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Smooth Controllability of the Navier–Stokes Equation with Navier Conditions: Application to Lagrangian Controllability

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Abstract

We deal with the 3D Navier–Stokes equation in a smooth simply connected bounded domain, with controls on a non-empty open part of the boundary and a Navier slip-with-friction boundary condition on the remaining, uncontrolled, part of the boundary. We extend the small-time global exact null controllability result in Coron et al. (J Eur Math Soc 22:1625–1673, 2020) from Leray weak solutions to the case of smooth solutions. Our strategy relies on a refinement of the method of well-prepared dissipation of the viscous boundary layers which appear near the uncontrolled part of the boundary, which allows to handle the multi-scale features in a finer topology. As a byproduct of our analysis we also obtain a small-time global approximate Lagrangian controllability result, extending to the case of the Navier–Stokes equations the recent results (Glass and Horsin in J Math Pures Appl (9) 93:61–90, 2010; Glass and Horsin in SIAM J Control Optim 50: 2726–2742, 2012; Horsin and Kavian in ESAIM Control Optim Calc Var 23:1179–1200, 2017) in the case of the Euler equations and the result (Glass and Horsin in ESAIM Control Optim Calc Var 22:1040–1053, 2016) in the case of the steady Stokes equations.

1. Introduction and Main Results

1.1. Setting

We consider an incompressible viscous fluid in a smooth bounded simply connected domain Ω in $\mathbb{R}^3$. We denote by u and p its velocity and its pressure respectively and we assume that they evolve according to the Navier–Stokes equations. We assume that we can act on a non-empty open part Σ of the boundary $\partial\Omega$. On the remaining part of the boundary, we assume the fluid satisfies a Navier-slip-with-friction boundary condition. To formalize this boundary condition we introduce the normal $\mathbf{n}$ pointing outward the domain, and for a vector field f, we define its tangential part $f_{\tan}$, the strain tensor $D(f)$ and the tangential Navier boundary operator $\mathcal{N}(f)$, respectively, as

$$\begin{aligned} f_{\tan} &:= f - (f \cdot \mathbf{n})\mathbf{n}, \quad D_{ij}(f) := \frac{1}{2}(\partial_i f_j + \partial_j f_i) \quad \text{and} \\ \mathcal{N}(f) &:= (D(f)\mathbf{n} + Mf)_{\tan}, \end{aligned} \tag{1.1}$$

where M is a given smooth symmetric matrix-valued function, describing the friction near the boundary. The Navier condition then reads $\mathcal{N}(u) = 0$; it dates back to [31]. Finally we prescribe an initial data u_0 for the fluid velocity u at time $t = 0$. Then the system at stake for the unknowns u and p is

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0 \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathcal{N}(u) = 0 \quad \text{on } \partial\Omega \backslash \Sigma, \\ u(t, 0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{1.2}$$

Let us highlight that, in (1.2), there is no boundary condition on the part Σ of the boundary $\partial\Omega$. This is typical of the controllability issue, when one chooses not to mention explicitly the controls. Indeed the controls which will be used in this paper are quite intricate, in particular because of their multi-scale feature. Let us only point out right now that this freedom of choice on Σ allows, in particular, for some fluid to go into and out the domain. Let us also mention here that we are not going to really use a control all the time in the sense that it will be relevant on some time intervals to choose as boundary condition on Σ the same Navier condition as on $\partial\Omega \setminus \Sigma$ so that the system then coincides with its uncontrolled counterpart for which $\Sigma = \emptyset$.

1.2. First main result: smooth small-time global exact null controllability

Our first main result is the following small-time global exact null controllability by solutions for which the velocity vector field u is in the class

$$C([0, T]; H^1(\Omega)) \cap L^2((0, T); H^2(\Omega)). \tag{1.3}$$

Theorem 1.1. *Let $T > 0$, and u_0 in $H^1(\Omega)$ satisfying* $\operatorname{div} u_0 = 0$ *in Ω and $u_0 \cdot n = 0$ on $\partial\Omega$. Then there exists u in the space (1.3) satisfying (1.2) and $u(T, \cdot) = 0$.*

Theorem 1.1 extends the result in [5] where the existence of u in the weaker class

$$C_w([0, T]; L^2(\Omega)) \cap L^2((0, T); H^1(\Omega)), \tag{1.4}$$

is obtained. Indeed the result in [5] deals with the case where the initial data u_0 has only a $L^2(\Omega)$ regularity but the proof developed there fails to guarantee that the constructed solution propagates higher regularity. One underlying reason is the multi-scale feature of the constructed solution which makes small scales more singular in a finer topology. Indeed the question of whether or not a result such as Theorem 1.1 holds true was explicitly raised in [5, Remark 2] and in [6, Perspective 1].

Remark 1.2. Theorem 1.1 is stated as an existence result. The lack of uniqueness comes from the fact that multiple controls can drive the initial state to zero, that is from the fact that there is no boundary condition on Σ for the initial boundary value problem (1.2). However, with some bookkeeping, it is possible to exhibit (though in a quite non-explicit way) from the proof of Theorem 1.1 below a boundary condition to be prescribed on Σ (which is inhomogeneous and depends on u_0) that generates a unique solution u in the space (1.3) to the corresponding initial boundary value problem, that is satisfying (1.2) and this boundary condition on Σ, and this unique solution u satisfies $u(T, \cdot) = 0$.

Remark 1.3. Controllability results such as the one obtained in [5] or in Theorem 1.1 should not be confused with results on the existence of wild solutions vanishing after a finite time, such as the ones obtained in [2–4]. The latter rely on the lack of regularity, in particular these solutions do not belong to $L^2((0,T); H^1(\Omega))$. On the other hand the setting of these papers does not allow any freedom of action, neither through a part of the boundary nor through an interior part of the domain. On the contrary, the controllability results of [5] and of Theorem 1.1 take advantage of the possibility to choose some appropriate boundary conditions on the permeable part Σ of the boundary to drive the fluid to rest in finite time. Since the controllability result of [5] holds for Leray's class of solutions (1.4), it concerns solutions which are more regular than in [2–4]. However, perhaps, one may think that the gap is narrow and perhaps only due to temporary technical limitations. The result of Theorem 1.1 shows that it is not the case and that the possibility of a localized action allows to drive a fluid to rest in finite time in a smooth setting as well. Indeed Theorem 1.1 is stated for H^1 initial data and for solutions in the regularity class (1.3), but it could be easily extended to higher regularity, as the H^1 norm is super-critical for the blow-up issue of the 3D Navier–Stokes equations.

Remark 1.4. Indeed, as in [5] for the case of weak solutions, the proof of Theorem 1.1 can be easily adapted to prove that one may intercept at any given positive time T any smooth uncontrolled solution to the Navier–Stokes system, that is any solution to the Navier–Stokes system with Navier condition on the whole boundary $\partial\Omega$, by the mean of a smooth controlled solution starting from any given initial data.

Remark 1.5. We deal here with the case of a simply connected domain just for simplicity. The multiply-connected domain could be covered by some simple modifications of our method in the case where Σ intersects all the connected components of $\partial\Omega$.

Remark 1.6. To simplify the exposition, Theorem 1.1 is stated in the case of an initial data which is tangent to the whole boundary. The result also holds in the case where the initial data is only tangent to the uncontrolled part $\partial\Omega \setminus \Sigma$ of the boundary. Indeed, to deduce this slightly more general statement from the one considered in

Theorem 1.1, it is sufficient to evolve the system on a short time interval with an appropriate control on Σ, smooth in time, initially compatible with the initial data and vanishing after some small positive time.

1.3. Second main result: lagrangian small-time global approximate controllability

The question that we now address is the possibility of prescribing the motion of a set of particles, following the Lagrangian description of fluids consisting in following fluid particles along the flow map associated with a velocity field satisfying the system (1.2). This type of Lagrangian controllability notion was raised in [18], where the authors showed that for the 2-D incompressible Euler equations, one can indeed prescribe approximately the motion of some specific sets of fluids, and extended in [19] to the case of the dimension 3. Let us also mention the paper [24] where an alternative approach was considered, the result [20] in the case of the steady Stokes equations and the result in [11] about the Lagrangian controllability of the 1-D Korteweg-de Vries equation.

Our second main result establishes the small-time global approximate Lagrangian controllability of (1.2) meaning that for two smooth contractible sets of fluid particles, surrounding the same volume, for any given smooth initial velocity field and any positive time interval, one can find a boundary control such that the corresponding solution of (1.2) makes the first of the two sets approximately reach the second one, while staying in the domain in the meantime.

Theorem 1.7. *Let* $T_0 > 0$, α *in* $(0, 1)$ *and* k *in* $\mathbb{N} \setminus \{0\}$. *Let* u_0 *in* $C^{k,\alpha}(\Omega; \mathbb{R}^3)$ *satisfy* $\operatorname{div} u_0 = 0$ *in* Ω *and* $u_0 \cdot n = 0$ *on* $\partial\Omega$. *Let* γ_0 *and* γ_1 *be two Jordan surfaces included in* Ω *such that* γ_0 *and* γ_1 *are isotopic in* Ω *and surrounding the same volume. Then for any* $\eta > 0$, *there are a time* T *in* $(0, T_0)$ *and a solution* (u, p) *in* $L^\infty(0, T; C^{k,\alpha}(\Omega; \mathbb{R}^4))$ *to (1.2) on* $[0, T]$ *such that*

$$\forall t \in [0, T], \ \phi^u(t, 0, \gamma_0) \subset \Omega, \tag{1.5}$$

$$\|\phi^u(T, 0, \gamma_0) - \gamma_1\|_{C^k} < \eta \tag{1.6}$$

hold (up to reparameterization), where ϕ^u *is the flow map associated with* u *by* $\partial_t \phi^u(t, s, x) = u(t, \phi^u(t, s, x))$ *for any* t, s *in* $[0, T]$ *and for any* x *in* Ω, *and* $\phi^u(s, s, x) = x$ *for any* s *in* $[0, T]$ *and for any* x *in* Ω.

Moreover the same result holds true in the case where u_0 *is only in* $H^1(\Omega; \mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *in* Ω *and* $u_0 \cdot n = 0$ *on* $\partial\Omega$, *with the two following modifications: one only guarantees the existence of a solution* u *in the class (1.3) and that (1.6) holds true with* $k = 0$.

Theorem 1.7 therefore extends to the case of the Navier–Stokes equations the results mentioned above for the case of the Euler equations and of the steady Stokes system; it answers, in the case of the Navier conditions, an open problem mentioned at the end of the introduction of [20], in [22, Section 3.3.3] and in [6, Perspective 2].

Remark 1.8. In Theorem 1.7 we only succeed to assert that there exists a time T in $(0, T_0)$ for which the conclusion holds, and we are not able to guarantee that $T = T_0$ is convenient. The difficulty is to prevent a possible blowup due to the vorticity associated with the surface. This difficulty is typical of the 3D case and was already observed in the case of the Euler equations, see [19].

Remark 1.9. The condition that γ_0 and γ_1 surround the same volume is well defined since by the Jordan-Brouwer separation theorem the set $\mathbb{R}^3 \setminus \gamma$ has two connected components, only one of which being bounded.

Remark 1.10. The conditions that γ_0 and γ_1 are isotopic and surround the same volume are necessary for the existence of a smooth volume-preserving flow driving γ_0 exactly to γ_1.

Remark 1.11. As in the previous result, see Remark 1.2, the boundary control is implicit in the statement of Theorem 1.7 as it is given as traces on $(0, T) \times \Sigma$ of the solution.

Remark 1.12. Let us mention that the controllability time T which is provided by the proof of Theorem 1.7 in Sect. 6 converges to 0 as η goes to 0, what is perhaps counterintuitive, as this means a more and more accurate achievement of the targeted final state in a shorter and shorter time. However, this corresponds to a larger and larger control force, as η goes to 0, and in particular the lack of compactness of these controlled solutions prevents from passing to the limit and from obtaining the absurd conclusion of an instant modification of a Jordan surface into another by a smooth flow map.

1.4. Organization of the rest of the paper

In Sect. 2 we give a scheme of the proof of Theorem 1.1. It will rely on two main intermediate results: Theorem 2.12, where an approximate solution is built thanks to a multi-scale asymptotic expansion involving some boundary layers correctors, and the *a priori* estimate (2.43) for the remainder term associated with this approximate solution. An auxiliary problem associated with the boundary layer is investigated in Sect. 3. Then the proof of Theorem 2.12 is given in Sect. 4. The proof of the *a priori* estimate (2.43) is given in Sect. 5. Finally Sect. 6 is devoted to the proof of Theorem 1.7.

2. Scheme of Proof of Theorem 1.1

This section is devoted to a scheme of proof of Theorem 1.1. We only highlight here the key steps of the proof, postponing to the next sections the proofs of several important intermediate results. As in [5,7,29] we will use the “well-prepared dissipation" method which consists in a rapid and violent stage where one makes use of the inviscid part of the system and of a second stage devoted to the dissipation of the boundary layers due to the discrepancy between the inviscid and the viscous case.

As in [5,7] this method is implemented by the means of multi-scale asymptotic expansions. The extension of this strategy to solutions of the Navier–Stokes equations in the space (1.3), rather than in the weaker class (1.4), requires much attention, in particular due to the fast scale associated with the boundary layer which leads to a more accurate asymptotic expansion and to a more involved preparation of the dissipation of various terms describing the fluid behaviour in the boundary layer.

2.1. Reduction to approximate controllability problem from a smooth data

In this section we reduce the proof of Theorem 1.1 to a combination of a regularisation result on the uncontrolled Navier–Stokes system, that is on the Navier–Stokes system with Navier condition on the whole boundary $\partial\Omega$, of a small-time local exact null controllability result and of a global approximate null controllability result.

(1) Let us first state the regularization result.

Theorem 2.1. *Let $T > 0$, p in $\mathbb{N}^*$ and $R > 0$. Then there exists a continuous function $C_{T,p,R}(\cdot)$ from $[0, +\infty)$ to $[0, +\infty)$ with $C_{T,p,R}(0) = 0$, such that there exists T_1 in $(0, T)$ and for any u_0 in $H^1(\Omega)$, with $\|u_0\|_{H^1(\Omega)} \leq R$, divergence free and tangent to $\partial\Omega$, there exists a unique strong solution u in $C([0, T_1]; H^1(\Omega)) \cap L^2([0, T_1]; H^2(\Omega))$ to (1.2) with $u \cdot \mathbf{n} = 0$ and $\mathcal{N}(u) = 0$ on $\partial\Omega$ such that u is in $C((0, T_1]; H^p(\Omega))$ with*

$$\|u(T_1, \cdot)\|_{H^p(\Omega)} \leq C_{T_1,p,R}(\|u_0\|_{H^1(\Omega)}). \tag{2.1}$$

In the case where the no-slip conditions is imposed on the boundary $\partial\Omega$, rather than the Navier conditions $\mathcal{N}(u) = 0$, such a result dates back to the pioneering work of Leray and Hopf, see [23,28]. In the case of the Navier conditions the part of Theorem 2.1 regarding the existence and uniqueness of local-in-time strong solutions with H^1 initial data is also very classical; we refer to the introduction of [5] for an overview of the literature on the subject. The part of Theorem 2.1 regarding the regularization, that is the bounds (2.1) for $p > 1$, is also part of the folklore on the Navier–Stokes equations with Navier boundary conditions, see for instance [5, Lemma 9]. As we will need a slight generalization of the result in [5] we present a detailed proof of Theorem 2.1 in the Appendix A. In fact, Theorem A.1 in the Appendix A will exhibit the exact singular behavior of the solution near the time zero.

(2) The second ingredient is the following small-time local exact null controllability result when the initial data is small in H^3 established in [21] by Guerrero.

Theorem 2.2. *Let $T > 0$. There exists $\eta > 0$ such that for any u_0 in $H^3(\Omega)$ divergence free, tangent to $\partial\Omega$ and satisfying $\|u_0\|_{H^3(\Omega)} < \eta$, there exists u in $C([0, T]; H^3(\Omega)) \cap L^2((0, T); H^4(\Omega))$ satisfying (1.2) and $u(T, \cdot) = 0$.*

(3) The third ingredient will be the following global approximate result:

Theorem 2.3. *Let $T > 0$, and u_0 in $H^{200}(\Omega)$ divergence free and tangent to $\partial\Omega$. For any $\delta > 0$, there exists u in $C([0, T]; H^1(\Omega)) \cap L^2((0, T); H^2(\Omega))$ satisfying (1.2) and $\|u(T, \cdot)\|_{H^1(\Omega)} < \delta$.*

This last result requires some hard work which will be done below.

On the other hand, with these three ingredients, the proof of Theorem 1.1 is plain sailing.

Proposition 2.4. *A combination of Theorem 2.1, Theorem 2.2 and Theorem 2.3 implies Theorem 1.1.*

Proof. The proof will make use of Theorem 2.2, of Theorem 2.3 and of Theorem 2.1 twice. We also need to care about the choice of the small parameters in the right order. Let $\eta > 0$ be associated with $T/4$ by Theorem 2.2. By Theorem 2.1, with $T/4$ instead of T, $p = 3$, and $R = 1$, there exists T_1 in $(0, T/4)$ and δ in $(0, 1)$ such that for any u_0 in $H^1(\Omega)$, with $\|u_0\|_{H^1(\Omega)} \leq \delta$, divergence free and tangent to $\partial\Omega$, there exists a unique strong solution u in $C([0, T_1]; H^1(\Omega)) \cap L^2([0, T_1]; H^2(\Omega))$ satisfying (1.2) with $u \cdot \mathbf{n} = 0$ and $\mathcal{N}(u) = 0$ on $\partial\Omega$, and with $\|u(T_1, \cdot)\|_{H^3(\Omega)} < \eta$. With these preliminaries at hand we can now proceed to the proof of Proposition 2.4 by chaining some appropriate applications of the three theorems. Let $T > 0$, and u_0 in $H^1(\Omega)$ satisfying $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot n = 0$ on $\partial\Omega$. We apply first Theorem 2.1 with $T/4$ instead of T, $p = 200$ and $R = \|u_0\|_{H^1(\Omega)}$, so that we obtain the existence of T_1' in $(0, T/4)$ and of a unique strong solution u in $C([0, T_1']; H^1(\Omega)) \cap L^2([0, T_1']; H^2(\Omega))$ satisfying (1.2) with $u \cdot \mathbf{n} = 0$ and $\mathcal{N}(u) = 0$ on $\partial\Omega$ and with $u(T_1', \cdot)$ in $H^{200}(\Omega)$. Then we apply Theorem 2.3 with $T/4$ instead of T, $\delta > 0$ as previously chosen and $u(T_1', \cdot)$ as initial data, so that we obtain the existence of u in $C([T_1', T_1' + T/4]; H^1(\Omega)) \cap L^2((T_1', T_1' + T/4); H^2(\Omega))$ satisfying (1.2) and $\|u(T_1' + T/4, \cdot)\|_{H^1(\Omega)} < \delta$. Now the choice of δ has been done to guarantee, by Theorem 2.1 again, that there exists a unique strong solution u in $C([T_1' + T/4, T_1 + T_1' + T/4]; H^1(\Omega)) \cap L^2([T_1' + T/4, T_1 + T_1' + T/4]; H^2(\Omega))$ satisfying (1.2) with $u \cdot \mathbf{n} = 0$ and $\mathcal{N}(u) = 0$ on $\partial\Omega$ and with $\|u(T_1 + T_1' + T/4, \cdot)\|_{H^3(\Omega)} < \eta$. Moreover the choice of η has been done to guarantee, by Theorem 2.2 the existence of u in $C([T_1 + T_1' + T/4, T_1 + T_1' + T/2]; H^1(\Omega)) \cap L^2([T_1 + T_1' + T/4, T_1 + T_1' + T/2]; H^2(\Omega))$ satisfying (1.2) and $u(T_1 + T_1' + T/2, \cdot) = 0$. Then extending u by 0 for t in $(T_1 + T_1' + T/2, T]$ provides the existence of u in $C([0, T]; H^1(\Omega)) \cap L^2((0, T); H^2(\Omega))$ satisfying (1.2) on $[0, T]$ and $u(T, \cdot) = 0$. □

2.2. *domain extension*

Let $\mathcal{O}$ be a smooth extension of the initial domain Ω such that $\Sigma \subset \mathcal{O}$ and $\partial\Omega \backslash \Sigma \subset \partial\mathcal{O}$. We denote $\mathbf{n}$ to be the outward pointing normal to the extended domain $\mathcal{O}$, which coincides with the outward pointing normal to Ω on the uncontrolled boundary $\partial\Omega\backslash\Sigma$. We also need to introduce a smooth function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that $\varphi = 0$ on $\partial\mathcal{O}$, $\varphi > 0$ in $\mathcal{O}$ and $\varphi < 0$ outside of $\overline{\mathcal{O}}$. Moreover, we assume that $|\varphi(x)| = \operatorname{dist}(x, \partial\mathcal{O})$ in a small neighborhood of $\partial\mathcal{O}$. Hence we can extend the normal $\mathbf{n}$ smoothly by $-\nabla\varphi$ to the full domain $\mathcal{O}$. We define

$\mathcal{V}_\delta := \{x \in \mathcal{O} : 0 \le \varphi(x) < \delta\}$. Thus there exists a $\delta_0 > 0$, such that $\varphi = 0$ on $\partial\mathcal{O}$ and $|\mathbf{n}| = 1$ in $\mathcal{V}_{\delta_0}$.

Theorem 2.3 follows from the following result:

Theorem 2.5. *Let $T > 0$ and u_* in $H^{200}(\mathcal{O})$ divergence free and tangent to $\partial\mathcal{O}$. Then for any $\delta > 0$, there are u in $C([0,T]; H^1(\mathcal{O})) \cap L^2((0,T); H^2(\mathcal{O}))$, ξ in $C([0,T]; H^1(\mathcal{O}))$, supported in $\overline{\mathcal{O}}\backslash\overline{\Omega}$ and σ a smooth scalar function supported in $(0,T) \times \overline{\mathcal{O}}\backslash\overline{\Omega}$, such that*

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = \xi \quad \textit{and} \quad \operatorname{div} u = \sigma \quad \textit{in } (0,T)\times\mathcal{O}, \\ u\cdot\mathbf{n} = 0 \quad \textit{and} \quad \mathcal{N}(u) = 0 \textit{ on } (0,T)\times\partial\mathcal{O}, \\ u(0,\cdot) = u_* \quad \textit{in } \mathcal{O}, \end{cases} \tag{2.2}$$

and $\|u(T,\cdot)\|_{H^1(\mathcal{O})} < \delta$.

We will see in the next section how the proof of Theorem 2.5 can be reduced to the proof of an asymptotic result, see Theorem 2.7 below. For the moment let us see how it allows us to conclude to the proof of Theorem 2.3.

Proposition 2.6. *Theorem 2.5 implies Theorem 2.3.*

Proof. Let $T > 0$, and u_0 in $H^{200}(\Omega)$ divergence free and tangent to $\partial\Omega$. Then there is an extension u_* in $H^{200}(\mathcal{O})$ of u_0 into a divergence free vector field on $\mathcal{O}$ tangent to $\partial\mathcal{O}$. Then applying Theorem 2.5 we are left with considering the restrictions of u to Ω to obtain a vector field in $C([0,T]; H^1(\Omega)) \cap L^2((0,T); H^2(\Omega))$ satisfying (1.2) and $\|u(T,\cdot)\|_{H^1(\Omega)} < \delta$. □

2.3. Time scaling and small viscosity

As mentioned above we will use the “well-prepared dissipation" method which consists in a rapid and violent stage followed by a longer one for which no control is applied, see [5,7,29] for earlier uses of this method. To implement this two-scales strategy, we introduce a positive small scale $\varepsilon \ll 1$ as in [5] and we perform the time scaling

$$u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x) \quad \text{and} \quad p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x). \tag{2.3}$$

Thus, we consider $(u^\varepsilon, p^\varepsilon)$ the solution to the following large time and slightly viscous problem:

$$\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon - \varepsilon\Delta u^\varepsilon + \nabla p^\varepsilon = \xi^\varepsilon \quad \text{in } (0,T/\varepsilon)\times\mathcal{O}, \tag{2.4a}$$

$$\operatorname{div} u^\varepsilon = \sigma^\varepsilon \quad \text{in } (0,T/\varepsilon)\times\mathcal{O}, \tag{2.4b}$$

$$u^\varepsilon\cdot\mathbf{n} = 0 \quad \text{on } (0,T/\varepsilon)\times\partial\mathcal{O}, \tag{2.4c}$$

$$\mathcal{N}(u^\varepsilon) = 0 \quad \text{on } (0,T/\varepsilon)\times\partial\mathcal{O}, \tag{2.4d}$$

$$u^\varepsilon(0,\cdot) = \varepsilon u_* \quad \text{in } \mathcal{O}. \tag{2.4e}$$

Observing the amplitude factor ε in the right hand side of (2.4e), we can deduce Theorem 2.3 from the following result:

Theorem 2.7. *Let $T > 0$ and u_* in $H^{200}(\mathcal{O})$ divergence free and tangent to $\partial\mathcal{O}$. Then there are some sequences, $\{u^\varepsilon\}_\varepsilon$, $\{\xi^\varepsilon\}_\varepsilon$ with u^ε in $C([0, T/\varepsilon]; H^1(\mathcal{O})) \cap L^2((0, T/\varepsilon); H^2(\mathcal{O}))$ and ξ^ε in $C([0, T/\varepsilon]; H^1(\mathcal{O}))$, and $\{\sigma^\varepsilon\}_\varepsilon$ a sequence of smooth scalar functions, for ε in $(0, 1)$, such that the mappings ξ^ε and σ^ε are supported in $\overline{\mathcal{O}}\backslash\overline{\Omega}$ as a function of x and compactly supported in $(0, T/\varepsilon)$ as a function of t. Furthermore, (2.4) holds true and*

$$\|u^\varepsilon(T/\varepsilon, \cdot)\|_{H^1(\mathcal{O})} = o(\varepsilon). \tag{2.5}$$

The proof of Theorem 2.7 is actually the core of the analysis and its proof will be detailed in the subsequent sections. Let us start to see here how it entails Theorem 2.5.

Proposition 2.8. *Theorem 2.7 implies Theorem 2.5.*

Proof. Let $T > 0$ and u_* in $H^{200}(\mathcal{O})$ divergence free. Then for any $\delta > 0$, according to Theorem 2.7, there is $\varepsilon > 0$ and there exist u^ε belongs to $C([0, T/\varepsilon]; H^1(\mathcal{O})) \cap L^2((0, T/\varepsilon); H^2(\mathcal{O}))$, ξ^ε belongs to $C([0, T/\varepsilon]; H^1(\mathcal{O}))$ and supported in $\overline{\mathcal{O}}\backslash\overline{\Omega}$, σ^ε is a smooth scalar function supported in $\overline{\mathcal{O}}\backslash\overline{\Omega}$ such that (2.4) holds true and $\|u^\varepsilon(T/\varepsilon, \cdot)\|_{H^1(\mathcal{O})} < \delta\varepsilon$. Let us set

$$(u, \sigma)(t, x) := \frac{1}{\varepsilon}(u^\varepsilon, \sigma^\varepsilon)\left(\frac{t}{\varepsilon}, x\right) \text{ and } (p, \xi)(t, x) := \frac{1}{\varepsilon^2}(p^\varepsilon, \xi^\varepsilon)\left(\frac{t}{\varepsilon}, x\right) \tag{2.6}$$

Then u belongs to $C([0, T]; H^1(\mathcal{O})) \cap L^2((0, T); H^2(\mathcal{O}))$, ξ and σ are compactly supported in $(0, T) \times \overline{\mathcal{O}}\backslash\overline{\Omega}$ so that (2.2) holds true and $\|u(T, \cdot)\|_{H^1(\mathcal{O})} < \delta$. □

2.4. An auxiliary euler solution due to the return method

When ε is small, it is expected that the analysis of the system (2.4) may be built on the small-time global exact controllability of Euler equations. We therefore consider the counterpart of the system (2.4) where the viscosity term has been dropped out. This involves the incompressible Euler equations. For these equations it is natural to prescribe the condition $u^\varepsilon \cdot \mathbf{n} = 0$ on an impermeable wall, and only this one. The natural inviscid counterpart of (2.4) is therefore

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \xi^\varepsilon \quad \text{in } (0, T/\varepsilon) \times \mathcal{O}, \tag{2.7a}$$

$$\operatorname{div} u^\varepsilon = \sigma^\varepsilon \quad \text{in } (0, T/\varepsilon) \times \mathcal{O}, \tag{2.7b}$$

$$u^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } (0, T/\varepsilon) \times \partial\mathcal{O}, \tag{2.7c}$$

$$u^\varepsilon(0, \cdot) = \varepsilon u_* \quad \text{in } \mathcal{O}. \tag{2.7d}$$

Considering an asymptotic expansion of the form $u^\varepsilon = \varepsilon u^1 + o(\varepsilon)$ would amount to considering the linearized Euler equations around the null state, an equation which is not controllable, unless the initial data u_* is the gradient of a harmonic function, which is not the case in general. In order to overcome this difficulty, we are going to use Coron's return method to take profit of the nonlinearity by forcing the amplitude of the solution thanks to the control. Indeed next result asserts that it is possible to guarantee the existence of a controlled solution to the Euler system with variations of order $O(1)$ on time interval of order $O(1)$, say $(0, T)$ (but observe that the allotted time in (2.5) is T/ε), vanishing at both ends of the time interval.

Lemma 2.9. *There exists a solution* $(u^0, p^0, \nu^0, \sigma^0)$ *in* $C^\infty([0, T] \times \overline{\mathcal{O}}; \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R})$ *to the system:*

$$\partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = \nu^0 \quad \textit{in } (0, T) \times \mathcal{O}, \tag{2.8a}$$
$$\operatorname{div} u^0 = \sigma^0 \quad \textit{in } (0, T) \times \mathcal{O}, \tag{2.8b}$$
$$u^0 \cdot \mathbf{n} = 0 \quad \textit{on } (0, T) \times \partial\mathcal{O}, \tag{2.8c}$$
$$u^0(0, \cdot) = 0 \quad \textit{in } \mathcal{O}, \tag{2.8d}$$
$$u^0(T, \cdot) = 0 \quad \textit{in } \mathcal{O}, \tag{2.8e}$$

such that the flow Φ^0 *defined by* $\partial_s \Phi^0(t, s, x) = u^0(s, \Phi^0(t, s, x))$ *and* $\Phi^0(t, t, x) = x$ *satisfies*

$$\forall x \in \overline{\mathcal{O}}, \ \exists t_x \in (0, T), \quad \Phi^0(0, t_x, x) \in \overline{\mathcal{O}} \setminus \overline{\Omega}. \tag{2.9}$$

Moreover, u^0 *can be chosen such that:*

$$\nabla \times u^0 = 0 \quad \textit{in } [0, T] \times \overline{\mathcal{O}}. \tag{2.10}$$

In addition, ν^0 *and* σ^0 *are supported in* $\overline{\mathcal{O}} \backslash \overline{\Omega}$*,* $(u^0, p^0, \nu^0, \sigma^0)$ *are compactly supported in* $(0, T)$*. In the sequel, when we need it, we will implicitly extend them by zero after T.*

Lemma 2.9 is the key argument of many papers concerning the small-time global exact controllability of Euler equations, cf. [8] for 2D simply connected domains, [9] for general 2D domains when Σ intersects all connected components of $\partial\Omega$, [15] for 3D simply connected domains, in [13] for general domains when Σ intersects all connected components of $\partial\Omega$. Let us also refer to [16,17] and to [5, Lemma 2].

With this particular auxiliary Euler solution in hands, Coron's return method consists in looking for solutions to (2.7) admitting asymptotic expansions of the form: $u^\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon)$ and $p^\varepsilon = p^0 + \varepsilon p^1 + o(\varepsilon)$, with some controls ξ^ε and σ^ε also admitting asymptotic expansions of the same form: $\xi^\varepsilon = \nu^0 + \varepsilon \nu^1 + o(\varepsilon)$ and $\sigma^\varepsilon = \sigma^0 + \varepsilon \sigma^1 + o(\varepsilon)$. Indeed, by gathering the terms of order $O(\varepsilon)$, we are led to the following equations for (u^1, p^1):

$$\begin{cases} \partial_t u^1 + u^0 \cdot \nabla u^1 + u^1 \cdot \nabla u^0 + \nabla p^1 = \nu^1 & \text{in } (0, T) \times \mathcal{O}, \\ \operatorname{div} u^1 = \sigma^1 & \text{in } (0, T) \times \mathcal{O}, \\ u^1 \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\mathcal{O}, \\ u^1|_{t=0} = u_0 & \text{in } \mathcal{O}. \end{cases}$$

This is the linearisation of the Euler equations around u^0, and the fact that the vector field u^0 satisfies (2.9) is a crucial gain with respect to the null state.

In the sequel we will use such equations only with zero control on the divergence (corresponding to setting $\sigma^1 = 0$) but also with a source term f supported in the whole domain $\mathcal{O}$ in the first equation. We therefore consider the linearized Euler system

$$\begin{cases} \partial_t u + u^0 \cdot \nabla u + u \cdot \nabla u^0 + \nabla p = \nu + f & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ u \cdot \mathbf{n} = 0 & \text{in } \mathbb{R}_+ \times \partial\mathcal{O}, \\ u(0, \cdot) = u_0 & \text{in } \mathcal{O}, \end{cases} \tag{2.11}$$

where f is a given source term whereas ν is a control force to be chosen supported in $\overline{\mathcal{O}} \backslash \overline{\Omega}$.

Lemma 2.10. *Let k, p in $\mathbb{N}_+$. Let u_0 in $H^p(\mathcal{O})$ with $\operatorname{div} u_0 = 0$ and $u_0 \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$. Let f in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}))$ (see Definition 3.2) and $\nabla \times f$ is supported in $[0, T]$ as a function of time t. Then there are $\nu(t, x)$ in $C^k(\mathbb{R}_+; H^{p-1}(\mathcal{O}))$, supported in $\overline{\mathcal{O}} \backslash \overline{\Omega}$ as a function of x and supported in $[0, T]$ as a function of time t, and u in $C^k(\mathbb{R}_+; H^p(\mathcal{O}))$, supported in $[0, T]$, such that (2.11) holds true. Moreover the unique pressure p, for which the integral condition $\int_{\mathcal{O}} p\, dx = 0$ is satisfied at any time, is in $C^{k-1}_\gamma(\mathbb{R}_+; H^p(\mathcal{O}))$.*

Remark 2.11. Though we do not require f to be supported in $[0, T]$, when $t \geq T$, since f is curl-free, f can be represented as a part of the pressure term and has decay. In this case, it will be used to solve u^4 below.

