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The Blow-Up Rate for a Non-Scaling Invariant Semilinear Heat Equation

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Abstract

We consider the semilinear heat equation

$$\partial_t u - \Delta u = f(u), \quad (x,t) \in \mathbb{R}^N \times [0,T), \tag{1}$$

with $f(u) = |u|^{p-1}u \log^a(2+u^2)$, where $p > 1$ is Sobolev subcritical and $a \in \mathbb{R}$. We first show an upper bound for any blow-up solution of (1). Then, using this estimate and the logarithmic property, we prove that the exact blow-up rate of any singular solution of (1) is given by the ODE solution associated with (1), namely $u' = |u|^{p-1}u \log^a(2+u^2)$. In other words, all blow-up solutions in the Sobolev subcritical range are Type I solutions. To the best of our knowledge, this is the first determination of the blow-up rate for a semilinear heat equation where the main nonlinear term is not homogeneous.

1. Introduction

1.1. Motivation of the Problem

This paper is devoted to the study of blow-up solutions for the following semilinear heat equation:

$$\begin{cases} \partial_t u = \Delta u + f(u), & (x,t) \in \mathbb{R}^N \times [0,T), \\ u(x,0) = u_0(x) \in L^\infty(\mathbb{R}^N). \end{cases} \tag{1.1}$$

Here $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ with focusing nonlinearity f defined by:

$$f(u) = |u|^{p-1}u \log^a(2+u^2), \quad p > 1, \quad a \in \mathbb{R}. \tag{1.2}$$

We assume in addition that $p > 1$ and if $N \geqq 3$, and we further assume that

$$p < p_S \equiv \frac{N+2}{N-2}. \tag{1.3}$$

Note that when $a \neq 0$, the nonlinear term is not homogeneous, and this is the focus of our paper.

By standard results the problem (1.1) has a unique solution for any $u_0 \in L^\infty(\mathbb{R}^N)$. More precisely, there is a unique maximal solution on $[0, T)$, with $T \leqq \infty$. If $T < \infty$, we say that the solution of (1.1) blows up in finite time. In that case, it holds that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \to T$. Such a solution u is called a blow-up solution of (1.1) with the blow-up time T.

In the case $a = 0$, equation (1.1) reduces to the semilinear heat equation with power nonlinearity:

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad (x,t) \in \mathbb{R}^N \times [0, T). \tag{1.4}$$

In the literature, the determination of the blow-up rate has been linked to the terminology of "Type I/Type II solutions", first introduced (up to our knowledge) by Matano and Merle in [19]. In that paper, if a solution u to (1.4) blows up at time T and satisfies for all $t \in [0, T)$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(T-t)^{-\frac{1}{p-1}}, \tag{1.5}$$

for some positive constant C, independent of time t, then u is called a Type I. If not, then u is said to be of Type II. Note that the bound given in (1.5) is (up to a multiplying factor) a solution of the associated ODE $u' = u^p$.

In the subcritical case under consideration (1.3), we know from Giga and Kohn [6–8], and also Giga, Matsui and Sasayama [9] that *all* blow-up solutions of (1.4) are of Type I. Moreover, from the construction provided by Nguyen and Zaag [22], we know that Type I solutions are available for any superlinear exponent p, not only in the subcritical case, despite what the authors noted at that time.

As for Type II solutions, we know that they are available in the critical range (see Schweyer [25], Harada [18], Del Pino, Musso and Wei [3], Collot, Merle and Raphaël [2], Filippas, Herrero and Velàzquez [5]), and also in the supercritical range (see Herrero and Velàzquez [17], Mizoguchi [20], Seki [26,27].

Going back to the proof given in [9] for the fact that all blow-up solutions for equation (1.4) in the subcritical range (1.3) are of Type I, we would like to mention that the following estimate is central in the argument:

$$\int_s^{s+1} \|w(\tau)\|_{L^{p+1}(\mathbf{B}_R)}^{(p+1)q} \mathrm{d}\tau \leqq K(q, R), \quad \forall q \geqq 2, \quad \forall R > 0, \quad \forall s > -\log T. \tag{1.6}$$

there w is the similarity variables version of the solution defined in (1.17) below and $\mathbf{B}_R \equiv B(0, R)$ is the open ball of radius R centered at the origin in $\mathbb{R}^N$.

Exploiting the non-trivial perturbative method introduced by the authors in [13,14] in the hyperbolic case and arguing as in the non perturbed case in [9], Nguyen proved in [21] a similar result to (1.5), valid in the subcritical case, for a class of strongly perturbed semilinear heat equations

$$\partial_t u = \Delta u + |u|^{p-1}u + h(u), \quad (x,t) \in \mathbb{R}^N \times [0,T), \tag{1.7}$$

under the assumptions $|h(u)| \leqq M(1+|u|^p)\log^{-a}(2+u^2)$, for some $M > 0$ and $a > 1$. Obtaining the same blow-up rate is reasonable, since the dynamics is still governed by the ODE $u' = |u|^{p-1}u$. Furthermore, the proof remains (non trivially) perturbative with respect to the homogeneous PDE (1.4), which is scale invariant.

This leaves unanswered an interesting question: is the scale invariance property crucial in deriving the blow-up rate?

In fact we *had the impression* that the answer was ”yes”, since the scaling invariance induces in similarity variables a PDE which is autonomous in the unperturbed case (1.4), and asymptotically autonomous in the perturbed case (1.7).

In this paper we *prove* that the answer is ”no” from the example of the non homogeneous PDE (1.4). In fact, our situation is different from (1.4) and (1.7). Indeed, the term $|u|^{p-1}u\log^a(2+u^2)$ is playing a fundamental role in the dynamics of the blow-up solution of (1.1). More precisely, we obtain an analogous result to (1.5) but with a logarithmic correction as shown in (1.28) below. In fact, the bow-up rate is given by the solution of the associated ODE $u' = |u|^{p-1}u\log^a(2+u^2)$.

In this paper, we study the blow-up rate of any singular solution of (1.1). Before handling the PDE, we first consider the ODE associated to (1.1),

$$v_T'(t) = |v_T(t)|^{p-1}v_T(t)\log^a\left(v_T^2(t)+2\right), \quad v(T) = \infty, \tag{1.8}$$

and show that the nonlinear term including the logarithmic factor gives rise to different dynamics. In fact, thanks to [4] (see Lemma A1), we can see that the solution v_T satisfies

$$v_T(t) \sim \kappa_a \psi_T(t), \text{ as } t \to T, \quad \text{where} \quad \kappa_a = \left(\frac{2^a}{(p-1)^{1-a}}\right)^{\frac{1}{p-1}}, \tag{1.9}$$

and

$$\psi_T(t) = (T-t)^{-\frac{1}{p-1}}(-\log(T-t))^{-\frac{a}{p-1}}. \tag{1.10}$$

Therefore, it is natural to extend the terminology “Type I/Type II solutions” for the blow-up of a solution $u(x,t)$ of (1.1) by the following:

$$(T-t)^{\frac{1}{p-1}}(-\log(T-t))^{\frac{a}{p-1}}\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqq C, \qquad \text{Type I} \tag{1.11}$$

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}(-\log(T-t))^{\frac{a}{p-1}}\|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty, \qquad \text{Type II.} \tag{1.12}$$

Let us mention that Duong, Nguyen and Zaag construct in [4] a solution of equation (1.1) which blows up in finite time T, only at one blow-up point x_0, according to the following asymptotic dynamics:

$$u(x,t) \sim v_T(t)\left(1 + \frac{(p-1)|x-x_0|^2}{4p(T-t)|\log(T-t)|}\right)^{-\frac{1}{p-1}}, \qquad as\ t \to T. \tag{1.13}$$

Here $v_T(t)$ is the solution of (1.8) with an equivalent given in (1.9). Note from (1.13) that the constructed solution is of Type I.

Concerning the blow-up rate for the hyperbolic equations with a non-homogeneous main term, we would like to mention that in [15] and [16], we consider the semilinear wave equation

$$\partial_t^2 u - \Delta u = |u|^{p-1} u \log^a(2+u^2), \quad (x,t) \in \mathbb{R} \times [0,T), \tag{1.14}$$

where $a \in \mathbb{R}$ and $p > 1$ is subconformal, in the sense that $(N-1)p < N+3$. We prove that the exact blow-up rate of any singular solution of (1.14) is given by the ODE solution associated with (1.14), namely

$$V_T''(t) = |V_T(t)|^{p-1} V_T(t) \log^a \left(V_T^2(t) + 2\right), \quad V(T) = \infty. \tag{1.15}$$

Let us mention that the nonlinear term involving the logarithmic factor gives raise to different dynamics. To be precise, the solution V_T satisfies

$$V_T(t) \sim C(a,p)(T-t)^{-\frac{2}{p-1}}(-\log(T-t))^{-\frac{a}{p-1}}, \text{ as } t \to T. \tag{1.16}$$

Since the blow-up rate is given by $V_T(t)$, we see that the effect of the nonlinearity is completely encapsulated in (1.16). Note that before [15,16], we could successfully implement our perturbative method in [10–14] to derive the blow-up rate for some classes of perturbed wave equations where the main nonlinear term is power-like (hence, homogeneous).

1.2. Strategy of the Proof

Going back to the equation under study in this paper (see (1.1) and (1.2)), we introduce the following similarity variables, defined for all $x_0 \in \mathbb{R}^N$:

$$y = \frac{x - x_0}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad u(x,t) = \psi_T(t) w_{x_0,T}(y,s). \tag{1.17}$$

Here $\psi_T(t)$ is the explicit rate given in (1.10). On may think that it would be more natural to replace $\psi_T(t)$ by $v_T(t)$ (defined in (1.8)) in this definition, since the latter is an exact solution of the ODE (1.8). That might be good, however, as $v_T(t)$ has no explicit expression, the calculations will immediately become too complicated. For that reason, we preferred to replace the non-explicit $v_T(t)$ by its explicit equivalent $\psi_T(t)$ in (1.10). The fact the latter is not an exact solution of (1.8) will have no incidence in our analysis.

From (1.1) and (1.17), the function $w_{x_0,T}$ (we write w for simplicity) satisfies, for all $y \in \mathbb{R}^N$ and $s \geqq \max(-\log T, 1)$,

$$\partial_s w = \frac{1}{\rho} \operatorname{div}(\rho \nabla w) - \frac{1}{p-1}(1 - \frac{a}{s})w + e^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}} f(\phi(s)w), \tag{1.18}$$

where

$$\rho(y) = e^{-\frac{|y|^2}{4}} \tag{1.19}$$

and

$$\phi(s) = e^{\frac{s}{p-1}} s^{-\frac{a}{p-1}}. \tag{1.20}$$

In the new set of variables (y, s), studying the behavior of u as $t \to T$ is equivalent to studying the behavior of w as $s \to +\infty$.

While reading Giga and Kohn [6–8] dedicated to the blow-up rate of the homogeneous case (1.4), one sees that the existence of a Lyapunov functional for the similarity variables' version (1.18) with $a = 0$ is central in the argument. Clearly, the invariance of equation (1.4) under the scaling transformation $u \mapsto u_\lambda(x,t) = \lambda^{\frac{1}{p-1}} u(\lambda x, \lambda^2 t)$ was crucial in the construction of the Lyapunov functional. The fact that equation (1.1) is not invariant under the last scaling transformation implies that the existence of a Lyapunov functional in similarity variables is far from being trivial (see [21,23] in the parabolic case and [10–15] in the hyperbolic case).

In this paper, we construct a Lyapunov functional in similarity variables for the problem (1.18). Then, we prove that the blow-up rate of any singular solution of (1.1) is given by the solution of (1.8).

Let us explain how we derive the Lyapunov functional. As we did for the perturbed wave equation with a conformal exponent in [10,12,13], we proceed in 2 steps:

- Step 1: we first introduce some functional (not a Lyapunov functional) for equation (1.18), which is bounded by s^α for some $\alpha > 0$, then show that w enjoys also a polynomial (in s) bound.
- Step 2: then, viewing equation (1.18) as a perturbation of the case of a pure power nonlinearity (case where $a = 0$ in (1.18)) by the following terms:

$$\frac{a}{(p-1)s} w \quad \text{and} \quad e^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}} f(\phi(s)w), \tag{1.21}$$

we use the rough estimates on w proved in the first step, in order to control the ≪ perturbative ≫ terms in (1.18). This way, we find a Lyapunov functional for (1.18), then use it to prove that the solution itself is bounded.

Specifically, in Step 1, we would like to add the following regarding the effect of the perturbation terms (1.21) and the way we handle them: The first term is a lower order term which was already handled in the Sobolev subcritical perturbative case treated in [21,23]. However, since the nonlinear term $e^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}} f(\phi(s)w)$ depends on time s, we expect the time derivatives to be delicate. Thanks to the fact that $uf(u) - (p+1)\int_0^u f(v)\mathrm{d}v \sim \frac{2a}{p+1}|u|^{p+1}\log^{a-1}(2+u^2)$, as $u \to \infty$, we construct a functional (in Section 2) satisfying the following kind of differential inequality:

$$\frac{d}{ds}h(s) \leqq -\frac{1}{2}\int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y)\mathrm{d}y + \frac{C}{s}h(s) + Ce^{-s}; \tag{1.22}$$

this implies a polynomial estimate.

In order to state our main result, we start by introducing the functionals

$$E(w(s),s)=\int_{\mathbb{R}^N}\Big(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}w^2-e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\Big)\rho(y)\mathrm{d}y, \tag{1.23}$$

$$L_0(w(s),s)=E(w(s),s)-\frac{1}{s\sqrt{s}}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y, \tag{1.24}$$

where

$$F(u)=\int_0^u f(v)\mathrm{d}v=\int_0^u |v|^{p-1}v\log^a(v^2+2)\mathrm{d}v. \tag{1.25}$$

Moreover, for all $s\geqq\max(-\log T,1)$, we define the functional

$$L(w(s),s)=\exp\Big(\frac{p+3}{\sqrt{s}}\Big)L_0(w(s),s)+\frac{\theta}{s^{\frac{3}{4}}}, \tag{1.26}$$

where θ is a sufficiently large constant that will be determined later. We derive that the functional $L(w(s),s)$ is a decreasing functional of time for equation (1.18), provided that s is large enough. Clearly, by (1.23), (1.24) and (1.26), the functional $L(w(s),s)$ is a small perturbation of the natural energy $E(w(s),s)$.

Our main theorem in this paper is as follows:

Theorem 1. (A Lyapunov functional in similarity variables) *Consider u a solution of (1.1), with blow-up time $T>0$. Then, there exists $t_1\in[0,T)$ such that, for all $s\geqq-\log(T-t_1)$ and $x_0\in\mathbb{R}^N$, we have*

$$L(w(s+1),s+1)-L(w(s),s)\leqq-\frac{1}{2}\int_s^{s+1}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y\mathrm{d}\tau, \tag{1.27}$$

where $w=w_{x_0,T}$ is defined in (1.17).

