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Global Solutions of the Compressible Euler Equations with Large Initial Data of Spherical Symmetry and Positive Far-Field Density

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Abstract

We are concerned with the global existence theory for spherically symmetric solutions of the multidimensional compressible Euler equations with large initial data of positive far-field density so that the total initial-energy is unbounded. The central feature of the solutions is the strengthening of waves as they move radially inward toward the origin. For the large initial data of positive far-field density, various examples have shown that the spherically symmetric solutions of the Euler equations blow up near the origin at a certain time. A fundamental unsolved problem is whether the density of the global solution would form concentration to become a measure near the origin for the case when the total initial-energy is unbounded and the wave propagation is not at finite speed starting initially. In this paper, we establish a global existence theory for spherically symmetric solutions of the compressible Euler equations with large initial data of positive far-field density and relative finite-energy. This is achieved by developing a new approach via adapting a class of degenerate density-dependent viscosity terms, so that a rigorous proof of the vanishing viscosity limit of global weak solutions of the Navier–Stokes equations with the density-dependent viscosity terms to the corresponding global solution of the Euler equations with large initial data of spherical symmetry and positive far-field density can be obtained. One of our main observations is that the adapted class of degenerate density-dependent viscosity terms not only includes the viscosity terms for the Navier–Stokes equations for shallow water (Saint Venant) flows but also, more importantly, is suitable to achieve the key objective of this paper. These results indicate that concentration is not formed in the vanishing viscosity limit for the Navier–Stokes approximations constructed in this paper even when the total initial-energy is unbounded, though the density may blow up near the origin at certain time and the wave propagation is not at finite speed.

1. Introduction

We are concerned with the global existence theory for spherically symmetric solutions of the multidimensional (M-D) compressible Euler equations with large initial data of positive far-field density, *that is*, a situation where, given constant density $\bar{\rho} > 0$ at infinity, the total initial-energy is unbounded. The study of spherically symmetric solutions dates back to the 1950s and is motivated by many important physical problems such as flow in a jet engine inlet manifold and stellar dynamics including gaseous stars and supernovae formation (*cf.* [19,28,52,55,59]). The central feature of the solutions is the strengthening of waves as they move radially inward toward the origin. An existence theory was established in Chen and Perepelitsa [17] and Chen and Schrecker [18] via an approach of vanishing artificial viscosity for the case when the initial data are of finite-energy, which requires that $\bar{\rho} = 0$. For the far-field density $\bar{\rho} > 0$, various physical examples have shown that the spherically symmetric solutions of the compressible Euler equations blow up more often near the origin at certain time (see [19,28,38,45,59] and the references cited therein). The fundamental unsolved problem is whether the density would form concentration to become a measure near the origin for the case when the total initial-energy is unbounded and the wave propagation is not at finite speed starting initially. In this paper, we establish a global existence theory for spherically symmetric solutions in L^p_{loc} of the compressible Euler equations with large initial data of positive far-field density $\bar{\rho} > 0$ and relative finite-energy in $\mathbb{R}^N$ for $N \geqq 2$. This is achieved by developing a new approach via adapting a class of degenerate density-dependent viscosity terms, so that a rigorous proof of the vanishing viscosity limit of global weak solutions of the compressible Navier–Stokes equations with the density-dependent viscosity terms to the corresponding global solution of the Euler equations with large initial data of spherical symmetry and positive far-field density can be obtained. One of our main observations is that the adapted class of degenerate density-dependent viscosity terms not only includes the viscosity terms for the Navier–Stokes equations for shallow water (Saint Venant) flows, among others (*cf.* Bresch and Dejardins [2,3], Bresch et al. [5], Lions [39], and Mallet and Vasseur [44]), but also, more importantly, is suitable to achieve the key objective of this paper. These results indicate that concentration is not formed in the vanishing viscosity limit for the Navier–Stokes approximations constructed in this paper even when the total initial-energy is unbounded, though the density may blow up near the origin at certain time and the wave propagation is not at finite speed.

More precisely, the M-D Euler equations for compressible isentropic fluids take the form

$$\begin{cases} \partial_t \rho + \mathrm{div}\mathcal{M} = 0, \\ \partial_t \mathcal{M} + \mathrm{div}\big(\frac{\mathcal{M}\otimes\mathcal{M}}{\rho}\big) + \nabla p = 0 \end{cases} \tag{1.1}$$

for $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^N$ with $N \geqq 2$, where ρ is the density, p is the pressure, and $\mathcal{M} \in \mathbb{R}^N$ represents the momentum; see also Chen and Feldman [14] and Dafermos [20]. When $\rho > 0$, $U = \frac{\mathcal{M}}{\rho} \in \mathbb{R}^N$ is the velocity. The constitutive pressure-density relation for polytropic gases is

$$p = p(\rho) = \kappa \rho^{\gamma},$$

where $\gamma > 1$ is the adiabatic exponent; by scaling, constant κ in the pressure-density relation may be chosen as $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ without loss of generality. We are concerned with the Cauchy problem for (1.1) with the Cauchy data

$$(\rho, \mathcal{M})|_{t=0} = (\rho_0, \mathcal{M}_0)(\mathbf{x}) \longrightarrow (\bar{\rho}, \mathbf{0}) \qquad \text{as } |\mathbf{x}| \to \infty, \tag{1.2}$$

where $(\bar{\rho}, \mathbf{0})$ is a constant far-field state, for which the initial far-field velocity has been assumed to be zero in (1.2) without loss of generality, owing to the Galilean invariance of system (1.1). Since a global solution of the Euler equations (1.1) normally contains the vacuum states $\{(t, \mathbf{x}) : \rho(t, \mathbf{x}) = 0\}$ where the fluid velocity $U(t, \mathbf{x})$ is not well-defined (even though the far-field density is positive), we will use the physical variables such as the momentum $\mathcal{M}(t, \mathbf{x})$, or $\frac{\mathcal{M}(t,\mathbf{x})}{\sqrt{\rho(t,\mathbf{x})}}$, which will be shown to be always well-defined, instead of $U(t, \mathbf{x})$, when the vacuum states are involved throughout this paper.

In order to construct global spherically symmetric solutions in L^p_{loc} of the Euler equations (1.1) with large initial data of positive far-field density, $\bar{\rho} > 0$, the approach of vanishing artificial viscosity developed in [17,18] is no longer applied directly, and the problem has been remained open. To solve this problem, in this paper, we develop a different approach by adapting a class of degenerate density-dependent viscosity terms so that the required uniform estimates in terms of the viscosity coefficients can be achieved for the vanishing viscosity limit. More precisely, we consider the M-D Navier–Stokes equations for compressible barotropic fluids with the adapted class of degenerate density-dependent viscosity terms:

$$\begin{cases} \partial_t \rho + \mathrm{div} \mathcal{M} = 0, \\ \partial_t \mathcal{M} + \mathrm{div}\big(\frac{\mathcal{M}\otimes\mathcal{M}}{\rho}\big) + \nabla p = \varepsilon \mathrm{div}\big(\mu(\rho) D(\frac{\mathcal{M}}{\rho})\big) + \varepsilon \nabla\big(\lambda(\rho)\mathrm{div}(\frac{\mathcal{M}}{\rho})\big). \end{cases} \tag{1.3}$$

Here $D(\frac{\mathcal{M}}{\rho}) = \frac{1}{2}\big(\nabla(\frac{\mathcal{M}}{\rho}) + (\nabla(\frac{\mathcal{M}}{\rho}))^\top\big)$ is the stress tensor, and the shear and bulk viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ depend on the density and may vanish on the vacuum. Indeed, in the derivation of the Navier–Stokes equations from the Boltzmann equation by the Chapman–Enskog expansions, the viscosity terms depend on the temperature, which are translated into the dependence on the density for barotropic flows (*cf.* [42]). Moreover, for the shallow water (Saint Venant) models, $N = 2, \gamma = 2$, and $(\mu(\rho), \lambda(\rho)) = (\rho, 0)$ (*cf.* Lions [39, §8.4]); also see [2,5] for such models in geophysical flows. This indicates that it is of independent interest and importance to analyze the Navier–Stokes equations (1.3) with the density-dependent viscosity terms. In particular, we are also interested in the inviscid limit of the Navier–Stokes equations (1.3). Formally, as $\varepsilon \to 0+$, the Navier–Stokes equations (1.3) converge to the Euler equations (1.1). A fundamental problem in mathematical fluid dynamics is whether a rigorous proof of the vanishing viscosity limit of the solutions of the Navier–Stokes equations (1.3) to the Euler equations (1.1) could be provided.

There is an extensive literature on the analysis of the vanishing artificial/numerical viscosity limit to the isentropic Euler equations. For the 1-D case with general L^∞ initial data, it has been analyzed by DiPerna [23], Ding et al. [22], Ding [21], Chen

[10,11], Lions et al. [40,41], and Huang and Wang [32] via the methods of entropy analysis and compensated compactness. Also see DiPerna [24], Morawetz [46], Perthame and Tzavaras [48], and Serre [54] for general 2×2 strictly hyperbolic systems of conservation laws. The vanishing artificial viscosity limit to general strictly hyperbolic systems of conservation laws with general small BV initial data was first established by Bianchini and Bressan [1] via direct BV estimates with small oscillation; see also [8,9] and the references cited therein for the rate of convergence.

For the study of spherically symmetric weak solutions, the local existence of such solutions outside a solid ball at the origin was discussed in Makino and Takeno [43] for the case $1 < \gamma \leqq \frac{5}{3}$; also see Yang [61,62]. A first global existence of spherically symmetric solutions in L^∞ including the origin was established in Chen [12] for a class of L^∞ Cauchy data of arbitrarily large amplitude, which model outgoing blast waves and large-time asymptotic solutions. A compactness framework was established in LeFloch and Westdickenberg [37] to construct finite-energy solutions to the isentropic Euler equations with spherical symmetry and finite-energy initial data for the case $1 < \gamma \leqq \frac{5}{3}$. As indicated earlier, the convergence of the vanishing artificial viscosity approximate solutions to the corresponding finite-energy entropy solution of the M-D Euler equations with large initial data of spherical symmetry was established in [17,18] for any $\gamma > 1$ for the case $\bar{\rho} = 0$.

For the compressible Navier–Stokes equations with constant viscosity coefficients (*that is*, μ and λ are constants), the global existence of solutions has been studied extensively; see [30,35] and the references cited therein for the 1-D case. For $\mathbf{x} \in \mathbb{R}^N$, $N \geq 2$, Lions [39] first obtained the global existence of renormalized solutions, provided that γ is suitably large, which was further extended by Feireisl et al. [25] to $\gamma > \frac{N}{2}$ and by Plotnikov and Weigant [49] to $\gamma = \frac{N}{2}$, and by Jiang and Zhang [34] to $\gamma > 1$ under the spherical symmetry. When μ and λ depend on the density, the Navier–Stokes equations (1.3) become degenerate when $\rho \to 0$. Such cases were analyzed in Bresch et al. [5] based on the new mathematical entropy—the BD entropy, first discovered by Bresch and Desjardins [2] for the particular case $(\mu, \lambda) = (\rho, 0)$, and later generalized by Bresch and Desjardins [3] to include the case of any viscosity coefficients (μ, λ) satisfying the BD relation: $\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho)$; also see Bresch and Desjardins [4]. When the initial data are of spherical symmetry, Guo et al. [29] obtained the global existence of spherically symmetric weak solutions of the system for $\gamma \in (1, 3)$ in a finite ball with Dirichlet boundary conditions. Also see [7,58].

The idea of regarding inviscid gases as viscous gases with vanishing physical viscosity can date back the seminal paper by Stokes [56] and the important contributions of Rankine [50], Hugoniot [33], and Rayleigh [51] (*cf.* Dafermos [20]). However, the first rigorous convergence analysis of the inviscid limit from the barotropic Navier–Stokes to Euler equations was made by Gilbarg [26] much later, in which the existence and vanishing viscous limit of the Navier–Stokes shock layers was established. For the convergence analysis confined in the framework of piecewise smooth solutions, see [27,31,60] and the references cited therein.

The key objective of this paper is to establish the global existence of spherically symmetric solutions of (1.1):

$$\rho(t,\mathbf{x})=\rho(t,r),\quad \mathcal{M}(t,\mathbf{x})=m(t,r)\frac{\mathbf{x}}{r}\qquad \text{for } r=|\mathbf{x}|, \tag{1.4}$$

subject to the initial condition that

$$(\rho,\mathcal{M})(0,\mathbf{x})=(\rho_0,\mathcal{M}_0)(\mathbf{x})=(\rho_0(r),m_0(r)\frac{\mathbf{x}}{r})\longrightarrow(\bar\rho,\mathbf{0})\qquad \text{as } r\to\infty \tag{1.5}$$

with $\bar\rho>0$ and relative finite-energy. To achieve this, we establish the vanishing viscosity limit of the corresponding spherically symmetric solutions of the Navier–Stokes equations (1.3) with the adapted class of degenerate density-dependent viscosity terms and approximate initial data of similar form to (1.5). For spherically symmetric solutions of form (1.4), systems (1.1) and (1.3) become

$$\begin{cases}\rho_t+m_r+\frac{N-1}{r}m=0,\\ m_t+\big(\frac{m^2}{\rho}+p\big)_r+\frac{N-1}{r}\frac{m^2}{\rho}=0,\end{cases} \tag{1.6}$$

and

$$\begin{cases}\rho_t+m_r+\frac{N-1}{r}m=0,\\ m_t+\big(\frac{m^2}{\rho}+p\big)_r+\frac{N-1}{r}\frac{m^2}{\rho}=\varepsilon\big((\mu+\lambda)((\frac{m}{\rho})_r+\frac{N-1}{r}\frac{m}{\rho})\big)_r-\varepsilon\frac{N-1}{r}\frac{m}{\rho}\mu_r.\end{cases} \tag{1.7}$$

respectively.

In Chen and Perepelitsa [15], the vanishing viscosity limit of smooth solutions for the 1-D Navier–Stokes equations to the corresponding relative finite-energy solution of the Euler equations has been established for $\bar\rho>0$; also see [16] for the 1-D shallow water case. In [17,18], the convergence of artificial viscosity approximate smooth solutions to the corresponding finite-energy entropy solution of the Euler equations (1.6) with spherical symmetry and large initial data has been established for $\bar\rho=0$ (also see [53]). As indicated earlier, in this paper, we develop a different approach to investigate the vanishing physical viscosity limit of the weak solutions of the M-D Navier–Stokes equations (1.3) with spherical symmetry to the corresponding relative finite-energy solution of the Euler equations (1.1) with large initial data of positive far-field density $\bar\rho>0$. Owing to the non-zero initial density at infinity so that the total initial-energy is unbounded, which may cause the possibility for additional nature of singularities at origin $r=0$ and far-field $r=\infty$, several key techniques for the previous uniform estimates as in [15,17,18] no longer apply. In particular, for the weak solutions of the Navier–Stokes equations, it is essential to ensure enough decay of solutions *a priori* as $r\to\infty$ so that integration by parts on unbounded regions can be performed for the key estimates in the proof.

We now describe some of our approach and techniques involved to solve the problem posed in this paper. Owing to the singularity at $r=0$, it has not been clear yet whether there always exists a global smooth solution of the Cauchy problem of the Navier–Stokes equations with smooth large initial data of spherical symmetry. To achieve our key objective, the main point of this paper is first to obtain global weak solutions of the compressible Navier–Stokes equations with some uniform estimates and the H^{-1}_{loc}-compactness, so that the compactness framework in [15] can be applied. For this purpose, we first construct smooth approximate solutions $(\rho^{\varepsilon,\delta,b}, m^{\varepsilon,\delta,b})$, depending on the three parameters (ε, δ, b), through the Navier–Stokes equations (1.7); see (3.1)–(3.4). Noting that the spherically symmetric Navier–Stokes equations (1.7) become singular at the origin, we first remove the origin in the approximate problem. For the smooth approximate solutions as designed, it is direct to obtain the basic energy estimate, Lemma 3.1. Under relation (2.20), we also obtain the BD entropy estimate, Lemma 3.2. Similar to that in [15], we can obtain the uniform higher integrability of the density; see Lemma 3.3.

To employ the compactness framework in [15], we still need the uniform higher integrability of the velocity, as described in Proposition 4.1, for all $\gamma > 1$. To prove this, we apply the relative entropy pair $(\tilde{\eta}, \tilde{q})$ of the spherically symmetric Euler equations (1.6) to obtain (4.54) in §4. The most difficult terms are the second and third terms on the right-hand side of (4.54), which are essential for the M-D case (these two terms do not appear for the 1-D case). By a careful analysis on the relative entropy pair, we see that

$$m\partial_\rho\tilde{\eta}(\rho,m) + \frac{m^2}{\rho}\partial_m\tilde{\eta}(\rho,m) - \tilde{q}(\rho,m) \leqq C_\gamma(\bar{\rho})\Big(\frac{m^2}{\rho} + e(\rho,\bar{\rho})\Big) \tag{1.8}$$

for some constant $C_\gamma(\bar{\rho}) > 0$, which implies that the third term on the right-hand side of (4.54) can be bounded by using the basic energy at least locally; see Lemma 4.8 for the details. In fact, estimate (1.8) is quite subtle. Since the left-hand side of (1.8) contains the terms on $\frac{|m|^3}{\rho^2}$ and $\rho^{\gamma+\theta}$, we have to deal with such terms; otherwise, the higher integrability of the velocity may not be obtained. This is achieved by our observation of underlying cancellation by dividing it into several cases; see (4.40)–(4.52) for the details of its proof.

