Arch. Rational Mech. Anal. (2023) 247:81
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-023-01914-4

Time-Asymptotic Expansion with Pointwise Remainder Estimates for 1D Viscous Compressible Flow

Kai Koike

Communicated by T.-P. Liu

Abstract

We construct a time-asymptotic expansion with pointwise remainder estimates for solutions to 1D compressible Navier–Stokes equations. The leading-order term is the well-known diffusion wave and the higher-order terms are a newly introduced family of waves which we call *higher-order diffusion waves*. In particular, these provide an accurate description of the power-law asymptotics of the solution around the origin $x = 0$, where the diffusion wave decays exponentially. The expansion is valid locally and also globally in the $L^p(\mathbb{R})$-norm for all $1 \leq p \leq \infty$. The proof is based on pointwise estimates of Green's function.

1. Introduction

The equations

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u_t + p(v)_x = \nu(u_x/v)_x, & x \in \mathbb{R},\ t > 0, \\ v(x,0) = v_0(x),\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1}$$

describe the motion of a 1D viscous compressible flow. Here $v(x,t)$ is the specific volume (the reciprocal of the density ρ) and $u(x,t)$ is the flow velocity; t is the time and x is the Lagrangian mass coordinate related to the Eulerian coordinate X by $x = \int_{X_0(t)}^{X} \rho(X',t)\,dX'$, where $X_0(t)$ is the trajectory of a particle moving with the fluid and initially placed at $X_0(0) = 0$. The system above models barotropic flow, that is, the pressure $p(v)$ does not depend on the temperature. We assume that $p'(v) < 0$ and $p''(v) \neq 0$ for $v > 0$ and that the viscous coefficient ν is a positive constant. The system is often called the p-system and is a typical example of quasilinear hyperbolic–parabolic viscous conservation laws.

The purpose of this paper is to construct a time-asymptotic expansion of the solution to (1) together with pointwise estimates for the remainder. We shall consider solutions close to the steady state $(v_S, u_S) \equiv (1, 0)$. To study the long-time asymptotics of such solutions, it is convenient to consider

$$u_1 = \frac{p''(1)}{4c}[-(v-1)+u/c], \quad u_2 = \frac{p''(1)}{4c}[(v-1)+u/c] \tag{2}$$

instead of (v, u). Here $c = \sqrt{-p'(1)}$ is the speed of sound for the state (v_S, u_S).

It is well-known that u_i has the *diffusion wave* θ_i as its asymptotic profile. Here θ_i $(i = 1, 2)$ is the self-similar solution to the convective viscous Burgers equation

$$\begin{cases} \partial_t\theta_i + \lambda_i\partial_x\theta_i + \partial_x(\theta_i^2/2) = \frac{\nu}{2}\partial_x^2\theta_i, & x \in \mathbb{R}, t > 0, \\ \lim_{t\searrow -1}\theta_i(x,t) = M_i\delta(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where $\lambda_i = (-1)^{i-1}c$, $M_i = \int_{-\infty}^{\infty} u_i(x, 0)\,dx$, and $\delta(x)$ is the Dirac delta function. An explicit formula for θ_i is available through the use of Cole–Hopf transformation:

$$\theta_i(x,t) = \frac{\sqrt{\nu}\left(e^{\frac{M_i}{\nu}}-1\right)}{\sqrt{2(t+1)}}e^{-\frac{(x-\lambda_i(t+1))^2}{2\nu(t+1)}}\left[\sqrt{\pi}+\left(e^{\frac{M_i}{\nu}}-1\right)\int_{\frac{x-\lambda_i(t+1)}{\sqrt{2\nu(t+1)}}}^{\infty} e^{-y^2}\,dy\right]^{-1}. \tag{4}$$

The diffusion wave θ_i describes the leading-order asymptotics in the $L^p(\mathbb{R})$-norm. In fact, we have the following optimal decay estimates [14]:

$$\|\theta_i(\cdot,t)\|_{L^p} \lesssim t^{-(1-1/p)/2} \quad \text{and} \quad \|(u_i-\theta_i)(\cdot,t)\|_{L^p} \lesssim t^{-(3/2-1/p)/2} \quad (1 \le p \le \infty).$$

The key to proving the L^p-decay estimates above—especially for $p = 1$—is the pointwise estimates for Green's function of the linearization of (1) around (v_S, u_S). These, in fact, allow us to obtain pointwise estimates for the solution itself [10]:

$$\begin{aligned} |(u_i-\theta_i)(x,t)| \lesssim\ & [(x-\lambda_i(t+1))^2+(t+1)]^{-3/4} \\ & + [|x+\lambda_i(t+1)|^3+(t+1)^2]^{-1/2}. \end{aligned} \tag{5}$$

The L^p-decay estimates are obtained by integrating this.

Pointwise estimates (5) allow us to deduce not just global L^p-estimates but also local ones. In particular, we have $|(u_i - \theta_i)(x,t)| \lesssim t^{-3/4}$ for $x = \lambda_i t + O(1)$. Since $\theta_i(x,t) \lesssim t^{-1/2}$ for $x = \lambda_i t + O(1)$, the diffusion wave θ_i also describes the leading-order asymptotics locally around the characteristic line $x = \lambda_i t$. However, the situation is different around the origin $x = 0$. As can be seen from (4), the diffusion wave θ_i decays exponentially fast around the origin $x = 0$ but (5) implies $|(u_i - \theta_i)(x,t)| \lesssim t^{-3/2}$ for $x = O(1)$. Thus the diffusion wave θ_i provides almost no information about the long-time asymptotics around $x = 0$; we need new waves to capture the asymptotic behavior there.

In [12], van Baalen, Popović, and Wayne constructed a time-asymptotic expansion of u_i in an L^2-framework. The leading-order term of the expansion is the diffusion wave θ_i but the first higher-order term beyond θ_i turns out to be a wave decaying algebraically as $t^{-3/2}$ around the origin. It is then natural to expect that this new wave captures the leading-order asymptotics of the flow around $x = 0$. However, the decay estimate for the remainder of the expansion is given in the $H^1(\mathbb{R})$-norm. This implies only a far from optimal decay estimate around $x = 0$. For this reason, we cannot conclude that the higher-order term describes the leading-order asymptotics of the flow around $x = 0$.

To overcome this issue, we construct a time-asymptotic expansion of u_i with pointwise estimates for the remainder. The leading-order term is the diffusion wave θ_i and the higher-order terms are *higher-order diffusion waves* $\xi_{i;n}$ $(n \geq 1)$ defined in the next section. It turns out that $|\xi_{i;n}(x,t)| \lesssim t^{-(2-1/2^n)}$ for $x = O(1)$ as $t \to \infty$. Setting $n = 1$, we see that $|\xi_{i;1}(x,t)| \lesssim t^{-3/2}$ for $x = O(1)$. The pointwise estimates for the remainder imply $|(u_i - \xi_{i;1})(x,t)| \lesssim t^{-7/4}$ for $x = O(1)$, thus it is rigorously proved that $\xi_{i;1}$ describes the leading-order asymptotics of u_i for $x = O(1)$. In addition, thanks to the pointwise estimates, our asymptotic expansion is valid not only in the $L^2(\mathbb{R})$-norm but also in the $L^1(\mathbb{R})$-norm.

The proof is based on pointwise estimates of Green's function, and the basic strategy follows that of [10]. The most non-trivial part of the proof is perhaps the definition of the higher-order diffusion waves $\xi_{i;n}$ $(n \geq 1)$; see (7). Although the differential equation defining $\xi_{i;n}$ does not seem to have a simple solution formula such as (4),[1] we use its structure (by the help of Lemma A.1) to analyze cancellation effects which are crucial in nonlinear estimates; see the proof of Lemma 3.6.

Before concluding the introduction, we briefly comment on related works. Diffusion wave approximations and pointwise estimates of solutions has been extensively studied for hyperbolic–parabolic systems [10], hyperbolic–elliptic systems [3], hyperbolic balance laws [13,15], the Boltzmann equation [8], and so on. In these works, nonlinear diffusion waves similar to θ_i were constructed and pointwise estimates of solutions were obtained. However, to the best of our knowledge, time-asymptotic expansions with pointwise estimates have not been obtained previously. We mention that the author already analyzed the second-order term $\xi_{i;1}$ in connection with a fluid–structure interaction problem in [7]; the complete asymptotic expansion, however, was not given. We also comment that for multidimensional incompressible Navier–Stokes equations, time-asymptotic expansions were studied for example in [1,2]. Because the nonlinearity is weaker compared to the 1D case, nonlinear waves similar to $\xi_{i;n}$ do not appear in these works.

In the next section, we state our main results. These are proven in Sect. 3.

2. Main Results

To state our main results (Theorem 2.1) we start by defining and discussing the properties of the higher-order diffusion waves $\xi_{i;n}$ $(n \geq 1)$ mentioned in the introduction.

[1] Nevertheless, we provide accurate asymptotic analysis in Propositions 2.1 and 2.2.

2.1. Higher-Order Diffusion Waves

Let (v, u) be the solution to (1). Then define u_i by (2) and set

$$M_i = \int_{-\infty}^{\infty} u_i(x, 0)\, dx. \tag{6}$$

Let $\xi_{i;0} = \theta_i/2$ with θ_i defined by (3). We then define the higher-order diffusion waves $\xi_{i;n}$ $(n \geq 1)$ inductively by the equations

$$\begin{cases} \partial_t \xi_{i;n} + \lambda_i \partial_x \xi_{i;n} + \partial_x(\theta_i \xi_{i;n}) + \partial_x(\theta_{i'} \xi_{i';n-1}) = \frac{\nu}{2}\partial_x^2 \xi_{i;n}, & x \in \mathbb{R}, t > 0, \\ \xi_{i;n}(x, 0) = 0, & x \in \mathbb{R}. \end{cases} \tag{7}$$

Here $\lambda_i = (-1)^{i-1}c$ and $i' = 3 - i$, that is, $1' = 2$ and $2' = 1$. We remind the reader that $c = \sqrt{-p'(1)} > 0$.

Although we do not have a simple explicit formula for $\xi_{i;n}$ like (4), we can still understand its asymptotic behavior quite well. To explain this, we introduce

$$\alpha_n = 2 - \frac{1}{2^{n+1}}, \quad \beta_n = \frac{3}{2} - \frac{1}{2^{n+1}} \quad (n \geq -1)$$

and

$$\begin{aligned} \psi_n(x, t; \lambda) &= [(x - \lambda(t+1))^2 + (t+1)]^{-\alpha_n/2}, \\ \tilde{\psi}_n(x, t; \lambda) &= [|x - \lambda(t+1)|^{\alpha_n} + (t+1)^{\beta_n}]^{-1}, \\ \Psi_{i;n}(x, t) &= \psi_n(x, t; \lambda_i) + \tilde{\psi}_n(x, t; \lambda_{i'}). \end{aligned} \tag{8}$$

Then we have the following decay estimates for $\xi_{i;n}$ (we postpone the proof until we later prove a finer version in Lemma 3.1):

Proposition 2.1. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. For $k \geq 0$, if ε is sufficiently small, we have*

$$|\partial_x^k \xi_{i;n}(x, t)| \leq C_{n,k} \varepsilon^{n+1} (t+1)^{-k/2} \Psi_{i;n-1}(x, t)$$

for some positive constant $C_{n,k}$. In particular, when $|x| \leq K$ for some fixed $K > 0$, we have

$$|\xi_{i;n}(x, t)| \leq C_{n,K}(t+1)^{-\alpha_{n-1}}$$

for some $C_{n,K} > 0$. Moreover, for any $1 \leq p \leq \infty$, there exists $C_{n,p} > 0$ such that

$$\|\xi_{i;n}(\cdot, t)\|_{L^p} \leq C_{n,p}(t+1)^{-(\alpha_{n-1} - 1/p)/2}.$$

We can also prove more detailed estimates if we focus on x with $(-1)^{i-1}x \geq 0$. Let

$$g(z) = \partial_x e^{-\frac{x^2}{2\nu}} = -(x/\nu)e^{-\frac{x^2}{2\nu}}, \tag{9}$$

$$f_{i;0}(z) = \frac{\sqrt{\nu}}{\sqrt{2}}\left(e^{\frac{M_i}{\nu}} - 1\right) e^{-\frac{z^2}{2\nu}} \left[\sqrt{\pi} + \left(e^{\frac{M_i}{\nu}} - 1\right)\int_z^{\infty} e^{-\xi^2}\, d\xi\right]^{-1}, \tag{10}$$

and

$$f_{i;n}(z) = \int_{(-1)^{i-1}z}^{\infty} [\xi - (-1)^{i-1}z]^{-(1-1/2^n)} \xi e^{-\frac{\xi^2}{2\nu}} \, d\xi. \tag{11}$$

We then have the following asymptotic formula. This is obtained from Lemma 3.2 proved in the next section.

Proposition 2.2. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. For any $K > 0$, if ε is sufficiently small, there exist $A_{i;n}$, $B_{i;n}$, and $C_n > 0$ such that*

$$\left| \left\{ \xi_{i;n}(x,t) - \frac{A_{i;n}}{(t+1)^{\alpha_{n-1}/2}} f_{i;n}\left(\frac{x-\lambda_i(t+1)}{\sqrt{t+1}}\right) - \frac{B_{i;n}}{(t+1)^{\alpha_{n-1}/2}} g\left(\frac{x-\lambda_i(t+1)}{\sqrt{t+1}}\right) \right\} \right| \\ \leq C_n \varepsilon^{n+1} \psi_n(x,t;\lambda_i)$$

for x with $-K \leq (-1)^{i-1}x$. The constants $A_{i;n}$ and $B_{i;n}$ are determined from (M_1, M_2) defined by (6).