Proof. The existence and uniqueness of a solution in $C^k(\mathbb{R}_+; H^p(\mathcal{O}))$ to the system (2.11) makes no debate, the point is here to choose an appropriate control function ν, supported in $\overline{\mathcal{O}} \backslash \overline{\Omega}$ as a function of x, such that the solution u of (2.11) vanishes when $t \geq T$. We can prove the Lemma by the argument in Lemma 3 of [5] and Duhamel formula. For sake of completeness let us quickly recall the key observation that $\omega := \nabla \times u$ satisfies

$$\begin{cases} \partial_t \omega + u^0 \cdot \nabla \omega - \omega \cdot \nabla u^0 + (\operatorname{div} u^0)\omega = \nabla \times \nu + \nabla \times f & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ \omega(0, \cdot) = \nabla \times u_0 & \text{in } \mathcal{O}. \end{cases} \tag{2.12}$$

By Duhamel's formula, we wish to find a solution

$$\omega(t, x) = \omega_1(t, x) + \int_0^t \omega_2(s, t, x) ds, \tag{2.13}$$

where ω_1 and ω_2 satisfy

$$\begin{cases} \partial_t \omega_1 + u^0 \cdot \nabla \omega_1 - \omega_1 \cdot \nabla u^0 + (\operatorname{div} u^0)\omega_1 = \nabla \times \nu_1, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ \omega_1(0, \cdot) = \nabla \times u_0, & \text{in } \mathcal{O}, \end{cases} \tag{2.14}$$

and

$$\begin{cases} \partial_s \omega_2 + u^0 \cdot \nabla \omega_2 - \omega_2 \cdot \nabla u^0 + (\operatorname{div} u^0)\omega_2 = \nabla \times \nu_2, & \{(s, t) / s \geq t\} \times \mathcal{O}, \\ \omega_2(t, t, \cdot) = \nabla \times f(t, \cdot), & \mathbb{R}_+ \times \mathcal{O}. \end{cases} \tag{2.15}$$

By the argument in Lemma 3 of [5] we can find control functions ν_1, ν_2 and solutions ω_1, ω_2 of (2.14) and (2.15). We take $\nu = \nu_1 + \int_0^t \nu_2(s, t, x)ds$, and define ω by (2.13). Then ω is a solution of (2.12). Since u^0 in $H^p(\mathcal{O})$, f in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}))$ and $\nabla \times f$ is supported in $[0, T]$, we can check from the proof of Lemma 3 of [5] that ν in $C^k(\mathbb{R}_+; H^{p-1}(\mathcal{O}))$ and is supported in $\overline{\mathcal{O}}\backslash\overline{\Omega}$ as a function of x and is supported in $[0, T]$ as a function of time t, ω in $C^k(\mathbb{R}_+; H^{p-1}(\mathcal{O}))$ and is supported in $[0, T]$. Since u satisfies $\nabla \times u = \omega$ in $\mathcal{O}$, div $u = 0$, in $\mathcal{O}$ and $u \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$, it is in $C^k(\mathbb{R}_+; H^p(\mathcal{O}))$ and supported in $[0, T]$. By the first equation of (2.11) and the Poincaré inequality we obtain the part of Lemma 2.10 concerning the pressure. □

2.5. Boundary layer and multi-scale asymptotic expansion

Since only the impermeability condition is considered in (2.7), a corrector has to be added to the Euler equation to guarantee the Navier slip-with-friction boundary condition (2.4d). The role of this corrector is to accurately describe the behaviour of the fluid close to the boundary in a layer which vanishes as ε goes to 0. For the Navier conditions, in the uncontrolled setting, it was highlighted in [25] that the thickness of this boundary layer is $\mathcal{O}(\sqrt{\varepsilon})$ and the the amplitude of the corrector term is also $\mathcal{O}(\sqrt{\varepsilon})$. Moreover, a multiscale asymptotic expansion of the solutions to the uncontrolled Navier–Stokes equations in the small viscosity limit involving a boundary layer term v, which involves an extra variable describing the fast variations of the fluid velocity in the normal direction near the boundary, is given. This corrector v is given as a solution to an initial boundary value problem with a boundary condition with respect to this extra variable, that is, in a informal way, an asymptotic expansion of the form

$$u^\varepsilon \sim u^0(t, x) + \sqrt{\varepsilon} v\left(t, x, \varphi(x)/\sqrt{\varepsilon}\right). \tag{2.16}$$

Indeed the boundary layer corrector is described by a smooth vector field v expressed in terms both of the slow space variable x in $\mathcal{O}$ and a fast scalar variable $z = \varphi(x)/\sqrt{\varepsilon}$, where $v(t, x, z)$ satisfies an equation of the form

$$\partial_t v + (u^0 \cdot \nabla)v - \partial_{zz} v = 0 \tag{2.17}$$

for x in $\bar{\mathcal{O}}$ and z in $\mathbb{R}_+$, with the following boundary condition at $z = 0$:

$$\partial_z v(t, x, 0) = 2\mathcal{N}(u^0)(t, x). \tag{2.18}$$

The interest in prescribing (2.18) is that the velocity vector field given by (2.16) satisfies the Navier condition (2.4d), up to an error term of order $o(1)$, due to the slow derivatives of v. Indeed it is more convenient to consider an evolution equation for v which is slightly more complicated than (2.17), and which in particular contains some extra-terms which are of lower order but allow to propagate the pointwise orthogonality condition

$$v(t, x, z) \cdot \mathbf{n}(x) = 0, \tag{2.19}$$

including the inside domain, not only on the boundary, from the initial and boundary data to positive times. For this type of linear hyperbolic-parabolic (focusing on t, x or t, z) equation, the Cauchy theory is now well-understood, see [22,33,34].

The analysis in [25] was performed for times of order $O(1)$, and in general this type of multiscale asymptotic expansions fails to describe the vanishing viscosity limit of the Navier–Stokes equation for large times of order $O(1/\varepsilon)$, even in the case where the Euler solution stays smooth for all times. However since the Euler solution u^0 at stake here vanishes after the time T, the equations (2.17) and (2.18), for $t \geq T$, reduce to

$$\partial_t v - \partial_{zz} v = 0, \text{ for } z \in \mathbb{R}_+, \text{ and } \partial_z v(t, x, 0) = 0, \tag{2.20}$$

where the dependence in the slow variable x only appears through the "initial" data $\overline{v}(x, z) := v(T, x, z)$. This heat system dissipates towards the null state for large times. However the decay at the final time $t = T/\varepsilon$ is only given by

$$\left\| \sqrt{\varepsilon} v \left(\frac{T}{\varepsilon}, \cdot, \frac{\varphi(\cdot)}{\sqrt{\varepsilon}} \right) \right\|_{L^2(\mathcal{O})} = \mathcal{O}(\varepsilon), \tag{2.21}$$

which is, unfortunately, not sufficient in view of the wished estimate (2.5) and of the tentative expansion (2.16).

2.6. Well-prepared dissipation method

This difficulty was already presented in [5,6], and there to overcome this difficulty, the authors make use of the well-prepared dissipation method, which was first introduced in [29] in the case of the 1D Burgers equation. The idea is to enhance the natural dissipation of the boundary layer after the time T by an appropriate control before, that is in guaranteeing that $\overline{v}$ satisfies a finite number of vanishing moment conditions for k in $\mathbb{N}$ of the form

$$\forall x \in \mathcal{O}, \quad \int_{\mathbb{R}_+} z^k \overline{v}(x, z)\, dz = 0, \tag{2.22}$$

so that the estimate (2.21) holds true but with $o(\varepsilon)$ in the right hand side. By linearity the moments of $\overline{v}$ in left hand side of (2.22) can be decomposed as the sum of an addend due to the free evolution of v and of an addend due to the control. Indeed due to the properties of the vector field u^0, see (2.9), it is possible to generate some moments outside, and to convect inside the physical original domain in the time interval $[0, T]$. This allows us to ensure the condition (2.22) for all x in $\mathcal{O}$.

2.7. Backflow

Thanks to the orthogonality condition (2.19), the divergence of the vector field $(t, x) \mapsto v\left(t, x, \varphi(x)/\sqrt{\varepsilon}\right)$ is not singular in ε. Still it is not zero, there is an error term of order $O(1)$, due to the slow derivatives of v. To compensate for this part, we set

$$w(t, x, z) := -\int_z^{\infty} \operatorname{div} v(t, x, z')dz', \tag{2.23}$$

and consider instead of the expansion (2.16) the refined asymptotic expansion

$$u^\varepsilon \sim u^0(t,x) + \sqrt{\varepsilon} v\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right) + \varepsilon w\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right)\mathbf{n}. \tag{2.24}$$

This expansion has the advantage over (2.16) to satisfy (2.7b) (observe that the right-hand-side has to be zero in Ω because of the support condition on σ^ε) up to an error of order $O(\varepsilon)$. The new term, the last one in (2.24), corresponds to a boundary layer on the normal velocity. The choice to integrate from infinity in (2.23) is precisely to guarantee that w vanishes as z goes to infinity. Then the new issue is that $w(t,x,0)$ is not zero so that the right-hand-side of (2.24) cannot satisfy the impermeability condtition (2.4c). Then a new correction is considered by the mean of what we call a backflow velocity. As w will be constructed with the integral condition

$$\int_{\partial\mathcal{O}} w(t,x,0)dx = 0,$$

there is a solution ϕ to the following Neumann problem:

$$\begin{cases} \Delta\phi = 0 \quad \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}}\phi = -w(\cdot,\cdot,0) \quad \text{on } \partial\mathcal{O}. \end{cases}$$

Thanks to (2.10), we observe that the so-called backflow velocity $\nabla\phi$ satisfies

$$\begin{cases} \partial_t\nabla\phi + u^0\cdot\nabla\nabla\phi + \nabla\phi\cdot\nabla u^0 + \nabla\Big(-\partial_t\phi - u^0\cdot\nabla\phi\Big) = 0, \quad \text{in } \mathbb{R}_+\times\mathcal{O}, \\ \operatorname{div}\nabla\phi = 0, \quad \text{in } \mathbb{R}_+\times\mathcal{O}, \\ (\nabla\phi)\cdot\mathbf{n} = -w(\cdot,\cdot,0), \quad \text{in } \mathbb{R}_+\times\partial\mathcal{O}; \end{cases} \tag{2.25}$$

that is, $\nabla\phi$ satisfies the Euler equations linearized around u^0. Then the asymptotic expansion

$$u^\varepsilon \sim u^0(t,x) + \sqrt{\varepsilon} v\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right) + \varepsilon\Big(w\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right)\mathbf{n} + \nabla\phi(t,x)\Big) \tag{2.26}$$

is better than the asymptotic expansion (2.24) in the sense that the impermeability condition (2.4c) is now satisfied up to error term $o(\varepsilon)$.

2.8. Approximate solutions

Indeed by expanding further the asymptotic expansion, in particular expanding the velocity into an expansion of the form

$$\begin{aligned} u_a^\varepsilon(t,x) &:= u^0(t,x) + \sqrt{\varepsilon} v^1(t,x,\varphi(x)/\sqrt{\varepsilon}) \\ &\quad + \sum_{j=2}^{4} \varepsilon^{\frac{j}{2}}\Big(u^j(t,x) + v^j\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right) + \nabla\phi^j(t,x) \\ &\quad + w^j\left(t,x,\varphi(x)/\sqrt{\varepsilon}\right)\mathbf{n}(x)\Big), \end{aligned} \tag{2.27}$$

with some profiles satisfying some PDEs of the previous types but with extra forcing terms due to error terms associated with the profiles which are already determined, we will be able to construct some approximate solutions u_a^ε, p_a^ε to the system (2.4) associated with some control forces ξ^ε and σ^0 (on the divergence the control given by Lemma 2.9 will be sufficient).

These solutions are approximate in the sense that

$$\partial_t u_a^\varepsilon - \varepsilon\Delta u_a^\varepsilon + u_a^\varepsilon\cdot\nabla u_a^\varepsilon + \nabla p_a^\varepsilon = \xi^\varepsilon + \varepsilon^2 F \quad \text{in } \mathcal{O}, \tag{2.28a}$$
$$\operatorname{div} u_a^\varepsilon = \sigma^0 + \varepsilon^2 H \quad \text{in } \mathcal{O}, \tag{2.28b}$$
$$u_a^\varepsilon\cdot\mathbf{n} = 0 \quad \text{on } \partial\mathcal{O}, \tag{2.28c}$$
$$\mathcal{N}(u_a^\varepsilon) = \varepsilon^2 G \quad \text{on } \partial\mathcal{O}, \tag{2.28d}$$
$$u_a^\varepsilon|_{t=0} = \varepsilon u_* - \varepsilon^2 R_0 \quad \text{in } \mathcal{O}, \tag{2.28e}$$

where H , G, F and R_0 are error terms which satisfy some uniform bounds in some appropriate spaces which we now define. Let us introduce a cut-off function χ in $C_0^\infty(\mathbb{R}^3)$ such that $\chi = 0$ when $|\varphi| \geq \delta_0$ and $\chi = 1$ when $|\varphi| < \frac{\delta_0}{2}$, where δ_0 is selected in Sect. 2.2, and the vector fields set

$$\begin{aligned}\mathfrak{W} := \Big\{ & w^0 := \varphi\mathbf{n},\ w^1 := \big(0, -\partial_3\varphi, \partial_2\varphi\big)^\top,\ w^2 := \big(\partial_3\varphi, 0, -\partial_1\varphi\big)^\top,\\ & w^3 := \big(-\partial_2\varphi, \partial_1\varphi, 0\big)^\top,\ w^4 := \big(\partial_3(x_3(1-\chi)), 0, -\partial_1(x_3(1-\chi))\big)^\top,\\ & w_0^5 := \big(\partial_2(x_1(1-\chi)), -\partial_1(x_1(1-\chi)), 0\big)^\top\Big\}.\end{aligned}$$

It is easy to observe that w^j are tangential to $\partial\mathcal{O}$, $0 \leq j \leq 5$. Moreover, for $1 \leq j \leq 5$, $w^j\cdot\mathbf{n} = 0$ in $\mathcal{V}_{\delta_0/2}$ and $\operatorname{div} w^j = 0$ in $\mathcal{O}$. Now we define the tangential derivatives

$$Z_j := w^j\cdot\nabla \ \text{ for } 0 \leq j \leq 5 \quad\text{and}\quad Z^\alpha := Z_0^{\alpha_0}\cdots Z_5^{\alpha_5} \text{ for } \alpha = (\alpha_0, \cdots, \alpha_5). \tag{2.29}$$

Let us observe that

$$\nabla Z_j = Z_j\nabla + \nabla w^j\cdot\nabla, \tag{2.30}$$
$$\Delta Z_j = Z_j\Delta + 2\nabla w^j : \nabla^2 + \Delta w^j\cdot\nabla. \tag{2.31}$$

Generally, for $|\alpha| = m$ in $\mathbb{N}_+$, we can use Leibniz formula to find that

$$[\Delta, Z^\alpha] = \sum_{|\beta|,|\gamma|\leq m-1} (c_\beta\nabla^2 Z^\beta + c_\gamma\nabla Z^\gamma), \tag{2.32}$$

for some smooth functions c_β and c_γ depended only on the vector field $\mathfrak{W}$.

Let us also observe that, for $1 \leq i, j \leq 5$,

$$\text{the commutators } [\partial_{\mathbf{n}}, Z_i], [Z_0, Z_i], [Z_i, Z_j] \text{ are tangential derivatives.} \tag{2.33}$$

Indeed, $[\partial_{\mathbf{n}}, Z_i] = (\mathbf{n}\cdot\nabla)w^i\cdot\nabla - (w^i\cdot\nabla)\mathbf{n}\cdot\nabla$, and, on one hand $(w^i\cdot\nabla\mathbf{n})\cdot\nabla$ is a tangential derivative since $w^i\cdot\nabla\mathbf{n}\cdot\mathbf{n}=0$ in $\mathcal{V}_{\delta_0}$, while on the other hand, due to $w^i\cdot\mathbf{n}=0$ in $\mathcal{V}_{\delta_0/2}$ and $\nabla\mathbf{n}$ is symmetric, we have

$$\mathbf{n}\cdot\nabla w^i\cdot\mathbf{n} = -\mathbf{n}\cdot\nabla\mathbf{n}\cdot w^i = -w^i\cdot\nabla\mathbf{n}\cdot\mathbf{n}=0$$

in $\mathcal{V}_{\delta_0/2}$, so that $\mathbf{n}\cdot\nabla w^i\cdot\nabla$ is also a tangential derivative. Moreover notice that for $1\le i\le 5$, $w^i\cdot\nabla\varphi = w^i\cdot\mathbf{n}=0$, we find that $[Z_0, Z_i]=\varphi[\partial_{\mathbf{n}}, Z_i]$ is also a tangential derivative. Finally, for $1\le i, j\le 5$, it holds that

$$[Z_i, Z_j] = (w^i\cdot\nabla w^j - w^j\cdot\nabla w^i)\cdot\nabla.$$

Since $w^i\cdot\mathbf{n}=w^j\cdot\mathbf{n}=0$ and $\nabla\mathbf{n}$ is symmetric, we have

$$w^i\cdot\nabla w^j\cdot\mathbf{n} - w^j\cdot\nabla w^i\cdot\mathbf{n} = -w^i\cdot\nabla\mathbf{n}\cdot w^j + w^j\cdot\nabla\mathbf{n}\cdot w^i = 0.$$

Thus $[Z_i, Z_j]$ is a tangential derivative and (2.33) holds true.

We define the Sobolev conormal spaces

$$H^m_{\rm co}(\mathcal{O}) := \Big\{u\in L^2(\mathcal{O}) : Z^\alpha u\in L^2(\mathcal{O}),\, |\alpha|\le m\Big\}$$

with norm

$$\|u\|_m := \Big(\sum_{|\alpha|\le m}\|Z^\alpha u\|_{L^2}^2\Big)^{\frac{1}{2}}. \tag{2.34}$$

In the same way, we set

$$\|u\|_{k,\infty} := \sum_{|\alpha|\le k}\|Z^\alpha u\|_{L^\infty},$$

and we say u in $W^{k,\infty}_{co}$ if $\|u\|_{k,\infty}$ is finite. Finally for $t\ge 0$, we denote $\langle t\rangle := \sqrt{1+t^2}$.

Theorem 2.12. *Let $\gamma>1$, k, p, s, q in $\mathbb{N}_+$ with $k\ge 2$, $p\ge 8$, $s, q\ge 4$. Assume u_* is smooth enough, say it satisfies (4.2) in Sect. 4.1. Then there exist u_a^ε, p_a^ε and ξ^ε satisfy (2.28a)-(2.28e) with F, G, H and R_0 satisfying, for $0\le j\le k$, $p_1+p_2\le p-3$, $p_2\le s-2$, $m\le p-3$,*

$$\Big\|\partial_t^j Z^{p_1}(\sqrt{\varepsilon}\partial_{\mathbf{n}})^{p_2}\begin{pmatrix}F\\H\end{pmatrix}\Big\|_{L^2(\mathcal{O})}\lesssim \varepsilon^{\frac14}\langle t\rangle^{-\gamma}, \tag{2.35}$$

$$\Big\|\partial_t^j Z^{p_1}(\sqrt{\varepsilon}\partial_{\mathbf{n}})^{p_2}\begin{pmatrix}F\\H\end{pmatrix}\Big\|_{L^\infty(\mathcal{O})}\lesssim \langle t\rangle^{-\gamma}, \tag{2.36}$$

$$\|H\|_{H^m(\partial\mathcal{O})} + \|\partial_t^j G\|_{H^{p-1}(\mathcal{O})}\lesssim\langle t\rangle^{-\gamma}, \tag{2.37}$$

$$\varepsilon^{-\frac14}\|Z^{p_1}(\sqrt{\varepsilon}\partial_{\mathbf{n}})^{p_2}R_0\|_{L^2(\mathcal{O})} + \|Z^{p_1}(\sqrt{\varepsilon}\partial_{\mathbf{n}})^{p_2}R_0\|_{L^\infty(\mathcal{O})}\lesssim\varepsilon^{-\frac12}, \tag{2.38}$$

Moreover u_a^ε satisfies,

$$\|u_a^\varepsilon\|_{W^{1,\infty}(\mathcal{O})} + \|\nabla u_a^\varepsilon\|_{m,\infty} + \sqrt{\varepsilon}\|\nabla^2 u_a^\varepsilon\|_{m-1,\infty}\lesssim\langle t\rangle^{-\gamma}, \tag{2.39}$$

$$\|u_a^\varepsilon - u^0\|_{m,\infty} + \sqrt{\varepsilon}\|\nabla(u_a^0-u^0)\|_{m,\infty}\lesssim\sqrt{\varepsilon}\langle t\rangle^{-\gamma}, \tag{2.40}$$

$$\|u_a^\varepsilon(T/\varepsilon,\cdot)\|_{H^1(\mathcal{O})} = o(\varepsilon). \tag{2.41}$$

The proof of Theorem 2.12 will be presented in Sect. 4.

2.9. Remainder estimate

It follows from the well-posedness of he Navier–Stokes equations with Navier boundary conditions (for fixed ε) that for every ε in $(0, 1)$, there is $T^\varepsilon \in (0, T/\varepsilon]$ and a solution $(u^\varepsilon, p^\varepsilon)$ to (2.4) with ξ^ε given by Theorem 2.12 and $\sigma^\varepsilon := \sigma^0$, for each ε, where σ^0 is given by Lemma 2.9.

We define a family of vector fields R, neglecting an index for the dependence on ε for sake of levity, by

$$u^\varepsilon = u_a^\varepsilon + \varepsilon^2 R. \tag{2.42}$$

The latter R stands for "remainder " as we hope to be able to find such a vector field with a nice behaviour in ε. Indeed we will prove in Sect. 5 the following *a priori* estimate:

$$\varepsilon^2 \sup_{t\in(0,T^\varepsilon)} \|R(t,\cdot)\|_{H^1(\mathcal{O})} \lesssim \varepsilon^{\frac{5}{4}}. \tag{2.43}$$

This entails that $T^\varepsilon = \frac{T}{\varepsilon}$ and, with (2.41), that (2.5) holds true. This concludes the scheme of proof of Theorem 2.7, and then according to Proposition 2.8, Proposition 2.6 and Proposition 2.4, this also concludes the scheme of proof of Theorem 1.1. To complete the proof of Theorem 2.7 it remains to prove the two main intermediate results which are Theorem 2.12 and the *a priori* estimate (2.43). In Sect. 3, we will study an auxiliary problem associated with the boundary layer on the tangential velocity. It will be instrumental in the proof of Theorem 2.12 which will be given in Sect. 4.

Compared to [5], we consider a more accurate asymptotic expansion, (2.27) rather than (2.26) and the large time behaviour of the higher order terms requires to adapt the well-prepared dissipation method. Moreover, the estimate of the remainder is performed in H^1 rather than in L^2, which requires much more work.

3. Well-Prepared Dissipation of Tangential Boundary Layers with Forcing

We set

$$u_\flat^0(t,x) := \frac{u^0(t,x)\cdot \mathbf{n}(x)}{\varphi(x)} \qquad \text{in } \mathbb{R}_+ \times \mathcal{O}, \tag{3.1}$$

where u^0 is given by Lemma 2.9 and we observe that $u_\flat^0$ is smooth in $\overline{\mathcal{O}}$. Let $B^0 = B^0(t,x)$ be a smooth field of 3×3 matrices such that for any v in $\mathbb{R}^3$,

$$B^0 v := v\cdot\nabla u^0 + (u^0\cdot\nabla\mathbf{n}\cdot v)\mathbf{n} - (v\cdot\nabla u^0\cdot\mathbf{n})\mathbf{n}. \tag{3.2}$$

The key property associated with B^0 is that for a smooth vector field $v(t,x)$,

$$(u^0\cdot\nabla v + B^0 v)\cdot\mathbf{n} = u^0\cdot\nabla(v\cdot\mathbf{n}) \quad \text{in } \mathcal{V}_{\delta_0}. \tag{3.3}$$

We are interested in this section by the following type of constrained initial-boundary value problem:

$$\begin{cases} \partial_t v + u^0 \cdot \nabla v + B^0 v - u^0_\flat z \partial_z v - \partial_z^2 v = \xi + f, & \text{in } \mathbb{R}_+ \times \partial\mathcal{O} \times \mathbb{R}_+, \\ \partial_z v|_{z=0} = g, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ v \cdot \mathbf{n} = 0, & \text{in } \mathbb{R}_+ \times \partial\mathcal{O} \times \mathbb{R}_+, \\ v|_{t=0} = v_0, & \text{in } \partial\mathcal{O} \times \mathbb{R}_+, \end{cases} \tag{3.4}$$

where f and g are given source terms whereas ξ is a control force to be chosen. Problem like (3.4) will be useful to construct such boundary layer correctors of the tangential velocity as that described in Sect. 2.5. As already mentioned, the Cauchy theory for this type of linear hyperbolic-parabolic (respectively in t, x and in t, z) equation is now well-understood, see [22,33,34], and our concern will rather be the large time asymptotics and in particular the implementation of the well-prepared dissipation method alluded in Sect. 2.6 in the presence of source terms. This will be useful in the next section in the course of constructing the higher order terms v^j for $j \geq 2$ alluded in (2.27).

Let us introduce the following weighted Sobolev spaces:

Definition 3.1. For z in $\mathbb{R}$, we denote $\langle z \rangle := \sqrt{1+z^2}$ and for s and q in $\mathbb{N}$, we set

$$H^s_q(\mathbb{R}_+) := \Big\{ f \in H^s(\mathbb{R}_+) : \sum_{j=0}^{s} \int_{\mathbb{R}_+} \langle z \rangle^{2q} |\partial_z^j f(z)|^2 dz < +\infty \Big\},$$

endowed with it natural associated norm. In the same way we define $H^s_q(\mathbb{R})$ and the norm

$$\| f \|_{H^s_q(\mathbb{R})} := \Big(\sum_{j=0}^{s} \int_{\mathbb{R}} \langle z \rangle^{2q} |\partial_z^j f(z)|^2 dz \Big)^{\frac{1}{2}}.$$

Observe that by the Plancherel theorem, we have the following equivalence of norms:

$$\| f \|_{H^s_q(\mathbb{R})} \sim \sum_{j=0}^{q} \Big(\int_{\mathbb{R}} \langle \zeta \rangle^{2s} |\partial_\zeta^j \hat{f}(\zeta)|^2 d\zeta \Big)^{\frac{1}{2}}. \tag{3.5}$$

Here $\hat{f}$ denotes the Fourier transform of f.

Definition 3.2. Let k in $\mathbb{N}$, $\gamma > 0$ and X a Banach space with norm $\| \cdot \|_X$. We define the space $C^k_\gamma(\mathbb{R}_+; X)$ of the functions f in $C^k(\mathbb{R}_+; X)$ such that

$$\| f \|_{C^k_\gamma(\mathbb{R}_+; X)} := \sup_{t \geq 0, 0 \leq j \leq k} \big(\| \partial_t^j f(t) \|_X \langle t \rangle^\gamma \big) < +\infty,$$

where

$$C^k(\mathbb{R}_+; X) := \big\{ f : \partial_t^j f \in C(\mathbb{R}_+; X),\ 0 \leq j \leq k \big\}.$$

Let $\mathcal{S}(\mathbb{R})$ the Schwartz space of smooth functions on $\mathbb{R}$ whose derivatives are rapidly decreasing. Let us denote by $\mathcal{S}(\mathbb{R}_+)$ the set of the restrictions to $\mathbb{R}_+$ of the functions of $\mathcal{S}(\mathbb{R})$.

The goal of this section is to prove the following result, where the notation $[x]$ designates the floor integer part of a real number x:

Proposition 3.3. *Let $\gamma > 0$ and s, q, k, p in $\mathbb{N}$ with $k \geq 1$. Set $n := [\frac{q}{2} + \gamma]$,*

$$\tilde{\gamma} := 2n + 3, \quad \tilde{s} := s + 2k + 2n, \quad \tilde{q} := 2n + 3, \tag{3.6}$$

$$k' := [\frac{s+1}{2}] + k + n, \quad \tilde{k} := k + k' - 1, \quad \tilde{p} := p + k' + 1. \tag{3.7}$$

Let

$$f \in C^{\tilde{k}}_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{p}}(\mathcal{O}; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}_+))) \text{ and } g \in C^{\tilde{k}}_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{p}}(\mathcal{O})),$$

such that $f(t, x, z)$ and $g(t, x)$ are supported in $\mathcal{V}_\delta$ as a function of x and such that $f(t, x, z) \cdot \mathbf{n}(x) = g(t, x) \cdot \mathbf{n}(x) = 0$, for any $t \geq 0$, x in $\mathcal{O}$ and z in $\mathbb{R}_+$. Let

$$v_0(x, z) = A(0, x, z) \in H^{p+2}(\mathcal{O}; C^\infty_0(\overline{\mathbb{R}_+})), \tag{3.8}$$

where $A(t, x, z)$ will be defined in (3.31) soon.

Then there are

$$\xi \in C^{k-1}(\mathbb{R}_+; H^p(\mathcal{O}; \mathcal{S}(\mathbb{R}_+))) \text{ and } v \in C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+))),$$

such that (3.4) holds true. Moreover there is a continuous function $\tilde{S} : \mathbb{R}_+ \to \mathbb{R}_+$, such that for any positive δ, $\delta \leq \tilde{S}(\delta)$, and ξ is supported in $(\overline{\mathcal{O}} \backslash \overline{\Omega}) \cap \mathcal{V}_{\tilde{S}(\delta)}$ as a function of x and is compactly supported in $(0, T)$ as a function of time t, and satisfies $\xi(t, x, z) \cdot \mathbf{n}(x) = 0$, for all t in $(0, T)$, x in $(\overline{\mathcal{O}} \backslash \overline{\Omega}) \cap \mathcal{V}_{\tilde{S}(\delta)}$ and z in $\mathbb{R}_+$, and v is supported in $\mathcal{V}_{\tilde{S}(\delta)}$ as a function of x . Moreover, if f and g are both supported away from $t = 0$ as a function of time t, then so does v.

The first key observation towards the proof of Proposition 3.3 is that for $t \geq T$, we have $u^0 = 0$, $u^0_\flat = 0$, $B^0 = 0$ and we look for a control ξ which is compactly supported in $(0, T)$, so the equations for v reduces to

$$\begin{cases} \partial_t v - \partial_z^2 v = f, & \text{in } [T, +\infty) \times \mathcal{O} \times \mathbb{R}_+, \\ \partial_z v|_{z=0} = g, & \text{in } [T, +\infty) \times \mathcal{O}, \\ v \cdot \mathbf{n} = 0, & \text{in } [T, +\infty) \times \partial\mathcal{O} \times \mathbb{R}_+, \end{cases}$$

with an "initial" data at $t = T$ which has no reason to be zero. To prepare the part of the proof of Proposition 3.3 regarding the decay in time, we first single out some well-prepared dissipation conditions for the heat equation on the full line (in space) with non-zero "initial" data at $t = T$ and non-zero source term:

$$\begin{cases} \partial_t v - \partial_z^2 v = f, & \text{in } [T, +\infty) \times \mathcal{O} \times \mathbb{R}_+, \\ v|_{t=T} = v(T, \cdot, \cdot), & \text{in } \mathcal{O} \times \mathbb{R}_+. \end{cases}$$

For n in $\mathbb{N}$ and x in $\mathbb{R}$, we set

$$s_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{3.9}$$

Lemma 3.4. *Let $\gamma > 0$ and k, s, q, n in $\mathbb{N}$ and*

$$n \geq \frac{q}{2} + \gamma - 1. \tag{3.10}$$

Let $\tilde{\gamma}$, $\tilde{s}$, and $\tilde{q}$ be as in (3.6). Let v_0 in $H^{s+2k}_{\tilde{q}}(\mathbb{R})$ and f in $C^0_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))$ when $k = 0$ and f in $C^{k-1}_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))$ when $k \geq 1$, such that

$$\Big(\partial^j_\zeta \big(\hat{v}_0(\zeta) + \int_0^\infty s_n(\tau\zeta^2) \hat{f}(\tau, \zeta) d\tau\big)\Big)\Big|_{\zeta=0} = 0, \quad \textit{for } 0 \leq j \leq 2n+1 \tag{3.11}$$

Then the Cauchy problem

$$\begin{cases} \partial_t v - \partial^2_z v = f, & \textit{in } [0, +\infty) \times \mathbb{R}, \\ v|_{t=0} = v_0, & \textit{in } \mathbb{R} \end{cases}$$

has a unique solution v in $C^k_\gamma(\mathbb{R}_+; H^s_q(\mathbb{R}))$.

Proof. We first observe that it is sufficient to deal with the case where $k = 0$, since the general case follows by using that for $0 \leq i \leq k$, for z in $\mathbb{R}$ and $t \geq 0$, $\partial^i_t v = \partial^2_z \partial^{i-1}_t v + \partial^{i-1}_t f$.