From the expression of $\tilde{q}$ in (4.60), in order to control the second term $r^{N-1}\tilde{q}$ on the right-hand side of (4.54), we need to obtain some decay rate estimate of $(\rho^{\varepsilon,\delta,b} - \bar{\rho}, m^{\varepsilon,\delta,b})(t,r)$ as $r \to \infty$. To achieve this, we first obtain the upper and lower bounds of density $\rho^{\varepsilon,\delta,b}$ so that they are independent of b. With these bounds of the density and property (4.1) satisfied by the approximate initial data, we can prove a better decay estimate for $(\rho^{\varepsilon,\delta,b} - \bar{\rho}, m^{\varepsilon,\delta,b})(t,r)$, uniformly in b; see Lemmas 4.6–4.7 in more detail. Then the decay estimate allows us to control $r^{N-1}\tilde{q}$. Since the boundary values of $(\rho^{\varepsilon,\delta,b}, u_r^{\varepsilon,\delta,b})(t,b)$ are determined by the equations and may depend on ε, we integrate (4.54) over $[0,T]\times[b-1,b]\times[d,D]$ to avoid the trace estimates, so that Proposition 4.1 is obtained. Then we take the limit, $b \to \infty$, to obtain the global existence of a strong solution $(\rho^{\varepsilon,\delta}, \mathcal{M}^{\varepsilon,\delta}) = (\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta}\,\frac{\mathbf{x}}{r})$ for (1.3) on $[0,\infty) \times (\mathbb{R}^N \setminus B_\delta(\mathbf{0}))$ for each fixed $\delta > 0$. Noting that the second term on the right-hand side of (4.3) vanishes when $b \to \infty$, we obtain the desired estimates in Proposition 5.2.

By similar arguments as in [29,44], we can then take the limit, $\delta \to 0+$, to obtain the global weak solution $(\rho^\varepsilon, \mathcal{M}^\varepsilon) = (\rho^\varepsilon, m^\varepsilon \frac{\mathbf{x}}{r})$ of the Cauchy problem for (1.3). To prove that

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon) \qquad \text{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+),$$

special care is required, since $(\rho^\varepsilon, m^\varepsilon)$ is only a weak solution and $\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon)$ is only a local bounded Radon measure for each fixed $\varepsilon > 0$. Moreover, since the viscosity coefficients depend on the density, we cannot say that $(\frac{m^\varepsilon}{\rho^\varepsilon})_r$ is a function due to the possible appearance of the vacuum in general so that it is not suitable to use the weak form of $(\rho^\varepsilon, m^\varepsilon)$ to prove the H^{-1}_{loc}-compactness. In fact, the H^{-1}_{loc}-compactness is achieved through smooth approximate solutions and their limits.

Based on the uniform estimates and the H^{-1}_{loc}-compactness, we then employ the compactness framework in [15] to take the vanishing viscosity $\varepsilon \to 0$ for all $\gamma > 1$. On the other hand, we have to be careful to pass the limit, $\varepsilon \to 0$, in the momentum equations (see (5.42)), since it is quite delicate to vanish the right-hand side of (5.42) by using the uniform estimates in Theorem 5.12. To overcome this difficulty, we employ underlying cancellations and introduce a new function V^ε, which is uniformly bounded in $L^2(0, T; L^2)$ so that the right-hand side of (5.42) is expressed by (5.43). Then we can vanish the viscosity terms by using the new expression.

The paper is organized as follows: in §2, we first introduce the notion of relative finite-energy solutions of the Cauchy problem (1.1)–(1.2) for the compressible Euler equations and then state Main Theorem I: Theorem 2.2 for the global existence of such solutions. To establish Theorem 2.2, we construct global weak solutions of the Cauchy problem (1.3) and (2.6) for the compressible Navier–Stokes equations and analyze their vanishing viscosity limit, as stated in Main Theorem II: Theorem 2.4. We also give several related remarks. In §3, we first construct global approximate smooth solutions $(\rho^{\varepsilon,\delta,b}, m^{\varepsilon,\delta,b})$ and make the basic energy estimate and the BD entropy estimate of $(\rho^{\varepsilon,\delta,b}, m^{\varepsilon,\delta,b})$, uniformly bounded in (ε, δ, b), for the Navier–Stokes equations (3.1). In §4, we derive the higher integrability of the approximate smooth solutions $(\rho^{\varepsilon,\delta,b}, m^{\varepsilon,\delta,b})$ uniformly in b. In §5, we first take the limit, $b \to \infty$, of $(\rho^{\varepsilon,\delta,b}, m^{\varepsilon,\delta,b})$ to obtain global strong solutions $(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta})$ of system (3.1) with some uniform bounds in (ε, δ), and then we take the limit, $\delta \to 0+$, to obtain global, spherically symmetric weak solutions of the Navier–Stokes equations (1.3) with some desired uniform bounds and the H^{-1}_{loc}-compactness, which are essential for us to employ the compensated compactness framework in §6 to establish Theorem 2.2. In the appendix, we construct the approximate initial data with desired estimates, which are used for the construction of the approximate solutions in §3.

Throughout this paper, we denote $L^p(\Omega)$, $W^{k,p}(\Omega)$, and $H^k(\Omega)$ as the standard Sobolev spaces on domain Ω for $p \in [1, \infty]$. We also use $L^p(\Omega; r^{N-1}\mathrm{d}r)$ or $L^p([0, T) \times \Omega; r^{N-1}\mathrm{d}r\mathrm{d}t)$ for $\Omega \subset \mathbb{R}_+$ with measure $r^{N-1}\mathrm{d}r$ or $r^{N-1}\mathrm{d}r\mathrm{d}t$ correspondingly, and $L^p_{\mathrm{loc}}([0, \infty); r^{N-1}\mathrm{d}r)$ to represent $L^p([0, R); r^{N-1}\mathrm{d}r)$ for any fixed $R > 0$.

2. Mathematical Problems and Main Theorems

In this section, we first introduce the notion of relative finite-energy solutions of the Cauchy problem (1.1)–(1.2) for the compressible Euler equations.

Definition 2.1. A pair $(\rho, \mathcal{M})$ is said to be a relative finite-energy solution of the Cauchy problem (1.1)–(1.2) if the following conditions hold:

(i) $\rho(t, \mathbf{x}) \geqq 0$ almost everywhere, and $(\mathcal{M}, \frac{\mathcal{M}}{\sqrt{\rho}})(t, \mathbf{x}) = \mathbf{0}$ almost everywhere on the vacuum states $\{(t, \mathbf{x}) : \rho(t, \mathbf{x}) = 0\}$;

(ii) For almost everywhere $t > 0$, the total relative energy with respect to the far-field state $(\bar{\rho}, \mathbf{0})$ is finite:

$$\int_{\mathbb{R}^N} \Big(\frac{1}{2}\big|\frac{\mathcal{M}}{\sqrt{\rho}}\big|^2 + e(\rho, \bar{\rho})\Big)(t, \mathbf{x})\, \mathrm{d}\mathbf{x} \leqq E_0, \tag{2.1}$$

where

$$E_0 := \int_{\mathbb{R}^N} \Big(\frac{1}{2}\big|\frac{\mathcal{M}_0}{\sqrt{\rho_0}}\big|^2 + e(\rho_0, \bar{\rho})\Big)(\mathbf{x})\, \mathrm{d}\mathbf{x} < \infty \tag{2.2}$$

is the finite total relative initial-energy, and $e(\rho, \bar{\rho})$ is the relative internal energy respective to $\bar{\rho} > 0$:

$$e(\rho, \bar{\rho}) := \frac{\kappa}{\gamma - 1}\big(\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1}(\rho - \bar{\rho})\big); \tag{2.3}$$

(iii) For any $\zeta(t, \mathbf{x}) \in C_0^1([0, \infty) \times \mathbb{R}^N)$,

$$\int_{\mathbb{R}^{N+1}_+} \big(\rho\zeta_t + \mathcal{M} \cdot \nabla\zeta\big)\, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathbb{R}^N} (\rho_0\zeta)(0, \mathbf{x})\, \mathrm{d}\mathbf{x} = 0; \tag{2.4}$$

(iv) For all $\psi(t, \mathbf{x}) = (\psi_1, \ldots, \psi_N)(t, \mathbf{x}) \in \big(C_0^1([0, \infty) \times \mathbb{R}^N)\big)^N$,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} \Big(\mathcal{M} \cdot \partial_t\psi + \frac{\mathcal{M}}{\sqrt{\rho}} \cdot \big(\frac{\mathcal{M}}{\sqrt{\rho}} \cdot \nabla\big)\psi + p(\rho)\,\mathrm{div}\,\psi\Big)\, \mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^N} \mathcal{M}_0(\mathbf{x}) \cdot \psi(0, \mathbf{x})\, \mathrm{d}\mathbf{x} = 0, \end{aligned} \tag{2.5}$$

where and whereafter we always use $\mathbb{R}^{N+1}_+ := \mathbb{R}_+ \times \mathbb{R}^N = (0, \infty) \times \mathbb{R}^N$ for $N \geq 2$.

Our first main theorem of this paper is

Theorem 2.2 (*Main Theorem I: Existence of Spherically Symmetric Solutions of the Euler Equations*). *Consider the Cauchy problem of the Euler equations* (1.1) *with large initial data of spherical symmetry of form* (1.5)*. Let* $(\rho_0, \mathcal{M}_0)(\mathbf{x})$ *satisfy* (2.2) *with the positive far-field density* $\bar{\rho} > 0$*. Then there exists a global relative finite-energy solution* $(\rho, \mathcal{M})(t, \mathbf{x})$ *of* (1.1) *and* (1.5) *with spherical symmetry of form* (1.4) *in the sense of Definition* 2.1*, where* $(\rho, m)(t, r)$ *is determined by the corresponding Cauchy problem of system* (1.6) *with the initial data* $(\rho_0, m_0)(r)$ *given in* (1.5)*.*

To establish Theorem 2.2, we first construct global weak solutions of the Cauchy problem of the compressible Navier–Stokes equations (1.3) with appropriately adapted degenerate density-dependent viscosity terms and approximate initial data

$$(\rho, \mathcal{M})|_{t=0} = (\rho_0^\varepsilon, \mathcal{M}_0^\varepsilon)(\mathbf{x}) \longrightarrow (\rho_0, \mathcal{M}_0)(\mathbf{x}) \qquad \text{as } \varepsilon \to 0, \tag{2.6}$$

constructed as in the appendix satisfying Lemmas A.1–A.2 and Lemma A.3(i).

For clarity, we adapt the viscosity terms with $(\mu, \lambda) = (\rho, 0)$ in (1.3), as the case for the shallow water (Saint Venant) models, and $\varepsilon \in (0, 1]$ without loss of generality throughout this paper. The arguments also work for a general class of degenerate density-dependent viscosity terms; see Remark 2.7 below for more details.

Definition 2.3. A pair $(\rho^\varepsilon, \mathcal{M}^\varepsilon)$ is said to be a weak solution of the Cauchy problem (1.3) and (2.6) with $(\mu, \lambda) = (\rho, 0)$ if the following conditions hold:

(i) $\rho^\varepsilon(t, \mathbf{x}) \geqq 0$ almost everywhere, and $(\mathcal{M}^\varepsilon, \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}})(t, \mathbf{x}) = \mathbf{0}$ almost everywhere on the vacuum states $\{(t, \mathbf{x}) \,:\, \rho^\varepsilon(t, \mathbf{x}) = 0\}$,

$$\rho^\varepsilon \in L^\infty(0, T; L^\gamma_{\mathrm{loc}}(\mathbb{R}^N)), \quad \nabla\sqrt{\rho^\varepsilon} \in \big(L^\infty(0, T; L^2(\mathbb{R}^N))\big)^N,$$
$$\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \in \big(L^\infty(0, T; L^2(\mathbb{R}^N))\big)^N;$$

(ii) For any $t_2 \geqq t_1 \geqq 0$ and any $\zeta(t, \mathbf{x}) \in C_0^1([0, \infty) \times \mathbb{R}^N)$, the mass equation $(1.3)_1$ holds in the sense:

$$\begin{aligned}&\int_{\mathbb{R}^N} (\rho^\varepsilon \zeta)(t_2, \mathbf{x})\, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^N} (\rho^\varepsilon \zeta)(t_1, \mathbf{x})\, \mathrm{d}\mathbf{x}\\ &\quad = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \big(\rho^\varepsilon \zeta_t + \mathcal{M}^\varepsilon \cdot \nabla\zeta\big)(t, \mathbf{x})\, \mathrm{d}\mathbf{x}\mathrm{d}t;\end{aligned}$$

(iii) For any $\psi = (\psi_1, \ldots, \psi_N) \in \big(C_0^2([0, \infty) \times \mathbb{R}^N)\big)^N$, the momentum equations $(1.3)_2$ hold in the sense:

$$\begin{aligned}&\int_{\mathbb{R}^{N+1}_+} \Big(\mathcal{M}^\varepsilon \cdot \psi_t + \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \nabla\big)\psi + p(\rho^\varepsilon)\, \mathrm{div}\, \psi\Big)\, \mathrm{d}\mathbf{x}\mathrm{d}t\\ &\qquad + \int_{\mathbb{R}^N} \mathcal{M}_0^\varepsilon(\mathbf{x}) \cdot \psi(0, \mathbf{x})\, \mathrm{d}\mathbf{x}\\ &\quad = -\varepsilon \int_{\mathbb{R}^{N+1}_+} \Big(\frac{1}{2}\mathcal{M}^\varepsilon \cdot \big(\Delta\psi + \nabla \mathrm{div}\, \psi\big) + \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \big(\nabla\sqrt{\rho^\varepsilon} \cdot \nabla\big)\psi\\ &\qquad\qquad + \nabla\sqrt{\rho^\varepsilon} \cdot \big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \nabla\big)\psi\Big)\, \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

Consider spherically symmetric solutions of form (1.4). Then systems (1.1) and (1.3) for such solutions become (1.6) and (1.7), respectively. A pair of functions $(\eta(\rho, m), q(\rho, m))$ is called an entropy pair of the 1-D Euler system (*that is*, system (1.6) with $N = 1$) if they satisfy

$$\partial_t \eta(\rho, m) + \partial_r q(\rho, m) = 0$$

for any smooth solution (ρ, m) of the 1-D Euler system; see Lax [36]. Furthermore, $\eta(\rho, m)$ is called a weak entropy if

$$\eta|_{\rho=0} = 0 \qquad \text{for any fixed } u = \frac{m}{\rho}.$$

From now on, we also use $u = \frac{m}{\rho}$ and m alternatively when $\rho > 0$.

From [41], it is well-known that any weak entropy pair (η, q) can be represented by

$$\begin{aligned} \eta(\rho, m) &= \int_{\mathbb{R}} \chi(\rho; s-u)\psi(s)\,\mathrm{d}s, \\ q(\rho, m) &= \int_{\mathbb{R}} (\theta s + (1-\theta)u)\chi(\rho; s-u)\psi(s)\,\mathrm{d}s \end{aligned} \tag{2.7}$$

when $\rho > 0$, where the kernel is

$$\chi(\rho; s-u) = [\rho^{2\theta} - (s-u)^2]_+^{\mathfrak{b}} \quad \text{for } \mathfrak{b} := \frac{3-\gamma}{2(\gamma-1)} > -\frac{1}{2} \text{ and } \theta := \frac{\gamma-1}{2}.$$

For instance, when $\psi(s) = \frac{1}{2}s^2$, the entropy pair consists of the mechanical energy and the associated energy flux

$$\eta^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + e(\rho), \quad q^*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + me'(\rho), \tag{2.8}$$

where $e(\rho) = \frac{\kappa}{\gamma-1}\rho^\gamma$ represents the internal energy. Since we expect that $(\rho, m)(t, r) \to (\bar{\rho}, 0)$ with $\bar{\rho} > 0$ as $r \to \infty$, we define the relative mechanical energy

$$\bar{\eta}^*(\rho, m) = \frac{m^2}{2\rho} + e(\rho, \bar{\rho}), \tag{2.9}$$

with $e(\rho, \bar{\rho})$ defined by (2.3) satisfying (see [15])

$$e(\rho, \bar{\rho}) \geqq C_\gamma \rho(\rho^\theta - \bar{\rho}^\theta)^2 \tag{2.10}$$

for some constant $C_\gamma > 0$.

Theorem 2.4 (*Main Theorem II: Existence and Inviscid Limit for the Navier–Stokes Equations*). *Consider the compressible Navier–Stokes equations* (1.3) *with* $N \geq 2$ *and the spherically symmetric approximate initial data* (2.6) *satisfying that, as* $\varepsilon \to 0$,

$$(\rho_0^\varepsilon, m_0^\varepsilon)(r) \to (\rho_0, m_0)(r) \qquad \text{in } L^1_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r), \tag{2.11}$$

$$E_0^\varepsilon := \omega_N \int_0^\infty \bar{\eta}^*(\rho_0^\varepsilon, m_0^\varepsilon)\, r^{N-1}\mathrm{d}r \to E_0, \tag{2.12}$$

$$E_1^\varepsilon := \varepsilon^2 \int_0^\infty \big|(\sqrt{\rho_0^\varepsilon})_r\big|^2\, r^{N-1}\mathrm{d}r \to 0, \tag{2.13}$$

and there exists a constant $C > 0$ *independent of* $\varepsilon \in (0, 1]$ *such that*

$$E_0^\varepsilon + E_1^\varepsilon \leqq C(E_0 + 1) \tag{2.14}$$

for E_0 defined in (2.2) and $\omega_N = 2\pi^{\frac{N}{2}}\Gamma(\frac{N}{2})^{-1}$ as the surface area of the unit ball in $\mathbb{R}^N$. Then the following statements hold:

Part I. Existence for the Navier–Stokes Equations (1.3): *For each $\varepsilon > 0$, there exists a global spherically symmetric weak solution*

$$(\rho^\varepsilon, \mathcal{M}^\varepsilon)(t, \mathbf{x}) = (\rho^\varepsilon(t, r), m^\varepsilon(t, r)\frac{\mathbf{x}}{r}) = (\rho^\varepsilon(t, r), \rho^\varepsilon(t, r)u^\varepsilon(t, r)\frac{\mathbf{x}}{r})$$

of the Cauchy problem of (1.3) *and* (2.6) *in the sense of Definition 2.3, where $u^\varepsilon(t, r) = \frac{m^\varepsilon(t,r)}{\rho^\varepsilon(t,r)}$ almost everywhere on $\{(t, r) : \rho^\varepsilon(t, r) \neq 0\}$ and $u^\varepsilon(t, r) = 0$ almost everywhere on $\{(t, r) : \rho^\varepsilon(t, r) = 0\}$. Moreover, $(\rho^\varepsilon, m^\varepsilon)(t, r)$ satisfies the following uniform bounds*:

$$\begin{aligned}&\int_0^\infty \bar{\eta}^*(\rho^\varepsilon, m^\varepsilon)(t, r)\, r^{N-1}\mathrm{d}r + \varepsilon\int_{\mathbb{R}^2_+} \rho^\varepsilon(s, r)|u^\varepsilon(s, r)|^2\, r^{N-3}\mathrm{d}r\mathrm{d}s\\
&\quad \leqq \frac{E_0^\varepsilon}{\omega_N} \leqq C(E_0 + 1), \qquad (2.15)\\
&\varepsilon^2\int_0^\infty \big|\big(\sqrt{\rho^\varepsilon(t, r)}\big)_r\big|^2\, r^{N-1}\mathrm{d}r + \varepsilon\int_{\mathbb{R}^2_+} \big|\big((\rho^\varepsilon(s, r))^{\frac{\gamma}{2}}\big)_r\big|^2\, r^{N-1}\mathrm{d}r\mathrm{d}s\\
&\quad \leqq C(E_0 + 1), \qquad (2.16)\end{aligned}$$

for any $t > 0$, and

$$\int_0^T\int_d^D \big(\rho^\varepsilon(t, r)\big)^{\gamma+1}\,\mathrm{d}r\mathrm{d}t \leqq C(d, D, T, E_0), \qquad (2.17)$$

$$\int_0^T\int_0^D \big(\rho^\varepsilon(t, r)|u^\varepsilon(t, r)|^3 + \big(\rho^\varepsilon(t, r)\big)^{\gamma+\theta}\big)\, r^{N-1}\mathrm{d}r\mathrm{d}t \leqq C(D, T, E_0) \quad (2.18)$$

for any fixed $T \in (0, \infty)$ and any compact subset $[d, D] \Subset (0, \infty)$, where and whereafter we denote $\mathbb{R}^2_+ := \{(t, r) : t \in (0, \infty),\ r \in (0, \infty)\}$, and $C > 0$ and $C(d, D, T, E_0) > 0$ as two universal constants independent of ε, but depending on (γ, N) and (d, D, T, E_0), respectively.

