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On Entropy Solutions of Scalar Conservation Laws with Discontinuous Flux

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Abstract

We introduce the notion of entropy solutions (e.s.) to a conservation law with an arbitrary jump continuous flux vector and prove the existence of the largest and the smallest e.s. to the Cauchy problem. The monotonicity and stability properties of these solutions are also established. In the case of a periodic initial function, we derive the uniqueness of e.s. Generally, the uniqueness property can be violated, which is confirmed by an example. Finally, we prove that in the case of a single space variable a weak limit of a sequence of e.s. is an e.s. as well (under the requirement of the spatial periodicity of the limit Young measure).

1. Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, where $\mathbb{R}_+ = (0, +\infty)$, we consider the conservation law

$$u_t + \operatorname{div}_x \varphi(u) = 0 \tag{1.1}$$

with a jump continuous (frequently called regulated) flux vector $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u))$. This means that at each point $u_* \in \mathbb{R}$ there exist one-sided limits $\lim\limits_{u\to u_*\pm} \varphi(u) \doteq \varphi(u_*\pm)$. For example, if the components $\varphi_i(u)$, $i = 1, \dots, n$, are BV-functions then the vector $\varphi(u)$ is jump continuous. Equations of such a type arise in numerous physical applications, for example in phase transitions [5], in elasticity [23], in models of material flow on conveyor belts [8] and in many others. In the one-dimensional case $n = 1$ Cauchy problem for Eq. (1.1) was first studied in [7] by the wave front tracking method. In this paper the author constructed a semigroup of weak solutions, however no entropy conditions were formulated. In the present study we will follow another approach, first developed by Carrillo [4] for an initial boundary value problem in a bounded domain; it is based on a relevant continuous parametrization of the curve $(u, \varphi(u))$, which allows us to reduce Eq. (1.1) to the well established case of conservation laws with continuous flux vector. The same approach was also exploited in papers [2,3,9] for the Cauchy problem. In these papers the authors supposed that $u(t, \cdot) \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and that the flux vector is Hölder continuous at zero with the exponent $\alpha \geqq (n-1)/n$. In the present paper we study the general case when $u \in L^\infty(\Pi)$ and when $\varphi(u)$ is an arbitrary jump continuous flux vector. Moreover, we also take into account the values $\varphi(u_*)$ at discontinuity points, which may be different from $\varphi(u_*\pm)$. In this general situation the uniqueness of e.s. may fail and it is useful to select e.s. with additional properties. We will prove the existence of the largest and the smallest e.s. This allows us to prove the uniqueness in the case of periodic initial data. In the one-dimensional situation we also establish that, despite of nonlinearity, a weak limit of e.s. is an e.s. as well. This extends results [19] to the case of discontinuous flux.

It is known that the set

$$D = \{\, u_* \in \mathbb{R} \mid |\varphi(u_*+) - \varphi(u_*)| + |\varphi(u_*) - \varphi(u_*-)| > 0 \,\}$$

of discontinuity points of the vector $\varphi(u)$ is at most countable (and may be an arbitrary at most countable set in $\mathbb{R}$). We use above and will use in the sequel the notation $|\cdot|$ for Euclidean finite-dimensional norms (including the absolute value in one-dimensional case). We will treat $\varphi(u)$ as a multi-valued vector function with values $\bar\varphi(u) = [\varphi(u-), \varphi(u)] \cup [\varphi(u), \varphi(u+)]$ being a union of two segments in $\mathbb{R}^n$. This set is different from a singleton only at discontinuity points $u_* \in D$. Let us demonstrate that the graph of $\bar\varphi(u)$ admits a continuous parametrization

$$u = b(v), b \in C(\mathbb{R}), \quad \bar\varphi(u) \ni g(v), g \in C(\mathbb{R}, \mathbb{R}^n), \tag{1.2}$$

such that the function $b(v)$ is non-strictly increasing and coercive, i.e., $b(v) \to \pm\infty$ as $v \to \pm\infty$, and that on each segment $b^{-1}(u_*)$, $u_* \in D$, $g(v)$ is the concatenation of (possibly non-strictly) monotone parametrizations of the linear paths $[\varphi(u_*-), \varphi(u_*)]$ and $[\varphi(u_*), \varphi(u+)]$ (this means that the distance from a point of such path to its starting point increases). We call parametrizations (1.2) with the indicated properties admissible. The existence of an admissible parametrization (1.2) was shown in paper [2], but only in the case when the set D admits monotone numeration $D = \{u_k\}$, $k \in \mathbb{N}$, $u_{k+1} > u_k$ $\forall k \in \mathbb{N}$, i.e., when D is a completely ordered subset of $\mathbb{R}$. In the following lemma we construct the required parametrization for the general case:

Lemma 1. *There exists an admissible parametrization (1.2) of the graph of* $\bar\varphi$.

Proof. We consider the more complicated case when D is infinite (in the case of finite D we only need to replace the set $\mathbb{N}$ in the proof below by its finite subset). We numerate set D: $D = \{u_k\}_{k\in\mathbb{N}}$ and choose positive numbers h_k such that $\sum_{k=1}^{\infty} h_k = c < \infty$ (in particular, we can take $h_k = 2^{-k}$). We define the finite discrete measure $\mu(u) = \sum_{k=1}^{\infty} h_k \delta(u - u_k)$, where by $\delta(u - u_k)$ we denote the

Dirac mass at the point u_k. Then we introduce the strictly increasing function $\alpha(u) = u + \mu((-\infty, u))$ with jumps at points in D. Notice that

$$u \leqq \alpha(u) \leqq u + c; \quad \alpha(u_2) - \alpha(u_1) \geqq u_2 - u_1 \ \forall u_1, u_2 \in \mathbb{R}, u_2 > u_1. \quad (1.3)$$

The function $b(v)$ is defined as the inverse to the function $\alpha(u)$ considered as maximal monotone graph, that is, the value $b(v)$ is such $u \in \mathbb{R}$ that $v \in [\alpha(u-), \alpha(u+)]$. It follows from (1.3) that $v - c \leqq b(v) \leqq v$. If $v_1 < v_2$ then denoting $u_i = b(v_i)$, $i = 1, 2$, we have $v_1 \leqq \alpha(u_1+) \leqq \alpha(u_2-) \leqq v_2$ whenever $u_1 < u_2$. This relation implies that $v_2 - v_1 \geqq \alpha(u_2-) - \alpha(u_1+) \geqq u_2 - u_1 = b(v_2) - b(v_1)$. Hence, $b(v_2) - b(v_1) \leqq v_2 - v_1$. In the case $u_1 = u_2$ we see that $b(v_2) = b(v_1)$ and the inequality $b(v_2) - b(v_1) \leqq v_2 - v_1$ is evident. The obtained inequality can be written in the form $|b(v_2) - b(v_1)| \leqq |v_2 - v_1|$. We find that $b(v)$ is Lipschitz continuous. Notice also that $b(v)$ takes values $u_k \in D$ on the segments $[a_k, b_k] = [\alpha(u_k-), \alpha(u_k+)]$ of length $h_k > 0$. To define the vector $g(v)$, we have to set $g(v) = \varphi(b(v))$ whenever $b(v) \notin D$. If $b(v) = u_k \Leftrightarrow v \in [a_k, b_k]$ we introduce the constants

$$l_k^{\pm} = |\varphi(u_k\pm) - \varphi(u_k)|, \quad c_k = (l_k^+ a_k + l_k^- b_k)/(l_k^+ + l_k^-) \in [a_k, b_k]$$

and set

$$g(v) = \begin{cases} \dfrac{(c_k - v)\varphi(u_k-) + (v - a_k)\varphi(u_k)}{c_k - a_k}, & a_k \leqq v \leqq c_k, c_k > a_k, \\ \dfrac{(b_k - v)\varphi(u_k) + (v - c_k)\varphi(u_k+)}{b_k - c_k}, & c_k \leqq v \leqq b_k, b_k > c_k, \end{cases} \quad (1.4)$$

so that $g(v)$ is a piecewise linear function on $[a_k, b_k]$. Let us show that the vector $g(v)$ is continuous on $\mathbb{R}$. We verify that $g(v)$ is continuous at each point $v_0 \in \mathbb{R}$. It is clear if $v_0 \in (a_k, b_k)$ for some $k \in \mathbb{N}$, in view of (1.4). Further, suppose that $v_0 \notin [a_k, b_k]$ for all $k \in \mathbb{N}$. This means that $u_0 = b(v_0) \notin D$ and $\varphi(u)$ is continuous at u_0. Therefore, for every $\varepsilon > 0$, there exists such a $\delta > 0$ that $|\varphi(u) - \varphi(u_0)| < \varepsilon$ in the interval $|u - u_0| < 2\delta$. This implies that

$$\begin{aligned} &\max(|\varphi(u) - \varphi(u_0)|, |\varphi(u-) - \varphi(u_0)|, |\varphi(u+) - \varphi(u_0)|) \\ &\leqq \varepsilon \quad \forall u \in \mathbb{R}, |u - u_0| < \delta. \end{aligned} \quad (1.5)$$

If $|v - v_0| < \delta$ then $|b(v) - u_0| \leqq |v - v_0| < \delta$ and taking into account (1.4) and (1.5) we conclude

$$\begin{aligned} |g(v) - g(v_0)| \leqq \max(&|\varphi(b(v)) - \varphi(u_0)|, |\varphi(b(v)-) \\ &- \varphi(u_0)|, |\varphi(b(v)+) - \varphi(u_0)|) \leqq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this means continuity of $g(v)$ at point v_0. By the similar reasons we prove that

$$\lim_{v \to a_k-} g(v) = \varphi(u_k-) = g(a_k), \quad \lim_{v \to b_k+} g(v) = \varphi(u_k+) = g(b_k) \ \forall k \in \mathbb{N}.$$

Since, in view of (1.4),

$$\lim_{v\to a_k+} g(v) = g(a_k), \quad \lim_{v\to b_k-} g(v) = g(b_k),$$

we find that the vector $g(v)$ is continuous at the remaining points $v = a_k, b_k, k \in \mathbb{N}$. The proof is complete. □

Remark 1. Notice that the parametrization described in Lemma 1, which will be referred as standard, is minimal in the sense that for any other admissible parametrization $u = b_1(w)$, $\bar{\varphi}(u) \ni g_1(w)$ there exists a continuous nondecreasing surjection $V(w)$ such that $b_1(w) = b(V(w))$, $g_1(w) = g(V(w))$. In fact, $V(w)$ is the unique value of the standard parameter v corresponding to the point $(b_1(w), g_1(w))$ of the graph of $\bar{\varphi}$ whenever this parameter is uniquely determined. Otherwise, necessarily $b_1(w) = u_k \in D$, both segments $I_- = [\varphi(u_k-), \varphi(u_k)]$, $I_+ = [\varphi(u_k+), \varphi(u_k)]$ are nontrivial, and one of them contains the other. We choose the segment $[\alpha, \beta] = b_1^{-1}(u_k)$ and a parameter $w = \gamma$ such that $g_1(\gamma) = \varphi(u_k)$. Since, as follows from the continuity of g_1, $g_1(\alpha) = \varphi(u_k-)$, $g_1(\beta) = \varphi(u_k+)$, then $\alpha < \gamma < \beta$. We set $V(w) = v_- \leqq c_k$, the smaller value of standard parameter v corresponding to the point $(b_1(w), g_1(w))$ if $w \in [\alpha, \gamma]$, $V(w) = v_+ \geqq c_k$, the larger value of standard parameter v corresponding to the point $(b_1(w), g_1(w))$ if $w \in [\gamma, \beta]$. Notice that $v_- = v_+ = c_k$ at the point γ, which readily implies that $V(w)$ is continuous on $[\alpha, \beta]$ as required. Since $g_1(w)$ forms increasing parametrizations of the segments I_-, I_+ when $w \in [\alpha, \gamma]$, respectively when $w \in [\gamma, \beta]$, we conclude that $V(w)$ is a nonstrictly increasing function. Evidently, the function $V(w)$ takes all values of the standard parameter $v \in \mathbb{R}$, i.e., it is a surjection.

At least formally, after the change $u = b(v)$, Eq. (1.1) reduces to the equation

$$b(v)_t + \operatorname{div}_x g(v) = 0 \tag{1.6}$$

with already continuous flux $(b(v), g(v)) \in \mathbb{R}^{n+1}$.

Recall that an entropy solution (e.s.) of Eq. (1.6) is a function $v = v(t, x) \in L^\infty(\Pi)$ satisfying the Kruzhkov entropy condition: $\forall k \in \mathbb{R}$

$$|b(v) - b(k)|_t + \operatorname{div}_x[\operatorname{sign}(v - k)(g(v) - g(k))] \leqq 0 \tag{1.7}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$). This means that for each test function $f = f(t, x) \in C_0^1(\Pi)$, $f \geqq 0$

$$\int_\Pi [|b(v) - b(k)| f_t + \operatorname{sign}(v - k)(g(v) - g(k)) \cdot \nabla_x f] \mathrm{d}t\mathrm{d}x \geqq 0. \tag{1.8}$$

Taking $k = \pm\|v\|_\infty$, we derive from (1.7) that $b(v)_t + \operatorname{div}_x g(v) = 0$ in $\mathcal{D}'(\Pi)$ and e.s. $v = v(t, x)$ of (1.6) is a weak solution of this equation. We study the Cauchy problem for Eqs. (1.1), (1.6) with initial condition

$$u(0, x) = b(v)(0, x) = u_0(x) \in L^\infty(\mathbb{R}^n). \tag{1.9}$$

This condition is understood in the sense of the relation

$$\operatorname*{ess\,lim}_{t\to 0} u(t, \cdot) = u_0 \ \text{ in } L^1_{loc}(\mathbb{R}^n). \tag{1.10}$$

It is well known (cf. [18, Proposition 2]) that conditions (1.7), (1.10) can be written in the form of single integral inequality similar to (1.8): for all $k \in \mathbb{R}$ and each non-negative test function $f = f(t,x) \in C_0^1(\bar{\Pi})$, where $\bar{\Pi} = [0,+\infty) \times \mathbb{R}^n$ is the closure of Π,

$$\int_\Pi [|b(v) - b(k)| f_t + \operatorname{sign}(v-k)(g(v) - g(k)) \cdot \nabla_x f] \mathrm{d}t \mathrm{d}x \\ + \int_{\mathbb{R}^n} |u_0(x) - b(k)| f(0,x) \mathrm{d}x \geqq 0. \tag{1.11}$$

Notice that any jump continuous function is Borel and locally bounded. Therefore, $\varphi(u) \in L^\infty(\Pi)$ for all $u = u(t,x) \in L^\infty(\Pi)$, and we can define the notion of e.s. of original problem (1.1), (1.9) by the standard Kruzhkov relation like (1.11)

$$\int_\Pi [|u-k| f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] \mathrm{d}t \mathrm{d}x \\ + \int_{\mathbb{R}^n} |u_0(x) - k| f(0,x) \mathrm{d}x \geqq 0 \tag{1.12}$$

for all $k \in \mathbb{R}$, $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geqq 0$. But such e.s. may not exist, see Example 2 below. For the correct definition we need multivalued extension of the flux at discontinuity points and the described above reduction to the well established case of continuous flux.

In the sequel, we need the more general class of measure-valued solutions. Recall (see [6,24,25]) that a measure-valued function (a Young measure) on Π is a weakly measurable map $(t,x) \to \nu_{t,x}$ of Π into the space $\operatorname{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$. The weak measurability of $\nu_{t,x}$ means that for each continuous function $p(v)$, the function $(t,x) \to \langle \nu_{t,x}, p(v) \rangle \doteq \int p(v) \mathrm{d}\nu_{t,x}(v)$ is Lebesgue-measurable on Π. We say that a measure-valued function $\nu_{t,x}$ is bounded if there exists such $R > 0$ that $\operatorname{supp} \nu_{t,x} \subset [-R, R]$ for almost all $(t,x) \in \Pi$. We shall denote by $\|\nu_{t,x}\|_\infty$ the smallest of such R. Finally, we say that measure-valued functions of the kind $\nu_{t,x}(v) = \delta(v - v(t,x))$, where $v(t,x) \in L^\infty(\Pi)$ and $\delta(v - v_*)$ is the Dirac measure at a point $v_* \in \mathbb{R}$, are regular. We identify these measure-valued functions and the corresponding functions $v(t,x)$, so that there is a natural embedding $L^\infty(\Pi) \subset \mathrm{MV}(\Pi)$, where by $\mathrm{MV}(\Pi)$ we denote the set of bounded measure-valued functions on Π. Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem by L. Tartar [25].

Theorem 1. *Let $v_k(t,x) \in L^\infty(\Pi)$, $k \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $v_k(t,x)$ for this subsequence) and a bounded measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ such that*

$$\forall p(v) \in C(\mathbb{R}) \quad p(v_k) \underset{k\to\infty}{\rightharpoonup} \langle \nu_{t,x}, p(v) \rangle \quad \text{weakly-}* \text{ in } L^\infty(\Pi). \tag{1.13}$$

Besides, $\nu_{t,x}$ is regular, i.e., $\nu_{t,x}(v) = \delta(v - v(t,x))$ if and only if $v_k(t,x) \underset{k\to\infty}{\to} v(t,x)$ in $L^1_{loc}(\Pi)$ (strongly).

