{3.109}$$

Then by applying (3.74)–(3.77) in Proposition 3.9 to $(\xi_{j+1}, \phi_{j+1}, \psi_{j+1})$, using (3.109) and $\alpha = n\sqrt{\varepsilon}$, we can deduce that

$$\|(m^{-1}\xi_{j+1}, \phi_{j+1}, \psi_{j+1})\|_{L^2} \lesssim \frac{1}{n(\text{Im}c)^2}\left(\alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right), \tag{3.110}$$

$$\alpha^{-1}\|\text{div}_\alpha(\phi_{j+1}, \psi_{j+1})\|_{H^1} + \alpha^{-1}\|m^{-2}\partial_Y\xi_{j+1}\|_{L^2} \lesssim \frac{1}{n(\text{Im}c)^2}\left(\alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right), \tag{3.111}$$

$$\|(\partial_Y\phi_{j+1}, \partial_Y\psi_{j+1})\|_{L^2} \lesssim \frac{1}{n^{\frac{1}{2}}(\text{Im}c)^{\frac{3}{2}}}\left(\alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right), \tag{3.112}$$

$$\|(\partial_Y^2\phi_{j+1}, \partial_Y^2\psi_{j+1})\|_{L^2} \lesssim \frac{1}{(\text{Im}c)^2}\left(\alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right). \tag{3.113}$$

Set

$$E_j \overset{\text{def}}{=} \|(m^{-1}\xi_j, \phi_j, \psi_j)\|_{L^2} + \alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{H^1} + \alpha^{-1}\|m^{-2}\partial_Y\xi_j\|_{L^2}, \quad j = 0, 1, 2, \cdots.$$

By the estimates (3.110) and (3.111), we have

$$\begin{aligned}E_{j+1} &\leqq \frac{C}{n(\text{Im}c)^2}\left(\alpha^{-1}\|\text{div}_\alpha(\phi_j, \psi_j)\|_{L^2} + \|\phi_j\|_{L^2}\right)\\ &\leqq \frac{C}{n(\text{Im}c)^2}E_j, \quad j = 0, 1, 2, \cdots. \end{aligned} \tag{3.114}$$

Recall (2.43) for the bounds on c and n when $\alpha = K\varepsilon^{\frac{1}{8}}$ and $c \in D_0$. By taking $\varepsilon_3 \in (0,1)$ suitably small so that $\frac{C}{n(\mathrm{Im}c)^2} \leqq C\varepsilon^{\frac{1}{8}} < \frac{1}{2}$ for any $\varepsilon \in (0, \varepsilon_3)$, we can deduce from (3.114) that

$$\sum_{j=0}^{\infty} E_j \leqq \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j E_0 \leqq C E_0. \tag{3.115}$$

Furthermore, by using the bounds obtained in (3.107), (3.108), (3.112)–(3.115) and $\frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{3}{2}}} \lesssim 1$ for any $c \in D_0$, we get

$$\sum_{j=1}^{\infty} \|(\partial_Y \phi_j, \partial_Y \psi_j)\|_{L^2} \lesssim \frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{3}{2}}} \left(\sum_{j=0}^{\infty} E_j\right) \lesssim E_0, \tag{3.116}$$

$$\sum_{j=1}^{\infty} \|(\partial_Y^2 \phi_j, \partial_Y^2 \psi_j)\|_{L^2} \lesssim \frac{1}{(\mathrm{Im}c)^2} \left(\sum_{j=0}^{\infty} E_j\right) \lesssim \frac{1}{(\mathrm{Im}c)^2} E_0, \tag{3.117}$$

$$\begin{aligned}\sum_{j=1}^{\infty} \|\mathfrak{u}_j\|_{H^1} + \sum_{j=1}^{\infty} \|(m^{-2}\varrho_j, \mathfrak{v}_j)\|_{H^2} + \alpha^{-1}\|\mathrm{div}_\alpha(\mathfrak{u}_j, \mathfrak{v}_j)\|_{H^1} \\ \lesssim \frac{1}{\mathrm{Im}c} \left(\sum_{j=0}^{\infty} E_j\right) \lesssim \frac{1}{\mathrm{Im}c} E_0,\end{aligned} \tag{3.118}$$

$$\sum_{j=1}^{\infty} \|\partial_Y^2 \mathfrak{u}_j\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}} \left(\sum_{j=0}^{\infty} E_j\right) \lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}} E_0. \tag{3.119}$$