Let (η, q) be an entropy pair defined in (2.7) *for a smooth compact supported function $\psi(s)$ on $\mathbb{R}$. Then, for $\varepsilon \in (0, 1]$,*

$$\partial_t\eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon) \quad \textit{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+), \qquad (2.19)$$

where $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+)$ represents $H^{-1}((0, T] \times \Omega)$ for any $T > 0$ and bounded open subset $\Omega \Subset (0, \infty)$.

Part II. Inviscid Limit to the Euler Equations (1.1): *For the global weak solutions $(\rho^\varepsilon, \mathcal{M}^\varepsilon)$ of the compressible Navier–Stokes equations* (1.3) *established in Part* I, *there exist a subsequence* (*still denoted*) $(\rho^\varepsilon, m^\varepsilon)$ *and a vector function (ρ, m) such that, as $\varepsilon \to 0$,*

$$\begin{aligned}&(\rho^\varepsilon, m^\varepsilon) \to (\rho, m)(t, r) \quad \textit{in } (L^p_{\mathrm{loc}} \times L^q_{\mathrm{loc}})([0, \infty); r^{N-1}\mathrm{d}r),\\
&\int_0^T\int_0^D \big|\big(\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big)(t, r) - \big(\frac{m}{\sqrt{\rho}}\big)(t, r)\big|^2\, r^{N-1}\mathrm{d}r\mathrm{d}t \to 0 \quad \textit{for any fixed } T, D \in (0, \infty),\end{aligned}$$

where $p \in [1, \gamma+1)$, $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$, *and* $(\rho, \mathcal{M})(t, \mathbf{x}) := (\rho(t,r), m(t,r)\frac{\mathbf{x}}{r})$ *is a global relative finite-energy solution of spherical symmetry of the Euler equations* (1.1) *with initial data* (1.5) *in the sense of Definition* 2.1.

Remark 2.5. In Theorem 2.4, the approximate initial data functions $(\rho_0^\varepsilon, m_0^\varepsilon)$ satisfying conditions (2.11)–(2.13) are constructed in Lemmas A.1–A.2 and Lemma A.3(i) in the appendix. Then Theorem 2.2 is a direct corollary of Theorem 2.4.

Remark 2.6. The main point of Theorem 2.4 is to construct suitable Navier–Stokes approximate solutions that converge strongly to a global relative finite-energy solution of spherical symmetry of the Euler equations (1.1) with initial data (1.5) in the sense of Definition 2.1 under the relative finite-energy condition (2.2) only. We can follow the same arguments as in §3–§6 to obtain a rigorous proof of the inviscid limit from the Navier–Stokes to Euler equations with fixed same initial data (ρ_0, m_0) of appropriate regularity and decay at infinity.

Remark 2.7. When both μ and λ are constants, it is still an open problem for the inviscid limit from (1.7) to (1.6), since the BD entropy estimate is invalid for this case so that the required uniform estimate for the derivative of the density has not obtained yet. On the other hand, our analysis in this paper applies to a class of more general viscosity coefficients $(\mu(\rho), \lambda(\rho))$. For instance, our results hold for the class of $(\mu(\rho), \lambda(\rho))$ that satisfy the BD relation (see [3,44]):

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho) \tag{2.20}$$

with some additional conditions; see also the approximate system (3.1)–(3.4).

3. Approximate Solutions and Basic Uniform Estimates

In this section, we first construct global approximate smooth solutions and make their basic energy estimate and the BD entropy estimate, uniformly bounded with respect to the approximation parameters.

The main difficulty is to obtain some uniform estimates directly for the exact solutions of the Navier–Stokes equations (1.3) with approximate initial data (1.5), owing to the potential appearance of the vacuum and singularity of their limits at both the origin, $r = 0$, and the far-field, $r = \infty$, generically. On the other hand, for our purpose, it suffices to obtain first uniform estimates for appropriately designed approximate solutions of the Navier–Stokes equations (1.3). To achieve these, we construct the approximate solutions as the solutions of the following approximate Navier–Stokes system with positive density (*that is*, $\rho > 0$ so that the velocity, $u = \frac{m}{\rho}$, is well-defined) in truncated domains:

$$\begin{cases} \rho_t + (\rho u)_r + \frac{N-1}{r}\rho u = 0, \\ (\rho u)_t + (\rho u^2 + p)_r + \frac{N-1}{r}\rho u^2 = \varepsilon\big((\mu+\lambda)(u_r + \frac{N-1}{r}u)\big)_r - \varepsilon\frac{N-1}{r}u\mu_r. \end{cases} \tag{3.1}$$

Here $t > 0$ and $r \in [\delta, b]$ with $\delta \in (0, 1]$ and $b \geq 1 + \delta^{-1}$, and

$$\mu(\rho) = \rho + \delta\rho^{\alpha}, \qquad \lambda(\rho) = \delta(\alpha - 1)\rho^{\alpha} \tag{3.2}$$

with $\alpha \in (\frac{N-1}{N}, 1)$. For concreteness, we take $\alpha = \frac{2N-1}{2N}$. It is easy to check that $(\mu(\rho), \lambda(\rho))$ in (3.2) satisfy relation (2.20).

We impose (3.1) with the approximate initial data

$$(\rho, u)(0, r) = (\rho_0^{\varepsilon,\delta,b}, u_0^{\varepsilon,\delta,b})(r) \quad \text{for } r \in [\delta, b], \tag{3.3}$$

and the boundary condition

$$u(t, \delta) = u(t, b) = 0 \quad \text{for } t > 0, \tag{3.4}$$

where $\rho_0^{\varepsilon,\delta,b}$ and $u_0^{\varepsilon,\delta,b}$ are smooth functions satisfying

$$0 < (\beta\varepsilon)^{\frac{1}{4}} \leqq \rho_0^{\varepsilon,\delta,b} \leqq (\beta\varepsilon)^{-\frac{1}{2}} < \infty \tag{3.5}$$

for some small constant β (determined in Lemma A.1).

Such approximate initial data functions in (3.3) have been constructed in the appendix, which satisfy all the properties in Lemmas A.1–A.3.

For $N = 2, 3$, the existence of global smooth solutions $(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b})$ of (3.1)–(3.4) with $0 < \rho^{\varepsilon,\delta,b}(t, r) < \infty$ can be established as in Guo et al. [29]. In fact, for any $N \geqq 2$, a similar global existence result for smooth solutions of the approximate system (3.1)–(3.4) can be obtained by using analogous arguments as in §3 and §4.1 of [29]; see also [30,34]. Since the upper and lower bounds of $\rho^{\varepsilon,\delta,b}$ in [29] depend on parameters (ε, δ, b), the key point of this section is to obtain some uniform estimates of $(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b})$ independent of (δ, b) so that both limits $b \to \infty$ and $\delta \to 0+$ can be taken to obtain the global weak solution of (1.3) and (2.6); see §5.

Throughout this section, for simplicity, we always fix parameters $\varepsilon, \delta \in (0, 1]$ and $b \geq 1 + \delta^{-1}$, use $u^{\varepsilon,\delta,b}$ or $m^{\varepsilon,\delta,b}$ alternatively since $\rho^{\varepsilon,\delta,b}$ is positive, and drop the superscripts of solution $(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b})(t, r)$ and the approximate initial data $(\rho_0^{\varepsilon,\delta,b}, u_0^{\varepsilon,\delta,b})$, when no confusion arises. We keep the superscripts when the initial data functions are involved.

Lemma 3.1 (Basic Energy Estimate). *The smooth solution (ρ, u) of (3.1)–(3.4) satisfies that, for any $t > 0$,*

$$\begin{aligned}
&\int_\delta^b \big(\frac{1}{2}\rho u^2 + e(\rho, \bar\rho)\big)(t, r)\, r^{N-1} \mathrm{d}r + \varepsilon \int_0^t \int_\delta^b \big(\rho u_r^2 + \frac{N-1}{r^2}\rho u^2\big)(s, r)\, r^{N-1} \mathrm{d}r\mathrm{d}s \\
&\quad + \varepsilon\delta \int_0^t \int_\delta^b \rho^{\alpha} \Big\{\alpha u_r^2 + 2(\alpha - 1)(N - 1)\frac{u u_r}{r} \\
&\qquad\qquad + \big(1 + (N - 1)(\alpha - 1)\big)(N - 1)\frac{u^2}{r^2}\Big\}(s, r)\, r^{N-1} \mathrm{d}r\mathrm{d}s \\
&= \int_\delta^b \big(\frac{1}{2}\rho_0 u_0^2 + e(\rho_0, \bar\rho)\big)(r)\, r^{N-1} \mathrm{d}r =: \frac{E_0^{\varepsilon,\delta,b}}{\omega_N},
\end{aligned} \tag{3.6}$$

where $E_0^{\varepsilon,\delta,b}$ satisfies the properties stated in Lemma A.3 in the appendix. In particular, there exists a positive constant $c_N > 0$ (depending only on N) such that

$$\begin{aligned}
&\int_\delta^b \big(\frac{1}{2}\rho u^2 + e(\rho,\bar\rho)\big)(t,r)\, r^{N-1}\mathrm{d}r + \varepsilon \int_0^t \int_\delta^b \Big(\rho u_r^2 + \frac{\rho u^2}{r^2}\Big)(s,r)\, r^{N-1}\mathrm{d}r\mathrm{d}s \\
&\quad + c_N\varepsilon\delta \int_0^t \int_\delta^b \Big(\rho^\alpha u_r^2 + \frac{\rho^\alpha u^2}{r^2}\Big)(s,r)\, r^{N-1}\mathrm{d}r\mathrm{d}s \\
&\quad \leqq \frac{E_0^{\varepsilon,\delta,b}}{\omega_N} \leqq C(E_0+1) \quad \text{for any } t>0,
\end{aligned} \tag{3.7}$$

for some constant $C > 0$ independent of (ε, δ, b), where we have used (A.37).

Proof. Multiplying $(3.1)_2$ by $r^{N-1}u$ and performing integration by parts, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b \frac{1}{2}\rho u^2\, r^{N-1}\mathrm{d}r + \int_\delta^b p_r u\, r^{N-1}\mathrm{d}r \\
&\quad = -\varepsilon \int_\delta^b \Big((\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)(r^{N-1}u)_r - (N-1)\mu(r^{N-2}u^2)_r\Big)\mathrm{d}r.
\end{aligned} \tag{3.8}$$

For the second term on the left-hand side of (3.8), it follows from $(3.1)_1$ and integration by parts that

$$\begin{aligned}
\int_\delta^b p_r u\, r^{N-1}\mathrm{d}r &= \frac{\kappa\gamma}{\gamma-1}\int_\delta^b \rho u(\rho^{\gamma-1})_r\, r^{N-1}\mathrm{d}r \\
&= -\frac{\kappa\gamma}{\gamma-1}\int_\delta^b (\rho u r^{N-1})_r \rho^{\gamma-1}\,\mathrm{d}r \\
&= \frac{\kappa}{\gamma-1}\int_\delta^b (\rho^\gamma)_t\, r^{N-1}\mathrm{d}r \\
&= \frac{\kappa}{\gamma-1}\int_\delta^b \big(\rho^\gamma - \bar\rho^\gamma - \gamma\bar\rho^{\gamma-1}(\rho-\bar\rho)\big)_t\, r^{N-1}\mathrm{d}r \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b e(\rho,\bar\rho)(t,r)\, r^{N-1}\mathrm{d}r.
\end{aligned} \tag{3.9}$$

For the viscous term, a direct calculation shows

$$\begin{aligned}
&(\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)(ur^{N-1})_r - (N-1)\mu(u^2r^{N-2})_r \\
&\quad = \mu\big(r^{N-1}u_r^2 + (N-1)r^{N-3}u^2\big) \\
&\qquad + \lambda\big(r^{N-1}u_r^2 + 2(N-1)r^{N-2}uu_r + (N-1)^2r^{N-3}u^2\big) \\
&\quad = \delta\rho^\alpha\Big(\alpha r^{N-1}u_r^2 + 2(\alpha-1)(N-1)r^{N-2}uu_r \\
&\qquad\qquad + (N-1)(1+(\alpha-1)(N-1))r^{N-3}u^2\Big) \\
&\qquad + \rho\big(r^{N-1}u_r^2 + (N-1)r^{N-3}u^2\big).
\end{aligned} \tag{3.10}$$

For the first term on the right-hand side of (3.10), we calculate its discriminant as

$$
\begin{aligned}
&4(\alpha-1)(N-1)^2-4\alpha(N-1)\big(1+(\alpha-1)(N-1)\big)\\
&\quad=4(N-1)^2\big(1-\frac{N}{N-1}\alpha\big)<0,
\end{aligned}
$$

since $\alpha\in(\frac{N-1}{N},1)$. Thus, there exists a positive constant $c_N>0$ such that

$$
\begin{aligned}
&(\mu+\lambda)\big(u_r+\frac{N-1}{r}u\big)(r^{N-1}u)_r-(N-1)\mu(r^{N-2}u^2)_r\\
&\quad\geqq\rho\big(u_r^2+\frac{u^2}{r^2}\big)r^{N-1}+c_N\delta\rho^{\alpha}\big(u_r^2+\frac{u^2}{r^2}\big)r^{N-1}.
\end{aligned}\tag{3.11}
$$

Integrating (3.8) over $[0,t]$ and using (3.9)–(3.11), we obtain (3.6)–(3.7). □

For (μ,λ) determined by (3.2), system (1.3) admits an additional *a priori* estimate for the density (via the BD entropy), as observed by Bresch and Desjardins [2,3] (see also Bresch et al. [6]) with the Dirichlet boundary conditions in the 3-D case. For the spherically symmetric problem, we have

Lemma 3.2 (BD Entropy Estimate). *The smooth solution of* (3.1)–(3.4) *satisfies*

$$
\begin{aligned}
&\varepsilon^2\int_\delta^b\big((1+\delta\rho^{\alpha-1}+\delta^2\rho^{2(\alpha-1)})\frac{\rho_r^2}{\rho}\big)(t,r)\,r^{N-1}\mathrm{d}r\\
&\quad+\varepsilon\int_0^t\int_\delta^b\big((1+\delta\rho^{\alpha-1})\rho^{\gamma-2}\rho_r^2\big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\leqq C(E_0+1),
\end{aligned}\tag{3.12}
$$

where we have used

$$
\sup_{0<\varepsilon,\delta\leqq1}\ \sup_{b\geq1+\delta^{-1}}\big(E_0^{\varepsilon,\delta,b}+E_1^{\varepsilon,\delta,b}\big)\leqq C(E_0+1),\tag{3.13}
$$

which follows from (A.38)*, with*

$$
E_1^{\varepsilon,\delta,b}:=\varepsilon^2\int_\delta^b\big(1+2\alpha\delta\rho_0^{\alpha-1}+\alpha^2\delta^2\rho_0^{2\alpha-2}\big)\big|(\sqrt{\rho_0})_r\big|^2\,r^{N-1}\mathrm{d}r.\tag{3.14}
$$

Proof. It is more convenient to deal with (3.1) in the Lagrangian coordinates for this proof. We divide the proof into four steps.