Remark 2.1. The function $f_{i;n}$ appears in [12, Section 4]. It is shown that $f_{i;n}(z)$ decays exponentially as $(-1)^{i-1}z \to \infty$ but decays algebraically as $f_{i;n}(z) \sim z^{-\alpha_{n-1}}$ in the limit $(-1)^{i-1}z \to -\infty$. In particular, if $|x| \leq K$ for some fixed $K > 0$, we have

$$(t+1)^{-\alpha_{n-1}/2} f_{i;n}\left(\frac{x-\lambda_i(t+1)}{\sqrt{t+1}}\right) \sim t^{-\alpha_{n-1}}$$

Remark 2.2. With some additional effort, we can show that, for $-K \leq (-1)^{i-1}x$, the higher-order diffusion waves $\xi_{i;n}$ ($n \geq 1$) are asymptotically equivalent to the higher-order terms of the asymptotic expansion constructed in [12].

2.2. Time-Asymptotic Expansion with Pointwise Remainder Estimates

Let $\boldsymbol{u}_0 = (v_0 - 1, u_0)$ and denote its anti-derivatives by $\boldsymbol{u}_0^{\pm}$, that is,

$$\boldsymbol{u}_0^-(x) = \int_{-\infty}^{x} \boldsymbol{u}_0(y)\, dy, \quad \boldsymbol{u}_0^+(x) = \int_{x}^{\infty} \boldsymbol{u}_0(y)\, dy.$$

Our main theorem is the following:

Theorem 2.1. *For $\boldsymbol{u}_0 = (v_0 - 1, u_0) \in H^6(\mathbb{R}) \times H^6(\mathbb{R})$, let (v, u) be the solution to (1). Define u_i, θ_i, and $\xi_{i;n}$ by (2), (3), and (7), respectively. Set*

$$u_{i;1} = \xi_{i;1} + \gamma_{i'} \partial_x \theta_{i'}, \quad u_{i;n} = \xi_{i;n} + \gamma_{i'} \partial_x \xi_{i';n-1} \quad (n \geq 2),$$

where $i' = 3 - i$ and $\gamma_i = (-1)^i \nu/(4c)$. Then for $n \geq 1$, there exist positive constants δ_n and C_n such that if

$$\delta := \|\boldsymbol{u}_0\|_6 + \sup_{x \in \mathbb{R}} [(|x|+1)^{\alpha_n} |\boldsymbol{u}_0(x)| + (|x|+1)^{5/4} |\boldsymbol{u}_0'(x)|] \\ + \sup_{x > 0} [(|x|+1)^{\beta_n} (|\boldsymbol{u}_0^-(-x)| + |\boldsymbol{u}_0^+(x)|)] \leq \delta_n, \tag{12}$$

the solution (v, u) satisfies the pointwise estimates

$$\left|\left(u_i - \theta_i - \sum_{k=1}^{n} u_{i;k}\right)(x,t)\right| \leq C_n \delta \Psi_{i;n}(x,t)$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Here $\Psi_{i;n}$ is defined by (8).

As a corollary, we obtain the following L^p-decay estimates. Combining this with Proposition 2.1, it follows that $u_i \sim \theta_i + \sum_{n=1}^{\infty} \xi_{i;n}$ is a time-asymptotic expansion in the $L^p(\mathbb{R})$-norm for all $1 \leq p \leq \infty$.

Corollary 2.1. *Under the assumptions of Theorem 2.1, we have the optimal L^p-decay estimate*

$$\left\|\left(u_i - \theta_i - \sum_{k=1}^{n} \xi_{i;k}\right)(\cdot,t)\right\|_{L^p} \leq C_n \delta (t+1)^{-(\alpha_n - 1/p)/2} \quad (1 \leq p \leq \infty).$$

Proof. The same bound for $u_i - \theta_i - \sum_{k=1}^{n} u_{i;k}$ easily follows from Theorem 2.1. We can replace $u_{i;k}$ by $\xi_{i;k}$ thanks to (4) and Proposition 2.1. □

We also obtain the following local-in-space decay estimates:

Corollary 2.2. *Under the assumptions of Theorem 2.1, when $|x| \leq K$ for some fixed $K > 0$, we have*

$$\left|\left(u_i - \sum_{k=1}^{n} \xi_{i;k}\right)(x,t)\right| \leq C_{n,K} \delta (t+1)^{-\alpha_n}.$$

Moreover, there exist constants $\{A_{i;k}\}_{k=1}^{n}$ determined from (M_1, M_2) such that

$$\left|u_i(x,t) - \sum_{k=1}^{n} \frac{A_{i;k}}{(t+1)^{\alpha_{k-1/2}}} f_{i;k}\left(\frac{x - \lambda_i(t+1)}{\sqrt{t+1}}\right)\right| \leq C_{n,K} \delta (t+1)^{-\alpha_n}.$$

Here M_i and $f_{i;k}$ are defined by (6) and (11), respectively.

Proof. Again, the same bound for $u_i - \theta_i - \sum_{k=1}^{n} u_{i;k}$ easily follows from Theorem 2.1. We can then replace $u_{i;k}$ by $\xi_{i;k}$ thanks to (4) and Proposition 2.1. The second inequality follows from Proposition 2.2. □

By Corollary 2.2, and also Remark 2.1, we now have a detailed picture of the power-law asymptotics of the solution around $x = 0$ where the diffusion waves decay exponentially.

Remark 2.3. The term $\gamma_{i'} \partial_x \theta_i$ and $\gamma_{i'} \partial_x \xi_{i;n}$ are both neglected in the two corollaries above. These are negligible in the $L^p(\mathbb{R})$-norm and locally around $x = 0$ but are important in the neighborhood of the other characteristic line $x = -\lambda_i t$. For this reason, these terms are required in the statement of Theorem 2.1.

Remark 2.4. The rather strong H^6-regularity is required to invoke pointwise estimates of $\partial_x(u_i - \theta_i)$ provided by [10, Theorem 2.6 and Remark 2.8]. The proof involves energy estimates up to the $H^6(\mathbb{R})$-norm. These also imply a unique global-in-time existence theorem in appropriate Sobolev spaces. Off course, global-in-time existence of solutions can be proved with much lower regularity [4,9], but proving detailed pointwise estimates for such data seems to be difficult at this point.

Remark 2.5. We add a comment on taking the limit $n \to \infty$ in Theorem 2.1 and also on a possible route to expand the solution to even higher order. A careful examination of the proof shows that the constant C_n in Theorem 2.1 grows as 2^n. So we cannot simply take the limit. However, as pointed out in [12, p. 1955], it might be possible to take the limit by adding a logarithmic weight:

$$\left|\left(u_i - \theta_i - \sum_{k=1}^{\infty} u_{i;k}\right)(x,t)\right| \leq C_\infty \delta \log(t+2) \Psi_{i;\infty}(x,t).$$

Here, C_∞ is a constant independent of n and $\Psi_{i;\infty}(x,t) = \lim_{n\to\infty} \Psi_{i;n}(x,t)$. Since $\|\Psi_{i;\infty}(\cdot,t)\|_{L^\infty} \lesssim t^{-1}$, to study an asymptotic expansion beyond the order $O(t^{-1}\log t)$, it seems that we need to identify waves describing this order. Such waves are identified for example in [5] for generalized Burgers equations. Analogous results for hyperbolic–parabolic systems are, as far as I know, not known. If such waves are identified, we might be able to expand the solution beyond the order $O(t^{-1}\log t)$. And drawing an analogy between the heat equation, terms beyond this order should also depend on higher-order moments $\int_{-\infty}^{\infty} x^k u_i(x,0)\,dx$ and not just on $M_i = \int_{-\infty}^{\infty} u_i(x,0)\,dx$.

3. Proof

The following function appears frequently in the subsequent part of the paper:

$$\Theta_\alpha(x,t;\lambda,\mu) = (t+1)^{-\alpha/2} e^{-\frac{(x-\lambda(t+1))^2}{\mu(t+1)}}. \tag{13}$$

Here $\lambda \in \mathbb{R}$ and $\alpha, \mu > 0$. Note that

$$|\theta_i(x,t)| \leq A_0|M_i|\Theta_1(x,t;\lambda_i,2\nu), \quad \Theta_{\alpha_n}(x,t;\lambda,\mu) \leq B_0\psi_n(x,t;\lambda) \tag{14}$$

for some positive constants A_0 and B_0. In what follows, the symbols C and ν^* denote sufficiently large constants.

3.1. Pointwise Estimates of the Higher-Order Diffusion Waves

We start with the proofs of Propositions 2.1 and 2.2.

Proposition 2.1 follows from the following finer version:

Lemma 3.1. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. If ε is sufficiently small, we have*

$$
\begin{aligned}
&|\partial_x^k \xi_{i;n}(x,t) - (-1)^i (2c)^{-1} \partial_x^k (\theta_{i'} \xi_{i';n-1})(x,t)| \\
&\quad \leq C_{n,k} \varepsilon^{n+1} (t+1)^{-k/2} \psi_{n-1}(x,t;\lambda_i)
\end{aligned}
\tag{15}
$$

for any integer $k \geq 0$. In particular, we have

$$
\begin{aligned}
|\partial_x^k \xi_{i;n}(x,t)| &\leq C_{n,k} \varepsilon^{n+1} (t+1)^{-k/2} [\psi_{n-1}(x,t;\lambda_i) + \Theta_{2\beta_{n-1}}(x,t;\lambda_{i'},\nu^*)] \\
&\leq C_{n,k} \varepsilon^{n+1} (t+1)^{-k/2} \Psi_{i;n-1}(x,t).
\end{aligned}
$$

Proof. We assume $t \geq 4$ in what follows (the lemma is otherwise easier to prove). The lemma is trivial for $n = 0$ if we set $\xi_{i;0} = \theta_i/2$ and $\xi_{i';-1} = 0$. So it suffices to prove the lemma for n assuming that it holds for $n - 1 \geq 0$. In what follows, we only prove the case of $i = 1$ and $k = 0$ since the other cases are similar. Note first that, by (7) and Duhamel's principle, we have $\xi_{i;n}(x,t) = \zeta_{1;n}(x,t) + \eta_{1;n}(x,t)$, where

$$
\zeta_{1;n}(x,t) = -(2\pi\nu)^{-1/2} \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \partial_x(\theta_2 \xi_{2;n-1})(y,s)\, dyds
\tag{16}
$$

and

$$
\eta_{1;n}(x,t) = -(2\pi\nu)^{-1/2} \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \partial_x(\theta_1 \xi_{1;n})(y,s)\, dyds.
\tag{17}
$$

We first consider $\zeta_{1;n}(x,t)$. Set $I(x,t) = -\sqrt{2\pi\nu}\zeta_{1;n}(x,t)$ and $f = \theta_2 \xi_{2;n-1}$. By Lemma A.1, we have

$$
I(x,t) = (2c)^{-1} \sqrt{2\pi\nu} f(x,t) + I_1(x,t) + I_2(x,t),
$$

where

$$
I_1(x,t) = \int_0^{t^{1/2}} \int_{-\infty}^{\infty} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \right\} f(y,s)\, dyds
$$

and

$$
\begin{aligned}
I_2(x,t) = &-(2c)^{-1} \int_{-\infty}^{\infty} (t-t^{1/2})^{-1/2} e^{-\frac{(x-y-c(t-\sqrt{t}))^2}{2\nu(t-\sqrt{t})}} f(y,t^{1/2})\, dy \\
&- (2c)^{-1} \int_{t^{1/2}}^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} L_2 f(y,s)\, dyds \\
=: &I_{21}(x,t) + I_{22}(x,t).
\end{aligned}
$$

Here $L_2 = \partial_t - c\partial_x - (\nu/2)\partial_x^2$. By the induction hypothesis, we have

$$
|f(x,t)| \leq C\varepsilon^{n+1} \Theta_{\alpha_{n-2}+1}(x,t;-c,\nu^*).
$$

By Lemmas A.2 and A.3, we obtain

$$|I_1(x,t)| + |I_{21}(x,t)| \le C\varepsilon^{n+1}\Theta_{\alpha_{n-1}}(x,t;c,\nu^*) \le C\varepsilon^{n+1}\psi_{n-1}(x,t;c).$$

Next, note that (3) and (7) imply

$$|L_2 f(x,t)| \le C\varepsilon^{n+1}\Theta_{\alpha_{n-2}+3}(x,t;-c,\nu^*).$$

Then by Lemma A.4, we obtain

$$|I_{22}(x,t)| \le C\varepsilon^{n+1}\psi_{n-1}(x,t;c).$$

We have thus proved that

$$|\zeta_{1;n}(x,t) + (2c)^{-1}(\theta_2\xi_{2;n-1})(x,t)| \le C\varepsilon^{n+1}\psi_{n-1}(x,t;c). \tag{18}$$

We next consider $\eta_{1;n}(x,t)$. Note that it is the solution to

$$\begin{cases} \partial_t\eta_{1;n} + c\partial_x\eta_{1;n} + \partial_x(\theta_1\eta_{1;n}) = \frac{\nu}{2}\partial_x^2\eta_{1;n} - \partial_x(\theta_1\zeta_{1;n}), & x\in\mathbb{R}, t>0,\\ \eta_{1;n}(x,0)=0, & x\in\mathbb{R}. \end{cases}$$