The Fourier transform $\hat{v}(t, \cdot)$ of $v(t, \cdot)$ is given, for $t \geq 0$ and ζ in $\mathbb{R}$, by

$$\hat{v}(t, \zeta) = e^{-t\zeta^2} \Big(\hat{v}_0(\zeta) + \int_0^t e^{\tau\zeta^2} \hat{f}(\tau, \zeta) d\tau\Big). \tag{3.12}$$

Let us observe that

$$\forall j \in \mathbb{N},\ \exists C_j > 0 \text{ such that } \forall t > 0,\ \forall \zeta \in \mathbb{R},\ |\partial^j_\zeta (e^{-t\zeta^2})| \leq C_j \langle t \rangle^{\frac{j}{2}} e^{-\frac{3}{4}t\zeta^2}. \tag{3.13}$$

Now we decompose the proof of Lemma 3.4 into the following two steps:
Step 1: we first prove that, for $0 \leq t \leq 1$, $\|u(t, \cdot)\|_{H^s_q(\mathbb{R})}$ is bounded. Indeed, for $0 \leq t \leq 1$ and $s < \tilde{s}$, $q < \tilde{q}$, it follows from (3.5), (3.12), the Leibniz formula and (3.13) that

$$\begin{aligned} \|v(t, \cdot)\|_{H^s_q(\mathbb{R})} &\lesssim \sum_{j=0}^{q} \|\langle \zeta \rangle^s \partial^j_\zeta \hat{v}(t, \zeta)\|_{L^2_\zeta} \\ &\lesssim \sum_{j=0}^{q} \sum_{j_1+j_2=j} \Big(\|\langle \zeta \rangle^s \partial^{j_1}_\zeta (e^{-t\zeta^2}) \partial^{j_2}_\zeta \hat{v}_0(\zeta)\|_{L^2_\zeta} \end{aligned}$$

$$+\|\langle\zeta\rangle^s \int_0^t \partial_\zeta^{j_1}(e^{-(t-\tau)\zeta^2})\partial_\zeta^{j_2}\hat{f}(\tau,\zeta)d\tau\|_{L^2_\zeta}\Big)$$
$$\lesssim \|v_0\|_{H^s_{\tilde{q}}(\mathbb{R})} + \|f\|_{C^0_{\tilde{\gamma}}(\mathbb{R}_+;H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))}.$$

Thus for $0 \le t \le 1$, $\|v(t,\cdot)\|_{H^s_q(\mathbb{R})}$ is bounded.

Step 2: It remains to prove that there exists $C > 0$ such that for $t \ge 1$, $\|v(t,\cdot)\|_{H^s_q} \le C\langle t\rangle^{-\gamma}$. Indeed, for $t \ge 1$, by (3.12), we write

$$\hat{v}(t,\zeta) = \sum_{i=1}^{4} I_i(t,\zeta), \tag{3.14}$$

where

$$I_1(t,\zeta) := e^{-t\zeta^2}\Big(\hat{v}_0(\zeta) + \int_0^{+\infty} s_n(\tau\zeta^2)\hat{f}(\tau,\zeta)d\tau\Big),$$
$$I_2(t,\zeta) := -e^{-t\zeta^2}\int_{\frac{t}{4}}^{+\infty} s_n(\tau\zeta^2)\hat{f}(\tau,\zeta)d\tau,$$
$$I_3(t,\zeta) := e^{-t\zeta^2}\int_0^{\frac{t}{4}} \big(e^{\tau\zeta^2} - s_n(\tau\zeta^2)\big)\hat{f}(\tau,\zeta)d\tau,$$
$$I_4(t,\zeta) := \int_{\frac{t}{4}}^{t} e^{-(t-\tau)\zeta^2}\hat{f}(\tau,\zeta)d\tau.$$

Thanks to (3.13), to conclude this second step, it is sufficient to show that, for $1 \leqslant i \leqslant 4$, $0 \le j \le q$ and $t \ge 1$,

$$\|\langle\zeta\rangle^s\partial_\zeta^j I_i(t,\zeta)\|_{L^2_\zeta} \lesssim \langle t\rangle^{-\gamma}.$$

We observe that for $t \ge 1$,

$$t \le \langle t\rangle \le \sqrt{2}t. \tag{3.15}$$

- <u>Estimate of I_1</u>

Since u_0 and f satisfies (3.11), we have, for $0 \le j_2 \le q$, by the Taylor formula,

$$|\partial_\zeta^{j_2}(\hat{v}_0(\zeta) + \int_0^{+\infty} s_n(\tau\zeta^2)\hat{f}(\tau,\zeta)d\tau)|$$
$$\lesssim |\zeta|^{2n+2-j_2}\|\partial_\zeta^{2n+2}\big(\hat{v}_0(\zeta) + \int_0^{+\infty} s_n(\tau\zeta^2)\hat{f}(\tau,\zeta)d\tau\big)\|_{L^\infty_\zeta}$$
$$\lesssim |\zeta|^{2n+2-j_2}\|\partial_\zeta^{2n+2}\big(\hat{v}_0(\zeta) + \int_0^{+\infty} s_n(\tau\zeta^2)\hat{f}(\tau,\zeta)d\tau\big)\|_{H^1_\zeta}$$
$$\lesssim |\zeta|^{2n+2-j_2}\big(\|v_0\|_{H^0_{\tilde{q}}(\mathbb{R})} + \|f\|_{C^0_{\tilde{\gamma}}(\mathbb{R}_+;H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))}\big)$$
$$\le C|\zeta|^{2n+2-j_2}.$$

This together with the Leibniz formula, (3.13) and (3.15) implies that for $0 \le j \le q$ and $t \ge 1$,

$$\|\langle\zeta\rangle^s \partial_\zeta^j I_1(t,\zeta)\|_{L^2_\zeta} \lesssim \sum_{j_1+j_2=j} \|\langle\zeta\rangle^s e^{-\frac{3}{4}t\zeta^2}\langle t\rangle^{\frac{j_1}{2}}|\zeta|^{2n+2-j_2}\|_{L^2_\zeta} \lesssim t^{-(n+\frac{5}{4}-\frac{j}{2})}.$$

Thus, thanks to (3.10), we achieve

$$\|\langle\zeta\rangle^s \partial_\zeta^j I_1(t,\zeta)\|_{L^2_\zeta} \le C\langle t\rangle^{-\gamma}. \tag{3.16}$$

- Estimate of I_2

By the Leibniz formula and (3.13), for $0 \le j \le q$, we find

$$\begin{aligned}\|\langle\zeta\rangle^s \partial_\zeta^j I_2(t,\zeta)\|_{L^2_\zeta} &\lesssim \sum_{j_1+j_2+j_3=j} \|\langle\zeta\rangle^s \int_{\frac{t}{4}}^{\infty} \partial_\zeta^{j_1}(e^{-t\zeta^2})\partial_\zeta^{j_2}(s_n(\tau\zeta^2))\partial_\zeta^{j_3}\hat{f}(\tau,\zeta)d\tau\|_{L^2_\zeta}\\ &\lesssim \sum_{j_1+j_2+j_3=j} \|\int_{\frac{t}{4}}^{\infty} \langle t\rangle^{\frac{j_1}{2}} e^{-\frac{3}{4}t\zeta^2}\langle\tau\rangle^n\langle\zeta\rangle^{s+2n}|\partial_\zeta^{j_2}\hat{f}(\tau,\zeta)|d\tau\|_{L^2_\zeta}\\ &\lesssim \int_{\frac{t}{4}}^{\infty} \langle t\rangle^{\frac{q}{2}}\langle\tau\rangle^n\|f(\tau,\cdot)\|_{H_q^{s+2n}}d\tau.\end{aligned}$$

Since $\|f(\tau,\cdot)\|_{H_q^{s+2n}} \lesssim \langle\tau\rangle^{-(2n+3)}$, by using (3.10), we deduce that

$$\|\langle\zeta\rangle^s \partial_\zeta^j I_2(t,\zeta)\|_{L^2_\zeta} \le C\langle t\rangle^{-\gamma}. \tag{3.17}$$

- Estimate of I_3

By Taylor's expansion and by induction on j, we prove that for all j in $\mathbb{N}$, there exists $C_{j,n} > 0$ such that for all $\tau > 0$, for all ζ in $\mathbb{R}$,

$$|\partial_\zeta^j(e^{\tau\zeta^2} - s_n(\tau\zeta^2))| \le C_{j,n}\tau^{n+1}|\zeta|^{2n+2-j}e^{(2-\frac{1}{j+1})\tau\zeta^2}.$$

Then, for $0 \le j \le q$, by the Leibniz formula, one has

$$\begin{aligned}&\|\langle\zeta\rangle^s \partial_\zeta^j I_3(t,\zeta)\|_{L^2_\zeta}\\ &\lesssim \sum_{j_1+j_2+j_3=j} \|\langle\zeta\rangle^s \int_0^{\frac{t}{4}} \partial_\zeta^{j_1}(e^{-t\zeta^2})\partial_\zeta^{j_2}(e^{\tau\zeta^2} - s_n(\tau\zeta^2))\partial_\zeta^{j_3}\hat{f}(\tau,\zeta)d\tau\|_{L^2_\zeta}\\ &\lesssim \sum_{j_1+j_2+j_3=j} \|\langle\zeta\rangle^s \int_0^{\frac{t}{4}} \langle t\rangle^{\frac{j_1}{2}} e^{-\frac{3}{4}t\zeta^2}\tau^{n+1}|\zeta|^{2n+2-j_2}e^{2\tau\zeta^2}|\partial_\zeta^{j_3}\hat{f}(\tau,\zeta)|d\tau\|_{L^2_\zeta}\\ &\lesssim \sum_{j_1+j_2+j_3=j} t^{\frac{j_1+j_2}{2}-n-1}\|\int_0^{\frac{t}{4}} \langle\zeta\rangle^s\tau^{n+1}|\partial_\zeta^{j_3}\hat{f}(\tau,\zeta)|d\tau\|_{L^2_\zeta}\\ &\lesssim t^{\frac{q}{2}-n-1}\int_0^{\frac{t}{4}} \tau^{n+1}\|f(\tau,\cdot)\|_{H_q^s}d\tau.\end{aligned}$$

Since $\|f(\tau,\cdot)\|_{H_q^s} \lesssim \langle\tau\rangle^{-\tilde{\gamma}}$ and (3.15), we obtain

$$\|\langle\zeta\rangle^s \partial_\zeta^j I_3(t,\zeta)\|_{L^2_\zeta} \le C\langle t\rangle^{\frac{q}{2}-n-1} \le C\langle t\rangle^{-\gamma}. \tag{3.18}$$

• Estimate of I_4
By (3.13), we find, for $0 \le j \le q$,

$$\begin{aligned}\|\langle\zeta\rangle^s \partial_\zeta^j I_4(t,\zeta)\|_{L^2_\zeta} &\lesssim \sum_{j_1+j_2=j} \Big\| \int_{\frac{t}{4}}^t \partial_\zeta^{j_1}(e^{-(t-\tau)\zeta^2})\langle\zeta\rangle^s \partial_\zeta^{j_2}\hat{f}(\tau,\zeta)d\tau\Big\|_{L^2_\zeta}\\ &\lesssim \sum_{j_1+j_2=j} \Big\| \int_{\frac{t}{4}}^t \langle t-\tau\rangle^{\frac{j_1}{2}} e^{-\frac{3(t-\tau)}{4}\zeta^2}\langle\zeta\rangle^s |\partial_\zeta^{j_2}\hat{f}(\tau,\zeta)|d\tau\Big\|_{L^2_\zeta}\\ &\lesssim \int_{\frac{t}{4}}^t \langle\tau\rangle^{\frac{q}{2}} \|f(\tau,\cdot)\|_{H^s_q}d\tau.\end{aligned}$$

Since $\|f(t,\cdot)\|_{H^s_q} \lesssim \langle\tau\rangle^{-(2n+3)}$, we infer

$$\|\langle\zeta\rangle^s \partial_\zeta^j I_4(\tau,\zeta)\|_{L^2_\zeta} \lesssim \langle t\rangle^{\frac{q}{2}-2n-2} \le \langle t\rangle^{-\gamma}. \tag{3.19}$$

By combining the estimates, (3.14), (3.16), (3.17), (3.18) and (3.19), we deduce that there exists $C > 0$ such that for $t \ge 1$, $\|v(t,\cdot)\|_{H^s_q} \le C\langle t\rangle^{-\gamma}$.

Finally by combining step 1 with step 2, we conclude that v belongs to $C^0_\gamma(\mathbb{R}_+; H^s_q(\mathbb{R}))$. □

We now turn to the following counterpart for the whole line $z \in \mathbb{R}$ of the initial-boundary value problem (3.4):

$$\begin{cases}\partial_t V + u^0\cdot\nabla V + B^0 V - u^0_\flat z\partial_z V - \partial_z^2 V = \Xi + F, & \text{in } \mathbb{R}_+\times\mathcal{O}\times\mathbb{R},\\ V|_{t=0} = 0, & \text{in } \mathcal{O}\times\mathbb{R}.\end{cases} \tag{3.20}$$

We recall that B^0 is defined in (3.2).

Lemma 3.5. *Let $\gamma > 0$, k, p, s, q, n in $\mathbb{N}$, $k \ge 1$ satisfying $n \ge \frac{q}{2} + \gamma - 1$. Let $\tilde{\gamma}, \tilde{s}, \tilde{q}$ be as in (3.6) and $\delta > 0$ be a small constant. Let*

$$F \in C^{k-1}_{\tilde{\gamma}}(\mathbb{R}_+; H^{p+1}(\mathcal{O}; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R})), \tag{3.21}$$

with $F(t,x,z)$ being supported in $\mathcal{V}_\delta$ as a function of x and $F(t,x,z)\cdot\mathbf{n}(x) = 0$, for all $t \ge 0$, x in $\mathcal{O}$ and z in $\mathbb{R}$.

Then there are

$$\Xi(t,x,z) \in C^{k-1}(\mathbb{R}_+; H^p(\mathcal{O}; \mathcal{S}(\mathbb{R}))) \text{ and } V \in C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}))),$$

such that (3.20) holds true, and there is a continuous function $\tilde{S}: \mathbb{R}_+ \to \mathbb{R}_+$, such that for any positive δ, $\delta \le \tilde{S}(\delta)$, and Ξ is supported in $(\overline{\mathcal{O}}\backslash\overline{\Omega}) \cap \mathcal{V}_{\tilde{S}(\delta)}$ as a function of x and is compactly supported in $(0,T)$ as a function of time t, and satisfies $\Xi(t,x,z)\cdot\mathbf{n}(x) = 0$, for all t in $(0,T)$, x in $(\overline{\mathcal{O}}\backslash\overline{\Omega}) \cap \mathcal{V}_{\tilde{S}(\delta)}$ and z in $\mathbb{R}$, and V is supported in $\mathcal{V}_{\tilde{S}(\delta)}$ as a function of x and satisfies $V(t,x,z)\cdot\mathbf{n}(x) = 0$, for all $t \ge 0$, x in $\mathcal{O}$ and z in $\mathbb{R}$.

Moreover, if F is supported away from $t = 0$ as a function of time t, then so does V.

Proof. For $0 \le j \le 2n+1$ and x in $\mathcal{O}$, let

$$\gamma_j(x) := \partial_\zeta^j \int_0^\infty s_n(\tau\zeta^2)\hat{F}(T+\tau, x, \zeta)d\tau|_{\zeta=0},$$

where $\hat{F}(t, x, \cdot)$ is the partial Fourier transform of $F(t, x, z)$ with respect to the z variable. We use ζ as dual variable of z by the partial Fourier transform. We also recall that s_n is defined in (3.9). By (3.21), for $0 \le j \le 2n+1$, γ_j in $H^{p+1}(\mathcal{O})$. We look for a control profile Ξ, with the properties mentioned in the statement of Lemma 3.5, such that there is a solution V in $C^k(\mathbb{R}_+; H^p(\mathcal{O}; H_q^s(\mathbb{R})))$ to (3.20) satisfying

$$(\partial_\zeta^j \hat{V}(T, x, \zeta) + \gamma_j(x))|_{\zeta=0} = 0, \quad \text{for } 0 \le j \le 2n+1 \text{ and } x \in \overline{\mathcal{O}}, \tag{3.22}$$

where $\hat{V}(t, x, \cdot)$ is the partial Fourier transform of $V(t, x, \cdot)$. Then, for $t \ge T$, as $u^0 = 0$, $u_\flat^0 = 0$ and $B^0 = 0$, the first equation in (3.20) reduces to

$$\partial_t V - \partial_z^2 V = F, \quad \text{in } \mathcal{O} \times \mathbb{R}. \tag{3.23}$$

Therefore it would follow from Lemma 3.4 that V in $C_\gamma^k(\mathbb{R}_+; H^p(\mathcal{O}; H_q^s(\mathbb{R})))$.

Indeed for a given control profile Ξ, with the properties mentioned in the statement of Lemma 3.5, the existence of a solution V in $C^k(\mathbb{R}_+; H^p(\mathcal{O}; H_q^s(\mathbb{R})))$ to (3.20), supported in a neighborhood of the boundary as a function of x and satisfying $V(t, x, z) \cdot \mathbf{n}(x) = 0$, for all $t \ge 0$, x in $\mathcal{O}$ and z in $\mathbb{R}$, can be proved along the same lines as [25, Proposition 5]. We therefore focus on the existence of a control profile Ξ for which the corresponding solution V to (3.20) satisfies the conditions (3.22). In this perspective we first observe that the Cauchy problem (3.20) for V translates into the following one for $\hat{V}$:

$$\begin{cases} \partial_t \hat{V} + u^0 \cdot \nabla \hat{V} + (B^0 + \zeta^2 - u_\flat^0)\hat{V} - u_\flat^0 \zeta \partial_\zeta \hat{V} = \hat{\Xi} + \hat{F}, \\ \hat{V}|_{t=0} = 0. \end{cases} \tag{3.24}$$

Let

$$H(x, \zeta) := \sum_{j=0}^{2n+1} \gamma_j(x) \frac{\zeta^j}{j!} \chi_1(\zeta).$$

Here χ_1 in $C_0^\infty(\mathbb{R})$ is a cut-off function satisfying $\chi_1(\zeta) = 1$ when $|\zeta| \le 1$ and $\chi_1(\zeta) = 0$ when $|\zeta| \ge 2$, so that H in $H^{p+1}(\mathcal{O}; C_0^\infty(\mathbb{R}))$ and

$$\partial_\zeta^j H(x, \zeta)|_{\zeta=0} = \gamma_j(x) \text{ for } 0 \le j \le 2n+1 \text{ and } x \in \overline{\mathcal{O}}. \tag{3.25}$$

Let

$$\tilde{F} := \hat{F} + u^0 \cdot \nabla H + (B^0 + \zeta^2 - u_\flat^0)H - u_\flat^0 \zeta \partial_\zeta H. \tag{3.26}$$

By (3.21), for $0 \le j \le 2n+1$, the function $\partial_\zeta^j \tilde{F}|_{\zeta=0}$ is in $C_{\tilde{\gamma}}^{k-1}(\mathbb{R}_+; H^p(\mathcal{O}))$.

Using (2.9), we can prove the existence of Ξ with the properties mentioned in the statement of Lemma 3.5, such that for $0 \leq j \leq 2n+1$, the unique solution Q_j to

$$\begin{cases} \partial_t Q_j + u^0 \cdot \nabla Q_j + B^0 Q_j - (j+1) u^0_\flat Q_j = -j(j-1)Q_{j-2} + \partial_\zeta^j \hat{\Xi}|_{\zeta=0} + \partial_\zeta^j \tilde{F}|_{\zeta=0}, \\ Q_j|_{t=0} = \gamma_j(x), \end{cases} \tag{3.27}$$

where $\hat{\Xi}(t, x, \cdot)$ is the Fourier transform of $\Xi(t, x, \cdot)$, satisfies

$$Q_j(T, x) = 0, \text{ for } 0 \leq j \leq 2n+1 \text{ and } x \in \overline{\mathcal{O}}. \tag{3.28}$$

We refer here to [5, Lemma 7], see also the discussion in Sect. 2.6. By differentiating (3.24), by (3.25) and by using the uniqueness of the Cauchy problem (3.27), we observe that the solution V to (3.20), for the control profile Ξ mentioned above, satisfies

$$Q_j(t, x) = \partial_\zeta^j \hat{V}(t, x, \zeta)|_{\zeta=0} + \gamma_j(x), \text{ for } 0 \leq j \leq 2n+1, \ t \in \mathbb{R}_+ \text{ and } x \in \overline{\mathcal{O}}. \tag{3.29}$$

By combining (3.28) and (3.29), we conclude that (3.22) is satisfied. From the construction of Ξ and Q_j we can see that, if F vanishes near $t = 0$, so does V.

Finally, thanks to the argument in [5, Section 3.4], there is a continuous function $\tilde{S} : \mathbb{R}_+ \to \mathbb{R}_+$, such that for any positive δ, $\delta \leq \tilde{S}(\delta)$, and V is supported in $\mathcal{V}_{\tilde{S}(\delta)}$. We can choose δ small enough such that $\tilde{S}(\delta) < \delta_0$. (Recall that δ_0 is defined in Sect. 2.2). □

Now we are in a position to complete the proof of Proposition 3.3.

Proof of Proposition 3.3. Let

$$\begin{aligned} g_1 &:= g, \\ g_{j+1} &:= \partial_t g_j + u^0 \cdot \nabla g_j \\ &\qquad + B^0 g_j - (2j-1) u^0_\flat g_j - (\partial_z^{2j-1} f)|_{z=0^+} \text{ for } 1 \leq j < k'. \end{aligned} \tag{3.30}$$

It is clear that g_j is supported in $\mathcal{V}_\delta$ as a function of x, $g_j \cdot \mathbf{n} = 0$ and g_j in $C_{\tilde{\gamma}}^{\tilde{k}+1-j}(\mathbb{R}_+; H^{\tilde{p}+1-j}(\mathcal{O}))$ for $1 \leq j \leq k'$.

For $z \geq 0$, we denote

$$A(t, x, z) := \sum_{j=1}^{k'} g_j(t, x) \frac{z^{2j-1}}{(2j-1)!} \chi_1(z), \tag{3.31}$$

where χ_1 in $C_0^\infty(\mathbb{R})$ is an even cut-off function as in the proof of Lemma 3.5. One can check that

$$A \in C_{\tilde{\gamma}}^k(\mathbb{R}_+; H^{p+2}(\mathcal{O}; C_0^\infty(\overline{\mathbb{R}_+}))),$$

and satisfies

$$\partial_z^{2j-1} A|_{z=0^+} = g_j, \quad \text{for } 1 \leq j \leq k'. \tag{3.32}$$

Let

$$F := f - (\partial_t A + u^0 \cdot \nabla A + B^0 A - u^0_\flat z \partial_z A - \partial_z^2 A). \tag{3.33}$$

It is easy to check that

$$F \in C^{k-1}_{\tilde{\gamma}}(\mathbb{R}_+; H^{p+1}(\mathcal{O}; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}_+))).$$

By combining (3.30), (3.32) and (3.33), we observe that $\partial_z^{2j-1} F|_{z=0} = 0$ for $1 \le j < k'$. Thus, extending F by $F(t, x, z) := F(t, x, |z|)$, and by the definition of k', we have

$$F \in C^{k-1}_{\tilde{\gamma}}(\mathbb{R}_+; H^{p+1}(\mathcal{O}; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))),$$

which is supported in $\mathcal{V}_\delta$ as a function of x. Thus we can use Lemma 3.5 to find Ξ and V, such that, in particular, (3.20) holds true. Let

$$v(t, x, z) := V(t, x, z) + A(t, x, z), \quad \text{in } \mathbb{R}_+ \times \partial\mathcal{O} \times \mathbb{R}_+. \tag{3.34}$$

Then v satisfies all the properties listed in Proposition 3.3. In particular it follows from (3.20), (3.30), (3.32) and (3.34) that (3.4) holds true, with $v_0 = A(0, x, z)$ in $H^{p+2}(\mathcal{O}; C^\infty_0(\overline{\mathbb{R}_+}))$. In particular, if f and g are both supported away from $t = 0$ as a function of time t, the so do A, V and v, and $v_0 = 0$. □

4. Proof of Theorem 2.12

let us first introduce a Lemma which handles multiplication in space $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$.

Lemma 4.1. *Let $\gamma > 0, k, p, s, q$ in $\mathbb{N}_+$ with $p \ge 4$ and $s \ge 2$. Let U in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}))$ and V, $\tilde{V}$ in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$ be scalar functions, then, one has*

$$UV, \tilde{V}V \in C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+))). \tag{4.1}$$

Proof. By Definition 3.2 and Sobolev imbedding, for $0 \le j \le k$, $0 \le |\alpha| \le p - 2$ and $0 \le \beta \le s - 1$,

$$\partial_t^j \partial_x^\alpha U \in L^\infty(\mathbb{R}_+ \times \mathcal{O}) \quad \text{and} \quad \partial_t^j \partial_x^\alpha \partial_z^\beta V \in L^\infty(\mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+).$$

Note that when $p \ge 4$ and $s \ge 2$,

$$\left[\frac{p}{2}\right] \le p - 2 \quad \text{and} \quad \left[\frac{s}{2}\right] \le s - 1.$$

Then we can easily check (4.1) by definition. □

4.1. Construction of profiles

Recall that u^0 is given by Lemma 2.9 which is smooth, curl-free and compactly supported in $(0, T)$ as a function of time t. Now we construct an approximate solution of form (2.27). Plug (2.27) into (2.28), and we can find the equation for u^i and v^i. For the equation of v^i, profiles v^j, u^j with $j < i$ will play roles as source terms. We use Proposition 3.3 to find profile v^j. But there will be some regularity loss. Thanks to Lemma 3.4, we need more regularity of the source term to gain decay of the solution.

Let $\gamma > 1, k, p, s, q$ in $\mathbb{N}_+$ and set $n := [\frac{q}{2} + \gamma]$. We define the mapping Γ by setting $\Gamma(\gamma, k, p, s, q) := (\tilde{\gamma}, \tilde{k}, \tilde{p}, \tilde{s}, \tilde{q})$, where $\tilde{\gamma}, \tilde{k}, \tilde{p}, \tilde{s}, \tilde{q}$ are given by (3.6) and (3.7).

From now on, we fix $\gamma > 1, k, p, s, q$ in $\mathbb{N}_+$ with $k \geq 2$, $p \geq 8$, $s, q \geq 4$, we denote

$$(\gamma_4, k_4, p_4, s_4, q_4) := (\gamma, k, p, s, q),$$
$$(\gamma_i, k_i, p_i - 1, s_i - 1, q_i - 2) := \Gamma(\gamma_{i+1}, k_{i+1}, p_{i+1}, s_{i+1}, q_{i+1}) \quad \text{for } 1 \leq i \leq 3.$$

We observe that, for $0 \leq i \leq 3$,

$$\begin{aligned}
n_{i+1} &= \left[\frac{q_{i+1}}{2} + \gamma_{i+1}\right] \geq 3, \quad k'_{i+1} = \left[\frac{s_{i+1}+1}{2}\right] + k_{i+1} + n_{i+1} \geq 7,\\
\gamma_i &= 2n_{i+1} + 3 \geq q_{i+1} + 2\gamma_{i+1} + 1 \geq \gamma_{i+1} + 6,\\
k_i &= k_{i+1} + k'_{i+1} - 1 \geq k_{i+1} + 6,\\
p_i &= p_{i+1} + k'_{i+1} + 1 \geq p_{i+1} + 8,\\
s_i &= s_{i+1} + 2k_{i+1} + 2n_{i+1} \geq s_{i+1} + 10,\\
q_i &= 2n_{i+1} + 3 \geq q_{i+1} + 2\gamma_{i+1} + 1 \geq q_{i+1} + 3.
\end{aligned}$$

Let

$$\delta_1 := \tilde{S}(\delta) \quad \text{and} \quad \delta_i := \tilde{S}(\delta_{i-1}) \ \text{ for } 2 \leq i \leq 4.$$

Recall that $\tilde{S} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying $\tilde{S}(0) = 0$ and $\tilde{S}(\delta) \geq \delta$ for any $\delta > 0$, we can choose and fix a small $\delta > 0$ such that $\delta_4 < \delta_0$, where δ_0 is defined in Sect. 2.2.

We assume that the initial data u_* satisfies

$$u_* \in H^{p_1 - 1}(\mathcal{O}). \tag{4.2}$$

• Main velocity boundary layer

Let χ_2 a cut-off function such that $\chi_2(x) = 1$ when x in $\mathcal{V}_{\delta/2}$, and $\chi_2(x) = 0$ when x in $\mathcal{O}\backslash\mathcal{V}_\delta$. Set

$$g^1 := 2\mathcal{N}(u^0)\chi_2(x). \tag{4.3}$$

Then g^1 is in $C^\infty(\mathbb{R}_+ \times \overline{\mathcal{O}})$, is supported in $\mathcal{V}_\delta$ as a function of x, is compactly supported in $(0, T)$ as a function of time, and $g^1 \cdot \mathbf{n} = 0$. By Proposition 3.3, there exist ξ^1 in $C^{k_1-1}(\mathbb{R}_+; H^{p_1}(\mathcal{O}; \mathcal{S}(\mathbb{R}_+))$ and v^1 in $C^{k_1}_{\gamma_1}(\mathbb{R}_+; H^{p_1}(\mathcal{O}; (H^{s_1}_{q_1}(\mathbb{R}_+)))$ such that

$$\begin{cases} \partial_t v^1 + u^0 \cdot \nabla v^1 + B^0 v^1 - u^0_\flat z \partial_z v^1 - \partial_z^2 v^1 = \xi^1, & \text{in } \mathbb{R}_+ \times \partial\mathcal{O} \times \mathbb{R}_+, \\ \partial_z v^1|_{z=0} = g^1, & \text{in } \mathbb{R}_+ \times \mathcal{O} \\ v^1|_{t=0} = 0, & \text{in } \mathcal{O} \times \mathbb{R}_+. \end{cases} \tag{4.4}$$

Moreover, ξ^1 is supported in $(\overline{\mathcal{O}}\backslash\overline{\Omega}) \cap \mathcal{V}_{\delta_1}$ as a function of x and is compactly supported in $(0, T)$ as a function of time t, and v^1 is supported in $\mathcal{V}_{\delta_1}$ as a function of x and is supported away from $t = 0$ as a function of time t, and $\xi^1 \cdot \mathbf{n} = v^1 \cdot \mathbf{n} = 0$, for any $t \geq 0$, x in $\mathcal{O}$ and $z \geq 0$.

- Main pressure boundary layer

We set

$$\pi^2(t, x, z) := -\int_z^{+\infty} \left(-u^0 \cdot \nabla \mathbf{n} \cdot v^1 + v^1 \cdot \nabla u^0 \cdot \mathbf{n}\right) dz'.$$

Then π^2 is in $C^{k_1}_{\gamma_1}(\mathbb{R}_+; H^{p_1}(\mathcal{O}; H^{s_1}_{q_1-2}(\mathbb{R}_+)))$ and

$$\partial_z \pi^2 = -u^0 \cdot \nabla \mathbf{n} \cdot v^1 + v^1 \cdot \nabla u^0 \cdot \mathbf{n}. \tag{4.5}$$

Moreover, π^2 is supported in $\mathcal{V}_{\delta_1}$ as a function of x, and is supported away from $t = 0$ as a function of time t.

- Main normal velocity boundary layer

We set

$$w^2(t, x, z) := -\int_z^{\infty} \operatorname{div} v^1(t, x, z') dz'. \tag{4.6}$$

Then $\partial_z w^2 = \operatorname{div} v^1$ and w^2 is $C^{k_1}_{\gamma_1}(\mathbb{R}_+; H^{p_1-1}(\mathcal{O}; H^{s_1}_{q_1-2}(\mathbb{R}_+)))$ is supported in $\mathcal{V}_{\delta_1}$ as a function of x and its t support is away from $t = 0$. Similar to the proof in Sect. 6.1 of [35], we find that

$$\int_{\partial\mathcal{O}} w^2(t, x, 0) dx = 0. \tag{4.7}$$

- Main backflow velocity

Let ϕ^2 be a solution of the following Neumann problem:

$$\begin{cases} \Delta\phi^2 = 0 & \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}}\phi^2 = -w^2(t, x, 0) & \text{on } \partial\mathcal{O}. \end{cases} \tag{4.8}$$

Thanks to (4.7), there exists a unique solution ϕ^2 in $C^{k_1}_{\gamma_1}(\mathbb{R}_+; H^{p_1}(\mathcal{O}))$ up to a constant and ϕ^2 is supported away from $t = 0$ as a function of time t.

- Linearized Euler flow

It follows from Lemma 2.9 that Δu^0 is supported in $\overline{\mathcal{O}}\backslash\Omega$ and is smooth. Thus, by Lemma 2.10 and (4.2), there are ν^2 in $C^{k_1}(\mathbb{R}_+; H^{p_1-2}(\mathcal{O}))$, supported in $\overline{\mathcal{O}} \backslash \overline{\Omega}$

as a function of x, u^2 in $C^{k_1}(\mathbb{R}_+; H^{p_1-1}(\mathcal{O}))$ and p^2 in $C^{k_1-1}_{\gamma_1}(\mathbb{R}_+; H^{p_1-1}(\mathcal{O}))$ such that

$$\begin{cases} \partial_t u^2 + u^0 \cdot \nabla u^2 + u^2 \cdot \nabla u^0 + \nabla p^2 = \nu^2 + \Delta u^0, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ u^2 \cdot \mathbf{n} = 0, & \text{in } \mathbb{R}_+ \times \partial\mathcal{O}, \\ \operatorname{div} u^2 = 0, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ u^2 = u_*, & \text{in } \mathcal{O}. \end{cases} \tag{4.9}$$

Moreover, ν^2, u^2 and p^2 are supported in $[0, T]$ as functions of time t.