1. For simplicity, denote $L_b:=\int_\delta^b\rho_0(r)r^{N-1}\,\mathrm{d}r$. Note that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b\rho(t,r)\,r^{N-1}\mathrm{d}r=-\int_\delta^b(\rho u r^{N-1})_r(t,r)\,\mathrm{d}r=0.
$$

Then

$$
\int_\delta^b\rho(t,r)\,r^{N-1}\mathrm{d}r=\int_\delta^b\rho_0(r)\,r^{N-1}\mathrm{d}r=L_b\qquad\text{for all }t>0.
$$

For $r \in [\delta, b]$ and $t \in [0, T]$, we define the Lagrangian transformation:

$$x = \int_\delta^r \rho(t, y)\, y^{N-1} \mathrm{d}y, \qquad \tau = t,$$

which translates domain $[0, T] \times [\delta, b]$ into $[0, T] \times [0, L_b]$ and satisfies

$$\begin{cases} \frac{\partial x}{\partial r} = \rho r^{N-1} > 0, \ \frac{\partial x}{\partial t} = -\rho u r^{N-1}, \ \frac{\partial \tau}{\partial r} = 0, \ \frac{\partial \tau}{\partial t} = 1, \\ \frac{\partial r}{\partial x} = \frac{1}{\rho r^{N-1}} > 0, \ \frac{\partial r}{\partial \tau} = u, \ \frac{\partial t}{\partial \tau} = 1, \ \frac{\partial t}{\partial x} = 0. \end{cases} \tag{3.15}$$

Applying the Lagrange transformation, system (3.1) becomes

$$\begin{cases} \rho_\tau + \rho^2 (r^{N-1} u)_x = 0, \\ u_\tau + r^{N-1} p_x = \varepsilon r^{N-1} \big(\rho(\mu + \lambda)(r^{N-1} u)_x\big)_x - \varepsilon (N-1) r^{N-2} \mu_x u, \end{cases} \tag{3.16}$$

and the boundary condition (3.4) becomes

$$u(\tau, 0) = u(\tau, L_b) = 0 \qquad \text{for } \tau > 0. \tag{3.17}$$

2. Multiplying $(3.16)_1$ by $\mu'(\rho)$ and using (2.20), we have

$$\mu_\tau + \rho(\mu + \lambda)(r^{N-1} u)_x = 0. \tag{3.18}$$

Substituting (3.18) into the viscous term of $(3.16)_2$ leads to

$$u_\tau + r^{N-1} p_x = -\varepsilon r^{N-1} (\mu_x)_\tau - \varepsilon (N-1) r^{N-2} \mu_x u. \tag{3.19}$$

Note from (3.15) that $\frac{\partial r}{\partial \tau} = u$. Then the last term of (3.19) is rewritten as

$$\varepsilon (N-1) r^{N-2} \mu_x u = (N-1) r^{N-2} r_\tau \mu_x = (r^{N-1})_\tau \mu_x,$$

which, with (3.19), yields

$$(u + \varepsilon r^{N-1} \mu_x)_\tau + r^{N-1} p_x = 0. \tag{3.20}$$

3. Multiplying (3.20) by $u + \varepsilon r^{N-1} \mu_x$, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{L_b} (u + \varepsilon r^{N-1} \mu_x)^2 \, \mathrm{d}x + \varepsilon \int_0^{L_b} p_x \mu_x \, r^{2N-2} \mathrm{d}x + \int_0^{L_b} p_x u \, r^{N-1} \mathrm{d}x = 0. \tag{3.21}$$

For the last term on the left-hand side of (3.21), it follows from integration by parts and $(3.16)_1$ that

$$\begin{aligned} \int_0^{L_b} p_x u r^{N-1} \, \mathrm{d}x &= \kappa \int_0^{L_b} \rho^{\gamma - 2} \rho_\tau \, \mathrm{d}x \\ &= \frac{\kappa}{\gamma - 1} \int_0^{L_b} (\rho^{\gamma - 1})_\tau \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{L_b} \frac{e(\rho, \bar{\rho})}{\rho} \, \mathrm{d}x. \end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{L_b} \Big(\frac{1}{2} (u + \varepsilon r^{N-1} \mu_x)^2 + \frac{e(\rho, \bar{\rho})}{\rho} \Big) \, \mathrm{d}x + \varepsilon \int_0^{L_b} p_x \mu_x \, r^{2N-2} \mathrm{d}x = 0. \tag{3.23}$$

Integrating (3.23) over $[0, \tau]$ yields

$$\int_0^{L_b} \Big(\frac{1}{2}(u+\varepsilon r^{N-1}\mu_x)^2 + \frac{e(\rho,\bar{\rho})}{\rho}\Big)\,\mathrm{d}x + \varepsilon\int_0^{\tau}\int_0^{L_b} p_x\mu_x\, r^{2N-2}\mathrm{d}x\mathrm{d}s$$
$$= \int_0^{L_b} \Big(\frac{1}{2}(u_0+\varepsilon r_0^{N-1}\mu_{0x})^2 + \frac{e(\rho_0,\bar{\rho})}{\rho_0}\Big)\,\mathrm{d}x. \tag{3.24}$$

4. Plugging (3.24) back to the Eulerian coordinates, we have

$$\int_a^b \Big(\frac{1}{2}\rho\big|u+\varepsilon\frac{\mu_r}{\rho}\big|^2 + e(\rho,\bar{\rho})\Big)\,r^{N-1}\mathrm{d}r + \varepsilon\int_0^{\tau}\int_a^b \frac{p_r}{\rho}\mu_r\, r^{N-1}\mathrm{d}r\mathrm{d}s$$
$$= \int_a^b \Big(\frac{1}{2}\rho_0\big|u_0+\varepsilon\frac{\mu_{0r}}{\rho_0}\big|^2 + e(\rho_0,\bar{\rho})\Big)\,r^{N-1}\mathrm{d}r,$$

which, with (3.7), leads to (3.12). □

Lemma 3.3. *For given d and D with $[d, D] \Subset [\delta, b]$, any smooth solution of* (3.1)–(3.4) *satisfies*

$$\int_0^T\int_K \rho^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t \leqq C(d, D, T, E_0), \tag{3.25}$$

where K is any compact subset of $[d, D]$.

Proof. We divide the proof into five steps.
1. Let $w(r)$ be a smooth compact support function with $\operatorname{supp} w \subseteq [d, D]$ and $w(r) \equiv 1$ for $r \in K$. Multiplying $(3.1)_2$ by $w(r)$, we have

$$\begin{aligned}
&(\rho u w)_t + \big((\rho u^2+p)w\big)_r + \frac{N-1}{r}\rho u^2 w\\
&= \varepsilon\big((\mu+\lambda)(u_r+\frac{N-1}{r}u)w\big)_r - \varepsilon\frac{N-1}{r}\mu_r u w\\
&\quad + \big(\rho u^2 + p - \varepsilon(\mu+\lambda)(u_r+\frac{N-1}{r}u)\big)w_r.
\end{aligned} \tag{3.26}$$

Integrating (3.26) over $[d, r)$ and multiplying the resultant equation by ρw, we have

$$\begin{aligned}
(\rho^2u^2+\rho p)w^2 &= -\rho w\Big(\int_d^r \rho u w\,\mathrm{d}y\Big)_t - \rho w\int_d^r \frac{N-1}{y}\rho u^2 w\,\mathrm{d}y\\
&\quad + \rho w\int_d^r \Big(\rho u^2 + p - \varepsilon(\mu+\lambda)\big(u_y+\frac{N-1}{y}u\big)\Big)w_y\,\mathrm{d}y\\
&\quad + \varepsilon\rho(\mu+\lambda)\big(u_r+\frac{N-1}{r}u\big)w^2 - \varepsilon\rho w\int_d^r \frac{N-1}{y}u\mu_y w\,\mathrm{d}y.
\end{aligned} \tag{3.27}$$

A direct calculation shows

$$
\begin{aligned}
\rho p w^2 = & -\Big(\rho w \int_d^r \rho u w \, \mathrm{d}y\Big)_t - \Big(\rho u w \int_d^r \rho u w \, \mathrm{d}y\Big)_r + \rho u w_r \int_d^r \rho u w \, \mathrm{d}y \\
& - \frac{N-1}{r} \rho u w \int_d^r \rho u w \, \mathrm{d}y - \rho w \int_d^r \frac{N-1}{y} \rho u^2 w \, \mathrm{d}y \\
& + \rho w \int_d^r \Big(\rho u^2 + p - \varepsilon \frac{\mu+\lambda}{y} \big(y u_y + (N-1)u\big)\Big) w_y \, \mathrm{d}y \\
& - \varepsilon \rho w \int_d^r \frac{N-1}{y} u \mu_y w \, \mathrm{d}y + \varepsilon \rho (\mu+\lambda) \big(u_r + \frac{N-1}{r} u\big) w^2 \\
:= & \sum_{j=1}^{8} I_j .
\end{aligned} \tag{3.28}
$$

To estimate the right-hand side of (3.28), we first note from (2.10) and (3.7) that

$$
\int_d^D \rho^\gamma \, r^{N-1} \mathrm{d}r \leqq C(D, E_0). \tag{3.29}
$$

Using (3.7) and (3.29), we see that

$$
\int_d^D \rho \, \mathrm{d}r \leqq \frac{C}{d^{N-1}} \int_d^D \rho \, r^{N-1} \mathrm{d}r \leqq C(d) \int_d^D (\rho^\gamma + 1) \, r^{N-1} \mathrm{d}r \leqq C(d, D, E_0), \tag{3.30}
$$

$$
\int_d^D \rho u^2 \, \mathrm{d}r \leqq \frac{C}{d^{N-1}} \int_d^D \rho u^2 \, r^{N-1} \mathrm{d}r \leqq C(d, E_0). \tag{3.31}
$$

2. Now it follows from (3.30)–(3.31) that

$$
\begin{aligned}
\Big| \int_0^T \int_d^D I_1 \, \mathrm{d}r \mathrm{d}t \Big| & \leqq \int_d^D \Big(\Big| \Big(\rho w \int_d^r \rho u w \, \mathrm{d}y\Big)(T, r) \Big| + \Big| \Big(\rho w \int_d^r \rho u w \, \mathrm{d}y\Big)(0, r) \Big| \Big) \mathrm{d}r \\
& \leqq C(d, D, T, E_0),
\end{aligned} \tag{3.32}
$$

$$
\int_0^T \int_d^D I_2 \, \mathrm{d}r \mathrm{d}t = \int_0^T \int_d^D \Big(\rho u w \int_d^r \rho u w \, \mathrm{d}y\Big)_r \mathrm{d}r \mathrm{d}t = 0, \tag{3.33}
$$

$$
\Big| \int_0^T \int_d^D I_3 \, \mathrm{d}r \mathrm{d}t \Big| = \Big| \int_0^T \int_d^D \Big(\rho u w_r \int_d^r \rho u w \, \mathrm{d}y\Big) \mathrm{d}r \mathrm{d}t \Big| \leqq C(d, D, T, E_0), \tag{3.34}
$$

$$
\Big| \int_0^T \int_d^D I_4 \, \mathrm{d}r \mathrm{d}t \Big| \leqq C(d) \Big| \int_0^T \int_d^D \Big(\rho u \int_d^r \rho u \, \mathrm{d}y\Big) \mathrm{d}r \mathrm{d}t \Big| \leqq C(d, D, T, E_0), \tag{3.35}
$$

$$
\Big| \int_0^T \int_d^D I_5 \, \mathrm{d}r \mathrm{d}t \Big| = \Big| \int_0^T \int_d^D \Big(\rho w \int_d^r \frac{N-1}{y} \rho u^2 w \, \mathrm{d}y\Big) \mathrm{d}r \mathrm{d}t \Big| \leqq C(d, D, T, E_0). \tag{3.36}
$$

3. We now estimate I_6. It follows from (3.7) that

$$
\Big| \int_0^T \int_d^D \Big(\rho w \int_d^r (\rho u^2 + p) w_y \, \mathrm{d}y\Big) \mathrm{d}r \mathrm{d}t \Big| \leqq C(d, D, T, E_0), \tag{3.37}
$$

$$
\Big| \int_0^T \int_d^D \varepsilon \rho w \Big(\int_d^r \frac{\rho + \alpha \delta \rho^\alpha}{y} \big(y u_y + (N-1)u\big) w_y \, \mathrm{d}y\Big) \mathrm{d}r \mathrm{d}t \Big|
$$

$$\leqq C(d,D,E_0)\Big\{\varepsilon\int_0^T\int_d^D(\rho+\delta\rho^\alpha)\big(u_r^2+\frac{u^2}{y^2}+1\big)\,y^{N-1}\mathrm{d}y\mathrm{d}t\Big\}$$
$$\leqq C(d,D,T,E_0). \tag{3.38}$$

Then it follows from (3.37)–(3.38) that

$$\Big|\int_0^T\int_d^D I_6\,\mathrm{d}r\mathrm{d}t\Big|\leqq C(d,D,T,E_0). \tag{3.39}$$

4. For I_7, it follows from (3.7) and integration by parts that

$$\begin{aligned}&\Big|\int_d^r\frac{1}{y}u\mu_y w\,\mathrm{d}y\Big|\\&\quad\leqq\Big|\frac{1}{r}(\mu u w)(t,r)\Big|+\Big|\int_d^r\frac{1}{y}\mu\big(-\frac{1}{y}uw+u_y w+uw_y\big)(t,y)\,\mathrm{d}y\Big|\\&\quad\leqq\frac{1}{r}\big((\rho+\delta\rho^\alpha)|uw|\big)(t,r)+C(d)\int_d^D\big(\rho u_r^2+\delta\rho^\alpha u_r^2\big)r^{N-1}\,\mathrm{d}r+C(d,D,E_0),\end{aligned}$$

which implies that

$$\begin{aligned}&\Big|\int_0^T\int_d^D I_7\,\mathrm{d}r\mathrm{d}t\Big|\\&\quad\leqq C(d,D,T,E_0)\Big(1+\varepsilon\int_0^T\int_d^D(\rho+\delta\rho^\alpha)u_r^2\,r^{N-1}\mathrm{d}r\mathrm{d}t\Big)+\varepsilon\int_0^T\int_d^D\rho^3w^2\,\mathrm{d}r\mathrm{d}t\\&\quad\leqq C(d,D,T,E_0)+\varepsilon\int_0^T\int_d^D\rho^3w^2\,\mathrm{d}r\mathrm{d}t,\end{aligned}\tag{3.40}$$

where we have used $\alpha<1$.
For I_8, it follows from (3.7) and the Cauchy inequality that

$$\begin{aligned}\Big|\int_0^T\int_d^D I_8\,\mathrm{d}r\mathrm{d}t\Big|&\leqq\varepsilon\Big|\int_0^T\int_d^D\rho^2\big(u_r+\frac{N-1}{r}u\big)w^2\,\mathrm{d}r\mathrm{d}t\Big|\\&\quad+\varepsilon\delta\Big|\int_0^T\int_d^D\rho^{1+\alpha}\big(u_r+\frac{N-1}{r}u\big)w^2\,\mathrm{d}r\mathrm{d}t\Big|\\&\leqq C(d)\int_0^T\int_d^D\varepsilon(\rho+\delta\rho^\alpha)\big(u_r^2+\frac{u^2}{r^2}\big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\&\quad+\frac{\varepsilon}{2}\int_0^T\int_d^D\big(\rho^3+\rho^{2+\alpha}\big)w^2\,\mathrm{d}r\mathrm{d}t\\&\leqq C(d,D,T,E_0)+\varepsilon\int_0^T\int_d^D\rho^3w^2\,\mathrm{d}r\mathrm{d}t.\end{aligned}\tag{3.41}$$

To close the estimate, we still need to bound the last term on the right-hand sides of (3.40)–(3.41).

We first consider the case: $\gamma\in(1,2]$. Notice that

$$\varepsilon\int_0^T\int_d^D\rho^3w^2\,\mathrm{d}r\mathrm{d}t$$

$$
\begin{aligned}
&\leqq \varepsilon \int_0^T \Big(\int_d^D \rho^\gamma \,\mathrm{d}r \Big) \sup_{r\in[d,D]} \big(\rho^{3-\gamma} w^2\big)\,\mathrm{d}t \\
&\leqq C(d, D, E_0) \int_0^T \varepsilon \sup_{r\in[d,D]} \big(\rho^{3-\gamma} w^2\big)\mathrm{d}t \\
&\leqq \hat{C}(d, D, E_0) \int_0^T \int_d^D \big(\varepsilon\rho^{2-\gamma}|\rho_r|w^2 + \varepsilon\rho^{3-\gamma} w|w_r|\big)\,\mathrm{d}r\mathrm{d}t, \qquad (3.42)
\end{aligned}
$$

where $\hat{C}(d, D, E_0)$ is a constant depending on (d, D, E_0). A direct calculation shows that

$$
\begin{aligned}
\int_0^T \int_d^D \varepsilon\rho^{2-\gamma}|\rho_r|w^2\,\mathrm{d}r\mathrm{d}t &\leqq \int_0^T \int_d^D \varepsilon\rho^{\gamma-2}\rho_r^2\,\mathrm{d}r\mathrm{d}t + \frac{\varepsilon}{2}\int_0^T \int_d^D \rho^{3(2-\gamma)}w^2\,\mathrm{d}r\mathrm{d}t \\
&\leqq C(d, D, E_0) + \frac{\varepsilon}{2\hat{C}(d, D, E_0)} \int_0^T \int_d^D \rho^3 w^2\,\mathrm{d}r\mathrm{d}t, \qquad (3.43) \\
\int_0^T \int_d^D \varepsilon\rho^{3-\gamma} w|w_r|\,\mathrm{d}r\mathrm{d}t &\leqq \int_0^T \varepsilon \sup_r(\rho w)(t,r) \Big(\int_d^D \rho^{2-\gamma}|w_r|\,\mathrm{d}r\Big)\,\mathrm{d}t \\
&\leqq C(d, D, E_0) \int_0^T \varepsilon \sup_r(\rho w)(t,r)\,\mathrm{d}t \\
&\leqq C(d, D, E_0) \int_0^T \int_d^D \varepsilon\,\big(|\rho_r|w + \rho|w_r|\big)\,\mathrm{d}r\mathrm{d}t \\
&\leqq C(d, D, E_0)\Big(\int_0^T \int_d^D \big(\varepsilon\rho^{\gamma-2}\rho_r^2 + \rho^{2-\gamma}w\big)\,\mathrm{d}r\mathrm{d}t + 1\Big) \\
&\leqq C(d, D, E_0). \qquad (3.44)
\end{aligned}
$$

Combining (3.42)–(3.44), we have

$$
\varepsilon \int_0^T \int_d^D \rho^3 w^2\,\mathrm{d}r\mathrm{d}t \leqq C(d, D, E_0) \qquad \text{for } \gamma \in (1, 2]. \qquad (3.45)
$$

For the case: $\gamma \in [2, 3]$, notice that

$$
\begin{aligned}
&\varepsilon \int_0^T \int_d^D \rho^3 w^2\,\mathrm{d}r\mathrm{d}t \\
&\quad\leqq \varepsilon \int_0^T \sup_{r\in[d,D]} \big(\rho^2 w\big) \int_d^D \rho w\,\mathrm{d}r\,\mathrm{d}t \\
&\quad\leqq C(d, D, E_0) \int_0^T \int_d^D \big(\varepsilon\rho|\rho_r|w + \varepsilon\rho^2|w_r|\big)\,\mathrm{d}r\mathrm{d}t \\
&\quad\leqq C(d, D, E_0) \int_0^T \int_d^D \big(\varepsilon^2\rho^{\gamma-2}|\rho_r|^2 w + \rho^2|w_r| + \rho^{4-\gamma}w\big)\,\mathrm{d}r\mathrm{d}t \\
&\quad\leqq C(d, D, E_0). \qquad (3.46)
\end{aligned}
$$

For case $\gamma \in [3, \infty)$, we can immediately see that

$$
\int_0^T \int_d^D \rho^3 w^2\,\mathrm{d}r\mathrm{d}t \leqq C(d, D) \int_0^T \int_d^D \big(1 + r^{N-1}\rho^\gamma\big)\,\mathrm{d}r\mathrm{d}t \leqq C(d, D, E_0). \qquad (3.47)
$$

Now substituting (3.45)–(3.47) into (3.40)–(3.41), we obtain

$$\Big|\int_0^T\int_d^D (I_7+I_8)\,\mathrm{d}r\mathrm{d}t\Big| \leqq C(d,D,T,E_0). \tag{3.48}$$

5. Integrating (3.28) over $[0,T]\times[d,D]$ and then using (3.32)–(3.36), (3.39), and (3.48), we conclude (3.25). □

4. Uniform Higher Integrability of the Approximate Solutions

To employ the compensated compactness framework in [15], we further require the higher integrability of the approximate solutions.