This variable coefficient equation can be solved by an iteration scheme. Let $\eta_{1;n}^{(1)}$ be the solution to

$$\begin{cases} \partial_t\eta_{1;n}^{(1)} + c\partial_x\eta_{1;n}^{(1)} = \frac{\nu}{2}\partial_x^2\eta_{1;n}^{(1)} - \partial_x(\theta_1\zeta_{1;n}), & x\in\mathbb{R}, t>0,\\ \eta_{1;n}^{(1)}(x,0)=0, & x\in\mathbb{R} \end{cases}$$

and $\eta_{1;n}^{(k)}$ $(k\ge 2)$ be the solution to

$$\begin{cases} \partial_t\eta_{1;n}^{(k)} + c\partial_x\eta_{1;n}^{(k)} = \frac{\nu}{2}\partial_x^2\eta_{1;n}^{(k)} - \partial_x(\theta_1\eta_{1;n}^{(k-1)}) & x\in\mathbb{R}, t>0,\\ \eta_{1;n}^{(k)}(x,0)=0, & x\in\mathbb{R}. \end{cases}$$

Then we can write $\eta_{1;n}$ as

$$\eta_{1;n}(x,t) = \sum_{k=1}^{\infty}\eta_{1;n}^{(k)}(x,t). \tag{19}$$

We now give bounds for $\eta_{1;n}^{(k)}$ $(k\ge 1)$ inductively. Note first that

$$\eta_{1;n}^{(1)}(x,t) = -(2\pi\nu)^{-1/2}\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}\partial_x(\theta_1\zeta_{1;n})(y,s)\,dyds \tag{20}$$

and that (18) implies

$$|(\theta_1\zeta_{1;n})(x,t)| \le A_1\varepsilon^{n+2}\Theta_{\alpha_{n-1}+1}(x,t;c,\nu')$$

for some positive constants A_1 and ν'. Then by [10, Lemma 3.2], we obtain

$$|\eta_{1;n}^{(1)}(x,t)| \le MA_1\varepsilon^{n+2}\Theta_{\alpha_{n-1}}(x,t;c,\nu')$$

for some $M > 0$. This means that the inequality

$$|\eta_{1;n}^{(l)}(x,t)| \leq MA_1(MA_0)^{l-1}\varepsilon^{n+l+1}\Theta_{\alpha_{n-1}}(x,t;c,\nu') \tag{21}$$

holds for $l = 1$. We then show that (21) holds for $l = k+1$ assuming that it holds for $l = k$. By the induction hypothesis and (14), we have

$$|(\theta_1\eta_{1;n}^{(k)})(x,t)| \leq A_1(MA_0)^k\varepsilon^{n+k+2}\Theta_{\alpha_{n-1}+1}(x,t;c,\nu').$$

Applying [10, Lemma 3.2] again, this time to the integral representation

$$\eta_{1;n}^{(k+1)}(x,t) = -(2\pi\nu)^{-1/2}\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}\partial_x(\theta_1\eta_{1;n}^{(k)})(y,s)\,dyds,$$

we obtain

$$|\eta_{1;n}^{(k+1)}(x,t)| \leq MA_1(MA_0)^k\varepsilon^{n+k+2}\Theta_{\alpha_{n-1}}(x,t;c,\nu').$$

Therefore, (21) holds for any $l \geq 1$, and by taking ε sufficiently small, we get

$$|\eta_{1;n}(x,t)| = \left|\sum_{k=1}^{\infty}\eta_{1;n}^{(k)}(x,t)\right| \leq C\varepsilon^{n+2}\Theta_{\alpha_{n-1}}(x,t;c,\nu') \leq C\varepsilon^{n+2}\psi_{n-1}(x,t;c).$$

Combining this with (18), we obtain (15). □

Remark 3.1. The proof above can be modified to show that

$$\begin{aligned}&|\partial_x^k\xi_{i;m}(x,t) - (-1)^i(2c)^{-1}\partial_x^k(\theta_{i'}\xi_{i';m-1})(x,t)|\\&\quad\leq C_{n,k}\varepsilon^{m+1}(t+1)^{-k/2}\psi_{n-1}(x,t;\lambda_i)\end{aligned}$$

holds for all $m \geq n$ with the smallness of ε depending only on n and k.

We next prove Proposition 2.2. (The proof is rather lengthy and may be skipped; the rest of the paper can be read independently.) Define $\zeta_{i;n}$ and $\eta_{i;n}$ by (16) and (17), respectively. Then Proposition 2.2 is a direct consequence of the following lemma.

Lemma 3.2. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. Fix $k \geq 0$. For any $K > 0$, if ε is sufficiently small, there exist $A_{i;n}$, $B_{i;n}$, and $C_{n,k} > 0$ such that*

$$\begin{aligned}&\left|\partial_x^k\left\{\zeta_{i;n}(x,t) - \frac{A_{i;n}}{(t+1)^{\alpha_{n-1}/2}}f_{i;n}\left(\frac{x-\lambda_i(t+1)}{\sqrt{t+1}}\right)\right\}\right|\\&\quad\leq C_{n,k}\varepsilon^{n+1}(t+1)^{-k/2}\psi_n(x,t;\lambda_i)\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\left|\partial_x^k\left\{\eta_{i;n}(x,t) - \frac{B_{i;n}}{(t+1)^{\alpha_{n-1}/2}}g\left(\frac{x-\lambda_i(t+1)}{\sqrt{t+1}}\right)\right\}\right|\\&\quad\leq C_{n,k}\varepsilon^{n+2}(t+1)^{-k/2}\psi_n(x,t;\lambda_i)\end{aligned} \tag{23}$$

when $-K \leq (-1)^{i-1}x$. Here g and $f_{i;n}$ are defined by (9) and (11), respectively.

Proof. The lemma is proved by induction in n.

We first consider the case of $n = 1$. Let $i = 1$ and $k = 0$ (the other cases are similar). We start with the proof of (22). Note that (16) implies

$$\zeta_{1;1}(x,t) = -\frac{1}{2\sqrt{2\pi\nu}}\partial_x \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \theta_2^2(y,s)\, dyds.$$

In addition, by (4) and (10), we have

$$\theta_i(x,t) = (t+1)^{-1/2} f_{i;0}\left(\frac{x-\lambda_i(t+1)}{\sqrt{t+1}}\right).$$

Hence we may write

$$\theta_2^2(x,t) = \frac{a_{1;1}}{(t+1)\sqrt{2\pi\nu}} e^{-\frac{(x+c(t+1))^2}{2\nu(t+1)}} + \partial_x r(x,t),$$

where

$$a_{1;1} = \int_{-\infty}^{\infty} f_{2;0}^2(z)\, dz$$

and

$$r(x,t) = \int_{-\infty}^{x} \left(\theta_2^2(z,t) - \frac{a_{1;1}}{(t+1)\sqrt{2\pi\nu}} e^{-\frac{(z+c(t+1))^2}{2\nu(t+1)}}\right) dz.$$

Noting that $\lim_{x\to\infty} r(x,t) = 0$, we can show that

$$|r(x,t)| \le C\varepsilon^2 \Theta_2(x,t;-c,\nu^*), \quad |L_2 r(x,t)| \le C\varepsilon^2 \Theta_4(x,t;-c,\nu^*),$$

where $L_2 = \partial_t - c\partial_x - (\nu/2)\partial_x^2$. We then have

$$\begin{aligned}
\zeta_{1;1}(x,t) = &-\frac{a_{1;1}}{4\pi\nu}\partial_x \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2}(s+1)^{-1} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} e^{-\frac{(y+c(s+1))^2}{2\nu(s+1)}}\, dyds \\
&- \frac{1}{2\sqrt{2\pi\nu}}\partial_x \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \partial_y r(y,s)\, dyds \\
= &-\frac{a_{1;1}}{2\sqrt{2\pi\nu}}\partial_x \int_0^t (t+1)^{-1/2}(s+1)^{-1/2} e^{-\frac{(x-c(t-s)+c(s+1))^2}{2\nu(t+1)}}\, ds \\
&- \frac{1}{2\sqrt{2\pi\nu}}\partial_x \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \partial_y r(y,s)\, dyds.
\end{aligned}$$

Concerning the second term, similar calculations leading to the bound of $\zeta_{1;n}(x,t)$ in Lemma 3.1 imply

$$\left|\partial_x \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}} \partial_y r(y,s)\, dyds\right| \le C\varepsilon^2 \psi_n(x,t;c)$$

for $-K \le x$. For the first term, note that a simple change of variable yields

$$-\frac{(t+1)^{-3/4}}{\nu\sqrt{2c}} f_{1;1}\left(\frac{x-c(t+1)}{\sqrt{t+1}}\right)$$
$$= \partial_x \int_{-1}^{\infty} (t+1)^{-1/2}(s+1)^{-1/2} e^{-\frac{(x-c(t-s)+c(s+1))^2}{2\nu(t+1)}}\, ds.$$

Therefore,

$$\zeta_{1;1}(x,t) - \frac{a_{1;1}}{2\nu\sqrt{4\pi\nu c}}(t+1)^{-3/4} f_{1;1}\left(\frac{x-c(t+1)}{\sqrt{t+1}}\right)$$
$$= O(\varepsilon^2)\partial_x \int_{s\in(-1,0)\cup(t,\infty)} (t+1)^{-1/2}(s+1)^{-1/2} e^{-\frac{(x-c(t-s)+c(s+1))^2}{2\nu(t+1)}}\, ds$$
$$+ O(\varepsilon^2)\psi_n(x,t;c).$$

We then set

$$I(x,t) = \partial_x \int_{s\in(-1,0)\cup(t,\infty)} (t+1)^{-1/2}(s+1)^{-1/2} e^{-\frac{(x-c(t-s)+c(s+1))^2}{2\nu(t+1)}}\, ds$$

and show that $|I(x,t)| \le C\Theta_2(x,t;c,\nu^*)$ for $-K \le x$. We first consider Case (i) $|x-c(t+1)| \le (t+1)^{1/2}$. In this case, we simply have

$$|I(x,t)| \le C(t+1)^{-1} \le C\Theta_2(x,t;c,\nu^*).$$

We next consider Case (ii) $-K \le x \le c(t+1)-(t+1)^{1/2}$. The integral over $(-1,0)$ is easy to handle. For $s \in (t,\infty)$ on the other hand, when t is large (the case when t is not large is easier), we have

$$0 \le c(t+1) - x - 2K \le x - c(t-s) + c(s+1),$$
$$0 \le c(s+1) - K \le x - c(t-s) + c(s+1).$$

Hence

$$\left|\partial_x \int_{t}^{\infty} (t+1)^{-1/2}(s+1)^{-1/2} e^{-\frac{(x-c(t-s)+c(s+1))^2}{2\nu(t+1)}}\, ds\right|$$
$$\le C\int_0^{\infty} (s+1)^{-1/2} e^{-\frac{s^2}{C(t+1)}}\, ds \cdot \Theta_2(x,t;c,\nu^*) \le C\Theta_2(x,t;c,\nu^*).$$

We end the analysis of $\zeta_{1;1}$ by considering Case (iii) $x \ge c(t+1)+(t+1)^{1/2}$. When $s > -1$, we have

$$0 \le x - c(t+1) \le x - c(t-s) + c(s+1) = x - c(t+1) + 2c(s+1),$$
$$0 \le 2c(s+1) \le x - c(t-s) + c(s+1).$$

From these, it follows that $|I(x,t)| \le C\Theta_2(x,t;c,\nu^*)$ as in Case (ii). These prove (22) for $n=1$ by setting

$$A_{1;1} = \frac{a_{1;1}}{2\nu\sqrt{4\pi\nu c}}.$$

We next prove (23) by using the series representation (19). We first consider

$$\eta_{1;1}^{(1)}(x,t) = -(2\pi\nu)^{-1/2}\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}\,\partial_x(\theta_1\zeta_{1;1})(y,s)\,dyds.$$

The bound (22) for $\zeta_{1;1}(x,t)$ implies

$$(\theta_1\zeta_{1;1})(x,t) = \theta_1(x,t)\frac{A_{1;1}}{(t+1)^{3/4}}f_{1;1}\left(\frac{x-c(t+1)}{\sqrt{t+1}}\right) + O(\varepsilon^3)\Theta_{\alpha_n+1}(x,t;c,\nu^*).$$