More generally, the following weak precompactness property holds for bounded sequences of measure valued function, see for instance [16, Theorem 2]:

Theorem 2. *Let $\nu_{t,x}^k \in MV(\Pi)$, $k \in \mathbb{N}$, be a bounded sequence (this means that the scalar sequence $\|\nu_{t,x}^k\|_\infty$ is bounded). Then there exists a subsequence $\nu_{t,x}^k$ (not relabeled) weakly convergent to a bounded measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ in the sense of relation*

$$\forall p(v) \in C(\mathbb{R}) \quad \langle \nu_{t,x}^k, p(v)\rangle \underset{k\to\infty}{\rightharpoonup} \langle \nu_{t,x}, p(v)\rangle \quad \text{weakly-}* \text{ in } L^\infty(\Pi). \tag{1.14}$$

Obviously, in the case when the sequence $\nu_{t,x}^k$ consists of regular functions v_k, relation (1.14) reduces to (1.13). Remark that in Theorems 1, 2 the half-space Π may be replaced by arbitrary finite-dimensional domain Ω.

Recall (see [6,17,24]) that a measure valued e.s. of (1.6), (1.9) is a bounded measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$, which satisfies the following averaged variant of entropy relation (1.11): for all $k \in \mathbb{R}$, $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geqq 0$

$$\begin{aligned}\int_\Pi \Bigg[\int |b(v) - b(k)| d\nu_{t,x}(v) f_t \\ + \int \operatorname{sign}(v-k)(g(v)-g(k)) d\nu_{t,x}(v) \cdot \nabla_x f\Bigg] dt dx \\ + \int_{\mathbb{R}^n} |u_0(x) - b(k)| f(0,x) dx \geqq 0.\end{aligned} \tag{1.15}$$

Now we are ready to define the notion of e.s. of original problem (1.1), (1.9).

Definition 1. (cf. [2]) A function $u = u(t,x) \in L^\infty(\Pi)$ is called an e.s. of problem (1.1), (1.9) if there exists a measure valued e.s. $\nu_{t,x}(v)$ of (1.6), (1.9) such that the push-forward measure $b^*\nu_{t,x}(u)$ coincides with the Dirac mass $\delta(u - u(t,x))$ for a.e. $(t,x) \in \Pi$.

In view of the requirement $b^*\nu_{t,x}(u) = \delta(u - u(t,x))$ entropy relation (1.15) can be written as

$$\begin{aligned}\int_\Pi \Bigg[|u - b(k)| f_t + \int \operatorname{sign}(v-k)(g(v)-g(k)) d\nu_{t,x}(v) \cdot \nabla_x f\Bigg] dt dx \\ + \int_{\mathbb{R}^n} |u_0(x) - b(k)| f(0,x) dx \geqq 0.\end{aligned} \tag{1.16}$$

Remark 2. If $u(t,x)$ is an e.s. of (1.1), (1.9), then $u = -u(t,x)$ is an e.s. of the problem

$$u_t - \operatorname{div}_x \varphi(-u) = 0, \quad u(0,x) = -u_0(x) \tag{1.17}$$

regarding the continuous parametrization $u = -b(-v)$, $-\bar{\varphi}(-u) \ni -g(-v)$ of the flux. In fact, let $\nu_{t,x}$ be a measure valued e.s. of (1.6), (1.9) such that $b^*\nu_{t,x}(u) = \delta(u - u(t,x))$. Then the measure valued function $\tilde{\nu}_{t,x} = l^*\nu_{t,x} \in \mathrm{MV}(\Pi)$, where $l(v) = -v$, is a measure valued e.s. of the problem (1.17). In fact, for each $k \in \mathbb{R}$,

$$\int |-b(-v) - (-b(-k))| d\tilde{\nu}_{t,x}(v) = \int |b(v) - b(-k)| d\nu_{t,x}(v) = |u - b(-k)|,$$

$$\int \operatorname{sign}(v-k)(-g(-v)-(-g(-k)))d\tilde{\nu}_{t,x}(v) = \int \operatorname{sign}(v+k)(g(v)-g(-k))\mathrm{d}\nu_{t,x}(v),$$
$$|-u_0(x)-(-b(-k))| = |u_0(x)-b(-k)|,$$

and these equalities imply that, for every $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geqq 0$,

$$\begin{aligned}
&\int_\Pi \left[\int |-b(-v)-(-b(-k))| d\tilde{\nu}_{t,x}(v) f_t \right. \\
&\quad \left. + \int \operatorname{sign}(v-k)(-g(-v)-(-g(-k)))d\tilde{\nu}_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} |-u_0(x)-(-b(-k))| f(0,x)\mathrm{d}x \\
&= \int_\Pi \left[\int |b(v)-b(-k)|\mathrm{d}\nu_{t,x}(v) f_t + \int \operatorname{sign}(v+k)(g(v)-g(-k))\mathrm{d}\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} |u_0(x)-b(-k)| f(0,x)\mathrm{d}x \geqq 0,
\end{aligned}$$

by the entropy relation (1.15) with k replaced by $-k$. Further more,

$$(-b(-\cdot))^*\tilde{\nu}_{t,x}(u) = (-b)^*\nu_{t,x}(u) = l^*\delta(u-u(t,x)) = \delta(u-(-u(t,x))).$$

We conclude that $-u(t,x)$ satisfies all the requirement of Definition 1 for the problem (1.6).

Remark 3. The notion of e.s. does not depend on the choice of admissible parametrization (1.2). In fact, let

$$u = b_1(w), \quad \bar{\varphi}(u) \ni g_1(w) \tag{1.18}$$

be an admissible parametrization of $\bar{\varphi}(u)$, and $u(t,x)$ be an e.s. of (1.1), (1.9) corresponding to this parametrization. According to Definition 1, there exists a measure valued e.s. $\tilde{\nu}_{t,x}(w)$ of the problem

$$b_1(w)_t + \operatorname{div}_x g_1(w) = 0, \quad b_1(w(0,x)) = u_0(x) \tag{1.19}$$

such that $(b_1^*\tilde{\nu}_{t,x})(u) = \delta(u-u(t,x))$. In view of entropy relation (1.16) for each $k \in \mathbb{R}$ and and all $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geqq 0$

$$\begin{aligned}
&\int_\Pi \left[|u-b_1(k)| f_t + \int \operatorname{sign}(w-k)(g_1(w)-g_1(k))d\tilde{\nu}_{t,x}(w)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} |u_0(x)-b_1(k)| f(0,x)\mathrm{d}x \geqq 0.
\end{aligned} \tag{1.20}$$

Further more, by Remark 1 there exists a continuous nondecreasing surjection $v = V(w)$ such that

$$b_1(w) = b(V(w)), \quad g_1(w) = g(V(w)),$$

where $u = b(v)$, $\bar{\varphi}(u) \ni g(v)$ is the standard parametrization of the graph of $\bar{\varphi}$ given in Lemma 1. Notice that by monotonicity of V

$$\operatorname{sign}(w-k)(g_1(w)-g_1(k)) = \operatorname{sign}(V(w)-V(k))(g(V(w))-g(V(k)))$$

and therefore relation (1.20) turns into

$$\int_{\Pi}\left[|u-b(l)|f_t+\int \operatorname{sign}(v-l)(g(v)-g(l))\mathrm{d}\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x$$
$$+\int_{\mathbb{R}^n}|u_0(x)-b(l)|f(0,x)\mathrm{d}x\geqq 0, \tag{1.21}$$

where $\nu_{t,x}(v)=(V^*\tilde{\nu}_{t,x})(v)$ is the push-forward measure and $l=V(k)$. Observe that

$$(b^*\nu_{t,x})(u)=(b(V)^*\tilde{\nu}_{t,x})(u)=(b_1^*\tilde{\nu}_{t,x})(u)=\delta(u-u(t,x)).$$

Since $l=V(k)$ takes all real values, it follows from (1.21) that $\nu_{t,x}(v)$ is a measure valued e.s. of (1.6), (1.9). According to Definition 1, $u(t,x)$ is an e.s. of (1.1), (1.9) corresponding to the standard parametrization.

Conversely, any such e.s. $u(t,x)$ satisfies $(b^*\nu_{t,x})(u)=\delta(u-u(t,x))$, where $\nu_{t,x}(v)$ is a measure valued e.s. of (1.6), (1.9). Since the function V is a surjection, we can find a Young measure $\tilde{\nu}_{t,x}(w)$ on Π such that $\nu_{t,x}(v)=(V^*\tilde{\nu}_{t,x})(v)$. In view of equivalence of relations (1.20) and (1.21), we find that (1.20) holds, that is, $\tilde{\nu}_{t,x}(w)$ is a measure valued e.s. of (1.19). Since $(b_1^*\tilde{\nu}_{t,x})(u)=(b^*\nu_{t,x})(u)=\delta(u-u(t,x))$, we find that $u(t,x)$ is an e.s. corresponding to the admissible parametrization (1.18).

In [2] (also see [3,9]) the existence and uniqueness of e.s. were established only in the case of integrable initial function $u_0\in L^1(\mathbb{R}^n)$ and under assumption of Hölder continuity of the flux vector $\varphi(u)$ at zero with the exponent $\alpha\geqq(n-1)/n$. Using methods of [11,12], one can prove the uniqueness under a weaker anisotropic conditions on the continuity moduli of the flux functions at a point c when $u_0\in c+L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. In a general situation the uniqueness may fail and our main result on existence of the largest and the smallest e.s. of (1.1), (1.9) seems to be useful.

The uniqueness of e.s. follows from our result in the particular case when initial function u_0 is periodic. This extends results of [18]. In the case $n=1$ we also prove the weak completeness of the set of spatially periodic e.s., generalizing results of [19] to the case of discontinuous flux.

In the next section we establish some important properties of e.s. including maximum/minimum and comparison principles.

2. Some Properties of e.s.

We denote $z^\pm=\max(\pm z,0)$, $\operatorname{sign}^+z=(\operatorname{sign}z)^+$, $\operatorname{sign}^-z=-\operatorname{sign}^+(-z)$ (so that $\operatorname{sign}^\pm z=\frac{d}{dz}z^\pm$).

Proposition 1. *If $u=u(t,x)$ is an e.s. of (1.1), (1.9), $c\in\mathbb{R}$, then for a.e. $t>0$*

$$\int_{\mathbb{R}^n}(u(t,x)-c)^\pm\mathrm{d}x\leqq\int_{\mathbb{R}^n}(u_0(x)-c)^\pm\mathrm{d}x$$

(these integrals are allowed to be infinite).

Proof. Without loss of generality we will suppose that $\int_{\mathbb{R}^n}(u_0(x)-c)^{\pm}\mathrm{d}x < \infty$, otherwise the required estimate is evident. It follows from (1.16) with $k = \pm M$, $M \geqq \|\nu_{t,x}\|_\infty$, that, for each $f = f(t,x) \in C_0^1(\bar{\Pi})$,

$$\int_\Pi \left[u f_t + \int g(v)\mathrm{d}\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n} u_0(x)f(0,x)\mathrm{d}x = 0. \quad (2.1)$$

Taking into account that, for every constant $k \in \mathbb{R}$,

$$\int_\Pi [b(k)f_t + g(k)\cdot\nabla_x f]\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n} b(k)f(0,x)\mathrm{d}x = 0,$$

we can rewrite the previous identity in the form

$$\int_\Pi \left[(u-b(k))f_t + \int (g(v)-g(k))\mathrm{d}\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n}(u_0(x)-b(k))f(0,x)\mathrm{d}x = 0.$$

Putting this equality together with entropy inequality (1.16) and taking into account that $|z| + z = 2z^+$, $\operatorname{sign} z + 1 = 2\operatorname{sign}^+ z$, we arrive at the relation

$$\int_\Pi \left[(u-b(k))^+ f_t + \int \operatorname{sign}^+(v-k)(g(v)-g(k))d\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n}(u_0(x)-b(k))^+ f(0,x)\mathrm{d}x \geqq 0. \quad (2.2)$$

By coercivity condition there is such $d \in \mathbb{R}$ that $c = b(d)$. Let $m \geqq n$, $\delta > 0$, $\beta(s) = \min((s/\delta)^+, 1)^m$. Integrating the inequality (2.2) over the measure $\beta'(b(k)-c)db(k)$, we arrive at the relation

$$\int_\Pi \left[\eta(u-c)f_t + \int q(v)\mathrm{d}\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n}\eta(u_0(x)-c)f(0,x)\mathrm{d}x \geqq 0, \quad (2.3)$$

where

$$\eta(b(v)-c) = \int_d^v (b(v)-b(k))^+\beta'(b(k)-c)db(k) = \int_d^v \beta(b(k)-c)db(k) = \begin{cases} ((b(v)-c)^+)^{m+1}/((m+1)\delta^m), & b(v)-c<\delta, \\ b(v)-c-m\delta/(m+1), & b(v)-c \geqq \delta, \end{cases}$$

$$q(v) = \int_d^v \operatorname{sign}^+(v-k)(g(v)-g(k))\beta'(b(k)-c)db(k).$$

In particular, if $\operatorname{supp}\nu_{t,x} \subset [-M, M]$ a.e. on Π, and $C = 2\max\limits_{|v|\leqq M+d}|g(v)|$ then for all $v \in [-M, M]$

$$|q(v)| \leqq C\int_d^v \beta'(b(k)-c)db(k) = C\beta(b(v)-c),$$

which implies that

$$\left|\int q(v)\mathrm{d}\nu_{t,x}(v)\right| \leqq C\int \beta(b(v)-c)\mathrm{d}\nu_{t,x}(v) = C\beta(u-c). \tag{2.4}$$

Now we fix $\varepsilon > 0$. Since $\beta(s) = 1$ for $s > \delta$, the function $\gamma(s) \doteq \dfrac{\beta(s)}{\eta(s)+\varepsilon}$ decreases on $[\delta, +\infty)$. This implies that

$$\begin{aligned}\max \gamma(s) &= \max_{s\in[0,\delta]} \gamma(s) \leqq \max_{s>0} \frac{(s/\delta)^m}{\delta(s/\delta)^{m+1}/(m+1)+\varepsilon}\\ &= \max_{\sigma=s/\delta>0} \frac{m+1}{\delta\sigma+(m+1)\varepsilon\sigma^{-m}}.\end{aligned}$$

By direct computations we find that

$$\min_{\sigma>0}(\delta\sigma+(m+1)\varepsilon\sigma^{-m}) = \frac{\delta(m+1)}{m}\left(\frac{m(m+1)\varepsilon}{\delta}\right)^{\frac{1}{m+1}}.$$

Therefore,

$$\gamma(s) \leqq \frac{m}{\delta}\left(\frac{\delta}{m(m+1)}\right)^{\frac{1}{m+1}}\varepsilon^{-\frac{1}{m+1}}.$$

This, together with estimate (2.4), implies that

$$\left|\int q(v)\mathrm{d}\nu_{t,x}(v)\right| \leqq N(\eta(u-c)+\varepsilon), \tag{2.5}$$

where

$$N = N(\varepsilon) = \frac{Cm}{\delta}\left(\frac{\delta}{m(m+1)}\right)^{\frac{1}{m+1}}\varepsilon^{-\frac{1}{m+1}}. \tag{2.6}$$

Since $\int_\Pi f_t \mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n} f(0,x)\mathrm{d}x = 0$ we can write (2.3) in the form

$$\begin{aligned}&\int_\Pi \left[(\eta(u-c)+\varepsilon)f_t + \int q(v)d\nu_{t,x}(v)\cdot\nabla_x f\right]\mathrm{d}t\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^n}(\eta(u_0(x)-c)+\varepsilon)f(0,x)\mathrm{d}x \geqq 0.\end{aligned} \tag{2.7}$$