• Subprincipal tangential boundary layer

Let

$$\begin{aligned} f^2 := &-\big[v^1 \cdot \nabla v^1 + 2n \cdot \nabla \partial_z v^1 - \Delta\varphi \partial_z v^1 - w^2 \partial_z v^1 + \nabla \pi^2\big]_{\tan} \\ &-(n \cdot \nabla u^0)_{\tan} w^2 - (u^0 \cdot \nabla \mathbf{n}) w^2, \end{aligned} \tag{4.10}$$

$$g^2 := 2\mathcal{N}(v^1)|_{z=0} \chi_2(x). \tag{4.11}$$

By Lemma 4.1, we find that f^2 is in $C^{k_1}_{\gamma_1}(\mathbb{R}_+; H^{p_1-1}(\mathcal{O}; H^{s_1-1}_{q_1-2}(\mathbb{R}_+)))$ and g^2 is in $C^{k_1}_{\gamma_1}(\mathbb{R}_+; H^{p_1-1}(\mathcal{O}))$ satisfy the conditions of Proposition 3.3, that is, f^2 and g^2 are supported in $\mathcal{V}_{\delta_1}$ as functions of x and are supported away from $t = 0$ as functions of time t, and satisfy $f^2(t, x, z) \cdot \mathbf{n}(x) = g^2(t, x) \cdot \mathbf{n}(x) = 0$ for any $t \geq 0$, x in $\mathcal{O}$ and $z \geq 0$. Therefore there exist ξ^2 in $C^{k_2-1}_{\gamma_2}(\mathbb{R}_+; H^{p_2}(\mathcal{O}; \mathcal{S}(\mathbb{R}_+)))$ and a solution v^2 in $C^{k_2}_{\gamma_2}(\mathbb{R}_+; H^{p_2}(\mathcal{O}; H^{s_2}_{q_2}(\mathbb{R}_+)))$ to

$$\begin{cases} \partial_t v^2 + u^0 \cdot \nabla v^2 + B^0 v^2 - u^0_\flat z \partial_z v^2 - \partial_z^2 v^2 = \xi^2 + f^2 & \text{in } \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+, \\ \partial_z v^2|_{z=0} = g^2 & \text{on } \mathbb{R}_+ \times \mathcal{O} \times \{z = 0\}, \\ v^2|_{t=0} = 0 & \text{on } \mathcal{O} \times \mathbb{R}_+. \end{cases} \tag{4.12}$$

Furthermore, ξ^2 is supported in $(\overline{\mathcal{O}} \backslash \overline{\Omega}) \cap \mathcal{V}_{\delta_2}$ as a function of x and is compactly supported in $(0, T)$ as a function of time t, and v^2 is supported in $\mathcal{V}_{\delta_2}$ as a function of x and is supported away from $t = 0$ as a function of time t, and $\xi^2 \cdot \mathbf{n} = v^2 \cdot \mathbf{n} = 0$.

• Subprincipal pressure boundary layer

We set

$$\begin{aligned} \pi^3(t, x, z) := -\int_z^{+\infty} \Big(&\partial_t w^2 + u^0 \cdot \nabla w^2 - u^0 \cdot \nabla \mathbf{n} \cdot v^2 + (v^2 + w^2 \mathbf{n}) \cdot \nabla u^0 \cdot \mathbf{n} \\ &- u^0_\flat z' \partial_z w^2 + v^1 \cdot \nabla v^1 \cdot \mathbf{n} - \partial_z^2 w^2 + \partial_{\mathbf{n}} \pi^2\Big)(t, x, z') dz'. \end{aligned}$$

Then it follows from Lemma 4.1 that π^3 is in $C^{k_2}_{\gamma_2}(\mathbb{R}_+; H^{p_2}(\mathcal{O}; H^{s_2}_{q_2-2}(\mathbb{R}_+))$ and

$$\begin{aligned} \partial_z \pi^3 =& \partial_t w^2 + u^0 \cdot \nabla w^2 - u^0 \cdot \nabla \mathbf{n} \cdot v^2 + (v^2 + w^2 \mathbf{n}) \cdot \nabla u^0 \cdot \mathbf{n} \\ &- u^0_\flat z \partial_z w^2 + v^1 \cdot \nabla v^1 \cdot \mathbf{n} - \partial_z^2 w^2 + \partial_{\mathbf{n}} \pi^2. \end{aligned} \tag{4.13}$$

Moreover, π^3 is supported in $\mathcal{V}_{\delta_2}$ as a function of x and is supported away from $t = 0$ as a function of time t.

• Subprincipal normal velocity boundary layer

Let

$$w^3(t,x,z) := -\int_z^\infty \operatorname{div}(v^2 + w^2\mathbf{n})(t,x,z')dz'. \tag{4.14}$$

Then $\partial_z w^3 = \operatorname{div}(v^2 + w^2\mathbf{n})$ and w^3 is in $C^{k_2}_{\gamma_2}(\mathbb{R}_+; H^{p_2-1}(\mathcal{O}; H^{s_2}_{q_2-2}(\mathbb{R}_+))$ is supported in $\mathcal{V}_{\delta_2}$ as a function of x and is supported away from $t = 0$ as a function of time t, furthermore

$$\int_{\partial\mathcal{O}} w^3(t,x,0)dx = 0. \tag{4.15}$$

• <u>Lower order backflow velocity</u>

Let ϕ^3 be a solution of the following Neumann problem:

$$\begin{cases} \Delta\phi^3 = 0 \quad \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}}\phi^3 = -w^3(t,x,0) \quad \text{on } \partial\mathcal{O}. \end{cases} \tag{4.16}$$

Thanks to (4.15), there exists a unique solution ϕ^3 in $C^{k_2}_{\gamma_2}(\mathbb{R}_+; H^{p_2}(\mathcal{O}))$ up to a constant and ϕ^3 is supported away from $t = 0$ as a function of time t.

• <u>Lower order interior flow</u>

We take

$$u^3(t,x) = v^3(t,x) = 0, \quad p^3(t,x) = 0 \quad \text{for } t \in \mathbb{R}_+, x \in \mathcal{O}. \tag{4.17}$$

• <u>Lower order tangential velocity boundary layer</u>

Let

$$\begin{aligned} \tilde{f}^3 :=& \nabla\pi^3 + v^1 \cdot \nabla(u^2 + \nabla\phi^2 + v^2 + w^2\mathbf{n}) \\ &+ (u^2 + \nabla\phi^2 + v^2 + w^2\mathbf{n}) \cdot \nabla v^1 \\ &- w^2\partial_z(v^2 + w^2\mathbf{n}) - w^3\partial_z v^1 - \Delta v^1 \\ &+ 2n \cdot \nabla\partial_z(v^2 + w^2\mathbf{n}) - \Delta\varphi\partial_z(v^2 + w^2\mathbf{n}), \end{aligned} \tag{4.18}$$

and

$$f^3 := -(\tilde{f}^3)_{\tan} - (n \cdot \nabla u^0)_{\tan} w^3 - (u^0 \cdot \nabla\mathbf{n})w^3, \tag{4.19}$$
$$g^3 := 2\mathcal{N}(u^2 + v^2 + \nabla\phi^2 + w^2\mathbf{n})|_{z=0}\chi_2(x). \tag{4.20}$$

Thanks to Lemma 4.1, f^3 in $C^{k_2}_{\gamma_2}(\mathbb{R}_+; H^{p_2-1}(\mathcal{O};, H^{s_2-1}_{q_2-2}(\mathbb{R}_+))$ and g^3 in $C^{k_2}_{\gamma_2}$ $(\mathbb{R}_+; H^{p_2-1}(\mathcal{O}))$ and satisfy $f^3(t,x,z)\cdot\mathbf{n}(x) = g^3(t,x)\cdot\mathbf{n}(x) = 0$ for any $t \geq 0$, x in $\mathcal{O}$ and $z \geq 0$. Moreover f^3 and g^3 are supported in $\mathcal{V}_{\delta_2}$ as functions of x. Then, by using Proposition 3.3, there exist ξ^3 in $C^{k_3-1}_{\gamma_3}(\mathbb{R}_+; H^{p_3}(\mathcal{O}; \mathcal{S}(\mathbb{R}_+)))$, v^3 in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3}(\mathcal{O}; H^{s_3}_{q_3}(\mathbb{R}_+))$ and v^3_0 in $H^{p_3+2}(\mathcal{O}; C^\infty_0(\overline{\mathbb{R}_+}))$ such that

$$\begin{cases} \partial_t v^3 + u^0\cdot\nabla v^3 + B^0 v^3 - u^0_\flat z\partial_z v^3 - \partial_z^2 v^3 = \xi^3 + f^3 \quad \text{in } \mathbb{R}_+\times\mathcal{O}\times\mathbb{R}_+, \\ \partial_z v^3|_{z=0} = g^3 \quad \text{on } \mathbb{R}_+\times\mathcal{O}\times\{z=0\}, \\ v^3|_{t=0} = v^3_0 \quad \text{on } \mathcal{O}\times\mathbb{R}_+. \end{cases} \tag{4.21}$$

Moreover, ξ^3 is supported in $(\overline{\mathcal{O}}\backslash\overline{\Omega}) \cap \mathcal{V}_{\delta_3}$ as a function of x, v^3 is supported in $\mathcal{V}_{\delta_3}$ as a function of x and $\xi^3 \cdot \mathbf{n} = v^3 \cdot \mathbf{n} = 0$.

- A lower order pressure boundary layer

  We set

$$\pi^4(t,x,z) := -\int_z^{+\infty} \Big(\partial_t w^3 + u^0 \cdot \nabla w^3 - u^0 \cdot \nabla \mathbf{n} \cdot v^3 - u^0_\flat z' \partial_z w^3 + (v^3 + w^3\mathbf{n}) \cdot \nabla u^0 \cdot \mathbf{n} - \partial_z^2 w^3 + \tilde{f}^3 \cdot \mathbf{n}\Big)(t,x,z')dz'.$$

Hence $\partial_z \pi^4$ in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3}(\mathcal{O}; H^{s_3}_{q_3-2}(\mathbb{R}_+))$ and

$$\partial_z \pi^4 := \partial_t w^3 + u^0 \cdot \nabla w^3 - u^0 \cdot \nabla \mathbf{n} \cdot v^3 + (v^3 + w^3\mathbf{n}) \cdot \nabla u^0 \cdot \mathbf{n} - u^0_\flat z \partial_z w^3 - \partial_z^2 w^3 + \tilde{f}^3 \cdot \mathbf{n}. \tag{4.22}$$

Furthermore, π^4 is supported in $\mathcal{V}_{\delta_3}$ as a function of x.

- A lower order normal velocity boundary layer

  Set

$$w^4(t,x,z) := -\int_z^{\infty} \operatorname{div}(v^3 + w^3\mathbf{n})(t,x,z')dz'. \tag{4.23}$$

Then $\partial_z w^4 = \operatorname{div}(v^3 + w^3\mathbf{n})$ and w^4 belongs to $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3-1}(\mathcal{O}; H^{s_3}_{q_3-2}(\mathbb{R}_+))$ and is supported in $\mathcal{V}_{\delta_3}$ as a function of x, with $w^4|_{t=0} = w^4_0$ in $H^{p_3+1}(\mathcal{O}; C^\infty_0(\overline{\mathbb{R}_+}))$. Moreover w^4 satisfies

$$\int_{\partial\mathcal{O}} w^4(t,x,0)dx = 0. \tag{4.24}$$

- A lower order backflow velocity

  Let ϕ^4 be a solution of the following Neumann problem:

$$\begin{cases} \Delta\phi^4 = 0 \quad \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}}\phi^4 = -w^4(t,x,0) \quad \text{on } \partial\mathcal{O}. \end{cases} \tag{4.25}$$

Thanks to (4.24), there exists a unique solution ϕ^4 in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3}(\mathcal{O}))$ up to a constant, with $\phi^4|_{t=0} = \phi^4_0$ in $H^{p_3+2}(\mathcal{O})$.

- A lower order interior flow

  Let

$$l^4 := -u^2 \cdot \nabla u^2 + \Delta u^2 \in C^{k_2}_{\gamma_2}(\mathbb{R}_+; H^{p_2-2}(\mathcal{O})), \tag{4.26}$$

and obeserve that curl l^4 is supported in $[0,T]$ as a function of time. By Lemma 2.10, there are ν^4 in $C^{k_3}(\mathbb{R}_+; H^{p_3-2}(\mathcal{O}))$, supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$ as a function of x, u^4 in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3-1}(\mathcal{O}))$ and p^4 in $C^{k_3-1}_{\gamma_3}(\mathbb{R}_+; H^{p_3-1}(\mathcal{O}))$ such that

$$\begin{cases} \partial_t u^4 + u^0 \cdot \nabla u^4 + u^4 \cdot \nabla u^0 + \nabla p^4 = \nu^4 + l^4 \quad \text{in } R_+ \times \mathcal{O}, \\ \operatorname{div} u^4 = 0 \quad \text{in } R_+ \times \mathcal{O}, \\ u^4 \cdot \mathbf{n} = 0 \quad \text{on } R_+ \times \partial\mathcal{O}, \\ u^4|_{t=0} = 0 \quad \text{in } \mathcal{O}. \end{cases} \tag{4.27}$$

Moreover, ξ^4, u^4 and p^4 are supported in $[0, T]$ as functions of time t.

• a lower order tangential velocity boundary layer

Set

$$
\begin{aligned}
\tilde{f}^4 :=& v^1 \cdot \nabla(u^3 + \nabla\phi^3 + v^3 + w^3\mathbf{n}) + (u^2 + \phi^2 + v^2 + w^2\mathbf{n}) \cdot \nabla(v^2 + w^2\mathbf{n}) \\
&+ (v^2 + w^2\mathbf{n}) \cdot \nabla(u^2 + \phi^2 + v^2 + w^2\mathbf{n}) + (u^3 + \nabla\phi^3 + v^3 + w^3\mathbf{n}) \cdot \nabla v^1 \\
&- w^2\partial_z(v^3 + w^3\mathbf{n}) - w^3\partial_z(v^2 + w^2\mathbf{n}) - w^4\partial_z v^1 - \Delta(v^2 + w^2\mathbf{n}) \\
&+ 2n \cdot \nabla\partial_z(v^3 + w^3\mathbf{n}) - \Delta\varphi\partial_z(v^3 + w^3\mathbf{n}) + \nabla\pi^4,
\end{aligned}
\tag{4.28}
$$

and

$$
f^4 := -\tilde{f}^4_{\tan} - (n \cdot \nabla u^0)_{\tan} w^4 - (u^0 \cdot \nabla\mathbf{n})w^4, \tag{4.29}
$$

$$
g^4 := 2\mathcal{N}(u^3 + \nabla\phi^3 + v^3 + w^3\mathbf{n})|_{z=0}\chi_2(x). \tag{4.30}
$$

Thanks to Lemma 4.1, one can check that f^4 in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3-1}(\mathcal{O}; H^{s_3-1}_{q_3-2}(\mathbb{R}_+))$ and g^4 in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3-1}(\mathcal{O}))$ and satisfy $f^4(t, x, z) \cdot \mathbf{n}(x) = g^4(t, x) \cdot \mathbf{n}(x) = 0$ for any $t \geq 0$, x in $\mathcal{O}$ and $z \geq 0$. Moreover f^4 and g^4 are supported in $\mathcal{V}_{\delta_3}$ as functions of x. Then by using Proposition 3.3, there exist ξ^4 in $C^{k_4-1}_{\gamma_4}(\mathbb{R}_+; H^{p_4}(\mathcal{O}; \mathcal{S}(\mathbb{R}_+)))$, v^4 in $C^{k_4}_{\gamma_4}(\mathbb{R}_+; H^{p_4}(\mathcal{O}; H^{s_4}_{q_4}(\mathbb{R}_+))$ and v^4_0 in $H^{p_4+2}(\mathcal{O}; C^\infty_0(\overline{\mathbb{R}_+}))$ such that

$$
\begin{cases}
\partial_t v^4 + u^0 \cdot \nabla v^4 + B^0 v^4 - u^0_\flat z\partial_z v^4 - \partial_z^2 v^4 = \xi^4 + f^4 & \text{in } \mathcal{O}, \\
\partial_z v^4|_{z=0} = g^4 & \text{in } \mathcal{O}, \\
v^4|_{t=0} = v^4_0 & \text{in } \mathcal{O}.
\end{cases}
\tag{4.31}
$$

Moreover ξ^4 is supported in $(\overline{\mathcal{O}}\backslash\overline{\Omega}) \cap \mathcal{V}_{\delta_4}$ as a function of x and is compactly supported in $(0, T)$ as a function of time t, and v^4 is supported in $\mathcal{V}_{\delta_4}$ as a function of x and $\xi^4 \cdot \mathbf{n} = v^4 \cdot \mathbf{n} = 0$.

• A last pressure boundary layer

We set

$$
\begin{aligned}
\pi^5(t, x, z) := -\int_z^{+\infty} \Big(&\partial_t w^4 + u^0 \cdot \nabla w^4 + (v^4 + w^4\mathbf{n}) \cdot \nabla u^0 \cdot \mathbf{n} \\
&- u^0 \cdot \nabla\mathbf{n} \cdot v^4 - u^0_\flat z'\partial_z w^4 - \partial_z^2 w^4 + \tilde{f}^4 \cdot \mathbf{n}\Big)(t, x, z')dz',
\end{aligned}
$$

Then π^5 in $C^{k_4}_{\gamma_4}(\mathbb{R}_+; H^{p_4}(\mathcal{O}; H^{s_4}_{q_4-2}(\mathbb{R}_+))$ and

$$
\begin{aligned}
\partial_z\pi^5 :=& \partial_t w^4 + u^0 \cdot \nabla w^4 - u^0 \cdot \nabla\mathbf{n} \cdot v^4 + (v^4 + w^4\mathbf{n}) \cdot \nabla u^0 \cdot \mathbf{n} \\
&- u^0_\flat z\partial_z w^4 - \partial_z^2 w^4 + \tilde{f}^4 \cdot \mathbf{n}.
\end{aligned}
\tag{4.32}
$$

Moreover, π^5 is supported in $\mathcal{V}_{\delta_4}$ as a function of x.

In summary, we have now constructed

$$
\begin{aligned}
u^j &\in C^{k_{j-1}}(\mathbb{R}_+; H^{p_{j-1}-1}(\mathcal{O})), p^j \in C^{k_{j-1}-1}(\mathbb{R}_+; H^{p_{j-1}-1}(\mathcal{O})), \quad 2 \le j \le 4,\\
\nu^j &\in C^{k_{j-1}}(\mathbb{R}_+; H^{p_{j-1}-2}(\mathcal{O})), \phi^j \in C^{k_{j-1}}_{\gamma_{j-1}}(\mathbb{R}_+; H^{p_{j-1}}(\mathcal{O})), \quad 2 \le j \le 4,\\
v^j &\in C^{k_j}_{\gamma_j}(\mathbb{R}_+; H^{p_j}(\mathcal{O}; H^{s_j}_{q_j}(\mathbb{R}_+))), \pi^{j+1} \in C^{k_j}_{\gamma_j}(\mathbb{R}_+; H^{p_j}(\mathcal{O}; H^{s_j}_{q_j-2}(\mathbb{R}_+))), \quad 1 \le j \le 4,\\
w^j &\in C^{k_{j-1}}_{\gamma_{j-1}}(\mathbb{R}_+; H^{p_{j-1}-1}(\mathcal{O}; H^{s_{j-1}}_{q_{j-1}-2}(\mathbb{R}_+))), \quad 2 \le j \le 4,\\
\xi^j &\in C^{k_j-1}_{\gamma_j}(\mathbb{R}_+; H^{p_j}(\mathcal{O}; \mathcal{S}(\mathbb{R}_+))), \quad 1 \le j \le 4.
\end{aligned}
$$

Moreover, u^j, p^j, ν^j and ξ^j are supported in $[0, T]$ as functions of time t, ν^j and ξ^j are supported in $\overline{\mathcal{O}}\backslash\overline{\Omega}$ as functions of x, v^j, w^{j+1}, π^{j+1} are supported in $\mathcal{V}_{\delta_j}$ as functions of x and $v^j \cdot \mathbf{n} = \xi^j \cdot \mathbf{n} = 0$. Furthermore, v^1, v^2, ϕ^2, ϕ^3, w^2 and w^3 are supported away from $t = 0$ as a function of time t.

4.2. Construction of the family of approximate solutions

Let us start with a notation: for a profile $f(t, x, z)$, we define

$$\{f\}_\varepsilon := f\Big(t, x, \frac{\varphi(x)}{\sqrt{\varepsilon}}\Big).$$

We define the approximate solutions via

$$u_a^\varepsilon := u^0 + \sqrt{\varepsilon}\{v^1\}_\varepsilon + \sum_{j=2}^{4} \varepsilon^{\frac{j}{2}}\big(u^j + \nabla\phi^j + \{v^j\}_\varepsilon + \{w^j\}_\varepsilon\mathbf{n}\big), \tag{4.33}$$

$$p_a^\varepsilon := p^0 + \sum_{j=2}^{4} \varepsilon^{\frac{j}{2}}\big(p^j - \partial_t\phi^j - u^0 \cdot \nabla\phi^j + \{\pi^j\}_\varepsilon\big), \tag{4.34}$$

$$\xi^\varepsilon := \nu^0 + \sqrt{\varepsilon}\{\xi^1\}_\varepsilon + \sum_{j=2}^{4} \varepsilon^{\frac{j}{2}}\big(\nu^j + \{\xi^j\}_\varepsilon\big). \tag{4.35}$$

4.3. Consistency estimates of the approximate solutions

Below we use a slight abuse of notations by denoting, for an integer m, by Z^m any iterated tangential derivatives Z^α where $\alpha = (\alpha_0, \cdots, \alpha_5)$ is in $\mathbb{N}^5$ with $|\alpha| = m$, where we recall that the notation Z^α is defined in (2.29) of Sect. 2.8.

Lemma 4.2. *Let $\gamma > 0$, k, p, s, q in $\mathbb{N}$ with $p \ge 3$ and $s \ge 1$. Let the profile V in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$ and is supported in $\mathcal{V}_{\delta_0}$. Then one has*

(1) for $0 \le j \le k$, $p_1 + p_2 \le p - 1$ and $p_2 \le s$,

$$\|\partial_t^j Z^{p_1}(\sqrt{\varepsilon}\partial_\mathbf{n})^{p_2}\{V\}_\varepsilon\|_{L^2(\mathcal{O})} \lesssim \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}, \tag{4.36}$$

(2) for $0 \le j \le k$, $p_1 + p_2 \le p - 2$ *and* $p_2 \le s - 1$,

$$\|\partial_t^j Z^{p_1}(\sqrt{\varepsilon}\partial_{\mathbf{n}})^{p_2}\{V\}_\varepsilon\|_{L^\infty(\mathcal{O})} \lesssim \langle t\rangle^{-\gamma}, \tag{4.37}$$

(3) for $m \le p - 1$,

$$\|\{V\}_\varepsilon\|_{H^m(\partial\mathcal{O})} \lesssim \langle t\rangle^{-\gamma}. \tag{4.38}$$

Proof. We first observe that

$$\sqrt{\varepsilon}\partial_{\mathbf{n}}\{V\}_\varepsilon = \sqrt{\varepsilon}\{\partial_{\mathbf{n}}V\}_\varepsilon + \{\partial_z V\}_\varepsilon,$$
$$Z^0\{V\}_\varepsilon = \{Z^0 V\}_\varepsilon + \{z\partial_z V\}_\varepsilon \text{ and } Z^j\{V\}_\varepsilon = \{Z^j V\}_\varepsilon \text{ for } 1 \le j \le 5.$$

We can take the normal derivatives p_2 times, take the tangential derivatives p_1 times and take the time derivatives j times and use [25, Lemma 3] to get (4.36). For (4.37), we use Sobolev imbedding $H^1(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$ for variable z and $H^2(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ for variable x. For (4.38), we use the trace theorem to get $H^{m+1}(\mathcal{O}) \hookrightarrow H^m(\partial\mathcal{O})$. □

Let us now turn to the justification of the consistence of the approximate solutions constructed in (4.33-4.35) with the system (2.28a-2.28e).

• Consistency of (2.28a). Definition and estimate of F.

By (4.33)-(4.35), (2.8a), (2.25), (4.4)-(4.6), (4.9), (4.10), (4.12), (4.13), (4.14), (4.17)-(4.19), (4.21), (4.22), (4.23), (4.26)-(4.29), (4.31) and (4.32), we find that u_a^ε satisfies (2.28b) with

$$\begin{aligned}
F :=& -\{n\partial_z\pi^5\}_\varepsilon + \sqrt{\varepsilon}\Big\{v^1\cdot\nabla(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\\
&+(u^2+\nabla\phi^2+v^2+w^2\mathbf{n})\cdot\nabla(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})\\
&-w^2\partial_z(v^4+w^4\mathbf{n})+(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})\cdot\nabla(u^2+\nabla\phi^2+v^2+w^2\mathbf{n})\\
&-w^3\partial_z(v^3+w^3\mathbf{n})+(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\cdot\nabla v^1-w^4\partial_z(v^2+w^2\mathbf{n})\\
&-\Delta(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})+2n\cdot\nabla\partial_z(v^4+w^4\mathbf{n})-\Delta\varphi\partial_z(v^4+w^4\mathbf{n})\Big\}_\varepsilon\\
&+\varepsilon\Big\{(u^2+\nabla\phi^2+v^2+w^2\mathbf{n})\cdot\nabla(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\\
&+(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})\cdot\nabla(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})-w^3\partial_z(v^4+w^4\mathbf{n})\\
&+(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\cdot\nabla(u^2+\nabla\phi^2+v^2+w^2\mathbf{n})-w^4\partial_z(v^3+w^3\mathbf{n})\\
&-\Delta(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\Big\}_\varepsilon\\
&+\varepsilon^{\frac{3}{2}}\Big\{(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})\cdot\nabla(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\\
&+(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\cdot\nabla(u^3+\nabla\phi^3+v^3+w^3\mathbf{n})-w^4\partial_z(v^4+w^4\mathbf{n})\Big\}_\varepsilon\\
&+\varepsilon^2\Big\{(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\cdot\nabla(u^4+\nabla\phi^4+v^4+w^4\mathbf{n})\Big\}_\varepsilon.
\end{aligned}\tag{4.39}$$

By the constructions of u^i, ϕ^i, v^i, w^i and the definition of γ_i, k_i, p_i, s_i, q_i, we have $u^i + \nabla\phi^i$ in $C_\gamma^k(\mathbb{R}_+; H^p(\mathcal{O}))$ and $v^i + w^i\mathbf{n}$ in $C_\gamma^k(\mathbb{R}_+; H^p(\mathcal{O}; H_q^s(\mathbb{R}_+)))$. Then (2.35) and (2.36) for the part of F is a direct consequence of Lemma 4.1 and Lemma 4.2.

• Consistency of (2.28b). Definition and estimate of H.

By (2.8b), (4.8), (4.9), (4.14), (4.16), (4.17), (4.23), (4.25) and (4.27), we find that u_a^ε satisfies (2.28b) with

$$H := \{\mathrm{div}\,(v^4 + w^4\mathbf{n})\}_\varepsilon. \tag{4.40}$$

By construction div $(v^4 + w^4\mathbf{n})$ in $C^k_\gamma(\mathbb{R}_+; H^{p-1}(\mathcal{O}; H^s_q(\mathbb{R}_+)))$, so Lemma 4.2 immediately leads to the estimates, (2.35), (2.36) and (2.37) for the part of H.

• Consistency of (2.28d). Definition and estimate of G.

By (4.3), (4.4), (4.11), (4.12), (4.20), (4.21), (4.30) and (4.31), u_a^ε satisfies (2.28d) with

$$G := \mathcal{N}(u^4 + \nabla\phi^4 + v^4 + w^4\mathbf{n})|_{z=0}. \tag{4.41}$$

By construction, $u^4 + \nabla\phi^4$ in $C^{k_3}_{\gamma_3}(\mathbb{R}_+; H^{p_3-1}(\mathcal{O}))$ and $v^4 + w^4\mathbf{n}$ in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$. By definition of $\gamma_i, k_i, p_i, s_i, q_i$, we find that G in $C^k_\gamma(\mathbb{R}_+; H^{p-1}(\mathcal{O}))$, which is exactly (2.37) for the part of G.

• Consistency of (2.28c) and (2.28e). Definition and estimate of R_0.

By (2.8c), (2.8d), (4.4), (4.6), (4.8), (4.9), (4.12), (4.14), (4.16), (4.17), (4.21), (4.23), (4.25), (4.27) and (4.31), (2.28c) and (2.28e) are satisfied with

$$R_0 = -\varepsilon^{\frac{1}{2}} v_0^3 - (v_0^4 + \nabla\phi_0^4 + w_0^4\mathbf{n}), \tag{4.42}$$

and (2.38)) is a direct consequences of Lemma 4.2.

4.4. Verification of (2.39)–(2.41)

Let us verify (2.39) and (2.40) first. Since u^0 is smooth and has compact support in t,

$$\|u^0\|_{W^{1,\infty}(\mathcal{O})} + \|\nabla u^0\|_{m-1,\infty} + \|\nabla^2 u^0\|_{m-1,\infty} \lesssim \chi_{[0,T]}(t). \tag{4.43}$$

By construction, v^j in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$, $1 \le j \le 4$. Then it follows from (4.37) of Lemma 4.2 that, for $1 \le j \le 4$, $m \le p - 3$,

$$\sqrt{\varepsilon}\|\{v^j\}_\varepsilon\|_{W^{1,\infty}} + \sqrt{\varepsilon}\|\nabla\{v^j\}_\varepsilon\|_{m-1,\infty} + \varepsilon\|\nabla^2\{v^j\}_\varepsilon\|_{m-1,\infty} \lesssim \langle t\rangle^{-\gamma}. \tag{4.44}$$

The same inequality holds for $w^j\mathbf{n}$ with $2 \le j \le 4$, since they also belong to the space $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$. For u^j, $2 \le j \le 4$, it belongs to $C^k(\mathbb{R}_+; H^p(\mathcal{O}))$ and is supported in $[0, T]$. Hence Sobolev imbedding Theorem ensures that, for $2 \le j \le 4$, $m \le p - 3$,

$$\|u^j\|_{W^{1,\infty}(\mathcal{O})} + \|\nabla u^j\|_{m-1,\infty} + \|\nabla^2 u^j\|_{m-1,\infty} \lesssim \chi_{[0,T]}(t). \tag{4.45}$$

For $\nabla\phi^j$, $2 \le j \le 4$, it belongs to $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}))$. Then it follows form Sobolev imbedding Theorem that, for $2 \le j \le 4$, $m \le p - 3$,

$$\|\nabla\phi^j\|_{W^{1,\infty}(\mathcal{O})} + \|\nabla^2\phi^j\|_{m-1,\infty} + \|\nabla^3\phi^j\|_{m-1,\infty} \lesssim \langle t\rangle^{-\gamma}. \tag{4.46}$$

Combine (4.43)-(4.46), we have verified (2.39) and (2.40).

Let us move on to (2.41). Since u^0 is smooth and u^j in $C^k(\mathbb{R}_+; H^p(\mathcal{O}))$ for $2 \le j \le 4$, and they both supported in $[0, T]$, one has

$$\|u^0\|_{H^1(\mathcal{O})} + \|u^j\|_{H^1(\mathcal{O})} \le \chi_{[0,T]}(t).$$

For $\nabla\phi^j$ in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}))$, $2 \le j \le 4$,

$$\|\nabla\phi^j\|_{H^1(\mathcal{O})} \lesssim \langle t\rangle^{-\gamma}.$$

For v^j in $C^k_\gamma(\mathbb{R}_+; H^p(\mathcal{O}; H^s_q(\mathbb{R}_+)))$, $1 \le j \le 4$, it follows from Lemma 4.2 that

$$\sqrt{\varepsilon}\|\{v^j\}_\varepsilon\|_{H^1(\mathcal{O})} \lesssim \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}.$$

The same estimates holds for $w^j\mathbf{n}$. By gathering the above estimates, we find that

$$\|u^\varepsilon_a(t,\cdot)\|_{H^1(\mathcal{O})} \lesssim \chi_{[0,T]}(t) + \varepsilon\langle t\rangle^{-\gamma} + \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}.$$

As a result, it comes out

$$\|u^\varepsilon_a(T/\varepsilon,\cdot)\|_{H^1(\mathcal{O})} \lesssim \varepsilon^{\gamma+\frac{1}{4}}.$$

Since $\gamma > 1$, (2.41) holds true.

5. Estimates of the Remainder R

The goal of this section is to establish the *a priori* estimate (2.43) for the remainder term R defined by (2.42). We also introduce the remainder pressure term π such that $p^\varepsilon = p^\varepsilon_a + \varepsilon^2\pi$. Then in view of (2.4), (2.28) and (2.42), we write

$$\partial_t R - \varepsilon\Delta R + u^\varepsilon\cdot\nabla R + R\cdot\nabla u^\varepsilon_a + \nabla\pi = -F \quad \text{and}$$
$$\operatorname{div} R = -H \quad \text{in } \mathbb{R}_+\times\mathcal{O}, \tag{5.1a}$$
$$R\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathcal{N}(R) = -G \quad \text{on } \mathbb{R}_+\times\partial\mathcal{O}, \tag{5.1b}$$
$$R|_{t=0} = R_0 \quad \text{in } \mathcal{O}. \tag{5.1c}$$

These equations are satisfied up to the time T^ε introduced in Sect. 2.9. At the end of this section, once the *a priori* estimate (2.43) in hands, we will deduce that $T^\varepsilon \ge \frac{T}{\varepsilon}$.

We will start with a L^2 estimate in Sect. 5.1, then we will turn to tangential derivatives estimates in Subsection 5.2. We will also need to handle the estimate of one normal derivative, and for that, we introduce an appropriate substitute to the vorticity, see (5.37), which is in the spirit of [30]. We will see in Sect. 5.3 that this quantity, as the vorticity, allows us to estimate one normal derivative. The advantage of this quantity over the vorticity is that its time evolution is easier to be investigated; this will be done in Sect. 5.4. The estimate of the terms involving the pressure are quite difficult and are therefore postponed to Sect. 5.5. An estimate of $\|R\|_{1,\infty}$ will be obtained in Sect. 5.6. The end of the proof of (2.43) will be given in Sect. 2.9.

5.1. L^2 Estimates

From now on, we simplify $\|\cdot\|_{L^2(\mathcal{O})}$ as $\|\cdot\|$.

Proposition 5.1. *There exist a constant $C>0$, such that the remainder R satisfies*

$$\|R(t)\|^2+\varepsilon\int_0^t\|\nabla R\|^2dt'\leq C\varepsilon^{-\frac{1}{4}}\quad \textit{for}\ 0\leq t\leq T^\varepsilon. \tag{5.2}$$

Proof. Let $\mathbb{P}$ the Leray projection operator to the divergence free vector field, we decompose R into $R=\mathbb{P}R+\nabla\phi$. Hence ϕ satisfies $\Delta\phi=\operatorname{div}R=-H$ in $\mathcal{O}$ and $\partial_{\mathbf{n}}\phi=R\cdot\mathbf{n}=0$ on $\partial\mathcal{O}$. By elliptic regularity and (2.35), one has

$$\|(I-\mathbb{P})R\|_{H^1(\mathcal{O})}\lesssim\|H\|_{L^2(\mathcal{O})}\lesssim\varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}. \tag{5.3}$$

Next we estimate $\mathbb{P}R$. Indeed by taking L^2 inner product of (5.1a) with $\mathbb{P}R$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbb{P}R(t)\|^2-\varepsilon\int_{\mathcal{O}}\Delta R\cdot\mathbb{P}R+\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla R)\cdot\mathbb{P}R\\ +\int_{\mathcal{O}}(R\cdot\nabla u_a^\varepsilon)\cdot\mathbb{P}R+\int_{\mathcal{O}}\nabla\pi\cdot\mathbb{P}R=-\int_{\mathcal{O}}F\cdot\mathbb{P}R.\end{aligned} \tag{5.4}$$

Let us now estimate each term of (5.4), from the right to the left.