This holds for all $x\in\mathbb{R}$ since $\theta_1(x,t)$ decays exponentially for $x\leq -K$. Plugging this into (20) and arguing similarly to the analysis of $\zeta_{1;1}$, we get

$$\begin{aligned}\eta_{1;1}^{(1)}(x,t) &= -\frac{A_{1;1}b_{1;1}}{2\pi\nu}\partial_x\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1/2}(s+1)^{-5/4}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}e^{-\frac{(y-c(s+1))^2}{2\nu(s+1)}}\,dyds\\ &\quad + O(\varepsilon^3)\psi_n(x,t;c)\\ &= -\frac{A_{1;1}b_{1;1}}{\sqrt{2\pi\nu}}\partial_x\int_0^t(t+1)^{-1/2}(s+1)^{-3/4}e^{-\frac{(x-c(t+1))^2}{2\nu(t+1)}}\,ds + O(\varepsilon^3)\psi_n(x,t;c)\\ &= -\frac{4A_{1;1}b_{1;1}}{\sqrt{2\pi\nu}}(t+1)^{-3/4}g\left(\frac{x-c(t+1)}{\sqrt{t+1}}\right) + O(\varepsilon^3)\psi_n(x,t;c),\end{aligned}$$

where

$$b_{1;1} = \int_{-\infty}^{\infty}(f_{1;0}f_1)(z)\,dz.$$

Similar analysis for $\eta_{1;1}^{(k)}(x,t)$ $(k\geq 2)$ shows that

$$\eta_{1;1}^{(k)}(x,t) = O(\varepsilon^{k+2})(t+1)^{-3/4}g\left(\frac{x-c(t+1)}{\sqrt{t+1}}\right) + O(\varepsilon^{k+2})\psi_n(x,t;c).$$

Taking the sum $\sum_{k=1}^{\infty}$, it follows that (23) holds for $n=1$ with

$$B_{1;1} = -\frac{4A_{1;1}b_{1;1}}{\sqrt{2\pi\nu}} + O(\varepsilon^4).$$

We next prove the lemma for n assuming that it holds for $n-1$. Let $i=1$ and $k=0$ (the other cases are similar). The induction hypothesis and Lemma 3.1 imply

$$\begin{aligned}h(x,t) &:= (\theta_2\xi_{2;n-1})(x,t) - \frac{A_{2;n-1}}{(t+1)^{\alpha_{n-2}/2}}\theta_2(x,t)f_{2;n-1}\left(\frac{x+c(t+1)}{\sqrt{t+1}}\right)\\ &\quad - \frac{B_{2;n-1}}{(t+1)^{\alpha_{n-2}/2}}\theta_2(x,t)g\left(\frac{x+c(t+1)}{\sqrt{t+1}}\right) = O(\varepsilon^{n+1})\Theta_{\alpha_{n-1}+1}(x,t;-c,\nu^*)\end{aligned}$$

for all $x\in\mathbb{R}$ (not just for $x\leq K$). We also have $\partial_x h(x,t) = O(\varepsilon^{n+1})\Theta_{\alpha_{n-1}+2}(x,t;-c,\nu^*)$. Using these, we can show that

$$L_2h(x,t) = O(\varepsilon^{n+1})\Theta_{\alpha_{n-1}+3}(x,t;-c,\nu^*),$$

where $L_2 = \partial_t - c\partial_x - (\nu/2)\partial_x^2$. Then similar calculations leading to the bound of $\zeta_{1;1}(x,t)$ above imply (22) with

$$A_{1;n} = \frac{1}{\nu\sqrt{4\pi\nu c}}(A_{2;n-1}a_{1;n} + B_{2;n-1}b_{1;n}),$$

where

$$a_{1;n} = \int_{-\infty}^{\infty}(f_{2;0}f_{2;n-1})(-z)\,dz, \quad b_{1;n} = \int_{-\infty}^{\infty}(gf_{2;0})(z)\,dz.$$

The bound (23) for $\eta_{1;n}(x,t)$ is proved in a way similar to that for $n = 1$. This ends the proof of the lemma. □

For the proof of Theorem 2.1, it is convenient to unify $(\xi_{i;n})_{n=1}^{\infty}$ into a single function

$$\Xi_i(x,t) = \sum_{n=1}^{\infty}\xi_{i;n}(x,t). \tag{24}$$

Taking the infinite sum of (7), we see that (Ξ_1, Ξ_2) is the solution to the system

$$\begin{cases} \partial_t\Xi_1 + c\partial_x\Xi_1 + \partial_x(\theta_2^2/2 + \theta_1\Xi_1 + \theta_2\Xi_2) = \frac{\nu}{2}\partial_x^2\Xi_1, & x \in \mathbb{R}, t > 0, \\ \partial_t\Xi_2 - c\partial_x\Xi_2 + \partial_x(\theta_1^2/2 + \theta_1\Xi_1 + \theta_2\Xi_2) = \frac{\nu}{2}\partial_x^2\Xi_2, & x \in \mathbb{R}, t > 0, \\ \Xi_1(x,0) = \Xi_2(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{25}$$

Then Lemma 3.1 and Remark 3.1 imply the following:

Lemma 3.3. *Let*

$$\Xi_{i;n}(x,t) = \sum_{m=n+1}^{\infty}\xi_{i;m}(x,t) \quad (n \geq -1).$$

Here $\xi_{i;0} = \theta_i/2$. *Then for* $n \geq 0$, *if* $\varepsilon = \max(M_1, M_2)$ *is sufficiently small, we have*

$$|\partial_x^k\Xi_{i;n}(x,t) - (-1)^i(2c)^{-1}\partial_x^k(\theta_{i'}\Xi_{i';n-1})(x,t)| \leq C_{n,k}\varepsilon^{n+2}(t+1)^{-k/2}\psi_n(x,t;\lambda_i)$$

for any integer $k \geq 0$. *In particular, we have*

$$|\partial_x^k\Xi_{i;n}(x,t)| \leq C_{n,k}\varepsilon^{n+2}(t+1)^{-k/2}\Psi_{i;n}(x,t).$$

3.2. Proof of Theorem 2.1

Let us explain the strategy to prove Theorem 2.1. Note first that by Lemma 3.3, it suffices to prove the following:

Theorem 3.1. *For $\boldsymbol{u}_0 = (v_0 - 1, u_0) \in H^6(\mathbb{R}) \times H^6(\mathbb{R})$, let (v, u) be the solution to (1). Define u_i, θ_i, and Ξ_i by (2), (3), and (25), respectively. Then for $n \geq 1$, there exist positive constants δ_n and C_n such that if $\delta \leq \delta_n$, where δ is defined by (12), the solution (v, u) satisfies the pointwise estimates*

$$|(u_i - \theta_i - \Xi_i - \gamma_{i'}\partial_x\theta_{i'} - \gamma_{i'}\partial_x\Xi_{i'})(x, t)| \leq C_n\delta\Psi_{i;n}(x, t)$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Here $\gamma_i = (-1)^i\nu/(4c)$ and $i' = 3 - i$.

To prove Theorem 3.1, we set

$$v_i = u_i - \theta_i - \Xi_i - \gamma_{i'}\partial_x\theta_{i'} - \gamma_{i'}\partial_x\Xi_{i'} \tag{26}$$

and define $P(t)$ by

$$P(t) := \sum_{i=1}^{2} \sup_{0\leq s\leq t} \left| v_i(\cdot, s)\Psi_{i;n}(\cdot, s)^{-1} \right|_{L^\infty}. \tag{27}$$

Our goal is then to prove the inequality

$$P(t) \leq C\delta + C(\delta + P(t))^2 \quad (t \geq 0). \tag{28}$$

From this inequality, taking δ sufficiently small, we can conclude that $P(t) \leq C\delta$ for all $t \geq 0$ by a standard argument (see Sect. 3.2.4).

Remark 3.2. For the argument above to work, we first need to show that $P(t)$ is finite. This can be proved, for example, by examining the iterative scheme in [6, Section 2.1] for the construction of the local-in-time solution to (1). The key step of the scheme consists of solving a variable coefficient parabolic equation, and by the Levi parametrix method, we can prove a gaussian upper bound for the fundamental solution. This bound allows us to control the spatial decay of each approximate solution, and by taking the limit, we can check that $P(t)$ is finite at least for a short period of time. By the calculations below, it follows that (28) and hence $P(t) \leq C\delta$ hold for this short duration. Then a standard continuity argument shows that $P(t) \leq C\delta$ actually holds for all $t \geq 0$.

The proof of (28) is based on pointwise estimates of Greens' function [10,11] which we shall explain in the next section. We also give an integral formulation of (1). In the remaining sections, we prove bounds for the terms appearing in the integral equations which yield (28).

3.2.1. Pointwise Estimates of Green's Function and Integral Equations Our equations (1) can be written in the form

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} \boldsymbol{u}_{xx} + \begin{pmatrix} 0 \\ N_x \end{pmatrix} \tag{29}$$

with

$$\boldsymbol{u} = \begin{pmatrix} v-1 \\ u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}, \quad N = -p(v) + p(1) - c^2(v-1) - \nu \frac{v-1}{v} u_x. \tag{30}$$

The matrix A has right and left eigenvectors r_i and l_i ($i = 1, 2$), corresponding to the eigenvalue $\lambda_i = (-1)^{i-1}c$, given by

$$r_i = \frac{2c}{p''(1)} \begin{bmatrix} (-1)^i \\ c \end{bmatrix}, \quad l_i = \frac{p''(1)}{4c} \left[(-1)^i \ 1/c \right].$$

We note that (2) can be written as $u_i = l_i(v-1\ u)^T$.

We define Green's function $G = G(x,t) \in \mathbb{R}^{2\times 2}$ for the linearization of (29) as the solution to

$$\begin{cases} \partial_t G + A\partial_x G = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} \partial_x^2 G, & x \in \mathbb{R},\ t > 0, \\ G(x,0) = \delta(x) I_2, & x \in \mathbb{R}, \end{cases}$$

where $\delta(x)$ is the Dirac delta function and I_2 is the 2×2 identity matrix. In addition, define $G^* = G^*(x,t) \in \mathbb{R}^{2\times 2}$ by

$$G^*(x,t) = \frac{1}{2(2\pi\nu t)^{1/2}} e^{-\frac{(x-ct)^2}{2\nu t}} \begin{pmatrix} 1 & -1/c \\ -c & 1 \end{pmatrix} + \frac{1}{2(2\pi\nu t)^{1/2}} e^{-\frac{(x+ct)^2}{2\nu t}} \begin{pmatrix} 1 & 1/c \\ c & 1 \end{pmatrix}.$$

The next theorem is of fundamental importance in our analysis.

Theorem 3.2. ([10, Theorem 5.8] and [11, Theorem 1.3]) *For any $k \geq 0$, we have*

$$\begin{aligned} &\left| \partial_x^k G(x,t) - \partial_x^k G^*(x,t) - e^{-\frac{c^2}{\nu}t} \sum_{j=0}^{k} \delta^{(k-j)}(x) Q_j(t) \right| \\ &\leq C(t+1)^{-\frac{1}{2}} t^{-\frac{k+1}{2}} \sum_{i=1}^{2} e^{-\frac{(x-\lambda_i t)^2}{Ct}}, \end{aligned}$$

where $\delta^{(k)}(x)$ is the k-th derivative of the Dirac delta function and $Q_j = Q_j(t)$ is a 2×2 polynomial matrix. In particular,

$$Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & -1/\nu \\ -c^2/\nu & 0 \end{pmatrix}.$$

Moreover, with $\gamma_i = (-1)^i \nu/(4c)$*, we have*

$$
\begin{aligned}
&\partial_x^k G(x,t) - \partial_x^k G^*(x,t) - \partial_x^{k+1} \sum_{i=1}^{2} \gamma_i \frac{e^{-\frac{(x-\lambda_i t)^2}{2\nu t}}}{(2\pi\nu t)^{1/2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad - e^{-\frac{c^2}{\nu}t} \sum_{j=0}^{k} \delta^{(k-j)}(x) Q_j(t) \\
&= O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{k+1}{2}} e^{-\frac{(x-ct)^2}{Ct}} \begin{pmatrix} 1 & -1/c \\ -c & 1 \end{pmatrix} \\
&\quad + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{k+1}{2}} e^{-\frac{(x+ct)^2}{Ct}} \begin{pmatrix} 1 & 1/c \\ c & 1 \end{pmatrix} \\
&\quad + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{k+2}{2}} \sum_{i=1}^{2} e^{-\frac{(x-\lambda_i t)^2}{Ct}}.
\end{aligned}
$$

Here $O(1)$ *is a bounded scalar function.*

For the analysis of u_i, we need pointwise estimates for

$$
g_i = \left(g_{i1}\ g_{i2}\right) := l_i G \left(r_1\ r_2\right), \quad g_i^* = \left(g_{i1}^*\ g_{i2}^*\right) := l_i G^* \left(r_1\ r_2\right).
$$

We note that

$$
g_{ij}^* = (2\pi\nu t)^{-1/2} e^{-\frac{(x-\lambda_i t)^2}{2\nu t}} \delta_{ij}, \tag{31}
$$

where δ_{ij} is the Kronecker delta. Then Theorem 3.2 implies

$$
\begin{aligned}
&\left| \partial_x^k g_i(x,t) - \partial_x^k g_i^*(x,t) - e^{-\frac{c^2}{\nu}t} \sum_{j=0}^{k} \delta^{(k-j)}(x) q_{ik}(t) \right| \\
&\quad \leq C(t+1)^{-\frac{1}{2}} t^{-\frac{k+1}{2}} \sum_{i=1}^{2} e^{-\frac{(x-\lambda_i t)^2}{Ct}},
\end{aligned} \tag{32}
$$

where

$$
q_{ik}(t) = l_i Q_k(t) \left(r_1\ r_2\right).
$$

Moreover, we have

$$
\begin{aligned}
&\partial_x^k g_i(x,t) - \partial_x^k g_i^*(x,t) - \gamma_{i'} \partial_x^{k+1} g_{i'}^*(x,t) - e^{-\frac{c^2}{\nu}t} \sum_{j=0}^{k} \delta^{(k-j)}(x) q_{ik}(t) \\
&= O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{k+1}{2}} e^{-\frac{(x-ct)^2}{Ct}} + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{k+2}{2}} e^{-\frac{(x+ct)^2}{Ct}}.
\end{aligned} \tag{33}
$$

We next write down an integral equation for v_i defined by (26). Let

$$
n = l_i \begin{pmatrix} 0 \\ N \end{pmatrix} = \frac{p''(1)}{4c^2} N, \quad n^* = -\theta_1^2/2 - \theta_2^2/2 - \theta_1 \Xi_1 - \theta_2 \Xi_2,
$$

where Ξ_i and N are defined by (24) and (30), respectively. Then by Duhamel's principle, we obtain

Lemma 3.4. *The function v_i defined by (26) satisfies the integral equation*

$$\begin{aligned} v_i(x,t) &= \int_{-\infty}^{\infty} g_i(x-y,t)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(y,0)\,dy - \int_{-\infty}^{\infty} g_i^*(x-y,t)\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}(y,0)\,dy \\ &\quad - \gamma_{i'}\int_{-\infty}^{\infty} \partial_x g_{i'}^*(x-y,t)\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}(y,0)\,dy \\ &\quad + \int_0^t\int_{-\infty}^{\infty} g_{ii}^*(x-y,t-s)\partial_x(n-n^*)(y,s)(y,s)\,dyds \\ &\quad + \sum_{j=1}^{2}\int_0^t\int_{-\infty}^{\infty}(g_{ij}-g_{ij}^*)(x-y,t-s)\partial_x n(y,s)\,dyds \\ &\quad - \gamma_{i'}\int_0^t\int_{-\infty}^{\infty}\partial_x g_{i'i'}^*(x-y,t-s)\partial_x n^*(y,s)(y,s)\,dyds. \end{aligned}$$

Here $\gamma_i = (-1)^i\nu/(4c)$ and $i' = 3-i$.