Let E be a set of $t > 0$ such that (t, x) is a Lebesgue point of $u(t, x)$ for almost all $x \in \mathbb{R}^n$. It is well-known (see for example [21, Lemma 1.2]) that E is a set of full measure and $t \in E$ is a common Lebesgue point of the functions $t \to \int_{\mathbb{R}^n} u(t,x)h(x)\mathrm{d}x$ for all $h(x) \in L^1(\mathbb{R}^n)$. Since every Lebesgue point of a bounded function u is also a Lebesgue point of $p(u)$ for an arbitrary function $p \in C(\mathbb{R})$, we may replace u in the above property by $p(u)$, and in particular by $\eta(u-c)+\varepsilon$. We choose a function $\omega(s) \in C_0^\infty(\mathbb{R})$ such that $\omega(s) \geqq 0$, $\operatorname{supp}\omega \subset [0,1]$, $\int \omega(s)ds = 1$, and define the sequences $\omega_r(s) = r\omega(rs)$, $\theta_r(s) = \int_{-\infty}^s \omega_r(\sigma)d\sigma = \int_{-\infty}^{rs}\omega(\sigma)d\sigma$, $r \in \mathbb{N}$. Obviously, the sequence $\omega_r(s)$ converges as $r \to \infty$ to the Dirac δ-measure weakly in $\mathcal{D}'(\mathbb{R})$ while the sequence $\theta_r(s)$ converges to the Heaviside function $\theta(s)$ pointwise and in $L^1_{loc}(\mathbb{R})$. Now we take the test function in the form

$$f = f(t,x) = h\theta_r(t_0 - t), \quad h = \rho(N(t-t_0) + |x| - R),$$

where $\rho(\sigma) \in C^\infty(\mathbb{R})$ is a decreasing function such that $\rho(\sigma) = 1$ for $\sigma \leqq 0$ and $\rho(\sigma) = 0$ for $\sigma \geqq 1$ (we can take $\rho(\sigma) = 1 - \theta_1(\sigma)$), $R > 0$, and $t_0 \in E$. Observe that $f = \theta_r(t_0 - t)$ in a vicinity $|x| < R$ of the singular point $x = 0$ and therefore $f \in C^\infty(\bar{\Pi})$, $f \geqq 0$. Applying (2.7) to the test function f, we arrive at the relation

$$\begin{aligned}&\int_{\mathbb{R}^n} (\eta(u_0(x) - c) + \varepsilon)h(0,x)\mathrm{d}x - \int_\Pi (\eta(u-c) + \varepsilon)h\omega_r(t_0 - t)\mathrm{d}t\mathrm{d}x \\ &\quad + \int_\Pi \left[N(\eta(u(x) - c) + \varepsilon) + \int q(v)\mathrm{d}\nu_{t,x}(v) \cdot \frac{x}{|x|}\right] \\ &\quad \times \rho'(N(t-t_0) + |x| - R)\theta_r(t_0 - t)\mathrm{d}t\mathrm{d}x \geqq 0 \end{aligned} \tag{2.8}$$

for sufficient large $r \in \mathbb{N}$ such that $rt_0 > 1$. In view of (2.5) and the condition $\rho'(\sigma) \leqq 0$, the last integral in (2.8) is non-positive and it follows that

$$\int_\Pi (\eta(u-c) + \varepsilon)h\omega_r(t_0 - t)\mathrm{d}t\mathrm{d}x \leqq \int_{\mathbb{R}^n} (\eta(u_0(x) - c) + \varepsilon)h(0,x)\mathrm{d}x.$$

Dropping ε in the left integral, we obtain the inequality

$$\int_0^\infty \left(\int_{\mathbb{R}^n} \eta(u(t,x) - c)h(t,x)\mathrm{d}x\right)\omega_r(t_0 - t)\mathrm{d}t \leqq \int_{\mathbb{R}^n} (\eta(u_0(x) - c) + \varepsilon)h(0,x)\mathrm{d}x.$$

Since $t_0 \in E$ is a Lebesgue point of the function $t \to \int_{\mathbb{R}^n} \eta(u(t,x) - c)h(t,x)\mathrm{d}x$, we can pass to the limit as $r \to \infty$ in the above inequality, resulting in

$$\int_{\mathbb{R}^n} \eta(u(t_0,x) - c)h(t_0,x)\mathrm{d}x \leqq \int_{\mathbb{R}^n} (\eta(u_0(x) - c) + \varepsilon)h(0,x)\mathrm{d}x.$$

Revealing this relation, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \eta(u(t_0,x) - c)\rho(|x| - R)\mathrm{d}x &\leqq \int_{\mathbb{R}^n} (\eta(u_0(x) - c) + \varepsilon)\rho(|x| - Nt_0 - R)\mathrm{d}x \\ &\leqq \int_{\mathbb{R}^n} \eta(u_0(x) - c)\mathrm{d}x + \varepsilon\int_{\mathbb{R}^n} \rho(|x| - Nt_0 - R)\mathrm{d}x. \end{aligned} \tag{2.9}$$

With the help of (2.6), we obtain that for some constants $c_1, c_2 = c_2(R,\delta)$

$$\varepsilon\int_{\mathbb{R}^n} \rho(|x| - N(\varepsilon)t_0 - R)\mathrm{d}x \leqq c_1\varepsilon(N(\varepsilon)t_0 + R + 1)^n \leqq c_2\varepsilon(1 + t_0\varepsilon^{-\frac{1}{m+1}})^n \underset{\varepsilon\to 0+}{\to} 0$$

(recall that $m + 1 > n$). Therefore, passing to the limit in (2.9) as $\varepsilon \to 0+$, we obtain that for all $t_0 \in E$

$$\int_{\mathbb{R}^n} \eta(u(t_0,x) - c)\rho(|x| - R)\mathrm{d}x \leqq \int_{\mathbb{R}^n} \eta(u_0(x) - c)\mathrm{d}x. \tag{2.10}$$

Now observe that $0 \leqq \eta(s) \leqq s^+$ and $\eta(s) \to s^+$ as $\delta \to 0$. By Lebesgue dominated convergence theorem it follows from (2.10) in the limit as $\delta \to 0$ that for a.e. $t = t_0 > 0$

$$\int_{\mathbb{R}^n} (u(t,x) - c)^+ \rho(|x| - R)\mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_0(x) - c)^+ \mathrm{d}x < +\infty.$$

By Fatou's lemma this implies, in the limit as $R \to \infty$, that

$$\int_{\mathbb{R}^n} (u(t,x) - c)^+ \mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_0(x) - c)^+ \mathrm{d}x, \tag{2.11}$$

as required. In view of Remark 2 the function $-u(t,x)$ is an e.s. of the problem $u_t - \operatorname{div}_x \varphi(-u)_x - 0, u(0,x) = -u_0(x)$. Applying (2.11) to this e.s. with c replaced by $-c$, we obtain the inequality

$$\int_{\mathbb{R}^n} (u(t,x) - c)^- \mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_0(x) - c)^- \mathrm{d}x \quad \forall t \in E. \tag{2.12}$$

□

Corollary 1. *Any e.s. $u = u(t,x)$ of (1.1), (1.9) satisfies the maximum/minimum principle*

$$a = \textit{ess inf}\, u_0(x) \leqq u(t,x) \leqq b = \textit{ess sup}\, u_0(x) \ \ \textit{for a.e.}\ (t,x) \in \Pi.$$

Proof. The maximum/minimum principles directly follows from (2.11) and (2.12) with $k = b$ and $k = a$, respectively. □

Putting inequalities (2.11), (2.12) together and using the known relation $|z| = z^+ + z^-$, we obtain the following:

Corollary 2. *If $u(t,x)$ is an e.s. of (1.1), (1.9) then for a.e. $t > 0$*

$$\int_{\mathbb{R}^n} |u(t,x) - c|\mathrm{d}x \leqq \int_{\mathbb{R}^n} |u_0(x) - c|\mathrm{d}x.$$

If u_1, u_2 is a pair of e.s. and $\nu_{t,x}^{(1)}, \nu_{t,x}^{(2)}$ are the corresponding measure valued e.s. of (1.6) then by a measure-valued analogue of the doubling variable method, developed in [24] (also see [17]), we have the relation

$$\frac{\partial}{\partial t} \iint (b(v) - b(w))^+ \mathrm{d}\nu_{t,x}^{(1)}(v)\mathrm{d}\nu_{t,x}^{(2)}(w)$$
$$+ \operatorname{div}_x \iint \operatorname{sign}^+(v - w)(g(v) - g(w))\mathrm{d}\nu_{t,x}^{(1)}(v)\mathrm{d}\nu_{t,x}^{(2)}(w) \leqq 0 \text{ in } \mathcal{D}'(\Pi).$$

Since $b(v) = u_1(t,x)$, $b(w) = u_2(t,x)$ on $\operatorname{supp} \nu_{t,x}^{(1)}$, $\operatorname{supp} \nu_{t,x}^{(2)}$, respectively, then the above relation can be written as

$$\frac{\partial}{\partial t}(u_1 - u_2)^+ + \operatorname{div}_x \iint \operatorname{sign}^+(v - w)(g(v)$$
$$-g(w))\mathrm{d}\nu_{t,x}^{(1)}(v)\mathrm{d}\nu_{t,x}^{(2)}(w) \leqq 0 \text{ in } \mathcal{D}'(\Pi). \tag{2.13}$$

Proposition 2. *Let u_1, u_2 be e.s. of (1.1), (1.9) with initial functions u_{10}, u_{20}, respectively. Assume that, for every $T > 0$,*

$$\text{meas}\{\ (t, x) \in (0, T) \times \mathbb{R}^n \mid u_1(t, x) \geqq u_2(t, x)\ \} < +\infty.$$

Then, for a.e. $t > 0$,

$$\int_{\mathbb{R}^n} (u_1(t, x) - u_2(t, x))^+ \mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{10}(x) - u_{20}(x))^+ \mathrm{d}x.$$

In particular, $u_1(t, x) \leqq u_2(t, x)$ a.e. in Π whenever $u_{10}(x) \leqq u_{20}(x)$ a.e. in $\mathbb{R}^n$ (the comparison principle).

Proof. Let, as above, $\nu^{(1)}_{t,x}$, $\nu^{(2)}_{t,x}$ be measure valued e.s. of (1.6) corresponding to u_1, u_2. Let $E \subset \mathbb{R}_+$ be a set of full measure similar to one in the proof of Proposition 1 consisting of values $t > 0$ such that (t, x) is a Lebesgue point of $(u_1(t, x) - u_2(t, x))^+$ for a.e. $x \in \mathbb{R}^n$. Then $t \in E$ is a common Lebesgue point of the functions $t \to \int (u_1(t, x) - u_2(t, x))^+ h(x)\mathrm{d}x$, $h(x) \in L^1(\mathbb{R}^n)$. Let $t_0, t_1 \in E$, $t_0 < t_1$, $\chi_r(t) = \theta_r(t - t_0) - \theta_r(t - t_1)$, where the sequence $\theta_r(t)$, $r \in \mathbb{N}$, was defined in the proof of Proposition 1. Applying (2.13) to the nonnegative test function $f(t, x) = \chi_r(t)q(x/R)$, where $q = q(y) \in C^1_0(\mathbb{R}^n)$, $0 \leqq q \leqq 1$, $q(0) = 1$, and $R > 0$, we get

$$\int_{\Pi} (u_1(t, x) - u_2(t, x))^+ (\omega_r(t - t_0) - \omega_r(t - t_1))q(x/R)\mathrm{d}t\mathrm{d}x$$
$$+ \frac{1}{R} \int_{\Pi} \iint \text{sign}^+(v - w)(g(v) - g(w))\mathrm{d}\nu^{(1)}_{t,x}(v)\mathrm{d}\nu^{(2)}_{t,x}(w) \cdot \nabla_y q(x/R)\chi_r(t)\mathrm{d}t\mathrm{d}x \geqq 0.$$

Since t_i, $i = 1, 2$, are Lebesgue points of the functions $\int_{\mathbb{R}^n} (u_1(t, x) - u_2(t, x))^+ q(x/R)\mathrm{d}x$ while the sequence $\chi_r(t)$ is uniformly bounded and converges pointwise to the indicator function of the interval $(t_0, t_1]$, we can pass to the limit as $r \to \infty$ in the above relation and get

$$\int_{\mathbb{R}^n} (u_1(t_1, x) - u_2(t_1, x))^+ q(x/R)\mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_1(t_0, x) - u_2(t_0, x))^+ q(x/R)\mathrm{d}x$$
$$+ \frac{1}{R} \int_{(t_0,t_1)\times\mathbb{R}^n} \iint \text{sign}^+(v - w)(g(v) - g(w))\mathrm{d}\nu^{(1)}_{t,x}(v)\mathrm{d}\nu^{(2)}_{t,x}(w) \cdot \nabla_y q(x/R)\mathrm{d}t\mathrm{d}x. \tag{2.14}$$

It follows from the inequality

$$|(u_1(t_0, x) - u_2(t_0, x))^+ - (u_{10}(x) - u_{20}(x))^+|$$
$$\leqq |u_1(t_0, x) - u_{10}(x)| + |u_2(t_0, x) - u_{20}(x)|$$

and initial relation (1.10) that

$$\operatorname*{ess\,lim}_{t_0 \to 0} (u_1(t_0, x) - u_2(t_0, x))^+ = (u_{10}(x) - u_{20}(x))^+ \ \text{ in } L^1_{loc}(\mathbb{R}^n).$$

This allows us to pass to the limit as $t_0 \to 0$ in (2.14), resulting in the relation: for a.e. $T = t_1 > 0$

$$\int_{\mathbb{R}^n} (u_1(T, x) - u_2(T, x))^+ q(x/R)\mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{10}(x) - u_{20}(x))^+ q(x/R)\mathrm{d}x$$

$$+\frac{1}{R}\int_{(0,T)\times\mathbb{R}^n}\iint \operatorname{sign}^+(v-w)(g(v)-g(w))d\nu_{t,x}^{(1)}(v)d\nu_{t,x}^{(2)}(w)\cdot\nabla_y q(x/R)dtdx$$
$$\leqq \int_{\mathbb{R}^n}(u_{10}(x)-u_{20}(x))^+dx+\frac{1}{R}\int_{(0,T)\times\mathbb{R}^n} G(t,x)\cdot\nabla_y q(x/R)dtdx, \quad (2.15)$$

where

$$G=G(t,x)\doteq \iint \operatorname{sign}^+(v-w)(g(v)-g(w))d\nu_{t,x}^{(1)}(v)d\nu_{t,x}^{(2)}(w).$$

By Definition 1 $b(v)\equiv u_1(t,x)$ on $\operatorname{supp}\nu_{t,x}^{(1)}$, $b(w)\equiv u_2(t,x)$ on $\operatorname{supp}\nu_{t,x}^{(2)}$ and if $u_1(t,x)<u_2(t,x)$ then $v<w$ whenever $v\in\operatorname{supp}\nu_{t,x}^{(1)}$, $w\in\operatorname{supp}\nu_{t,x}^{(2)}$, and it follows that $G(t,x)=0$. Therefore, the vector-function G can be different from zero vector only on the set $\{u_1(t,x)\geqq u_2(t,x)\}$, which has finite measure in any layer $\Pi_T=(0,T)\times\mathbb{R}^n$. Thus, denoting $D_T=\{\,(t,x)\in\Pi_T \mid u_1(t,x)\geqq u_2(t,x)\,\}$, we find

$$\left|\int_{(0,T)\times\mathbb{R}^n} G(t,x)\cdot\nabla_y q(x/R)dtdx\right|$$
$$=\left|\int_{D_T} G(t,x)\cdot\nabla_y q(x/R)dtdx\right|\leqq \|G\|_\infty\|\nabla_y q\|_\infty \operatorname{meas} D_T<\infty$$

(notice that $\|G\|_\infty\leqq 2\max\limits_{|v|\leqq M}|g(v)|$, where $M=\max(\|\nu_{t,x}^{(1)}\|_\infty,\|\nu_{t,x}^{(2)}\|_\infty)$). We see that the last term in (2.15) disappears in the limit as $R\to\infty$ due to the factor $1/R$. Hence, passing to the limit as $R\to\infty$ and using Fatou's lemma (observe that $q(x/R)\underset{R\to\infty}{\to}q(0)=1$), we arrive at the desired relation: for all $T\in E$,

$$\int_{\mathbb{R}^n}(u_1(T,x)-u_2(T,x))^+dx\leqq\int_{\mathbb{R}^n}(u_{10}(x)-u_{20}(x))^+dx.$$

□

The nest result asserts the strong completeness of the set of e.s. of the problem (1.1), (1.9). More precisely, we consider the approximate problem

$$u_t+\operatorname{div}_x g(v)=0,\ u=b_r(v);\quad u(0,x)=u_{r0}(x), \quad (2.16)$$

where $b_r(u)\in C(\mathbb{R})$, $r\in\mathbb{N}$, is a sequence of non-strictly increasing functions approximating $b(u)$ (in the sense indicated in the next proposition).