- Since F satisfies (2.35), we have

$$|\int_{\mathcal{O}}F\cdot\mathbb{P}R|\lesssim\|F\|\|\mathbb{P}R\|\lesssim\varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}\big(\|\mathbb{P}R\|^2+1\big). \tag{5.5}$$

- On the other hand, in view of (5.1b), we get, by an integration by parts, that

$$\int_{\mathcal{O}}\nabla\pi\cdot\mathbb{P}R=0.$$

- Let us now move to the term before in (5.4). We first deduce from (5.3) that

$$\|R\|\lesssim\|\mathbb{P}R\|+\varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}, \tag{5.6}$$

  which, together with (2.39), ensures that

$$|\int_{\mathcal{O}}(R\cdot\nabla u_a^\varepsilon)\cdot\mathbb{P}R|\lesssim\|\nabla u_a^\varepsilon\|_{L^\infty(\mathcal{O})}\|R\|\|\mathbb{P}R\|\lesssim\langle t\rangle^{-\gamma}\big(\|\mathbb{P}R\|^2+\varepsilon^{\frac{1}{2}}\langle t\rangle^{-2\gamma}\big). \tag{5.7}$$

- To deal with the third term in (5.4), we start with using again the Helmholtz-Leray decomposition to deduce that

$$\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla R)\cdot\mathbb{P}R=\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla\mathbb{P}R)\cdot\mathbb{P}R+\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla(I-\mathbb{P})R)\cdot\mathbb{P}R. \tag{5.8}$$

Thanks to (2.4b), (2.4c), and σ^0 is supported on $[0, T]$, we get, by using integration by parts, that

$$|\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla\mathbb{P}R)\cdot\mathbb{P}R|\lesssim\langle t\rangle^{-\gamma}\|\mathbb{P}R\|^2. \tag{5.9}$$

Moreover, to deal with the last term in (5.8), we first use the decomposition (2.42) to obtain

$$|\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla(I-\mathbb{P})R)\cdot\mathbb{P}R|\lesssim\|\nabla(I-\mathbb{P})R\|(\|u_a^\varepsilon\|_{L^\infty}\|\mathbb{P}R\|+\varepsilon^2\|R\|_{L^4}^2).$$

Observing from Korn's inequality and (5.6) that

$$\|R\|_{H^1}\lesssim\|D(R)\|+\|\mathbb{P}R\|+\varepsilon^{\frac14}\langle t\rangle^{-\gamma}. \tag{5.10}$$

Then recalling that $\|R\|_{L^4}\lesssim\|R\|^{\frac14}\|\nabla R\|^{\frac34}$, and using again (5.6), we find

$$\begin{aligned}\|R\|_{L^4}^2&\lesssim(\|\mathbb{P}R\|+\varepsilon^{\frac14}\langle t\rangle^{-\gamma})^{\frac12}(\|D(R)\|+\|\mathbb{P}R\|+\varepsilon^{\frac14}\langle t\rangle^{-\gamma})^{\frac32}\\&\lesssim\lambda\|D(R)\|^2+C_\lambda(\|\mathbb{P}R\|^2+\varepsilon^{\frac12}\langle t\rangle^{-2\gamma}),\end{aligned}$$

for a small constant $\lambda>0$, where in the last step, we used Young's inequality. As a consequence, we deduce from (5.3) and (2.39) that

$$\begin{aligned}|\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla(I-\mathbb{P})R)\cdot\mathbb{P}R|&\lesssim\varepsilon^{\frac14}\langle t\rangle^{-\gamma}(\langle t\rangle^{-\gamma}\|\mathbb{P}R\|\\&\quad+\varepsilon^2\lambda\|D(R)\|^2+\varepsilon^2C_\lambda(\|\mathbb{P}R\|^2+\varepsilon^{\frac12}\langle t\rangle^{-2\gamma}))\\&\lesssim\varepsilon^2\lambda\|D(R)\|^2+C_\lambda\varepsilon^{\frac14}\langle t\rangle^{-\gamma}(\|\mathbb{P}R\|^2+1).\end{aligned} \tag{5.11}$$

- For the second term of the energy equality, (5.4), we start with the following integration by parts:

$$-\int_{\mathcal{O}}\Delta R\cdot\mathbb{P}R=2\int_{\mathcal{O}}D(R)\cdot D(\mathbb{P}R)+2\int_{\partial\mathcal{O}}(D(R)\cdot\mathbf{n})_{\tan}\cdot\mathbb{P}R.$$

Then, on the one hand, it follows from (5.3) that

$$\begin{aligned}2\int_{\mathcal{O}}D(R)\cdot D(\mathbb{P}R)=&2\|D(R)\|^2-2\int_{\mathcal{O}}D(R)\cdot D((I-\mathbb{P})R)\\\geq&\|D(R)\|^2-C\|D((I-\mathbb{P})R)\|^2\\\geq&\|D(R)\|^2-C\varepsilon^{\frac14}\langle t\rangle^{-\gamma},\end{aligned} \tag{5.12}$$

and, on the other hand, by using boundary condition $\mathcal{N}(R)=G$ on $\partial\mathcal{O}$, one has

$$\begin{aligned}\int_{\partial\mathcal{O}}(D(R)\cdot\mathbf{n})_{\tan}\cdot\mathbb{P}R&=\int_{\partial\mathcal{O}}\big(G-(MR)_{\tan}\big)\cdot\mathbb{P}R\\&=\int_{\mathcal{O}}\operatorname{div}\big(n(G-(MR)_{\tan})\cdot\mathbb{P}R\big),\end{aligned}$$

so that thanks to (5.6), (5.10) and (2.37), for $\lambda > 0$, we get, by applying Young's inequality, that

$$\left|\int_{\partial\mathcal{O}} (D(R)\cdot\mathbf{n})_{\tan}\cdot\mathbb{P}R\right| \lesssim \lambda\|D(R)\|^2 + C_\lambda(\|\mathbb{P}R\|^2 + \langle t\rangle^{-2\gamma}). \quad (5.13)$$

By inserting the estimates, (5.5), (5.7), (5.9), (5.11) and (5.13), into (5.4), we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\mathbb{P}R(t)\|^2 + \varepsilon\|D(R)\|^2 \le C\varepsilon\lambda\|D(R)\|^2 + C_\lambda(\varepsilon + \langle t\rangle^{-\gamma})\|\mathbb{P}R\|^2 + \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}.$$

Choosing λ small enough such that $C\lambda < \frac{1}{2}$ and note that (2.38) implies

$$\|\mathbb{P}R_0\| \le \|R_0\| \lesssim \varepsilon^{-\frac{1}{4}}, \quad (5.14)$$

then we use Gronwall inequality to find that

$$\|\mathbb{P}R(t)\|^2 + \varepsilon\int_0^t \|D(R)\|^2 dt' \le C\varepsilon^{-\frac{1}{4}} \quad \text{for } 0 \le t \le T^\varepsilon.$$

Together with (5.3) and (5.10), we thus conclude the proof of (5.2). □

5.2. Tangential Derivatives Estimates

We now estimate the tangential derivatives of the remainder. Recall that the tangential derivatives Z^α are defined in (2.29) of Sect. 2.8 and the conormal Sobolev norm $\|\cdot\|_m$ is defined in (2.34). Let us start by estimating ∇R on the boundary.

Lemma 5.2. *Let $m \ge 1$. It holds that*

$$\|\nabla R\|_{H^{m-1}(\partial\mathcal{O})} \lesssim \|R\|_{H^m(\partial\mathcal{O})} + \langle t\rangle^{-\gamma}. \quad (5.15)$$

Proof. Indeed we only need to estimate $\|\partial_{\mathbf{n}}R\|_{H^{m-1}(\partial\mathcal{O})}$. On the one hand, we deduce from the boundary conditions: $\mathcal{N}(R) = -G$, $R\cdot\mathbf{n} = 0$ on $\partial\mathcal{O}$, that

$$(\partial_{\mathbf{n}}R - \nabla\mathbf{n}\cdot R + 2MR)_{\tan} = -2G \quad \text{on } \partial\mathcal{O},$$

from which, with (2.37), we infer

$$\|(\partial_{\mathbf{n}}R)_{\tan}\|_{H^{m-1}(\partial\mathcal{O})} \lesssim \|R\|_{H^m(\partial\mathcal{O})} + \|G\|_{H^{m-1}(\partial\mathcal{O})} \lesssim \|R\|_{H^m(\partial\mathcal{O})} + \langle t\rangle^{-\gamma}. \quad (5.16)$$

On the other hand, $\operatorname{div} R = -H$ gives us

$$\partial_{\mathbf{n}}R\cdot\mathbf{n} + \sum_j c_j Z_j R = -H \quad (5.17)$$

for some smooth functions c_j, which depends only on vector field w^j. Thus, by (2.37),

$$\|\partial_{\mathbf{n}}R\cdot\mathbf{n}\|_{H^{m-1}(\partial\mathcal{O})} \lesssim \|R\|_{H^m(\partial\mathcal{O})} + \|H\|_{H^{m-1}(\partial\mathcal{O})} \lesssim \|R\|_{H^m(\partial\mathcal{O})} + \langle t\rangle^{-\gamma}. \quad (5.18)$$

Combining the estimate (5.16) with (5.18), we have proved the part of (5.15) regarding $\|\partial_{\mathbf{n}}R\|_{H^{m-1}(\partial\mathcal{O})}$. The other part of the estimate is straightforward. □

Proposition 5.3. *Let $1 \leq m \leq p-3$ be an integer. Then there exists a constant $C_1 > 0$ such that for any t in $[0, T^\varepsilon]$,*

$$\begin{aligned}\frac{d}{dt}\|R(t)\|_m^2 + C_1\varepsilon\|\nabla R\|_m^2 \lesssim \varepsilon\|\nabla R\|_{m-1}^2 + \sum_{|\alpha|\leq m} \Big|\int_{\mathcal{O}} Z^\alpha\nabla\pi\cdot Z^\alpha R\Big| + \varepsilon^{\frac14}\langle t\rangle^{-\gamma} \\ + \|R\|_m^2(\varepsilon + \langle t\rangle^{-\gamma}) + \varepsilon^2(\|R\|_{1,\infty}\|\nabla R\|_{m-1}\|R\|_m + \|\nabla R\|_{L^\infty}\|R\|_m^2).\end{aligned} \tag{5.19}$$

Proof. Let $1 \leq \ell \leq p-3$ be an integer and α be a multi-index with $|\alpha| = \ell$. By applying Z^α to (5.1a), we obtain

$$\begin{aligned}&\partial_t Z^\alpha R - \varepsilon\Delta Z^\alpha R + u^\varepsilon\cdot\nabla Z^\alpha R + Z^\alpha(R\cdot\nabla u_a^\varepsilon) + Z^\alpha\nabla\pi \\ &\quad = Z^\alpha F - \varepsilon[\Delta, Z^\alpha]R + [u^\varepsilon\cdot\nabla, Z^\alpha]R,\end{aligned}$$

Taking L^2 inner product of the above equation with $Z^\alpha R$ gives rise to

$$\begin{aligned}&\frac12\frac{d}{dt}\|Z^\alpha R(t)\|^2 - \varepsilon\int_{\mathcal{O}}\Delta Z^\alpha R\cdot Z^\alpha R + \int_{\mathcal{O}}(u^\varepsilon\cdot\nabla Z^\alpha R)\cdot Z^\alpha R + \int_{\mathcal{O}} Z^\alpha\nabla\pi\cdot Z^\alpha R \\ &= -\int_{\mathcal{O}} Z^\alpha(R\cdot\nabla u_a^\varepsilon)\cdot Z^\alpha R + \int_{\mathcal{O}} Z^\alpha F\cdot Z^\alpha R - \varepsilon\int_{\mathcal{O}}[\Delta, Z^\alpha]R\cdot Z^\alpha R \\ &\quad + \int_{\mathcal{O}}[u^\varepsilon\cdot\nabla, Z^\alpha]R\cdot Z^\alpha R.\end{aligned} \tag{5.20}$$

In what follows, we shall handle term by term above in (5.20).

- We start with estimating the second term in (5.20), which relies on the following lemma:

Lemma 5.4. *Let $1 \leq |\alpha| \leq m$. There exist constants $C_1, C > 0$ such that*

$$-\int_{\mathcal{O}}\Delta Z^\alpha R\cdot Z^\alpha R \geq C_1\|\nabla Z^\alpha R\|^2 - C\|R\|_m^2 - C\langle t\rangle^{-2\gamma}. \tag{5.21}$$

We postpone its proof to the end of this subsection.

- For the third term of (5.20), since $\operatorname{div} u^\varepsilon = \sigma^0$ in $\mathcal{O}$, $u^\varepsilon\cdot\mathbf{n} = 0$ on $\partial\mathcal{O}$, and σ^0 is supported on $[0, T]$, we get, by using integration by parts, that

$$\Big|\int_{\mathcal{O}}(u^\varepsilon\cdot\nabla Z^\alpha R)\cdot Z^\alpha R\Big| \lesssim \langle t\rangle^{-\gamma}\|R\|_m^2. \tag{5.22}$$

.

- By using the Leibniz formula and (2.39), we find

$$\begin{aligned}\Big|\int_{\mathcal{O}} Z^\alpha(R\cdot\nabla u_a^\varepsilon)\cdot Z^\alpha R\Big| &\lesssim \sum_{\alpha_1+\alpha_2=\alpha}\Big|\int_{\mathcal{O}} Z^{\alpha_1}R\cdot Z^{\alpha_2}\nabla u_a^\varepsilon\cdot Z^\alpha R\Big| \\ &\lesssim \langle t\rangle^{-\gamma}\|R\|_m^2.\end{aligned} \tag{5.23}$$

- (2.35) ensures that

$$\Big|\int_{\mathcal{O}} Z^\alpha F\cdot Z^\alpha R\Big| \lesssim \varepsilon^{\frac14}\langle t\rangle^{-\gamma}\big(\|R\|_m^2 + 1\big). \tag{5.24}$$

- Thanks to (2.32) and $Z^\alpha(R \cdot \mathbf{n}) = 0$ on $\partial\mathcal{O}$, we get, by using integration by parts, that

$$
\begin{aligned}
|\int_{\mathcal{O}} \varepsilon[\Delta, Z^\alpha]R \cdot Z^\alpha R| \lesssim & |\int_{\mathcal{O}} \sum_{|\beta|,|\gamma|\le m-1} \varepsilon(c_\beta \nabla^2 Z^\beta R + c_\gamma \nabla Z^\gamma R) \cdot Z^\alpha R| \\
\lesssim & \varepsilon(\|\nabla R\|_m + \|R\|_m)\|\nabla R\|_{m-1} \\
& + \varepsilon\|\nabla R\|_{H^{m-1}(\partial\mathcal{O})}\|R\|_{H^m(\partial\mathcal{O})}.
\end{aligned}
$$

Moreover, due to trace Theorem (see (87) of [30] for instance) that

$$
\|R\|^2_{H^m(\partial\mathcal{O})} \lesssim \|R\|_m^2 + \|R\|_m\|\nabla R\|_m, \tag{5.25}
$$

and Lemma 5.2, for any $\lambda > 0$, we infer

$$
\begin{aligned}
\|\nabla R\|_{H^{m-1}(\partial\mathcal{O})}\|R\|_{H^m(\partial\mathcal{O})} \lesssim & \|R\|^2_{H^m(\partial\mathcal{O})} + \langle t\rangle^{-2\gamma} \\
\lesssim & \|R\|_m^2 + \|R\|_m\|\nabla R\|_m + \langle t\rangle^{-2\gamma} \\
\lesssim & \lambda\|\nabla R\|_m^2 + C_\lambda\|R\|_m^2 + \langle t\rangle^{-2\gamma}.
\end{aligned} \tag{5.26}
$$

As a result, it comes out

$$
\varepsilon|\int_{\mathcal{O}} [\Delta, Z^\alpha]R \cdot Z^\alpha R| \le \lambda\varepsilon\|\nabla R\|_m^2 + C_\lambda\varepsilon(\|\nabla R\|_{m-1}^2 + \|R\|_m^2 + \langle t\rangle^{-2\gamma}), \tag{5.27}
$$

- For the last term, we use the decomposition (2.42) to get

$$
\begin{aligned}
\int_{\mathcal{O}} [u^\varepsilon \cdot \nabla, Z^\alpha]R \cdot Z^\alpha R = & \int_{\mathcal{O}} [u^0 \cdot \nabla, Z^\alpha]R \cdot Z^\alpha R \\
& + \int_{\mathcal{O}} [(u_a^\varepsilon - u^0) \cdot \nabla, Z^\alpha]R \cdot Z^\alpha R \\
& + \varepsilon^2 \int_{\mathcal{O}} [R \cdot \nabla, Z^\alpha]R \cdot Z^\alpha R.
\end{aligned}
$$

We write

$$
u^0 \cdot \nabla = \sum_j c_j Z_j + (u^0 \cdot \mathbf{n})\partial_{\mathbf{n}} = \sum_j c_j Z_j + u^0_\flat Z_0, \tag{5.28}
$$

for some smooth functions c_j.
Thanks to (2.33), we can easily show by induction that $[Z_j, Z^\alpha]$, $0 \le j \le 5$, is a tangential derivative of order m. Note that u^0 is supported in $[0, T]$, we have

$$
|\int_{\mathcal{O}} [u^0 \cdot \nabla, Z^\alpha]R \cdot Z^\alpha R| \lesssim \chi_{[0,T]}(t)\|R\|_m^2. \tag{5.29}
$$

On the other hand, applying the Leibniz formula yields

$$
[(u_a^\varepsilon - u^0) \cdot \nabla, Z^\alpha]R = \sum_{\alpha_1+\alpha_2=\alpha, \alpha_1\ne 0} c_{\alpha_1} Z^{\alpha_1}(u_a^\varepsilon - u^0) Z^{\alpha_2}\nabla R + (u_a^\varepsilon - u^0)[\nabla, Z^\alpha]R,
$$

for some smooth functions c_{α_1} depended on the vector field $\mathfrak{W}$.

It follows from (2.40) that

$$\|[(u_a^\varepsilon - u^0)\cdot\nabla, Z^\alpha]R\| \lesssim \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\|\nabla R\|_{m-1},$$

which implies

$$\begin{aligned}|\int_{\mathcal{O}}[(u_a^\varepsilon - u^0)\cdot\nabla, Z^\alpha]R\cdot Z^\alpha R| &\lesssim \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\|\nabla R\|_{m-1}\|R\|_m\\ &\lesssim \varepsilon\|\nabla R\|_{m-1}^2 + \langle t\rangle^{-2\gamma}\|R\|_m^2. \quad (5.30)\end{aligned}$$

Applying the Leibniz formula once again gives

$$[R\cdot\nabla, Z^\alpha]R = \sum_{\alpha_1+\alpha_2=\alpha,\alpha_1\neq 0} c_{\alpha_1}Z^{\alpha_1}R\cdot Z^{\alpha_2}\nabla R + R[\nabla, Z^\alpha]R.$$

Yet it follows from generalized Sobolev-Gagliardo-Nirenberg-Morse inequality that

$$\|[R\cdot\nabla, Z^\alpha]R\| \lesssim \|R\|_{1,\infty}\|\nabla R\|_{m-1} + \|\nabla R\|_{L^\infty}\|R\|_m,$$

so that we infer

$$|\int_{\mathcal{O}}[R\cdot\nabla, Z^\alpha]R\cdot Z^\alpha R| \lesssim \|R\|_{1,\infty}\|\nabla R\|_{m-1}\|R\|_m + \|\nabla R\|_{L^\infty}\|R\|_m^2. \quad (5.31)$$

Combining (5.29), (5.30) with (5.31), we arrive at

$$\begin{aligned}|\int_{\mathcal{O}}[u^\varepsilon\cdot\nabla, Z^\alpha]R\cdot Z^\alpha R| \lesssim& \varepsilon\|\nabla R\|_{m-1}^2 + \langle t\rangle^{-2\gamma}\|R\|_m^2\\ &+ \varepsilon^2(\|R\|_{1,\infty}\|\nabla R\|_{m-1}\|R\|_m + \|\nabla R\|_{L^\infty}\|R\|_m^2).\end{aligned} \quad (5.32)$$

By inserting the estimates, (5.21), (5.22), (5.23), (5.24), (5.27) and (5.32), into (5.20), and then by summing up (5.2) with the resulting inequality over all the multi-indices α with $1\leq|\alpha|\leq m$, finally choosing λ to be sufficiently small, we arrive to (5.19). □

Let us now present the proof of Lemma 5.4.

Proof of Lemma 5.4. We first get, by using integration by parts, that

$$-\int_{\mathcal{O}}\Delta Z^\alpha R\cdot Z^\alpha R = 2\int_{\mathcal{O}}|D(Z^\alpha R)|^2 + 2\int_{\partial\mathcal{O}}(D(Z^\alpha R)\cdot\mathbf{n})_{\tan}\cdot Z^\alpha R. \quad (5.33)$$

It follows from Korn's inequality that

$$\|D(Z^\alpha R)\|^2 \geq C_1\|\nabla Z^\alpha R\|^2 - C_2\|Z^\alpha R\|^2. \quad (5.34)$$

As $\mathcal{N}(R) = G$ on $\partial\mathcal{O}$, we have

$$(D(Z^\alpha R)\cdot\mathbf{n})_{\tan} = -(MZ^\alpha R)_{\tan} + Z^\alpha G + [\mathcal{N}, Z^\alpha]R,$$

so that there holds

$$\int_{\partial\mathcal{O}}(D(Z^\alpha R)\cdot\mathbf{n})_{\tan}\cdot Z^\alpha R=\int_{\partial\mathcal{O}}(Z^\alpha G-(MZ^\alpha R)_{\tan})\cdot Z^\alpha R + \int_{\partial\mathcal{O}}[\mathcal{N},Z^\alpha]R\cdot Z^\alpha R. \tag{5.35}$$

We are going to estimate each term of the right hand side of (5.35).

On the one hand, by virtue of (2.37) and for any $\lambda>0$, we get, by applying Young's inequality, that

$$\begin{aligned}
&\Big|\int_{\partial\mathcal{O}}(Z^\alpha G-(MZ^\alpha R)_{\tan})\cdot Z^\alpha R\Big|\\
&=\Big|\int_{\mathcal{O}}\operatorname{div}\big(n(Z^\alpha G-(MZ^\alpha R)_{\tan})\cdot Z^\alpha R\big)\Big|\\
&\lesssim\|Z^\alpha G\|_{H^1}\|Z^\alpha R\|+(\|Z^\alpha G\|+\|Z^\alpha R\|)\|Z^\alpha R\|_{H^1}\\
&\lesssim\lambda\|\nabla Z^\alpha R\|^2+C_\lambda(\|Z^\alpha G\|_{H^1}^2+\|Z^\alpha R\|^2)\\
&\lesssim\lambda\|\nabla Z^\alpha R\|^2+C_\lambda(\langle t\rangle^{-2\gamma}+\|Z^\alpha R\|^2).
\end{aligned}$$

On other hand, we deduce from (5.26) that

$$\begin{aligned}
\Big|\int_{\partial\mathcal{O}}[N,Z^\alpha]R\cdot Z^\alpha R\Big|&\lesssim\|\nabla R\|_{H^{m-1}(\partial\mathcal{O})}\|R\|_{H^m(\partial\mathcal{O})}\\
&\le\lambda\|\nabla R\|_m^2+C_\lambda\big(\|R\|_m^2+\langle t\rangle^{-2\gamma}\big).
\end{aligned}$$

By substituting the above inequalities into (5.35), we achieve

$$\Big|\int_{\partial\mathcal{O}}(D(Z^\alpha R)\cdot\mathbf{n})_{\tan}\cdot Z^\alpha R\Big|\le 2\lambda\|\nabla R\|_m^2+2C_\lambda\big(\|R\|_m^2+\langle t\rangle^{-2\gamma}\big). \tag{5.36}$$

By inserting (5.34) and (5.36) into (5.33) and choosing λ to be sufficiently small, we arrive at (5.21). □

5.3. An Appropriate Substitute to the Vorticity

We observe that the right hand side of (5.19) involves $\|\nabla R\|_{m-1}$ and $\|\sqrt{\varepsilon}\nabla R\|_\infty$, so that we need to estimate at least one normal derivative of R. We define

$$\eta:=\sqrt{\varepsilon}(\mathcal{N}(R)+G)\chi(x), \tag{5.37}$$

where χ is a cut-off function defined in Sect. 2. From the definition of η, we know that $\eta=0$ on the boundary $\partial\mathcal{O}$. Observe that this property is not satisfied by the vorticity curl R; this is indeed the reason why we would rather use η following [30] than curl R.

Lemma 5.5. *Let $m\ge1$. The following equivalences hold true:*

$$\|\eta\|_{m-1}+\|R\|_m+\sqrt{\varepsilon}\langle t\rangle^{-\gamma}\approx\|\sqrt{\varepsilon}\nabla R\|_{m-1}+\|R\|_m+\sqrt{\varepsilon}\langle t\rangle^{-\gamma}, \tag{5.38}$$

$$\|\eta\|_{L^\infty}+\|R\|_{1,\infty}+\sqrt{\varepsilon}\langle t\rangle^{-\gamma}\approx\|\sqrt{\varepsilon}\nabla R\|_\infty+\|R\|_{1,\infty}+\sqrt{\varepsilon}\langle t\rangle^{-\gamma}. \tag{5.39}$$

Proof. Let us focus on the proof of (5.38). We first deduce from the definitions (1.1) and (5.37), and the estimate of G in (2.37) that

$$\|\eta\|_{m-1} \lesssim \|\sqrt{\varepsilon}\nabla R\|_{m-1} + \|R\|_m + \sqrt{\varepsilon}\langle t\rangle^{-\gamma},$$

which implies

$$\|\eta\|_{m-1} + \|R\|_m + \sqrt{\varepsilon}\langle t\rangle^{-\gamma} \lesssim \|\sqrt{\varepsilon}\nabla R\|_{m-1} + \|R\|_m + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}. \quad (5.40)$$

To prove the other side of the inequality (5.40), we introduce

$$\Pi f := f_{\tan}. \quad (5.41)$$

Then notice that

$$D(R)n = \frac{1}{2}\left(\partial_{\mathbf{n}} R + \nabla R \cdot n\right)_{\tan} \quad \text{and} \quad \left(\nabla R_j n_j\right)_{\tan} = (\nabla R_j)_{\tan} n_j,$$

we have

$$\sqrt{\varepsilon}\|\chi\Pi\partial_{\mathbf{n}} R\|_{m-1} \lesssim \|\eta\|_{m-1} + \|R\|_m + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}.$$

On the other hand, by definitions of χ and of the norm $\|\cdot\|_m$, one has

$$\|(1-\chi)\Pi\partial_{\mathbf{n}} R\|_{m-1} \lesssim \|R\|_m.$$

And it follows from (5.17) and (2.35) that

$$\|\partial_{\mathbf{n}} R \cdot \mathbf{n}\|_{m-1} \lesssim \|R\|_m + \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}.$$

As a consequence, we obtain

$$\begin{aligned}\|\partial_{\mathbf{n}} R\|_{m-1} &\lesssim \|\chi\Pi\partial_{\mathbf{n}} R\|_{m-1} + \|(1-\chi)\Pi\partial_{\mathbf{n}} R\|_{m-1} + \|\partial_{\mathbf{n}} R\cdot\mathbf{n}\|_{m-1} \\ &\lesssim \|\eta\|_{m-1} + \|R\|_m + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}.\end{aligned} \quad (5.42)$$

(5.42) shows that the other side of the inequality (5.40) holds, which leads to (5.38). The equivalence (5.39) can be proved along the same line. □

By virtue of (5.38) and (5.39), we can rewrite (5.19) as

$$\begin{aligned}\frac{d}{dt}\|R(t)\|_m^2 + C_1\varepsilon\|\nabla R\|_m^2 \lesssim& \varepsilon\|\nabla R\|_{m-1}^2 + \sum_{|\alpha|\le m}\Big|\int_{\mathcal{O}} Z^\alpha\nabla\pi\cdot Z^\alpha R\Big| + \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma} \\ &+ (\|\eta\|_{m-1}^2 + \|R\|_m^2)\big(\varepsilon + \langle t\rangle^{-\gamma} \\ &+ \varepsilon^2(\|\eta\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\big).\end{aligned} \quad (5.43)$$

5.4. Time Evolution of the Auxiliary Quantity

Let us now estimate the time evolution of $\|\eta(t)\|_{m-1}$, which appears in the right hand side of (5.43).

Proposition 5.6. *Let $1 \leq m \leq p-3$. Then there exist a constant $C_1 > 0$ such that for any $t \in [0, T^\varepsilon]$,*

$$\begin{aligned}
&\frac{d}{dt}\|\eta(t)\|_{m-1}^2 + C_1\varepsilon\|\nabla\eta\|_{m-1}^2 \\
&\quad \lesssim \varepsilon\|\nabla\eta\|_{m-2}^2 + \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma} + \sum_{|\beta|\leq m-1}\sqrt{\varepsilon}\Big|\int_{\mathcal{O}} Z^\beta\left(\chi\mathcal{N}(\nabla\pi)\right)\cdot Z^\beta\eta\Big| \qquad (5.44)\\
&\qquad + \left(\varepsilon + \langle t\rangle^{-\gamma} + \varepsilon^2(\|\eta\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\right)\left(\|\eta\|_{m-1}^2 + \|R\|_m^2\right).
\end{aligned}$$

Here the term $\varepsilon\|\nabla\eta\|_{m-2}^2$ does not appear on the right-hand side of (5.44) when $m = 1$.

Proof. In view of (5.1), η satisfies

$$\begin{aligned}
\partial_t\eta - \varepsilon\Delta\eta + u^\varepsilon\cdot\nabla\eta = &-\sqrt{\varepsilon}\chi\mathcal{N}(F + \nabla\pi + R\cdot\nabla u_a^\varepsilon)\\
&+\sqrt{\varepsilon}\left(\partial_t - \varepsilon\Delta + u^\varepsilon\cdot\nabla\right)(\chi G)\\
&-\varepsilon^{\frac{3}{2}}[\Delta, \chi\mathcal{N}]R + \sqrt{\varepsilon}[u^0\cdot\nabla, \chi\mathcal{N}]R\\
&+\sqrt{\varepsilon}[(u^\varepsilon - u^0)\cdot\nabla, \chi\mathcal{N}]R.
\end{aligned}$$

Applying Z^β with $|\beta| = m-1$ to the above equation yields

$$\begin{aligned}
\partial_t Z^\beta\eta - \varepsilon\Delta Z^\beta\eta + u^\varepsilon\cdot\nabla Z^\beta\eta = &-\sqrt{\varepsilon}Z^\beta\left(\chi\mathcal{N}(F + \nabla\pi + R\cdot\nabla u_a^\varepsilon)\right)\\
&+\sqrt{\varepsilon}Z^\beta(\partial_t - \varepsilon\Delta + u^\varepsilon\cdot\nabla)(\chi G)\\
&-\varepsilon^{\frac{3}{2}}[\Delta, Z^\beta(\chi\mathcal{N})]R\\
&+\sqrt{\varepsilon}[u^0\cdot\nabla, Z^\beta(\chi\mathcal{N})]R\\
&+\sqrt{\varepsilon}[(u^\varepsilon - u^0)\cdot\nabla, Z^\beta(\chi\mathcal{N})]R.
\end{aligned}$$

Note that $\eta = 0$ on $\partial\mathcal{O}$ and Z^β is tangential derivative, we have $Z^\beta\eta = 0$ on $\partial\mathcal{O}$. Then we get, by taking L^2 inner product of the above equation with $Z^\beta\eta$, that

$$\frac{1}{2}\frac{d}{dt}\|Z^\beta\eta(t)\|^2 + 2\varepsilon\|D(Z^\beta\eta)\|^2 \lesssim \sum_{i=1}^{8}|I_i|, \qquad (5.45)$$

where

$$\begin{aligned}
I_1 &:= \int_{\mathcal{O}} u^\varepsilon\cdot\nabla Z^\beta\eta\cdot Z^\beta\eta,\\
I_2 &:= \sqrt{\varepsilon}\int_{\mathcal{O}} Z^\beta(\chi\mathcal{N}(F))\cdot Z^\beta\eta,\\
I_3 &:= \sqrt{\varepsilon}\int_{\mathcal{O}} Z^\beta(\chi\mathcal{N}(\nabla\pi))\cdot Z^\beta\eta,
\end{aligned}$$

$$
\begin{aligned}
I_4 &:= \sqrt{\varepsilon}\int_{\mathcal{O}} Z^\beta\left(\chi\mathcal{N}(R\cdot\nabla u_a^\varepsilon)\right)\cdot Z^\beta\eta,\\
I_5 &:= \sqrt{\varepsilon}\int_{\mathcal{O}} (\partial_t-\varepsilon\Delta+u^\varepsilon\cdot\nabla)(Z^\beta(\chi G))\cdot Z^\beta\eta,\\
I_6 &:= \varepsilon^{\frac{3}{2}}\int_{\mathcal{O}} [\Delta, Z^\beta(\chi\mathcal{N})]R\cdot Z^\beta\eta,\\
I_7 &:= \sqrt{\varepsilon}\int_{\mathcal{O}} [u^0\cdot\nabla, Z^\beta(\chi\mathcal{N})]R\cdot Z^\beta\eta,\\
I_8 &:= \sqrt{\varepsilon}\int_{\mathcal{O}} [(u^\varepsilon-u^0)\cdot\nabla, Z^\beta(\chi\mathcal{N})]R\cdot Z^\beta\eta.
\end{aligned}
$$

First, regarding the second term in the left hand side of (5.45), we observe from Korn's inequality that

$$
\|D(Z^\beta\eta)\|\geq C_1\|\nabla Z^\beta\eta\|-C_2\|Z^\beta\eta\|. \tag{5.46}
$$

Let us now handle term by term in the right-hand side of (5.45).