From now on, we denote

$$\begin{aligned} &M_1 := E_0+\bar\rho+\bar\rho^{-1}+\delta^{-1}+\varepsilon^{-1}+\sup_{b\geq 1+\delta^{-1}} E_2^{\varepsilon,\delta,b}<\infty,\\ &M_2 := M_1+\sup_{b\geq 1+\delta^{-1}} \tilde E_0^{\varepsilon,\delta,b}<\infty, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &E_2^{\varepsilon,\delta,b} := \int_\delta^b \rho_0\Big(u_0^{2N}+\big|\frac{\mu_{0r}}{\rho_0}\big|^{2N}\Big)\,r^{N-1}\mathrm{d}r,\\ &\tilde E_0^{\varepsilon,\delta,b} := \int_\delta^b \big(\frac12\rho_0u_0^2+e(\rho_0,\bar\rho)\big)\,r^{2(N-1)+\vartheta}\mathrm{d}r \end{aligned} \tag{4.2}$$

for some $\vartheta\in(0,1)$. From Lemma A.3, we note that $E_2^{\varepsilon,\delta,b}$ and $\tilde E_0^{\varepsilon,\delta,b}$ are uniformly bounded with respect to b, while the upper bounds may depend on (ε,δ), so that M_1 and M_2 are finite for any fixed (ε,δ), independent of $b>0$.

Proposition 4.1. *Let* $[d,D]\Subset[\delta,b]$. *Then the smooth solution of* (3.1)–(3.4) *satisfies*

$$\int_0^T\int_d^D \big(\rho|u|^3+\rho^{\gamma+\theta}\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t \leqq C(d,D,T,E_0)+C(T,M_2)b^{-\frac{\vartheta}{2}}, \tag{4.3}$$

where $\vartheta\in(0,1)$ *given in* (4.2).

To prove (4.3), we need to integrate the equations from the far-field, so that the asymptotic behavior of $(\rho-\bar\rho)(t,r)$ and $u(t,r)$ near boundary $r=b$ must be known. Indeed, the key point for Proposition 4.1 is that a decay rate of $(\rho-\bar\rho)(t,r)$ and $u(t,r)$ can be derived, and the positive constant $C(T,M_2)$ in (4.3) is independent of b so that this term vanishes when $b\to\infty$. In order to prove Proposition 4.1, we require the next six lemmas.

To obtain the asymptotic behavior of $\rho(t,r)$ near boundary $r=b$, we first need the lower and upper bounds of $\rho(t,r)$, which are independent of b.

Lemma 4.2 (Upper Bound of the Density). *There exists a constant $C(M_1) > 0$ such that the smooth solution of* (3.1)–(3.4) *satisfies*

$$0 < \rho(t,r) \leqq C(M_1) \qquad \text{for } t \geqq 0 \text{ and } r \in [\delta, b]. \tag{4.4}$$

Proof. Notice that

$$\begin{cases} e(\rho,\bar\rho) \cong |\rho-\bar\rho|^2 & \text{if } \rho \in [\frac{\bar\rho}{2}, 2\bar\rho], \\ e(\rho,\bar\rho) \cong |\rho-\bar\rho|^\gamma & \text{if } \rho \in \mathbb{R}_+\backslash[\frac{\bar\rho}{2}, 2\bar\rho]. \end{cases}$$

Denote

$$A(t) := \{r \,:\, r \in [\delta, b],\ \rho(t,r) \geqq 2\bar\rho\} \tag{4.5}$$

with $A_1(t) := \{r \,:\, r \in [1, b],\ r \in A(t)\} \subset A(t)$ and $A_2(t) := A(t)\backslash A_1(t)$. It is easy to see that

$$e(\rho,\bar\rho) \geqq C(\bar\rho)^{-1} \qquad \text{for } r \in A(t), \tag{4.6}$$

which, along with (3.7), yields

$$E_0^{\varepsilon,\delta,b} \geqq \int_\delta^b e(\rho,\bar\rho)\, r^{N-1} \mathrm{d}r \geqq \int_{A_1(t)} e(\rho,\bar\rho)\, \mathrm{d}r \geqq C(\bar\rho)^{-1}|A_1(t)|.$$

Since $E_0^{\varepsilon,\delta,b} \leqq C(E_0+1)$, we have

$$|A(t)| \leqq |A_1(t)| + |A_2(t)| \leqq C(\bar\rho, E_0). \tag{4.7}$$

Since $\rho(t,r)$ is a continuous function on $[\delta, b]$, then, for any $r \in A(t)$, there exists $r_0 \in A(t)$ such that $\rho(t, r_0) = 2\bar\rho$ and $|r - r_0| \leqq C(\bar\rho, E_0)$, which implies that

$$\begin{aligned} \sqrt{\rho(t,r)} &\leqq \sqrt{\rho(t,r_0)} + \int_{r_0}^r \frac{|\rho_y(t,y)|}{2\sqrt{\rho(t,y)}} \mathrm{d}y \\ &\leqq \sqrt{2\bar\rho} + C(\bar\rho, E_0)\Big(\int_\delta^b \frac{\rho_r^2}{\rho}\mathrm{d}r\Big)^{\frac12} \\ &\leqq \sqrt{2\bar\rho} + \frac{C(\bar\rho, E_0)}{\delta^{\frac{N-1}{2}}\varepsilon} \\ &\leqq C(\bar\rho, \varepsilon, \delta, E_0). \end{aligned}$$

This completes the proof. □

Lemma 4.3. *The smooth solution of* (3.1)–(3.4) *satisfies*

$$\int_\delta^b \frac{\rho_r^{2N}}{\rho^{2N}} r^{N-1} \mathrm{d}r \leqq C(T, M_1) \qquad \text{for any } t \in [0, T]. \tag{4.8}$$

Proof. We divide the proof into three steps.
1. We rewrite (3.20) as

$$(\varepsilon r^{N-1}\mu_x)_\tau = -u_\tau - r^{N-1} p_x \tag{4.9}$$

in the Lagrangian coordinates. Integrating (4.9) over $[0, \tau]$ leads to

$$\varepsilon(r^{N-1}\mu_x)(\tau, x) = \varepsilon(r^{N-1}\mu_x)(0, x) - \big(u(x, \tau) - u_0(x)\big) - \int_0^\tau (r^{N-1} p_x)(s, x)\,\mathrm{d}s. \tag{4.10}$$

Multiplying (4.10) by $(r^{N-1}\mu_x)^{2N-1}$ and integrating the resultant equation yield

$$\begin{aligned}
&\varepsilon \int_0^{L_b} \big|(r^{N-1}\mu_x)(\tau)\big|^{2N}\,\mathrm{d}x \\
&\quad\leqq \left(\int_0^{L_b} |(r^{N-1}\mu_x)(\tau)|^{2N}\mathrm{d}x\right)^{\frac{2N-1}{2N}} \\
&\qquad \times \Big\{\|(u(\tau), u_0, (r^{N-1}\mu_x)(0))\|_{L^{2N}} + C_T\|r^{N-1}(\rho^\gamma)_x\|_{L^{2N}((0,\tau)\times(0,L_b))}\Big\},
\end{aligned}$$

which leads to

$$\begin{aligned}
&\int_0^{L_b} |(r^{N-1}\mu_x)(\tau)|^{2N}\mathrm{d}x \\
&\quad\leqq C(\varepsilon)\Big\{\big\|(u(\tau), u_0, (r^{N-1}\mu_x)(0))\big\|_{L^{2N}}^{2N} + C_T\|r^{N-1}(\rho^\gamma)_x\|^{2N}_{L^{2N}((0,\tau)\times(0,L_b))}\Big\}.
\end{aligned} \tag{4.11}$$

Notice that $|\mu_x| = \big|\big(\frac{1}{\alpha}\rho^{1-\alpha} + \delta\big)(\rho^\alpha)_x\big| \geqq \delta\big|(\rho^\alpha)_x\big|$ and $(\rho^\gamma)_x = \frac{\gamma}{\alpha}\rho^{\gamma-\alpha}(\rho^\alpha)_x$. It follows from (4.4) and (4.11) that

$$\begin{aligned}
&\int_0^{L_b} \big|\big(r^{N-1}(\rho^\alpha)_x\big)(\tau)\big|^{2N}\mathrm{d}x \\
&\quad\leqq C(T, \varepsilon, \delta, E_0)\Big\{\|(u(\tau), u_0, r^{N-1}\mu_x\big)(0))\|^{2N}_{L^{2N}} \\
&\qquad\qquad + \|r^{N-1}(\rho^\alpha)_x\|^{2N}_{L^{2N}((0,\tau)\times(0,L_b))}\Big\}.
\end{aligned} \tag{4.12}$$

Plugging (4.12) back to the Eulerian coordinates and noting $\alpha = \frac{2N-1}{2N}$, we see that, for $t \in [0, T]$,

$$\begin{aligned}
&\int_\delta^b \Big(\frac{\rho_r^{2N}}{\rho^{2N}}\Big)(t)\, r^{N-1}\mathrm{d}r \\
&\quad\leqq C(T, \varepsilon, \delta, E_0)\Big\{E_2^{\varepsilon,\delta,b} + \int_\delta^b (\rho u^{2N})(t)\, r^{N-1}\mathrm{d}r \\
&\qquad\qquad + \int_0^t\int_\delta^b \Big(\frac{\rho_r^{2N}}{\rho^{2N}}\Big)(s)\, r^{N-1}\mathrm{d}r\mathrm{d}s\Big\}.
\end{aligned} \tag{4.13}$$

2. In order to close the above estimate, we need to bound $\int_\delta^b \rho u^{2N} r^{N-1}\,\mathrm{d}r$. Multiplying $(3.1)_2$ by $r^{N-1}u^{2N-1}$ and then integrating by parts, we have

$$\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b \rho u^{2N}\, r^{N-1}\mathrm{d}r - \int_\delta^b p(r^{N-1}u^{2N})_r\,\mathrm{d}r$$

$$= -\varepsilon \int_\delta^b \Big\{(\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)(r^{N-1}u^{2N-1})_r - (N-1)\mu(r^{N-2}u^{2N})_r\Big\}\,\mathrm{d}r. \quad (4.14)$$

By similar arguments as in (3.10)–(3.11), we obtain

$$\begin{aligned}&(\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)(r^{N-1}u^{2N-1})_r - (N-1)\mu(r^{N-2}u^{2N})_r\\ &\quad \geqq \rho u^{2N-2}\Big\{(2N-1)u_r^2 + (N-1)\frac{u^2}{r^2}\Big\}r^{N-1}. \end{aligned}\quad (4.15)$$

For the pressure term, it follows from (4.4) and the Hölder inequality that

$$\begin{aligned}&\Big|\int_\delta^b p(r^{N-1}u^{2N-1})_r\mathrm{d}r\Big|\\ &\quad = \Big|\int_\delta^b p\big((2N-1)r^{N-1}u^{2N-2}u_r + (N-1)r^{N-2}u^{2N-1}\big)\,\mathrm{d}r\Big|\\ &\quad \leqq \frac{\varepsilon}{2}\int_\delta^b \rho u^{2N-2}\big(u_r^2 + \frac{u^2}{r^2}\big)\,r^{N-1}\mathrm{d}r + C\int_\delta^b \rho^{2\gamma-1}u^{2N-2}\,r^{N-1}\mathrm{d}r\\ &\quad \leqq \frac{\varepsilon}{2}\int_\delta^b \rho u^{2N-2}\big(u_r^2 + \frac{u^2}{r^2}\big)\,r^{N-1}\mathrm{d}r + C(M_1)\Big(1 + \int_\delta^b \rho u^{2N}\,r^{N-1}\mathrm{d}r\Big). \end{aligned}\quad (4.16)$$

Substituting (4.15)–(4.16) into (4.14), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b \rho u^{2N}\,r^{N-1}\mathrm{d}r \leqq C(M_1)\Big(1 + \int_\delta^b \rho u^{2N}\,r^{N-1}\mathrm{d}r\Big),$$

which, with the Gronwall inequality, implies that

$$\int_\delta^b \rho u^{2N}\,r^{N-1}\mathrm{d}r \leqq C(T, M_1) \qquad \text{for } t \in [0, T]. \quad (4.17)$$

3. Now substituting (4.17) into (4.13) yields that

$$\int_\delta^b \Big(\frac{\rho_r^{2N}}{\rho^{2N}}\Big)(t)\,r^{N-1}\mathrm{d}r \leqq C(T, M_1)\Big(1 + \int_0^t\int_\delta^b \Big(\frac{\rho_r^{2N}}{\rho^{2N}}\Big)(s)\,r^{N-1}\mathrm{d}r\mathrm{d}s\Big). \quad (4.18)$$

Applying the Gronwall inequality to (4.18), we conclude (4.8). □

With the above preparation, we have the following lower bound of the density:

Lemma 4.4 (Lower Bound of the Density). *There exists $C(T, M_1) > 0$ depending only on (T, M_1) such that the smooth solution of (3.1)–(3.4) satisfies*

$$\rho(t, r) \geqq C(T, M_1)^{-1} > 0 \qquad \textit{for } (t, r) \in [0, T] \times [\delta, b]. \quad (4.19)$$

Proof. Define

$$B(t) := \{r : r \in [\delta, b],\ 0 \leqq \rho(t,r) \leqq \frac{\bar\rho}{2}\} \tag{4.20}$$

with $B_1(t) := \{r : r \in [1,b],\ r \in B(t)\} \subset B(t)$ and $B_2(t) := B(t)\backslash B_1(t)$. Similar to (4.6)–(4.7), we have

$$|B(t)| \leqq C(\bar\rho, E_0). \tag{4.21}$$

Since $\rho(t,r)$ is a continuous function on $[\delta, b]$, then, for any $r \in B(t)$, there exists $r_0 \in B(t)$ such that $\rho(t, r_0) = \frac{\bar\rho}{2}$ and $|r - r_0| \leqq C(\bar\rho, E_0)$. Thus, for $\beta > 0$,

$$\begin{aligned}
\rho(t,r)^{-\beta} &\leqq \rho(t,r_0)^{-\beta} + \beta\Big|\int_{r_0}^{r} \rho^{-\beta-1}|\rho_r|\,\mathrm{d}y\Big| \\
&\leqq C(\bar\rho) + \beta\Big(\int_\delta^b \frac{|\rho_r|^{2N}}{\rho^{2N}}\,\mathrm{d}r\Big)^{\frac{1}{2N}}\Big(\int_{B(t)} \rho^{-\frac{2\beta N}{2N-1}}\,\mathrm{d}r\Big)^{\frac{2N-1}{2N}} \\
&\leqq C(\bar\rho) + \beta\hat C(T, M_1)\max_{r\in B(t)} \rho(t,r)^{-\beta},
\end{aligned}$$

where (4.8) has been used in the last inequality. Then we have

$$\max_{r\in B(t)} \rho(t,r)^{-\beta} \leqq C(\bar\rho) + \beta\hat C(T, M_1)\max_{r\in B(t)} \rho(t,r)^{-\beta}.$$

Taking $\beta > 0$ small enough such that $\beta\hat C(T, M_1) \leqq \frac{1}{2}$, we obtain

$$\max_{r\in B(t)} \rho(t,r)^{-\beta} \leqq C(\bar\rho).$$

Therefore, we conclude

$$\rho(t,r) \geqq C(\bar\rho)^{-\frac{1}{\beta}} = C(T, M_1)^{-1} \qquad \text{for all } r \in B(t),$$

which leads to (4.19). □

Remark 4.5. Since M_1 is independent of b, the key point of Lemmas 4.2 and 4.4 is that the lower and upper bounds of the density are independent of b.

With the above lower and upper bounds of the density, even though they depend on (ε, δ), we can have the following weighted estimate:

Lemma 4.6. *Let* $\vartheta \in (0, 1)$ *be some positive constant. Then the smooth solution of* (3.1)–(3.4) *satisfies*

$$\int_\delta^b \big(\frac{1}{2}\rho u^2 + e(\rho,\bar\rho)\big)\, r^{2(N-1)+\vartheta}\,\mathrm{d}r + \varepsilon\int_0^T\int_\delta^b (\rho + \alpha\delta\rho^\alpha)u_r^2\, r^{2(N-1)+\vartheta}\,\mathrm{d}r\mathrm{d}s \leqq C(T, M_2). \tag{4.22}$$

Proof. The proof consists of five steps.