Proposition 3. *Let $u_{r0}=u_{r0}(x)$, $r\in\mathbb{N}$, be a bounded sequence in $L^\infty(\mathbb{R}^n)$, and $u_r=u_r(t,x)$ be a sequence of e.s. of (2.16). Assume that $b_r(u)\to b(u)$ as $r\to\infty$ uniformly on any segment, and that the sequences $u_{r0}\to u_0=u_0(x)$, $u_r\to u=u(t,x)$ in $L^1_{loc}(\mathbb{R}^n)$, $L^1_{loc}(\Pi)$, respectively. Then u is an e.s. of (1.1), (1.9) with initial data u_0.*

Proof. Let $M = \sup_{r\in\mathbb{N}} \|u_{r0}\|_\infty$. By Corollary 1 we see that $\|u_r\|_\infty \leqq M$ for all $r \in \mathbb{N}$. By Definition 1 there exists a sequence $\nu^r_{t,x} \in \mathrm{MV}(\Pi)$ such that

$$b_r^* \nu^r_{t,x}(u) = \delta(u - u_r(t,x)), \tag{2.17}$$

and that, for all $k \in \mathbb{R}$ for every $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geqq 0$,

$$\int_\Pi \left[|u_r - b_r(k)| f_t + \int \operatorname{sign}(v-k)(g(v) - g(k)) \mathrm{d}\nu^r_{t,x}(v) \cdot \nabla_x f \right] \mathrm{d}t \mathrm{d}x \\ + \int_{\mathbb{R}^n} |u_{r0}(x) - b_r(k)| f(0,x) \mathrm{d}x \geqq 0. \tag{2.18}$$

By the coercivity assumption, there exist such a constant $R > 0$ that $b(-R) < -M$, $b(R) > M$. Since $b_r(\pm R) \to b(\pm R)$ as $r \to \infty$, we find that $b_r(-R) < -M$, $b_r(R) > M$ for sufficiently large r. Without loss of generality we can suppose that these inequalities hold for all $r \in \mathbb{N}$. Then, in view of (2.17), $\operatorname{supp} \nu^r_{t,x} \subset [-R, R]$. Therefore, the sequence of measure valued functions $\nu^r_{t,x}$ is bounded and by Theorem 2 some subsequence of $\nu^r_{t,x}$ converges weakly to a bounded measure valued function $\nu_{t,x}$ (in the sense of relation (1.14)). We replace the original sequences u_{r0}, u_r, $\nu^r_{t,x}$ by the corresponding subsequences (keeping the notations), and pass to the limit as $r \to \infty$ in (2.18). As a result, we get

$$\int_\Pi \left[|u - b(k)| f_t + \int \operatorname{sign}(v-k)(g(v) - g(k)) \mathrm{d}\nu_{t,x}(v) \cdot \nabla_x f \right] \mathrm{d}t \mathrm{d}x \\ + \int_{\mathbb{R}^n} |u_0(x) - b(k)| f(0,x) \mathrm{d}x \geqq 0 \tag{2.19}$$

for all $k \in \mathbb{R}$ and each $f = f(t,x) \in C_0^1(\bar{\Pi})$, $f \geqq 0$. Moreover, passing to the limit as $r \to \infty$ in the relation (following from (2.17))

$$\int q(b_r(v)) \mathrm{d}\nu^r_{t,x}(v) = q(u_r(t,x)) \quad \forall q(u) \in C(\mathbb{R}),$$

with the help of the relation $q(b_r(v)) - q(b(v)) \rightrightarrows 0$ uniformly on $[-R, R]$, we obtain that for a.e. $(t,x) \in \Pi$

$$\int q(b(v)) \mathrm{d}\nu_{t,x}(v) = q(u(t,x)). \tag{2.20}$$

A set of full measure E of points (t,x), for which relation (2.20) holds can be chosen common for all q from a countable dense subset of $C(\mathbb{R})$. By the density, this relation remains valid for all $q \in C(\mathbb{R})$, which evidently means that $b^* \nu_{t,x}(u) = \delta(u - u(t,x))$ for all $(t,x) \in E$. In particular, it follows from (2.19) that the entropy relation (1.15) is fulfilled, and $\nu_{t,x}$ is a measure valued e.s. of (1.6), (1.9). In correspondence with Definition 1, we conclude that u is an e.s. of (1.1), (1.9), as required. □

3. Existence of e.s.: The Case of Integrable Initial Data

In this section we assume that the initial function is integrable, $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The general case will be treated in the next section, where we will establish existence of the largest and the smallest e.s. for arbitrary $u_0 \in L^\infty(\mathbb{R}^n)$.

We introduce the approximations $b_r(u) = b(u)+u/r, r \in \mathbb{N}$, of $b(u)$ by strictly increasing functions. Then the equation in (2.16) can be written in the standard form

$$u_t + \operatorname{div}_x \varphi_r(u) = 0, \tag{3.1}$$

where $\varphi_r(u) = g((b_r)^{-1}(u)) \in C(\mathbb{R}, \mathbb{R}^n)$. As was established in [1], there exists the unique largest e.s. $u_r = u_r(t, x)$ of the Cauchy problem for Eq. (3.1) with initial data $u_0(x)$. Moreover, after possible correction on a set of null measure, $u_r(t, \cdot) \in C([0, +\infty), L^1(\mathbb{R}))$, and for each fixed $r \in \mathbb{N}$ the maps $u_0 \to u_r(t, \cdot), t \geqq 0$, are nonexpansive in $L^1(\mathbb{R}^n)$. It is clear that for every $\Delta x \in \mathbb{R}^n$ the shifted functions $u_r(t, x + \Delta x)$ are the largest e.s. of (3.1) with the initial functions $u_0(x + \Delta x)$. This implies the uniform estimate

$$\int_{\mathbb{R}^n} |u_r(t_0, x + \Delta x) - u_r(t_0, x)|\mathrm{d}x \leqq \int_{\mathbb{R}^n} |u_0(x + \Delta x) - u_0(x)|\mathrm{d}x \quad \forall t_0 > 0.$$

It follows from this estimate that

$$\int_{\mathbb{R}^n} |u_r(t_0, x + \Delta x) - u_r(t_0, x)|\mathrm{d}x \leqq \omega^x(|\Delta x|), \tag{3.2}$$

where $\omega^x(h) = \sup\limits_{|\Delta x|<h} \int_{\mathbb{R}^n} |u_0(x + \Delta x) - u_0(x)|\mathrm{d}x$ is the continuity modulus of u_0 in $L^1(\mathbb{R}^n)$. We then proceed as in [10] to get a similar estimate for shifts of the time variable. For the sake of completeness we provide the details. We choose an averaging kernel $\beta(y) \in C_0^1(\mathbb{R}^n)$ with the properties: $\beta(y) \geqq 0$, $\operatorname{supp} \beta(y) \subset B_1(0) = \{y \in \mathbb{R}^n \mid |y| \leqq 1\}$, $\int_{\mathbb{R}^n} \beta(y)\mathrm{d}y = 1$. For a function $q(x) \in L^\infty(\mathbb{R}^n)$ we consider the corresponding averaged functions

$$q^h(x) = h^{-n} \int q(y)\beta((x - y)/h)\mathrm{d}y, \quad h > 0,$$

which are the convolutions $q * \beta^h(x)$, where $\beta^h(x) = h^{-n}\beta(x/h)$. It is clear that $q^h(x) \in C^1(\mathbb{R}^n)$ for each $h > 0$, $\|q^h\|_\infty \leqq \|q\|_\infty$, and $q^h \to q$ as $h \to 0$ a.e. in $\mathbb{R}^n$. Moreover, since $\nabla q^h = q * \nabla\beta^h(x)$, we have

$$\|\nabla q^h\|_\infty \leqq \|q\|_\infty \|\nabla\beta^h\|_1 = \frac{c}{h}\|q\|_\infty, \quad c = \|\nabla_y \beta\|_1. \tag{3.3}$$

Applying (3.1) with $u = u_r$ to the test function

$$f = (\theta_\nu(t - t_0) - \theta_\nu(t - t_0 - \Delta t))p(x),$$

where $t_0, \Delta t > 0$, $p = p(x) \in C_0^1(\mathbb{R}^n)$, $\nu \in \mathbb{N}$, and passing to the limit as $\nu \to \infty$, we get

$$\int_{\mathbb{R}^n} (u_r(t_0 + \Delta t) - u_r(t_0, x))p(x)\mathrm{d}x = \int_{(t_0,t_0+\Delta t)\times\mathbb{R}^n} \varphi_r(u_r) \cdot \nabla p\mathrm{d}x. \tag{3.4}$$

By Corollary 1 $\|u_r\|_\infty \leqq M = \|u_0\|_\infty$ for every $r \in \mathbb{N}$. It follows from the coercivity assumption that there is such $R > 0$ that $b(-R) < -M, b(R) > M$. All the more, $b_r(-R) < b(R) < -M, b_r(R) > b(R) > M$ for all $r \in \mathbb{N}$. This implies that $(b_r)^{-1}([-M, M]) \subset (-R, R)$ and therefore for a.e. $(t, x) \in \Pi$

$$|\varphi_r(u_r)| = g((b_r)^{-1}(u_r)) \leqq N \doteq \max_{|v| \leqq R} |g(v)|.$$

It now follows from (3.4) that

$$\left|\int_{\mathbb{R}^n} (u_r(t_0 + \Delta t) - u_r(t_0, x))p(x)\mathrm{d}x\right| \leqq N\|\nabla p\|_1 \Delta t. \tag{3.5}$$

Further more, we make use of the following variant of Kruzhkov's lemma [10, Lemma 1] (for the sake of completeness, we provide it with the proof):

Lemma 2. *Let $w(x) \in L^1(\mathbb{R}^n)$. Then, for each $h > 0$,*

$$\int_{\mathbb{R}^n} ||w(x)| - w(x)(\mathrm{sign}\, w)^h(x)|\mathrm{d}x \leqq 2\omega_w(h),$$

where $\omega_w(h) = \sup\limits_{|\Delta x| < h} \int_{\mathbb{R}^n} |w(x + \Delta x) - w(x)|\mathrm{d}x$ is the continuity modulus of w in $L^1(\mathbb{R}^n)$.

Proof. First, notice that, for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
||w(x)| - w(x)\,\mathrm{sign}\, w(y)| &= ||w(x)| - (w(x) - w(y))\,\mathrm{sign}\, w(y) - w(y)\,\mathrm{sign}\, w(y)| \\
&= ||w(x)| - |w(y)| - (w(x) - w(y))\,\mathrm{sign}\, w(y)| \\
&\leqq ||w(x)| - |w(y)|| + |w(x) - w(y)| \leqq 2|w(x) - w(y)|.
\end{aligned}$$

With the help of above inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} ||w(x)| - w(x)(\mathrm{sign}\, w)^h(x)|\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (|w(x)| - w(x)\,\mathrm{sign}\, w(x-y))\beta_h(y)\mathrm{d}y\right| \mathrm{d}x \\
&\quad \leqq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} ||w(x)| - w(x)\,\mathrm{sign}\, w(x-y)|\beta_h(y)\mathrm{d}y\mathrm{d}x \\
&\quad \leqq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2|w(x) - w(x-y)|\beta_h(y)\mathrm{d}y\mathrm{d}x \\
&\quad = 2\int_{|y| \leqq h} \left(\int_{\mathbb{R}^n} |w(x) - w(x-y)|\mathrm{d}x\right) \beta_h(y)\mathrm{d}y \leqq 2\omega_w(h),
\end{aligned}$$

as was to be proven. □

As it readily follows from Lemma 2, for any $\rho = \rho(x) \in C_0^1(\mathbb{R}^n)$

$$\left|\int_{\mathbb{R}^n} |w(x)|\rho(x)\mathrm{d}x - \int_{\mathbb{R}^n} w(x)\rho(x)(\mathrm{sign} w)^h(x)\mathrm{d}x\right| \leqq \int_{\mathbb{R}^n} ||w(x)| - w(x)(\mathrm{sign} w)^h(x)|\rho(x)\mathrm{d}x \leqq 2\|\rho\|_\infty \omega_w(h). \tag{3.6}$$

We apply this relation to the function $w(x) = u_r(t_0 + \Delta t, x) - u_r(t_0, x)$ for fixed $t_0, \Delta t > 0, r \in \mathbb{N}$. In view of estimate (3.2), for every $\Delta x \in \mathbb{R}^n$, $|\Delta x| < h$,

$$\int_{\mathbb{R}^n} |w(x + \Delta x) - w(x)|\mathrm{d}x \leqq \int_{\mathbb{R}^n} |u_r(t_0, x + \Delta x) - u_r(t_0, x)|\mathrm{d}x + \int_{\mathbb{R}^n} |u_r(t_0 + \Delta t, x + \Delta x) - u_r(t_0 + \Delta t, x)|\mathrm{d}x \leqq 2\omega_x(h),$$

so that $\omega_w(h) \leqq 2\omega^x(h)$. It follows from (3.6), (3.5), and (3.3) that

$$\begin{aligned}\int_{\mathbb{R}^n} |w(x)|\rho(x)\mathrm{d}x &\leqq \left|\int_{\mathbb{R}^n} w(x)\rho(x)(\mathrm{sign}\, w)^h(x)\mathrm{d}x\right| + 4\|\rho\|_\infty \omega^x(h) \\ &= \left|\int_{\mathbb{R}^n} (u_r(t_0 + \Delta t, x) - u_r(t_0, x))\rho(x)(\mathrm{sign}\, w)^h(x)\mathrm{d}x\right| + 4\|\rho\|_\infty \omega^x(h) \\ &\leqq N\|\nabla(\rho(x)(\mathrm{sign}\, w)^h(x))\|_1 \Delta t + 4\|\rho\|_\infty \omega^x(h) \leqq c_\rho(\Delta t/h + \omega^x(h)),\end{aligned} \tag{3.7}$$

where $0 < h < 1$, and c_ρ is a constant depending only on ρ. Since the left hand side of this estimate does not depend on h, we arrive at the estimate

$$\int_{\mathbb{R}^n} |u_r(t_0 + \Delta t, x) - u_r(t_0, x)|\rho(x)\mathrm{d}x \leqq c_\rho \omega^t(\Delta t), \tag{3.8}$$

where $\omega^t(\Delta t) = \inf\limits_{0<h<1}(\Delta t/h + \omega^x(h))$. Taking $h = (\Delta t)^{1/2}$, we find $\omega^t(\Delta t) \leqq (\Delta t)^{1/2} + \omega^x((\Delta t)^{1/2})$ for all $\Delta t \in (0, 1)$. Thus, $\omega^t(\Delta t) \to 0$ as $\Delta t \to 0$. Both estimates (3.2), (3.8) are uniform with respect to $t_0 > 0$ and $r \in \mathbb{N}$. By the known compactness criterium they imply pre-compactness of the sequence u_r in $L^1_{loc}(\Pi)$. Therefore, passing to a subsequence, we can assume that $u_r \to u$ as $r \to \infty$ in $L^1_{loc}(\Pi)$. We conclude that all the requirements of Proposition 3 are satisfied (with the constant sequence $u_{r0} = u_0$), and by this proposition $u = u(t, x)$ is an e.s. of (1.1), (1.9).

For more general initial functions $u_0(x) \in (c + L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n)$, where $c \in \mathbb{R}$, one can make the change $\tilde{u} = u - c$. As is easy to verify, u is an e.s. of (1.1), (1.9) if and only if $\tilde{u}$ is an e.s. to the problem

$$u_t + \mathrm{div}_x\, \varphi(c + u) = 0, \quad u(0, x) = u_0(x) - c,$$

corresponding to the parametrization $u = b(v) - c, \bar{\varphi}(c+u) \ni g(v)$. The existence of such an e.s. has been just shown. This yields the existence of e.s. to the original problem. Thus, we have the following result:

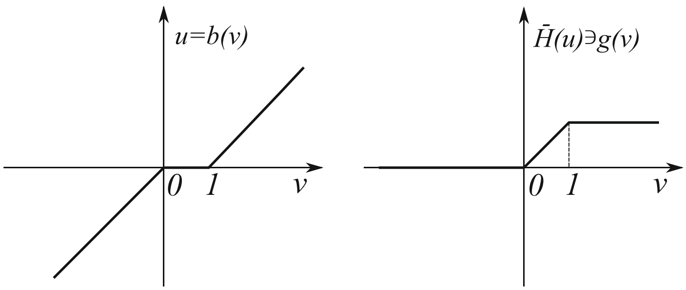

Fig. 1. Parametrization of $\bar{H}(u)$

Theorem 3. *For every initial function* $u_0 \in (c + L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n)$, *where* $c \in \mathbb{R}$, *there exists an e.s. of problem (1.1), (1.9).*

Concerning the uniqueness, it may fail even if $n = 1$ and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Example 1. We will study the problem

$$u_t + H(u)_x = 0, \quad u(0, x) = u_0(x) \doteq \frac{1}{1 + x^2}, \tag{3.9}$$

where $H(u) = \operatorname{sign}^+ u$ is the Heaviside function. The natural solution of this problem is the stationary solution $u(t, x) \equiv u_0(x)$. To construct other e.s., we choose the appropriate continuous parametrization of the flux (it corresponds (1.4) if we set $H(0) = 1/2$)

$$u = b(v) = \begin{cases} v, & v < 0, \\ 0, & 0 \leqq v \leqq 1, \\ v - 1, & v > 1, \end{cases} \quad \bar{H}(u) \ni g(v) = \begin{cases} 0, & v < 0, \\ v, & 0 \leqq v \leqq 1, \\ 1, & v > 1, \end{cases} \tag{3.10}$$

where $\bar{H}(u) = H(u)$, $u \neq 0$, $\bar{H}(0) = [0, 1]$, see Fig. 1.