• Estimate of I_1.

Since u^ε satisfies (2.4b), (2.4c), and $\sigma^\varepsilon=\sigma^0$ is supported in $[0,T]$, by using integration by parts, we find

$$
|I_1|\lesssim \langle t\rangle^{-\gamma}\|\eta\|_{m-1}^2. \tag{5.47}
$$

• Estimate of I_2.

By virtue of (2.35), we get, by applying the Cauchy-Schwarz inequality, that

$$
|I_2|\lesssim \varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}\left(1+\|\eta\|_{m-1}^2\right). \tag{5.48}
$$

• Estimate of I_3.

We simply estimate I_3 by the third term on the right-hand side of (5.44). We remark that we do not try to get rid of the pressure at this step. Indeed this delicate issue will be postponed to Sect. 5.5.

• Estimate of I_4.

Recalling (5.41) and in view of (1.1), we write

$$
\sqrt{\varepsilon}\chi\mathcal{N}(R\cdot\nabla u_a^\varepsilon)=\sqrt{\varepsilon}\chi\Pi\Big(\frac{1}{2}(\partial_{\mathbf{n}}(R\cdot\nabla u_a^\varepsilon)+\nabla(R\cdot\nabla u_a^\varepsilon)\cdot\mathbf{n})+M(R\cdot\nabla u_a^\varepsilon)\Big).
$$

Since M is a smooth matrix-valued function and $m\leq p-3$, we get, by applying Leibniz formula and (2.39), that

$$
\|\sqrt{\varepsilon}\chi\mathcal{N}(R\cdot\nabla u_a^\varepsilon)\|_{m-1}\lesssim\sqrt{\varepsilon}\langle t\rangle^{-\gamma}\|\nabla R\|_{m-1}+\langle t\rangle^{-\gamma}\|R\|_{m-1},
$$

which, together with (5.38), ensures that

$$
|I_4|\lesssim\langle t\rangle^{-\gamma}\left(\|R\|_m^2+\|\eta\|_{m-1}^2\right)+\sqrt{\varepsilon}\langle t\rangle^{-3\gamma}. \tag{5.49}
$$

• Estimate of I_5.
We split it into two terms

$$I_5 = I_{51} + I_{52} \quad \text{with} \quad I_{51} := \sqrt{\varepsilon}\int_{\mathcal{O}} (\partial_t - \varepsilon\Delta + u_a^\varepsilon \cdot \nabla)\big(Z^\beta(\chi G)\big)\cdot Z^\beta\eta,$$
$$I_{52} := \varepsilon^{\frac{5}{2}}\int_{\mathcal{O}} R\cdot\nabla(Z^\beta(\chi G))\cdot Z^\beta\eta.$$

Thanks to (2.37) and (2.39), χ is a smooth function, for $m \le p-3$, we infer

$$\begin{aligned}|I_{51}| &\lesssim \sqrt{\varepsilon}\big(\|\partial_t G\|_{m-1} + \varepsilon\|\nabla^2 G\|_{m-1} + \|u_a^\varepsilon\|_{L^\infty(\mathcal{O})}\|\nabla G\|_{m-1}\big)\|\eta\|_{m-1}\\ &\lesssim \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\|\eta\|_{m-1},\end{aligned}$$

and

$$|I_{52}| \lesssim \varepsilon^{\frac{5}{2}}\|R\|\|\nabla G\|_{m-1}\|\eta\|_{m-1} \lesssim \varepsilon^{\frac{5}{2}}\langle t\rangle^{-\gamma}\|R\|_m\|\eta\|_{m-1},$$

so that we achieve

$$|I_{51} \lesssim \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\big(1+\varepsilon^2\|R\|_m\big)\|\eta\|_{m-1}. \tag{5.50}$$

• Estimate of I_6.
In view of (5.1b), we decompose $I_6 = I_{61} + I_{62}$ with

$$I_{61} := \varepsilon\int_{\mathcal{O}} [\Delta, Z^\beta](\eta - \sqrt{\varepsilon}\chi G)\cdot Z^\beta\eta \quad \text{and} \quad I_{62} := \varepsilon^{\frac{3}{2}}\int_{\mathcal{O}} Z^\beta[\Delta, \chi\mathcal{N}]R\cdot Z^\beta\eta.$$

On the one hand, thanks to (2.32), we write

$$\begin{aligned}\int_{\mathcal{O}} [\varepsilon\Delta, Z^\beta](\eta - \sqrt{\varepsilon}\chi G)\cdot Z^\beta\eta = \varepsilon\int_{\mathcal{O}} &\sum_{|\beta_1|,|\beta_2|<m-1}\big(c_{\beta_1}\nabla^2 Z^{\beta_1}\eta + c_{\beta_2}\nabla Z^{\beta_2}\eta\big)\cdot Z^\beta\eta\\ &-\varepsilon^{\frac{3}{2}}\int_{\mathcal{O}}[\Delta, Z^\beta](\chi G)\cdot Z^\beta\eta,\end{aligned}$$

where c_{β_1}, c_{β_2} are smooth functions depend only on vector field $\mathfrak{W}$. Due to $Z^\beta\eta = 0$ on $\partial\mathcal{O}$, by using integration by parts and (2.37), we infer

$$\begin{aligned}|I_{61}| &\lesssim \varepsilon\|\nabla\eta\|_{m-1}\|\nabla\eta\|_{m-2} + \varepsilon\|\nabla\eta\|_{m-2}\|\eta\|_{m-1} + \varepsilon^{\frac{3}{2}}\langle t\rangle^{-\gamma}\|\eta\|_{m-1}\\ &\le \lambda\varepsilon\|\nabla\eta\|_{m-1}^2 + C_\lambda\varepsilon(\|\nabla\eta\|_{m-2}^2 + \|\eta\|_{m-1}^2) + C\varepsilon^{\frac{3}{2}}\langle t\rangle^{-\gamma},\end{aligned} \tag{5.51}$$

where $\lambda > 0$ is a small constant.

On the other hand, we write

$$\begin{aligned}[\Delta, \chi\mathcal{N}]R = {} & \Delta\chi\mathcal{N}(R) + 2\sqrt{\varepsilon}\nabla\chi : \nabla\mathcal{N}(R)\\ &+\chi(\Delta\Pi)\big(D(R)\cdot\mathbf{n} + MR\big) + 2\sqrt{\varepsilon}\chi(\nabla\Pi) : \nabla\big(D(R)\cdot\mathbf{n} + MR\big)\\ &+\chi\Pi\big(D(R)\cdot\Delta\mathbf{n} + 2\nabla D(R) : \nabla\mathbf{n} + (\Delta M)R + 2\nabla M : \nabla R\big).\end{aligned}$$

Corresponding to the second and the forth term above, we use integration by parts in I_{62}. Then we deduce from (5.38) that

$$\begin{aligned}|I_{62}| \lesssim &\varepsilon^{\frac{3}{2}}\|\nabla R\|_{m-1}\big(\|\nabla\eta\|_{m-1}+\|\eta\|_{m-1}\big)\\ \leq&\lambda\varepsilon\|\nabla\eta\|_{m-1}^2+C_\lambda\varepsilon\big(\|\eta\|_{m-1}^2+\|R\|_m^2\big)+C_\lambda\varepsilon^2\langle t\rangle^{-2\gamma}.\end{aligned} \tag{5.52}$$

• Estimate of I_7.

We write

$$[\,u^0\cdot\nabla, Z^\beta(\chi\mathcal{N})]R=[u^0\cdot\nabla, Z^\beta](\chi\mathcal{N}(R))+Z^\beta[u^0\cdot\nabla,\chi\mathcal{N}]R.$$

It follows from (5.28) that $u^0\cdot\nabla$ is a tangential derivative. So that thanks to the observation (2.33), we find that $[u^0\cdot\nabla, Z^\beta]$ is an operator of linear combination of tangential derivatives of order $m-1$. Then due to the fact that u^0 is supported in $[0,T]$, we infer

$$\sqrt{\varepsilon}\|[u^0\cdot\nabla, Z^\beta(\chi\mathcal{N})]R\|\lesssim\sqrt{\varepsilon}\chi_{[0,T]}(t)\|\nabla R\|_{m-1}\lesssim\langle t\rangle^{-\gamma}\|\nabla R\|_{m-1},$$

which together with (5.38) implies

$$|I_7|\lesssim\langle t\rangle^{-\gamma}\big(\|R\|_m^2+\|\eta\|_{m-1}^2\big)+\varepsilon\langle t\rangle^{-3\gamma}. \tag{5.53}$$

• Estimate of I_8.

We first decompose I_8 as

$$I_8=\sum_{i=1}^{6}I_{8i},$$

with

$$\begin{aligned}I_{81}&:=\int_{\mathcal{O}}[(u_a^\varepsilon-u^0)\cdot\nabla, Z^\beta]\eta\cdot Z^\beta\eta,\\ I_{82}&:=-\sqrt{\varepsilon}\int_{\mathcal{O}}[(u_a^\varepsilon-u^0)\cdot\nabla, Z^\beta](\chi G)\cdot Z^\beta\eta\\ I_{83}&:=\sqrt{\varepsilon}\int_{\mathcal{O}}Z^\beta[(u_a^\varepsilon-u^0)\cdot\nabla,\chi\mathcal{N}]R\cdot Z^\beta\eta\\ I_{84}&:=\varepsilon^2\int_{\mathcal{O}}[R\cdot\nabla, Z^\beta]\eta\cdot Z^\beta\eta\\ I_{85}&:=-\varepsilon^{\frac{5}{2}}\int_{\mathcal{O}}[R\cdot\nabla, Z^\beta](\chi G)\cdot Z^\beta\eta\\ I_{86}&:=\varepsilon^{\frac{5}{2}}\int_{\mathcal{O}}Z^\beta[R\cdot\nabla,\chi\mathcal{N}]R\cdot Z^\beta\eta\end{aligned}$$

Next we deal with all the terms above.

- *Estimate of* I_{81}. In view of (2.30), we write

$$[(u_a^\varepsilon - u^0)\cdot\nabla, Z^\beta]\eta = \sum_{\beta_1+\beta_2=\beta,\beta_1\neq 0} c_{\beta_1} Z^{\beta_1}(u_a^\varepsilon - u^0)\cdot Z^{\beta_2}\nabla\eta + (u_a^\varepsilon - u^0)\cdot[\nabla, Z^\beta]\eta,$$

where c_{β_1}, c_{β_2} are smooth functions depend only on vector filed $\mathfrak{W}$. Then thanks to (2.40), we infer

$$|I_{81}| \lesssim \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\|\nabla\eta\|_{m-2}\|\eta\|_{m-1} \lesssim \varepsilon\|\nabla\eta\|_{m-2}^2 + \langle t\rangle^{-2\gamma}\|\eta\|_{m-1}^2. \tag{5.54}$$

- *Estimate of* I_{82}. It follows from (2.37) and (2.40) that

$$|I_{82}| \lesssim \sqrt{\varepsilon}\|u_a^\varepsilon - u^0\|_{m-1,\infty}\|G\|_{H^{m-1}}\|\eta\|_{m-1} \lesssim \varepsilon^{\frac{3}{4}}\langle t\rangle^{-2\gamma}\|\eta\|_{m-1}. \tag{5.55}$$

- *Estimate of* I_{83}. In view of (1.1), we write

$$\begin{aligned}[(u_a^\varepsilon - u^0)\cdot\nabla, \chi\mathcal{N}]R = {} & [(u_a^\varepsilon - u^0)\cdot\nabla, \chi\Pi]\big(D(R)\cdot\mathbf{n} + MR\big)\\ & +\chi\Pi\big((u_a^\varepsilon - u^0)\cdot\nabla(D(R)\cdot\mathbf{n})\\ & -D((u_a^\varepsilon - u^0)\cdot\nabla R)\cdot\mathbf{n}) + ((u_a^\varepsilon - u^0)\cdot\nabla M)R\big).\end{aligned}$$

Notice that the second order derivatives of R vanish on the right hand side above, we deduce that

$$|I_{83}| \lesssim \sqrt{\varepsilon}\big(\|u_a^\varepsilon - u^0\|_{m-1,\infty} + \|\nabla(u_a^\varepsilon - u^0)\|_{m-1,\infty}\big)\|\nabla R\|_{m-1}\|\eta\|_{m-1},$$

which together with (2.40) and (5.38) ensures that

$$|I_{83}| \lesssim \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\|\nabla R\|_{m-1}\|\eta\|_{m-1} \lesssim \langle t\rangle^{-\gamma}(\|\eta\|_{m-1}^2 + \|R\|_m^2) + \varepsilon\langle t\rangle^{-3\gamma}. \tag{5.56}$$

- *Estimate of* I_{84}. In view of (2.30), we write

$$I_{84} = \varepsilon^2\int_{\mathcal{O}}\Big(\sum_{\beta_1+\beta_2=\beta,\beta_1\neq 0} c_{\beta_1} Z^{\beta_1}R\cdot Z^{\beta_2}\nabla\eta + R\cdot[\nabla, Z^\beta]\eta\Big)\cdot Z^\beta\eta.$$

We remark that if we use directly use the generalized Sobolev-Gagliardo-Nirenberg-Morse inequality above, there appears the term, $\|\nabla\eta\|_{L^\infty}$, which we do not have the estimate. To overcome this difficulty, we use integrations by parts to transfer the ∇ on terms like $Z^{\beta_2}\nabla\eta$ into other terms. Notice that $Z^\beta\eta = 0$ on $\partial\mathcal{O}$, no boundary term appears during this process. Then by applying the generalized Sobolev-Gagliardo-Nirenberg-Morse inequality, we find

$$\begin{aligned}|I_{84}| \lesssim {} & \varepsilon^2\big(\|ZR\|_{L^\infty}\|\eta\|_{m-2} + \|R\|_{m-1}\|\eta\|_{L^\infty}\big)\|\nabla\eta\|_{m-1}\\ & + \varepsilon^2\big(\|\nabla R\|_{L^\infty}\|\eta\|_{m-1} + \|\nabla R\|_{m-1}\|\eta\|_{L^\infty}\big)\|\eta\|_{m-1}\\ & + \varepsilon^2\|R\|_{L^\infty}\|\nabla\eta\|_{m-1}\|\eta\|_{m-1},\end{aligned}$$

from which, with (5.38) and (5.39), we infer

$$|I_{84}| \leq \lambda\varepsilon\|\nabla\eta\|_{m-1}^2 + C_\lambda\varepsilon(\|\eta\|_{m-1}^2 + \|R\|_m^2)(\varepsilon\|\eta\|_{L^\infty}^2 + \varepsilon\|R\|_{1,\infty}^2 + 1) + C\varepsilon^2\langle t\rangle^{-\gamma}. \tag{5.57}$$

- *Estimate of* I_{85}. Along the same line to the estimate of I_{84}, we write

$$[R\cdot\nabla, Z^\beta](\chi G)=\sum_{\beta_1+\beta_2=\beta,\beta_1\neq 0} c_{\beta_1} Z^{\beta_1}R\cdot Z^{\beta_2}\nabla(\chi G)\cdot Z^\beta\eta$$
$$+R\cdot[\nabla, Z^\beta](\chi G)\cdot Z^\beta\eta,$$

from which, with (2.37) and $m\leq p-3$, we infer

$$|I_{85}|\lesssim \varepsilon^{\frac{5}{2}}\langle t\rangle^{-\gamma}\|R\|_{m-1}\|\eta\|_{m-1}. \tag{5.58}$$

- *Estimate of* I_{86}. In view of (1.1), we write

$$R=[R\cdot\nabla, \chi\Pi](D(R)\cdot\mathbf{n}+MR)$$
$$+\chi\Pi\big(R\cdot\nabla(D(R)\cdot\mathbf{n})-D(R\cdot\nabla R)\cdot\mathbf{n}+(R\cdot\nabla M)R\big).$$

Notice that the second order derivatives of R vanish on the right-hand side above, we deduce from the generalized Sobolev-Gagliardo-Nirenberg-Morse inequality, that

$$\varepsilon^{\frac{5}{2}}\|[R\cdot\nabla, \chi\mathcal{N}]R\|_{m-1}\lesssim \varepsilon^{\frac{5}{2}}\big(\|\nabla R\|_{L^\infty}+\|R\|_{L^\infty}\big)\big(\|\nabla R\|_{m-1}+\|R\|_{m-1}\big),$$

which, together with (5.38) and (5.39), ensures that

$$|I_{86}|\lesssim\varepsilon\big(\|\eta\|_{m-1}^2+\|R\|_m^2\big)\big(1+\varepsilon\|\eta\|_{L^\infty}^2+\varepsilon\|R\|_{1,\infty}^2\big)+\varepsilon^2\langle t\rangle^{-\gamma}. \tag{5.59}$$

By summing up the estimates, (5.54-5.59), we arrive at

$$\begin{aligned}|I_8|\leq&\lambda\varepsilon\|\nabla\eta\|_{m-1}^2+C\big(\varepsilon\|\nabla\eta\|_{m-2}^2+\varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}\big)\\&+C_\lambda\big(\varepsilon+\langle t\rangle^{-\gamma}+\varepsilon^2(\|\eta\|_{L^\infty}^2+\|R\|_{1,\infty}^2)\big)\left(\|\eta\|_{m-1}^2+\|R\|_m^2\right).\end{aligned} \tag{5.60}$$

By inserting the estimates, (5.47), (5.48), (5.49), (5.50), (5.50), (5.51), (5.52), (5.53) and (5.60), into (5.45) and summing over the resulting inequalities with the multi-indices α with $|\alpha|\leq m$, and finally choosing λ to be sufficiently small, we obtain (5.44). This ends the proof of Proposition 5.6. □

By summing up (5.43) and (5.44), we achieve

$$\begin{aligned}&\frac{d}{dt}\big(\|R(t)\|_m^2+\|\eta(t)\|_{m-1}^2\big)+\varepsilon\big(\|\nabla R\|_m^2+\|\nabla\eta\|_{m-1}^2\big)\\&\quad\lesssim\varepsilon\big(\|\nabla R\|_{m-1}^2+\|\nabla\eta\|_{m-2}^2\big)+\varepsilon^{\frac{1}{4}}\langle t\rangle^{-\gamma}\\&\quad+\big(\varepsilon+\langle t\rangle^{-\gamma}+\varepsilon^2(\|\eta\|_{L^\infty}^2+\|R\|_{1,\infty}^2)\big)\big(\|\eta\|_{m-1}^2+\|R\|_m^2\big)\\&\quad+\sum_{|\alpha|\leq m}\Big|\int_{\mathcal{O}}Z^\alpha\nabla\pi\cdot Z^\alpha R\Big|+\sqrt{\varepsilon}\sum_{|\beta|\leq m-1}\Big|\int_{\mathcal{O}}Z^\beta\chi\mathcal{N}(\nabla\pi)\cdot Z^\beta\eta\Big|.\end{aligned} \tag{5.61}$$

To estimate the two integrals in (5.61), we will have to deal with the pressure estimates in the coming subsection.

5.5. Estimate of the Pressure Term

In view of (5.1), the pressure π satisfies

$$\begin{cases} \Delta\pi = -\mathrm{div}\, F - \mathrm{div}\,(u^\varepsilon \cdot \nabla R + R \cdot \nabla u_a^\varepsilon) + \partial_t H - \varepsilon\Delta H & \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}}\pi = -F \cdot \mathbf{n} - (u^\varepsilon \cdot \nabla R + R \cdot \nabla u_a^\varepsilon) \cdot \mathbf{n} + \varepsilon\Delta R \cdot \mathbf{n} & \text{on } \partial\mathcal{O}. \end{cases}$$

We start the estimate of $\nabla\pi$ with the following toy model:

Lemma 5.7. *Let π_1 be determined by*

$$\begin{cases} \Delta\pi_1 = -\mathrm{div}\, F & \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}}\pi_1 = -F \cdot \mathbf{n} & \text{on } \partial\mathcal{O}, \end{cases} \tag{5.62}$$

Then for any non-negative integer ℓ, one has

$$\|\nabla\pi_1\|_\ell \lesssim \|F\|_\ell. \tag{5.63}$$

Proof. We proceed by induction on ℓ. By taking L^2 inner product of the first equation of (5.62) and using integrations by parts, we find

$$\|\nabla\pi_1\|^2 = -\int_{\mathcal{O}} F \cdot \nabla\pi_1,$$

which implies $\|\nabla\pi_1\| \le \|F\|$; therefore (5.63) holds for $\ell = 0$.

Next we assume that (5.63) holds for $\ell = m - 1$ with $\ell \ge 1$. We are going to prove that (5.63) holds for $\ell = m$. Indeed by applying Z^α with $|\alpha| = m$ to (5.62), we get

$$-\Delta Z^\alpha\pi_1 - [Z^\alpha, \Delta]\pi_1 = Z^\alpha \mathrm{div}\, F.$$

By taking L^2 inner product of the above equation with $Z^\alpha\pi_1$ and using integration by parts, we obtain

$$\|\nabla Z^\alpha\pi_1\|^2 = \sum_{i=1}^{4} J_i, \tag{5.64}$$

where

$$J_1 := \int_{\partial\mathcal{O}} [\partial_{\mathbf{n}}, Z^\alpha]\pi_1 Z^\alpha\pi_1, \quad J_2 := -\int_{\mathcal{O}} [\Delta, Z^\alpha]\pi_1 Z^\alpha\pi_1,$$
$$J_3 := -\int_{\partial\mathcal{O}} Z^\alpha(F \cdot \mathbf{n}) Z^\alpha\pi_1, \quad J_4 := \int_{\mathcal{O}} Z^\alpha \mathrm{div}\, F Z^\alpha\pi_1.$$

Let us now handle term by term the quantities above.

• Estimate of J_1.

Notice that $Z_0 = 0$ on $\partial\mathcal{O}$, so that if Z^α contains the tangential vector field Z_0, $J_1 = 0$. Without loss of generality, we may assume that Z^α is composed of Z_i with $1 \le i \le 5$. We write, $Z^\alpha = Z_{i_1} Z^{\alpha_1}$, $|\alpha_1| = \ell - 1$, then

$$[\partial_{\mathbf{n}}, Z^\alpha] = [\partial_{\mathbf{n}}, Z_{i_1}]Z^{\alpha_1} + Z_{i_1}[\partial_{\mathbf{n}}, Z^{\alpha_1}].$$

As presented in Sect. 2.8, the vector fields, $[\partial_{\mathbf{n}}, Z_{i_1}]$, are also tangential derivatives. By induction, $[\partial_{\mathbf{n}}, Z^\alpha]$ is a tangential derivative operator of order m. By trace inequality, (5.25), we infer

$$|J_1| \lesssim \|\pi_1\|^2_{H^m(\partial\mathcal{O})} \lesssim \|\pi_1\|_m^2 + \|\pi_1\|_m \|\nabla\pi_1\|_m \lesssim \|\nabla\pi_1\|_{m-1}\|\nabla\pi_1\|_m. \tag{5.65}$$

- Estimate of J_2.

  To deal with the commutator, we use (2.32) to write

$$\int_{\mathcal{O}} [\Delta, Z^\alpha]\pi_1 Z^\alpha \pi_1 = \sum_{|\alpha_1|,|\alpha_2|\le m-1} \int_{\mathcal{O}} \big(c_{\alpha_1}\nabla^2 Z^{\alpha_1}\pi_1 + c_{\alpha_2}\nabla Z^{\alpha_2}\pi_1\big) Z^\alpha \pi_1, \tag{5.66}$$

where $c_{\alpha_1}, c_{\alpha_2}$ are some smooth functions. Yet we do not want the second order normal derivative of π_1 to appear in (5.66). The idea is to use integration by parts. The cost is that boundary terms like $\int_{\partial\mathcal{O}} \mathbf{n}\cdot c_{\alpha_1}\cdot \nabla Z^{\alpha_1}\pi_2 Z^\alpha \pi_2$ will appear. In general, we can not guarantee that $\mathbf{n}\cdot c_{\alpha_1}\cdot\nabla$ is a tangential derivative. One attempt is to use the boundary condition, $\partial_{\mathbf{n}}\pi_1 = -F\cdot\mathbf{n}$, and then the boundary terms will be bounded by $\|F\cdot\mathbf{n}\|_{H^m(\partial\mathcal{O})}$. Although Lemma 4.2 gives $\|F\|_{H^m(\partial\mathcal{O})} \lesssim \langle t\rangle^{-\gamma}$, so that $\|F\cdot\mathbf{n}\|_{H^m(\partial\mathcal{O})}$ will gives rise to an appropriate estimate of π_1. But when we apply similar estimate to deal with π_3, term like $\|R\cdot\nabla R\|_{H^m(\partial\mathcal{O})}$ will appear, which is out of control.

To overcome the above mentioned difficulty, we distinguish the terms in (5.66) into two cases.

- If Z^α contains a field Z_0, then $Z^\alpha = 0$ on $\partial\mathcal{O}$. In this case, we use integration by parts to get

$$|J_2| \lesssim \|\nabla\pi_2\|_{m-1}\|\nabla\pi_2\|_m.$$

- If Z^α does not contain any Z_0, we write

$$Z^\alpha = Z_{k_1}Z_{k_2}\cdots Z_{k_m} \quad \text{with} \quad Z_{k_i} = w^{k_i}\cdot\nabla, \quad k_i \in \{1,2,3,4,5\},\ 1\le i\le m,$$

  for w^{k_i} given in Sect. 2.8.

  As a convention, let $Z^{\alpha_0} = Z^{\beta_{m+1}}$ be the identity operators, we denote

$$Z^{\alpha_i} := Z_{k_1}\cdots Z_{k_i} \quad \text{and} \quad Z^{\beta_i} := Z_{k_i}\cdots Z_{k_m} \quad \text{with} \quad 1\le i\le m.$$

  Then by (2.31), we write

$$\begin{aligned}[\Delta, Z^\alpha]\pi_1 &= \sum_{i=1}^m Z^{\alpha_{i-1}}[\Delta, Z_{k_i}]Z^{\beta_{i+1}}\pi_1\\ &= \sum_{i=1}^m Z^{\alpha_{i-1}}\big(\Delta w^{k_i}\cdot\nabla Z^{\beta_{i+1}}\pi_1 + 2\nabla w^{k_i} : \nabla^2 Z^{\beta_{i+1}}\pi_1\big).\end{aligned}$$

Notice that for $k_i \neq 0$, $w^{k_i} \cdot \mathbf{n} = 0$ in $\mathcal{V}_{\delta_0/2}$, $|\mathbf{n}| = 1$ in $\mathcal{V}_{\delta_0}$ and ∇n is symmetric, we have

$$\mathbf{n} \cdot \nabla w^{k_i} \cdot \mathbf{n} = -\mathbf{n} \cdot \nabla \mathbf{n} \cdot w^{k_i} = -w^{k_i} \cdot \nabla \mathbf{n} \cdot \mathbf{n} = 0, \quad \text{in } \mathcal{V}_{\delta_0/2}.$$

So that $\nabla w^{k_i} : \nabla^2$ contains at most one normal derivative and this implies

$$\|[\Delta, Z^\alpha]\pi_1\| \lesssim \|\nabla \pi_1\|_m.$$

As a result, it turns out that

$$|J_2| \lesssim \|\nabla \pi_1\|_{m-1} \|\nabla \pi_1\|_m. \tag{5.67}$$

- Estimate of $J_3 + J_4$

  Again we distinguish to the following two cases:

  - If Z^α contains Z_0, then $Z^\alpha = 0$ on $\partial\mathcal{O}$. In this case $J_3 = 0$. For J_4, we use integration by parts to get

$$\begin{aligned} J_4 &= \sum_{|\alpha_1| \leq m} \int_{\mathcal{O}} c_{\alpha_1} \cdot \nabla Z^{\alpha_1} F Z^\alpha \pi_1 \\ &= \sum_{|\alpha_1| \leq m} \int_{\mathcal{O}} Z^{\alpha_1} F (\operatorname{div} c_{\alpha_1} Z^\alpha \pi_1 + c_{\alpha_1} \cdot \nabla Z^\alpha \pi_1), \end{aligned}$$

  from which, we infer

$$|J_4| \lesssim \|\nabla \pi_1\|_m \|F\|_m.$$

  - If Z^α does not contain Z_0, notice that for $1 \leq i \leq 5$, $Z_i = w^i \cdot \nabla$ and $w^i \cdot \mathbf{n} = 0$ in $\mathcal{V}_{\delta_0/2}$ and $\operatorname{div} w^i = 0$, we get, by using integration by parts, that

$$\begin{aligned} \int_{\mathcal{O}} Z^\alpha \operatorname{div} F Z^\alpha \pi_1 &= (-1)^m \int_{\mathcal{O}} \operatorname{div} F Z^{2\alpha} \pi_1 \\ &= (-1)^m \int_{\partial\mathcal{O}} F \cdot \mathbf{n} Z^{2\alpha} \pi_1 - (-1)^m \int_{\mathcal{O}} F \cdot \nabla Z^{2\alpha} \pi_1 \\ &= \int_{\partial\mathcal{O}} Z^\alpha (F \cdot \mathbf{n}) Z^\alpha \pi_1 + \sum_{|\alpha_1| \leq m} \int_{\mathcal{O}} c_{\alpha_1} F \cdot Z^{\alpha_1} \nabla Z^\alpha \pi_1 \\ &= \int_{\partial\mathcal{O}} Z^\alpha (F \cdot \mathbf{n}) Z^\alpha \pi_1 + \sum_{|\alpha_1| \leq m} \int_{\mathcal{O}} \nabla Z^\alpha \pi_1 \cdot Z^{\alpha_1} (c_{\alpha_1} F), \end{aligned}$$

  where c_{α_1} are some smooth functions depend only on the vector field in $\mathfrak{W}$. As a consequence, we obtain

$$J_3 + J_4 = \sum_{|\alpha_1| \leq m} \int_{\mathcal{O}} \nabla Z^\alpha \pi_1 \cdot Z^{\alpha_1} (c_{\alpha_1} F),$$

  which implies

$$|J_3 + J_4| \lesssim \|\nabla \pi_1\|_m \|F\|_m. \tag{5.68}$$

In view of (5.64), by summarizing the estimates, (5.65), (5.67) and (5.68), we conclude the proof of (5.63). □

Proposition 5.8. *For* $1 \leq m \leq p-4$ *and for* $t \in [0, T^\varepsilon]$*, we have*

$$\|\nabla \pi\|_m \lesssim \varepsilon^{\frac{1}{4}} \langle t\rangle^{-\gamma} + \varepsilon \|\nabla R\|_m + \langle t\rangle^{-\gamma} \|R\|_m + \varepsilon^2 \big(\|R\|_{L^\infty} \|\nabla R\|_m + \|R\|_m \|\nabla R\|_{L^\infty}\big). \tag{5.69}$$

Proof. We first decompose π into four terms $\pi = \pi_1 + \pi_2 + \pi_3 + \pi_4$, where π_1, π_2, π_3 and π_4 are determined respectively by (5.62) and

$$\begin{cases} \Delta \pi_2 = \partial_t H \quad \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}} \pi_2 = 0 \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{5.70}$$

$$\begin{cases} \Delta \pi_3 = -\operatorname{div}(u^\varepsilon \cdot \nabla R + R \cdot \nabla u_a^\varepsilon) \quad \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}} \pi_3 = -(u^\varepsilon \cdot \nabla R + R \cdot \nabla u_a^\varepsilon) \cdot \mathbf{n} \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{5.71}$$

and

$$\begin{cases} \Delta \pi_4 = -\varepsilon \Delta H \quad \text{in } \mathcal{O}, \\ \partial_{\mathbf{n}} \pi_4 = \varepsilon \Delta R \cdot \mathbf{n} \quad \text{on } \partial\mathcal{O}. \end{cases} \tag{5.72}$$

- The estimate of$\nabla \pi_1$.

The estimate $\nabla \pi_1$ relies on Lemma 5.7. Indeed we deduce from Lemma 5.7 and (2.35) that

$$\|\nabla \pi_1\|_\ell \lesssim \varepsilon^{\frac{1}{4}} \langle t\rangle^{-\gamma} \quad \text{for } 0 \leq \ell \leq p-3. \tag{5.73}$$

- The estimate of $\nabla \pi_2$.

We claim that for $0 \leq \ell \leq p-3$,

$$\|\nabla \pi_2\|_\ell \lesssim \varepsilon^{\frac{1}{4}} \langle t\rangle^{-\gamma}. \tag{5.74}$$

Without losing generality, we may assume that $\int_{\mathcal{O}} \pi_2 = 0$. Again we proceed by induction on ℓ. Indeed by taking L^2 inner product of the (5.70) with π_2 and then using integrations by parts and the Poincaré inequality, we find

$$\|\nabla \pi_2\|^2 = -\int_{\mathcal{O}} (\Delta \pi_2) \pi_2 = -\int_{\mathcal{O}} (\partial_t H) \pi_2 \lesssim \|\partial_t H\| \|\nabla \pi_2\|,$$

which together with (2.35) yields (5.74) for $\ell = 0$.

Next let us assume that (5.74) holds for $\ell \leq m-1 \leq p-4$, we are going to prove that (5.74) holds for $\ell = m$. In order to do it, we apply Z^α with $|\alpha| \leq m$ to (5.70) and then taking L^2 inner product of the resulting equation with $Z^\alpha \pi_2$ and using integration by parts, we obtain

$$\begin{aligned} \|\nabla Z^\alpha \pi_2\|^2 &= \int_{\partial\mathcal{O}} (\partial_{\mathbf{n}} Z^\alpha \pi_2) Z^\alpha \pi_2 - \int_{\mathcal{O}} (\Delta Z^\alpha \pi_2) Z^\alpha \pi_2 \\ &= \int_{\partial\mathcal{O}} [\partial_{\mathbf{n}}, Z^\alpha] \pi_2 Z^\alpha \pi_2 - \int_{\mathcal{O}} (Z^\alpha \partial_t H) Z^\alpha \pi_2 + \int_{\mathcal{O}} [\Delta, Z^\alpha] \pi_2 Z^\alpha \pi_2, \end{aligned} \tag{5.75}$$

where we used $\partial_{\mathbf{n}}\pi_2 = 0$ on $\partial\mathcal{O}$, so that $Z^\alpha \partial_{\mathbf{n}}\pi_2 = 0$ on $\partial\mathcal{O}$.