Remark 1.1. We choose to put forward this result proving the existence of a Lyapunov functional and state it as the first result of our paper (namely Theorem 1), mainly because we consider it as the crucial step in our argument, and also because its proof is far from being trivial.

The existence of this Lyapunov functional $L(w(s),s)$ together with a blow-up criterion for equation (1.18) make a crucial step in the derivation of the blow-up rate for equation (1.1). Indeed, with the functional $L(w(s),s)$, we are able to adapt the analysis performed in [6–8] for equation (1.4) and obtain the following result:

Theorem 2. (Blow-up rate for equation (1.1)) *Consider u a solution of (1.1), with blow-up time $T>0$. Then, there exists $t_2\in[t_1,T)$ such that for all $t\in[t_2,T)$, we have*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)}\leqq K(T-t)^{-\frac{1}{p-1}}(-\log(T-t))^{-\frac{a}{p-1}}, \tag{1.28}$$

where $K=K(p,a,T,t_2,\|u(\tilde{t_2})\|_{L^\infty})$, for some $\tilde{t_2}\in[0,t_2)$.

Remark 1.2. Note that the blow-up rate in this upper bound is sharp, since we have, from a simple comparison argument, the lower bound

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \geqq v_T(t) \sim \kappa_a (T-t)^{-\frac{1}{p-1}} (-\log(T-t))^{-\frac{a}{p-1}},$$

where the last equivalence was given in (1.9).

Remark 1.3. Let us remark that we can obtain the same blow-up rate for the more general equation

$$\partial_t u = \partial_x^2 u + |u|^{p-1} u \log^a(2+u^2) + k(u), \quad (x,t) \in \mathbb{R} \times [0,T), \tag{1.29}$$

under the assumption that $|k(u)| \leqq M(1+|u|^p \log^b(2+u^2))$, for some $M > 0$ and $b < a-1$. More precisely, under this hypothesis, we can construct a suitable Lyapunov functional for this equation. Then, we can prove a similar result to (1.28). However, the case where $a-1 \leqq b < a$ seems to be out of reach of our technics, though we think we may obtain the same rate as in the unperturbed case.

This paper is organized as follows: in Section 2, we obtain a rough control of the solution w. In Section 3, thanks to that result, we prove that the functional $L(w(s), s)$ is a Lyapunov functional for equation (1.18). Thus, we get Theorem 1. Finally, by applying this last theorem, we give the proof of Theorem 2.

Throughout this paper, C denotes a generic positive constant depending only on p, N and a, which may vary from line to line. As for M, it will be used for constants depending on initial data, in addition to p, N and a. We may also use K_1, K_2, K_3... M_1, M_2, M_3... Q_1, Q_2, Q_3 for constants having the same dependence as M. If necessary, we may write explicitly the dependence of the constants we use.

Moreover, we denote by $\mathbf{B}_R$ the open ball in $\mathbb{R}^N$ with center 0 and radius R. Finally, note that we use the notation $f(s) \sim g(s)$ when $\lim_{s\to\infty} \frac{f(s)}{g(s)} = 1$.

2. A Polynomial Bound for Solutions of Equation (1.18)

This section is devoted to the derivation of a polynomial bound for a global solution of equation (1.18). More precisely, we have

Proposition 2.1. *Let $R > 0$. Consider w a global solution of (1.18). Then, there exist $\widehat{S}_1 = \widehat{S}_1(a, p, N, R) \geqq 1$ and $\mu = \mu(a, p, N, R) > 0$ such that, for all $s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1)$, we have*

$$\|w(s)\|_{H^1(\mathbf{B}_R)} \leqq K_1(R) s^\mu, \tag{2.1}$$

where K_1 depends on p, a, N, R and $\|w(\widehat{s}_1)\|_{H^1}$.

Remark 2.1. By using the Sobolev's embedding and the above proposition, we can deduce that for all $r \in [2, 2^*)$, where $2^* = \frac{2N}{N-2}$, if $N \geqq 3$ and $2^* = \infty$, if $N = 2$,

$$\|w(s)\|_{L^r(\mathbf{B}_R)} \leqq K_2(R) s^\mu, \quad \text{for all} \quad s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1), \tag{2.2}$$

where $K_2(R)$ depends on p, a, N, R and $\|w(\widehat{s}_1)\|_{H^1}$.

In order to prove this proposition, we need to construct a Lyapunov functional for equation (1.18). Accordingly, we start by recalling from (1.23) the functional

$$E(w(s),s)=\int_{\mathbb{R}^N}\Big(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}w^2-e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\Big)\rho(y)\mathrm{d}y, \tag{2.3}$$

where F is given by (1.25). Then, we introduce the functionals

$$J(w(s),s)=-\frac{1}{2s}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y, \tag{2.4}$$

$$H_m(w(s),s)=E(w(s),s)+mJ(w(s),s), \tag{2.5}$$

where $m>0$ is a sufficiently large constant that will be fixed later.

In fact, the main target of this section is to prove, for some m_0 large enough, that the energy $H_{m_0}(w(s),s)$ satisfies the inequality

$$\frac{d}{ds}H_{m_0}(w(s),s)\leqq-\frac{1}{2}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y+\frac{m_0(p+3)}{2s}H_{m_0}(w(s),s)+Ce^{-s}, \tag{2.6}$$

which implies that $H_{m_0}(w(s),s)$ satisfies the following polynomial estimate:

$$H_{m_0}(w(s),s))\leqq A_0 s^{\mu_0}, \tag{2.7}$$

for some $A_0>0$ and $\mu_0>0$.

2.1. Classical Energy Estimates

In this subsection, we state two lemmas which are crucial for the construction of a Lyapunov functional. We begin with bounding the time derivative of $E(w(s),s)$ in the following lemma:

Lemma 2.2. *For all* $s\geqq\max(-\log T,1)$*, we have*

$$\frac{d}{ds}E(w(s),s)\leqq-\frac{1}{2}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y+\frac{C}{s^{a+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y+\Sigma_1(s), \tag{2.8}$$

where $\Sigma_1(s)$ *satisfies*

$$\Sigma_1(s)\leqq\frac{C}{s^2}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y+Ce^{-s}. \tag{2.9}$$

Proof. Consider $s \geqq \max(-\log T, 1)$. Multiplying (1.18) by $\partial_s w\, \rho(y)$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
\frac{d}{ds}E(w(s),s) = &-\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y + \underbrace{\frac{a}{(p-1)s}\int_{\mathbb{R}^N} w\partial_s w\rho(y)\mathrm{d}y}_{\Sigma_1^1(s)} \\
&+ \underbrace{\frac{p+1}{p-1}e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}}\int_{\mathbb{R}^N}\big(F(\phi w) - \frac{\phi w f(\phi w)}{p+1}\big)\rho(y)\mathrm{d}y}_{\Sigma_1^2(s)} \\
&\underbrace{-\frac{2a}{p-1}e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}-1}\int_{\mathbb{R}^N}\big(F(\phi w) - \frac{\phi w f(\phi w)}{2}\big)\rho(y)\mathrm{d}y}_{\Sigma_1^3(s)}.
\end{aligned} \tag{2.10}$$

Now, we control the terms $\Sigma_1^1(s)$, $\Sigma_1^2(s)$ and $\Sigma_1^3(s)$. By using the following basic inequality

$$ab \leqq \varepsilon a^2 + \frac{1}{\varepsilon}b^2,\ \forall \varepsilon > 0, \tag{2.11}$$

we write

$$\Sigma_1^1(s) \leqq \frac{1}{2}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y + \frac{C}{s^2}\int_{\mathbb{R}^N} w^2\rho(y)\mathrm{d}y. \tag{2.12}$$

Let us introduce the functions F_1 and F_2 defined by

$$F_1(x) = -\frac{2a}{(p+1)^2}|x|^{p+1}\log^{a-1}(2+x^2), \tag{2.13}$$

and

$$F_2(x) = F(x) - \frac{xf(x)}{p+1} - F_1(x). \tag{2.14}$$

By the expressions of F_1, F_2 given by (2.13) and (2.14) and the estimates (B.5) and (B.6), we obtain

$$F(\phi w) - \frac{\phi w f(\phi w)}{p+1} \leqq C + C\frac{\phi w}{s}f(\phi w), \tag{2.15}$$

which implies

$$\Sigma_1^2(s) \leqq Ce^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}-1}\int_{\mathbb{R}^N}\phi w f(\phi w)\rho(y)\mathrm{d}y + Ce^{-s}. \tag{2.16}$$

From the expression of $\phi = \phi(s)$ defined in (1.20), we have

$$e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}}\phi w f(\phi w) = \frac{1}{s^a}|w|^{p+1}\log^a(2+\phi^2 w^2). \tag{2.17}$$

Thus, using (2.16) and (2.17), we obtain

$$\Sigma_1^2(s) \leqq \frac{C}{s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + Ce^{-s}. \tag{2.18}$$

Similarly, by (B.4) and (2.17), we easily obtain

$$\Sigma_1^3(s) \leqq \frac{C}{s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + Ce^{-s}. \tag{2.19}$$

The results (2.8) and (2.9) follows immediately from (2.10), (2.12), (2.18) and (2.19), which ends the proof of Lemma 2.2.

Remark 2.2. By showing the estimate proved in Lemma 2.2, related to the so called natural functional $E(w(s), s)$, we have some nonnegative terms in the right-hand side of (2.8) and this does not allow to construct a decreasing functional (unlike the case of a pure power nonlinearity). The main problem is related to the nonlinear term

$$\frac{1}{s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2(s)w^2)\rho(y)\mathrm{d}y = \frac{1}{s} \int_{\mathbb{R}^N} we^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}} f(\phi(s)w)\rho(y)\mathrm{d}y.$$

To overcome this problem, we adapt the strategy used in [10–14,21]. Indeed, by using the identity obtained by multiplying equation (1.1) by $w\rho(y)$, then integrating over $\mathbb{R}^N$, we can introduce a new functional $H_m(w(s), s)$ defined in (2.5), where $m > 0$ is sufficiently large and will be fixed such that $H_m(w(s), s)$ satisfies a differential inequality similar to (1.22).

We will prove the following estimate on the functional $J(w(s), s)$:

Lemma 2.3. *For all* $s \geqq \max(-\log T, 1)$*, we have*

$$\begin{aligned} \frac{d}{ds} J(w(s), s) \ \leqq\ & \frac{p+3}{2s} E(w(s), s) - \frac{p-1}{4s} \int_{\mathbb{R}^N} |\nabla w|^2 \rho(y)\mathrm{d}y - \frac{1}{4s} \int_{\mathbb{R}^N} w^2 \rho(y)\mathrm{d}y \\ & - \frac{p-1}{2(p+1)s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + \Sigma_2(s), \end{aligned} \tag{2.20}$$

where $\Sigma_2(s)$ *satisfies*

$$\Sigma_2(s) \leqq \frac{C}{s^{a+2}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + \frac{C}{s^2} \int_{\mathbb{R}^N} w^2 \rho(y)\mathrm{d}y + Ce^{-s}. \tag{2.21}$$

Proof. Consider $s \geqq \max(-\log T, 1)$. Note that $J(w(s), s)$ is a differentiable function and that we get

$$\frac{d}{ds} J(w(s), s) = -\frac{1}{s} \int_{\mathbb{R}^N} w \partial_s w \rho(y)\mathrm{d}y + \frac{1}{2s^2} \int_{\mathbb{R}^N} w^2 \rho(y)\mathrm{d}y.$$

From equation (1.18) and the identity (2.17), we conclude that

$$\frac{d}{ds}J(w(s),s) = \frac{1}{s}\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y + \frac{1}{(p-1)s}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y - \frac{1}{s^{a+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y + \frac{1}{2s^2}\big(1-\frac{2a}{p-1}\big)\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y.$$

According to the expressions of $E(w(s),s)$, $\phi(s)$ defined in (2.3) and (1.20) and the identity (2.17) with some straightforward computation, we obtain (2.20) where

$$\Sigma_2(s) = \Sigma_2^1(s) + \Sigma_2^2(s), \tag{2.22}$$

and

$$\Sigma_2^1(s) = \frac{p+3}{2}e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}-1}\int_{\mathbb{R}^N}\Big(F(\phi w) - \frac{\phi w f(\phi w)}{p+1}\Big)\rho(y)\mathrm{d}y,$$
$$\Sigma_2^2(s) = \frac{1}{2s^2}\big(1-\frac{2a}{p-1}\big)\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y.$$

Thanks to (2.17) and (2.15), we deduce

$$\Sigma_2^1(s) \leqq \frac{C}{s^{a+2}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y + Ce^{-s}. \tag{2.23}$$

Hence, collecting (2.22) and (2.23), one easily obtains that $\Sigma_2(s)$ satisfies (2.21), which ends the proof of Lemma 2.3.

2.2. Existence of a Decreasing Functional for Equation (1.18)

In this subsection, by using Lemmas 2.2 and 2.3 , we will construct a decreasing functional for equation (1.18). Let us define the functional

$$N_m(w(s),s) = s^{-\frac{m(p+3)}{2}}H_m(w(s),s) + A(m)e^{-s}, \tag{2.24}$$

where $H_m(w(s),s)$ is defined in (2.5), and m together with $A = A(m)$ are constants that will be determined later.