In view of (3.115)–(3.119), we have justified the convergence of $\vec{\Xi} = (\rho, u, v) = \sum_{j=0}^{\infty} \vec{\Xi}_j$ in $H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)^2$. This gives the existence of solution. Moreover, Recall (3.95). By applying (3.74)–(3.77) to $\vec{\Xi}_0$ with $q_0 = 0$, $q_1 = f_u$ and $q_2 = f_v$, we derive the following estimates:

$$E_0 \lesssim \frac{1}{\alpha \mathrm{Im}c} \|(f_u, f_v)\|_{L^2}, \tag{3.120}$$

$$\|(\partial_Y \phi_0, \partial_Y \psi_0)\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{\alpha (\mathrm{Im}c)^{\frac{1}{2}}} \|(f_u, f_v)\|_{L^2}, \tag{3.121}$$

$$\|(\partial_Y^2 \phi_0, \partial_Y^2 \psi_0)\|_{L^2} \lesssim \frac{n}{\alpha \mathrm{Im}c} \|(f_u, f_v)\|_{L^2}. \tag{3.122}$$

By summarizing the estimates (3.115)–(3.122), we have

$$\|(m^{-1}\rho, u, v)\|_{L^2} \lesssim \sum_{j=0}^{\infty} \|(m^{-1}\xi_j, \phi_j, \psi_j)\|_{L^2} + \sum_{j=1}^{\infty} \|(m^{-1}\varrho_j, \mathfrak{u}_j, \mathfrak{v}_j)\|_{L^2}$$

$$\lesssim \left(1 + \frac{1}{\mathrm{Im}c}\right) E_0 \lesssim \frac{1}{\alpha(\mathrm{Im}c)^2} \|(f_u, f_v)\|_{L^2}, \tag{3.123}$$

$$\|m^{-2}\partial_Y \rho\|_{L^2} + \|\mathrm{div}_\alpha(u, v)\|_{H^1} \lesssim \alpha \sum_{j=0}^{\infty} E_j + \sum_{j=1}^{\infty} \|m^{-2}\partial_Y \varrho_j\|_{L^2}$$

$$+ \sum_{j=1}^{\infty} \|\mathrm{div}_\alpha(\mathfrak{u}_j, \mathfrak{v}_j)\|_{H^1}$$

$$\lesssim \left(\alpha + \frac{1}{\mathrm{Im}c} + \frac{\alpha}{\mathrm{Im}c}\right) E_0 \lesssim \frac{1}{\alpha(\mathrm{Im}c)^2} \|(f_u, f_v)\|_{L^2}, \tag{3.124}$$

$$\|(\partial_Y u, \partial_Y v)\|_{L^2} \lesssim \|\partial_Y \phi_0, \partial_Y \psi_0\|_{L^2} + \sum_{j=1}^{\infty} \|\partial_Y \phi_j, \partial_Y \psi_j\|_{L^2} + \sum_{j=1}^{\infty} \|(\partial_Y \mathfrak{u}_j, \partial_Y \mathfrak{v}_j)\|_{L^2}$$

$$\lesssim \frac{n^{\frac{1}{2}}}{\alpha(\mathrm{Im}c)^{\frac{1}{2}}} \|(f_u, f_v)\|_{L^2} + \left(1 + \frac{1}{\mathrm{Im}c}\right) E_0$$

$$\lesssim \frac{n^{\frac{1}{2}}}{\alpha(\mathrm{Im}c)^{\frac{1}{2}}} \left(1 + \frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{3}{2}}}\right) \|(f_u, f_v)\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{\alpha(\mathrm{Im}c)^{\frac{1}{2}}} \|(f_u, f_v)\|_{L^2}, \tag{3.125}$$

and

$$\|(\partial_Y^2 u, \partial_Y^2 v)\|_{L^2} \lesssim \|(\partial_Y^2 \phi_0, \partial_Y^2 \psi_0)\|_{L^2} + \sum_{j=1}^{\infty} \|(\partial_Y^2 \phi_j, \partial_Y^2 \psi_j)\|_{L^2}$$