As the estimate of J_1 in the proof of Lemma 5.7, if Z^α contains Z_0, the first term of the right hand side of (5.75) disappear. Otherwise, $[\partial_n, Z^\alpha]$ is a tangential differential operator of order m. Then we get, by applying the trace inequality (5.25), that

$$|\int_{\partial\mathcal{O}} [\partial_{\mathbf{n}}, Z^\alpha]\pi_2 Z^\alpha \pi_2| \lesssim \|\pi_2\|^2_{H^m(\partial\mathcal{O})} \lesssim \|\pi_2\|_m^2 + \|\pi_1\|_m \|\nabla\pi_2\|_m \lesssim \|\nabla\pi_2\|_{m-1}\|\nabla\pi_2\|_m. \tag{5.76}$$

On the other hand, it follows from (2.35) that

$$|\int_{\mathcal{O}} (Z^\alpha \partial_t H) Z^\alpha \pi_2| \lesssim \varepsilon^{\frac14} \langle t\rangle^{-\gamma} \|\pi_2\|_m. \tag{5.77}$$

For the last term in the right hand side of (5.75), we deduce along the same line to that of J_2 in the proof of Lemma 5.7 that

$$|\int_{\mathcal{O}} [\Delta, Z^\alpha]\pi_2 Z^\alpha \pi_2| \lesssim \|\nabla\pi_2\|_{m-1}\|\pi_2\|_m. \tag{5.78}$$

On the other hand, it follows from the boundary condition $\partial_{\mathbf{n}}\pi_2 = 0$ that

$$\|\nabla\pi_2\|_{H^{m-1}(\mathcal{O})} \approx \|\pi_2\|_{H^m(\mathcal{O})}.$$

Then by inserting the estimates (5.76),(5.77) and (5.78) into (5.75) and then summing up the resulting inequalities for $|\alpha| \le m$, we obtain

$$\|\nabla\pi_2\|_m^2 \le C\big(\|\nabla\pi_2\|_{m-1}\|\nabla\pi_2\|_m + \varepsilon^{\frac14}\langle t\rangle^{-\gamma}\|\nabla\pi_2\|_{m-1}\big),$$

which, together with the inductive assumption, ensures (5.74) for $\ell = m$. This proves (5.74).

- The estimate of $\nabla\pi_3$.

Due to $\operatorname{div} u^\varepsilon = \sigma^0$ and $\operatorname{div} R = -H$, we write

$$\operatorname{div}(u^\varepsilon \cdot \nabla R) = \operatorname{div}\big(R\cdot\nabla u^\varepsilon - Hu^\varepsilon - \sigma^0 R\big).$$

On the other hand, due to $u^\varepsilon \cdot \mathbf{n} = R\cdot\mathbf{n} = 0$ on $\partial\mathcal{O}$ and $\nabla\mathbf{n}$ is symmetric, one has

$$(u^\varepsilon\cdot\nabla R)\cdot\mathbf{n} = -(u^\varepsilon\cdot\nabla\mathbf{n})\cdot R = -(R\cdot\nabla\mathbf{n})\cdot u^\varepsilon = (u^\varepsilon\cdot\nabla\mathbf{n})\cdot R.$$

In view of (5.71), π_3 verifies

$$\begin{cases} \Delta\pi_3 = -\operatorname{div}\big(R\cdot\nabla(u^\varepsilon + u_a^\varepsilon) - \sigma^0 R - Hu^\varepsilon\big) & \text{in } \mathcal{O},\\ \partial_{\mathbf{n}}\pi_3 = -R\cdot\nabla(u^\varepsilon + u_a^\varepsilon)\cdot\mathbf{n} & \text{on } \partial\mathcal{O}. \end{cases}$$

From Lemma 5.7 and the generalized Sobolev-Gagliardo-Nirenberg-Morse inequality, we infer that

$$\begin{aligned}\|\nabla\pi_3\|_m \lesssim& \|R\cdot\nabla(u^\varepsilon + u_a^\varepsilon) - \sigma^0 R - Hu^\varepsilon\|_m\\ \lesssim& \|R\|_m\|\nabla u_a^\varepsilon\|_{m,\infty} + \varepsilon^2\big(\|R\|_{L^\infty}\|\nabla R\|_m + \|R\|_m\|\nabla R\|_{L^\infty}\big)\\ &+ \|\sigma^0\|_{m,\infty}\|R\|_m + \|H\|_m\|u_a^\varepsilon\|_{m,\infty} + \varepsilon^2\|H\|_{m,\infty}\|R\|_m,\end{aligned}$$

which, together (2.35), (2.36), (2.39), and the fact that σ^0 is smooth and supported in $[0, T]$, ensures that for $m \le p-3$,

$$\|\nabla\pi_3\|_m \lesssim \langle t\rangle^{-\gamma}\|R\|_m + \varepsilon^{\frac14}\langle t\rangle^{-\gamma} + \varepsilon^2\big(\|R\|_{L^\infty}\|\nabla R\|_m + \|R\|_m\|\nabla R\|_{L^\infty}\big). \tag{5.79}$$

- The estimate of $\nabla\pi_4$.

In view of (5.72), we write

$$\Delta(\pi_4+\varepsilon H) = 0 \text{ in } \mathcal{O} \quad\text{and}\quad \partial_{\mathbf{n}}(\pi_4+\varepsilon H) = -\varepsilon\Delta R\cdot\mathbf{n} + \varepsilon\partial_{\mathbf{n}}H \text{ on } \partial\mathcal{O},$$

from which, we deduce that for $m \ge 1$

$$\|\nabla(\pi_4+\varepsilon H)\|_m \lesssim \varepsilon\|\Delta R\cdot\mathbf{n} - \partial_{\mathbf{n}}H\|_{H^{m-\frac12}(\partial\mathcal{O})}.$$

yet it follows from (2.35) and trace theorem that, for $m \le p-4$,

$$\varepsilon\|\Delta H\|_m \lesssim \varepsilon^{\frac14}\langle t\rangle^{-\gamma},$$
$$\varepsilon\|\partial_{\mathbf{n}}H\|_{H^{m-\frac12}(\mathcal{O})} \lesssim \varepsilon\|\nabla^2 H\|_m \lesssim \varepsilon^{\frac14}\langle t\rangle^{-\gamma}.$$

As a result, it turns out that

$$\|\nabla\pi_4\|_m \lesssim \varepsilon\|\Delta R\cdot\mathbf{n}\|_{H^{m-\frac12}(\partial\mathcal{O})} + \varepsilon^{\frac14}\langle t\rangle^{-\gamma}.$$

The term $\|\Delta R\cdot\mathbf{n}\|_{H^{m-\frac12}(\partial\mathcal{O})}$ above can be handled exactly as that in Proposition 19 of [30] so that

$$\|\Delta R\cdot\mathbf{n}\|_{H^{m-\frac12}(\partial\mathcal{O})} \lesssim \|\nabla R\|_m.$$

Then we obtain, for $1 \le m \le p-4$,

$$\|\nabla\pi_4\|_m \lesssim \varepsilon\|\nabla R\|_m + \varepsilon^{\frac14}\langle t\rangle^{-\gamma}. \tag{5.80}$$

By summarizing the estimates (5.73), (5.74), (5.79) and (5.80), we arrive at (5.69). This completes the proof of Proposition 5.8. □

With Proposition 5.8, we now turn to the estimate of the two integrals involving the pressure term in (5.61).

Corollary 5.9. *Let $2 \le m \le p-4$. Then for α, β satisfying $|\alpha| \le m$, $|\beta| \le m-1$, and any $\lambda > 0$ there exists C_λ so that*

$$\begin{aligned} \Big|\int_{\mathcal{O}} Z^\alpha\nabla\pi\cdot Z^\alpha R\Big| \le{}& \lambda\varepsilon\|\nabla R\|_m^2 + C\varepsilon^{\frac14}\langle t\rangle^{-\gamma} \\ &+ C_\lambda\big(\varepsilon + \langle t\rangle^{-\gamma} + \varepsilon^2(\|\eta\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\big)\|R\|_m^2, \end{aligned} \tag{5.81}$$

and

$$\begin{aligned} \sqrt{\varepsilon}\Big|\int_{\mathcal{O}} Z^\beta\chi\mathcal{N}(\nabla\pi)\cdot Z^\beta\eta\Big| \le{}& \lambda\varepsilon\|\nabla\eta\|_{m-1}^2 + C_\lambda\big(\varepsilon^{\frac12}\langle t\rangle^{-2\gamma} + \varepsilon^4\langle t\rangle^{-2\gamma}\|R\|_{L^\infty}^2\big) \\ &+ C_\lambda\big(\varepsilon + \langle t\rangle^{-2\gamma} + \varepsilon^3(\|\eta\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\big) \\ &(\|R\|_m^2 + \|\eta\|_{m-1}^2). \end{aligned} \tag{5.82}$$

Proof. Thanks to (5.69), for any $\lambda > 0$, we get, by applying Young's inequality, that

$$|\int_{\mathcal{O}} Z^\alpha \nabla\pi \cdot Z^\alpha R| \leq \lambda\varepsilon \|\nabla R\|_m^2 + \varepsilon^{\frac{1}{4}} \langle t\rangle^{-\gamma} + C_\lambda \left(\varepsilon + \langle t\rangle^{-\gamma}\right) \|R\|_m^2 + \left(C_\lambda \varepsilon^3 \|R\|_{L^\infty}^2 + \varepsilon^2 \|\nabla R\|_{L^\infty}\right) \|R\|_m^2,$$

which together with (5.39) ensures (5.81).

On the other hand, due to $\eta = 0$ on $\partial\mathcal{O}$, by using integration by parts and Young's inequality, we find that for any $\lambda > 0$,

$$\sqrt{\varepsilon}|\int_{\mathcal{O}} Z^\beta \chi \mathcal{N}(\nabla\pi) \cdot Z^\beta \eta| \leq \lambda\varepsilon \|\nabla\eta\|_{m-1}^2 + C_\lambda \|\nabla\pi\|_{m-1}^2, \tag{5.83}$$

Yet it follows from (5.69), (5.38) and (5.39) that

$$\|\nabla\pi\|_{m-1}^2 \lesssim \varepsilon^{\frac{1}{2}} \langle t\rangle^{-2\gamma} + \varepsilon^4 \langle t\rangle^{-2\gamma} \|R\|_{L^\infty}^2 + \big(\varepsilon + \langle t\rangle^{-2\gamma} + \varepsilon^3 (\|\eta\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\big)(\|R\|_m^2 + \|\eta\|_{m-1}^2).$$

Substituting the above estimate into (5.83) leads to (5.82). □

By inserting the estimates (5.81) and (5.82) into (5.61) and choosing λ to be sufficiently small, we deduce that for $2 \leq m \leq p - 4$ and for t in $[0, T^\varepsilon]$,

$$\begin{aligned}&\frac{d}{dt}(\|R(t)\|_m^2 + \|\eta(t)\|_{m-1}^2) + \varepsilon\big(\|\nabla R\|_m^2 + \|\nabla\eta\|_{m-1}^2\big)\\ &\quad\lesssim \varepsilon\big(\|\nabla R\|_{m-1}^2 + \|\nabla\eta\|_{m-2}^2\big) + \varepsilon^{\frac{1}{4}} \langle t\rangle^{-\gamma} + \varepsilon^4 \langle t\rangle^{-2\gamma} \|R\|_{L^\infty}^2\\ &\qquad + \big(\varepsilon + \langle t\rangle^{-\gamma} + \varepsilon^2 (\|\eta\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\big)(\|\eta\|_{m-1}^2 + \|R\|_m^2).\end{aligned} \tag{5.84}$$

In order to close the estimate of (5.84), we still need the estimate of $\|R\|_{1,\infty}$ and $\|\eta\|_{L^\infty}$, which will be the content of the next section.

5.6. *Estimate of* $\|R\|_{1,\infty}$ *and* $\|\eta\|_{L^\infty}$

Proposition 5.10. *Let $m > 3$ be an integer. Then one has*

$$\varepsilon \|R(t)\|_{1,\infty}^2 \leq C\big(\|R(t)\|_m^2 + \|\eta(t)\|_{m-1}^2 + \varepsilon \langle t\rangle^{-2\gamma}\big). \tag{5.85}$$

Proof. We first deduce from Proposition 20 of [30] that for $m_0 > 1$,

$$\|R(t)\|_{L^\infty}^2 \leq C\big(\|\partial_{\mathbf{n}} R(t)\|_{m_0} \|R(t)\|_{m_0} + \|R(t)\|_{m_0}^2\big),$$

which, together with (5.38), implies

$$\begin{aligned}\varepsilon \|R(t)\|_{L^\infty}^2 &\leq C\big(\varepsilon \|\partial_{\mathbf{n}} R(t)\|_{m_0} \|R(t)\|_{m_0} + \varepsilon \|R(t)\|_{m_0}^2\big)\\ &\leq C\big(\|\eta(t)\|_{m-1}^2 + \|R(t)\|_m^2 + \varepsilon \langle t\rangle^{-2\gamma}\big) \quad \text{if } m \geq m_0 + 1.\end{aligned} \tag{5.86}$$

Along the same lines, we can prove similar estimate for $\|ZR\|_{L^\infty}$ if $m \geq m_0 + 2$. □

In order to estimate $\|\eta\|_{L^\infty}$, we introduce

$$\tilde{\eta} := \sqrt{\varepsilon}\nabla \wedge R. \tag{5.87}$$

Lemma 5.11. *Let η and $\tilde{\eta}$ be determined respectively by (5.37) and (5.87). Then one has*

$$\|\eta\|_{L^\infty} + \|R\|_{1,\infty} + \sqrt{\varepsilon}\langle t\rangle^{-\gamma} \approx \|\tilde{\eta}\|_{L^\infty} + \|R\|_{1,\infty} + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}.$$

Proof. On the one hand, it follows from (5.39) that

$$\|\tilde{\eta}\|_{L^\infty} \lesssim \sqrt{\varepsilon}\|\nabla R\|_{L^\infty} \lesssim \|\eta\|_{L^\infty} + \|R\|_{1,\infty} + \sqrt{\varepsilon}\langle t\rangle^{-\gamma},$$

which implies

$$\|\tilde{\eta}\|_{L^\infty} + \|R\|_{1,\infty} + \sqrt{\varepsilon}\langle t\rangle^{-\gamma} \lesssim \|\eta\|_{L^\infty} + \|R\|_{1,\infty} + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}. \tag{5.88}$$

On the other hand, due to $n \wedge (\nabla \wedge R) = \nabla R \cdot \mathbf{n} - \partial_{\mathbf{n}} R$, we have

$$\sqrt{\varepsilon}\|\partial_{\mathbf{n}} R\|_{L^\infty} \lesssim \|\tilde{\eta}\|_{L^\infty} + \sqrt{\varepsilon}\|ZR\|_{L^\infty} + \sqrt{\varepsilon}\|\partial_{\mathbf{n}} R \cdot \mathbf{n}\|_{L^\infty}.$$

However, it follows from (5.17) and (2.36) that

$$\|\partial_{\mathbf{n}} R \cdot \mathbf{n}\|_{L^\infty} \lesssim \|ZR\|_{L^\infty} + \langle t\rangle^{-\gamma},$$

so that

$$\sqrt{\varepsilon}\|\partial_{\mathbf{n}} R\|_{L^\infty} \lesssim \|\tilde{\eta}\|_{L^\infty} + \|R\|_{1,\infty} + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}.$$

This together with (5.37) shows that the other side of the inequality (5.88) holds. This concludes the proof of Lemma 5.11. □

Now let us set

$$\mathfrak{N}_m(t) := \|R(t)\|_m^2 + \|\eta(t)\|_{m-1}^2 + \varepsilon\|\tilde{\eta}(t)\|_{L^\infty}^2. \tag{5.89}$$

Note that (2.38) implies

$$\|R_0\|_m \lesssim \varepsilon^{-\frac{1}{4}}, \|\nabla R_0\|_{m-1} \lesssim \varepsilon^{-\frac{3}{4}}, \|\nabla^2 R_0\|_{m-2} \lesssim \varepsilon^{-\frac{5}{4}}. \tag{5.90}$$

Hence

$$\begin{aligned}\|\eta_0\|_{m-1} \lesssim \sqrt{\varepsilon}\|\nabla R_0\|_{m-1} \lesssim \varepsilon^{-\frac{1}{4}} \text{ and } \|\tilde{\eta}_0\|_{L^\infty(\mathcal{O})} \\ \lesssim \sqrt{\varepsilon}\|\nabla R_0\|_{H^1(\mathcal{O})} \lesssim \varepsilon^{-\frac{3}{4}}.\end{aligned} \tag{5.91}$$

Therefore

$$\mathfrak{N}_m(0) \lesssim \varepsilon^{-\frac{1}{2}}. \tag{5.92}$$

Proposition 5.12. *Let $\mathfrak{N}_m(t)$ be determined by (5.89). Then there exist constant ε_0, C so that for $\varepsilon \leq \varepsilon_0$, $4 \leq m \leq p-4$*

$$\mathfrak{N}_m(t) + \varepsilon \int_0^t \big(\|\nabla R\|_m^2 + \|\eta\|_{m-1}^2\big)\, ds \leq C\varepsilon^{-\frac{1}{2}} \quad \textit{for} \ t \leq T^\varepsilon. \tag{5.93}$$

Proof. In view of (5.1), $\tilde{\eta}$ satisfies

$$\partial_t \tilde{\eta} - \varepsilon\Delta\tilde{\eta} + u^\varepsilon \cdot \nabla\tilde{\eta} + \sqrt{\varepsilon}\nabla u^\varepsilon \wedge \nabla R + \sqrt{\varepsilon}\nabla \wedge (R \cdot \nabla u_a^\varepsilon) = \sqrt{\varepsilon}\nabla \wedge F.$$

Maximum principle for the transport-diffusion equation ensures that

$$\|\tilde{\eta}(t)\|_{L^\infty} \leq \|\tilde{\eta}_0\|_{L^\infty} + \sqrt{\varepsilon}\int_0^t \big(\|\nabla \wedge F\|_{L^\infty} + \|\nabla u^\varepsilon \wedge \nabla R\|_{L^\infty} + \|\nabla \wedge (R \cdot \nabla u_a^\varepsilon)\|_{L^\infty}\big)\, ds. \tag{5.94}$$

Applying (2.36) gives

$$\sqrt{\varepsilon}\|\nabla \wedge F(s)\|_{L^\infty} \lesssim \langle s\rangle^{-\gamma}.$$

On the other hand, it follows from (2.39) that

$$\sqrt{\varepsilon}\|\nabla u^\varepsilon \wedge \nabla R(s)\|_{L^\infty} \lesssim \sqrt{\varepsilon}\langle s\rangle^{-\gamma}\|\nabla R(s)\|_{L^\infty} + \varepsilon^{\frac{5}{2}}\|\nabla R(s)\|_{L^\infty}^2.$$

Notice that

$$\sqrt{\varepsilon}\nabla \wedge (R \cdot \nabla u_a^\varepsilon) = \sqrt{\varepsilon}\left(\partial_i R \cdot \nabla(u_a^\varepsilon)^j - \partial_j R \cdot \nabla(u_a^\varepsilon)^i\right)_{3\times 3} + R \cdot \nabla(\sqrt{\varepsilon}\nabla \wedge u_a^\varepsilon),$$

we infer

$$\sqrt{\varepsilon}\|\nabla \wedge (R \cdot \nabla u_a^\varepsilon)(s)\|_{L^\infty} \lesssim \langle s\rangle^{-\gamma}\big(\sqrt{\varepsilon}\|\nabla R(s)\|_{L^\infty} + \|R(s)\|_{L^\infty}\big).$$

By inserting the above estimates into (5.94) and then using (5.38), (5.39) and (5.91), we achieve

$$\begin{aligned}
\|\tilde{\eta}(t)\|_{L^\infty} &\lesssim \varepsilon^{-\frac{3}{4}} + \int_0^t \Big(\langle s\rangle^{-\gamma} + \sqrt{\varepsilon}\big(\langle s\rangle^{-\gamma}\|\nabla R\|_{L^\infty} + \varepsilon^2\|\nabla R\|_{L^\infty}^2\big) \\
&\qquad + \langle s\rangle^{-\gamma}\big(\|\tilde{\eta}\|_{L^\infty} + \|R\|_{L^\infty}\big)\Big)\, ds \\
&\lesssim \varepsilon^{-\frac{3}{4}} + \int_0^t \langle s\rangle^{-\gamma}\big(\|\tilde{\eta}\|_{L^\infty} + \|R\|_{1,\infty} + \varepsilon^{\frac{3}{2}}(\|\tilde{\eta}\|_{L^\infty}^2 + \|R\|_{1,\infty}^2)\big)\, ds,
\end{aligned}$$

from which, with (5.85) and (5.89), we deduce

$$\varepsilon\|\tilde{\eta}(t)\|_{L^\infty}^2 \lesssim \varepsilon^{-\frac{1}{2}} + \int_0^t \langle s\rangle^{-\gamma}\big(\mathfrak{N}_m + \varepsilon^2\mathfrak{N}_m^2\big)\, ds.$$

For any $t \leq T^\varepsilon$, by integrating (5.84) over $[0, t]$ and then summing up the resulting inequality with the above inequality, we obtain for $2 \leq m \leq p-4$ that

$$\begin{aligned}
\mathfrak{N}_m(t) + \varepsilon\big(\|\nabla R\|_{L_t^2(H_{co}^m)}^2 + \|\nabla\eta\|_{L_t^2(H_{co}^{m-1})}^2\big) &\leq C\Big(\varepsilon\big(\|\nabla R\|_{L_t^2(H_{co}^{m-1})}^2 + \|\nabla\eta\|_{L_t^2(H_{co}^{m-2})}^2\big) \\
&\quad + \varepsilon^{-\frac{1}{2}} + \int_0^t \big((\varepsilon + \langle s\rangle^{-\gamma})\mathfrak{N}_m + \varepsilon^2\mathfrak{N}_m^2\big)\, ds\Big).
\end{aligned} \tag{5.95}$$

On the other hand, thanks to Propositions 5.3 and 5.6, we get, by a similar derivation of (5.95), that

$$\mathfrak{N}_1(t) + \varepsilon\big(\|\nabla R\|^2_{L^2_t(H^1_{\rm co})} + \|\nabla\eta\|^2_{L^2_t(L^2)}\big) \leq C\Big(\varepsilon^{-\frac{1}{2}} + \varepsilon\|\nabla R\|^2_{L^2_t(L^2)} + \int_0^t \big((\varepsilon + \langle s\rangle^{-\gamma})\mathfrak{N}_m + \varepsilon^2\mathfrak{N}_m^2\big)\,ds\Big),$$

which, together with Proposition 5.1, ensures that

$$\mathfrak{N}_1(t) + \varepsilon\big(\|\nabla R\|^2_{L^2_t(H^1_{\rm co})} + \|\nabla\eta\|^2_{L^2_t(L^2)}\big) \leq C\Big(\varepsilon^{-\frac{1}{2}} + \int_0^t \big((\varepsilon + \langle s\rangle^{-\gamma})\mathfrak{N}_m + \varepsilon^2\mathfrak{N}_m^2\big)\,ds\Big). \tag{5.96}$$

By virtue of (5.95) and (5.96), we get by an inductive argument that

$$\mathfrak{N}_m(t) + \varepsilon\big(\|\nabla R\|^2_{L^2_t(H^1_{\rm co})} + \|\nabla\eta\|^2_{L^2_t(L^2)}\big) \leq C\Big(\varepsilon^{-\frac{1}{2}} + \int_0^t \big((\varepsilon + \langle s\rangle^{-\gamma})\mathfrak{N}_m + \varepsilon^2\mathfrak{N}_m^2\big)\,ds\Big),$$

from which and a comparison argument, we infer

$$\begin{aligned}&\mathfrak{N}_m(t) + \varepsilon\big(\|\nabla R\|^2_{L^2_t(H^1_{\rm co})} + \|\nabla\eta\|^2_{L^2_t(L^2)}\big)\\ &\quad\leq C\varepsilon^{-\frac{1}{2}}\Big(1 - C^2\varepsilon^{\frac{3}{2}}t\Big)^{-1}\exp\Big(C\int_0^t(\varepsilon + \langle s\rangle^{-\gamma})\,ds\Big) \quad \text{for } t \leq T^\varepsilon \leq \frac{T}{\varepsilon}.\end{aligned} \tag{5.97}$$

In particular, if we take ε to be so small that $\varepsilon \leq (2TC^2)^{-\frac{2}{3}}$, we deduce from (5.97) that

$$\mathfrak{N}_m(t) + \varepsilon\big(\|\nabla R\|^2_{L^2_t(H^1_{\rm co})} + \|\nabla\eta\|^2_{L^2_t(L^2)}\big) \leq Ce^{CT}\varepsilon^{-\frac{1}{2}},$$

which yields (5.93). This completes the proof of Proposition 5.12. □

5.7. End of the Proof of (2.43)

For our purpose, we can take $(\gamma, k, p, s, q) = (2, 2, 8, 4, 4)$ in Sect. 4 and $m = 4$. By an iteration argument, we find that $(\gamma_1, k_1, p_1, s_1, q_1) = (107, 166, 178, 252, 107)$ and u_0 and u_* belongs to $H^{177}(\mathcal{O})$ are sufficient.

Then for any $t \in (0, T^\varepsilon)$, we deduce from (5.38) that

$$\varepsilon^2\|R(t)\|_{H^1(\mathcal{O})} \lesssim \varepsilon^{\frac{3}{2}}\big(\|R(t)\|_1 + \|\eta(t)\| + \sqrt{\varepsilon}\langle t\rangle^{-\gamma}\big),$$

from which, with (5.89) and (5.93), we infer

$$\varepsilon^2\|R(t)\|_{H^1(\mathcal{O})} \lesssim \varepsilon^{\frac{3}{2}}\mathfrak{N}_4^{\frac{1}{2}}(t) + \varepsilon^2 \leq C\varepsilon^{\frac{5}{4}}.$$

This concludes the proof of (2.43).

6. Proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. The scheme of the proof of Theorem 1.7 is very similar to that of Theorem 1.1 with some simplifications due to the facts that the statement of Theorem 1.7 only promises approximate controllability (see [19, Remark 3]), and for one positive time before the imparted time, which can be chosen arbitrarily small (recall Remark 1.8). Therefore there is no need of the well-prepared dissipation of the boundary layers as we did in Sect. 2.1 in the course of proving Theorem 1.1. Again we make use of a rapid and violent control so that the behavior of the system will follow from the one of its inviscid counterpart. Let us therefore recall a few ingredients used in [19] to tackle the inviscid case. We recall the notation for the flow map already used in the statement of Theorem 1.7: with a vector field u depending on t in $[0, T]$ and on the space variable x, we associate, when it makes sense (below we will only need flow maps in some cases where the classical Cauchy-Lipschitz theorem applies), the flow map ϕ^u such that $\partial_t \phi^u(t, s, x) = u(t, \phi^u(t, s, x))$ for any t, s in $[0, T]$ and for any x in Ω, and $\phi^u(s, s, x) = x$ for any s in $[0, T]$ and for any x in Ω. First thanks to a construction due to Krygin [27], given γ_0 and γ_1 two Jordan surfaces included in Ω such that γ_0 and γ_1 are isotopic in Ω and surrounding the same volume, there exists a volume-preserving diffeotopy h in $C^\infty([0, 1] \times \Omega; \Omega)$ such that $\partial_t h$ is compactly supported in $(0, 1) \times \Omega$, $h(0, \gamma_0) = \gamma_0$ and $h(1, \gamma_0) = \gamma_1$. Then the smooth vector field $X(t, x) := \partial_t h(t, h^{-1}(x))$ is compactly supported in $(0, 1) \times \Omega$ and satisfies for all t in $[0, 1]$, $\phi^X(t, 0, \gamma_0) \subset \Omega$, $\phi^X(1, 0, \gamma_0) = \gamma_1$ and $\operatorname{div} X = 0$ in $(0, 1) \times \Omega$. Then, thanks to [19, Proposition 2.2], for any $\nu > 0$ and k in $\mathbb{N}$, there exists θ^0 in $C_0^\infty((0, 1) \times \overline{\Omega}; \mathbb{R})$ such that

$$\begin{cases} \forall\, t \in [0, 1], \ \Delta_x \theta^0 = 0 \ \text{ in } \ \Omega, \\ \dfrac{\partial \theta^0}{\partial n} = 0 \ \text{ on } \ [0, 1] \times (\partial\Omega \setminus \Sigma), \\ \forall\, t \in [0, 1], \ \phi^{\nabla\theta^0}(t, 0, \gamma_0) \subset \Omega, \\ \|\phi^{\nabla\theta^0}(1, 0, \gamma_0) - \gamma_1\|_{C^k(\mathbb{S}^2)} \leq \nu, \end{cases} \tag{6.1}$$

up to a reparameterization. Above $\mathbb{S}^2$ is the two-dimensional torus.

With these ingredients of the inviscid case in hands, let us now start the proof of Theorem 1.7; it is split into two parts, depending on the regularity of the initial data.

Proof of the first part of Theorem 1.7. Case where u_0 is in $C^{k,\alpha}(\Omega; \mathbb{R}^3)$

We first consider the case where u_0 is in $C^{k,\alpha}(\Omega; \mathbb{R}^3)$, with α in $(0, 1)$ and k in $\mathbb{N} \setminus \{0\}$, and satisfies $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot n = 0$ on $\partial\Omega$. One also assumes that $T_0 > 0$, γ_0 and γ_1 two Jordan surfaces included in Ω such that γ_0 and γ_1 are isotopic in Ω and surrounding the same volume, are given.

We first use the scaling transformation (2.3) to transform our original problem (1.2) to (2.4). Then we consider the same expansion as in the proof of Theorem 1.1, that is, (2.42), with u_a^ε being given by (2.27) and $u^0 := \mathfrak{e}(\nabla\theta^0)$, where θ^0 is given by (6.1) and $\mathfrak{e}$ is a linear continuous extension operator from $C^{k,\beta}(\overline{\Omega}; \mathbb{R}^3) \to$

$C_0^{k,\beta}(\mathcal{O};\mathbb{R}^3)$. Of course, u^0 thus constructed verifies Lemma 2.9 except (2.9), which is unnecessary here.

Let us first focus on proving (1.6) for $k=0$, while maintaining the condition (1.5). It follows from (5.86) and (5.93) that

$$\begin{aligned}\varepsilon^2\int_0^1 \|R(t)\|_{L^\infty(\mathcal{O})}\,dt &\le C\varepsilon^{\frac32}\int_0^1\big(\|\eta(t)\|_{m-1}+\|R(t)\|_m+\varepsilon\big)\,dt \quad \text{if } m>2\\ &\le C\varepsilon\big(\|\varepsilon^{\frac12}\eta\|_{L^2((0,1);H^{m-1}_{\mathrm{co}})}+\varepsilon^{\frac12}\|R\|_{L^\infty((0,1);H^{m-1}_{\mathrm{co}})}+\varepsilon\big)\\ &\le C\sqrt{\varepsilon}. \end{aligned}\tag{6.2}$$

We remark that the choice of 1 is quite arbitrary but the fact that we consider here times of order $O(1)$, not of order $O(1/\varepsilon)$ as in the proof of (2.42), makes the use of the well-prepared dissipation of the boundary layers unnecessary here.

With thus obtained u^ε, we define u via (2.6) and we denote by p the corresponding pressure. Then (u,p) is in $L^\infty(0,T;C^{k,\alpha}(\Omega;\mathbb{R}^4))$ and satisfies (1.2) on $[0,\varepsilon]$. We denote by $\phi^u(t,s,x)$ and $\phi^{u^0}(t,s,x)$ the flow maps associated with u and u^0 respectively. Then in view of (2.6) and (2.42), we write

$$\begin{aligned}\partial_t\big(\phi^u(t,s,x)-\phi^{u^0}(t/\varepsilon,s,x)\big) &= \frac{1}{\varepsilon}\big(u^\varepsilon(t/\varepsilon,\phi^u(t,s,x))-u^0(t/\varepsilon,\phi^{u^0}(t/\varepsilon,s,x))\big)\\ &= \frac{1}{\varepsilon}\big(u^0(t/\varepsilon,\phi^u(t,s,x))-u^0(t/\varepsilon,\phi^{u^0}(t/\varepsilon,s,x))\big)\\ &\quad+\frac{1}{\varepsilon}\mathfrak{R}^\varepsilon(t/\varepsilon,\phi^u(t,s,x)) \quad \text{with} \quad \mathfrak{R}^\varepsilon := u_a^\varepsilon-u^0+\varepsilon^2 R,\end{aligned}$$

from which we get, by applying Gronwall's inequality, that

$$\begin{aligned}&\big\|\phi^u(t,s,\cdot)-\phi^{u^0}(t/\varepsilon,s,\cdot)\big\|_{L^\infty(\mathcal{O})}\\ &\quad\le \varepsilon^{-1}\|\mathfrak{R}^\varepsilon(t/\varepsilon)\|_{L^1((s,t);L^\infty(\mathcal{O}))}\exp\Big(\frac{1}{\varepsilon}\int_s^t\|\nabla u^0(t')\|_{L^\infty(\mathcal{O})}\,dt'\Big).\end{aligned}$$

On the other hand, it follows from (2.40) and (6.2) that

$$\begin{aligned}&\|u_\varepsilon^a-u^0\|_{L^\infty((0,\varepsilon)\times\mathcal{O})}\le C\varepsilon^{\frac12},\quad \frac{1}{\varepsilon}\int_0^\varepsilon\|\nabla u^0(t')\|_{L^\infty(\mathcal{O})}\,dt'\le\|\nabla u^0\|_{L^\infty((0,1)\times\mathcal{O})},\\ &\varepsilon\|\mathfrak{R}^\varepsilon(t/\varepsilon)\|_{L^1((0,\varepsilon);L^\infty(\mathcal{O}))}=\varepsilon^2\|\mathfrak{R}^\varepsilon\|_{L^1((0,1);L^\infty(\mathcal{O}))}\le C\sqrt{\varepsilon},\end{aligned}$$

so that for any $t,s\in[0,\varepsilon]$ it holds that

$$\big\|\phi^u(t,s,\cdot)-\phi^{u^0}(t/\varepsilon,s,\cdot)\big\|_{L^\infty(\mathcal{O})}\le C\sqrt{\varepsilon}.\tag{6.3}$$

Then (6.1), together with (6.3), ensures that

$$\begin{aligned}\big\|\phi^u(\varepsilon,0,\gamma_0)-\gamma_1\big\|_{L^\infty(\mathbb{S}^2)} &\le \big\|\phi^u(\varepsilon,0,\cdot)-\phi^{\nabla\theta^0}(1,0,\cdot)\big\|_{L^\infty(\Omega)}\\ &\quad+\big\|\phi^{\nabla\theta^0}(1,0,\gamma_0)-\gamma_1\big\|_{L^\infty(\mathbb{S}^2)}\\ &\le C\big(\sqrt{\varepsilon}+\nu\big).\end{aligned}\tag{6.4}$$

This entails (1.5) and (1.6) for $k = 0$, with the time $T := \varepsilon \in (0, T_0)$, by appropriate choices of ν and ε. Now to prove (1.6) for $k > 0$ it is sufficient to use the counterpart of (6.3) for higher order derivatives, see for instance [26, Equation (23)]. This estimate is performed in a compact set K such that an open neighborhood of $\cup_{t\in[0,\varepsilon]} \phi^u(t, 0, \gamma_0)$ is contained in K and such that K is included in Ω, the existence of such a compact set is granted by the condition (1.5). The higher order estimates of the velocity field on K are deduced, by Sobolev embedding, from the estimate of $\|R(t)\|_m$ in Proposition 5.12, since on K, $\|R(t)\|_m$ is equivalent to the usual Sobolev norm of order m, by the very definition of the the Sobolev conormal spaces in (2.34). The details are left to the reader.