We now state the following proposition:

Proposition 2.4. *There exist $m_0 > 1$, $A(m_0) > 0$, $S_1 \geqq 1$ and $\lambda_1 > 0$, such that for all $s = s_1 \geqq \max(-\log T, S_1)$, we have*

$$N_{m_0}(w(s+1),s+1) - N_{m_0}(w(s),s) \leqq -\frac{\lambda_1}{s^b}\int_s^{s+1}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y\mathrm{d}\tau - \frac{\lambda_1}{s^{b+1}}\int_s^{s+1}\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y\mathrm{d}\tau$$

$$
\begin{aligned}
&- \frac{\lambda_1}{s^{a+b+1}} \int_s^{s+1} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y \mathrm{d}\tau \\
&- \frac{\lambda_1}{s^{b+1}} \int_s^{s+1} \int_{\mathbb{R}^N} w^2 \rho(y) \mathrm{d}y \mathrm{d}\tau, \qquad (2.25)
\end{aligned}
$$

where

$$
b = \frac{m_0(p+3)}{2}. \qquad (2.26)
$$

Moreover, there exists $S_2 \geqq S_1$ *such that for all* $s \geqq \max(-\log T, S_2)$*, we have*

$$
N_{m_0}(w(s), s) \geqq -1. \qquad (2.27)
$$

Proof. From the definition of $H_m(w(s), s)$ given in (2.5), Lemmas 2.2 and 2.3, we can write, for all $s \geqq \max(-\log T, 1)$,

$$
\begin{aligned}
\frac{d}{ds} H_m(w(s), s) \leqq & \frac{m(p+3)}{2s} H_m(w(s), s) - \frac{1}{2} \int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y) \mathrm{d}y \\
& - \Big(\frac{m(p-1)}{2(p+1)} - C_0 - \frac{C_0 m}{s} \Big) \frac{1}{s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y \\
& - \frac{m(p-1)}{4s} \int_{\mathbb{R}^N} |\nabla w|^2 \rho(y) \mathrm{d}y - \Big(\frac{m}{4s} - \frac{C_0 m}{s^2} - \frac{C_0}{s^2} \Big) \int_{\mathbb{R}^N} w^2 \rho(y) \mathrm{d}y \\
& + (C_0 m + C_0) e^{-s}, \qquad (2.28)
\end{aligned}
$$

where C_0 stands for some universal constant depending only on N, p and a. We first choose m_0 such that $\frac{m_0(p-1)}{4(p+1)} - C_0 = 0$, so

$$
\frac{m_0(p-1)}{2(p+1)} - C_0 - \frac{C_0 m_0}{s} = m_0 \Big(\frac{p-1}{4(p+1)} - \frac{C_0}{s} \Big).
$$

We now choose $S_1 = S_1(a, p, N)$ large enough ($S_1 \geqq 1$), so that for all $s \geqq S_1$, we have

$$
\frac{p-1}{8(p+1)} - \frac{C_0}{s} \geqq 0, \qquad \frac{m_0}{8} - \frac{C_0 m_0}{s} - \frac{C_0}{s} \geqq 0.
$$

Then, we deduce that for all $s \geqq \max(-\log T, S_1)$,

$$
\begin{aligned}
\frac{d}{ds} H_{m_0}(w(s), s) \leqq & \frac{m_0(p+3)}{2s} H_{m_0}(w(s), s) - \frac{1}{2} \int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y) \mathrm{d}y \\
& - \frac{\lambda_0}{s} \int_{\mathbb{R}^N} |\nabla w|^2 \rho(y) \mathrm{d}y - \frac{\lambda_0}{s} \int_{\mathbb{R}^N} w^2 \rho(y) \mathrm{d}y \\
& - \frac{\lambda_0}{s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y \\
& + (C_0 m_0 + C_0) e^{-s}, \qquad (2.29)
\end{aligned}
$$

where $\lambda_0 = \inf(\frac{m_0}{8}, \frac{m_0(p-1)}{4(p+1)})$.

By using the definition of $N_{m_0}(w(s), s)$ given in (2.24) together with the estimate (2.29), we easily prove that for all $s \geqq \max(-\log T, S_1)$,

$$\begin{aligned}\frac{d}{ds}N_{m_0}(w(s),s) \;\leqq\; & -\frac{1}{2s^b}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y \\ & -\frac{\lambda_0}{s^{b+1}}\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y-\frac{\lambda_0}{s^{b+1}}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y \\ & -\frac{\lambda_0}{s^{a+b+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y \\ & -e^{-s}\Big(A(m_0)-C_0(m_0+1)\frac{1}{s^b}\Big). \end{aligned} \tag{2.30}$$

We now choose $A(m_0) = C_0(m_0+1){S_1}^{-b}$, so we have

$$A(m_0)-\frac{C_0(m_0+1)}{s^b}\geqq 0, \qquad \forall s\geqq S_1. \tag{2.31}$$

By integrating in time between s and $s+1$ the inequality (2.30) and using (2.31), we easily obtain (2.25). This concludes the proof of the first part of Proposition 2.4.

We prove (3.38) here. Arguing by contradiction, we assume that there exists $\tilde{s_1} \geqq \max(-\log T, S_2)$ such that $N_{m_0}(w(\tilde{s_1}), \tilde{s_1}) < -1$, where $S_2 \geqq S_1$ is large enough.

Now, we consider

$$I(w(s),s)=s^{-b}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y, \qquad \forall s\geqq \max(-\log T,1),$$

where b is defined in (2.26). Thanks to (2.20) and (2.5), we have for all $s \geqq \max(-\log T, 1)$

$$\begin{aligned}\frac{d}{ds}I(w(s),s) \;\geqq\; & -(p+3)s^{-b}H_{m_0}(w(s),s)+\frac{1}{2s^b}(1-\frac{C_1}{s})\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y \\ & +\frac{p-1}{(p+1)s^{a+b}}(1-\frac{C_1}{s})\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y. \end{aligned} \tag{2.32}$$

Let us choose $S_2 = S_2(a, p, N)$ is large enough, such that $1 - \frac{C_1}{S_2} \geqq \frac{1}{2}$. So, we write for all $s \geqq \max(-\log T, S_2)$

$$\frac{d}{ds}I(w(s),s)\geqq -(p+3)N_{m_0}(w(s),s)+\frac{p-1}{2(p+1)s^{a+b}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y. \tag{2.33}$$

Since the energy $N_{m_0}(w(s), s)$ decreases in time, we have $N_{m_0}(w(s), s) < -1$, for all $s \geqq \tilde{s_1}$. Then, for all $s \geqq \tilde{s_1}$

$$\frac{d}{ds}I(w(s),s)\geqq p+3+\frac{p-1}{2(p+1)s^{a+b}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y. \tag{2.34}$$

Thanks to (B.4), (B.10) and (2.17), we get for all $s \geqq \tilde{s_1}$

$$\frac{1}{s^a}\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \geqq C\int_{\mathbb{R}^N}|w|^{\frac{p+3}{2}}\rho(y)\mathrm{d}y - C. \quad (2.35)$$

Therefore, by using (2.34) and (2.35), there exist $\tilde{S}_2 \geqq S_2$ large enough such that $p+2-\frac{C}{(\tilde{S}_2)^b} > 0$, we have for all $s \geqq \max(\tilde{s_1}, \tilde{S}_2)$

$$\frac{d}{ds}I(w(s),s) \geqq 1 + \frac{C}{s^b}\int_{\mathbb{R}^N}|w|^{\frac{p+3}{2}}\rho(y)\mathrm{d}y. \quad (2.36)$$

Thanks to Jensen's inequality, we infer

$$\frac{d}{ds}I(w(s),s) \geqq 1 + Cs^{\frac{b(p-1)}{4}}\Big(I(w(s),s)\Big)^{\frac{p+3}{4}}. \quad (2.37)$$

This quantity must then tend to ∞ in finite time, which is a contradiction. Thus (3.38) holds. This concludes the proof of Proposition 2.4.

2.3. *Proof of Proposition 2.5.*

Based on Proposition 2.4, a bootstrap argument given in [24], we are able to adapt the analysis performed in [9], to prove the following key proposition:

Proposition 2.5. *For all $q \geqq 2$, $\varepsilon > 0$ and $R > 0$ there exist $\varepsilon_1 = \varepsilon_1(q,R) > 0$, $\mu_1(q,R,\varepsilon) > 0$ and $S_3(q,R,\varepsilon) \geqq S_2$ such that, for all $s \geqq \max(-\log T, S_3)$, we have*

$$(A_{q,R,\varepsilon}) \quad \int_s^{s+1} \|w(\tau)\|^{(p-\varepsilon+1)q}_{L^{p-\varepsilon+1}(\mathbf{B}_R)}\mathrm{d}\tau \leqq K_3(q,R,\varepsilon)s^{\mu_1(q,R,\varepsilon)}, \qquad \forall \varepsilon \in (0,\varepsilon_1],$$

where $K_3(q,R,\varepsilon)$ depends on $p, a, N, q, R, \varepsilon, s_1 = \max(-\log T, S_1)$ and $\|w(s_1)\|_{H^1}$.

To prove Proposition 2.5, we will proceed as in [9]. In fact, by using Proposition 2.4, we easily obtain the following Corollary

Corollary 3. *For all $s \geqq \max(-\log T, S_2)$, we have*

$$-1 \leqq N_{m_0}(w(s),s) \leqq K_4, \quad (2.38)$$

$$-K_5 s^b \leqq H_{m_0}(w(s),s) \leqq K_5 s^b, \quad (2.39)$$

$$\int_s^{s+1}\int_{\mathbb{R}^N}\Big(|\nabla w|^2 + (\partial_s w)^2 + w^2\Big)\rho(y)\mathrm{d}y\mathrm{d}\tau \leqq K_6 s^{b+1}, \quad (2.40)$$

$$\frac{1}{s^a}\int_s^{s+1}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\mathrm{d}\tau \leqq K_6 s^{b+1}, \quad (2.41)$$

$$\int_{\mathbb{R}^N} w^2\rho(y)\mathrm{d}y \leqq K_7 s^{b+1}, \quad (2.42)$$

$$\frac{1}{s^a}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y \leqq C\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y + K_8 s^{b+1}, \quad (2.43)$$

$$\int_{\mathbb{R}^N} |\nabla w|^2 \rho(y) \mathrm{d}y \leqq \frac{C}{s^a} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y + K_9 s^{b+1}, \quad (2.44)$$

$$\int_{\mathbb{R}^N} |\nabla w|^2 \rho(y) \mathrm{d}y \leqq C s^{\frac{b+1}{2}} \sqrt{\int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y) \mathrm{d}y} + K_{10} s^{b+1}, \quad (2.45)$$

$$\int_s^{s+1} \Big(\int_{\mathbb{R}^N} |\nabla w|^2 \rho(y) \mathrm{d}y \Big)^2 \leqq K_{11} s^{2b+2}, \quad (2.46)$$

$$\frac{1}{s^{2a}} \int_s^{s+1} \Big(\int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y \Big)^2 \mathrm{d}\tau \leqq K_{12} s^{2b+2}, \quad (2.47)$$

where b is defined in (2.26) and where $K_4, K_5, K_6, \ldots K_{12}$ depend on $p, a, N, s_1 = \max(-\log T, S_1)$ and $\|w(s_1)\|_{H^1}$.

Remark 2.3. Let us mention that, the estimates obtained in the above corollary are similar to the ones obtained in the pure power case treated in [9] except for the following features:

- The presence of the term $K s^{b+1}$ instead of K.
- In some estimates, we have the term $F(u)$ instead of $\frac{|u|^{p+1}}{p+1}$ in the pure power case. We easily overcome this problem thanks to the fact that $uf(u) - (p+1)\int_0^u f(v)\mathrm{d}v \sim \frac{2a}{p+1}|u|^{p+1}\log^{a-1}(2+u^2)$, as $u \to \infty$.

In order to prove Proposition 2.5, we introduce the following local functional:

$$\mathscr{E}_\psi(w(s), s) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)} w^2 - e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \Big) \psi^2(y) \rho(y) \mathrm{d}y, \quad (2.48)$$

where $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$
satisfies

$$0 \leqq \psi(y) \leqq 1, \quad \psi(y) = \begin{cases} 1 & \text{on} \quad \mathbf{B}_R \\ 0 & \text{on} \quad \mathbb{R}^N \backslash \mathbf{B}_{2R} \end{cases}, \quad (2.49)$$

where $R > 0$. An argument similar to that in [9], implies the following estimate:

Proposition 2.6. *There exist positive constants $K_{13} = K_{13}(R) > 0$ and $S_4 \geqq S_2$ such that, for all $s \geqq \max(-\log T, S_4)$, we have*

$$-K_{13}(R) s^{b+1} \leqq \mathscr{E}_\psi(w(s), s) \leqq K_{13}(R) s^{b+1}, \quad (2.50)$$

where K_{13} depends on $p, a, N, R, s_1 = \max(-\log T, S_1)$ and $\|w(s_1)\|_{H^1}$.

Proof. Most of the steps of the proof are the same as in the pure power case treated in [9] and some others are more delicate. For that reason, we leave the proof to Appendix C.

With Proposition 2.6, we are in a position to claim the following:

Lemma 2.7. *There exists a positive constant* $K_{14}(R,\varepsilon) > 0$ *such that, for all* $s \geqq \max(-\log T, S_4)$

$$\|w(s)\|^{(p-\varepsilon+1)}_{L^{p-\varepsilon+1}(\mathbf{B}_R)} \leqq K_{14}(R,\varepsilon)\|\nabla w\|^2_{L^2(\mathbf{B}_{2R})} + K_{14}(R,\varepsilon)s^{b+1}, \quad \forall \varepsilon \in (0, p-1), \quad (2.51)$$

where $K_{14}(R,\varepsilon)$ *depends on* p, a, N, R, ε, $s_1 = \max(-\log T, S_1)$ *and* $\|w(s_1)\|_{H^1}$.

Proof. From (2.50) and the definition of $\mathscr{E}_\psi$ in (2.48), we have for all $s \geqq \max(-\log T, S_4)$,

$$e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} \int_{\mathbb{R}^N} F(\phi w)\psi^2\rho(y)\mathrm{d}y \leqq C\int_{\mathbb{R}^N} |\nabla w|^2\psi^2\rho(y)\mathrm{d}y + K_{13}(R)s^{b+1}. \quad (2.52)$$

By exploiting (B.10), we write for all $s \geqq \max(-\log T, S_4)$,

$$\int_{\mathbb{R}^N} |w|^{p-\varepsilon+1}\psi^2\rho(y)\mathrm{d}y \leqq C\int_{\mathbb{R}^N} |\nabla w|^2\psi^2\rho(y)\mathrm{d}y + K_{13}(R)s^{b+1}$$
$$+C(\varepsilon)e^{-s}, \quad \forall \varepsilon \in (0, p-1). \quad (2.53)$$

Thus, (2.51) follows from (2.53) and the property of ψ. This conclude the proof of Lemma 2.7

By (2.51), the proof of estimate $(A_{q,R,\varepsilon})$ is available when we have

$$\int_s^{s+1} \|\nabla w(\tau)\|^{2q}_{L^2(\mathbf{B}_R)}\mathrm{d}\tau \leqq K_{15}(q,R,\varepsilon)s^{\mu_2(q,R,\varepsilon)}, \qquad \forall s \geqq \max(-\log T, S_3), \quad (2.54)$$

for some $\mu_2(q,R,\varepsilon) > (b+1)q$. Note from (2.46) that (2.54) already holds in the case $q = 2$.