1. Let $L \in [0, N]$. Multiplying $(3.1)_2$ by $r^{N-1+L}u$ and then integrating by parts yield,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b \frac{1}{2}\rho u^2\, r^{N-1+L}\mathrm{d}r + \int_\delta^b p_r u\, r^{N-1+L}\mathrm{d}r\\
&\quad = \frac{L}{2}\int_\delta^b \rho u^3\, r^{N-2+L}\mathrm{d}r\\
&\qquad - \varepsilon\int_\delta^b (\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)\big(u_r + \frac{N-1+L}{r}u\big)\, r^{N-1+L}\mathrm{d}r\\
&\qquad + \varepsilon(N-1)\int_\delta^b \mu u\big(2u_r + \frac{N-2+L}{r}u\big)\, r^{N-2+L}\mathrm{d}r.
\end{aligned}
\tag{4.23}
$$

2. It follows from integration by parts, (4.4), and (4.19) that

$$
\begin{aligned}
&\int_\delta^b p_r u\, r^{N-1+L}\mathrm{d}r\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b e(\rho,\bar\rho)\, r^{N-1+L}\mathrm{d}r - \frac{\kappa\gamma L}{\gamma-1}\int_\delta^b \rho u\big(\rho^{\gamma-1} - \bar\rho^{\gamma-1}\big)\, r^{N-2+L}\mathrm{d}r\\
&\geqq -C(T, M_1)\int_\delta^b \big(\rho u^2 + e(\rho,\bar\rho)\big)\, r^{N-2+L}\mathrm{d}r + \frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b e(\rho,\bar\rho)\, r^{N-1+L}\mathrm{d}r.
\end{aligned}
\tag{4.24}
$$

Using the Sobolev inequality:

$$
\|u(t)\|_{L^\infty} \leqq C\|u(t)\|_{L^2}^{\frac{1}{2}}\,\|u_r(t)\|_{L^2}^{\frac{1}{2}},
\tag{4.25}
$$

we have

$$
\begin{aligned}
&\frac{L}{2}\left|\int_\delta^b \rho u^3\, r^{N-2+L}\mathrm{d}r\right|\\
&\quad\leqq C\|u\|_{L^2}^{\frac{1}{2}}\|u_r\|_{L^2}^{\frac{1}{2}}\int_\delta^b \rho u^2\, r^{N-2+L}\mathrm{d}r\\
&\quad\leqq C(T, M_1)\Big\{\int_\delta^b \rho u_r^2\, r^{N-1}\mathrm{d}r + \Big(\int_\delta^b \rho u^2\, r^{N-2+L}\mathrm{d}r\Big)^{\frac{4}{3}}\Big\},
\end{aligned}
\tag{4.26}
$$

where we have used (4.4), (4.19), and

$$
\|u\|_{L^2} \leqq C(T, M_1)\Big(\int_\delta^b \rho u^2\, r^{N-1}\mathrm{d}r\Big)^{\frac{1}{2}} \leqq C(T, M_1).
$$

3. For the viscous term, a direct calculation shows that

$$
\begin{aligned}
&-(\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)\big(u_r + \frac{N-1+L}{r}u\big)r^{N-1+L}\\
&\quad + (N-1)\mu u\big(2u_r + \frac{N-2+L}{r}u\big)r^{N-2+L}
\end{aligned}
$$

$$\leqq -\frac{1}{2}(\rho+\alpha\delta\rho^{\alpha})u_r^2 r^{N-1+L} + C(T, M_1) r^{N-3+L}\rho u^2. \tag{4.27}$$

4. Substituting (4.24) and (4.26)–(4.27) into (4.23) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\delta}^{b}\big(\frac{1}{2}\rho u^2 + e(\rho,\bar\rho)\big)\, r^{N-1+L}\mathrm{d}r + \frac{\varepsilon}{2}\int_{\delta}^{b}(\rho+\alpha\delta\rho^{\alpha})u_r^2\, r^{N-1+L}\mathrm{d}r\\
&\quad\leqq C(T, M_1)\Big\{\int_{\delta}^{b}\big(\rho u^2 + e(\rho,\bar\rho)\big)\, r^{N-2+L}\mathrm{d}r + \Big(\int_{\delta}^{b}\rho u^2 r^{N-2+L}\,\mathrm{d}r\Big)^{\frac{4}{3}}\\
&\qquad\qquad + \int_{\delta}^{b}\rho u_r^2\, r^{N-1}\mathrm{d}r\Big\}.
\end{aligned} \tag{4.28}$$

5. Taking $L = 1$ in (4.28), integrating the resultant inequality over $[0, t]$, and using (3.7) yield

$$\begin{aligned}
&\int_{\delta}^{b}\big(\frac{1}{2}\rho u^2 + e(\rho,\bar\rho)\big)\, r^{N}\mathrm{d}r + \frac{\varepsilon}{2}\int_0^t\int_{\delta}^{b}(\rho+\alpha\delta\rho^{\alpha})u_r^2\, r^{N}\mathrm{d}r\mathrm{d}s\\
&\quad\leqq \int_{\delta}^{b}\big(\frac{1}{2}\rho_0 u_0^2 + e(\rho_0,\bar\rho)\big)\, r^{N}\mathrm{d}r + C(T, M_1)\\
&\quad\leqq C(T, M_2) \qquad\qquad \text{for all } t\in[0,T].
\end{aligned}$$

Then, taking $L = 2, 3, \ldots, N-1$ in (4.28) step by step, we have

$$\begin{aligned}
&\int_{\delta}^{b}\big(\frac{1}{2}\rho u^2 + e(\rho,\bar\rho)\big)\, r^{2N-2}\mathrm{d}r + \frac{\varepsilon}{2}\int_0^t\int_{\delta}^{b}(\rho+\alpha\delta\rho^{\alpha})u_r^2\, r^{2N-2}\mathrm{d}r\mathrm{d}s\\
&\quad\leqq \int_{\delta}^{b}\big(\frac{1}{2}\rho_0 u_0^2 + e(\rho_0,\bar\rho)\big)\, r^{2N-2}\mathrm{d}r + C(T, M_2)\\
&\quad\leqq C(T, M_2) \qquad\qquad \text{for all } t\in[0,T].
\end{aligned} \tag{4.29}$$

Finally, taking $L = N - 1 + \vartheta$ with $\vartheta\in(0,1)$ in (4.28) and integrating it over $[0, t]$, then it follows from (4.29) that

$$\begin{aligned}
&\int_{\delta}^{b}\big(\frac{1}{2}\rho u^2 + e(\rho,\bar\rho)\big)\, r^{2N-2+\vartheta}\mathrm{d}r + \varepsilon\int_0^t\int_{\delta}^{b}(\rho+\alpha\delta\rho^{\alpha})u_r^2\, r^{2N-2+\vartheta}\mathrm{d}r\mathrm{d}s\\
&\quad\leqq \int_{\delta}^{b}\big(\frac{1}{2}\rho_0 u_0^2 + e(\rho_0,\bar\rho)\big)\, r^{2N-2+\vartheta}\mathrm{d}r + C(T, M_2)\\
&\quad\leqq C(T, M_2) \qquad\qquad \text{for all } t\in[0,T].
\end{aligned}$$

This completes the proof. □

Lemma 4.7 (Decay Estimates). *Any smooth solution of* (3.1)–(3.4) *satisfies that, for all* $r\in[1,b]$,

$$|(\rho-\bar\rho)(t,r)| \leqq C(T, M_2) r^{-\frac{3}{4}(N-1)-\frac{\vartheta}{4}}, \tag{4.30}$$

$$\int_0^T \big(|u(t,r)| + |u(t,r)|^3\big)\,\mathrm{d}t \leqq C(T, M_2) r^{-N+1-\frac{\vartheta}{2}} \quad \textit{for any } T>0. \tag{4.31}$$

Proof. It follows from (3.12), (4.4), (4.19), and (4.22) that, for all $t \in [0, T]$,

$$\int_1^b \left((|(\rho-\bar{\rho})(t,r)|^2 + |u(t,r)|^2)r^{N-1+\vartheta} + |\rho_r(t,r)|^2\right) r^{N-1}\mathrm{d}r \\ + \int_0^T \int_1^b |u_r(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r\mathrm{d}t \leqq C(T, M_2). \tag{4.32}$$

For any $r \in [n, n+1] \cap [1, b]$ with $n+1 \leqq [b]$, it follows from (4.32) and the Sobolev inequality that

$$\begin{aligned} |(\rho-\bar{\rho})(t,r)|^2 &\leqq 2\Big(\int_n^{n+1} |(\rho-\bar{\rho})(t,r)|^2 \mathrm{d}r\Big)^{\frac{1}{2}}\Big(\int_n^{n+1} |\rho_r(t,r)|^2\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\quad + \int_n^{n+1} |(\rho-\bar{\rho})(t,r)|^2 \mathrm{d}r \\ &\leqq C n^{-\frac{3}{2}(N-1)-\frac{\vartheta}{2}} \Big(\int_n^{n+1} |(\rho-\bar{\rho})(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\quad \times \Big(\int_n^{n+1} |\rho_r(t,r)|^2 \, r^{N-1}\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\quad + n^{-2(N-1)-\vartheta} \int_n^{n+1} |(\rho-\bar{\rho})(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d} \\ &\leqq C(T, M_2) r^{-\frac{3}{2}(N-1)-\frac{\vartheta}{2}}. \end{aligned}$$

Similarly, for $r \in [n, n+1] \cap [1, b]$ with $n+1 \leqq [b]$, it follows from (4.25) and (4.32) that

$$\begin{aligned} |u(t,r)|^2 &\leqq C r^{-2(N-1)-\vartheta} \int_n^{n+1} |u(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r \\ &\quad + C r^{-2(N-1)-\vartheta}\Big(\int_n^{n+1} |u(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\quad \times \Big(\int_n^{n+1} |u_r(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\leqq C(T, M_2) r^{-2(N-1)-\vartheta}\Big(\Big(\int_n^{n+1} |u_r(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r\Big)^{\frac{1}{2}} + 1\Big), \end{aligned}$$

which yields

$$|u(t,r)| + |u(t,r)|^3 \\ \leqq C(T, M_2) r^{-N+1-\frac{\vartheta}{2}}\Big(\int_n^{n+1} |u_r(t,r)|^2 \, r^{2(N-1)+\vartheta}\mathrm{d}r + 1\Big). \tag{4.33}$$

Integrating (4.33) over $[0, T]$, we obtain

$$\int_0^T \left(|u(t,r)| + |u(t,r)|^3\right) \mathrm{d}t \leqq C(T, M_2) r^{-N+1-\frac{\vartheta}{2}} \qquad \text{for any } r \in [1, [b]].$$

Finally, we consider the case that $r \in [b-1, b]$. Then, by the same arguments as above, we see that, for $r \in [b-1, b]$,

$$|(\rho-\bar{\rho})(r)| \leqq C(T, M_2) b^{-\frac{3}{2}(N-1)-\frac{\vartheta}{2}},$$
$$\int_0^T \big(|u(t,r)| + |u(t,r)|^3\big)\mathrm{d}t \leqq C(T, M_2) b^{-N+1-\frac{\vartheta}{2}}.$$

Combining all the above estimates, we prove (4.30)–(4.31). This completes the proof. □

Choosing $\psi(s) = \frac{1}{2}s|s|$ in (2.7) leads to the corresponding entropy pair as

$$\begin{cases} \eta^{\#}(\rho, m) = \dfrac{1}{2}\rho \displaystyle\int_{-1}^{1} (u+\rho^\theta s)|u+\rho^\theta s|[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s, \\ q^{\#}(\rho, m) = \dfrac{1}{2}\rho \displaystyle\int_{-1}^{1} (u+\theta\rho^\theta s)(u+\rho^\theta s)|u+\rho^\theta s|\,[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s, \end{cases} \tag{4.34}$$

where $\mathfrak{b} = \frac{3-\gamma}{2(\gamma-1)}$, $\theta = \frac{\gamma-1}{2}$, and $m = \rho u$ as indicated earlier. Then a direct calculation shows

$$|\eta^{\#}(\rho, m)| \leqq C_\gamma\big(\rho|u|^2 + \rho^\gamma\big), \quad q^{\#}(\rho, m) \geq C_\gamma^{-1}\big(\rho|u|^3 + \rho^{\gamma+\theta}\big), \tag{4.35}$$

where and whereafter $C_\gamma > 0$ is a universal constant depending only on $\gamma > 1$.

Moreover, notice that

$$\begin{aligned} \partial_\rho \eta^{\#} &= \int_{-1}^{1} \big(-\frac{1}{2}u + (\theta+\frac{1}{2})\rho^\theta s\big)|u+\rho^\theta s|\,[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s, \\ \partial_m \eta^{\#} &= \int_{-1}^{1} |u+\rho^\theta s|\,[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s. \end{aligned} \tag{4.36}$$

Then

$$\begin{aligned} &|\eta^{\#}_m| \leqq C_\gamma(|u| + \rho^\theta), \quad |\eta^{\#}_\rho| \leqq C_\gamma(|u|^2 + \rho^{2\theta}), \\ &\eta^{\#}_\rho(\rho, 0) = 0, \quad \eta^{\#}_m(\rho, 0) = 2\rho^\theta \int_0^1 s[1-s^2]_+^{\mathfrak{b}}\mathrm{d}s. \end{aligned} \tag{4.37}$$

Now we define the relative entropy pair as

$$\begin{cases} \tilde{\eta}(\rho, m) = \eta^{\#}(\rho, m) - \eta^{\#}(\bar{\rho}, 0) - \eta^{\#}_m(\bar{\rho}, 0)m, \\ \tilde{q}(\rho, m) = q^{\#}(\rho, m) - q^{\#}(\bar{\rho}, 0) - \eta^{\#}_m(\bar{\rho}, 0)\big(\dfrac{m^2}{\rho} + p(\rho) - p(\bar{\rho})\big). \end{cases} \tag{4.38}$$

With these, we have the following useful lemma:

Lemma 4.8. *The relative entropy pair* $(\tilde{\eta}, \tilde{q})$ *satisfies*

$$m\partial_\rho\tilde{\eta}(\rho, m) + \frac{m^2}{\rho}\partial_m\tilde{\eta}(\rho, m) - \tilde{q}(\rho, m) \leqq C_\gamma(\bar{\rho})\big(\frac{m^2}{\rho} + e(\rho, \bar{\rho})\big), \tag{4.39}$$

where $C_\gamma(\bar{\rho}) > 0$ *is a positive constant depending only on* $(\gamma, \bar{\rho})$.

Proof. The estimate for (4.39) is very subtle, which will be used to overcome the singularity from the far-field in the M-D case, different from the 1-D case. The proof is divided into three steps.

1. *Claim*: $(\eta^\#, q^\#)$ satisfies

$$
\begin{aligned}
&m\partial_\rho \eta^\#(\rho, m) + \frac{m^2}{\rho}\partial_m \eta^\#(\rho, m) - q^\#(\rho, m)\\
&\leqq \min\left\{0, -q^\#(\rho, 0) + C_\gamma \rho^{\theta-1} m^2\right\},
\end{aligned}
\tag{4.40}
$$

where $q^\#(\rho, 0) = \theta\rho^{3\theta+1}\int_0^1 s^3[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s$.
A direct calculation shows that

$$
\begin{aligned}
&m\partial_\rho \eta^\#(\rho, m) + \frac{m^2}{\rho}\partial_m \eta^\#(\rho, m) - q^\#(\rho, m)\\
&= \frac{\theta}{2}\rho^{1+\theta}\int_{-1}^1 (u - \rho^\theta s)s|u + \rho^\theta s|[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s.
\end{aligned}
\tag{4.41}
$$

Now we divide the proof into three cases.

Case 1. $u \geqq 0$ and $|u| \geqq \rho^\theta$. For this case, $u + \rho^\theta s \geqq 0$ for $s \in [-1, 1]$. Then

$$
m\partial_\rho \eta^\# + \frac{m^2}{\rho}\partial_m \eta^\# - q^\# = 0. \tag{4.42}
$$

On the other hand, we have

$$
\begin{aligned}
m\partial_\rho \eta^\# + \frac{m^2}{\rho}\partial_m \eta^\# - q^\# &= 0 = -q^\#(\rho, 0) + q^\#(\rho, 0)\\
&= -q^\#(\rho, 0) + \theta\int_0^1 s^3[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s\rho^{1+3\theta}\\
&\leqq -q^\#(\rho, 0) + C_\gamma \rho^{\theta-1} m^2,
\end{aligned}
\tag{4.43}
$$

where we have used that $\rho^\theta \leqq |u|$ in the last inequality.

Case 2. $u \geqq 0$ and $|u| < \rho^\theta$. For this case, $s_0 := -\frac{u}{\rho^\theta} \in (-1, 0]$, which implies that $u^2 - s^2\rho^{2\theta} \leqq 0$ for $s \geqq |s_0|$. Then

$$
m\partial_\rho \eta^\# + \frac{m^2}{\rho}\partial_m \eta^\# - q^\# = \theta\rho^{1+\theta}\int_{|s_0|}^1 (u^2 - s^2\rho^{2\theta})s[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s \leqq 0. \tag{4.44}
$$

On the other hand, we have

$$
\begin{aligned}
&m\partial_\rho \eta^\# + \frac{m^2}{\rho}\partial_m \eta^\# - q^\#\\
&= \theta\rho^{1+\theta}\int_{|s_0|}^1 (u^2 - s^2\rho^{2\theta})s[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \theta\rho^{1+\theta}u^2 \int_{|s_0|}^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s - \theta\rho^{1+3\theta}\int_0^1 s^3[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s \\
&\quad + \theta\rho^{1+3\theta}\int_0^{|s_0|} s^3[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s \\
&\leqq -q^{\#}(\rho,0) + C_\gamma\rho^{1+\theta}u^2 + C_\gamma\rho^{1+3\theta}|s_0|^2 \\
&\leqq -q^{\#}(\rho,0) + C_\gamma\rho^{\theta-1}m^2.
\end{aligned}
\tag{4.45}
$$

Case 3. $u \leqq 0$. Similar to (4.42)–(4.45), we also obtain (4.40).
Combining Cases 1–3, we conclude the claim for (4.40).