We set

$$\begin{aligned} \mathcal{I}_i(x,t) &= \int_{-\infty}^{\infty} g_i(x-y,t)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(y,0)\,dy - \int_{-\infty}^{\infty} g_i^*(x-y,t)\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}(y,0)\,dy \\ &\quad - \gamma_{i'}\int_{-\infty}^{\infty} \partial_x g_{i'}^*(x-y,t)\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}(y,0)\,dy \end{aligned} \tag{34}$$

and

$$\begin{aligned} \mathcal{N}_i(x,t) &= \int_0^t\int_{-\infty}^{\infty} g_{ii}^*(x-y,t-s)\partial_x(n-n^*)(y,s)(y,s)\,dyds \\ &\quad + \sum_{j=1}^{2}\int_0^t\int_{-\infty}^{\infty}(g_{ij}-g_{ij}^*)(x-y,t-s)\partial_x n(y,s)\,dyds \\ &\quad - \gamma_{i'}\int_0^t\int_{-\infty}^{\infty}\partial_x g_{i'i'}^*(x-y,t-s)\partial_x n^*(y,s)(y,s)\,dyds. \end{aligned} \tag{35}$$

Lemma 3.4 may then be written as

$$v_i(x,t) = \mathcal{I}_i(x,t) + \mathcal{N}_i(x,t).$$

In the next two sections, we prove pointwise estimates for $\mathcal{I}_i(x,t)$ and $\mathcal{N}_i(x,t)$.

3.2.2. Contribution from the Initial Data Our goal in this section is to prove the following pointwise estimates for $\mathcal{I}_i(x,t)$ defined by (34):

Lemma 3.5. *For any $n \geq 1$, there exist positive constants δ_n and C_n such that if (12) holds, then we have*

$$|\mathcal{I}_i(x,t)| \leq C\delta\Psi_{i;n}(x,t)$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

Proof. We assume $t \geq 1$ below (the case when $t < 1$ is easier to handle). Let

$$\mathcal{I}_{i,1}(x,t) = \int_{-\infty}^{\infty} g_i^*(x-y,t) \begin{pmatrix} u_1 - \theta_1 \\ u_2 - \theta_2 \end{pmatrix} (y,0)\, dy$$

and

$$\mathcal{I}_{i,2}(x,t) = \int_{-\infty}^{\infty} (g_i - g_i^*)(x-y,t) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (y,0)\, dy - \gamma_{i'} \int_{-\infty}^{\infty} \partial_x g_{i'}^*(x-y,t) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} (y,0)\, dy.$$

Then of course $\mathcal{I}_i(x,t) = \mathcal{I}_{i,1}(x,t) + \mathcal{I}_{i,2}(x,t)$.

We first show

$$|\mathcal{I}_{i,1}(x,t)| \leq C\delta \Psi_{i;n}(x,t).$$

For this purpose, set

$$\eta_j(x) = \int_{-\infty}^{x} (u_j - \theta_j)(y,0)\, dy$$

and $\eta = (\eta_1\ \eta_2)^T$. We then have

$$\mathcal{I}_{i,1}(x,t) = \int_{-\infty}^{\infty} g_i^*(x-y,t) \partial_x \eta(y)\, dy.$$

By the definition of M_i, see (6), we have

$$\eta_j(x) := \int_{-\infty}^{x} (u_j - \theta_j)(y,0)\, dy = -\int_{x}^{\infty} (u_j - \theta_j)(y,0)\, dy.$$

This and (12) imply

$$|\eta_j(x)| \leq C\delta (|x|+1)^{-\beta_n}.$$

We first consider Case (i) $|x - \lambda_i t| \leq (t+1)^{1/2}$. In this case, integration by parts and (31) yield

$$|\mathcal{I}_{i,1}(x,t)| = \left| \int_{-\infty}^{\infty} \partial_x g_i^*(x-y,t) \eta(y)\, dy \right| \leq C(t+1)^{-1} \int_{-\infty}^{\infty} |\eta_i(x)|\, dx \leq C\delta (t+1)^{-1} \leq C\delta \Psi_{i;n}(x,t).$$

We next consider Case (ii) $(t+1)^{1/2} < |x - \lambda_i t| < t+1$ with $x - \lambda_i t > 0$ (the case when $x - \lambda_i t < 0$ is similar). Again, by integration by parts,

$$\begin{aligned} |\mathcal{I}_{i,1}(x,t)| &\leq C(t+1)^{-1} e^{-\frac{(x-\lambda_i t)^2}{Ct}} \int_{-\infty}^{(x-\lambda_i t)/2} |\eta_i(y)|\, dy \\ &\quad + C\delta (t+1)^{-1} \int_{(x-\lambda_i t)/2}^{\infty} e^{-\frac{(x-y-\lambda_i t)^2}{Ct}} (y+1)^{-\beta_n}\, dy \\ &\leq C\delta (t+1)^{-1} e^{-\frac{(x-\lambda_i t)^2}{Ct}} + C\delta (t+1)^{-1/2} (|x - \lambda_i t| + 1)^{-\beta_n} \leq C\delta \Psi_{i;n}(x,t). \end{aligned}$$

We finally consider Case (iii) $|x-\lambda_i t| \geq t+1$. For brevity, we assume $x-\lambda_i t > 0$. In this case, by (12), we have

$$
\begin{aligned}
|\mathcal{I}_{i,1}(x,t)| &\leq C(t+1)^{-1/2} e^{-\frac{(x-\lambda_i t)^2}{Ct}} \int_{-\infty}^{(x-\lambda_i t)/2} |(u_i-\theta_i)(y,0)|\,dy \\
&\quad + C\delta(t+1)^{-1/2} \int_{(x-\lambda_i t)/2}^{\infty} e^{-\frac{(x-y-\lambda_i t)^2}{2\nu t}} (y+1)^{-\alpha_n}\,dy \\
&\leq C\delta e^{-\frac{t}{C}} e^{-\frac{(x-\lambda_i t)^2}{Ct}} + C\delta(|x-\lambda_i t|+1)^{-\alpha_n} \leq C\delta \Psi_{i;n}(x,t).
\end{aligned}
$$

We next show

$$
|\mathcal{I}_{i,2}(x,t)| \leq C\delta\Psi_{i;n}(x,t).
$$

Writing

$$
\begin{aligned}
\mathcal{I}_{i,2}(x,t) &= \int_{-\infty}^{\infty} (g_i - g_i^* - \gamma_{i'}\partial_x g_{i'}^*)(x-y,t) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (y,0)\,dy \\
&\quad + \gamma_{i'} \int_{-\infty}^{\infty} \partial_x g_{i'}^*(x-y,t) \begin{pmatrix} u_1-\theta_1 \\ u_2-\theta_2 \end{pmatrix} (y,0)\,dy
\end{aligned}
$$

and applying (33), we see that it suffices to show that

$$
\begin{aligned}
A(x,t) &= \gamma_{i'}\partial_x \int_{-\infty}^{\infty} g_{i'}^*(x-y,t) \begin{pmatrix} u_1-\theta_1 \\ u_2-\theta_2 \end{pmatrix} (y,0)\,dy, \\
B(x,t) &= (t+1)^{-1} \int_{-\infty}^{\infty} e^{-\frac{(x-y-\lambda_i t)^2}{Ct}} \left| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right| (y,0)\,dy, \\
C(x,t) &= (t+1)^{-3/2} \int_{-\infty}^{\infty} e^{-\frac{(x-y-\lambda_{i'} t)^2}{Ct}} \left| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right| (y,0)\,dy \\
D(x,t) &= e^{-\frac{c^2}{\nu}t} \left| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right| (x,0)
\end{aligned}
$$

are all bounded by $C\delta\Psi_{i;n}(x,t)$. First, this is trivial for $D(x,t)$. Next, since $A(x,t) = \gamma_{i'}\partial_x \mathcal{I}_{i',1}(x,t)$, modifying the calculations above for $\mathcal{I}_{i,1}(x,t)$ yield the bound for $A(x,t)$. The bound for $B(x,t)$ is also obtained in a way similar to that for $\mathcal{I}_{i,1}(x,t)$ (except that we don't need η_i in the analysis).

Let us finally consider $C(x,t)$. First, Case (i) $|x-\lambda_{i'}t| \leq (t+1)^{1/2}$ is easy:

$$
|C(x,t)| \leq C\delta(t+1)^{-3/2} \leq C\delta\Psi_{i;n}(x,t).
$$

Case (ii) $|x-\lambda_{i'}t| > (t+1)^{1/2}$ with $x-\lambda_{i'}t > 0$ is as follows:

$$
\begin{aligned}
|C(x,t)| &\leq C(t+1)^{-3/2} e^{-\frac{(x-\lambda_{i'} t)^2}{Ct}} \int_{-\infty}^{(x-\lambda_{i'} t)/2} \left| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right| (y,0)\,dy \\
&\quad + C\delta(t+1)^{-3/2} \int_{(x-\lambda_{i'} t)/2}^{\infty} e^{-\frac{(x-y-\lambda_{i'} t)^2}{Ct}} (y+1)^{-\alpha_n}\,dy \\
&\leq C\delta(t+1)^{-3/2} e^{-\frac{(x-\lambda_{i'} t)^2}{Ct}} + C\delta(t+1)^{-1}(|x-\lambda_{i'}t|+1)^{-\alpha_n} \leq C\delta\Psi_{i;n}(x,t).
\end{aligned}
$$

The case when $x-\lambda_{i'}t < 0$ is similar. This ends the proof of the lemma. □

3.2.3. Contribution from the Nonlinear Terms Our goal in this section is to prove the following pointwise estimates for $\mathcal{N}_i(x,t)$ defined by (35):

Lemma 3.6. *For any $n \geq 1$, there exist positive constants δ_n and C_n such that if (12) holds, then we have*

$$|\mathcal{N}_i(x,t)| \leq C(\delta + P(t))^2 \Psi_{i;n}(x,t)$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

To prove this lemma, we first prove some preparatory lemmas. To state these, we introduce the notation

$$\mathcal{I}_i[f](x,t) := \int_0^t \int_{-\infty}^{\infty} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-y-\lambda_i(t-s))^2}{2\nu(t-s)}} \right\} f(y,s)\, dyds$$

for a function $f = f(x,t)$.