In order to derive (2.54) for all $q > 2$, we need the following result:

Lemma 2.8. *There exist positive constants* $K_{16}(R) > 0$ *and* $S_5 \geqq S_4$ *such that, we have*

$$\|\nabla w\|^2_{L^2(\mathbf{B}_R)} \leqq C\|w\partial_s w\psi^2\|_{L^1(\mathbf{B}_{2R})} + K_{16}(R)s^{b+1}, \quad \forall s \geqq \max(-\log T, S_5). \quad (2.55)$$

Proof. Multiplying equation (1.18) with $w\rho(y)\psi^2$, integrating over $\mathbb{R}^N$ and using the definition of $\mathscr{E}_\psi(w(s),s)$ given in (2.48), we write

$$\int_{\mathbb{R}^N} |\nabla w|^2\psi^2\rho(y)\mathrm{d}y = \frac{4}{p-1}\int_{\mathbb{R}^N} w\partial_s w\psi^2\rho(y)\mathrm{d}y + \underbrace{\frac{2(p+3)}{p-1}\mathscr{E}_\psi(w(s),s)}_{\Sigma_2^1(s)}$$
$$-\frac{2}{(p+1)s^a}\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2w^2)\psi^2\rho(y)\mathrm{d}y$$
$$+\underbrace{\frac{8}{p-1}\int_{\mathbb{R}^N} w\nabla w.\nabla\psi\psi\rho(y)\mathrm{d}y}_{\Sigma_2^2(s)} \underbrace{-\frac{1}{p-1}(1+\frac{4a}{(p-1)s})\int_{\mathbb{R}^N} w^2\psi^2\rho(y)\mathrm{d}y}_{\Sigma_2^3(s)}$$

$$+\underbrace{\frac{2(p+3)}{p-1}e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}}\int_{\mathbb{R}^N}\Big(F(\phi w)-\frac{\phi w f(\phi w)}{p+1}\Big)\psi^2 w\mathrm{d}y}_{\Sigma_2^5(s)}. \tag{2.56}$$

From (2.50), (C.12) and (2.42) we infer for all $s \geqq \max(-\log T, S_4)$,

$$\Sigma_2^1(s)+\Sigma_2^2(s)+\Sigma_2^3(s) \leqq K_{17}(R)s^{b+1}. \tag{2.57}$$

According to the the estimates (2.15) and the identity (2.17), we get for all $s \geqq \max(-\log T, S_4)$,

$$\Sigma_2^5(s) \leqq \frac{C_2}{s^{a+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y + Ce^{-s}. \tag{2.58}$$

Hence, using (2.56), (2.57) and (2.58), yields for all $s \geqq \max(-\log T, S_4)$,

$$\int_{\mathbb{R}^N}|\nabla w|^2\psi^2\rho(y)\mathrm{d}y \leqq -\frac{2}{(p+1)s^a}\Big(1-\frac{(p+1)C_2}{2s}\Big)\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\psi^2\rho(y)\mathrm{d}y$$
$$+\frac{4}{p-1}\int_{\mathbb{R}^N}w\partial_s w\psi^2\rho(y)\mathrm{d}y + K_{17}s^{b+1}+Ce^{-s}. \tag{2.59}$$

Taking $S_5 \geqq S_4$ large enough such that $1-\frac{(p+1)C_2}{2S_5} > 0$, we have for all $s \geqq \max(-\log T, S_5)$,

$$\int_{\mathbb{R}^N}|\nabla w|^2\psi^2\rho(y)\mathrm{d}y \leqq \frac{4}{p-1}\int_{\mathbb{R}^N}w\partial_s w\ \psi^2 w\mathrm{d}y + K_{17}(R)s^{b+1}+Ce^{-s}.$$

Thus, (2.55) follows from the property of ψ. This ends the proof of Lemma 2.8.

Now, we are ready to give the proof of Proposition 2.5.

Proof of Proposition 2.5:. [**Proof of (2.54) for all $q \geqq 2$ by a bootstrap argument**] The proof is obtained by following the same part in [9]. However, as explained before (see Remarks 2.3), in our case we have two additional problems. Let $R > 0$ and suppose that we have

$$\int_s^{s+1}\|\nabla w(\tau)\|^{2q}_{L^2(\mathbf{B}_{4R})}\mathrm{d}\tau \leqq K_{15}(q,4R,\varepsilon)s^{\mu_2(q,4R,\varepsilon)}, \qquad \forall s \geqq \max(-\log T, S_3), \tag{2.60}$$

for some $\mu_2(q,4R,\varepsilon) > 0$ and for some $q \geqq 2$.

Combining (2.60) and (2.51), we write for all $s \geqq \max(-\log T, \tilde{S_3})$,

$$\int_s^{s+1}\|w(\tau)\|^{q(p-\varepsilon+1)}_{L^{p-\varepsilon+1}(\mathbf{B}_{2R})}d\tau \leqq K_{18}(q,R,\varepsilon)s^{\mu_3(q,R,\varepsilon)}, \ \forall\varepsilon\in(0,p-1) \tag{2.61}$$

for some $\mu_3(q,R,\varepsilon) > \mu_2(q,R,\varepsilon)$. where $\tilde{S_3} = \max(S_3, S_5)$. Thus, we use (2.40), (2.61) and apply Lemma A.1 with $\alpha = q(p-\varepsilon+1)$, $\beta = p-\varepsilon+1$, $\gamma=\delta=2$ to get that for all $s \geqq \max(-\log T, \tilde{S_3})$,

$$\|w(s)\|_{L^\lambda(\mathbf{B}_{2R})} \leqq K_{19}(q,R,\varepsilon)s^{\mu_4(q,R,\varepsilon)}, \ \ \forall\lambda < p-\varepsilon+1-\frac{p-\varepsilon-1}{q+1}. \ \forall\varepsilon\in(0,p-1), \tag{2.62}$$

for some $\mu_4(q,R,\varepsilon) > \mu_3(q,R,\varepsilon)$. Thanks to the Holder's inequality,

$$\|\psi^2 w\partial_s w\|_{L^1(\mathbf{B}_{2R})} \leqq \|\psi w\|_{L^\lambda(\mathbf{B}_{2R})} \times \|\psi\partial_s w\|_{L^{\lambda'}(\mathbf{B}_{2R})}, \quad \frac{1}{\lambda}+\frac{1}{\lambda'}=1, \tag{2.63}$$

with Lemma 2.8, (2.63) and (2.62), we have for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\|\nabla w\|^2_{L^2(\mathbf{B}_R)} \leqq K_{20}(q,r,\varepsilon)s^{\mu_5(q,R,\varepsilon)}\|\psi\partial_s w\|_{L^{\lambda'}(\mathbf{B}_{2R})} + K_{20}(q,r,\varepsilon)s^{b+1}. \tag{2.64}$$

From now, we take $\lambda > 2$ and we choose $\varepsilon\in(0,\varepsilon_0]$ small enough. Observe that $\lambda' > \frac{p+1}{p}$ since $\lambda < p+1$. Let us now bound $\|\psi\partial_s w\|_{L^{\lambda'}(\mathbf{B}_{2R})}$. By using Holder's inequality, we have

$$\|\psi\partial_s w\|_{L^{\lambda'}(B_{2R})} \leqq \|\psi\partial_s w\|^{1-\theta}_{L^2(B_{2R})} \times \|\psi\partial_s w\|^{\theta}_{L^{p_1-\varepsilon}(B_{2R})}, \quad \frac{1}{\lambda'} = \frac{1-\theta}{2}+\frac{\theta}{p_1-\varepsilon}, \tag{2.65}$$

$p_1 = \frac{p+1}{p}$ and where

$$\theta = \frac{(\lambda-2)(p+1-\varepsilon p)}{\lambda(p-1+\varepsilon p)} \in (0,1). \tag{2.66}$$

Putting (2.64) and (2.65) together, we get for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\begin{aligned}\|\nabla w\|^2_{L^2(\mathbf{B}_R)} \leqq & K_{20}(q,R,\varepsilon)s^{\mu_5(q,R,\varepsilon)}\|\psi\partial_s w\|^{1-\theta}_{L^2(\mathbf{B}_{2R})} \times \|\psi\partial_s w\|^{\theta}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})}\\ &+ K_{20}(q,R,\varepsilon)s^{b+1}.\end{aligned} \tag{2.67}$$

By integrating inequality (2.67) between s and $s+1$, we obtain for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\begin{aligned}\int_s^{s+1} \|\nabla w(\tau)\|^{2\tilde{q}}_{L^2(\mathbf{B}_R)}\mathrm{d}\tau \leqq & K_{21}(q,R,\varepsilon)s^{\mu_6(q,R,\varepsilon)\tilde{q}}\\ & \underbrace{\int_s^{s+1}\|\psi\partial_s w\|^{\tilde{q}(1-\theta)}_{L^2(\mathbf{B}_{2R})} \times \|\psi\partial_s w\|^{\tilde{q}\theta}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})}\mathrm{d}\tau}_{\Gamma(s)}\\ & + K_{21}(q,R,\varepsilon)s^{(b+1)\tilde{q}},\end{aligned} \tag{2.68}$$

for some $\tilde{q} > q$. Let $\alpha = \frac{2}{(1-\theta)\tilde{q}}$ and use Holder's inequality in time, we obtain for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\Gamma(s) \leqq \left(\int_s^{s+1} \|\psi\partial_s w\|^2_{L^2(\mathbf{B}_{2R})} \mathrm{d}\tau\right)^{\frac{1}{\alpha}} \left(\int_s^{s+1} \|\psi\partial_s w\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau\right)^{\frac{1}{\alpha'}}, \quad \frac{1}{\alpha}+\frac{1}{\alpha'}=1. \tag{2.69}$$

From the inequalities (2.40), (2.68) and (2.69), we infer that for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\int_s^{s+1} \|\nabla w(\tau)\|^{2\tilde{q}}_{L^2(\mathbf{B}_R)} \mathrm{d}\tau \leqq K_{22}(q,R,\varepsilon) s^{\mu_6(q,R,\varepsilon)\tilde{q}} \left(\int_s^{s+1} \|\psi\partial_s w\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau\right)^{\frac{1}{\alpha'}} + K_{22}(q,R,\varepsilon) s^{(b+1)\tilde{q}}. \tag{2.70}$$

Equipped with the arguments presented in the proof of Lemmas 6.5 and 6.6 in [9] and by exploiting Corollary 3, it is straightforward to get, for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau \leqq K_{23}(q,R,\varepsilon) \int_s^{s+1} \left\| \frac{1}{\tau^a} |w|^p \log^a(2+\psi^2 w^2)\right\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau + K_{23}(q,R,\varepsilon) s^{b+1}. \tag{2.71}$$

By combining (2.71), (B.7) and the identity $e^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}} |f(\phi w)| = \frac{1}{s^a}|w|^p \log^a(2+\phi^2 w^2)$, we deduce that for all $s \geqq \max(-\log T, \tilde{S}_3)$,

$$\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau \leqq K_{24}(q,R,\varepsilon) \int_s^{s+1} \left\||w|^{p+\tilde{\varepsilon}}\right\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau + K_{24}(q,R,\varepsilon) s^{b+1}, \tag{2.72}$$

where $\tilde{\varepsilon} = \frac{p(p-1)\varepsilon}{p+1-\varepsilon p}$. Therefore,

$$\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau \leqq K_{24}(q,R,\varepsilon) \int_s^{s+1} \left(\int_{\mathbf{B}_{2R}} |w|^{p+1-\varepsilon} \mathrm{d}y\right)^{\frac{p\tilde{q}\theta\alpha'}{p+1-\varepsilon p}} \mathrm{d}\tau + K_{24}(q,R,\varepsilon) s^{b+1}. \tag{2.73}$$

Using together (2.51) and (2.73), we obtain

$$\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau \leqq K_{25}(q,R,\varepsilon) \int_s^{s+1} \|\nabla w\|^{\frac{2p\tilde{q}\theta\alpha'}{p+1-\varepsilon p}}_{L^2(\mathbf{B}_{4R})} d\tau + K_{25}(q,R,\varepsilon)\left(s^{\frac{(b+1)p\tilde{q}\theta\alpha'}{p+1-\varepsilon p}} + s^{b+1}\right).$$

By Proposition 6.4 in [9], we have $\frac{2p\tilde{q}\theta\alpha'}{p+1-\varepsilon_1 p} < 2q$, (for ε_1 small enough) for all $\tilde{q} \in [q, q+\frac{2}{p+1}]$. Then, by using the inequality, for all $r \in [1, 2q]$, $X^r \leqq C + CX^{2q}$, for all $X > 0$, we write for all $s \geqq \max(-\log T, \tilde{S}_3)$, for all $\varepsilon \in (0, \varepsilon_1]$,

$$\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1-\varepsilon}(\mathbf{B}_{2R})} \mathrm{d}\tau \leqq K_{26}(q,R,\varepsilon) \int_s^{s+1} \|\nabla w\|^{2q}_{L^2(\mathbf{B}_{4R})} \mathrm{d}\tau + K_{26}(q,R,\varepsilon) s^{2q(b+1)}. \tag{2.74}$$

From (2.70) and (2.74), we have for all $s \geqq \max(-\log T, \tilde{S_3})$,

$$\int_s^{s+1} \|\nabla w(\tau)\|^{2\tilde{q}}_{L^2(\mathbf{B}_R)} \mathrm{d}\tau \leqq K_{27}(q, R, \varepsilon) s^{\mu_7(q,R,\varepsilon)\tilde{q}} \left(\int_s^{s+1} \|\nabla w\|^{2q}_{L^2(\mathbf{B}_{4R})} d\tau \right)^{\frac{1}{\alpha'}} + K_{27}(q, R, \varepsilon) s^{\mu_8(R,\varepsilon,q,\tilde{q})}. \tag{2.75}$$

Therefore, estimates (2.60) and (2.75) lead to the following:

$$\int_s^{s+1} \|\nabla w(\tau)\|^{2\tilde{q}}_{L^2(\mathbf{B}_R)} \mathrm{d}\tau \leqq K_{28}(q, R, \varepsilon) s^{\mu_9(q,R,\varepsilon,q,\tilde{q})}. \tag{2.76}$$

Thus, inequality (2.54) is valid for all $\tilde{q} \in [q, q + \frac{2}{p+1}]$. Repeating this argument, we would obtain that (2.54) holds for all $q \geqq 2$. This concludes the proof of Proposition 2.5. □

2.4. A Polynomial Bound for the $H^1(\mathbf{B}_R)$ Norm of Solution of Equation (1.18)

Based on Proposition 2.5, we are in position to derive a polynomial bound for the $H^1(\mathbf{B}_R)$ norm. More precisely, the aim of this subsection is to prove Proposition 2.1,

Proof of Proposition. 2.1. First, we use (2.40), Proposition 2.5 and apply Lemma A.1 with $\alpha = q(p - \frac{\varepsilon}{2} + 1)$, $\beta = p - \frac{\varepsilon}{2} + 1$, $\gamma = \delta = 2$ to get that, for all $s \geqq \max(-\log T, \tilde{S_3})$,

$$\|w(s)\|_{L^\lambda(\mathbf{B}_R)} \leqq K_{29}(q, R, \varepsilon) s^{\mu_{10}(q,R,\varepsilon)}, \quad \forall \lambda < p - \frac{\varepsilon}{2} + 1 - \frac{p - \frac{\varepsilon}{2} - 1}{q+1}, \quad \forall \varepsilon \in (0, \varepsilon_1], \tag{2.77}$$

where $\tilde{S_3} = \max(S_3, S_5)$. Clearly, there exists $\varepsilon_2 = \varepsilon_2(p, N, q) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_2]$, we have $q = \frac{2p-\varepsilon}{\varepsilon} - 1 \geqq 2$. Therefore, for all $\varepsilon \in (0, , \varepsilon_2]$, for all $s \geqq \max(-\log T, \tilde{S_3})$ we have

$$\int_{\mathbf{B}_R} |w(y, s)|^{p+1-\varepsilon} \mathrm{d}y \leqq K_{30}(\varepsilon, R) s^{\mu_{11}(R,\varepsilon)}. \tag{2.78}$$

□

We are now ready to Control of ∇w in $L^2(\mathbf{B}_R)$. In fact, we use the Gagliardo-Nirenberg inequality in order to claim the following:

Lemma 2.9. *There exists $\varepsilon_3 = \varepsilon_3(p, N) \in (0, \varepsilon_2]$ such that, for all $\varepsilon \in (0, \varepsilon_3]$, for all $s \geqq \max(-\log T, \tilde{S_3})$ we have*

$$\int_{\mathbb{R}^N} \psi^2 |w(y, s)|^{p+1+\varepsilon} \mathrm{d}y \leqq K_{31}(R, \varepsilon) s^{\mu_{12}(R,\varepsilon)} \Big(\int_{\mathbb{R}^N} \psi^2 |\nabla w(y, s)|^2 \mathrm{d}y \Big)^\beta + K_{31}(R, \varepsilon) s^{\mu_{12}(R,\varepsilon)}, \tag{2.79}$$

where $\beta = \beta(p, N, \varepsilon) \in (0, 1)$ and $\mu_{12} = \mu_{12}(R, \varepsilon) > 0$.