We are going to find an e.s. of (3.9) in the form

$$u(t, x) = \begin{cases} 1/(1 + x^2), & x > x(t), \\ 0, & x < x(t), \end{cases}$$

where $x(t) \in C^1((\alpha, \beta))$, $0 \leqq \alpha < \beta \leqq +\infty$; $x'(t) > 0$, $\lim\limits_{t \to \alpha+} x(t) = -\infty$, $\lim\limits_{t \to \beta-} x(t) = +\infty$ if $\beta < +\infty$. The corresponding measure valued e.s. $\nu_{t,x}$ is assumed being regular, i.e., it is an e.s. $v = v(t, x) \in L^\infty(\Pi)$ of the conservation law $b(v)_t + g(v)_x = 0$ such that $u = b(v)$. In particular, $v(t, x) = 1 + 1/(1 + x^2)$ if $x > x(t)$ and $v(t, x) \in [0, 1]$ if $x < x(t)$. Since in the latter case $v_x = b(v)_t + g(v)_x = 0$ in the sense of distributions, we claim that v does not depend on x, i.e., $v = v(t)$ in the domain $x < x(t)$. As is easy to realize, both the Rankine-Hugoniot and the Oleinik conditions (see [15]) should be fulfilled on the discontinuity line $x = x(t)$. These mean, respectively, that $x'(t)$ coincides with the slope of the chord connected the points $(b(v-), g(v-))$, $(b(v+), g(v+))$ of the graph of the flux function $\varphi \in \bar{H}(u)$, and that this graph

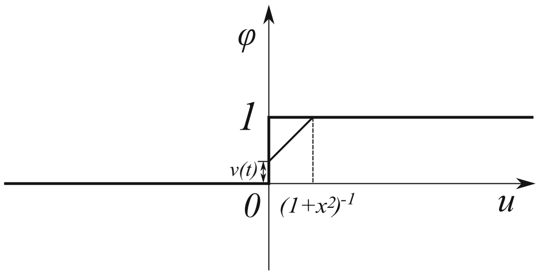

Fig. 2. The Rankine–Hugoniot and the Oleinik conditions

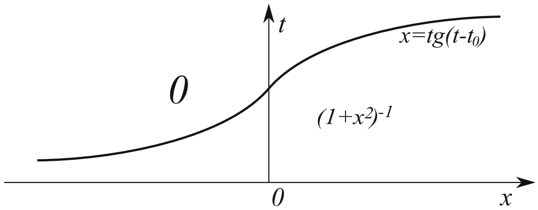

Fig. 3. Nonstationary entropy solution

lies above of the indicated chord then v runs between $v- = \lim\limits_{x\to x(t)-} v(t,x) = v(t)$ and $v+ = \lim\limits_{x\to x(t)+} v(t,x) = 1 + 1/(1+x(t)^2) > v-$, see Fig. 2.

Notice that the Oleinik condition is automatically satisfied while the Rankine-Hugoniot condition provides the differential equation $x'(t) = (1+x^2)(1-v(t))$. In particular, taking $v(t) \equiv 0$ and solving the above equation, we obtain the discontinuity curve $x = x(t) = \mathrm{tg}(t - t_0)$, $t_0 - \pi/2 < t < t_0 + \pi/2$ with the required properties for all $t_0 \geqq \pi/2$, see Fig. 3.

Varying $v(t)$, we can construct many other e.s. For example, choosing $v(t) = t^2/(1+t^2)$ and a particular solution $x = -1/t$ of the differential equation $x'(t) = (1+x^2)(1-v(t)) = (1+x^2)/(1+t^2)$, we find the e.s. $u = 1/(1+x^2)$ if $xt > -1$, $u = 0$ if $xt < -1$. We conclude that an e.s. of (3.9) is not unique. In the case of merely continuous flux vector an e.s. of the problem (1.1), (1.9) may also be non-unique but only if $n > 1$, see [11,12]. If $n = 1$ and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then the uniqueness can be proved under the assumption on continuity of the flux function at zero. Hence, it is essential for our example that 0 is a discontinuity point of the flux $H(u)$.

4. Existence of the Largest and the Smallest e.s. The General Case $u_0 \in L^\infty(\mathbb{R}^n)$

To construct the largest e.s., we choose a strictly decreasing sequence $d_r > d = \operatorname{ess\,sup} u_0(x)$, $r \in \mathbb{N}$, and the corresponding sequence u_r of e.s. of (1.1), (1.9) with initial functions

$$u_{0r}(x)=\begin{cases}u_0(x), & |x|\leqq r,\\ d_r, & |x|>r.\end{cases}$$

Since $u_{0r}\in(d_r+L^1(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n)$ an e.s. u_r actually exists by Theorem 3. Observe that $\forall r\in\mathbb{N}$

$$u_0(x)\leqq u_{0r+1}(x)\leqq u_{0r}(x)\leqq d_r \text{ a.e. on } \mathbb{R}^n, \text{ and } \lim_{r\to\infty}u_{0r}(x)=u_0(x).$$

Denote $\delta_r=d_r-d_{r+1}>0$. By the maximum principle $u_r\leqq d_r$ for all $r\in\mathbb{N}$. Therefore,

$$\begin{aligned}&\{(t,x)\mid u_{r+1}(t,x)\geqq u_r(t,x)\}\subset\{(t,x)\mid d_{r+1}\geqq u_r(t,x)\}\\&\quad=\{(t,x)\mid d_r-u_r(t,x)\geqq\delta_r\}.\end{aligned}$$

By Chebyshev's inequality and Corollary 2, for each $T>0$,

$$\begin{aligned}&\text{meas}\{\ (t,x)\in(0,T)\times\mathbb{R}^n\mid u_{r+1}(t,x)\geqq u_r(t,x)\ \}\\&\quad\leqq\text{meas}\{\ (t,x)\in(0,T)\times\mathbb{R}^n\mid d_r-u_r(t,x)\geqq\delta_r\ \}\\&\quad\leqq\frac{1}{\delta_r}\int_{(0,T)\times\mathbb{R}^n}|d_r-u_r|\mathrm{d}t\mathrm{d}x\leqq\frac{T}{\delta_r}\int_{\mathbb{R}^n}|d_r-u_{0r}|\mathrm{d}x\\&\quad=\frac{T}{\delta_r}\int_{|x|<r}(d_r-u_0)\mathrm{d}x<+\infty.\end{aligned}$$

We see that the assumption of Proposition 2 regarded to the e.s. u_{r+1} and u_r is satisfied and by this proposition $u_{r+1}\leqq u_r$ a.e. on Π. Since $u_{0r}\geqq u_0\geqq a\doteq$ ess $\inf u_0(x)$ then $u_r\geqq a$, by the minimum principle. Hence, the sequence

$$u_r(t,x)\underset{r\to\infty}{\to}u_+(t,x)\doteq\inf_{r>0}u_r(t,x)$$

a.e. on Π, as well as in $L^1_{loc}(\Pi)$. By Proposition 3 the limit function u_+ is an e.s. of original problem (1.1), (1.9). Let us demonstrate that u_+ is the largest e.s. of this problem. For that, we choose an arbitrary e.s. $u=u(t,x)$ of (1.1), (1.9). By the maximum principle, $u\leqq d$. Therefore, for each $r\in\mathbb{N}$

$$\begin{aligned}&\{(t,x)\in\Pi_T=(0,T)\times\mathbb{R}^n\mid u\geqq u_r\}\subset\{(t,x)\in\Pi_T\mid d\geqq u_r\}\\&\quad=\{(t,x)\in\Pi_T\mid d_r-u_r\geqq d_r-d\}\end{aligned}$$

and consequently

$$\begin{aligned}&\text{meas}\{(t,x)\in\Pi_T\mid u\geqq u_r\}\leqq\frac{1}{d_r-d}\int_{\Pi_T}|d_r-u_r|\mathrm{d}x\\&\quad\leqq\frac{T}{d_r-d}\int_{|x|<r}(d_r-u_0)\mathrm{d}x<+\infty,\end{aligned}$$

where we use again Chebyshev's inequality and Corollary 2. Hence, the requirement of Proposition 2, applied to the e.s. u and u_r, is satisfied and, by the comparison principle, the inequality $u_0 \leqq u_{0r}$ implies that $u \leqq u_r$ a.e. on Π. In the limit as $r \to \infty$ we conclude that $u \leqq u_+$ a.e. on Π. Hence, u_+ is the unique largest e.s. The smallest e.s. u_- can be found as $u_- = -\tilde{u}_+$, where $\tilde{u}_+$ is the largest e.s. to the problem (1.6).

We have established the existence of the largest and the smallest e.s. Let us demonstrate that these e.s. satisfy the stability and monotonicity properties with respect to initial data.

Theorem 4. *Let $u_{1+}, u_{2+} \in L^\infty(\Pi)$ be the largest e.s. of (1.1), (1.9) with initial functions u_{10}, u_{20}, respectively. Then for a.e. $t > 0$*

$$\int_{\mathbb{R}^n} (u_{1+}(t,x) - u_{2+}(t,x))^+ \mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{10}(x) - u_{20}(x))^+ \mathrm{d}x.$$

In particular, if $u_{10} \leqq u_{20}$ a.e. in $\mathbb{R}^n$ then $u_{1+} \leqq u_{2+}$ a.e. in Π.

Proof. We choose a decreasing sequence $d_r > d = \max(\operatorname{ess\,sup} u_{10}(x), \operatorname{ess\,sup} u_{20}(x))$, $r \in \mathbb{N}$, and define the following sequences of initial functions:

$$u_{1r}^0(x) = \begin{cases} u_{10}(x), & |x| \leqq r, \\ d_r, & |x| > r, \end{cases} \qquad u_{2r}^0(x) = \begin{cases} u_{20}(x), & |x| \leqq r, \\ d_r + 1, & |x| > r. \end{cases}$$

Let $u_{1r} = u_{1r}(t,x)$, $u_{2r} = u_{2r}(t,x)$ be e.s. of problem (1.1), (1.9) with initial functions u_{1r}^0, u_{2r}^0, respectively. As was demonstrated above, the sequences u_{1r}, u_{2r} decrease and converge in $L^1_{loc}(\Pi)$ to the largest e.s. u_{1+}, u_{2+}, respectively. By the maximum principle $u_{1r} \leqq d_r$ a.e. in Π and therefore, for each $T > 0$,

$$\{(t,x) \in \Pi_T \mid u_{1r}(t,x) \geqq u_{2r}(t,x)\} \subset \{(t,x) \in \Pi_T \mid d_r \geqq u_{2r}(t,x)\}$$
$$\subset \{(t,x) \in \Pi_T \mid d_r + 1 - u_{2r}(t,x) \geqq 1\}.$$

By Chebyshev inequality and Corollary 2

$$\begin{aligned} &\operatorname{meas}\{(t,x) \in \Pi_T \mid u_{1r}(t,x) \geqq u_{2r}(t,x)\} \\ &\quad \leqq \operatorname{meas}\{(t,x) \in \Pi_T \mid d_r + 1 - u_{2r}(t,x) \geqq 1\} \\ &\quad \leqq \int_{\Pi_T} |d_r + 1 - u_{2r}(t,x)| \mathrm{d}t \mathrm{d}x \leqq T \int_{\mathbb{R}^n} |d_r + 1 - u_{2r}^0(x)| \mathrm{d}x \\ &\quad = T \int_{|x|<r} (d_r + 1 - u_{20}(x)) \mathrm{d}x < \infty, \end{aligned}$$

which allows us to apply Proposition 2 and conclude that, for a.e. $t > 0$ and all $r \in \mathbb{N}$,

$$\begin{aligned} &\int_{\mathbb{R}^n} (u_{1r}(t,x) - u_{2r}(t,x))^+ \mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{1r}^0(x) - u_{2r}^0(x))^+ \mathrm{d}x \\ &\quad = \int_{|x|<r} (u_{10}(x) - u_{20}(x))^+ \mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{10}(x) - u_{20}(x))^+ \mathrm{d}x. \end{aligned}$$

To complete the proof, it remains only to pass to the limit as $r \to \infty$ in above relation with the help of Fatou's lemma. □

Corollary 3. *With notations of Theorem 4 for a.e.* $t > 0$

$$\int_{\mathbb{R}^n} |u_{1+}(t,x) - u_{2+}(t,x)|\mathrm{d}x \leqq \int_{\mathbb{R}^n} |u_{10}(x) - u_{20}(x)|\mathrm{d}x.$$

Proof. By Theorem 4 we find that, for a.e. $t > 0$,

$$\int_{\mathbb{R}^n} (u_{1+}(t,x) - u_{2+}(t,x))^+\mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{10}(x) - u_{20}(x))^+\mathrm{d}x,$$
$$\int_{\mathbb{R}^n} (u_{2+}(t,x) - u_{1+}(t,x))^+\mathrm{d}x \leqq \int_{\mathbb{R}^n} (u_{20}(x) - u_{10}(x))^+\mathrm{d}x.$$

Putting these inequalities together, we derive the desired result. □

The analogues of Theorem 4 and Corollary 3 for the smallest e.s. follows from the results for the largest e.s. to the problem (1.17) after the change $u \to -u$.

Let us return to the problem (3.9) from Example 1 and find the largest and the smallest e.s. explicitly. First, we demonstrate that the largest e.s. u_+ coincides with the stationary solution $u_0 = 1/(1+x^2)$. Since the e.s. u_+ is the largest one, then $u_+ \geqq u_0$. Further more, by Proposition 1, for a.e. $t > 0$,

$$\int_{\mathbb{R}} u_+(t,x)\mathrm{d}x = \int_{\mathbb{R}} (u_+(t,x) - 0)^+\mathrm{d}x \leqq \int_{\mathbb{R}} (u_0(x) - 0)^+\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\mathrm{d}x,$$

which implies the inequality

$$\int_{\mathbb{R}} (u_+(t,x) - u_0(x))\mathrm{d}x \leqq 0.$$

Since $u_+ \geqq u_0$, we conclude that $u_+ = u_0(x)$ a.e. in Π, as was claimed.

Let us show that the smallest e.s. of (3.9) is given by the expression

$$u_-(t,x) = \tilde{u}(t,x) \doteq \begin{cases} 1/(1+x^2), & x > \mathrm{tg}(t-\pi/2), \\ 0, & x < \mathrm{tg}(t-\pi/2), \end{cases}$$

where we agree that $\mathrm{tg}\, z = \pm\infty$ if $\pm z \geqq \pi/2$. In particular, $\tilde{u} \equiv 0$ for $t \geqq \pi$. As was shown in Example 1, $\tilde{u}$ is indeed an e.s. of (3.9). Therefore, the smallest e.s. $u_- \leqq \tilde{u}$. By the minimum principle we also claim that $u_- \geqq 0$. Direct calculation shows that

$$\int \tilde{u}(t,x)\mathrm{d}x = \int_{\mathrm{tg}(t-\pi/2)}^{+\infty} \frac{\mathrm{d}x}{1+x^2} = (\pi - t)^+. \tag{4.1}$$

Let $\nu_{t,x}^-$ be a measure valued e.s. to the problem (1.6), (1.9) corresponding to the smallest e.s. u_- of problem (3.9). In view of identity (2.1) we find that $(u_-)_t + G_x = 0$ in $\mathcal{D}'(\Pi)$, where $G = G(t,x) = \int g(v)\mathrm{d}\nu_{t,x}^-(v) \in [0,1]$ because $g(v) = \max(0, \min(1,v)) \in [0,1]$, see (3.10). This easily implies that, for a.e. $r > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-r}^{r} u_-(t,x)\mathrm{d}x = G(t,-r) - G(t,r) \geqq -1 \ \text{ in } \mathcal{D}'(\mathbb{R}),$$

which, in turn, implies the estimate $\int_{-r}^{r} u_-(t,x)\mathrm{d}x \geqq \int_{-r}^{r} u_0(x)\mathrm{d}x - t$. Passing in this estimate to the limit as $r \to +\infty$, we find that $\int u_-(t,x)\mathrm{d}x \geqq \int u_0(x)\mathrm{d}x - t = \pi - t$. Taking also into account that $u_- \geqq 0$, we see that for a.e. $t > 0$

$$\int u_-(t,x)\mathrm{d}x \geqq (\pi - t)^+.$$

Comparing this inequality with (4.1), we get

$$\int (\tilde{u}(t,x) - u_-(t,x))\mathrm{d}x \leqq 0$$

for a.e. $t > 0$. Since $\tilde{u} \geqq u_-$, this implies the desired identity $u_- = \tilde{u}(t,x)$.

In the end of this section we put the example promised in Introduction, which shows the necessity of the multi-valued extension of the flux.