$$+ \sum_{j=1}^{\infty} \|(\partial_Y^2 \mathfrak{u}_j, \partial_Y^2 \mathfrak{v}_j)\|_{L^2}$$

$$\lesssim \frac{n}{\alpha \mathrm{Im}c} \|(f_u, f_v)\|_{L^2} + \left(\frac{1}{(\mathrm{Im}c)^2} + \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\right) E_0$$

$$\lesssim \frac{n}{\alpha \mathrm{Im}c} \left(1 + \frac{1}{n(\mathrm{Im}c)^2} + \frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{1}{2}}}\right) \|(f_u, f_v)\|_{L^2}$$

$$\lesssim \frac{n}{\alpha \mathrm{Im}c} \|(f_u, f_v)\|_{L^2}. \tag{3.126}$$

Putting (3.123)–(3.126) together yields the estimates (3.3)–(3.6). The analytic dependence on c of the solution (ρ, u, v) follows from the uniformly convergence. Therefore, the proof of the first part of Proposition 3.1 is completed.

Now we assume that $f_u, f_v \in H^1(\mathbb{R}_+)$ and $\|\Omega(f_u, f_v)\|_{L^2_w} < \infty$. As discussed in the formal presentation of the iteration scheme, we can decompose the solution (ρ, u, v) into $(\rho, u, v) = (\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0) + \vec{\Upsilon} = (\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0) + (\tilde{\rho}, \tilde{u}, \tilde{v})$, where $(\varrho_0, \mathfrak{u}_0, \mathfrak{v}_0)$ is the solution to (3.101) that generates an error $\vec{\mathcal{E}}_{-1}$ defined in (3.102), and $\vec{\Upsilon}$ solves $\mathcal{L}(\vec{\Upsilon}) = -\vec{\mathcal{E}}_{-1}$. By (3.59) and (3.60) in Proposition 3.7, we have

$$\begin{aligned}\|\mathfrak{u}_0\|_{H^1} + \|(m^{-2}\varrho_0, \mathfrak{v}_0)\|_{H^2} + \alpha^{-1}\|\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\|_{H^1} \\ \lesssim \frac{1}{\mathrm{Im}c}\|\Omega(f_u, f_v)\|_{L^2_w} + \frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|\mathrm{div}_\alpha(f_u, f_v)\|_{L^2},\end{aligned} \tag{3.127}$$

and

$$\|\partial_Y^2\mathfrak{u}_0\|_{L^2} \lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|\Omega(f_u, f_v)\|_{L^2_w} + \frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|\mathrm{div}_\alpha(f_u, f_v)\|_{L^2}. \tag{3.128}$$

Then we can estimate the L^2-bound of the error $\vec{\mathcal{E}}_{-1}$ by

$$\|\vec{\mathcal{E}}_{-1}\|_{L^2} \lesssim \sqrt{\varepsilon}\left(\|\varrho_0\|_{H^2} + \|\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\|_{H^1}\right). \tag{3.129}$$

By (3.129), applying (3.3)–(3.6) to $\vec{\Upsilon} = (\tilde{\rho}, \tilde{u}, \tilde{v})$ leads to

$$\begin{aligned}\|(m^{-1}\tilde{\rho}, \tilde{u}, \tilde{v})\|_{L^2} &\lesssim \frac{1}{\alpha(\mathrm{Im}c)^2}\|\vec{\mathcal{E}}_{-1}\|_{L^2} \\ &\lesssim \frac{1}{n(\mathrm{Im}c)^2}\left(\|\varrho_0\|_{H^2} + \|\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\|_{H^1}\right),\end{aligned} \tag{3.130}$$

$$\begin{aligned}\|m^{-2}\partial_Y\tilde{\rho}\|_{L^2} + \|\mathrm{div}_\alpha(\tilde{u}, \tilde{v})\|_{L^2} &\lesssim \frac{1}{\alpha(\mathrm{Im}c)^2}\|\vec{\mathcal{E}}_{-1}\|_{L^2} \\ &\lesssim \frac{1}{n(\mathrm{Im}c)^2}\left(\|\varrho_0\|_{H^2} + \|\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\|_{H^1}\right),\end{aligned} \tag{3.131}$$