Proof. Let $\varepsilon \in (0, \varepsilon_2)$. By interpolation, we write

$$\int_{\mathbb{R}^N} \psi^2 |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leqq \Big(\int_{\mathbb{R}^N} \psi^{\nu} |w(y,s)|^{p+1-\varepsilon} \mathrm{d}y \Big)^{\eta} \Big(\int_{\mathbb{R}^N} |\psi w(y,s)|^r \mathrm{d}y \Big)^{1-\eta}, \quad (2.80)$$

where

$$\nu = 2\frac{r(1-\varepsilon)-(p+1+\varepsilon)}{r-(p+1+\varepsilon)}, \quad \eta = \frac{r-(p+1+\varepsilon)}{r-(p+1-\varepsilon)},$$

where

$$r = \begin{cases} \frac{2N}{N-2}, & if\, N \geqq 3, \\ p+2, & if\, N = 2, \end{cases} \quad (2.81)$$

and where $\varepsilon < r - p - 1$. Exploiting the fact that there exists $\tilde{\varepsilon}_2 = \tilde{\varepsilon}_2(p, N) \in (0, r-p-1)$ small enough such that for $\varepsilon \in (0, \tilde{\varepsilon}_2]$, we have $\nu = \nu(p, \varepsilon) \in [1, 2)$. Therefore, by using the properties of ψ given by (2.49) and the estimate (2.78) we get

$$\int_{\mathbb{R}^N} \psi^{\mu} |w(y,s)|^{p+1-\varepsilon} \mathrm{d}y \leqq \int_{\mathbf{B}_{2R}} |w(y,s)|^{p+1-\varepsilon} \mathrm{d}y \leqq K_{30}(\varepsilon, 2R) s^{\mu_{11}(\varepsilon,2R)}. \quad (2.82)$$

Thanks to (2.80), (2.82) and the Sobolev embedding, we conclude

$$\int_{\mathbb{R}^N} \psi^2 |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leqq K_{32}(R,\varepsilon) s^{\mu_{13}(\varepsilon,R)} \Big(\int_{\mathbb{R}^N} |\nabla\big(\psi w(y,s)\big)|^2 \mathrm{d}y \Big)^{\beta}, \quad (2.83)$$

where

$$\beta = \frac{r\varepsilon}{r-(p+1-\varepsilon)}.$$

Note that, by exploiting the inequality $|\nabla\big(\psi w\big)|^2 \leqq 2\psi^2 |\nabla w|^2 + 2|\nabla \psi|^2 w^2$, the properties of ψ given by (2.49) and the fact that $\|\nabla \psi\|_{L^\infty} \leqq C$, we obtain

$$\int_{\mathbb{R}^N} |\nabla\big(\psi w(y,s)\big)|^2 \mathrm{d}y \leqq C \int_{\mathbb{R}^N} \psi^2 |\nabla w(y,s)|^2 \mathrm{d}y + C \int_{\mathbf{B}_{2R}} w^2(y,s) \mathrm{d}y \quad (2.84)$$

From (2.83), (2.84)and (2.42), we conclude

$$\int_{\mathbb{R}^N} \psi^2 |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leqq K_{33}(\varepsilon, R) s^{\mu_{14}(\varepsilon,R)} \Big(\int_{\mathbb{R}^N} \psi^2 |\nabla w(y,s)|^2 \mathrm{d}y \Big)^{\beta} + K_{33}(\varepsilon, R) s^{\mu_{14}(\varepsilon,R)}. \quad (2.85)$$

Now, if $\varepsilon_3 \leqq \tilde{\varepsilon}_2$ is chosen small enough such that $\beta = \frac{r\varepsilon_3}{r-(p+1-\varepsilon_3)} \in (0, 1)$, the estimate (2.85) implies (2.79). This ends the proof of Lemma 2.9.

Proof of Proposition 2.1:. From (2.50), the definition (2.48) of the local functional: $\mathscr{E}_\psi(w(s), s)$, we see that for all $s \geqq \max(-\log T, S_4)$,

$$\int_{\mathbb{R}^N} \psi^2 |\nabla w|^2 \rho(y) \mathrm{d}y \leqq 2 \int_{\mathbb{R}^N} e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} \psi^2 F(\phi w) \rho(y) \mathrm{d}y + 2K_{13}(R) s^{b+1}. \tag{2.86}$$

Thanks to (B.9) and (2.86) and the fact that $\rho(2R) \leqq \rho(y) \leqq 1$, for all $y \in B_{2R}$, we conclude for all $s \geqq \max(-\log T, S_4)$

$$\int_{\mathbb{R}^N} \psi^2 |\nabla w|^2 \mathrm{d}y \leqq K_{34}(R, \varepsilon) \int_{\mathbb{R}^N} \psi^2 |w(y, s)|^{p+\varepsilon+1} \mathrm{d}y + K_{34}(R, \varepsilon) s^{b+1}. \tag{2.87}$$

According to (2.87) together with Lemma 2.9 in the particular case when $\varepsilon = \varepsilon_3$, we have for all $s \geqq \max(-\log T, \tilde{S_3})$

$$\int_{\mathbb{R}^N} \psi^2 |\nabla w|^2 \mathrm{d}y \leqq K_{35}(R, \varepsilon_3) s^{\mu_{12}(R,\varepsilon_3)} \Big(\int_{\mathbb{R}^N} \psi^2 |\nabla w|^2 \mathrm{d}y \Big)^\beta + K_{35}(R, \varepsilon_3) s^{\mu_{12}(R,\varepsilon_3)}, \tag{2.88}$$

where $\beta = \beta(p, N, \varepsilon_3) \in (0, 1)$. It suffices to combine (2.88) and the fact that $\beta < 1$, to obtain that for all $s \geqq \max(-\log T, \tilde{S_3})$

$$\int_{\mathbb{R}^N} \psi^2 |\nabla w|^2 \mathrm{d}y \leqq K_{36}(R, \varepsilon_3) s^{\frac{\mu_{12}(R,\varepsilon_3)}{1-\beta}}. \tag{2.89}$$

Clearly, by combining (2.89), (2.42) and (2.49), we conclude (2.1), where $\mu = \frac{\mu_{12}(R,\varepsilon_3)}{2-2\beta}$, which yields the conclusion of Proposition 2.1. □

3. Proof of Theorems 1 and 2

In this section, thanks to polynomial estimate obtained in Proposition 2.1, we prove Theorems 1 and 2 here. This section is divided into two parts:

- In subsection 3.1, we prove Theorem 1. More precisely, based upon Proposition 2.1, we construct a Lyapunov functional for equation (1.18) and a blow-up criterion involving this functional.
- In subsection 3.2, we prove Theorem 2.

3.1. A Lyapunov Functional

In this subsection, our aim is to construct a Lyapunov functional for equation (1.18). Note that this functional is far from being trivial and makes our main contribution. More precisely, thanks to the rough estimate obtained in the Proposition 2.1, we derive here that the functional $L(w(s), s)$ defined in (1.26) is a decreasing functional of time for equation (1.18), provided that is s large enough.

Let us remark that in Section 2, we construct a Lyapunov functional $N_{m_0}(w(s), s)$ defined in (2.24), but we obtain just a rough estimate because the multiplier is not bounded. Nevertheless, the multiplier related to the functional $L(w(s), s)$ is bounded. Then, as we said above, the natural energy $E(w(s), s)$ defined in (2.40) is a small perturbation of $L(w(s), s)$.

In order to prove that the functional $L(w(s), s)$ is a Lyapunov functional, we start by using the additional information obtained in Section 2, to write several useful lemmas which play key roles in our analysis. More precisely, we start by stating the following:

Lemma 3.1. *For all $r \in [2, 2^*)$, for all $s \geqq \widehat{s_1} = \max(-\log T, \widehat{S_1})$, we have*

$$\int_{\mathbb{R}^N} |w(y,s)|^r \rho(y)\mathrm{d}y \leqq M_1 s^{\sigma r}, \tag{3.1}$$

where $\sigma = \mu(a, p, N, \frac{1}{2})$, M_1 depends on p, a, N, r and $\|w(\widehat{s_1})\|_{H^1}$ and where $2^* = \frac{2N}{N-2}$, if $N \geqq 3$ and $2^* = \infty$, if $N = 2$.

Throughout the proof we employ the following notations:

The ball in $\mathbb{R}^N$ with radius R around the point z is denoted $\mathbf{D}(z, R) = \{x \in \mathbb{R}^N, \|x-z\|_\infty \leqq R\}$, where the infinity norm is given by the formula $\|x\|_\infty = \sup\limits_{1\leqq i\leqq N} |x_i|$. Also, the ball in $\mathbb{R}^N$ with radius R around the point z is denoted $\mathbf{B}(z, R) = \{x \in \mathbb{R}^N, |x - z| \leqq R\}$, where the norm is given by $|x| = \sqrt{\sum_{i=1}^{N} x_i^2}$. Finally, let us recall that theses norms on $\mathbb{R}^N$ are equivalent. In fact, we have

$$\|x\|_\infty \leqq |x| \leqq \sqrt{N}\|x\|_\infty, \qquad \forall x \in \mathbb{R}^N. \tag{3.2}$$

Proof. In order to obtain the estimate (3.1), we combine a covering technique and the result obtained in Proposition 2.1.

First, we claim that $\mathbb{R}^N = \cup_{z\in Z^N} \mathbf{D}(z, \frac{1}{2})$ and the sequence $\left(\mathbf{D}(z, \frac{1}{2})\right)_{z\in Z^N}$ are arbitrary pairwise sets are negligible. Let $r \in [2, 2^*]$. As an immediate consequence, we write

$$\begin{aligned}\int_{\mathbb{R}^N} |w_{x_0}(y,s)|^r \rho(y)\mathrm{d}y &= \sum_{z\in Z^N} \int_{\mathbf{D}(z,\frac{1}{2})} |w_{x_0}(y,s)|^r \rho(y)\mathrm{d}y \\ &\leqq \sum_{z\in Z^N} \Big(\sup_{y\in\mathbf{D}(z,\frac{1}{2})} \rho(y)\Big) \int_{\mathbf{D}(z,\frac{1}{2})} |w_{x_0}(y,s)|^r \mathrm{d}y. \end{aligned} \tag{3.3}$$

Note that using the definition (1.17) of w_{x_0}, we see that

$$\text{for all } \ y, z \in \mathbb{R}^N, w_{x_0}(y+z, s) = w_{x_0+ze^{-s/2}}(y, s) \tag{3.4}$$

From (3.2) and (3.4), for all $z \in \mathbb{R}^N$, $s \geqq \widehat{s}_1 = \max(-\log T, , \widehat{S}_1)$

$$\int_{\mathbf{D}(z,\frac{1}{2})} |w_{x_0}(y,s)|^r \mathrm{d}y \leqq \int_{\mathbf{B}(z,\frac{\sqrt{N}}{2})} |w_{x_0}(y,s)|^r \mathrm{d}y = \int_{\mathbf{B}(0,\frac{\sqrt{N}}{2})} |w_{x_0}(y+z,s)|^r \mathrm{d}y$$
$$= \int_{\mathbf{B}(0,\frac{\sqrt{N}}{2})} |w_{x_0+ze^{-s/2}}(y,s)|^r \mathrm{d}y. \quad (3.5)$$

Thanks to (2.2) and (3.5), we have for all $z \in \mathbb{R}^N$, $s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1)$

$$\int_{\mathbf{D}(z,\frac{1}{2})} |w_{x_0}(y,s)|^r \mathrm{d}y \leqq M_2 s^{r\sigma}, \quad (3.6)$$

where $\sigma = \mu(a, p, N, \frac{1}{2})$ and where M_2 depends on p, a, N and $\|w(\widehat{s}_1)\|_{H^1}$. By exploiting (3.6) and (3.3), we have for all $x_0, z \in \mathbb{R}^N$, $s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1)$

$$\int_{\mathbb{R}^N} |w_{x_0}(y,s)|^r \rho(y) \mathrm{d}y \leqq M_2 s^{r\mu} \sum_{z\in Z^N} \sup_{y\in\mathbf{D}(z,\frac{1}{2})} \rho(y). \quad (3.7)$$

To complete the proof, it remains to control the right-hand side of (3.7). More precisely, the term $\sum_{z\in Z^N} \sup_{y\in\mathbf{D}(z,\frac{1}{2})} \rho(y)$. Using the fact that for all $z \in \mathbb{R}^N$, for all $y \in \mathbf{D}(z, \frac{1}{2})$, we have

$$\|z\|_\infty \leqq \|y\|_\infty + \|y-z\|_\infty \leqq \|y\|_\infty + \frac{1}{2}. \quad (3.8)$$

Therefore, by using the basic inequality $(a+b)^2 \leqq 2a^2 + 2b^2$, for all $a, b > 0$, we set

$$\|z\|_\infty^2 \leqq \big(\|y\|_\infty + \frac{1}{2}\big)^2 \leqq 2\|y\|_\infty^2 + \frac{1}{2}. \quad (3.9)$$

In view of (3.9), (3.2), we have, for all $z \in \mathbb{R}^N$, for all $y \in \mathbf{D}(z, \frac{1}{2})$, we have

$$|y|^2 \geqq \|y\|_\infty^2 \geqq \frac{1}{2}\|z\|_\infty^2 - \frac{1}{4} \geqq \frac{1}{2N}|z|^2 - \frac{1}{4}. \quad (3.10)$$

Due to (3.10) and to the definition of ρ given by (1.19), we conclude for all $z \in \mathbb{R}^N$,

$$\sup_{y\in\mathbf{D}(z,\frac{1}{2})} \rho(y) \leqq Ce^{-\frac{|z|^2}{8N}}. \quad (3.11)$$

Thank to (3.11), we get

$$\sum_{z\in Z^N} \sup_{y\in\mathbf{D}(z,\frac{1}{2})} \rho(y) \leqq C \sum_{z\in Z^N} e^{-\frac{|z|^2}{8N}} \leqq C \prod_{i=1}^{N} \sum_{z_i\in Z} e^{-\frac{z_i^2}{8N}} \leqq C. \quad (3.12)$$

By combining (3.12) and (3.7), we easily obtain (3.1). This concludes the proof of Lemma 3.1.