2. Claim: $(\eta^{\#}, q^{\#})$ satisfies

$$
\begin{aligned}
&\eta_m^{\#}(\bar\rho,0)(p(\rho)-p(\bar\rho)) - q^{\#}(\rho,0) + q^{\#}(\bar\rho,0) \\
&= 2\bar\rho^{\theta}\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\big(p(\rho)-p(\bar\rho)-p'(\bar\rho)(\rho-\bar\rho)\big) \\
&\quad - \frac{4\theta^2}{3\gamma-1}\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\big(\rho^{1+3\theta}-\bar\rho^{1+3\theta}-(1+3\theta)\bar\rho^{3\theta}(\rho-\bar\rho)\big),
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
&\eta_m^{\#}(\bar\rho,0)\big(p(\rho)-p(\bar\rho)\big) + q^{\#}(\bar\rho,0) \\
&= \int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\Big(2\bar\rho^{\theta}p(\rho) - \frac{4\theta^3}{\gamma(3\gamma-1)}\bar\rho^{\gamma+\theta}\Big).
\end{aligned}
\tag{4.47}
$$

A direct calculation shows that

$$
\begin{aligned}
\int_0^1 s^3[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s &= \frac{1}{2}B(2,1+\mathfrak{b}) = \frac{1}{2(2+\mathfrak{b})}B(1,1+\mathfrak{b}) \\
&= \frac{1}{2+\mathfrak{b}}\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s,
\end{aligned}
\tag{4.48}
$$

where we have used the properties of the beta function $B(\cdot,\cdot)$. Using (4.48), we have

$$
\begin{aligned}
&\eta_m^{\#}(\bar\rho,0)\big(p(\rho)-p(\bar\rho)\big) - q^{\#}(\rho,0) + q^{\#}(\bar\rho,0) \\
&= 2\bar\rho^{\theta}\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\big(p(\rho)-p(\bar\rho)-p'(\bar\rho)(\rho-\bar\rho)\big) \\
&\quad - \frac{\theta}{2+\mathfrak{b}}\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\Big(\rho^{1+3\theta}-\bar\rho^{1+3\theta}-\gamma\kappa\frac{2(2+\mathfrak{b})}{\theta}\bar\rho^{3\theta}(\rho-\bar\rho)\Big).
\end{aligned}
\tag{4.49}
$$

Combining $2+\mathfrak{b} = \frac{3\gamma-1}{4\theta}$ and $\gamma\kappa\frac{2(2+\mathfrak{b})}{\theta} = 1+3\theta$ with (4.49), we conclude (4.46).
For (4.47), we note that

$$
\begin{aligned}
&\eta_m^{\#}(\bar\rho,0)\big(p(\rho)-p(\bar\rho)\big) + q^{\#}(\bar\rho,0) \\
&= \int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\Big(2\bar\rho^{\theta}p(\rho) + (\frac{\theta}{2+\mathfrak{b}} - 2\kappa)\bar\rho^{\gamma+\theta}\Big)
\end{aligned}
$$

$$= \int_0^1 s[1-s^2]_+^{\mathfrak{b}} \, \mathrm{d}s \Big(2p(\rho) - \frac{4\theta^3}{\gamma(3\gamma-1)} \bar{\rho}^{\gamma} \Big) \bar{\rho}^{\theta},$$

which implies (4.47).

3. Noting (2.10) and (4.5), we have

$$\begin{aligned} e(\rho,\bar{\rho}) I_{A(t)}(r) &\geqq C_\gamma \rho (\rho^\theta - \bar{\rho}^\theta)^2 \, I_{A(t)}(r) \\ &\geqq C_\gamma \rho (1 - \frac{1}{2^\theta})^2 \rho^{2\theta} \, I_{A(t)}(r) \\ &\geqq C_\gamma \, p(\rho) \, I_{A(t)}(r). \end{aligned} \tag{4.50}$$

If $r \in A(t)$, then it follows from (4.40) and (4.47) that

$$\begin{aligned} &\big(m\partial_\rho \tilde{\eta}(\rho,m) + \frac{m^2}{\rho} \partial_m \tilde{\eta}(\rho,m) - \tilde{q}(\rho,m)\big) I_{A(t)}(r) \\ &= \Big\{ \big(m\partial_\rho \eta^\#(\rho,m) + \frac{m^2}{\rho}\partial_m \eta^\#(\rho,m) - q^\#(\rho,m)\big) \\ &\qquad + \eta^\#_m(\bar{\rho},0) \cdot (p(\rho) - p(\bar{\rho})) + q^\#(\bar{\rho},0) \Big\} I_{A(t)}(r) \\ &\leqq \int_0^1 s[1-s^2]_+^{\mathfrak{b}} \, \mathrm{d}s \Big(2\bar{\rho}^\theta p(\rho) - \frac{4\theta^3}{\gamma(3\gamma-1)} \bar{\rho}^{\gamma+\theta} \Big) I_{A(t)}(r) \\ &\leqq C_\gamma(\bar{\rho}) p(\rho) I_{A(t)}(r) \\ &\leqq C_\gamma(\bar{\rho}) e(\rho,\bar{\rho}) \, I_{A(t)}(r), \end{aligned} \tag{4.51}$$

where (4.50) has been used in the last inequality.

On the other hand, for $r \in A^c(t) = [\delta, b] \setminus A(t)$, it follows from (4.40) and (4.46) that

$$\begin{aligned} &\big(m\partial_\rho \tilde{\eta}(\rho,m) + \frac{m^2}{\rho}\partial_m \tilde{\eta}(\rho,m) - \tilde{q}(\rho,m)\big) I_{A^c(t)}(r) \\ &= \Big\{ \big(m\partial_\rho \eta^\#(\rho,m) + \frac{m^2}{\rho}\partial_m \eta^\#(\rho,m) - q^\#(\rho,m)\big) \\ &\qquad + \eta^\#_m(\bar{\rho},0)\,(p(\rho) - p(\bar{\rho})) + q^\#(\bar{\rho},0) \Big\} I_{A^c(t)}(r) \\ &\leqq \Big\{ q^\#(\bar{\rho},0) + \eta^\#_m(\bar{\rho},0)\,(p(\rho)-p(\bar{\rho})) - q^\#(\rho,0) \Big\} I_{A^c(t)}(r) + C_\gamma \rho^{\theta-1} m^2 I_{A^c(t)}(r) \\ &= 2\bar{\rho}^\theta \int_0^1 s[1-s^2]_+^{\mathfrak{b}} \, \mathrm{d}s \, \big(p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \big) I_{A^c(t)}(r) \\ &\quad - \frac{4\theta^2}{3\gamma-1} \int_0^1 s[1-s^2]_+^{\mathfrak{b}} \, \mathrm{d}s \, \big(\rho^{1+3\theta} - \bar{\rho}^{1+3\theta} - (1+3\theta)\bar{\rho}^{3\theta}(\rho-\bar{\rho}) \big) I_{A^c(t)}(r) \\ &\quad + C_\gamma \rho^{\theta-1} m^2 I_{A^c(t)}(r) \\ &\leqq 2\bar{\rho}^\theta \int_0^1 s[1-s^2]_+^{\mathfrak{b}} \, \mathrm{d}s \, \big(p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho-\bar{\rho}) \big) I_{A^c(t)}(r) \\ &\quad + C_\gamma \rho^{\theta-1} m^2 I_{A^c(t)}(r) \\ &\leqq C_\gamma \big(\rho^{\theta-1} m^2 + e(\rho,\bar{\rho}) \big) I_{A^c(t)}(r) \\ &\leqq C_\gamma(\bar{\rho}) \big(\frac{m^2}{\rho} + e(\rho,\bar{\rho}) \big) I_{A^c(t)}(r), \end{aligned} \tag{4.52}$$

where we have used (2.3) and $\rho^\theta(t,r) \leqq (2\bar{\rho})^\theta$ for $r \in A^c(t)$. Combining (4.51) with (4.52), we conclude (4.39). $\square$

Now we are in the position to prove the key estimate, Proposition 4.1.

Proof of Proposition 4.1. We divide the proof into six steps.

1. For $\tilde{\eta}(\rho, m)$ defined in (4.38), we multiply $(3.1)_1$ by $r^{N-1}\partial_\rho\tilde{\eta}(\rho,m)$ and $(3.1)_2$ by $r^{N-1}\partial_m\tilde{\eta}(\rho,m)$ to obtain

$$
\begin{aligned}
&(r^{N-1}\tilde{\eta})_t + (r^{N-1}\tilde{q})_r + (N-1)r^{N-2}\big(-\tilde{q} + m\partial_\rho\tilde{\eta} + \frac{m^2}{\rho}\partial_m\tilde{\eta}\big)\\
&\quad = \varepsilon r^{N-1}\partial_m\tilde{\eta}\Big\{\big((\rho+\alpha\delta\rho^\alpha)(u_r + \frac{N-1}{r}u)\big)_r - \frac{N-1}{r}(\rho+\delta\rho^\alpha)_r u\Big\}. \qquad (4.53)
\end{aligned}
$$

Let $y \in [b-1, b]$ and $r \in [d, D]$. Integrating (4.53) over $[r, y]$ leads to

$$
\begin{aligned}
&\tilde{q}(t,r)r^{N-1}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_r^y \tilde{\eta}(t,z)\,z^{N-1}\mathrm{d}z + \tilde{q}(t,y)y^{N-1}\\
&\quad + (N-1)\int_r^y \big(-\tilde{q} + m\partial_\rho\tilde{\eta} + \frac{m^2}{\rho}\partial_m\tilde{\eta}\big)(t,z)\,z^{N-2}\mathrm{d}z\\
&\quad - \varepsilon\int_r^y \partial_m\tilde{\eta}\,\Big\{\big((\rho+\alpha\delta\rho^\alpha)(u_z + \frac{N-1}{z}u)\big)_z - \frac{N-1}{z}(\rho+\delta\rho^\alpha)_z u\Big\}\,z^{N-1}\mathrm{d}z. \qquad (4.54)
\end{aligned}
$$

Integrating (4.54) over $[0,T]\times[b-1,b]\times[d,D]$, we have

$$
\begin{aligned}
&\int_0^T\int_d^D \tilde{q}(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&\quad = (N-1)\int_0^T\int_{b-1}^b\int_d^D\int_r^y \big(m\partial_\rho\tilde{\eta} + \frac{m^2}{\rho}\partial_m\tilde{\eta} - \tilde{q}\big)(t,z)\,z^{N-2}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\qquad + \int_{b-1}^b\int_d^D\int_r^y \big(\tilde{\eta}(T,z) - \tilde{\eta}(0,z)\big)\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\\
&\qquad + (D-d)\int_0^T\int_{b-1}^b \tilde{q}(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\qquad - \varepsilon\int_0^T\int_{b-1}^b\int_d^D\int_r^y \partial_m\tilde{\eta}\,\Big\{\big((\rho+\alpha\delta\rho^\alpha)(u_z + \frac{N-1}{z}u)\big)_z\\
&\qquad\qquad\qquad - \frac{N-1}{z}(\rho+\delta\rho^\alpha)_z u\Big\}\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\quad =: \sum_{j=1}^4 J_j. \qquad (4.55)
\end{aligned}
$$

2. For J_1 in (4.55), it follows from (3.7) and Lemma 4.8 that

$$J_1 \leqq C_\gamma(\bar\rho)\frac{D}{d}\int_0^T\int_d^b \big(\rho u^2+e(\rho,\bar\rho)\big)(t,z)\,z^{N-1}\mathrm{d}z\mathrm{d}t$$
$$\leqq C_\gamma(\bar\rho)\frac{DT}{d}(E_0+1). \tag{4.56}$$

3. For J_2 in (4.55), we first note that $|\partial_{mm}\eta^\#(\rho,m)| \leqq \frac{2}{\rho}\int_0^1[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s$. This, combining (4.36) and (4.37) with the Taylor expansion of $\eta^\#(\rho,m)$ at $m=0$, yields

$$\eta^\#(\rho,m) = 2\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\rho^\theta m + R_1(\rho,m) \tag{4.57}$$

with

$$|R_1(\rho,m)| \leqq C_\gamma\frac{m^2}{\rho}. \tag{4.58}$$

Then it follows from (2.10), (4.37)–(4.38), and (4.57)–(4.58) that

$$|\tilde\eta(\rho,m)| \leqq 2\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,|m(\rho^\theta-\bar\rho^\theta)| + |R_1(\rho,m)| \leqq C_\gamma\big(\frac{m^2}{\rho}+e(\rho,\bar\rho)\big),$$

which, along with (3.7), implies

$$|J_2| = \left|\int_{b-1}^b\int_d^D\int_r^y \big(\tilde\eta(T,z)-\tilde\eta(0,z)\big)\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\right| \leqq C_\gamma D(E_0+1). \tag{4.59}$$

4. For the third term J_3 in (4.55), we need to use the decay properties obtained in Lemma 4.7. A direct calculation shows that

$$|q^\#(\rho,m)-q^\#(\rho,0)| \leqq C_\gamma\big(\frac{|m|^3}{\rho^2}+\rho^{2\theta}|m|\big),$$

which, with (4.46), yields

$$\tilde q(\rho,m) = \frac{4\theta^2}{3\gamma-1}\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\big(\rho^{1+3\theta}-\bar\rho^{1+3\theta}-(1+3\theta)\bar\rho^{3\theta}(\rho-\bar\rho)\big)$$
$$-2\bar\rho^\theta\int_0^1 s[1-s^2]_+^{\mathfrak{b}}\,\mathrm{d}s\,\big(\rho u^2+p(\rho)-p(\bar\rho)-p'(\bar\rho)(\rho-\bar\rho)\big)$$
$$+\big(q^\#(\rho,\rho u)-q^\#(\rho,0)\big)$$
$$\leqq C(T,M_2)\big(|\rho-\bar\rho|^2+|u|^3+|u|\big), \tag{4.60}$$

where we have used the Taylor expansion, (4.4), and (4.19) in the last inequality. Now it follows from (4.60) and Lemma 4.7 that

$$|J_3| \leqq C(T,M_2)D\int_{b-1}^b\int_0^T \big(|\rho-\bar\rho|^2+|u|^3+|u|\big)(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}t$$
$$\leqq C(T,M_2)b^{-\frac{\vartheta}{2}}. \tag{4.61}$$

5. For J_4 in (4.55), we regard $\tilde{\eta}_m(\rho, \rho u)$ as a function of (ρ, u) to obtain

$$
\begin{aligned}
&|\partial_m \tilde{\eta}(\rho, \rho u)| \leqq C_\gamma\big(|u| + |\rho^\theta - \bar{\rho}^\theta|\big),\\
&|\partial_{mu} \tilde{\eta}(\rho, \rho u)| \leqq C_\gamma, \quad |\partial_{m\rho} \tilde{\eta}(\rho, \rho u)| \leqq C_\gamma \rho^{\theta-1},
\end{aligned}
\tag{4.62}
$$

which, with integration by parts, leads to

$$
\begin{aligned}
|J_4| &\leqq \varepsilon \int_0^T \int_{b-1}^b \int_d^D \Big| \int_r^y z^{N-1} \partial_m \tilde{\eta} \Big\{ \big((\rho + \alpha\delta\rho^\alpha) u_z\big)_z + (\rho + \alpha\delta\rho^\alpha)(\frac{N-1}{z} u)_z \\
&\qquad\qquad + (\alpha - 1)\delta(\rho^\alpha)_z \frac{N-1}{z} u \Big\} \,\mathrm{d}z \Big| \,\mathrm{d}y \mathrm{d}r \mathrm{d}t \\
&\leqq C\varepsilon \int_0^T \int_{b-1}^b \int_d^D \int_r^y \Big((\rho + \delta\rho^\alpha)\big(|u_z (z^{N-1} \partial_m \tilde{\eta})_z| + |\partial_m \tilde{\eta}\, (\frac{u}{z})_z| z^{N-1}\big) \\
&\qquad\qquad + \delta\rho^\alpha \big|(z^{N-1} \partial_m \tilde{\eta})_z \frac{u}{z}\big| \Big) \,\mathrm{d}z \mathrm{d}r \mathrm{d}y \mathrm{d}t \\
&\quad + C\varepsilon \int_0^T \int_d^D \Big(\big|\big(r^{N-1} (\rho + \delta\rho^\alpha) \partial_m \tilde{\eta} u_r\big)(t, r)\big| \\
&\qquad\qquad + \delta \big|\big(r^{N-2} \rho^\alpha u \partial_m \tilde{\eta}\big)(t, r)\big| \Big) \,\mathrm{d}r \mathrm{d}t \\
&\quad + CD\varepsilon \int_0^T \int_{b-1}^b \Big(\big|\big(y^{N-1} (\rho + \delta\rho^\alpha) \partial_m \tilde{\eta} u_y\big)(t, y)\big| \\
&\qquad\qquad + \delta \big|\big(y^{N-2} \rho^\alpha u \partial_m \tilde{\eta}\big)(t, y)\big| \Big) \,\mathrm{d}y \mathrm{d}t.
\end{aligned}
\tag{4.63}
$$

In order to estimate the terms on the right-hand side of (4.63), we notice that

$$
e(\rho, \bar{\rho})\, I_{B(t)}(r) \geqq C(\bar{\rho})^{-1},
\tag{4.64}
$$

where $B(t)$ is defined in (4.20). Then combining (4.64) with (2.10), (3.7), and (4.21) implies that

$$
\begin{aligned}
&\int_d^b \big(\rho^\alpha (\rho^\theta - \bar{\rho}^\theta)^2\big)(t, r)\, r^{N-1} \mathrm{d}r \\
&\leqq C(\bar{\rho}) \int_d^b I_{B^c(t)}(r) \big(\rho (\rho^\theta - \bar{\rho}^\theta)^2\big)(t, r)\, r^{N-1} \mathrm{d}r \\
&\quad + \int_d^b I_{B(t)}(r) \big(\rho^\alpha (\rho^\theta - \bar{\rho}^\theta)^2\big)(t, r)\, r^{N-1} \mathrm{d}r \\
&\leqq C(\bar{\rho}) \int_d^b e(\rho, \bar{\rho})(t, r)\, r^{N-1} \mathrm{d}r + C(\bar{\rho}) \int_d^b I_{B(t)}(r)\, r^{N-1} \mathrm{d}r \\
&\leqq C(\bar{\rho}) \int_d^b e(\rho, \bar{\rho})(t, r)\, r^{N-1} \mathrm{d}r + C(\bar{\rho}) \int_d^b I_{B(t)}(r)\, e(\rho, \bar{\rho})(t, r)\, r^{N-1} \mathrm{d}r \\
&\leqq C(\bar{\rho}, E_0).
\end{aligned}
\tag{4.65}
$$