$$\begin{aligned}\|(\partial_Y\tilde{u}, \partial_Y\tilde{v})\|_{L^2} &\lesssim \frac{n^{\frac{1}{2}}}{\alpha(\mathrm{Im}c)^{\frac{1}{2}}}\|\vec{\mathcal{E}}_{-1}\|_{L^2} \\ &\lesssim \frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{1}{2}}}\left(\|\varrho_0\|_{H^2} + \|\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\|_{H^1}\right),\end{aligned} \tag{3.132}$$

$$\begin{aligned}\|(\partial_Y^2\tilde{u}, \partial_Y^2\tilde{v})\|_{L^2} &\lesssim \frac{n}{\alpha\mathrm{Im}c}\|\vec{\mathcal{E}}_{-1}\|_{L^2} \\ &\lesssim \frac{1}{\mathrm{Im}c}\left(\|\varrho_0\|_{H^2} + \|\mathrm{div}_\alpha(\mathfrak{u}_0, \mathfrak{v}_0)\|_{H^1}\right).\end{aligned} \tag{3.133}$$

By summarizing the estimates (3.127), (3.128), (3.130)–(3.133) and using the fact that $n(\mathrm{Im}c)^2 \gtrsim 1$ and $n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{3}{2}} \gtrsim 1$ for $c \in D_0$, we derive the following estimates:

$$\|(m^{-1}\rho, u, v)\|_{H^1} \lesssim \frac{1}{\mathrm{Im}c}\|\Omega(f_u, f_v)\|_{L^2_w} + \frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|\mathrm{div}_\alpha(f_u, f_v)\|_{L^2},$$

$$\begin{aligned}\|\partial_Y^2 u, \partial_Y^2 v)\|_{L^2} &\lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\left(1 + \frac{1}{n^{\frac{1}{2}}(\mathrm{Im}c)^{\frac{3}{2}}}\right)\|\Omega(f_u, f_v)\|_{L^2_w} \\ &\quad + \frac{1}{\mathrm{Im}c}\left(\frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|\mathrm{div}_\alpha(f_u, f_v)\|_{L^2}\right) \\ &\lesssim \frac{n^{\frac{1}{2}}}{(\mathrm{Im}c)^{\frac{1}{2}}}\|\Omega(f_u, f_v)\|_{L^2_w} \\ &\quad + \frac{1}{\mathrm{Im}c}\left(\frac{1}{\alpha}\|f_u\| + \|f_v\|_{L^2} + \|\mathrm{div}_\alpha(f_u, f_v)\|_{L^2}\right).\end{aligned}$$

Thus, the improved estimates (3.7) and (3.8) are proved. And this completes the proof of the proposition. □

4. Proof of Theorem 1.1

Finally, in this section, we prove Theorem 1.1. We construct the solution to linearized system (1.3) with no-slip boundary condition (1.4) in the following form

$$\vec{\Xi}(Y; c) = \vec{\Xi}_{\mathrm{app}}(Y; c) - \vec{\Xi}_{\mathrm{sm}}(Y; c) - \vec{\Xi}_{\mathrm{re}}(Y; c), \tag{4.1}$$

Here $\vec{\Xi}_{\mathrm{app}}$ is the approximate solution obtained in (2.32) which satisfies (2.44), $\vec{\Xi}_{\mathrm{sm}} = (\rho_{\mathrm{sm}}, u_{\mathrm{sm}}, v_{\mathrm{sm}})$ and $\vec{\Xi}_{\mathrm{re}} = (\rho_{\mathrm{re}}, u_{\mathrm{re}}, v_{\mathrm{re}})$ solve the remainder system

$$\mathcal{L}(\vec{\Xi}_{\mathrm{sm}}) = (0, E_{u,\mathrm{sm}}, E_{v,\mathrm{sm}}),\ \ v_{\mathrm{sm}}|_{Y=0} = 0,$$

and

$$\mathcal{L}(\vec{\Xi}_{\mathrm{re}}) = (0, 0, E_{v,\mathrm{re}}),\ \ v_{\mathrm{re}}|_{Y=0} = 0,$$