Thanks to of Lemma 3.1, we are in position to state the following:

Lemma 3.2. *For all* $s \geqq \widehat{s_1} = \max(-\log T, \widehat{S}_1)$*, we have*

$$\int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\log(2+w^2)\rho(y)\mathrm{d}y \leq M_3 s^{\frac{1}{4}} \int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + M_3 s^{a+\frac{1}{4}}, \tag{3.13}$$

where, M_3 *depends on* p, a, N *and* $\|w(\widehat{s_1})\|_{H^1}$.

Remark 3.1. Let us mention that, in the first term on the right-hand side the choice of the power $\frac{1}{4}$ is not optimal. In fact, with the same proof, one can show the same estimate with the power ν, for any $\nu > 0$, instead of the power $\frac{1}{4}$. Let us denote that, we can construct a Lyapunov functional, when we have the estimate above for some power ν such that $\nu \in (0,1)$ instead of the power $\frac{1}{4}$.

Proof. Let $\varepsilon \in (0,1)$. By using the inequality $\log(2+z^2) \leqq C + |z|^{\varepsilon^2}$, for all $z \in \mathbb{R}$, we conclude that

$$\int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\log(2+w^2)\rho(y)\mathrm{d}y \leq C\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + \int_{\mathbb{R}^N} |w|^{p+1+\varepsilon^2}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y. \tag{3.14}$$

Furthermore, we apply the interpolation in Lebesgue spaces to get

$$\int_{\mathbb{R}^N} |w|^{p+1+\varepsilon^2}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \leqq \Big(\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y\Big)^{1-\varepsilon} \Big(\int_{\mathbb{R}^N} |w|^{p+1+\varepsilon}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y\Big)^{\varepsilon}. \tag{3.15}$$

By combining(B.4), (B.9) and the inequality $|z|^{p+1+\varepsilon} \leqq 1 + |z|^{p+1+2\varepsilon}$, for all $z \in \mathbb{R}$, we obtain

$$\frac{1}{s^a}\int_{\mathbb{R}^N} |w|^{p+1+\varepsilon}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \leqq C + C\int_{\mathbb{R}^N} |w|^{p+1+2\varepsilon}\rho(y)\mathrm{d}y. \tag{3.16}$$

Since $p < p_S = \frac{N+2}{N-2}$, we then choose ε_4 small enough, such that for all $\varepsilon \in (0,\varepsilon_4]$ we have $p+1+2\varepsilon < 2^*$ where $2^* = \frac{2N}{N-2}$, if $N \geqq 3$ and $2^* = \infty$, if $N = 2$. Therefore, estimate (3.1) implies that, for all $s \geqq \widehat{s_1} = \max(-\log T, \widehat{S}_1)$, for all $\varepsilon \in [0,\varepsilon_4]$,

$$\int_{\mathbb{R}^N} |w|^{p+1+2\varepsilon}\rho(y)\mathrm{d}y \leqq \int_{\mathbb{R}^N} |w|^{p+1}\rho(y)\mathrm{d}y + \int_{\mathbb{R}^N} |w|^{p+1+2\varepsilon_4}\rho(y)\mathrm{d}y \leqq M_4 s^{\sigma_3}, \tag{3.17}$$

where σ_3 depends on p, a, N, ε_4 and M_4 depends on p, a, N, ε_4 and $\|w(\widehat{s_1})\|_{H^1}$.

By combining (3.15), (3.16) and (3.17), we deduce that, for all $s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1)$, for all $\varepsilon \in (0, \varepsilon_4]$.

$$\int_{\mathbb{R}^N} |w|^{p+1+\varepsilon^2} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \leqq M_5 s^{(\sigma_3+a)\varepsilon}\Big(\int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y\Big)^{1-\varepsilon}. \tag{3.18}$$

Thanks to the basic inequality $|a_1|^\nu |a_2|^{1-\nu} \leqq C|a_1| + C|a_2|$, for all $a_1, a_2 \in \mathbb{R}$, for all $\nu \in (0,1)$, we conclude that, for all $s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1)$, for all $\varepsilon \in (0, \varepsilon_4]$,

$$\int_{\mathbb{R}^N} |w|^{p+1+\varepsilon^2} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \leqq M_6 s^{\sigma_3\varepsilon}\Big(s^a + \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y\Big). \tag{3.19}$$

Now, we choose $\varepsilon_5 \in (0, \varepsilon_4]$, such that $\sigma_3\varepsilon_5 < \frac{1}{4}$. Then, by (3.14) and (3.19), we easily obtain (3.13). This concludes the proof of Lemma 3.2.

Thanks to estimate (3.13), we can improve the estimate (2.8) related to the control of the time derivative of the functional $E(w(s), s)$. More precisely, we prove the following lemma:

Lemma 3.3. *There exists $\widehat{S}_2 > \widehat{S}_1$ such that for all $s \geqq \widehat{s}_2 = \max(-\log T, \widehat{S}_2)$, we have*

$$\begin{aligned}\frac{d}{ds}E(w(s),s) \leqq &-\frac{1}{2}\int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y)\mathrm{d}y + \frac{M_7}{s^{a+\frac{7}{4}}}\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \\ &+ \frac{C}{s^2}\int_{\mathbb{R}^N} w^2\rho(y)\mathrm{d}y + \frac{M_7}{s^{\frac{7}{4}}},\end{aligned} \tag{3.20}$$

where, M_7 depends on p, a, N and $\|w(\widehat{s}_1)\|_{H^1}$.

Proof. By using the additional information obtained in (3.13), we are going to refine the estimate related to $\Sigma_1^2(s)$ and $\Sigma_1^3(s)$ defined in (2.10). Let us mention that the estimate (2.12) related to $\Sigma_1^1(s)$ defined in (2.10) is acceptable and does not need any improvement. More precisely, we write

$$\begin{aligned}\Sigma_1^2(s) + \Sigma_1^3(s) = &\frac{p+1}{p-1} e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} \int_{\mathbb{R}^N} \Big(F(\phi w) - \frac{\phi(s) w f(\phi w)}{p+1}\Big)\rho(y)\mathrm{d}y \\ &- \frac{2a}{p-1} e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}-1} \int_{\mathbb{R}^N} \Big(F(\phi w) - \frac{\phi w f(\phi w)}{2}\Big)\rho(y)\mathrm{d}y.\end{aligned}$$

We attempt to group the main terms together. A straightforward computations implies that

$$\Sigma_1^2(s) + \Sigma_1^3(s) = \chi_1(s) + \chi_2(s), \tag{3.21}$$

where

$$\chi_1(s) = \frac{a}{(p+1)s^{a+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^{a-1}(2+\phi^2 w^2)\Big(\log(2+\phi^2 w^2) - \frac{2s}{p-1}\Big)\rho(y)\mathrm{d}y, \quad (3.22)$$

$$\chi_2(s) = \frac{e^{-\frac{(p+1)s}{p-1}}}{p-1} s^{\frac{2a}{p-1}} \int_{\mathbb{R}^N} \Big((p+1)F_2(\phi w) - \frac{a}{s}F_1(\phi w) - \frac{a}{s}F_2(\phi w)\Big)\rho(y)\mathrm{d}y, \quad (3.23)$$

where F_1 and F_2 are defined by (2.13) and (2.14).

Note that, in (3.21) we grouped the main terms together. In fact, it is easy to control the terms $\chi_2(s)$. However, the control of the term $\chi_1(s)$ needs the use of the additional information obtained in Lemma 3.2. More precisely, for all $s \geqq \widehat{s}_1 = \max(-\log T, \widehat{S}_1)$, we divide $\mathbb{R}^N$ into two parts

$$A_1(s) = \{y \in B \mid \phi(s)w^2(y,s) \leqq 1\} \text{ and } A_2(s) = \{y \in B \mid \phi(s)w^2(y,s) \geqq 1\}. \quad (3.24)$$

Accordingly, we write $\chi_1(s) = \chi_1^1(s) + \chi_1^2(s)$, where

$$\chi_1^1(s) = \frac{a}{(p+1)s^{a+1}} \int_{A_1(s)} |w|^{p+1} \log^{a-1}(2+\phi^2 w^2)\Big(\log(2+\phi^2 w^2) - \frac{2s}{p-1}\Big)\rho(y),$$

$$\chi_1^2(s) = \frac{a}{(p+1)s^{a+1}} \int_{A_2(s)} |w|^{p+1} \log^{a-1}(2+\phi^2 w^2)\Big(\log(2+\phi^2 w^2) - \frac{2s}{p-1}\Big)\rho(y)\mathrm{d}y.$$

On the one hand, by using the definition of the set $A_1(s)$ given in (3.24), we get, for all $s \geqq \widehat{s}_1$,

$$|w|^{p+1} \log^a(2+\phi^2 w^2) \leqq C\phi^{-\frac{p+1}{2}}(s) \log^{|a|}(2+\phi(s)) \leqq Ce^{-\frac{s}{2}}. \quad (3.25)$$

From (3.25) and the fact that $1 - \frac{2s}{(p-1)\log(2+\phi^2 w^2)} \leqq 1$, we get

$$\chi_1^1(s) \leqq Ce^{-\frac{s}{2}}. \quad (3.26)$$

On the other hand, by using the definition of the $\phi(s)$ given by (1.20), we write the identity

$$\log(2+\phi^2 w^2) - \frac{2s}{p-1} = \log(2\phi^{-2} + w^2) - \frac{2a\log s}{p-1}. \quad (3.27)$$

Now, by using the inequality $\phi(s) \geqq 1$ and (3.27), we write for all for all $s \geqq \widehat{s}_1$,

$$\log(2+\phi^2 w^2) - \frac{2s}{p-1} \leqq \log(2+w^2) + C\log s. \quad (3.28)$$

Also, by using the definition of the set $A_2(s)$ defined in (3.24), we can write for all $s \geqq \widehat{s}_1$, if $y \in A_2(s)$, we have

$$\log(2+\phi^2 w^2) \geqq \log(\phi(s)) \geqq \frac{2s}{p-1} - \frac{a\log s}{p-1}. \quad (3.29)$$

Clearly, the exists $S_2 > S_1$ such that for all $s \geqq S_2$, we have $\frac{2s}{p-1} - \frac{a\log s}{p-1} \geqq \frac{s}{p-1}$. Therefore, by exploiting (3.28) and (3.29) we have for all $s \geqq \widehat{s}_2 = \max(-\log T, \widehat{S}_2)$,

$$\chi_1^2(s) \leqq \frac{C}{s^{a+2}} \int_B |w|^{p+1} \log^a(2+\phi^2 w^2) \log(2+w^2) \rho(y) \mathrm{d}y + \frac{C \log s}{s^{a+2}} \int_B |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y. \quad (3.30)$$

Note that, by using the fact $\chi_1(s) = \chi_1^1(s) + \chi_1^2(s)$, (3.13), (3.26) and (3.30), we get for all $s \geqq \widehat{s}_2 = \max(-\log T, \widehat{S}_2)$,

$$\chi_1(s) \leqq \frac{M_8}{s^{a+\frac{7}{4}}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y + \frac{M_8}{s^{\frac{7}{4}}}. \quad (3.31)$$

Thanks to (B.5) and (B.6), we write

$$\frac{1}{s}|F_1(\phi w)| + |F_2(\phi w)| \leqq C + C\frac{\phi w}{s^2} f(\phi w). \quad (3.32)$$

By (2.10), (3.32) and (2.17), we have, for all $s \geqq \widehat{s}_1$,

$$\chi_2(s) \leqq \frac{C}{s^{a+2}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y + Ce^{-\frac{s}{2}}. \quad (3.33)$$

The result (3.20) derives immediately from (2.10), (2.12), (3.31), (3.33), and the identity (3.21), which ends the proof of Lemma 3.3

With Lemmas 2.3 and 3.3, we are in a position to prove Theorem 1.

Proof of Theorem 1:. By exploiting the defintion of $L_0(w(s), s)$ in (2.4), we can write easily, for all $s \geqq \widehat{s}_2 = \max(-\log T, \widehat{S}_2)$,

$$\frac{d}{ds} L_0(w(s), s) = \frac{d}{ds} E(w(s), s) + \frac{1}{\sqrt{s}} \frac{d}{ds} J(w(s), s) - \frac{1}{2s\sqrt{s}} J(w(s), s), \quad (3.34)$$

where $J(w(s), s) = \frac{1}{s} \int_{\mathbb{R}^N} w^2 \rho(y) \mathrm{d}y$. Lemmas 2.3 and 3.3 allows to prove that for all $s \geqq \widehat{s}_2 = \max(-\log T, \widehat{S}_2)$, we have

$$\begin{aligned}\frac{d}{ds} L_0(w(s), s) \leqq & -\frac{1}{2} \int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y) \mathrm{d}y + \frac{p+3}{2s\sqrt{s}} L_0(w(s), s) \\ & - \frac{1}{s^{a+\frac{3}{2}}} \Big(\frac{p-1}{2(p+1)} - \frac{M_7}{s^{\frac{1}{4}}} - \frac{C}{s} \Big) \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2+\phi^2 w^2) \rho(y) \mathrm{d}y \\ & - \frac{1}{s\sqrt{s}} \Big(\frac{p+1}{2(p-1)} - \frac{C}{\sqrt{s}} \Big) \int_{\mathbb{R}^N} w^2 \rho(y) \mathrm{d}y + \frac{M_7}{s^{\frac{7}{4}}} + Ce^{-s}.\end{aligned}$$

Again, choosing $\widehat{S}_3 > \widehat{S}_2$ large enough, this implies that for all for all $s \geqq \max(-\log T, \widehat{S}_3)$, we have

$$\frac{d}{ds} L_0(w(s), s) \leqq -\frac{1}{2} \int_{\mathbb{R}^N} (\partial_s w)^2 \rho(y) \mathrm{d}y + \frac{p+3}{2s\sqrt{s}} L_0(w(s), s) + \frac{M_9}{s^{\frac{7}{4}}}. \quad (3.35)$$

Recalling that,

$$L(w(s),s) = \exp\Big(\frac{p+3}{\sqrt{s}}\Big)L_0(w(s),s) + \frac{\theta}{s^{\frac{3}{4}}}.$$

we get from straightforward computations

$$\frac{d}{ds}L(w(s),s) = -\frac{p+3}{2s\sqrt{s}}\exp\Big(\frac{p+3}{\sqrt{s}}\Big)L_0(w(s),s) + \exp\Big(\frac{p+3}{\sqrt{s}}\Big)\frac{d}{ds}L_0(w(s),s) - \frac{4\theta}{3s^{\frac{7}{4}}}. \tag{3.36}$$

Therefore, estimates (3.35) and (3.36) lead to the following crucial estimate:

$$\frac{d}{ds}L(w(s),s) \leqq -\frac{1}{2}\exp\Big(\frac{p+3}{\sqrt{s}}\Big)\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y + \Big(M_9\exp\Big(\frac{p+3}{\sqrt{s}}\Big) - \frac{4\theta}{3}\Big)\frac{1}{s^{\frac{7}{4}}}. \tag{3.37}$$

Since we have $1 \leqq \exp\Big(\frac{p+3}{\sqrt{s}}\Big) \leqq \exp\Big(\frac{p+3}{\sqrt{\widehat{S}_3}}\Big)$, we then choose θ large enough, so that $M_9\exp\Big(\frac{p+3}{\sqrt{s}}\Big) - \frac{4\theta}{3} \leqq 0$, which yields, for all $s \geqq s_3 = \max(-\log T, \widehat{S}_3)$,

$$\frac{d}{ds}L(w(s),s) \leqq -\frac{1}{2}\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y.$$

A simple integration between s and $s+1$ ensures the result. This concludes the proof of Theorem 1. □

We now claim the following lemma:

Lemma 3.4. *There exist $M_{10} > 0$ and $\hat{S}_4 \geqq \hat{S}_3$ such that, we have for all $s \geqq \max(\hat{S}_4, -\log T)$*

$$N_{m_0}(w(s),s) \geqq -M_{10}. \tag{3.38}$$

Proof. The argument is the same as the similar part in Proposition 2.4.