Lemma 3.7. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. If ε is sufficiently small, we have*

$$|\mathcal{I}_i[\theta_{i'}\Xi_i](x,t)| \leq C\varepsilon^3 \Psi_{i;n}(x,t).$$

Proof. We only prove the lemma for $i = 1$ (the other case is similar). By Lemma 3.3, we have

$$|(\theta_2\Xi_1)(x,t) + (2c)^{-1}(\theta_2^3/2 + \theta_2^2\Xi_2)(x,t)| \leq C\varepsilon^3\Theta_4(x,t;-c,\nu^*).$$

Since

$$\Theta_4(x,t;-c,\nu^*) \leq C(t+1)^{-(2+1/2^{n+1})/2}\psi_n(x,t;-c),$$

Lemma A.7 implies

$$|\mathcal{I}_1[\Theta_4(\cdot,\cdot;-c,\nu^*)](x,t)| \leq C\Psi_{1;n}(x,t).$$

Next, note that (3) and (25) imply

$$L_2\theta_2 = -\partial_x(\theta_2^2/2), \quad L_2\Xi_2 = -\partial_x(\theta_1^2/2 + \theta_1\Xi_1 + \theta_2\Xi_2),$$

where $L_2 = \partial_t - c\partial_x - (\nu/2)\partial_x^2$. Using these, similar to the bound for $\zeta_{1;n}$ in the proof of Lemma 3.1, we can show that

$$|\mathcal{I}_1[\theta_2^3/2 + \theta_2^2\Xi_2](x,t)| \leq C\varepsilon^3\Psi_{1;n}(x,t).$$

Combining these, we obtain the lemma. □

We next show the following:

Lemma 3.8. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. If ε is sufficiently small, we have*

$$|\mathcal{I}_i[\partial_x\theta_{i'}^2](x,t)| \leq C\varepsilon^2\Psi_{i;n}(x,t).$$

Proof. We only prove the lemma for $i = 1$ (the other case is similar). Applying Lemma A.1 yields

$$\mathcal{I}_1[\partial_x\theta_2^2](x,t) = (2c)^{-1}\sqrt{2\pi\nu}\partial_x\theta_2^2(x,t) + I_1(x,t) + I_2(x,t),$$

where

$$I_1(x,t) = \int_0^{t^{1/2}}\int_{-\infty}^{\infty}\partial_x\left\{(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}\right\}\partial_x\theta_2^2(y,s)\,dyds$$

and

$$\begin{aligned} I_2(x,t) &= -(2c)^{-1}\int_{-\infty}^{\infty}(t-t^{1/2})^{-1/2}e^{-\frac{(x-y-c(t-\sqrt{t}))^2}{2\nu(t-\sqrt{t})}}\partial_x\theta_2^2(y,t^{1/2})\,dy \\ &\quad - (2c)^{-1}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}\partial_x L_2\theta_2^2(y,s)\,dyds \\ &=: I_{21}(x,t) + I_{22}(x,t). \end{aligned}$$

By Lemmas A.2 and A.3, we obtain

$$|I_1(x,t)| + |I_{21}(x,t)| \le C\varepsilon^2\log(t+2)\Theta_2(x,t;c,\nu^*) \le C\varepsilon^2\Psi_{1;n}(x,t).$$

For $I_{22}(x,t)$, we apply Lemma A.1 (without the integral on $[0, t^{1/2}]$), which yields

$$\mathcal{I}_{22}(x,t) = -(2c)^{-2}\sqrt{2\pi\nu}L_2\theta_2^2(x,t) + J_2(x,t),$$

where

$$\begin{aligned} J_2(x,t) &= (2c)^{-2}\int_{-\infty}^{\infty}(t-t^{1/2})^{-1/2}e^{-\frac{(x-y-c(t-\sqrt{t}))^2}{2\nu(t-\sqrt{t})}}L_2\theta_2^2(y,t^{1/2})\,dy \\ &\quad + (2c)^{-2}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}L_2^2\theta_2^2(y,s)\,dyds \\ &=: J_{21}(x,t) + J_{22}(x,t). \end{aligned}$$

By some tedious calculations, we obtain

$$L_2\theta_2^2 = -2\partial_x(\theta_2^3/3) - \nu(\partial_x\theta_2)^2$$

and

$$L_2^2\theta_2^2 = \partial_x^2(\theta_2^4/2) + \nu\partial_x[(\partial_x\theta_2)\partial_x\theta_2^2] + \nu(\partial_x\theta_2)\partial_x^2\theta_2^2 + \nu^2(\partial_x^2\theta_2)^2.$$

Since $|L_2\theta_2^2(x,t)| \le C\varepsilon^2\Theta_4(x,t;-c,\nu^*)$, Lemma A.3 yields

$$|J_{21}(x,t)| \le C\varepsilon^2\Theta_{5/2}(x,t;c,\nu^*) \le C\varepsilon^2\Psi_{1;n}(x,t).$$

And since $|L_2^2\theta_2^2(x,t)| \le C\varepsilon^2\Theta_6(x,t;-c,\nu^*)$, Lemma A.4 implies

$$|J_{22}(x,t)| \le C\varepsilon^2[(x-c(t+1))^2 + (t+1)]^{-5/4} \le C\varepsilon^2\Psi_{1;n}(x,t;c).$$

This proves the lemma. □

Similarly, we can now show the following:

Lemma 3.9. *Let $n \geq 1$ and $\varepsilon = \max(M_1, M_2)$. If ε is sufficiently small, we have*

$$|\mathcal{I}_i[\partial_x(\theta_{i'}\Xi_{i'})](x,t)| \leq C\varepsilon^3\Psi_{i;n}(x,t).$$

We move on to prove the following:

Lemma 3.10. *Let $n \geq 1$. If δ defined by (12) is sufficiently small, we have*

$$|\mathcal{I}_i[\theta_{i'}v_{i'}](x,t)| \leq C(\delta + P(t))^2\Psi_{i;n}(x,t).$$

Here $P(t)$ is defined by (27).

Proof. We only prove the lemma for $i = 1$ (the other case is similar). Set $f = \theta_2 v_2$. Then Lemma A.1 implies

$$\mathcal{I}_1[\theta_2 v_2](x,t) = (2c)^{-1}\sqrt{2\pi\nu}f(x,t) + I_1(x,t) + I_2(x,t),$$

where

$$I_1(x,t) = \int_0^{t^{1/2}}\int_{-\infty}^{\infty}\partial_x\left\{(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}\right\}f(y,s)\,dyds$$

and

$$\begin{aligned}I_2(x,t) &= -(2c)^{-1}\int_{-\infty}^{\infty}(t-t^{1/2})^{-1/2}e^{-\frac{(x-y-c(t-\sqrt{t}))^2}{2\nu(t-\sqrt{t})}}f(y,t^{1/2})\,dy\\&\quad-(2c)^{-1}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{2\nu(t-s)}}L_2 f(y,s)\,dyds\\&=: I_{21}(x,t) + I_{22}(x,t).\end{aligned}$$

By Lemmas A.2 and A.3, we obtain

$$|I_1(x,t)| + |I_{21}(x,t)| \leq C\delta P(t)\Theta_{\alpha_{n+1}}(x,t;c,\nu^*) \leq C\delta P(t)\Psi_{1;n}(x,t).$$

To bound $I_{22}(x,t)$, we first note that

$$\begin{aligned}L_2 v_2 &= L_2(u_2 - \theta_2 - \Xi_2 - \gamma_1\partial_x\theta_1 - \gamma_1\partial_x\Xi_1)\\&= \frac{\nu}{2}\partial_x^2 u_1 + \partial_x n + \partial_x(\theta_2^2/2) + \partial_x(\theta_1^2/2 + \theta_1\Xi_1 + \theta_2\Xi_2)\\&\quad+\gamma_1\partial_x^2(\theta_1^2/2 + 2c\theta_1) + \gamma_1\partial_x^2(\theta_2^2/2 + \theta_1\Xi_1 + \theta_2\Xi_2 + 2c\Xi_1)\\&= \frac{\nu}{2}\partial_x^2(u_1 - \theta_1 - \Xi_1) + \partial_x(n - n_*) - \gamma_1\partial_x^2 n_*.\end{aligned}$$

Then set

$$F = \frac{\nu}{2}\partial_x(u_1 - \theta_1 - \Xi_1) + n - n_* - \gamma_1\partial_x n_*$$

and

$$G = \theta_2 F - \nu v_2\partial_x\theta_2.$$

We note that

$$L_2 f = -v_2 \partial_x(\theta_2^2/2) + \partial_x G - (\partial_x \theta_2) F + \nu v_2 \partial_x^2 \theta_2.$$

By [10, Theorem 2.6 and Remark 2.8], we have

$$|\partial_x(u_1 - \theta_1)(x,t)| \le C\delta(t+1)^{-1/2}\Psi_{1;0}(x,t), \quad |\partial_x^2 u_1(x,t)| \le C\delta(t+1)^{-3/2}.$$ [2]

In addition, applying Taylor's theorem, we see that

$$|(n-n_*)(x,t)| \le C(\delta + P(t))^2 (t+1)^{-1/2}[\psi_n(x,t;c) + \psi_n(x,t;-c)].$$

These imply

$$|G(x,t)| \le C(\delta + P(t))^2 (t+1)^{-1}\Theta_{\alpha_n}(x,t;-c,\nu^*)$$

and

$$|L_2 f(x,t) - \partial_x G(x,t)| \le C(\delta + P(t))^2 (t+1)^{-3/2}\Theta_{\alpha_n}(x,t;-c,\nu^*).$$

Using these and integration by parts, we get

$$\begin{aligned} |I_{22}(x,t)| \le C(\delta + P(t))^2 \int_{t^{1/2}}^{t/2}\int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{C(t-s)}} (s+1)^{-3/2}\psi_n(y,s;-c)\,dyds \\ + C(\delta + P(t))^2 \int_{t^{1/2}}^{t}\int_{-\infty}^{\infty} (t-s)^{-1} e^{-\frac{(x-y-c(t-s))^2}{C(t-s)}} (s+1)^{-1}\psi_n(y,s;-c)\,dyds. \end{aligned}$$

Applying Lemmas A.7 and A.9, we obtain

$$|I_{22}(x,t)| \le C(\delta + P(t))^2 \Psi_{1;n}(x,t).$$

This proves the lemma. □

The lemma below can be shown in a similar manner.

Lemma 3.11. *Let* $n \ge 1$. *If* δ *defined by (12) is sufficiently small, we have*

$$|\mathcal{I}_i[\Xi_{i'}^2](x,t)| + |\mathcal{I}_i[\Xi_{i'} v_{i'}](x,t)| \le C(\delta + P(t))^3 \Psi_{i;n}(x,t).$$

Set

$$n_a = -\frac{p''(1)^2}{8c^2}(v-1)^2, \quad n_b = -\frac{\nu p''(1)}{4c^2}(v-1)u_x, \quad n_c = n - n_a - n_b. \tag{36}$$

[2] The decay estimate for $\partial_x^2 u$ is not explicitly stated in the theorem but is shown in its proof (see [10, p. 107]). Note that this is where the H^6-regularity is used.

Of course $n = n_a + n_n + n_c$. Correspondingly, set

$$\begin{aligned}\mathcal{N}_{i,a}(x,t) = &\int_0^t \int_{-\infty}^{\infty} g_{ii}^*(x-y,t-s)\partial_x(n_a - n^*)(y,s)(y,s)\,dyds\\ &+\sum_{j=1}^{2}\int_0^t \int_{-\infty}^{\infty}(g_{ij}-g_{ij}^*)(x-y,t-s)\partial_x n_a(y,s)\,dyds\\ &-\gamma_{i'}\int_0^t \int_{-\infty}^{\infty}\partial_x g_{i'i'}^*(x-y,t-s)\partial_x n^*(y,s)(y,s)\,dyds,\end{aligned}$$

$$\begin{aligned}\mathcal{N}_{i,b}(x,t) = &\int_0^t \int_{-\infty}^{\infty} g_{ii}^*(x-y,t-s)\partial_x n_b(y,s)(y,s)\,dyds\\ &+\sum_{j=1}^{2}\int_0^t \int_{-\infty}^{\infty}(g_{ij}-g_{ij}^*)(x-y,t-s)\partial_x n_b(y,s)\,dyds,\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_{i,c}(x,t) = &\int_0^t \int_{-\infty}^{\infty} g_{ii}^*(x-y,t-s)\partial_x n_c(y,s)(y,s)\,dyds\\ &+\sum_{j=1}^{2}\int_0^t \int_{-\infty}^{\infty}(g_{ij}-g_{ij}^*)(x-y,t-s)\partial_x n_c(y,s)\,dyds.\end{aligned}$$

Then $\mathcal{N}_i(x,t) = \mathcal{N}_{i,a}(x,t) + \mathcal{N}_{i,b}(x,t) + \mathcal{N}_{i,c}(x,t)$; see (35).