This completes the proof of the first part of Theorem 1.7. □

Proof of the Second Part of Theorem 1.7. *Case where u_0 is in $H^1(\Omega; \mathbb{R}^3)$*

Let us now tackle the case where the initial data u_0 is only in $H^1(\Omega; \mathbb{R}^3)$, with still the compatibility conditions: $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot n = 0$ on $\partial\Omega$. In this case we first use the regularization result of Theorem 2.1, or more precisely of Theorem A.1 in the Appendix A. More precisely, for $\nu > 0$, which will be chosen small enough later on, we consider u to be the unique solution in $u \in C([0, \nu]; H^1(\Omega)) \cap L^2([0, \nu]; H^2(\Omega))$ of (A.1) on $[0, \nu]$ with initial data u_0. In particular, for any s_0 in $(2, 3)$, we deduce from interpolation inequality and (A.2) that

$$\left\|t^{\frac{s_0}{2}-1}u\right\|_{L^2((0,\nu);H^{s_0}(\Omega))} \leq C\left\|t^{\frac{1}{2}}u\right\|^{s_0-2}_{L^2((0,\nu);H^3(\Omega))}\|u\|^{3-s_0}_{L^2((0,\nu);H^2(\Omega))} \leq C(\|u_0\|_{H^1}),$$

from which, along with the Sobolev imbedding theorem, we infer that for any s_0 in $(5/2, 3)$,

$$\|\nabla u\|_{L^1((0,\nu);L^\infty(\Omega))} \leq C\left\|t^{\frac{s_0}{2}-1}u\right\|_{L^2((0,\nu);H^{s_0}(\Omega))}\left\|t^{1-\frac{s_0}{2}}\right\|_{L^2(0,\nu)} \leq C(\|u_0\|_{H^1})\nu^{\frac{3-s_0}{2}}.$$

Consequently, according to the classical Cauchy-Lipschitz theorem, the vector field u generates a unique flow map $\phi^u(t, s, x)$ on $[0, \nu]$. Furthermore, for any t, s in $[0, \nu]$, it holds that

$$\begin{aligned}\left\|\phi^u(t, s, x) - x\right\|_{L^\infty(\Omega)} &\leq \int_0^\nu \|u(t, \cdot)\|_{L^\infty(\Omega)}\,dt \\ &\leq \left\|t^{\frac{1}{2}}u\right\|_{L^\infty((0,\nu);H^2(\Omega))}\int_0^\nu t^{-\frac{1}{2}}\,dt \leq C(\|u_0\|_{H^1})\sqrt{\nu}.\end{aligned} \tag{6.5}$$

In particular, this entails that for any t in $[0, \nu]$, $\phi^u(t, 0, \gamma_0) \subset \Omega$ and that the Jordan surface $\gamma_* := \phi^u(\nu, 0, \gamma_0)$ satisfies

$$\left\|\gamma_* - \gamma_0\right\|_{L^\infty(\mathbb{S}^2)} \leq C(\|u_0\|_{H^1})\sqrt{\nu}. \tag{6.6}$$

Moreover it follows from (A.2) that $u_* := u(\nu, \cdot)$ belongs to $H^\infty(\Omega)$. Thus we can use the first part of Theorem 1.7, in particular the estimate (6.4) on the time interval $[\nu, \nu + \varepsilon]$, so that there exists an extension of u, which we still denote by u, to the time interval $[\nu, \nu + \varepsilon]$ such that u is in $C([0, \nu + \varepsilon]; H^1(\Omega))$ and in

$L^2([0, \nu+\varepsilon]; H^2(\Omega))$ and generates a flow ϕ^u such that for any t in $[\nu, \nu+\varepsilon]$, $\phi^u(t, \varepsilon, \gamma_0) \subset \Omega$, such that

$$\left\|\phi^u(\nu+\varepsilon, \nu, \gamma_0) - \gamma_1\right\|_{L^\infty(\mathbb{S}^2)} \leq C\sqrt{\varepsilon}. \tag{6.7}$$

Furthermore, $\phi^u(\nu+\varepsilon, \nu, \cdot)$ is Lipschitz. Thus combining these three last properties with (6.6), and choosing ε and ν small enough, we arrive at

$$\begin{aligned}\|\phi^u(\nu+\varepsilon, 0, \gamma_0) - \gamma_1\|_{L^\infty(\mathbb{S}^2)} \leq & \|\phi^u(\nu+\varepsilon, \nu, \gamma_*) - \phi^u(\nu+\varepsilon, \nu, \gamma_0)\|_{L^\infty(\mathbb{S}^2)} \\ & + \|\phi^u(\nu+\varepsilon, \nu, \gamma_0) - \gamma_1\|_{L^\infty(\mathbb{S}^2)} \\ \leq & C(\|u_0\|_{H^1})\big(\sqrt{\varepsilon}+\sqrt{\nu}\big),\end{aligned}$$

while maintaining the condition that for any t in $[0, \nu+\varepsilon]$, $\phi^u(t, 0, \gamma_0) \subset \Omega$.

This completes the proof of the second part of Theorem 1.7. □

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Appendix A. On the Regularization of the Uncontrolled Strong Solutions to the Navier–Stokes Equations with Navier Boundary Conditions

In this appendix we prove a regularization result of the uncontrolled strong solutions to the Navier–Stokes equations with Navier boundary conditions on the whole boundary $\partial\Omega$, that is, to the following system:

$$\begin{cases}\partial_t u + u\cdot\nabla u - \Delta u + \nabla p = 0, \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega,\\ u\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathcal{N}(u) = 0 \quad \text{on } \partial\Omega,\\ u = u_0 \quad \text{at } t = 0.\end{cases} \tag{A.1}$$

Theorem A.1. *Let $T > 0$, p in $\mathbb{N}^*$ and $R > 0$. Then there exists a continuous function $C_{T,p,R}$ from $[0, +\infty)$ to $[0, +\infty)$ with $C_{T,p,R}(0) = 0$, such that there exists T_1 in $(0, T)$ and for any u_0 in $H^1(\Omega)$, with $\|u_0\|_{H^1(\Omega)} \leq R$, divergence free and tangent to $\partial\Omega$, the unique strong solution u in $C([0, T_1]; H^1(\Omega)) \cap L^2([0, T_1]; H^2(\Omega))$ to (A.1) satisfies*

$$\sum_{0\leq j\leq \frac{p}{2}} \left\|t^{\frac{p-1}{2}}\partial_t^j u\right\|_{L^\infty_{T_1}(H^{p-2j}(\Omega))} + \sum_{0\leq j\leq \frac{p+1}{2}} \left\|t^{\frac{p-1}{2}}\partial_t^j u\right\|_{L^2_{T_1}(H^{p+1-2j}(\Omega))} \leq C_{p,T_1,R}(\|u_0\|_{H^1(\Omega)}). \tag{A.2}$$

As recalled in Sect. 2.1 The goal of this section is to present the proof of Theorem 2.1. The local-in-time existence and uniqueness of strong solutions with H^1 initial data is classical. The interest of Theorem A.1 is to detail the regularization in time of this strong solution near the time zero. In particular it implies the part of Theorem 2.1 regarding the regularization.

Proof. We will proceed by induction on p. We start with recalling how to prove the case $p = 1$, by proving first a $L^2(\Omega)$ energy estimate and then a $H^1(\Omega)$ energy estimate.

• $\underline{L^2(\Omega)\text{ energy estimate}}$

Indeed, we first get, by taking $L^2(\Omega)$ inner product of the u equation in (A.1) with u, that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + (u\cdot\nabla u|u)_{L^2(\Omega)} - (\Delta u|u)_{L^2(\Omega)} + (\nabla p|u)_{L^2(\Omega)} = 0. \tag{A.3}$$

Here and in all that follows, we always denote $(f|g)_{L^2(\Omega)} := \int_\Omega fg\,dx$.
Due to $\operatorname{div} u = 0$ and $u\cdot\mathbf{n}|_{\partial\Omega} = 0$, we have

$$(u\cdot\nabla u|u)_{L^2(\Omega)} = 0 = (\nabla p|u)_{L^2(\Omega)}\,.$$

Moreover it follows from Stokes formula that

$$-(\Delta u|u)_{L^2(\Omega)} = \int_{\partial\Omega}[(\nabla\times u)\times u]\cdot\mathbf{n}\,dS + \int_\Omega|\nabla\times u|^2\,dx.$$

By inserting the above equalities into (A.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla\times u\|^2_{L^2(\Omega)} = \int_{\partial\Omega}[u\times(\nabla\times u)]\cdot\mathbf{n}\,dS. \tag{A.4}$$

Let us denote by M_{w} the shape operator associated with Ω. Recall that, since Ω is smooth, the shape operator M_{w} is smooth and for any x in $\partial\Omega$, it defines a self-adjoint operator with values in the tangent space T_x. Then we have the following result, see [1,12].

Lemma A.2. *For any smooth divergence free vector field u satisfying* $u\cdot\mathbf{n} = 0$ *on* $\partial\Omega$, *we have*

$$[D(u)\mathbf{n} + M_{\mathrm{w}}u]_{tan} = \frac{1}{2}(\nabla\times u)\times\mathbf{n}. \tag{A.5}$$

However, due to $\mathcal{N}(u)|_{\partial\Omega} = 0$, we deduce from Lemma A.2 that

$$\begin{aligned}[u\times(\nabla\times u)]\cdot\mathbf{n}\big|_{\partial\Omega} =& u\cdot[(\nabla\times u)\times\mathbf{n}]\big|_{\partial\Omega}\\ =&2\,[(M_{\mathrm{w}}-M)u]_{\tan}\cdot u\big|_{\partial\Omega}\\ =&2\,[(M_{\mathrm{w}}-M)u]\cdot u\big|_{\partial\Omega},\end{aligned} \tag{A.6}$$

where we used $u\cdot \mathbf{n}\big|_{\partial\Omega}=0$ in the last step. Then by applying Stokes formula and Young's inequality, we find that for any $\lambda>0$, there exists C_λ so that

$$\begin{aligned}\Big|\int_{\partial\Omega}[(\nabla\times u)\times u]\cdot\mathbf{n}\,dS\Big| =&2\Big|\int_{\Omega}\operatorname{div}\,\big[\big((M_{\rm w}-M)u\cdot u\big)\mathbf{n}\big]\,dx\Big|\\ \leq&\lambda\|\nabla u\|_{L^2(\Omega)}^2+C_\lambda\|u\|_{L^2(\Omega)}^2,\end{aligned}\tag{A.7}$$

On the other hand, due to $\operatorname{div} u=0$ in Ω and $u\cdot\mathbf{n}|_{\partial\Omega}=0$, we deduce from Korn's type inequality (see [10] for instance) that there exists a positive constant C_Ω so that

$$\|\nabla\times u\|_{L^2(\Omega)}^2\geq\frac{1}{C_\Omega}\|u\|_{H^1(\Omega)}^2-\|u\|_{L^2(\Omega)}^2.\tag{A.8}$$

By inserting the estimates, (A.7) and (A.8), into (A.4) and taking $\lambda=\frac{1}{2C_\Omega}$ in the resulting inequality, we achieve

$$\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2+\frac{1}{C_\Omega}\|u\|_{H^1(\Omega)}^2\leq C\|u\|_{L^2(\Omega)}^2.\tag{A.9}$$

Applying Gronwall's inequality gives rise to

$$\|u\|_{L_t^\infty(L^2(\Omega))}^2+\frac{1}{C_\Omega}\|u\|_{L_t^2(H^1(\Omega))}^2\leq\|u_0\|_{L^2(\Omega)}^2e^{Ct}.\tag{A.10}$$

• <u>$H^1(\Omega)$ energy estimate</u>

By taking $L^2(\Omega)$ inner product of the u equation of (A.1) with $\partial_t u$, we get

$$\|\partial_t u\|_{L^2(\Omega)}^2-(\Delta u|\partial_t u)_{L^2(\Omega)}+(\nabla p|\partial_t u)_{L^2(\Omega)}=-(u\cdot\nabla u|\partial_t u)_{L^2(\Omega)}.\tag{A.11}$$

Notice that $\partial_t u\cdot\mathbf{n}|_{\partial\Omega}=0$, by applying Stokes formula and along the same line to the proof of (A.6), we obtain

$$\begin{aligned}-(\Delta u|\partial_t u)_{L^2(\Omega)}=&\int_{\partial\Omega}[(\nabla\times u)\times\partial_t u]\cdot\mathbf{n}\,dS+\int_\Omega(\nabla\times u)\cdot(\nabla\times\partial_t u)\,dx\\ =&2\int_{\partial\Omega}\partial_t u(M-M_{\rm w})u\,dS+\frac12\frac{d}{dt}\int_\Omega|\nabla\times u|^2\,dx,\end{aligned}$$

which together with the facts: M is a symmetric matrix and $M_{\rm w}$ is a self-adjoint operator on T_x, ensures that

$$-(\Delta u|\partial_t u)_{L^2(\Omega)}=\frac{d}{dt}\Big(\int_{\partial\Omega}u(M-M_{\rm w})u\,dS+\frac12\int_\Omega|\nabla\times u|^2\,dx\Big).$$

Again due to $\partial_t u\cdot\mathbf{n}|_{\partial\Omega}=0$, one has

$$(\nabla p|\partial_t u)_{L^2(\Omega)}=0.$$

By inserting the above equalities into (A.11), we achieve

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\partial\Omega} u(M-M_{\rm w})u\,dS+\frac{1}{2}\int_\Omega |\nabla\times u|^2\,dx\Big)+\|\partial_t u\|^2_{L^2(\Omega)}=-\,(u\cdot\nabla u|\partial_t u)_{L^2(\Omega)}\\&\le \|u\|_{L^6(\Omega)}\|\nabla u\|_{L^3(\Omega)}\|\partial_t u\|_{L^2(\Omega)}\\&\le C\|u\|_{H^1(\Omega)}\|\nabla u\|^{\frac12}_{L^2(\Omega)}\|\nabla u\|^{\frac12}_{H^1(\Omega)}\|\partial_t u\|_{L^2(\Omega)}.\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\partial\Omega} u(M-M_{\rm w})u\,dS+\frac{1}{2}\int_\Omega |\nabla\times u|^2\,dx\Big)+\frac34\|\partial_t u\|^2_{L^2(\Omega)}\\&\quad\le C_\lambda\big(1+\|u\|^4_{H^1(\Omega)}\big)\|\nabla u\|^2_{L^2(\Omega)}+\lambda\|\nabla^2 u\|^2_{L^2(\Omega)}.\end{aligned}\tag{A.12}$$

Moreover in view of (A.1), we write

$$\begin{cases}-\Delta u+\nabla p=-\partial_t u-u\cdot\nabla u\\ \operatorname{div} u=0\quad \text{in }\Omega,\\ u\cdot\mathbf{n}=0\quad\text{and}\quad \mathcal{N}(u)=0\quad\text{on }\partial\Omega.\end{cases}\tag{A.13}$$

The following type of Cattabriga-Solonnikov estimate can be proved along the same line to that of Theorem 2.2 in [32]:

Lemma A.3. *Let k be a non-negative integer and Ω be a bounded domain with sufficiently smooth boundary. Let f in $H^k(\Omega)$ and g in $H^{k+1}(\Omega)$ with $\int_\Omega g\,dx=0$. Then the non-homogeneous Stokes problem*

$$\begin{cases}-\Delta u+\nabla p=f\\ \operatorname{div} u=g\quad \textit{in }\Omega,\\ u\cdot\mathbf{n}=0\quad\textit{and}\quad \mathcal{N}(u)=0\quad\textit{on }\partial\Omega\end{cases}$$

has a unique solution (u,p) so that

$$\|\nabla^2 u\|_{H^k(\Omega)}+\|\nabla p\|_{H^k(\Omega)}\le C\big(\|f\|_{H^k(\Omega)}+\|\nabla g\|_{H^k(\Omega)}\big).\tag{A.14}$$

Then it follows from Lemma A.3 and (A.13) that

$$\begin{aligned}\|\nabla^2 u\|_{L^2(\Omega)}&\le C\big(\|\partial_t u\|_{L^2(\Omega)}+\|u\cdot\nabla u\|_{L^2(\Omega)}\big)\\&\le C\big(\|\partial_t u\|_{L^2(\Omega)}+\|u\|_{H^1(\Omega)}\|\nabla u\|^{\frac12}_{L^2(\Omega)}\|\nabla u\|^{\frac12}_{H^1(\Omega)}\big),\end{aligned}$$

from which, we infer

$$\|\nabla u\|_{H^1(\Omega)}\le C\big(\|\partial_t u\|_{L^2(\Omega)}+(1+\|u\|^2_{H^1(\Omega)})\|\nabla u\|_{L^2(\Omega)}\big).\tag{A.15}$$

By substituting (A.15) into (A.12) and then taking $\lambda=\frac{1}{4C}$, we achieve

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\partial\Omega} u(M-M_{\rm w})u\,dS+\frac{1}{2}\int_\Omega |\nabla\times u|^2\,dx\Big)+\frac12\|\partial_t u\|^2_{L^2(\Omega)}\\&\qquad\le C\big(1+\|u\|^4_{H^1(\Omega)}\big)\|\nabla u\|^2_{L^2(\Omega)}.\end{aligned}\tag{A.16}$$

On the other hand, it follows from trace inequality (5.25) that

$$\Big|\int_{\partial\Omega} u(M-M_{\rm w})u\,dS\Big| \leq C\|u\|_{L^2(\partial\Omega)}^2 \leq C\big(\|u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\big)$$
$$\leq \frac{1}{4C_\Omega}\|u\|_{H^1(\Omega)}^2 + C\|u\|_{L^2(\Omega)}^2,$$

so that in view of (A.8), there exists a large enough constant K which satisfies

$$E_1(u) := K\|u\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} u(M-M_{\rm w})u\,dS + \frac{1}{2}\int_\Omega |\nabla\times u|^2\,dx \geq \frac{1}{4C_\Omega}\|u\|_{H^1(\Omega)}^2. \tag{A.17}$$

Then we get, by summing up $K\times$(A.9) and (A.16), that

$$\frac{d}{dt}E_1(u) + \frac{1}{2}\|\partial_t u\|_{L^2(\Omega)}^2 \leq CE_1(u)\big(1+E_1^2(u)\big), \tag{A.18}$$

from which, we deduce by a comparison argument that for any $T > 0$ and $R > 0$, there exists a continuous function $C_{T,p,R}$ from $[0,+\infty)$ to $[0,+\infty)$ with $C_{T,1,R}(0) = 0$, such that there exists T_1 in $(0,T)$ and such that for any u_0 in $H^1(\Omega)$, with $\|u_0\|_{H^1(\Omega)} \leq R$, divergence free and tangent to $\partial\Omega$, the unique strong solution u in $C([0,T_1]; H^1(\Omega)) \cap L^2([0,T_1]; H^2(\Omega))$ to (A.1) satisfies (A.2) holds true for $p = 1$.

• Higher energy estimates

Inductively, we assume that (A.2) holds for $p \leq \ell - 1$, we are going to show that (A.2) holds for $p = \ell$. Without loss of generality, we may assume that ℓ is an even integer. The odd integer case can be proved along the same line. Indeed we first get, by applying $\partial_t^{\ell/2}$ to (A.1), that

$$\begin{cases} \partial_t^{1+\frac{\ell}{2}}u + \partial_t^{\frac{\ell}{2}}(u\cdot\nabla u) - \Delta\partial_t^{\frac{\ell}{2}}u + \nabla\partial_t^{\frac{\ell}{2}}p = 0, \\ \operatorname{div}\partial_t^{\frac{\ell}{2}}u = 0 \quad \text{in } (0,T_1)\times\Omega, \\ \partial_t^{\frac{\ell}{2}}u\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathcal{N}(\partial_t^{\frac{\ell}{2}}u) = 0 \quad \text{on } (0,T_1)\times\partial\Omega, \end{cases} \tag{A.19}$$

from which, we get, by a similar derivation of (A.4) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big(t^{\ell-1}\|\partial_t^{\frac{\ell}{2}}u(t)\|_{L^2(\Omega)}^2\big) + t^{\ell-1}\|\nabla\times\partial_t^{\frac{\ell}{2}}u\|_{L^2(\Omega)}^2 &= \frac{\ell-1}{2}t^{\ell-2}\|\partial_t^{\frac{\ell}{2}}u\|_{L^2(\Omega)}^2 \\ + t^{\ell-1}\int_{\partial\Omega}\big[\partial_t^{\frac{\ell}{2}}u\times(\nabla\times\partial_t^{\frac{\ell}{2}}u)\big]\cdot\mathbf{n}\,dS &- t^{\ell-1}\big(\partial_t^{\frac{\ell}{2}}(u\cdot\nabla u)|\partial_t^{\frac{\ell}{2}}u\big)_{L^2(\Omega)}. \end{aligned} \tag{A.20}$$

Similarly to (A.7), we have

$$t^{\ell-1}\Big|\int_{\partial\Omega}\big[\partial_t^{\frac{\ell}{2}}u\times(\nabla\times\partial_t^{\frac{\ell}{2}}u)\big]\cdot\mathbf{n}\,dS\Big| \leq \lambda\big\|t^{\frac{\ell-1}{2}}\nabla\partial_t^{\frac{\ell}{2}}u\big\|_{L^2(\Omega)}^2 + C_\lambda\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{L^2(\Omega)}^2.$$

On the other hand, due to $u \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\operatorname{div} u = 0$, we get, by using integration by parts, that

$$\begin{aligned}\big(\partial_t^{\frac{\ell}{2}}(u\cdot\nabla u)|\partial_t^{\frac{\ell}{2}}u\big)_{L^2(\Omega)} &= \big(\partial_t^{\frac{\ell}{2}}(u\cdot\nabla u) - u\cdot\nabla\partial_t^{\frac{\ell}{2}}u|\partial_t^{\frac{\ell}{2}}u\big)_{L^2(\Omega)}\\ &= -\sum_{\substack{\ell_1+\ell_2=\frac{\ell}{2}\\ \ell_1\geq 1}} C_{\frac{\ell}{2}}^{\ell_1}\big(\partial_t^{\ell_1}u\otimes\partial_t^{\ell_2}u|\nabla\partial_t^{\frac{\ell}{2}}u\big)_{L^2(\Omega)},\end{aligned}$$

from which we infer

$$\begin{aligned}&t^{\ell-1}\big|\big(\partial_t^{\frac{\ell}{2}}(u\cdot\nabla u)|\partial_t^{\frac{\ell}{2}}u\big)_{L^2(\Omega)}\big|\\ &\lesssim \sum_{\substack{\ell_1+\ell_2=\frac{\ell}{2}\\ \ell_1\geq 1}} t^{\ell-1}\|\partial_t^{\ell_1}u\|_{L^3(\Omega)}\|\partial_t^{\ell_2}u\|_{L^6(\Omega)}\|\nabla\partial_t^{\frac{\ell}{2}}u\|_{L^2(\Omega)}\\ &\lesssim \sum_{\substack{\ell_1+\ell_2=\frac{\ell}{2}\\ \ell_1\geq 1}} t^{\ell-1}\|\partial_t^{\ell_1}u\|_{L^2(\Omega)}^{\frac12}\|\partial_t^{\ell_1}u\|_{H^1(\Omega)}^{\frac12}\|\partial_t^{\ell_2}u\|_{H^1(\Omega)}\|\nabla\partial_t^{\frac{\ell}{2}}u\|_{L^2(\Omega)}\\ &\leq \lambda\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{H^1(\Omega)}^2 + C_\lambda\|u\|_{H^1(\Omega)}^4\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{L^2(\Omega)}^2\\ &\quad + C_\lambda\sum_{\substack{\ell_1+\ell_2=\frac{\ell}{2}\\ 1\leq\ell_1\leq\frac{\ell}{2}-1}}\big\|t^{\ell_1-\frac12}\partial_t^{\ell_1}u\big\|_{H^1(\Omega)}^2\big\||t^{\ell_2}\partial_t^{\ell_2}u\big\|_{H^1(\Omega)}^2.\end{aligned}$$

By substituting the above estimates into (A.20) and using Korn's type inequality (A.8), we find

$$\begin{aligned}&\frac12\frac{d}{dt}\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u(t)\big\|_{L^2(\Omega)}^2 + \frac{1}{C_\Omega}\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{H^1(\Omega)}^2\\ &\quad\leq \frac{\ell-1}{2}\big\|t^{\frac{\ell}{2}-1}\partial_t^{\frac{\ell}{2}}u\big\|_{L^2(\Omega)}^2 + C_\lambda\big(1+\|u\|_{H^1(\Omega)}^4\big)\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{L^2(\Omega)}^2\\ &\qquad + 2\lambda\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{H^1(\Omega)}^2 + C_\lambda\sum_{\substack{\ell_1+\ell_2=\frac{\ell}{2}\\ 1\leq\ell_1\leq\frac{\ell}{2}-1}}\big\|t^{\ell_1-\frac12}\partial_t^{\ell_1}u\big\|_{H^1(\Omega)}^2\big\||t^{\ell_2}\partial_t^{\ell_2}u\big\|_{H^1(\Omega)}^2.\end{aligned}$$

By taking $\lambda = \frac{1}{4C_\Omega}$ in the above inequality and then applying Gronwall's inequality to the resulting inequality, we achieve

$$\begin{aligned}&\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{L_t^\infty(L^2(\Omega))}^2 + \frac{1}{C_\Omega}\big\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\big\|_{L_t^2(H^1(\Omega))}^2 \leq C\exp\Big(C\big(1+t\|u\|_{L_t^\infty(H^1(\Omega))}^4\big)\Big)\\ &\quad\times\Big(\big\|t^{\frac{\ell}{2}-1}\partial_t^{\frac{\ell}{2}}u\big\|_{L_t^2(L^2(\Omega))}^2 + \sum_{\substack{\ell_1+\ell_2=\frac{\ell}{2}\\ 1\leq\ell_1\leq\frac{\ell}{2}-1}}\big\|t^{\ell_1-\frac12}\partial_t^{\ell_1}u\big\|_{L_t^2(H^1(\Omega))}^2\big\||t^{\ell_2}\partial_t^{\ell_2}u\big\|_{L_t^\infty(H^1(\Omega))}^2\Big),\end{aligned}$$

from which, with the inductive assumption, we deduce that

$$\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\|^2_{L^\infty_{T_1}(L^2(\Omega))}+\frac{1}{C_\Omega}\|t^{\frac{\ell-1}{2}}\partial_t^{\frac{\ell}{2}}u\|^2_{L^2_{T_1}(H^1(\Omega))}\leq C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)})\tag{A.21}$$

On the other hand, for any non-negative integer $j\leq\frac{\ell}{2}-1$, we infer from the inductive assumption that

$$\begin{aligned}\|t^{\frac{\ell-1}{2}}\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2j}(\Omega))}=&\|t^{\frac{\ell-1}{2}}\nabla^2\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}+\|t^{\frac{\ell-1}{2}}\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-1-2j}(\Omega))}\\ \leq&\|t^{\frac{\ell-1}{2}}\nabla^2\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}+C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}).\end{aligned}$$

Moreover in view of (A.1), we write

$$-\Delta\partial_t^j u+\nabla\partial_t^j p=-\partial_t^{j+1}u-\partial_t^j(u\cdot\nabla u),$$

from which, with Lemma A.3, we infer

$$\begin{aligned}\|t^{\frac{\ell-1}{2}}\nabla^2\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}\lesssim&\|t^{\frac{\ell-1}{2}}\partial_t^{j+1}u\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}\\ &+\|t^{\frac{\ell-1}{2}}\partial_t^j(u\cdot\nabla u)\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}.\end{aligned}$$

As a result, we get that

$$\begin{aligned}\|t^{\frac{\ell-1}{2}}\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2j}(\Omega))}\leq&C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)})+\|t^{\frac{\ell-1}{2}}\partial_t^{j+1}u\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}\\ &+\|t^{\frac{\ell-1}{2}}\partial_t^j(u\cdot\nabla u)\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))},\quad\forall\, j\leq\frac{\ell}{2}-1.\end{aligned}\tag{A.22}$$

However, it follows from Moser type inequality and the inductive assumption that

$$\begin{aligned}\|t^{\frac{\ell-1}{2}}\partial_t^j\nabla(u\otimes u)\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}\lesssim&\sum_{j_1+j_2=j}\|t^{j_1+\frac{1}{2}}\partial_t^{j_1}u\|_{L^\infty_{T_1}(H^2(\Omega))}\\ &\times\|t^{\frac{\ell-2}{2}-j+j_2}\partial_t^{j_2}u\|_{L^\infty_{T_1}(H^{\ell-2j-1}(\Omega))}\\ \leq&C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}).\end{aligned}$$

Substituting the above estimates into (A.22) gives rise to

$$\|t^{\frac{\ell-1}{2}}\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2j}(\Omega))}\leq C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)})+\|t^{\frac{\ell-1}{2}}\partial_t^{j+1}u\|_{L^\infty_{T_1}(H^{\ell-2-2j}(\Omega))}.$$

We deduce from this inequality and from (A.21), by an iterative argument, that

$$\sum_{0\leq j\leq\frac{\ell}{2}}\|t^{\frac{\ell-1}{2}}\partial_t^j u\|_{L^\infty_{T_1}(H^{\ell-2j}(\Omega))}\leq C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}).\tag{A.23}$$

Exactly along the same line to the proof of (A.23), for any non-negative integer $j\leq\frac{\ell}{2}-1$, we infer from the inductive assumption that

$$\|t^{\frac{\ell-1}{2}}\partial_t^j u\|_{L^2_{T_1}(H^{\ell+1-2j}(\Omega))}\leq\|t^{\frac{\ell-1}{2}}\nabla^2\partial_t^j u\|_{L^2_{T_1}(H^{\ell-1-2j}(\Omega))}+C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}).$$

On the other hand it follows from Lemma A.3 that

$$\begin{aligned}\left\|t^{\frac{\ell-1}{2}}\nabla^2\partial_t^j u\right\|_{L^2_{T_1}(H^{\ell-1-2j}(\Omega))} \lesssim & \left\|t^{\frac{\ell-1}{2}}\partial_t^{j+1}u\right\|_{L^2_{T_1}(H^{\ell-1-2j}(\Omega))} \\ & + \left\|t^{\frac{\ell-1}{2}}\partial_t^j(u\otimes u)\right\|_{L^\infty_{T_1}(H^{\ell-2j}(\Omega))}.\end{aligned}$$

For, any $j \leq \frac{\ell}{2}-1$, it follows from Moser type inequality and the inductive assumption that

$$\begin{aligned}\left\|t^{\frac{\ell-1}{2}}\partial_t^j(u\otimes u)\right\|_{L^2_{T_1}(H^{\ell-2j}(\Omega))} \lesssim & \sum_{j_1+j_2=j}\left\|t^{j_1+\frac{1}{2}}\partial_t^{j_1}u\right\|_{L^\infty_{T_1}(H^2(\Omega))} \\ & \times \left\|t^{\frac{\ell-2}{2}-j+j_2}\partial_t^{j_2}u\right\|_{L^2_{T_1}(H^{\ell-2j}(\Omega))} \\ \leq & C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}).\end{aligned}$$

As a result, for any $j \leq \frac{\ell-1}{2}$, we arrive at

$$\left\|t^{\frac{\ell-1}{2}}\partial_t^j u\right\|_{L^2_{T_1}(H^{\ell+1-2j}(\Omega))} \leq C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}) + \left\|t^{\frac{\ell-1}{2}}\partial_t^{j+1}u\right\|_{L^2_{T_1}(H^{\ell-1-2j}(\Omega))},$$

from which, with (A.21), we deduce by an iterative argument that

$$\sum_{0\leq j\leq\frac{\ell}{2}}\left\|t^{\frac{\ell-1}{2}}\partial_t^j u\right\|_{L^2_{T_1}(H^{\ell+1-2j}(\Omega))} \leq C_{\ell,T_1}(\|u_0\|_{H^1(\Omega)}). \tag{A.24}$$

By combining (A.23) and (A.24), we obtain that (A.2) holds for $p=\ell$. This finishes the proof of (A.2) and therefore the proof of Theorem 2.1. □

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