3.2. Proof of Theorem 2

As in [9], by combining Theorem 1 and Lemma 3.4 we get the following bounds:

Corollary 4. *For all $s \geqq \max(-\log T, \widehat{S}_4)$, we have*

$$-M_{11} \leqq L(w(s),s) \leqq M_{11}, \tag{3.39}$$

$$\int_s^{s+1}\int_{\mathbb{R}^N}\Big(|\nabla w|^2 + (\partial_s w)^2 + w^2\Big)\rho(y)\mathrm{d}y\mathrm{d}\tau \leqq M_{12}, \tag{3.40}$$

$$\frac{1}{s^a}\int_s^{s+1}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\mathrm{d}\tau \leqq M_{13}. \tag{3.41}$$

$$\int_{\mathbb{R}^N} w^2\rho(y)\mathrm{d}y \leqq M_{14}, \tag{3.42}$$

$$\frac{1}{s^a}\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y \leqq C\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y + M_{15}, \tag{3.43}$$

$$\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y \leqq C\sqrt{\int_{\mathbb{R}^N}(\partial_s w)^2\rho(y)\mathrm{d}y} + M_{16}, \tag{3.44}$$

$$\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y \leqq \frac{C}{s^a}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y + M_{17}, \tag{3.45}$$

$$\int_s^{s+1}\Big(\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y\Big)^2 \leqq M_{18}, \tag{3.46}$$

$$\frac{1}{s^{2a}}\int_s^{s+1}\Big(\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\Big)^2\mathrm{d}\tau \leqq M_{19}, \tag{3.47}$$

where $M_{11}, M_{12}, M_{13}, ...M_{19}$ *depend on* $p, a, N, s_3 = \max(-\log T, \widehat{S}_3)$ *and* $\|w(s_3)\|_{H^1}$.

Let us denote that, the estimates obtained in the above corollary are similar to the Corollary (3) except for the presence of the term $K_i s^{b+1}$ instead of M_i. Consequently, following the proof of Proposition 2.5 line by line we are in position to prove the following:

Proposition 3.5. *For all* $q \geqq 2$, $\varepsilon > 0$ *and* $R > 0$ *there exist* $\varepsilon_6 = \varepsilon_6(q, R) > 0$, *there exists a time* $\widehat{S}_5(q, R, \varepsilon) \geqq \widehat{S}_4$, *such that for all* $s \geqq \max(-\log T, \widehat{S}_5)$, *we have*

$$(E_{q,R,\varepsilon}) \quad \int_s^{s+1}\|w(\tau)\|^{(p-\varepsilon+1)q}_{L^{p-\varepsilon+1}(\mathbf{B}_R)}\mathrm{d}\tau \leqq M_{20}(q, R, \varepsilon),$$

where $M_{20}(q, R, \varepsilon)$ *depends on* $p, a, N, q, R, \varepsilon, s_3 = \max(-\log T, \widehat{S}_3)$ *and* $\|w(s_3)\|_{H^1}$.

Finally, we are in position to prove Theorem 2 by exploiting Lemma A.1 and Lemma A.2.

Proof of Theorem 2. First, we use (3.40), Proposition 3.5 and apply Lemma A.1 with $\alpha = q(p - \frac{\varepsilon}{2} + 1)$, $\beta = p - \frac{\varepsilon}{2} + 1$, $\gamma = \delta = 2$ to get that, for all $s \geqq \max(-\log T, \widehat{S}_5)$,

$$\|w(s)\|_{L^\lambda(\mathbf{B}_R)} \leqq M_{21}(q, R, \varepsilon), \quad \forall\lambda < p - \frac{\varepsilon}{2} + 1 - \frac{p - \frac{\varepsilon}{2} - 1}{q+1}, \quad \forall\varepsilon \in (0, p-1), \quad \forall q \geqq 2. \tag{3.48}$$

Hence, for all $\varepsilon \in (0, p-1)$, we have $q = \frac{4p-4-\varepsilon}{\varepsilon} \geqq 2$. Therefore, the estimate (3.48) implies

$$\sup_{\tau\in[s,s+1]}\|w(\tau)\|_{L^{p+1-\varepsilon}(\mathbf{B}_R)} \leqq M_{22}(R, \varepsilon)., \qquad \forall\varepsilon \in (0, p-1). \tag{3.49}$$

Let us recall the equation in w:

$$\partial_s w = \Delta w - \frac{1}{2}y.\nabla w - \frac{1}{p-1}(1-\frac{a}{s})w + e^{-\frac{ps}{p-1}}s^{\frac{a}{p-1}}f(\phi(s)w), \tag{3.50}$$

where $\phi(s)$ and f are given in (1.20) and (1.2).

We now apply Lemma A.2 to w, with $b = b(y) = \frac{1}{2}y$ and

$$H(y,s,w) = -\frac{1}{p-1}(1-\frac{a}{s})w + e^{-\frac{ps}{p-1}}s^{\frac{a}{p-1}}f(\phi(s)w).$$

From (B.7), we see that, for all $\varepsilon \in (0, p-1)$, we have

$$|H(y,s,w)| \leqq C(\varepsilon)(|w|^{p-1+\varepsilon}+1)(|w|+1), \quad \forall s \geqq \max(-\log T, \widehat{S}_5).$$

Let $\lambda_1 = p+1-\varepsilon$, $\alpha_1 = \frac{\lambda_1}{p-1+\varepsilon}$ and $\beta_1 = \frac{1}{\varepsilon}$. Thus, the first identity in (A.2) holds with $g(y,s,w) = C(\varepsilon)(|w(y,s)|^{p-1+\varepsilon}+1)$. Since $p < \frac{N+2}{N-2}$, then we can choose $\varepsilon_7 \leqq \varepsilon_6$ small enough, such that the conditions $\frac{1}{\beta_1} + \frac{N}{2\alpha_1} < 1$ and $\alpha_1 \geqq 1$ hold. Moreover, for all $s \geqq \max(-\log T, \widehat{S}_5)$ we have

$$\begin{aligned}\int_s^{s+1} \|g(\tau)\|^{\beta_1}_{L^{\alpha_1}(\mathbf{B}_R)} \mathrm{d}\tau &\leqq C + C\int_s^{s+1}\left(\int_{\mathbf{B}_R} |w(y,\tau)|^{\lambda_1} dy\right)^{\frac{1}{\varepsilon_7\alpha_1}} \mathrm{d}\tau \\ &\leqq C + C\Big(\sup_{\tau\in[s,s+1]} \|w(\tau)\|_{L^{\lambda_1}(\mathbf{B}_R)}\Big)^{\frac{p-1+\varepsilon_7}{\varepsilon_7}}. \end{aligned}\tag{3.51}$$

By exploiting (3.51) and (3.49), we deduce that

$$\int_s^{s+1} \|g(\tau)\|^{\beta_1}_{L^{\alpha_1}(\mathbf{B}_R)} \mathrm{d}\tau \leqq M_{23}(R, \varepsilon_7). \tag{3.52}$$

Then the second condition in (A.2) holds. Therefore,

$$\|w(s)\|_{L^\infty(\mathbf{B}_{\frac{R}{4}})} \leqq M_{24}(R), \quad \forall s \geqq \max(\tau_0 - \log T, \tau_0 + \widehat{S}_5)), \tag{3.53}$$

for some $\tau_0 \in (0,1)$. By (3.53), we write

$$|w_{x_0}(0,s)| \leqq M_{25}, \quad \forall s \geqq \max(\tau_0 - \log T, \tau_0 + \widehat{S}_5), \tag{3.54}$$

for some $\tau_0 \in (0,1)$. From the fact that the above estimate is independent of x_0 and the definition of w_{x_0} given by (1.17), we infer

$$|w(y,s)| \leqq M_{25}, \quad \forall y \in \mathbb{R}^N \quad \forall s \geqq \max(1 - \log T, 1 + \widehat{S}_5). \tag{3.55}$$

This concludes the proof of Theorem 2. □

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A. Appendix

We recall the interpolation result from Cazenave and Lions [1] and the interior regularity theorem in [6].

Lemma A.1. (Interpolation technique, Cazenave and Lions [1]) *Let* $t_0 > 0$. *Assume that*

$$v \in L^\alpha \left([t_0, t_0+1]; L^\beta(\mathbf{B}_R)\right), \ \partial_t v \in L^\gamma \left([t_0, t_0+1]; L^\delta(\mathbf{B}_R)\right)$$

for some $1 < \alpha, \beta, \gamma, \delta < \infty$. *Then*

$$v \in \mathscr{C}\left([t_0, t_0+1]; L^\lambda(\mathbf{B}_R)\right)$$

for all $\lambda < \lambda_0 = \frac{(\alpha+\gamma')\beta\delta}{\gamma'\beta+\alpha\delta}$ *with* $\gamma' = \frac{\gamma}{\gamma-1}$, *and satisfies*

$$\sup_{t\in[t_0,t_0+1]} \|v(t)\|_{L^\lambda(\mathbf{B}_R)} \leqq C \int_{t_0}^{t_0+1} \left(\|v(\tau)\|^\alpha_{L^\beta(\mathbf{B}_R)} + \|\partial_\tau v(\tau)\|^\gamma_{L^\delta(\mathbf{B}_R)}\right) \mathrm{d}\tau$$

for $\lambda < \lambda_0$. *The positive constant* C *depends only on* $\alpha, \beta, \gamma, \delta, N$ *and* R.

The second one is an interior regularity result for a nonlinear parabolic equation:

Lemma A.2. (Interior regularity) *Let* $v(x,t) \in L^\infty\left((0,+\infty), L^2(\mathbf{B}_R)\right) \cap L^2\left((0,+\infty), H^1(\mathbf{B}_R)\right)$ *which satisfies*

$$v_t - \Delta v + b.\nabla v = H, \quad (x,t) \in Q_R = \mathbf{B}_R \times (0,+\infty), \tag{A.1}$$

where $R > 0$, $|b(x,t)| \leqq \mu_1$ *in* Q_R *and* $|H(x,t,v)| \leqq g(x,t)(|v|+1)$ *with*

$$\int_t^{t+1} \|g(\tau)\|^{\beta'}_{L^{\alpha'}(\mathbf{B}_R)} d\tau \leqq \mu_2, \quad \forall t \in (0,+\infty), \tag{A.2}$$

and $\frac{1}{\beta'} + \frac{N}{2\alpha'} < 1$, *and* $\alpha' \geqq 1$. *If*

$$\int_t^{t+1} \|v(\tau)\|^2_{L^2(\mathbf{B}_R)} d\tau \leqq \mu_3, \quad \forall t \in (0,+\infty), \tag{A.3}$$

and μ_1, μ_2 *and* μ_3 *are uniformly bounded in* t, *then there exists a positive constant* C *depending only on* $\mu_1, \mu_2, \mu_3, \alpha', \beta', N, R$ *and* $\tau \in (0,1)$ *such that*

$$|v(x,t)| \leqq C, \quad \forall (x,t) \in \mathbf{B}_{R/4} \times (\tau, +\infty).$$

B. Some Elementary Lemmas

Let f, F, F_2 be the functions defined in (1.2), (1.25) and (2.14). Clearly, we have

Lemma B.1. *Let* $q > 1$*,*

$$\int_0^u |v|^{q-1} v \log^a(2+v^2)\mathrm{d}v \sim \frac{|u|^{q+1}}{q+1}\log^a(2+u^2), \quad as \quad |u| \to \infty, \tag{B.1}$$

$$F(u) \sim \frac{uf(u)}{p+1} \quad as \quad |u| \to \infty, \tag{B.2}$$

$$F_2(u) \sim \frac{Cuf(u)}{\log^2(2+u^2)} \quad as \quad |u| \to \infty. \tag{B.3}$$

Proof. See Lemma A.1 in [15]. □

Thanks to (B.1), (B.2) and (B.3), we will give the first and the second order terms in the expansion of the nonlinearity $F(x)$ defined in (1.25), when $|x|$ is large enough. More precisely, we now state the following estimates:

Lemma B.2. *For all* $s \geqq 1$*, for all* $z \in \mathbb{R}$*,*

$$C^{-1}\phi(s)zf(\phi(s)z)) \leqq C + F\left(\phi(s)z\right) \leqq C(1+\phi(s)zf(\phi(s)z)), \tag{B.4}$$

$$F_1(\phi(s)z) \leqq C + C\frac{\phi(s)z}{s}f(\phi(s)z), \tag{B.5}$$

$$F_2(\phi(s)z) \leqq C + C\frac{\phi(s)z}{s^2}f(\phi(s)z), \tag{B.6}$$

$$e^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}}|f(\phi(s)z)| \leqq C(\varepsilon) + C|z|^{p+\varepsilon}, \qquad \forall \varepsilon \in (0, p-1), \tag{B.7}$$

$$|z|^{p-\varepsilon} \leqq Ce^{-\frac{ps}{p-1}} s^{\frac{a}{p-1}}|f(\phi(s)z)| + C(\varepsilon), \qquad \forall \varepsilon \in (0, p-1), \tag{B.8}$$

$$e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi(s)z) \leqq C(\varepsilon) + C|z|^{p+\varepsilon+1}, \qquad \forall \varepsilon \in (0, p-1), \tag{B.9}$$

$$|z|^{p-\varepsilon+1} \leqq e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi(s)z) + C(\varepsilon), \qquad \forall \varepsilon \in (0, p-1), \tag{B.10}$$

where ϕ*,* F*,* F_1 *and* F_2 *are given in (1.20), (1.25), (2.13) and (2.14).*

Proof. Note that (B.4) obviously follows from (B.2). In order to derive estimates (B.5) and (B.6), considering the first case $z^2\phi(s) \geqq 4$, then the case $z^2\phi(s) \leqq 4$, we would obtain (B.5) and (B.6) by using (B.1), (B.2) and(B.3). Similarly, by taking into account the inequality $\log^a(2+u^2) \leqq C(\varepsilon) + C(\varepsilon)|u|^{\varepsilon}$, we conclude easily (B.7), (B.8), (B.9) and (B.10). This ends the proof of Lemma B.2. □

C. Proof of Proposition 2.6

Let us first derive the upper bound for $\mathscr{E}_\psi$.