We next prove the following:

Lemma 3.12. *Let $n \geq 1$. If δ defined by (12) is sufficiently small, we have*

$$|\mathcal{N}_{i,a}(x,t)| \leq C(\delta + P(t))^2 \Psi_{i;n}(x,t).$$

Proof. Let $i = 1$ (the case of $i = 2$ is similar). By integration by parts, we have

$$\begin{aligned}\mathcal{N}_{1,a}(x,t) = &\int_0^t \int_{-\infty}^{\infty} \partial_x g_{11}^*(x-y,t-s)(n_a - n^*)(y,s)\,dyds\\ &+\sum_{j=1}^{2}\int_0^t \int_{-\infty}^{\infty}\partial_x(g_{1j}-g_{1j}^*)(x-y,t-s)n_a(y,s)\,dyds\\ &-\gamma_2\int_0^t \int_{-\infty}^{\infty}\partial_x^2 g_{22}^*(x-y,t-s)n^*(y,s)\,dyds.\end{aligned}$$

By some tedious calculations, we can show that

$$\begin{aligned}&|(n_a - n^*)(x,t) - [\theta_2\Xi_1 + \gamma_2\partial_x(\theta_2^2/2) + \gamma_2\partial_x(\theta_2\Xi_2) - \theta_2 v_2 - \Xi_2^2/2 - \Xi_2 v_2](x,t)|\\ &\leq C(\delta + P(t))^2[(t+1)^{-1/2}\psi_n(x,t;c) + (t+1)^{-\alpha_n/2}\psi_n(x,t;-c)].\end{aligned}$$

Then Lemmas 3.7–3.11, A.6, and A.7 yield

$$\left| \int_0^t \int_{-\infty}^{\infty} \partial_x g_{11}^*(x-y, t-s)(n_a - n^*)(y, s)\, dyds \right| \leq C(\delta + P(t))^2 \Psi_{i;n}(x, t).$$

It remains to show that

$$|I(x,t)| \leq C(\delta + P(t))^2 \Psi_{1;n}(x,t),$$

where

$$I(x,t) = \sum_{j=1}^{2} \int_0^t \int_{-\infty}^{\infty} \partial_x (g_{1j} - g_{1j}^*)(x-y, t-s) n_a(y,s)\, dyds \\ - \gamma_2 \int_0^t \int_{-\infty}^{\infty} \partial_x^2 g_{22}^*(x-y, t-s) n^*(y,s)\, dyds.$$

We define the decomposition $I(x,t) = I_1(x,t) + I_2(x,t)$ by

$$I_1(x,t) = \sum_{j=1}^{2} \int_0^t \int_{-\infty}^{\infty} \partial_x (g_{1j} - g_{1j}^*)(x-y, t-s)(n_a - n^*)(y,s)\, dyds$$

and

$$I_2(x,t) = \sum_{j=1}^{2} \int_0^t \int_{-\infty}^{\infty} \partial_x (g_{1j} - g_{1j}^*)(x-y, t-s) n^*(y,s)\, dyds \\ - \gamma_2 \int_0^t \int_{-\infty}^{\infty} \partial_x^2 g_{22}^*(x-y, t-s) n^*(y,s)\, dyds.$$

We first consider $I_1(x,t)$. By (32), to show that $|I_1(x,t)|$ is bounded by $C(\delta + P(t))^2 \Psi_{1;n}(x,t)$, it suffices to prove the same bound for

$$\int_0^t \int_{-\infty}^{\infty} (t-s)^{-1}(t-s+1)^{-1/2} e^{-\frac{(x-y-\lambda_j(t-s))^2}{C(t-s)}} |(n_a - n^*)(y,s)|\, dyds \quad (j = 1, 2)$$

and

$$\int_0^t e^{-\frac{c^2}{\nu}(t-s)} |(n_a - n^*)(x,s)|\, ds.$$

The term corresponding to $\delta^{(1)}(x)$ is not needed since $q_{10} = (1/2 \; -1/2)^T$. Noting that

$$|(n_a - n^*)(x,t)| \leq C(\delta + P(t))^2 (t+1)^{-1/2} [\psi_n(x,t;c) + \psi_n(x,t;-c)],$$

Lemmas A.6, A.7, and A.10 imply the desired bounds for the two integrals above.

We next consider $I_2(x,t)$. We have $I_2(x,t) = I_{21}(x,t) + I_{22}(x,t)$ with

$$I_{21}(x,t) = -\sum_{j=1}^{2} \int_0^t \int_{-\infty}^{\infty} \partial_x (g_{1j} - g_{1j}^*)(x-y,t-s)(\theta_1^2/2 + \theta_1 \Xi_1)(y,s)\,dyds \\ + \gamma_2 \int_0^t \int_{-\infty}^{\infty} \partial_x^2 g_{22}^*(x-y,t-s)(\theta_1^2/2 + \theta_1 \Xi_1)(y,s)\,dyds$$

and

$$I_{22}(x,t) = -\sum_{j=1}^{2} \int_0^t \int_{-\infty}^{\infty} \partial_x (g_{1j} - g_{1j}^*)(x-y,t-s)(\theta_2^2/2 + \theta_2 \Xi_2)\,dyds \\ + \gamma_2 \int_0^t \int_{-\infty}^{\infty} \partial_x^2 g_{22}^*(x-y,t-s)(\theta_2^2/2 + \theta_2 \Xi_2)\,dyds.$$

Taking into account (32) and (33), Lemmas A.6, A.7, and A.10 yield $|I_{21}(x,t)| \leq C(\delta + P(t))^2 \Psi_{1;n}(x,t)$ (we divide the domain of temporal integration into $[0, t/2]$ and $[t/2, t]$ then use integration by parts before applying the lemmas). For $I_{22}(x,t)$, we proceed as follows: using the technique in the proof of Lemma A.1, we obtain

$$\begin{aligned} I_{22}(x,t) = &-\sum_{j=1}^{2} \int_0^{t^{1/2}} \int_{-\infty}^{\infty} \partial_x (g_{1j} - g_{1j}^*)(x-y,t-s)(\theta_2^2/2 + \theta_2 \Xi_2)(y,s)\,dyds \\ &+ \gamma_2 \int_0^{t^{1/2}} \int_{-\infty}^{\infty} \partial_x^2 g_{22}^*(x-y,t-s)(\theta_2^2/2 + \theta_2 \Xi_2)(y,s)\,dyds \\ &- \frac{1}{2c} \sum_{j=1}^{2} \int_{t^{1/2}}^t \int_{-\infty}^{\infty} L_1 (g_{1j} - g_{1j}^*)(x-y,t-s)(\theta_2^2/2 + \theta_2 \Xi_2)(y,s)\,dyds \\ &+ \frac{\nu}{4c} \int_{t^{1/2}}^t \int_{-\infty}^{\infty} \partial_x^2 g_{22}^*(x-y,t-s)(\theta_2^2/2 + \theta_2 \Xi_2)(y,s)\,dyds \\ &+ \frac{1}{2c} \sum_{j=1}^{2} \int_{t^{1/2}}^t \int_{-\infty}^{\infty} (g_{1j} - g_{1j}^*)(x-y,t-s) L_2 (\theta_2^2/2 + \theta_2 \Xi_2)(y,s)\,dyds \\ &+ \frac{1}{2c} \sum_{j=1}^{2} \int_{-\infty}^{\infty} (g_{1j} - g_{1j}^*)(x-y,t-t^{1/2})(\theta_2^2/2 + \theta_2 \Xi_2)(y,t^{1/2})\,dyds, \end{aligned}$$

where $L_i = \partial_t + \lambda_i \partial_x - (\nu/2)\partial_x^2$. Here we used $\lim_{t \to 0}(g_{1j} - g_{1j}^*)(x,t) = 0$. The sum of the first two terms on the right-hand side can be bounded using Lemmas A.6, A.8, and A.10; the sum of the third and the fourth term can be bounded using Lemmas A.6, A.7, A.10, and the relation

$$L_1(g_{1j} - g_{1j}^*) = (\nu/2)\partial_x^2 g_{2j}.$$

To bound the sum of the fifth and the sixth term, noting that

$$|L_2(\theta_2^2/2 + \theta_2 \Xi_2)(x,t)| \leq C\delta^2 \Theta_4(x,t; -c, \nu^*),$$

it suffices to show that the following integrals are bounded by $C(\delta+P(t))^2\Psi_{1;n}(x,t)$:

$$A(x,t) = \int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}(t-s)^{-1/2}(t-s+1)^{-1/2}e^{-\frac{(x-y-c(t-s))^2}{C(t-s)}}\Theta_4(y,s;-c,\nu^*)\,dyds,$$

$$B(x,t) = \int_{-\infty}^{\infty}(t-t^{1/2})^{-1}e^{-\frac{(x-y-c(t-\sqrt{t}))^2}{C(t-\sqrt{t})}}\Theta_2(y,t^{1/2};-c,\nu^*)\,dy,$$

$$C(x,t) = \int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}(t-s)^{-1}(t-s+1)^{-1/2}e^{-\frac{(x-y+c(t-s))^2}{C(t-s)}}\Theta_4(y,s;-c,\nu^*)\,dyds,$$

$$D(x,t) = \int_{-\infty}^{\infty}(t-t^{1/2})^{-1}(t-t^{1/2}+1)^{-1/2}e^{-\frac{(x-y+c(t-\sqrt{t}))^2}{C(t-\sqrt{t})}}\Theta_2(y,t^{1/2};-c,\nu^*)\,dy,$$

and

$$E(x,t) = \int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}\partial_x g_{22}^*(x-y,t-s)L_2(\theta_2^2/2+\theta_2\Xi_2)(y,s)\,dyds + \int_{-\infty}^{\infty}\partial_x g_{22}^*(x-y,t-t^{1/2})(\theta_2^2/2+\theta_2\Xi_2)(y,t^{1/2})\,dy.$$

We can bound $A(x,t)$ using Lemma A.4, $B(x,t)$ and $D(x,t)$ by Lemma A.3, and $C(x,t)$ by Lemma A.6. Finally, we consider $E(x,t)$. Taking into account $L_2 g_{22}^* = 0$ and $\lim_{t\to 0}\partial_x g_{22}^*(x-y,t) = \delta^{(1)}(x-y)$, integration by parts applied to the operator L_2 yields

$$E(x,t) = \int_{-\infty}^{\infty}\delta^{(1)}(x-y)(\theta_2^2/2+\theta_2\Xi_2)(y,t)\,dy = \partial_x(\theta_2^2/2+\theta_2\Xi_2)(x,t).$$

Hence $|E(x,t)| \le C(\delta+P(t))^2\Psi_{1;n}(x,t)$. This ends the proof of the lemma. □

The lemma below can be proved similarly. Note that $\mathcal{N}_{i,b}(x,t)$ is related to the nonlinear term $(v-1)u_x$ as opposed to $(v-1)^2$ for $\mathcal{N}_{i,a}(x,t)$; see (36). As in the bound of $\mathcal{N}_{i,a}(x,t)$, the term $(v-1)$ is dealt with the inequality $|v_i(x,s)| \le P(t)\Psi_{i;n}(x,s)$ $(0\le s\le t)$; on the other hand, the first derivative u_x is handled using [10, Theorem 2.6 and Remark 2.8] as in the proof of Lemma 3.10. The term $\mathcal{N}_{i,c}(x,t)$ can be handled in a similar manner.

Lemma 3.13. *Let $n\ge 1$. If δ defined by (12) is sufficiently small, we have*

$$|\mathcal{N}_{i,b}(x,t)| + |\mathcal{N}_{i,c}(x,t)| \le C(\delta+P(t))^2\Psi_{i;n}(x,t).$$

Combining Lemmas 3.12 and 3.13, the proof of Lemma 3.6 is complete.

3.2.4. Final Step of the Proof The remaining step of the proof is standard. By Lemma 3.5 and 3.6, we obtain

$$P(t) \le C\delta + C(\delta+P(t))^2 \le C_1\delta + C_2P(t)^2 \tag{37}$$

for some $C_1, C_2 > 0$. Here $P(t)$ is defined by (27). When δ is sufficiently small, the line $y = p$ and the parabola $y = C_1\delta + C_2p^2$ intersect at $p = p_1$ and p_2, where

$$p_1 = \frac{1-\sqrt{1-4C_1C_2\delta}}{2C_2}, \quad p_2 = \frac{1+\sqrt{1-4C_1C_2\delta}}{2C_2}.$$

Note that $C_1\delta \le p_1 < p_2$. By (37), we either have $P(t) \le p_1$ or $P(t) \ge p_2$. Since $P(t)$ is continuous in t, if $P(0) \le p_1$, then $P(t) \le p_1$ for all $t \ge 0$. By (12), taking C_1 sufficiently large, we indeed have $P(0) \le C_1\delta \le p_1$. Therefore, we conclude that

$$P(t) \le p_1 \le \frac{1-(1-4C_1C_2\delta)}{2C_2} \le 2C_1\delta.$$

This ends the proof of Theorem 2.1.

Acknowledgements. This work was supported by Grant-in-Aid for JSPS Research Fellow (Grant Number 20J00882) and JSPS Grant-in-Aid for Early-Career Scientists (Grant Number 22K13938).

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest I declare that I have no conflict of interest.

Appendix A. Lemmas on Convolutions Involving a Heat Kernel

Lemma A.1. *Suppose that $f = f(x,t)$ is a C^2-smooth function on $\mathbb{R}\times(0,\infty)$. Let $\lambda \ne \lambda'$ and $\nu > 0$, and set $L_{\lambda'} = \partial_t + \lambda'\partial_x - (\nu/2)\partial_x^2$. Then for $t \ge 1$, the function $I(x,t)$ defined by*

$$I(x,t) = \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \partial_x f(y,s)\,dyds$$

can be written as

$$I(x,t) = (\lambda-\lambda')^{-1}\sqrt{2\pi\nu}\, f(x,t) + I_1(x,t) + I_2(x,t),$$

where

$$I_1(x,t) = \int_0^{t^{1/2}} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \partial_x f(y,s)\,dyds$$

and

$$I_2(x,t) = -(\lambda-\lambda')^{-1}\int_{-\infty}^{\infty}(t-t^{1/2})^{-1/2}e^{-\frac{(x-y-\lambda(t-\sqrt{t}))^2}{2\nu(t-\sqrt{t})}}f(y,t^{1/2})\,dy$$
$$-(\lambda-\lambda')^{-1}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}(t-s)^{-1/2}e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}}L_{\lambda'}f(y,s)\,dyds.$$

Proof. Let

$$g_\lambda(x,t) = t^{-1/2}e^{-\frac{(x-\lambda t)^2}{2\nu t}}.$$

Dividing the domain of temporal integration, we get

$$I(x,t) = \int_0^{t^{1/2}}\int_{-\infty}^{\infty}g_\lambda(x-y,t-s)\partial_x f(y,s)\,dyds$$
$$+\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}g_\lambda(x-y,t-s)\partial_x f(y,s)\,dyds.$$

The first term on the right-hand side is $I_1(x,t)$. For the second term, integration by parts yields

$$\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}g_\lambda(x-y,t-s)\partial_x f(y,s)\,dyds$$
$$= -(\lambda-\lambda')^{-1}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}g_\lambda(x-y,t-s)$$
$$\cdot[-\partial_s+\partial_s-\lambda\partial_y+\lambda'\partial_y+(\nu/2)\partial_y^2-(\nu/2)\partial_y^2]f(y,s)\,dyds$$
$$= (\lambda-\lambda')^{-1}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}L_\lambda g_\lambda(x-y,t-s)f(y,s)\,dyds$$
$$-(\lambda-\lambda')^{-1}\int_{t^{1/2}}^{t}\int_{-\infty}^{\infty}g_\lambda(x-y,t-s)L_{\lambda'}f(y,s)\,dyds$$
$$+(\lambda-\lambda')^{-1}\sqrt{2\pi\nu}f(x,t)-(\lambda-\lambda')^{-1}\int_{-\infty}^{\infty}g_\lambda(x-y,t-t^{1/2})f(x,t^{1/2})\,dyds,$$

where $L_\lambda = \partial_s+\lambda\partial_y-(\nu/2)\partial_y^2$. Here we used $\lim_{s\to t}g_\lambda(x-y,t-s)=\sqrt{2\pi\nu}\delta(x-y)$. The lemma follows from the equality above by noting that $L_\lambda g_\lambda = 0$. □

In the lemmas below, C and ν^* denote generic large constants. We remind the reader that $\Theta_\alpha(x,t;\lambda,\mu)$ is defined by (13).