respectively. By Proposition 2.3 and 3.1, both $\vec{\Xi}_{\mathrm{sm}}$ and $\vec{\Xi}_{\mathrm{re}}$ are well-defined. Moreover, it is straightforward to check that $\vec{\Xi} = (\rho, u, v)$ satisfies

$$\mathcal{L}(\vec{\Xi}) = \vec{0},\ \ v|_{Y=0} = 0.$$

To recover the no-slip boundary condition on the tangential component, we introduce the mapping

$$\mathcal{F} : D_0 \to \mathbb{C},\ \ \mathcal{F}(c) \stackrel{\mathrm{def}}{=} u(0; c) = \mathcal{F}_{\mathrm{app}}(c) - u_{\mathrm{sm}}(0; c) - u_{\mathrm{re}}(0; c).$$

On one hand, from Proposition 2.2, $\mathcal{F}_{\mathrm{app}}(c)$ is analytic and has a unique zero point in D_0. On the other hand, according to Remark 3.2 (a), both $u_{\mathrm{sm}}(0; c)$ and $u_{\mathrm{de}}(0; c)$ are analytic in D_0. Then by applying estimates (3.3), (3.5) to $\vec{\Xi}_{\mathrm{sm}}$ with

$(f_u, f_v) = (E_{u,\mathrm{sm}}, E_{v,\mathrm{sm}})$, using the bound in (2.49) and the Sobolev inequality, we deduce that

$$\begin{aligned} |u_{\mathrm{sm}}(0; c)| &\leqq \|u_{\mathrm{sm}}(\cdot\,; c)\|_{L^2}^{\frac{1}{2}} \|\partial_Y u_{\mathrm{sm}}(\cdot\,; c)\|_{L^2}^{\frac{1}{2}} \\ &\leqq \frac{Cn^{\frac{1}{4}}}{\alpha(\mathrm{Im}c)^{\frac{5}{4}}} \|(E_{u,\mathrm{sm}}, E_{v,\mathrm{sm}})\|_{L^2} \leqq \frac{Cn^{\frac{1}{4}}\varepsilon^{\frac{7}{16}}}{\alpha(\mathrm{Im}c)^{\frac{5}{4}}} \leqq C\varepsilon^{\frac{1}{16}}, \ \forall c \in D_0. \end{aligned} \tag{4.2}$$

Here we have used (2.43) in the last inequality. For $u_{\mathrm{re}}(0; c)$, we use the bounds given in (3.7) for $\|u_{\mathrm{re}}\|_{H^1}$ with $(f_u, f_v) = (0, E_{v,\mathrm{re}})$ and (2.48) to get that

$$\begin{aligned} |u_{\mathrm{re}}(0; c)| \leqq C\|u_{\mathrm{re}}(\cdot\,; c)\|_{H^1} &\leqq \frac{C}{\mathrm{Im}c}\|E_{v,\mathrm{re}}\|_{L^2_w} + \|E_{v,\mathrm{re}}\|_{H^1} \\ &\leqq C\left(1 + \frac{1}{\mathrm{Im}c}\right)\varepsilon^{\frac{3}{16}} \leqq C\varepsilon^{\frac{1}{16}}. \end{aligned} \tag{4.3}$$

Thus, by recalling the lower bound of $|\mathcal{F}_{\mathrm{app}}(c)|$ on the circle ∂D_0 in (2.36), and by using the bounds in (4.2) and (4.3), it holds that

$$\begin{aligned} |\mathcal{F}(c) - \mathcal{F}_{\mathrm{app}}(c)| &\leqq |u_{\mathrm{re}}(0; c)| + |u_{\mathrm{sm}}(0; c)| \leqq C\varepsilon^{\frac{1}{16}} \leqq \frac{1}{4}K^{-\theta} \\ &\leqq \frac{1}{2}|\mathcal{F}_{\mathrm{app}}(c)|, \ \forall c \in D_0, \end{aligned}$$

by taking $\varepsilon \in (0, 1)$ suitably small. Therefore, by Rouché's Theorem, $\mathcal{F}(c)$ and $\mathcal{F}_{\mathrm{app}}(c)$ have the same number of zero points in D_0. This justifies the existence of a unique $c \in D_0$ such that $\vec{\Xi}(Y; c)$ defined in (4.1) solves the linear equation (1.3) with the no-slip boundary condition (1.4). The proof of Theorem 1.1 is completed. □

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5. Appendix

Recall (3.14) the definition of operator Λ.