Example 2. Let $n = 1$ and $\chi_0(u)$ be a function that is different from zero only at the zero point, where it equals 1, i.e. $\chi_0(u)$ is the indicator function of the singleton $\{0\}$. We consider the Riemann problem

$$u_t + (\chi_0(u))_x = 0, \quad u(0,x) = H(x),$$

where $H(x)$ is the Heaviside function. Assume that there exists an e.s. $u = u(t,x)$ of this problem in the sense of relation (1.12). Putting this entropy relation

$$|u - k|_t + [\mathrm{sign}(u-k)(\chi_0(u) - \chi_0(k))]_x \leqq 0$$

together with the identities

$$\pm((u-k)_t + (\chi_0(u) - \chi_0(k))_x) = 0,$$

we get that for each $k \in \mathbb{R}$

$$((u-k)^{\pm})_t + [\mathrm{sign}^{\pm}(u-k)(\chi_0(u) - \chi_0(k))]_x \leqq 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{4.2}$$

It follows from this inequality that $((u-1)^+)_t \leqq 0$, $((u+\varepsilon)^-)_t \leqq 0$ in $\mathcal{D}'(\Pi)$ for each $\varepsilon > 0$ and since $0 \leqq u(0,x) \leqq 1$, we find that $(u-1)^+ = (u+\varepsilon)^- = 0$, that is, $-\varepsilon \leqq u \leqq 1$ a.e. in Π. In view of arbitrariness of $\varepsilon > 0$, we see that $0 \leqq u \leqq 1$ a.e. in Π. It again follows from (4.2) that $((u-\varepsilon)^+)_t \leqq 0$ in $\mathcal{D}'(\Pi)$ for every $\varepsilon > 0$. This implies that $(u-\varepsilon)^+ \leqq (H(x)-\varepsilon)^+ = 0$ a.e. in the quarter-plane $t > 0$, $x < 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $u(t,x) = 0$ in this quarter-plane. Now, we will demonstrate that $u = 1$ a.e. in the quarter-plane $t > 0$, $x > 0$. For that, we apply the relation $(1-u)_t - \chi_0(u)_x = 0$ to the test function $f = p(\min(R+T-t-x, x))h(t)$, where $T > 0$, $R > 2$, $p(v) \in C^1(\mathbb{R})$ is a function with the properties $p' \geqq 0$, $p(v) = 0$ for $v \leqq 0$, $p(v) > 0$ for $v > 0$, $p(v) = 1$ for $v \geqq 1$; $h(t) \in C_0^1((0,T))$, $h \geqq 0$ (notice that $p \equiv 1$ in a neighborhood of a singular line $x = R+T-t-x$, $t < T$, which implies that $f \in C_0^1(\Pi)$). As a result, we get

$$\int_\Pi (1-u)ph'(t)\mathrm{d}t\mathrm{d}x + \int_{x>R+T-t-x} (-(1-u)+\chi_0(u))p'h\mathrm{d}t\mathrm{d}x \\ + \int_{x<R+T-t-x} (-\chi_0(u))p'h\mathrm{d}t\mathrm{d}x = 0. \quad (4.3)$$

Observing that $0 \leqq \chi_0(u) \leqq 1-u$ for $u = u(t,x) \in [0,1]$, and that $p' = p'(\min(R+T-t-x,x)) \geqq 0$, we find that the last two integrals in (4.3) are non-positive and therefore for all $h = h(t) \in C_0^1((0,T))$, $h \geqq 0$

$$\int_0^T \left(\int_{\mathbb{R}^n} (1-u)p(\min(R+T-t-x,x))\mathrm{d}x\right) h'(t)\mathrm{d}t \\ = \int_\Pi (1-u)ph'(t)\mathrm{d}t\mathrm{d}x \geqq 0.$$

This means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} (1-u)p(\min(R+T-t-x,x))\mathrm{d}x \leqq 0 \ \text{ in } \mathcal{D}'((0,T)).$$

Taking into account the initial condition, we find that for a.e. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (1-u)p(\min(R+T-t-x,x))\mathrm{d}x \\ \leqq \int_{\mathbb{R}^n} (1-u_0(x))p(\min(R+T-x,x))\mathrm{d}x = 0$$

since $u_0(x) = 1$ for $x > 0$ while $p(\min(R+T-x,x)) = p(x) = 0$ for $x \leqq 0$. In the limit as $R \to +\infty$, this relation implies that for a.e. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (1-u(t,x))p(x)\mathrm{d}x = 0.$$

Since $p(x) > 0$ for $x > 0$, and $T > 0$ is arbitrary, we conclude that $u(t,x) = 1$ for a.e. $(t,x) \in \Pi$, $x > 0$. We have established that our solution $u = H(x)$. But this function is not even a weak solution of our equation because the Rankine-Hugoniot relation $0 = \chi_0(1) = \chi_0(0) = 1$ is violated on the shock line $x = 0$. Hence, our Riemann problem has no e.s. in the Kruzhkov sense. As we already know, there exists an e.s. of our problem in the sense of Definition 1, corresponding to the multi-valued extension $\bar{\chi}_0(0) = [0,1]$ of the flux. The corresponding continuous parametrization can be given by the functions

$$u = b(v) = \begin{cases} v+1, & v < -1, \\ 0, & -1 \leqq v \leqq 1, \\ v-1, & v > 1, \end{cases} \qquad \bar{\chi}_0(u) \ni g(v) = \begin{cases} 0, & |v| > 1, \\ 1-|v|, & |v| \leqq 1; \end{cases}$$

see Fig. 4.

Let us show that the stationary solution $u = H(x)$ is an e.s. of our problem. The corresponding e.s. $v = v(t,x)$ of the equation $b(v)_t + g(v)_x = 0$ can be chosen regular. For $x > 0$ it is uniquely determined by the requirement $b(v) = u = 1$ and therefore $v = 2$. For $x < 0$ one can chose $v \equiv -1$ or $v \equiv 1$ (it is even

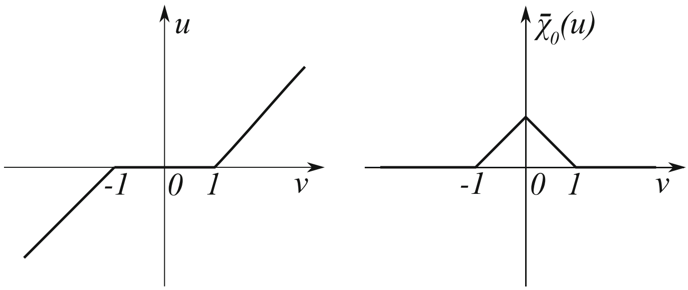

Fig. 4. Parametrization of the flux $\bar{\chi}_0$

possible to take measure valued function $\nu_{t,x}(v) = (1-\alpha)\delta(v+1) + \alpha\delta(v-1)$, $\alpha = \alpha(t,x) \in [0,1]$). By the construction both the Rankine-Hugoniot and the Oleinik conditions are satisfied in the shock line $x = 0$. Hence $H(x) = b(v)$ is the required e.s.

5. The Case of Periodic Initial Functions

Let us study the particular case when the initial function $u_0(x)$ is periodic, $u_0(x+e) = u_0(x)$ a.e. in $\mathbb{R}^n$ for all $e \in L$, where $L \subset \mathbb{R}^n$ is a lattice of periods. Without loss of generality we may suppose that L is the standard lattice $\mathbb{Z}^n$.

Theorem 5. *The largest e.s. u_+ and the smallest e.s. u_- of the problem (1.1), (1.9) are space-periodic and coincide: $u_+ = u_-$.*

Proof. Let $e \in L$. In view of periodicity of the initial function it is obvious that $u(t, x+e)$ is an e.s. of (1.1), (1.9) if and only if $u(t,x)$ is an e.s. of the same problem. Therefore, $u_+(t, x+e)$ is the largest e.s. of (1.1), (1.9) together with u_+. By the uniqueness $u_+(t,x+e) = u_+(t,x)$ a.e. on Π for all $e \in L$, that is, u_+ is a space periodic function. In the same way we prove space periodicity of the minimal e.s. u_-. Let $\nu^{\pm}_{t,x}(v)$ be measure valued e.s. of (1.6) corresponding to the e.s. $u_\pm$. In view of (2.1), we have

$$(u_+ - u_-)_t + \operatorname{div}_x \int g(v)d(\nu^+_{t,x} - \nu^-_{t,x})(v) = 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{5.1}$$

Let $\alpha(t) \in C_0^1(\mathbb{R}_+)$, $\beta(y) \in C_0^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \beta(y)\mathrm{d}y = 1$. Applying (5.1) to the test function $k^{-n}\alpha(t)\beta(x/k)$, with $k \in \mathbb{N}$, we arrive at the relation

$$k^{-n}\int_\Pi (u_+ - u_-)\alpha'(t)\beta(x/k)\mathrm{d}t\mathrm{d}x + k^{-n-1}\int_\Pi Q\cdot\nabla_y\beta(x/k)\alpha(t)\mathrm{d}t\mathrm{d}x = 0, \tag{5.2}$$

where the vector $Q = Q(t,x) = \int g(v)d(\nu_{t,x}^+ - \nu_{t,x}^-)(v) \in L^\infty(\Pi, \mathbb{R}^n)$. We observe that

$$k^{-n-1}\left|\int_\Pi Q\cdot\nabla_y\beta(x/k)\alpha(t)\mathrm{d}t\mathrm{d}x\right| \leqq k^{-n-1}\|Q\|_\infty\int_\Pi|\nabla_y\beta(x/k)|\alpha(t)\mathrm{d}t\mathrm{d}x$$
$$= k^{-1}\|Q\|_\infty\int_\Pi|\nabla_y\beta(y)|\alpha(t)\mathrm{d}t\mathrm{d}y = c/k, \quad c = \text{const}.$$

Therefore, in the limit as $k \to \infty$ the second integral in (5.2) disappears while (see for example [22, Lemma 2.1])

$$k^{-n}\int_\Pi(u_+ - u_-)\alpha'(t)\beta(x/k)\mathrm{d}t\mathrm{d}x \to \int_{\mathbb{R}_+\times\mathbb{T}^n}(u_+ - u_-)(t,x)\alpha'(t)\mathrm{d}t\mathrm{d}x,$$

where $\mathbb{T}^n = [0,1)^n$ is the periodicity cell (or, the same, the thorus $\mathbb{R}^n/\mathbb{Z}^n$). Hence, after the passage to the limit we get

$$\int_{\mathbb{R}_+\times\mathbb{T}^n}(u_+ - u_-)(t,x)\alpha'(t)\mathrm{d}t\mathrm{d}x = 0 \quad \forall\alpha(t) \in C_0^1(\mathbb{R}_+).$$

This identity means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n}(u_+(t,x) - u_-(t,x))\mathrm{d}x = 0 \text{ in } \mathcal{D}'(\mathbb{R}_+),$$

and implies, with the help of initial condition (1.10), that, for a.e. $t > 0$,

$$\int_{\mathbb{T}^n}(u_+(t,x) - u_-(t,x))\mathrm{d}x = \int_{\mathbb{T}^n}(u_0(x) - u_0(x))\mathrm{d}x = 0.$$

Since $u_+ \geqq u_-$, we conclude that $u_+ = u_-$ a.e. on Π. □

Since any e.s. of (1.1), (1.9) is situated between u_- and u_+, we deduce

Corollary 4. *An e.s. of (1.1), (1.9) is unique and coincides with* u_+.

6. Weak Completeness of e.s.

In the one-dimensional case $n = 1$ we consider a bounded sequence $u_r = u_r(t,x) \in L^\infty(\Pi)$, $r \in \mathbb{N}$, of e.s. of equation (1.1) (without a prescribed initial condition). Passing to a subsequence, we can suppose that this sequence converges weakly-$*$ in $L^\infty(\Pi)$ to a function $u = u(t,x)$. In the case of continuous flux function it was proved in [19] that $u(t,x)$ is an e.s. of problem (1.1), (1.9) with some initial function $u_0(x)$ (in [20] this result was even extended to the case of a degenerate parabolic equation $u_t + \varphi(u)_x = A(u)_{xx}$). Certainly, this property is purely one-dimensional, in the case $n > 1$ it is no longer valid, see [19, Remark 3]. We are going to extend the described weak completeness property of e.s. to the case of jump-continuous flux function. Due to the lack of uniqueness we use the additional spatial periodicity assumption on a limit Young measure corresponding to the sequence u_r. Let us formulate the main result of this section.

Theorem 6. *Let u_r, $r \in \mathbb{N}$, be a bounded sequence of e.s. of (1.1) weakly convergent to $u(t,x)$. Assume that a limit Young measure $\bar{\nu}_{t,x}$ corresponding to some subsequence of u_r (in accordance with Theorem 1), is space periodic, $\bar{\nu}_{t,x+1} = \bar{\nu}_{t,x}$ a.e. in Π. Then the limit function $u(t,x)$ is an e.s. of problem (1.1), (1.9) with some periodic initial function $u_0(x)$.*

Remark 4. The periodicity of $\bar{\nu}_{t,x}$ holds in the particular case when all e.s. u_r are space periodic. Also observe that the limit function $u(t,x)$ is spatially periodic, this readily follows from the equality $u(t,x) = \int u d\bar{\nu}_{t,x}(u)$.

Remark 5. Applying Theorem 6 to the constant sequence $u_r = u$, we obtain that any space periodic e.s. of Eq. (1.1) admits a strong trace u_0 at the initial line $t = 0$ in the sense of relation (1.10).

To prove Theorem 6, we will follow the scheme of paper [19]. First of all, we will modify the technical lemma [19, Lemma 2.3].

Lemma 3. *Let ν be a Borel measure with compact support in $\mathbb{R}$ and $p(v) \in C(\mathbb{R})$ be such a function that*

$$\int \operatorname{sign}^+(v-k)(p(v)-p(k))\mathrm{d}\nu(v) = 0 \quad \forall k \in [a,b], \tag{6.1}$$

where $a < b = \max \operatorname{supp} \nu$. Then $p(v) \equiv \text{const}$ on $[a,b]$.

Proof. We choose values $k_1, k_2 \in [a,b]$ such that $p(k_1) = \min\limits_{[a,b]} p(v)$, $p(k_2) = \max\limits_{[a,b]} p(v)$. If $p(k_1) < p(b)$ then $k_1 < b$. Taking $k = k_1$, we find that the integrand in (6.1) is not negative and strictly positive in an interval $(b-\delta, b]$, $\delta > 0$. Since $b = \max \operatorname{supp} \nu$ then $\nu((b-\delta,b]) > 0$, therefore, the integral in (6.1) is strictly positive, which contradicts to this condition. Hence, $p(k_1) = p(b)$. Similarly, assuming that $p(k_2) > p(b)$ and taking $k = k_2$ in (6.1), we come to a contradiction. Thus, $p(k_2) = p(k_1) = p(b)$, that is, $\min\limits_{[a,b]} p(v) = \max\limits_{[a,b]} p(v)$. We conclude that $p(v) \equiv \text{const}$ on $[a,b]$. □

Corollary 5. *Suppose that*

$$\int \operatorname{sign}^-(v-k)(p(v)-p(k))\mathrm{d}\nu(v) = 0 \quad \forall k \in [a,b], \tag{6.2}$$

where $a = \min \operatorname{supp} \nu < b$. Then $p(v) \equiv \text{const}$ on $[a,b]$.

Proof. After the change $v \to -v$, $k \to -k$, requirement (6.2) reduces to the following one: $\forall k \in [-b,-a]$

$$\int \operatorname{sign}^+(v-k)(p(-v)-p(-k))d\tilde{\nu}(v) = \\ -\int \operatorname{sign}^-(-v+k)(p(-v)-p(-k))d\tilde{\nu}(v) = 0,$$

where $\tilde{\nu}$ is the push-forward measure $l^*\nu$ under the map $l(v) = -v$. Notice that $-a = \max \operatorname{supp} \tilde{\nu}$. By Theorem 6 we conclude that $p(-v) \equiv \text{const}$ on $[-b,-a]$, which is equivalent to the desired statement. □

Passing to a subsequence, we can suppose that the sequence of e.s. u_r weakly converges to a Young measure $\bar{\nu}_{t,x}$ in the sense of relation (1.13). Let $\nu^r_{t,x}$, $r \in \mathbb{N}$, be a measure valued e.s. of (1.6) corresponding to the e.s. u_r. Then the sequence $\nu^r_{t,x}$, $r \in \mathbb{N}$, is bounded and, by Theorem 2, passing to a subsequence if necessary, we can suppose that this sequence converges weakly as $r \to \infty$ to a bounded measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$. Since $b^*\nu^r_{t,x}(u) = \delta(u - u_r(t,x))$ then for each $p(u) \in C(\mathbb{R})$

$$p(u_r) = \int p(b(v))\mathrm{d}\nu^r_{t,x}(v) \underset{r\to\infty}{\rightharpoonup} \int p(b(v))\mathrm{d}\nu_{t,x}(v) \ \text{ weakly-} * \ \text{ in } L^\infty(\Pi).$$

This relation implies that the push-forward measure $b^*(\nu_{t,x})(u)$ coincides with the measure valued function $\bar{\nu}_{t,x}$. Notice that, in correspondence with (1.13), the weak limit function $u(t,x) = \int u d\bar{\nu}_{t,x}(u) = \int b(v)\mathrm{d}\nu_{t,x}(v)$.

Passing to the limit as $r \to \infty$ in the entropy relation

$$\int_\Pi \left[\int |b(v) - b(k)|\mathrm{d}\nu^r_{t,x}(v) f_t \right.$$
$$\left. + \int \mathrm{sign}(v-k)(g(v) - g(k))\mathrm{d}\nu^r_{t,x}(v) f_x\right]\mathrm{d}t\mathrm{d}x \geqq 0,$$

$k \in \mathbb{R}$, $f = f(t,x) \in C^1_0(\Pi)$, $f \geqq 0$, we obtain the relation

$$\int_\Pi \left[\int |b(v) - b(k)|\mathrm{d}\nu_{t,x}(v) f_t \right.$$
$$\left. + \int \mathrm{sign}(v-k)(g(v) - g(k))\mathrm{d}\nu_{t,x}(v) f_x\right]\mathrm{d}t\mathrm{d}x \geqq 0,$$

which shows that $\nu_{t,x}$ is a measure valued e.s. of (1.6).