Lemma A.2. *Let* $\lambda\neq\lambda'$, $\mu>0$, *and* $0<\alpha\leq 3$. *Then we have*

$$\int_0^{t^{1/2}}\int_{-\infty}^{\infty}(t-s)^{-1}e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}}\Theta_\alpha(y,s;\lambda',\mu)\,dyds$$
$$\leq\begin{cases}C\Theta_{(\alpha+1)/2}(x,t;\lambda,\nu^*) & \text{if}\ \alpha\neq 3,\\ C\log(t+2)\Theta_{(\alpha+1)/2}(x,t;\lambda,\nu^*) & \text{if}\ \alpha=3.\end{cases}$$

Proof. See the analysis of $I_1(x,t)$ in the proof of [10, Lemma 3.4]. □

Lemma A.3. *Let $\lambda, \lambda' \in \mathbb{R}$, $\mu > 0$, and $\alpha > 0$ (not necesarily $\lambda \neq \lambda'$). Then for $t \geq 1$, we have*

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y-\lambda(t-\sqrt{t}))^2}{\mu(t-\sqrt{t})}} \Theta_\alpha(y, t^{1/2}; \lambda', \mu)\, dyds \leq C\Theta_{(\alpha-1)/2}(x, t; \lambda, \nu^*).$$

Proof. See the analysis of $I_{21}(x, t)$ in the proof of [10, Lemma 3.4]. □

Lemma A.4. *Let $\lambda \neq \lambda'$, $\mu > 0$, and $\alpha > 1$. Then we have*

$$\begin{aligned}&\int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} \Theta_\alpha(y, s; \lambda', \mu)\, dyds \\ &\quad \leq C[(x-\lambda(t+1))^2 + (t+1)]^{-(\alpha-1)/4}.\end{aligned}$$

Proof. See the analysis of $I_{22}^{(1)}(x, t)$ in the proof of [10, Lemma 3.4]. □

Lemma A.5. ([10, Lemma 3.2]) *Let $\lambda \in \mathbb{R}$, $\mu > 0$, $\alpha \geq 0$, and $\beta > 0$. Then we have*

$$\begin{aligned}&\int_{0}^{t/2} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} \Theta_\beta(y, s; \lambda, \mu)\, dyds \\ &\quad \leq \begin{cases} C\Theta_\gamma(x, t; \lambda, \nu^*) & \text{if } \beta \neq 3, \\ C\log(t+2)\Theta_\gamma(x, t; \lambda, \nu^*) & \text{if } \beta = 3, \end{cases}\end{aligned}$$

where $\gamma = \alpha + \min(\beta, 3) - 1$. We also have

$$\begin{aligned}&\int_{t/2}^{t} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} \Theta_\beta(y, s; \lambda, \mu)\, dyds \\ &\quad \leq \begin{cases} C\Theta_\gamma(x, t; \lambda, \nu^*) & \text{if } \alpha \neq 1, \\ C\log(t+2)\Theta_\gamma(x, t; \lambda, \nu^*) & \text{if } \alpha = 1, \end{cases}\end{aligned}$$

where $\gamma = \min(\alpha, 1) + \beta - 1$.

We remind the reader that $\psi_n(x, t; \lambda)$ is defined in (8).

Lemma A.6. *Let $\lambda \in \mathbb{R}$, $\mu > 0$, and $\alpha, \beta \geq 0$. Then we have*

$$\begin{aligned}&\int_{0}^{t/2} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} (s+1)^{-\beta/2}\psi_n(y, s; \lambda)\, dyds \\ &\quad \leq \begin{cases} C(t+1)^{-\gamma_1/2}\psi_n(x, t; \lambda) & \text{if } \beta \neq 3-\alpha_n, \\ C\log(t+2)(t+1)^{-\gamma_1/2}\psi_n(x, t; \lambda) & \text{if } \beta = 3-\alpha_n, \end{cases}\end{aligned}$$

where $\gamma_1 = \alpha + \min(\beta, 3-\alpha_n) - 1$. We also have

$$\begin{aligned}&\int_{t/2}^{t} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} (s+1)^{-\beta/2}\psi_n(y, s; \lambda)\, dyds \\ &\quad \leq \begin{cases} C(t+1)^{-\gamma_2/2}\psi_n(x, t; \lambda) & \text{if } \alpha \neq 1, \\ C\log(t+2)(t+1)^{-\gamma_2/2}\psi_n(x, t; \lambda) & \text{if } \alpha = 1, \end{cases}\end{aligned}$$

where $\gamma_2 = \min(\alpha, 1) + \beta - 1$.

Proof. A straightforward (but lengthy) adaptation of the proof of [7, Lemma A.7] proves the lemma. □

For $\lambda, \lambda' \in \mathbb{R}$ and $K > 0$, let

$$\chi_K(x, t; \lambda, \lambda') := \mathrm{char}\left\{\min(\lambda, \lambda')(t+1) + K\sqrt{t+1} \le x \le \max(\lambda, \lambda')(t+1) - K\sqrt{t+1}\right\},$$

where char$\{S\}$ is the indicator function of a set S.

Lemma A.7. *Let $\lambda \neq \lambda'$, $\mu > 0$, $\alpha \ge 0$, and $0 \le \beta \le 2\alpha_n$ ($\beta \neq 2$).*[3] *Then for $K > 0$ large enough, we have*

$$\int_0^{t/2}\int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} (s+1)^{-\beta/2}\psi_n(y, s; \lambda')\, dyds$$
$$\le \begin{cases} C[(t+1)^{-\gamma_1/2}\psi_n(x, t; \lambda) + (t+1)^{-\gamma_1'/2}\psi_n(x, t; \lambda')] & \text{if } \beta \neq 3-\alpha_n \\ C[\log(t+2)(t+1)^{-\gamma_1/2}\psi_n(x, t; \lambda) + (t+1)^{-\gamma_1'/2}\psi_n(x, t; \lambda')] & \text{if } \beta = 3-\alpha_n \end{cases}$$
$$+ C|x-\lambda(t+1)|^{-(\alpha_n+\min(\beta,\alpha_n+1)-1)/2}|x-\lambda'(t+1)|^{-\alpha/2-1/2}\chi_K(x, t; \lambda, \lambda'),$$

where $\gamma_1 = \alpha + \min(\beta, 3-\alpha_n) - 1$ and $\gamma_1' = \alpha + \min(\beta, 2) - 1$. We also have

$$\int_{t/2}^{t}\int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} (s+1)^{-\beta/2}\psi_n(y, s; \lambda')\, dyds$$
$$\le \begin{cases} C(t+1)^{-\gamma_2/2}[\psi_n(x, t; \lambda) + \psi_n(x, t; \lambda')] & \text{if } \alpha \neq 1 \\ C\log(t+2)(t+1)^{-\gamma_2/2}[\psi_n(x, t; \lambda) + \psi_n(x, t; \lambda')] & \text{if } \alpha = 1 \end{cases}$$
$$+ C|x-\lambda(t+1)|^{-(\alpha_n+\beta-1)/2}|x-\lambda'(t+1)|^{-\min(\alpha,1)/2-1/2}\chi_K(x, t; \lambda, \lambda'),$$

where $\gamma_2 = \min(\alpha, 1) + \beta - 1$.

Proof. This is also proved by an adaptation of the proof of [7, Lemma A.8]. □

Lemma A.8. *Let $\lambda \neq \lambda'$, $\mu > 0$, $\alpha \ge 0$, and $0 \le \beta < 3-\alpha_n$. Then we have*

$$\int_0^{t^{1/2}}\int_{-\infty}^{\infty} (t-s)^{-1-\alpha} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} (s+1)^{-\beta/2}\psi_n(y, s; \lambda')\, dyds$$
$$\le C(t+1)^{-\alpha-(\beta-\alpha_n+1)/4}\psi_n(x, t; \lambda).$$

Proof. This is a simple generalization of [7, Lemma A.2]. □

Lemma A.9. *Let $\lambda \neq \lambda'$, $\mu > 0$, and $\alpha \ge 0$. Then for $t \ge 4$, we have*

$$\int_{t^{1/2}}^{t/2}\int_{-\infty}^{\infty} (t-s)^{-1/2-\alpha} e^{-\frac{(x-y-\lambda(t-s))^2}{\mu(t-s)}} (s+1)^{-\beta_{n+1}}\psi_n(y, s; \lambda')\, dyds$$
$$\le C(t+1)^{-\alpha}\psi_n(x, t; \lambda).$$

Proof. A simple adaptation of the proof of [7, Lemma A.6] proves the lemma. □

Lemma A.10. *Let $\lambda \in \mathbb{R}$, $\mu > 0$, and $\alpha \ge 0$. Then*

$$\int_0^t e^{-\frac{t-s}{\mu}} (s+1)^{-\alpha}\psi_n(x, s; \lambda)\, ds \le C(t+1)^{-\alpha}\psi_n(x, t; \lambda).$$

Proof. The lemma can be proved by slightly modifying the proof of [10, Lemma 3.9]. □

[3] The case of $\beta = 2$ is excluded just for simplicity.

References

1. Carpio, A.: Large-time behavior in incompressible Navier–Stokes equations. *SIAM J. Math. Anal.* **27**, 449–475, 1996
2. Fujigaki, Y., Miyakawa, T.: Asymptotic profiles of nonstationary incompressible Navier–Stokes flows in the whole space. *SIAM J. Math. Anal.* **33**, 523–544, 2001
3. Iguchi, T., Kawashima, S.: On space-time decay properties of solutions to hyperbolic-elliptic coupled systems. *Hiroshima Math. J.* **32**, 229–308, 2002
4. Kanel', Y.I.: A model system of equations for the one-dimensional motion of a gas. *Differ. Uravn.* **4**, 721–734, 1968
5. Kato, M.: Large time behavior of solutions to the generalized Burgers equations. *Osaka J. Math.* **44**, 923–943, 2007
6. Koike, K.: Long-time behavior of a point mass in a one-dimensional viscous compressible fluid and pointwise estimates of solutions. *J. Differ. Equ.* **271**, 356–413, 2021
7. Koike, K.: Refined pointwise estimates for solutions to the 1D barotropic compressible Navier–Stokes equations: an application to the long-time behavior of a point mass. *J. Math. Fluid Mech.*, 2022. https://doi.org/10.1007/s00021-022-00732-0
8. Liu, T.-P., Yu, S.-H.: The Green's function and large-time behavior of solutions for the one-dimensional Boltzmann equation. *Commun. Pure Appl. Math.* **57**, 1543–1608, 2004
9. Liu, T.-P., Yu, S.-H.: Navier-Stokes equations in gas dynamics: Green's function, singularity, and well-posedness. *Commun. Pure Appl. Math.* **75**, 223–348, 2022
10. Liu, T.-P., Zeng, Y.: Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. *Mem. Amer. Math. Soc.* **125**(599), 1997
11. Liu, T.-P., Zeng, Y.: On Green's function for hyperbolic-parabolic systems. Acta Math. Sci. Ser. B (Engl. Ed.) 29, 1556–1572, 2009
12. Van Baalen, G., Popović, N., Wayne, E.: Long tails in the long-time asymptotics of quasi-linear hyperbolic-parabolic systems of conservation laws. *SIAM J. Math. Anal.* **39**, 1951–1977, 2008
13. Wang, W., Yang, T.: Pointwise estimates and L_p convergence rates to diffusion waves for p-system with damping. *J. Differ. Equ.* **187**, 310–336, 2003
14. Zeng, Y.: L^1 asymptotic behavior of compressible, isentropic, viscous 1-D flow. *Commun. Pure Appl. Math.* **47**, 1053–1082, 1994
15. Zeng, Y., Chen, J.: Pointwise time asymptotic behavior of solutions to a general class of hyperbolic balance laws. *J. Differ. Equ.* **260**, 6745–6786, 2016

Kai Koike
Department of Mathematics,
Tokyo Institute of Technology,
Tokyo
152-8551 Japan.
e-mail: koike.k@math.titech.ac.jp

(*Received November 26, 2022 / Accepted July 27, 2023*)
Published online August 17, 2023