Lemma 5.1. (The operator Λ^{-1}) *Let $m \in (0,1)$ and $\alpha \in (0,1)$. For small $|c| \ll 1$, if $\|h\|_{L^2} < \infty$, there exists a unique solution $\psi \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$ to $\Lambda(\psi) = h$ in $\mathbb{R}_+$, $\psi|_{Y=0} = 0$ which satisfies the following estimates*

$$\|\partial_Y^2 \psi\|_{L^2} + \|(\partial_Y \psi, \alpha\psi)\|_{L^2} \lesssim \min\{\alpha^{-1}\|h\|_{L^2}, \|(1+Y)h\|_{L^2}\}. \tag{5.1}$$

Moreover, the solution mapping $\Lambda^{-1}(\cdot\,; c) : L^2(\mathbb{R}_+) \to H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$ is analytic in c.

Proof. We first establish the a priori estimate (5.1). Denote $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(\mathbb{R}_+)$. Taking inner product with $-\bar{\psi}$ in the equation $\Lambda(\psi) = h$ gives that

$$\int_0^\infty A^{-1}|\partial_Y \psi|^2 + \alpha^2|\psi|^2 \mathrm{d}Y = \langle h, -\bar{\psi}\rangle. \tag{5.2}$$

Note that

$$A^{-1} = (1 - m^2 U_s^2)^{-1} + O(1)|c|. \tag{5.3}$$

Then for suitably small $|c|$, the real part of (5.2) yields

$$\|\partial_Y \psi\|_{L^2}^2 + \alpha^2\|\psi\|_{L^2}^2 \leqq C|\mathrm{Re}\langle h, \bar{\psi}\rangle| \leqq C\|h\|_{L^2}\|\psi\|_{L^2} \leqq C\alpha^{-2}\|h\|_{L^2}^2,$$

which gives H^1-estimate of ψ. The estimate of $\partial_Y^2\psi$ can be obtained by using the equation to have

$$\begin{aligned}\|\partial_Y^2 \psi\|_{L^2} &\lesssim \|A^{-1}\partial_Y A\|_{L^\infty}\|\partial_Y \psi\|_{L^2} + \alpha^2\|A\psi\|_{L^2} + \|Ah\|_{L^2} \\ &\lesssim \|(\partial_Y\psi, \alpha\psi)\|_{L^2} + \|h\|_{L^2} \lesssim \alpha^{-1}\|h\|_{L^2}.\end{aligned} \tag{5.4}$$

If in addition, $(1+Y)h \in L^2(\mathbb{R}_+)$, we can obtain $|\langle h, \bar{\psi}\rangle| \leqq \|Y^{-1}\psi\|_{L^2}\|Yh\|_{L^2} \leqq C\|\partial_Y\psi\|_{L^2}\|Yh\|_{L^2}$ by using the Hardy inequality for the term $Y^{-1}\psi$. Thus real part of (5.2) yields the H^1-estimate

$$\|(\partial_Y\psi, \alpha\psi)\|_{L^2} \leqq C\|Yh\|_{L^2}. \tag{5.5}$$

The estimate on $\partial_Y^2\psi$ then follows from the equation and H^1-estimate (5.5) similar to the estimation in (5.4). Hence the estimate (5.1) holds. The uniqueness of solution follows from the a priori estimate (5.1).
Set

$$\Lambda_0 : H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+) \to L^2(\mathbb{R}_+),\ \ \Lambda_0(\psi) = \partial_Y\left[(1 - m^2U_s^2)^{-1}\partial_Y\psi\right] - \alpha^2\psi.$$

For $m \in (0,1)$, the operator Λ_0 is clearly uniformly elliptic and invertible in $L^2(\mathbb{R}_+)$. Then by (5.3), the existence and analytic dependence on c of Λ^{-1} follow from the standard perturbative argument. Hence, we omit the details for brevity. The proof of the lemma is completed. □

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