Using compensated compactness arguments, we establish the formulated below one more important property of the limit measure valued e.s. $\nu_{t,x}$. We consider even the more general case of equations

$$\varphi_0(v)_t + \varphi_1(v)_x = 0, \tag{6.3}$$

where $\varphi_0(v)$, $\varphi_1(v)$ are arbitrary continuous functions. A measure valued e.s. $\nu_{t,x} \in \mathrm{MV}(\Pi)$ of this equation is characterized by the usual Kruzhkov entropy relation: for all $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\int \mathrm{sign}(v-k)(\varphi_0(v) - \varphi_0(k))\mathrm{d}\nu_{t,x}(v)$$
$$+ \frac{\partial}{\partial x}\int \mathrm{sign}(v-k)(\varphi_1(v) - \varphi_1(k))\mathrm{d}\nu_{t,x}(v) \leqq 0 \tag{6.4}$$

in $\mathcal{D}'(\Pi)$. Taking $k = \pm R$, $R \geqq \|\nu_{t,x}\|_\infty$, we derive the identity

$$\frac{\partial}{\partial t}\int (\varphi_0(v) - \varphi_0(k))\mathrm{d}\nu_{t,x}(v) + \frac{\partial}{\partial x}\int (\varphi_1(v) - \varphi_1(k))\mathrm{d}\nu_{t,x}(v)$$
$$= \frac{\partial}{\partial t}\int \varphi_0(v)\mathrm{d}\nu_{t,x}(v) + \frac{\partial}{\partial x}\int \varphi_1(v)\mathrm{d}\nu_{t,x}(v) = 0 \ \text{ in } \mathcal{D}'(\Pi)$$

for all $k \in \mathbb{R}$. Putting this identity multiplied by ± 1 together with (6.4), we get another (equivalent) form of entropy relation (6.4)

$$\frac{\partial}{\partial t}\int \psi_{0k}^{\pm}(v)\mathrm{d}\nu_{t,x}(v) + \frac{\partial}{\partial x}\int \psi_{1k}^{\pm}(v)\mathrm{d}\nu_{t,x}(v) \leqq 0 \ \text{ in } \mathcal{D}'(\Pi), \tag{6.5}$$

where

$$\psi_{ik}^{\pm}(v) = \operatorname{sign}^{\pm}(v-k)(\varphi_i(v) - \varphi_i(k)), \quad i = 0, 1, \ k \in \mathbb{R}.$$

Denote by $\operatorname{co} A$ the convex hull of a set $A \subset \mathbb{R}^n$. In the case when A is a compact subset of $\mathbb{R}$, $\operatorname{co} A = [\min A, \max A]$.

Proposition 4. *Let $\nu_{t,x}^r$, $r \in \mathbb{N}$, be a sequence of measure valued e.s. of Eq. (6.3) such that for a.e. $(t,x) \in \Pi$ and all $r \in \mathbb{N}$ the function $\varphi_0(v)$ is constant on* $\operatorname{co}\operatorname{supp} \nu_{t,x}^r$ *(in particular, this condition is always satisfied when the measure valued functions $\nu_{t,x}^r$ are regular). Suppose that this sequence converges weakly to a measure valued function $\nu_{t,x}$ (in the sense of relation (1.14)). Then for a.e. $(t,x) \in \Pi$ there exists a nonzero vector $(\xi_0, \xi_1) \in \mathbb{R}^2$ such that $\xi_0\varphi_0(v) + \xi_1\varphi_1(v) = \mathrm{const}$ on* $\operatorname{co}\operatorname{supp} \nu_{t,x}$.

Proof. Since $\nu_{t,x}^r$ are measure valued e.s. of (6.3) then in view of (6.5) for all $k \in \mathbb{R}$ the distributions

$$\alpha_{kr}^{\pm} \doteq \frac{\partial}{\partial t}\int \psi_{0k}^{\pm}(v)\mathrm{d}\nu_{t,x}^r(v) + \frac{\partial}{\partial x}\int \psi_{1k}^{\pm}(v)\mathrm{d}\nu_{t,x}^r(v) \leqq 0 \ \text{ in } \mathcal{D}'(\Pi).$$

By the known representation of nonnegative distributions $\alpha_{kr}^{\pm} = -\mu_{kr}$, where μ_{kr} are nonnegative locally finite measures on Π. We use also that $\alpha_{kr}^{+} = \alpha_{kr}^{-}$ because

$$\alpha_{kr}^{+} - \alpha_{kr}^{-} = \frac{\partial}{\partial t}\int \varphi_0(v)\mathrm{d}\nu_{t,x}^r(v) + \frac{\partial}{\partial x}\int \varphi_1(v)\mathrm{d}\nu_{t,x}^r(v) = 0 \ \text{ in } \mathcal{D}'(\Pi).$$

It is clear that $\mu_{kr} = 0$ for $|k| > M = \sup\limits_r \|\nu_{t,x}^r\|_\infty$ while for $|k| \leqq M$

$$\begin{aligned} < \mu_{kr}, f >&= \int_\Pi \left[\int \psi_{0k}^{\pm}(v)\mathrm{d}\nu_{t,x}^r(v) f_t + \int \psi_{1k}^{\pm}(v)\mathrm{d}\nu_{t,x}^r(v) f_x\right]\mathrm{d}t\mathrm{d}x \\ &\leqq 2\max_{|v|\leqq M}(|\varphi_0(v)| + |\varphi_1(v)|)\int_\Pi \max(|f_t|, |f_x|)\mathrm{d}t\mathrm{d}x \doteq C_f \end{aligned}$$

for each $f = f(t,x) \in C_0^1(\Pi)$, $f \geqq 0$. Since the constants C_f do not depend on r, the sequences of nonnegative measures μ_{kr}, $r \in \mathbb{N}$, are bounded in the space $\mathrm{M}_{loc}(\Pi)$ of locally finite measures in Π endowed with the standard locally convex topology. By the Murat interpolation lemma [14] the sequences of distributions $\alpha_{kr}^{\pm}$, $r \in \mathbb{N}$ are pre-compact in the Sobolev space $H_{loc}^{-1}(\Pi)$. Recall that this space consists of distributions α on Π such that for each $f \in C_0^\infty(\Pi)$ the distribution $f\alpha$ lies in the space $H^{-1}(\mathbb{R}^2)$, which is dual to the Sobolev space $H^1(\mathbb{R}^2)$. The topology of $H^{-1}_{loc}(\Pi)$ is generated by seminorms $\|f\alpha\|_{H^{-1}}$. We fix $k, l \in \mathbb{R}$ and denote

$$P^+_{kr} = \int \psi^+_{0k}(v)\mathrm{d}\nu^r_{t,x}(v), \quad Q^+_{kr} = \int \psi^+_{1k}(v)\mathrm{d}\nu^r_{t,x}(v),$$
$$P^-_{lr} = \int \psi^-_{0l}(v)\mathrm{d}\nu^r_{t,x}(v), \quad Q^-_{lr} = \int \psi^-_{1l}(v)\mathrm{d}\nu^r_{t,x}(v).$$

As we already demonstrated, the sequences

$$\alpha^+_{kr} = \frac{\partial}{\partial t}P^+_{kr} + \frac{\partial}{\partial x}Q^+_{kr}, \quad \alpha^-_{lr} = \frac{\partial}{\partial t}P^-_{lr} + \frac{\partial}{\partial x}Q^-_{lr}$$

are precompact in $H^{-1}_{loc}(\Pi)$. By the compensated compactness theory (see [13,25]), the quadratic functional $\Phi(\lambda) = \lambda_1\lambda_4 - \lambda_2\lambda_3$, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$, is weakly continuous on the sequence $(P^+_{kr}, Q^+_{kr}, P^-_{lr}, Q^-_{lr})$. By the definition of the measure valued limit function $\nu_{t,x}$ we find that as $r \to \infty$

$$P^+_{kr} \rightharpoonup P^+_k \doteq \int \psi^+_{0k}(v)\mathrm{d}\nu_{t,x}(v), \quad Q^+_{kr} \rightharpoonup Q^+_k \doteq \int \psi^+_{1k}(v)\mathrm{d}\nu_{t,x}(v),$$
$$P^-_{lr} \rightharpoonup P^-_l \doteq \int \psi^-_{0l}(v)\mathrm{d}\nu_{t,x}(v), \quad Q^+_{lr} \rightharpoonup Q^-_l \doteq \int \psi^-_{1l}(v)\mathrm{d}\nu_{t,x}(v)$$

weakly-$*$ in $L^\infty(\Pi)$. By our assumption the function $\varphi_0(v)$ is constant on the segment co supp $\nu^r_{t,x}$ for all $r \in \mathbb{N}$. Therefore, $\psi^+_{0k}(v) \equiv P^+_{kr}$, $\psi^-_{0l}(v) \equiv P^-_{lr}$ on this segment. It follows from this observation that

$$P^+_{kr}Q^-_{lr} - Q^+_{kr}P^-_{lr} = \int (\psi^+_{0k}(v)\psi^-_{1l}(v) - \psi^+_{1k}(v)\psi^-_{0l}(v))\mathrm{d}\nu^r_{t,x}(v) \underset{r\to\infty}{\rightharpoonup}$$
$$\int (\psi^+_{0k}(v)\psi^-_{1l}(v) - \psi^+_{1k}(v)\psi^-_{0l}(v))\mathrm{d}\nu_{t,x}(v) \text{ weakly-}* \text{ in } L^\infty(\Pi).$$

On the other hand, this limit equals $P^+_k Q^-_l - Q^+_k P^-_l$ in view of the mentioned above weak continuity of the functional $\Phi(\lambda)$. Hence, we arrive at the relation

$$\int (\psi^+_{0k}(v)\psi^-_{1l}(v) - \psi^+_{1k}(v)\psi^-_{0l}(v))\mathrm{d}\nu_{t,x}(v) = \int \psi^+_{0k}(v)\mathrm{d}\nu_{t,x}(v)\int \psi^-_{1l}(v)\mathrm{d}\nu_{t,x}(v)$$
$$- \int \psi^+_{1k}(v)\mathrm{d}\nu_{t,x}(v)\int \psi^-_{0l}(v)\mathrm{d}\nu_{t,x}(v). \tag{6.6}$$

Notice that $\psi^+_{ik}(v) = 0$ for $v \leqq k$ while $\psi^-_{il}(v) = 0$ for $v \geqq l$, where $i = 0, 1$. Therefore, the integrand in the left hand side of (6.6) is identically zero whenever $l \leqq k$. For all such pairs (k, l) we have

$$\int \psi^+_{0k}(v)\mathrm{d}\nu_{t,x}(v)\int \psi^-_{1l}(v)\mathrm{d}\nu_{t,x}(v) = \int \psi^+_{1k}(v)\mathrm{d}\nu_{t,x}(v)\int \psi^-_{0l}(v)\mathrm{d}\nu_{t,x}(v). \tag{6.7}$$

Let Ω be the set of common Lebesgue points of the functions $(t, x) \to \int p(v)\mathrm{d}\nu_{t,x}(v)$, $p(v) \in F$, where $F \subset C(\mathbb{R})$ is a countable dense set. Since the set F is countable, Ω

is a set of full measure in Π. By the density of F any point $(t,x)\in\Omega$ is a Lebesgue points of the functions $\int p(v)d\nu_{t,x}(v)$ for all $p(v)\in C(\mathbb{R})$. In particular, for each fixed $(t,x)\in\Omega$ the measure $\nu_{t,x}$ is uniquely determined. Since identity (6.7) fulfils a.e. in Π, it holds at each point of Ω. We fix such a point $(t,x)\in\Omega$ and denote $\nu=\nu_{t,x}$, $[a,b]=\operatorname{co}\operatorname{supp}\nu$. We have to show that $\xi_0\varphi_0(v)+\xi_1\varphi_1(v)=\text{const}$ on $[a,b]$ for some $\xi=(\xi_0,\xi_1)\in\mathbb{R}^2$, $\xi\neq 0$. If $\varphi_0(v)\equiv\text{const}$ on $[a,b]$, we can take $\xi=(1,0)$, thus completing the proof. So, assume that $\varphi_0(v)$ is not constant on $[a,b]$ and, in particular, that $a<b$. We define a smaller segment $[a_1,b_1]$, where

$$a_1=\max\{c\in[a,b]\mid \varphi_0(v)=\varphi_0(a)\ \forall v\in[a,c]\},$$
$$b_1=\min\{c\in[a,b]\mid \varphi_0(v)=\varphi_0(b)\ \forall v\in[c,b]\}.$$

If $a_1\geqq b_1$ then $\varphi_0(v)\equiv\text{const}$ on $[a,b]$, which contradicts to our assumption. Therefore, $a\leqq a_1<b_1\leqq b$ and we can choose such $a_2,b_2\in(a_1,b_1)$ that $a_2<b_2$. Observe that $\varphi_0(v)$ cannot be constant on segments $[a,a_2]$, $[b_2,b]$ (otherwise, $a_1\geqq a_2$, $b_1\leqq b_2$, respectively). Therefore, there exist such $l_0\in[a,a_2]$, $k_0\in[b_2,b]$ that $\varphi_0(l_0)$, $\varphi_0(k_0)$ are extreme values of $\varphi_0(u)$ on the segments $[a,a_2]$, $[b_2,b]$, which are different from $\varphi_0(a)$, $\varphi_0(b)$, respectively. Then, the functions $\psi^+_{0k_0}(v)$, $\psi^-_{0l_0}(v)$ keep their sign and different from zero in neighborhoods of points b, a, respectively. This implies that

$$\int\psi^+_{0k_0}(v)d\nu(v)\neq 0,\quad \int\psi^-_{0l_0}(v)d\nu(v)\neq 0.$$

Then, by relation (6.7) (with $\nu_{t,x}=\nu$)

$$\int\psi^-_{1l}(v)d\nu(v)=c\int\psi^-_{0l}(v)d\nu(v)\quad\forall l\in[a,b_2],\tag{6.8}$$

where

$$c=\int\psi^+_{1k_0}(v)d\nu(v)/\int\psi^+_{0k_0}(v)d\nu(v).$$

By relation (6.7) again

$$\int\psi^+_{1k}(v)d\nu(v)=c_1\int\psi^+_{0k}(v)d\nu(v)\quad\forall k\in[a_2,b],\tag{6.9}$$

where

$$c_1=\int\psi^-_{1l_0}(v)d\nu(v)/\int\psi^-_{0l_0}(v)d\nu(v).$$

Moreover, $c_1=c$ in view of (6.8). Introducing the function $p(v)=\varphi_1(v)-c\varphi_0(v)$, we can write equalities (6.8), (6.9) in the form

$$\int\operatorname{sign}^-(v-l)(p(v)-p(l))d\nu(v)=0\quad\forall l\in[a,b_2];$$
$$\int\operatorname{sign}^+(v-k)(p(v)-p(k))d\nu(v)=0\quad\forall k\in[a_2,b].$$

By Lemma 3 and its Corollary 5, we conclude that $p(v)$ is constant on each segment $[a, b_2]$, $[a_2, b]$. Since $a_2 < b_2$, these segments intersect and therefore $p(v) = -c\varphi_0(v) + \varphi_1(v) \equiv \text{const}$ on $[a, b] = \operatorname{co\,supp} \nu$, $\nu = \nu_{t,x}$. This completes the proof. $\square$

Notice, that the sequence $\nu^r_{t,x}$ of measure valued e.s. of Eq. (1.6) satisfies the requirements of Proposition 4 and we conclude that for a.e. $(t, x) \in \Pi$ there is a vector $\xi = (\xi_0, \xi_1) \in \mathbb{R}^2$, $\xi \neq 0$, such that $\xi_0 b(v) + \xi_1 g(v) \equiv \text{const}$ on $\operatorname{co\,supp} \nu_{t,x}$. In the case of linearly non-degenerate flux Proposition 4 implies the strong convergence of the sequence u_r, even without the periodicity requirement.

Corollary 6. *Assume that the function $\varphi(u)$ is not affine on nondegenerate intervals. Then the sequence $u_r \to u$ as $r \to \infty$ in $L^1_{loc}(\Pi)$ (strongly), and $u = u(t, x)$ is an e.s. of (1.1).*

Proof. By Proposition 4 for a.e. $(t, x) \in \Pi$ there is a vector $\xi = (\xi_0, \xi_1) \in \mathbb{R}^2$, $\xi \neq 0$, such that $\xi_0 b(v)+\xi_1 g(v) \equiv \text{const}$ on $\operatorname{co\,supp} \nu_{t,x}$. Let us show that for such (t, x) the function $b(v) \equiv \text{const}$ on the segment $\operatorname{co\,supp} \nu_{t,x}$. In fact, assuming the contrary, we realize that the component $\xi_1 \neq 0$ and consequently $g(v) = cb(v) + \text{const}$ for all $v \in \operatorname{co\,supp} \nu_{t,x}$, where $c = -\xi_0/\xi_1$. This means that $\varphi(u) = cu+\text{const}$ on the interior of the non-degenerate interval $\{u = b(v) | v \in \operatorname{co\,supp} \nu_{t,x}\}$, but this contradicts our assumption. We conclude that $b(v)$ is constant (equaled $u(t, x)$) on $\operatorname{co\,supp} \nu_{t,x}$. Therefore, the measure valued function $\bar{\nu}_{t,x} = b^*\nu_{t,x}$ is regular, $\bar{\nu}_{t,x}(u) = \delta(u - u(t, x))$. In correspondence with Theorem 1 the sequence u_r converges to $u(t, x)$ strongly. Moreover, like in the proof of Proposition 3, we conclude that the limit function $u = u(t, x)$ is an e.s. of (1.1). $\square$

Below, we prove Theorem 6 for an arbitrary jump continuous flux. Recall that in this general case we use the periodicity requirement $\bar{\nu}_{t,x+1} = \bar{\nu}_{t,x}$ a.e. in Π.