Proof. (**Proof of the upper bound for $\mathscr{E}_\psi$**) Multiplying (1.18) by $\partial_s w\psi^2\rho(y)$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}\frac{d}{ds}\mathscr{E}_\psi(w(s),s) = &-\int_{\mathbb{R}^N}(\partial_s w)^2\psi^2\rho(y)\mathrm{d}y - 2\int_{\mathbb{R}^N}\partial_s w\nabla w.\nabla\psi\psi\rho(y)\mathrm{d}y\\ &+\underbrace{\frac{a}{(p-1)s}\int_{\mathbb{R}^N} w\partial_s w\psi^2\rho(y)\mathrm{d}y}_{\Sigma_2^1(s)}\\ &+\underbrace{\frac{p+1}{p-1}e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}}\int_{\mathbb{R}^N}\big(F(\phi w)-\frac{\phi w f(\phi w)}{p+1}\big)\psi^2\rho(y)\mathrm{d}y}_{\Sigma_2^2(s)}\\ &\underbrace{-\frac{2a}{p-1}e^{-\frac{(p+1)s}{p-1}}s^{\frac{2a}{p-1}-1}\int_{\mathbb{R}^N}\big(F(\phi w)-\frac{\phi w f(\phi w)}{2}\big)\psi^2\rho(y)\mathrm{d}y}_{\Sigma_2^3(s)}.\end{aligned} \tag{C.1}$$

Proceeding similarly as for the terms $\Sigma_1^1(s)$, $\Sigma_1^2(s)$ and $\Sigma_1^3(s)$ defined in (2.10), we get

$$\begin{aligned}\frac{d}{ds}\mathscr{E}_\psi(w(s),s) \leqq &-\frac{1}{2}\int_{\mathbb{R}^N}\psi^2(\partial_s w)^2\rho(y)\mathrm{d}y - 2\int_{\mathbb{R}^N}\partial_s w\psi\nabla\psi.\nabla w\rho(y)\mathrm{d}y\\ &+\frac{C}{s^{a+1}}\int_{\mathbb{R}^N}\psi^2|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\\ &+\frac{C}{s^2}\int_{\mathbb{R}^N}\psi^2w^2\rho(y)\mathrm{d}y + Ce^{-s}.\end{aligned} \tag{C.2}$$

Using the fact that $2ab \leqq \frac{a^2}{4} + 4b^2$, we obtain

$$-2\partial_s w\psi\nabla\psi.\nabla w \leqq \frac{1}{4}\psi^2(\partial_s w)^2 + 4|\nabla\psi|^2|\nabla w|^2,$$

which implies, for all $s \geqq \max(-\log T, 1)$,

$$\begin{aligned}\frac{d}{ds}\mathscr{E}_\psi(w(s),s) \leqq\ & C\int_{\mathbb{R}^N}|\nabla w|^2\rho(y)\mathrm{d}y + \frac{C}{s^{a+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\\ &+\frac{C}{s^2}\int_{\mathbb{R}^N}w^2\rho(y)\mathrm{d}y + Ce^{-s},\end{aligned} \tag{C.3}$$

where $C = C(a, p, N, \|\psi\|_{L^\infty}, \|\nabla\psi\|_{L^\infty})$.
By combining (C.3), (2.40) and (2.41), we infer for all $s \geqq \max(-\log T, S_2)$

$$\int_s^{s+1}\frac{d}{ds}\mathscr{E}_\psi(w(\tau),\tau)\mathrm{d}\tau \leqq Q_1 s^{b+1}. \tag{C.4}$$

From the definition of $\mathscr{E}_\psi$ given in (2.48), using the fact that, $F(\phi w) \geqq 0$, we have

$$\mathscr{E}_\psi(w(s),s) \leqq \|\psi\|_{L^\infty}^2\int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2\right)\rho(y)\mathrm{d}y.$$

By the definition of $H_m(w(s), s)$ given in (2.5), exploiting (2.39), we write for all $s \geqq \max(-\log T, S_2)$

$$\mathscr{E}_\psi(w(s), s) \leqq C\left\{H_{m_0}(w(s), s) + \frac{m_0}{2s}\int_{\mathbb{R}^N} w^2\rho(y)\mathrm{d}y + e^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} \int_{\mathbb{R}^N} F(\phi w)\rho(y)\mathrm{d}y\right\}$$
$$\leqq Q_2 s^{b+1} + Ce^{-\frac{(p+1)s}{p-1}} s^{\frac{2a}{p-1}} \int_{\mathbb{R}^N} F(\phi w)\rho(y)\mathrm{d}y. \tag{C.5}$$

Integrating the inequality (C.5) from s to $s+1$ and using (2.17), (B.4) and (2.41) we get, for all $s \geqq \max(-\log T, S_2)$

$$\int_s^{s+1} \mathscr{E}_\psi(w(\tau), \tau)\mathrm{d}\tau \leqq Q_3 s^{b+1}.$$

By using the mean value theorem, we derive the existence of $\sigma(s) \in [s, s+1]$ such that

$$\mathscr{E}_\psi(w(\sigma(s)), \sigma(s)) = \int_s^{s+1} \mathscr{E}_\psi(w(\tau), \tau)\mathrm{d}\tau. \tag{C.6}$$

Let us write the identity, for all $s \geqq \max(-\log T, S_2)$

$$\mathscr{E}_\psi(w(s), s) = \mathscr{E}_\psi(w(\sigma(s)), \sigma(s)) + \int_{\sigma(s)}^{s} \frac{d}{d\tau}\mathscr{E}_\psi(w(\tau), \tau)\mathrm{d}\tau. \tag{C.7}$$

By combining (C.6), (C.7) and (C.4), we infer, for all $s \geqq \max(-\log T, S_2)$

$$\mathscr{E}_\psi(w(s), s) = \leqq Q_4 s^{b+1}. \tag{C.8}$$

This concludes the proof of the upper bound for $\mathscr{E}_\psi$. □

It remains to prove the lower bound.
[**Proof of the lower bound for $\mathscr{E}_\psi$**]
Consider now, for all $s \geqq \max(-\log T, 1)$,

$$\mathscr{I}_\psi(w(s), s) = \frac{1}{s^{b+1}}\int_{\mathbb{R}^N} w^2\psi^2\rho(y)\mathrm{d}y.$$

Multiplying equation (1.18) with $\psi^2 w$, integrating on $\mathbb{R}^N$ and using the same argument as in the proof of Lemma 2.3 yields

$$\frac{d}{ds}\mathscr{I}_\psi(w(s), s) \geqq -\frac{p+3}{s^{b+1}}\mathscr{E}_\psi(w(s), s) + \frac{1}{2s^{b+1}}(1 - \frac{C_4}{s})\int_{\mathbb{R}^N} w^2\psi^2\rho(y)\mathrm{d}y$$
$$+ \frac{p-1}{(p+1)s^{a+b+1}}(1 - \frac{C_4}{s})\int_{\mathbb{R}^N} |w|^{p+1}\log^a(2 + \phi^2 w^2)\psi^2\rho(y)\mathrm{d}y$$
$$- \frac{4}{s^{b+1}}\int_{\mathbb{R}^N} w\nabla w.\nabla\psi\,\psi\rho(y)\mathrm{d}y. \tag{C.9}$$

Therefore, there exists $\tilde{S}_2 > S_2$ large enough, such that for all $s \geqq \max(-\log T, \tilde{S}_2)$, we have

$$\frac{d}{ds}\mathscr{I}_\psi(w(s),s) \geqq \frac{p-1}{2(p+1)s^{a+b+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\psi^2\rho(y)\mathrm{d}y - \frac{p+3}{s^{b+1}}\mathscr{E}_\psi(w(s),s) - \frac{4}{s^{b+1}}\int_{\mathbb{R}^N} w\nabla w.\nabla\psi\,\psi\rho(y)\mathrm{d}y. \tag{C.10}$$

Furthermore, after some integration by parts, we write

$$-4\int_{\mathbb{R}^N} w\nabla w.\nabla\psi\,\psi\rho(y)\mathrm{d}y = 2\int_{\mathbb{R}^N} w^2\,\mathrm{div}\,(\psi\rho(y)\nabla\psi)\mathrm{d}y$$
$$= 2\int_{\mathbb{R}^N} w^2|\nabla\psi|^2\rho(y)\mathrm{d}y + 2\int_{\mathbb{R}^N} w^2\psi\Delta\psi\rho(y)\mathrm{d}y - \int_{\mathbb{R}^N} w^2\psi y.\nabla\psi\rho(y)\mathrm{d}y. \tag{C.11}$$

Thanks to the estimates $\|\psi\|^2_{L^\infty}+\|\Delta\psi\|^2_{L^\infty}+\|\nabla\psi\|^2_{L^\infty}+\|y.\nabla\psi\|^2_{L^\infty}\leqq C$, (C.11) and (2.42), we have for all $s\geqq\max(-\log T,\tilde{S}_2)$,

$$\Big|-4\int_{\mathbb{R}^N} w\nabla w.\nabla\psi\,\psi\rho(y)\mathrm{d}y\Big| \leqq C\int_{\mathbb{R}^N} w^2\rho(y)\mathrm{d}y \leqq Q_5 s^{b+1}. \tag{C.12}$$

Using (C.10) and (C.12), we obtain for all $s\geqq\max(-\log T,\tilde{S}_2)$,

$$\frac{d}{ds}\mathscr{I}_\psi(w(s),s) \geqq \frac{p-1}{2(p+1)s^{a+b+1}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\psi^2\rho(y)\mathrm{d}y - \frac{p+3}{s^{b+1}}\mathscr{E}_\psi(w(s),s) - Q_5. \tag{C.13}$$

Let us define the following functional:

$$\mathscr{G}_\psi(w(s),s) = \frac{p+3}{s^{b+1}}\mathscr{E}_\psi(w(s),s) + Q_5, \tag{C.14}$$

where $\mathscr{G}_\psi(w(s),s)$ is defined in (2.48).
We claim that the function of $\mathscr{G}_\psi(w(s),s)$ is bounded from below by some constant M, where M is a sufficiently large constant that will be determined later. Arguing by contradiction, we suppose that there exists a time $s^*\geqq\max(-\log T,\tilde{S}_2)$ such that $\mathscr{G}_\psi(w(s^*),s^*)\leqq -Q$, for some $Q>0$. Then, we write

$$\mathscr{G}_\psi(w(s),s) \leqq -Q + \int_{s^*}^{s}\frac{d}{d\tau}\mathscr{G}_\psi(w(\tau),\tau)\mathrm{d}\tau, \qquad \forall s\geqq s^*. \tag{C.15}$$

If we now compute the time derivative of $\mathscr{G}_\psi(w(s),s)$ we get for all $s\geqq s^*$,

$$\frac{d}{ds}\mathscr{G}_\psi(w(s),s) = \frac{p+3}{s^{b+1}}\frac{d}{ds}\mathscr{E}_\psi(w(s),s) - \frac{(b+1)(p+3)}{s^{b+2}}\mathscr{E}_\psi(w(s),s). \tag{C.16}$$

From the definition of $\mathscr{E}_\psi$ given in (2.48), using (B.4) and (2.17) we have for all $s\geqq s^*$,

$$-\frac{(b+1)(p+3)}{s^{b+2}}\mathscr{E}_\psi(w(s),s) \leqq \frac{C}{s^{a+b+2}}\int_{\mathbb{R}^N}|w|^{p+1}\log^a(2+\phi^2w^2)\psi^2\rho(y)\mathrm{d}y + Ce^{-s}. \tag{C.17}$$

Thanks to (C.4) we conclude for all $s \geqq s^*$,

$$\int_{s^*}^{s} \frac{1}{\tau^{b+1}} \frac{d}{ds} \mathscr{E}_\psi(w(\tau), \tau) \mathrm{d}\tau \leqq Q_6(s - s^*). \tag{C.18}$$

Moreover, from (2.41), we obtain for all $s \geqq s^*$,

$$\int_{s^*}^{s} \frac{1}{\tau^{a+b+2}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2 + \phi^2 w^2) \psi^2 \rho(y) \mathrm{d}y \mathrm{d}\tau \leqq Q_7(s - s^*). \tag{C.19}$$

Integrating the identity (C.16) over $[s^*, s]$ and combining (C.17), (C.18) and (C.19) we deduce that

$$\int_{s^*}^{s} \frac{d}{d\tau} \mathscr{G}_\psi(w(\tau), \tau) \mathrm{d}\tau \leqq Q_8(s - s^*), \qquad \forall s \geqq s^*. \tag{C.20}$$

Combining (C.13), (C.15) and (C.20) we infer for all $s \geqq s^*$,

$$\frac{d}{ds} \mathscr{I}_\psi(w(s), s) \geqq Q - Q_8(s - s^*) + \frac{C}{s^{a+b+1}} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2 + \phi^2 w^2) \psi^2 \rho(y) \mathrm{d}y. \tag{C.21}$$

Thanks to (B.4) and (B.10), we have for all $s \geqq s^*$, that

$$\frac{1}{s^a} \int_{\mathbb{R}^N} |w|^{p+1} \log^a(2 + \phi^2 w^2) \psi^2 \rho(y) \mathrm{d}y \geqq C \int_{\mathbb{R}^N} |w|^{\frac{p+3}{2}} \psi^2 \rho(y) \mathrm{d}y - C_5. \tag{C.22}$$

Due to Jensen inequality, (C.21) and (C.22) we find for all $s \geqq s^*$,

$$\frac{d}{ds} \mathscr{I}_\psi(w(s), s) \geqq \tilde{Q} - Q_9(s - s^*) + C_6 \Big(\mathscr{I}_\psi(w(s), s) \Big)^{\frac{p+3}{4}}, \tag{C.23}$$

where $\tilde{Q} = Q - C_5$.
It is interesting to denote that we easily prove that the solution of the differential inequality

$$\begin{cases} h'(s) \geqq 1 + C_6 h^{\frac{p+3}{4}}(s), & s > s^*, \\ h(s^*) \geqq 0 \end{cases}$$

blows up in finite time before

$$s = s^* + \int_0^{+\infty} \frac{d\xi}{1 + C_6 \xi^{\frac{p+3}{4}}} = s^* + T^*.$$

Now, we choose $Q = Q_9 T^* + C_5 + 1$ to get $\tilde{Q} - Q_9(s - s^*) \geqq 1$ for all $s \in [s^*, s^* + T^*]$.
Therefore, $\mathscr{I}_\psi(w(s), s)$ blows up in some finite time before $s^* + T^*$. But this contradicts with the global existence of w. This implies (2.50), and we complete the proof of Proposition 2.6. □

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