6.1. Proof of Theorem 6.

Let E be the set of full measure in $\mathbb{R}_+$, introduced in the proof of Proposition 1, consisting of such $t > 0$ that (t, x) is a Lebesgue point of $u(t, x)$ for almost all $x \in \mathbb{R}$. We remind ourselves that $t \in E$ is a common Lebesgue point of all functions $\int_{\mathbb{R}} u(t, x)\rho(x)\mathrm{d}x$, $\rho(x) \in L^1(\mathbb{R})$. We can choose a sequence $t_m \in E$ such that $t_m \to 0$ as $m \to \infty$, and $u(t_m, x) \rightharpoonup u_0(x) \in L^\infty(\mathbb{R})$ weakly-$*$ in $L^\infty(\mathbb{R})$. It is clear that $u_0(x)$ is a periodic function, and that $u(t, x) \rightharpoonup u_0(x)$ as $E \ni t \to 0$. Let $\tilde{u} = \tilde{u}(t, x)$ be a unique (by Corollary 4) e.s. of (1.1), (1.9) with initial function u_0, and $\tilde{\nu}_{t,x}$ be a corresponding measure valued e.s. of Eq. (1.6). We are going to demonstrate that $u = \tilde{u}$. Clearly, this will complete the proof. Applying the equalities

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\int g(v)\mathrm{d}\nu_{t,x}(v) = \frac{\partial}{\partial t}\tilde{u} + \frac{\partial}{\partial x}\int g(v)d\tilde{\nu}_{t,x}(v) = 0 \ \text{ in } \mathcal{D}'(\Pi)$$

to the test functions $f = k^{-n}\alpha(t)\beta(x/k)$ and passing to the limit as $k \to \infty$, we derive, like in the proof of Theorem 5, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u(t,x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{u}(t,x)\mathrm{d}x = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+).$$

This implies that, for a.e. $t > 0$,

$$\int_0^1 u(t,x)\mathrm{d}x = \int_0^1 \tilde{u}(t,x)\mathrm{d}x = I \doteq \int_0^1 u_0(x)\mathrm{d}x, \tag{6.10}$$

where we used the initial condition for e.s. $\tilde{u}$ and the fact that $\forall t \in E$

$$\int_0^1 u(t,x)\mathrm{d}x = \int_0^1 u(t_m,x)\mathrm{d}x \underset{m\to\infty}{\to} \int_0^1 u_0(x)\mathrm{d}x.$$

Moreover, it follows from the distributional relation $\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\int g(v)\mathrm{d}\nu_{t,x}(v) = 0$ that, for each $t, \tau \in E$, $t > \tau$ and every $\rho(x) \in C_0^1(\mathbb{R})$,

$$\int_{\mathbb{R}} u(t,x)\rho(x)\mathrm{d}x - \int_{\mathbb{R}} u(\tau,x)\rho(x)\mathrm{d}x = \int_{(\tau,t)\times\mathbb{R}}\int g(v)\mathrm{d}\nu_{s,x}(v)\rho'(x)ds\mathrm{d}x,$$

which implies, in the limit as $\tau = t_m \to 0$, the relation

$$\int_{\mathbb{R}} u(t,x)\rho(x)\mathrm{d}x - \int_{\mathbb{R}} u_0(x)\rho(x)\mathrm{d}x = \int_{(0,t)\times\mathbb{R}}\int g(v)\mathrm{d}\nu_{s,x}(v)\rho'(x)ds\mathrm{d}x \underset{t\to 0}{\to} 0.$$

Therefore, $u(t,\cdot) \rightharpoonup u_0$ as $E \ni t \to 0$, that is, u_0 is a weak trace of $u(t,x)$. Since in $\mathcal{D}'(\Pi)$

$$\frac{\partial}{\partial t}(u-\tilde{u}) + \frac{\partial}{\partial x}\left(\int g(v)\mathrm{d}\nu_{t,x}(v) - \int g(v)d\tilde{\nu}_{t,x}(v)\right) = 0,$$

there exists a Lipschitz function $P = P(t,x)$ (a potential) such that

$$P_x = u - \tilde{u}, \quad P_t = \int g(v)d\tilde{\nu}_{t,x}(v) - \int g(v)\mathrm{d}\nu_{t,x}(v) \quad \text{in } \mathcal{D}'(\Pi).$$

By the Lipschitz condition, this function admits continuous extension on the closure $\bar{\Pi}$. Since P is defined up to an additive constant, we can assume that $P(0,0) = 0$. It is clear that $P_x(t,x) \to P_x(0,x)$ weakly in $\mathcal{D}'(\mathbb{R})$ as $t \to 0$. Taking into account that $P_x(t,x) = u(t,x) - \tilde{u}(t,x) \rightharpoonup 0$ as $t \to 0$, running over a set of full measure, we find that $P_x(0,x) = 0$ and therefore $P(0,x) \equiv P(0,0) = 0$. Further, by the spatial periodicity of $u - \tilde{u}$ and the condition

$$\int_0^1 (u-\tilde{u})(t,x)\mathrm{d}x = 0$$

(following from (6.10)), we find that the function $P(t,x)$ is spatially periodic as well, $P(t,x+1) = P(t,x)$. Applying the doubling variables method [17] to the pair of measure valued e.s. $\nu_{t,x}$, $\tilde{\nu}_{t,x}$ of Eq. (1.6), we arrive at the relation

$$\frac{\partial}{\partial t}\iint |b(v) - b(w)|\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)$$

$$+\frac{\partial}{\partial x}\iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w) \leqq 0 \ \text{ in } \mathcal{D}'(\Pi). \quad (6.11)$$

Since $b(w) = \tilde{u}(t,x)$ on $\operatorname{supp}\tilde{\nu}_{t,x}$ and $b^*\nu_{t,x} = \bar{\nu}_{t,x}$, we can simplify the first integral

$$\iint |b(v)-b(w)|\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w) = \int |b(v)-\tilde{u}(t,x)|\mathrm{d}\nu_{t,x}(v)$$
$$= \int |u-\tilde{u}(t,x)|d\bar{\nu}_{t,x}(u). \quad (6.12)$$

We need the following key relation:

$$\iint |b(v)-b(w)|\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)P_t(t,x)$$
$$+\iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)P_x(t,x) = 0 \ \text{ a.e. in } \Pi. \quad (6.13)$$

We remind ourselves that

$$P_t(t,x) = \int g(w)d\tilde{\nu}_{t,x}(w) - \int g(v)\mathrm{d}\nu_{t,x}(v)$$
$$= -\iint (g(v)-g(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w),$$
$$P_x(t,x) = u-\tilde{u} = \iint (b(v)-b(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w),$$

and (6.13) can be written in the more symmetric form

$$\iint |b(v)-b(w)|\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)\iint (g(v)-g(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)$$
$$= \iint (b(v)-b(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)$$
$$\times \iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w). \quad (6.14)$$

To prove (6.14), we introduce, like in the proof of Proposition 4, the set of full measure $\Omega \subset \Pi$ consisting of common Lebesgue points of the functions

$$(t,x) \to \int p(v)\mathrm{d}\nu_{t,x}(v), \quad (t,x) \to \int p(v)d\tilde{\nu}_{t,x}(v), \quad p(v) \in C(\mathbb{R}).$$

For a fixed $(t,x) \in \Omega$ we then denote $\nu = \nu_{t,x}$, $\tilde{\nu} = \tilde{\nu}_{t,x}$, $[a,b] = \operatorname{co}\operatorname{supp}\nu$, $[a_1,b_1] = \operatorname{co}\operatorname{supp}\tilde{\nu}$, and consider the following four cases:

(i) $[a,b]\cap[a_1,b_1] = \emptyset$. In this case $\operatorname{sign}(v-w) \equiv s$ is constant on $[a,b] \times [a_1,b_1]$. Therefore,

$$\iint |b(v)-b(w)|\mathrm{d}\nu(v)d\tilde{\nu}(w) = s\iint (b(v)-b(w))\mathrm{d}\nu(v)d\tilde{\nu}(w),$$

$$\iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) = s\iint (g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w)$$

and (6.14) follows;

(ii) $[a, b] \subset [a_1, b_1]$. Since $b(w)$ is constant on $[a_1, b_1]$, we find

$$\iint (b(v)-b(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) = \iint |b(v)-b(w)|\mathrm{d}\nu(v)d\tilde{\nu}(w) = 0 \quad (6.15)$$

and (6.14) is trivial;

(iii) $[a_1, b_1] \subset [a, b]$. In correspondence with Proposition 4 for some nonzero vector (ξ_0, ξ_1) the function $\xi_0 b(v) + \xi_1 g(v) = \text{const}$ on $[a, b]$. If $\xi_1 = 0$ then $b(v) \equiv \text{const}$ on $[a, b]$, which implies (6.15), and (6.14) is trivially satisfied. For $\xi_1 \neq 0$ we find that $g(v) = cb(v) + \text{const}$ on $[a, b]$, $c = -\xi_0/\xi_1$. Therefore,

$$\iint (g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) = c\iint (b(v)-b(w))\mathrm{d}\nu(v)d\tilde{\nu}(w),$$
$$\iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) = c\iint |b(v)-b(w)|\mathrm{d}\nu(v)d\tilde{\nu}(w),$$

and (6.14) follows;

(iv) The remaining case: $a < a_1 \leqq b < b_1$ or $a_1 < a \leqq b_1 < b$. We consider only the former subcase $a < a_1 \leqq b < b_1$, the latter subcase is treated similarly. Since $b(w) \equiv b(b_1)$ on $[a_1, b_1]$ while $b(v) \leqq b(b_1)$ for all $v \in [a, b]$, we find that

$$\iint |b(v)-b(w)|\mathrm{d}\nu(v)d\tilde{\nu}(w) = -\iint (b(v)-b(w))\mathrm{d}\nu(v)d\tilde{\nu}(w). \quad (6.16)$$

Besides, if $b(v) \equiv \text{const}$ on $[a, b]$ then $b(v) \equiv \text{const}$ on $[a, b_1] = [a, b] \cup [a_1, b_1]$ and we again arrive at (6.15), which readily implies the desired relation (6.14). Thus, assume that $b(v)$ is not constant on $[a, b]$. In view of (6.16) relation (6.14) will follow from the equality

$$\iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w)$$
$$= -\iint (g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w). \quad (6.17)$$

By Proposition 4 we have $g(v) = cb(v) + \text{const}$ on $[a, b]$, where $c = -\xi_0/\xi_1$ (remark that $\xi_1 \neq 0$, otherwise $b(v) \equiv \text{const}$ on $[a, b]$, which contradicts our assumption). Therefore,

$$\iint \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w)$$
$$= \iint_{[a,b]\times[a_1,b]} \operatorname{sign}(v-w)(g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w)$$
$$-\iint_{[a,b]\times(b,b_1]} (g(v)-g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w)$$
$$= c\iint_{[a,b]\times[a_1,b]} |b(v)-b(w)|\mathrm{d}\nu(v)d\tilde{\nu}(w)$$

$$
\begin{aligned}
&- \iint_{[a,b]\times(b,b_1]} (g(v) - g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) \\
&= -c \iint_{[a,b]\times[a_1,b]} (b(v) - b(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) \\
&- \iint_{[a,b]\times(b,b_1]} (g(v) - g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w),
\end{aligned}
$$

where we use that $b(v) - b(w) = b(v) - b(b_1) \leqq 0$ for $v \in [a, b]$, $w \in [a_1, b]$. On the other hand,

$$
\begin{aligned}
&\iint (g(v) - g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) \\
&= \iint_{[a,b]\times[a_1,b]} (g(v) - g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) \\
&+ \iint_{[a,b]\times(b,b_1]} (g(v) - g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) \\
&= c \iint_{[a,b]\times[a_1,b]} (b(v) - b(w))\mathrm{d}\nu(v)d\tilde{\nu}(w) \\
&+ \iint_{[a,b]\times(b,b_1]} (g(v) - g(w))\mathrm{d}\nu(v)d\tilde{\nu}(w),
\end{aligned}
$$

and (6.17) follows. This completes the proof of relation (6.14).

Let $\rho(r) = r^2/(1 + r^2)$. Then the function $q = \rho(P(t, x))$ is nonnegative and Lipschitz. Moreover, by the chain rule for Sobolev derivatives $q_t = \rho'(P)P_t$, $q_x = \rho'(P)P_x$. Applying (6.11) to the test function qf, where $f = f(t, x) \in C_0^\infty(\Pi)$, $f \geqq 0$, we obtain the relation

$$
\int_\Pi [BP_t + GP_x] f\rho'(P)\mathrm{d}t\mathrm{d}x + \int_\Pi [Bf_t + Gf_x]q\mathrm{d}t\mathrm{d}x \geqq 0, \tag{6.18}
$$

where we denote

$$
\begin{aligned}
B = B(t, x) &= \iint |b(v) - b(w)|\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w), \\
G = G(t, x) &= \iint \operatorname{sign}(v - w)(g(v) - g(w))\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w).
\end{aligned}
$$

By (6.12)

$$
B(t, x) = \int |u - \tilde{u}(t, x)| d\bar{\nu}_{t,x}(u),
$$

and it follows from the periodicity of the Young measure $\bar{\nu}_{t,x}$ and the e.s. $\tilde{u}(t, x)$ that the function $B(t, x)$ is spatially periodic. In view of relation (6.13) $BP_t + GP_x = 0$ a.e. on Π and the first integral in (6.18) disappears. Therefore,

$$
\int_\Pi [Bf_t + Gf_x]q\mathrm{d}t\mathrm{d}x \geqq 0.
$$

Taking in this relation $f=k^{-1}\alpha(t)\beta(x/k)$, where $\alpha(t)\in C_0^1(\mathbb{R}_+)$, $\beta(y)\in C_0^1(\mathbb{R})$ are nonnegative functions, $\int\beta(y)\mathrm{d}y=1$, we arrive at the relation

$$k^{-1}\int_\Pi B(t,x)q(t,x)\alpha'(t)\beta(x/k)\mathrm{d}t\mathrm{d}x \\ +k^{-2}\int_\Pi G(t,x)q(t,x)\alpha(t)\beta'(x/k)\mathrm{d}t\mathrm{d}x \geqq 0.$$

In the limit, as $k\to\infty$, the second term in this relation disappears, while the first one is

$$k^{-1}\int_\Pi B(t,x)q(t,x)\alpha'(t)\beta(x/k)\mathrm{d}t\mathrm{d}x \underset{k\to\infty}{\to} \int_{\mathbb{R}^+\times[0,1]} B(t,x)q(t,x)\alpha'(t)\mathrm{d}t\mathrm{d}x,$$

where we utilize the x-periodicity of $B(t,x)q(t,x)$, which allows us to apply [22, Lemma 2.1]. As a result, we get

$$\int_{\mathbb{R}^+\times[0,1]} B(t,x)q(t,x)\alpha'(t)\mathrm{d}t\mathrm{d}x \geqq 0 \quad \forall\alpha(t)\in C_0^1(\mathbb{R}_+),\ \alpha(t)\geqq 0.$$

This inequality means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 B(t,x)q(t,x)\mathrm{d}x \leqq 0 \text{ in } \mathcal{D}'(\mathbb{R}_+),$$

and implies that, for $t,\tau\in E$, $t>\tau$,

$$0\leqq \int_0^1 B(t,x)q(t,x)\mathrm{d}x \leqq \int_0^1 B(\tau,x)q(\tau,x)\mathrm{d}x, \tag{6.19}$$

where $E\subset\mathbb{R}_+$ is a set of full measure. Observe that $0\leqq q(\tau,x)\leqq |P(\tau,x)| = |P(\tau,x)-P(0,x)|\leqq L\tau$, where L is a Lipschitz constant of P while the function $B(t,x)$ is bounded. Therefore,

$$\int_0^1 B(\tau,x)q(\tau,x)\mathrm{d}x \underset{E\ni\tau\to 0}{\to} 0$$

and it follows from (6.19) that $\int_0^1 B(t,x)q(t,x)\mathrm{d}x=0$. Since $B,q\geqq 0$, we find that $Bq=0$ a.e. on Π. Let $E\subset\Pi$ be the set where $q=0\Leftrightarrow P=0$, that is, $E=P^{-1}(0)$. By the known properties of Lipschitz functions, $\nabla P=0$ a.e. on E. In particular, $P_x=u-\tilde{u}=0$ a.e. in E. On the other hand, for $(t,x)\in\Pi\setminus E$ the function $q>0$ and therefore $B=0$ a.e. on this set. Since

$$B(t,x)=\iint |b(v)-b(w)|\mathrm{d}\nu_{t,x}(v)d\tilde{\nu}_{t,x}(w)=\int |b(v)-\tilde{u}|\mathrm{d}\nu_{t,x}(v),$$

we find that $b(v)=\tilde{u}$ on supp $\nu_{t,x}$. In particular, again $u(t,x)=\int b(v)\mathrm{d}\nu_{t,x}(v)=\tilde{u}(t,x)$. We conclude that $u=\tilde{u}$ a.e. in Π, which completes the proof.

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