Combining (4.62) and (4.65) with (3.7), (3.12), and the Cauchy inequality, we conclude that the first term on the right-hand side of (4.63) are bounded by

$$
\begin{aligned}
&C(d)\int_0^T\int_{b-1}^b\int_d^D\int_r^y \Big\{\varepsilon z^2(\rho+\delta\rho^\alpha)(u_z^2+\rho^{\gamma-3}\rho_z^2)+\varepsilon\delta\rho^\alpha u^2\\
&\qquad +\rho^\alpha(\rho^\theta-\bar\rho^\theta)^2+z^2\big(\rho u^2+e(\rho,\bar\rho)\big)\Big\}\,z^{N-3}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\leqq C(\bar\rho,d,D,T,E_0).
\end{aligned}
\tag{4.66}
$$

Using (3.7), (3.12), (4.65), and the Cauchy inequality, we can bound the second term on the right-hand side of (4.63) by

$$
\begin{aligned}
&C(D,d)\int_0^T\int_d^D \Big\{\varepsilon(\rho+\delta\rho^\delta)u_r^2+\varepsilon\delta\rho^\alpha\frac{u^2}{r^2}+(\rho u^2+e(\rho,\bar\rho))\\
&\qquad\qquad +\rho^\alpha(\rho^\theta-\bar\rho^\theta)^2\Big\}\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&\leqq C(\bar\rho,d,D,T,E_0).
\end{aligned}
\tag{4.67}
$$

Using (3.7), (3.12), (4.4), (4.19), (4.65), the Cauchy inequality, and Lemma 4.7, the last term on the right-hand side of (4.63) can be bounded by

$$
\begin{aligned}
&\frac{CD}{d}\int_0^T\int_{b-1}^b \Big(\varepsilon(\rho+\delta\rho^\alpha)|u_y|^2+\big(\rho u^2+e(\rho,\bar\rho)\big)\Big)\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\qquad +C(M_2,T)\int_0^T\int_{b-1}^b |u(t,y)|^2\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\leqq C(\bar\rho,d,D,T,E_0)+C(T,M_2)b^{-\frac{\vartheta}{2}}.
\end{aligned}
\tag{4.68}
$$

6. Substituting (4.56), (4.59), (4.61), and (4.63)–(4.68), we have

$$
\int_0^T\int_d^D r^{N-1}\tilde q(t,r)\,\mathrm{d}r\mathrm{d}t \leqq C(\bar\rho,d,D,T,E_0)+C(T,M_2)b^{-\frac{\vartheta}{2}}. \tag{4.69}
$$

Then (4.3) follows from (3.7), (4.35), and (4.69). This completes the proof. □

Employing Proposition 4.1, we can obtain the following higher integrability estimate up to the origin:

Lemma 4.9. *The smooth solution of* (3.1)–(3.4) *satisfies*

$$
\int_0^T\int_\delta^1 \big(\rho|u|^3+\rho^{\gamma+\theta}\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t \leqq C(T,E_0)+C(T,M_2)\,b^{-\frac{\vartheta}{2}}. \tag{4.70}
$$

Proof. Let $w(r)$ be a smooth non-negative cut-off function with $\operatorname{supp} w\subset[0,2]$ and $w(r)\equiv 1$ for $r\in[0,1]$. Multiplying $(3.1)_1$ by $w\partial_\rho\eta^\#(\rho,m)\,r^{N-1}$ and $(3.1)_2$ by $w\partial_m\eta^\#(\rho,m)r^{N-1}$, we have

$$
\begin{aligned}
&(w\eta^{\#}r^{N-1})_t + (wq^{\#}r^{N-1})_r - w_r q^{\#}r^{N-1}\\
&\quad + (N-1)w\big(-q^{\#} + m\partial_\rho\eta^{\#} + \frac{m^2}{\rho}\partial_m\eta^{\#}\big)r^{N-2}\\
&= \varepsilon w\partial_m\eta^{\#}\left\{\big((\rho+\alpha\delta\rho^{\alpha})(u_r + \frac{N-1}{r}u)\big)_r - \frac{N-1}{r}(\rho+\delta\rho^{\alpha})_r u\right\}r^{N-1}.
\end{aligned}
\tag{4.71}
$$

Integrating (4.71) over $[r, 2]$ with $r \leqq 2$, and then integrating the resultant equation over $[0, T] \times [\delta, 2]$ and using (4.40), we have

$$
\begin{aligned}
&\int_0^T \int_\delta^2 w(r)q^{\#}(t,r)\,r^{N-1}\mathrm{d}r\\
&\leqq \left|\int_\delta^2\int_r^2 w(y)\eta^{\#}(T,y)\,y^{N-1}\mathrm{d}y\mathrm{d}r - \int_\delta^2\int_r^2 y^{N-1}w(y)\eta^{\#}(0,y)\,y^{N-1}\mathrm{d}y\mathrm{d}r\right|\\
&\quad + \int_0^T\int_\delta^2\int_r^2 w_y(y)q^{\#}(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&\quad - \varepsilon\int_0^T\int_\delta^2\int_r^2 w(y)\partial_m\eta^{\#}\,\big((\rho+\alpha\delta\rho^{\alpha})u_y\big)_y\,y^{N-1}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&\quad - (N-1)\varepsilon\int_0^T\int_\delta^2\int_r^2 w(y)\partial_m\eta^{\#}\,(\rho+\alpha\delta\rho^{\alpha})\,(\frac{u}{y})_y\,y^{N-1}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&\quad - (N-1)(\alpha-1)\varepsilon\delta\int_0^T\int_\delta^2\int_r^2 w(y)\partial_m\eta^{\#}\,(\rho^{\alpha})_y\,\frac{u}{y}\,y^{N-1}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&:= \sum_{j=1}^{5} I_j.
\end{aligned}
\tag{4.72}
$$

For I_1, it follows from (4.35) and Lemma 3.1 that

$$
\begin{aligned}
I_1 &\leqq C\int_\delta^2 \big(\rho|u|^2 + \rho^{\gamma}\big)(T,y)\,y^{N-1}\mathrm{d}y + \int_\delta^2\big(\rho_0|u_0|^2 + \rho_0^{\gamma}\big)(y)\,y^{N-1}\mathrm{d}y\\
&\leqq C\int_\delta^2\big(1 + \frac{1}{2}\rho|u|^2 + e(\rho,\bar{\rho})\big)(t,y)\,y^{N-1}\mathrm{d}y\\
&\quad + C\int_\delta^2\big(1 + \frac{1}{2}\rho_0|u_0|^2 + e(\rho_0,\bar{\rho})\big)(y)\,y^{N-1}\mathrm{d}y\\
&\leqq C\big(E_0 + 1\big).
\end{aligned}
\tag{4.73}
$$

For I_2, we use Proposition 4.1 with $d = 1$ and $D = 2$ to obtain

$$
I_2 \leqq C\int_0^T\int_1^2 q^{\#}(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}t \leqq C(T,E_0) + C(T,M_2)\,b^{-\frac{\vartheta}{2}}. \tag{4.74}
$$

For I_3, we integrate by parts to obtain

$$
I_3 = (N-1)\varepsilon\int_0^T\int_\delta^2\int_r^2 (\rho+\alpha\delta\rho^{\alpha})u_y\,\partial_m\eta^{\#}w(y)\,y^{N-2}\mathrm{d}y\mathrm{d}r\mathrm{d}t
$$

$$
\begin{aligned}
&+\varepsilon \int_0^T \int_\delta^2 \int_r^2 (\rho+\alpha\delta\rho^\alpha)u_y\, \partial_m\eta^\# w_y(y)\, y^{N-1}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&+\varepsilon \int_0^T \int_\delta^2 \int_r^2 (\rho+\alpha\delta\rho^\alpha)u_y\, (\partial_m\eta^\#)_y w(y)\, y^{N-1}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&+\varepsilon \int_0^T \int_\delta^2 (\rho+\alpha\delta\rho^\alpha)u_r\, \partial_m\eta^\# w(r)\, r^{N-1}\mathrm{d}r\mathrm{d}t\\
:=&\sum_{j=1}^4 I_{3j}.
\end{aligned}
\tag{4.75}
$$

We regard $\eta_m^\#(\rho,\rho u)$ as a function of (ρ,u) to see that

$$
|\partial_{mu}\eta^\#(\rho,\rho u)|+\rho^{1-\theta}|\partial_{m\rho}\eta^\#(\rho,\rho u)|\leqq C_\gamma,
$$

which, with (4.37) and Lemmas 3.1–3.2, leads to

$$
\begin{aligned}
\sum_{j=2}^4 I_{3j} \leqq\,& C\int_0^T \int_\delta^2 \varepsilon(\rho+\delta\rho^\alpha)\big(|u_y|^2+\rho^{\gamma-3}\rho_y^2\big)\, y^{N-1}\mathrm{d}y\mathrm{d}t\\
&+\int_0^T \int_\delta^2 \big(\varepsilon(\rho+\delta\rho^\alpha)|u|^2+(\rho^\gamma+\rho^{\alpha+\gamma-1})\big)\, y^{N-1}\mathrm{d}y\mathrm{d}t\\
\leqq\,& C(T,E_0).
\end{aligned}
\tag{4.76}
$$

To estimate I_{31}, we have to be more careful, since the weight is y^{N-2} that may not be enough. Fortunately, we can gain a weight y by changing the order of integration:

$$
\begin{aligned}
I_{31} &= (N-1)\varepsilon\int_0^T \int_\delta^2 (\rho+\alpha\delta\rho^\alpha)u_y\, \partial_m\eta^\# w(y)\,(y-\delta)y^{N-2}\mathrm{d}y\mathrm{d}t\\
&\leqq C\varepsilon\int_0^T \int_\delta^2 (\rho+\alpha\delta\rho^\alpha)|u_y|(|u|+\rho^\theta)\, y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\leqq C(T,E_0).
\end{aligned}
\tag{4.77}
$$

Combining (4.75)–(4.76) with (4.77) yields

$$
I_3 \leqq C(T,E_0). \tag{4.78}
$$

For I_4, using (4.37) and changing the order of integration as in (4.77), we have

$$
\begin{aligned}
I_4 &\leqq C\varepsilon\int_0^T \int_\delta^2 \int_r^2 (|u|+\rho^\theta)(\rho+\alpha\delta\rho^\alpha)\,\Big(|u_y|+\frac{|u|}{y}\Big)\, y^{N-2}\mathrm{d}y\mathrm{d}r\mathrm{d}t\\
&\leqq C(T,E_0)+C\int_0^T \int_\delta^2 \Big(\varepsilon(\rho+\delta\rho^\alpha)\frac{|u|^2}{y^2}+(\rho^\gamma+\rho^{\alpha+\gamma-1})\Big)\, y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\leqq C(T,E_0).
\end{aligned}
\tag{4.79}
$$

Finally, for I_5, we first integrate by parts and then change the order of integration as in (4.77) to obtain

$$\begin{aligned} I_5 &\leqq C(T, E_0) + C\varepsilon\delta \int_0^T \int_\delta^2 \rho^\alpha \big(|u_y|^2 + \rho^{\gamma-3}|\rho_y|^2 + \frac{u^2}{y^2} + \rho^{\gamma-1}\big)\, y^{N-1}\mathrm{d}y\mathrm{d}t \\ &\leqq C(T, E_0). \end{aligned} \tag{4.80}$$

Substituting (4.73)–(4.74) and (4.78)–(4.80) into (4.72), and using (4.35), we conclude (4.70). □

We now prove a lemma which is needed when we take the limit $b \to \infty$.

Lemma 4.10. *The smooth solution of* (3.1)–(3.4) *satisfies that, for any* $t \in [0, T]$,

$$\|u_r(t)\|_{L^2}^2 + \int_0^T \big(\|u_t(t)\|_{L^2}^2 + \|u_{rr}(t)\|_{L^2}^2\big)\, \mathrm{d}t \leqq C(T, \|u_{0r}\|_{L^2}, M_2). \tag{4.81}$$

Proof. It follows from $(3.1)_1$ that

$$-\varepsilon\big((\mu+\lambda)u_r\big)_r + \rho u_t = H, \tag{4.82}$$

where $H := -\rho u u_r - p_r + \varepsilon(\mu+\lambda)\big(\frac{N-1}{r}u\big)_r + \varepsilon\frac{N-1}{r}u\lambda_r$. Multiplying (4.82) by u_t and integrating it over $[\delta, b]$, we have

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b (\mu+\lambda)|u_r|^2\,\mathrm{d}r + \int_\delta^b \rho u_t^2\,\mathrm{d}r = \frac{\varepsilon}{2}\int_\delta^b (\mu+\lambda)_t|u_r|^2\,\mathrm{d}t + \int_\delta^b H u_t\,\mathrm{d}r. \tag{4.83}$$

Using (3.7), (3.12), (4.4), (4.19), and the Sobolev inequality:

$$\|u_r\|_{L^\infty} \leqq C\big(\|u_r\|_{L^2} + \|u_r\|_{L^2}^{\frac12}\|u_{rr}\|_{L^2}^{\frac12}\big),$$

we obtain

$$\begin{aligned} &\frac{\varepsilon}{2}\int_\delta^b (\mu+\lambda)_t|u_r|^2\,\mathrm{d}r \\ &\quad\leqq C(T, M_2)\int_\delta^b \big(|\rho_r u| + |u_r| + |u|\big)|u_r|^2\,\mathrm{d}r \\ &\quad\leqq C(T, M_2)\Big\{\|u\|_{L^2}^{\frac12}\|\rho_r\|_{L^2}\big(\|u_r\|_{L^2}^2\|u_{rr}\|_{L^2}^{\frac12} + \|u_r\|_{L^2}^{\frac52}\big) \\ &\qquad\qquad + \|u\|_{L^2}^{\frac12}\|u_r\|_{L^2}^{\frac52} + \|u_r\|_{L^2}^2\big(\|u_r\|_{L^2} + \|u_r\|_{L^2}^{\frac12}\|u_{rr}\|_{L^2}^{\frac12}\big)\Big\} \\ &\quad\leqq C(T, M_2)\Big\{\big(\|u_r\|_{L^2}^2 + \|u_r\|_{L^2}^{\frac52}\big)\|u_{rr}\|_{L^2}^{\frac12} + \|u_r\|_{L^2}^3 + 1\Big\}, \end{aligned} \tag{4.84}$$

$$\begin{aligned} &\Big|\int_\delta^b H u_t\,\mathrm{d}r\Big| \\ &\quad\leqq \frac18\int_\delta^b \rho|u_t|^2\,\mathrm{d}r + C\int_\delta^b \rho^{-1}|H|^2\,\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&\leqq C(T, M_2)\Big\{\big(\|u\|_{L^\infty}^2+1\big)\|(\rho_r, u_r)\|_{L^2}^2+\|u\|_{L^2}^2\Big\}+\frac{1}{8}\int_\delta^b \rho|u_t|^2\,\mathrm{d}r\\
&\leqq \frac{1}{8}\int_\delta^b \rho|u_t|^2\mathrm{d}r + C(T, M_2)\big(\|u_r\|_{L^2}^3+1\big).
\end{aligned}\tag{4.85}
$$

To close the above estimate, we combine (4.82) with (3.7), (3.12), (4.4), and (4.19) to obtain

$$
\begin{aligned}
\|u_{rr}\|_{L^2}^2 &\leqq C(T, M_2)\Big\{\|\sqrt{\rho}u_t\|_{L^2}^2+\|\rho_r\|_{L^2}^2\|u_r\|_{L^2}\|u_{rr}\|_{L^2}+\|H\|_{L^2}^2\Big\}\\
&\leqq C(T, M_2)\Big\{\|\sqrt{\rho}u_t\|_{L^2}^2+\|u_r\|_{L^2}\|u_{rr}\|_{L^2}+\|u_r\|_{L^2}^3+1\Big\}\\
&\leqq C(T, M_2)\Big\{\|\sqrt{\rho}u_t\|_{L^2}^2+\|u_r\|_{L^2}^3+1\Big\}.
\end{aligned}\tag{4.86}
$$

Combining (4.83)–(4.86), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^b(\mu+\lambda)|u_r|^2\,\mathrm{d}r+\int_\delta^b\rho u_t^2\,\mathrm{d}r \leqq C(T, M_2)\Big\{1+\|u_r\|_{L^2}^2\int_\delta^b(\mu+\lambda)|u_r|^2\,\mathrm{d}r\Big\}.
$$

Applying the Gronwall inequality, we have

$$
\int_\delta^b(\mu+\lambda)|u_r|^2\,\mathrm{d}r+\int_0^t\int_\delta^b\rho u_t^2\,\mathrm{d}r\mathrm{d}s \leqq C(T, \|u_{0r}\|_{L^2}, M_2),
$$

which, with (4.86), implies (4.81). □

5. Limits of the Approximate Solutions for the Navier–Stokes Equations

In this section, we first take the limit, $b \to \infty$, to obtain global strong solutions $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ of the Navier–Stokes equations with some uniform bounds. Then we take the limit, $\delta \to 0+$, to obtain global, spherically symmetric weak solutions of the Navier–Stokes equations (1.3) with some desired uniform bounds on $[0, T]\times[0, \infty)$, which are essential for us to employ the compensated compactness framework in §6.

5.1. Passage the Limit: $b \to \infty$

In this subsection, we fix parameters (ε, δ) and denote the solution of (3.1)–(3.4) as $(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b})$. It follows from (A.31)–(A.32) and Lemmas A.1–A.3 in the appendix that there exist sequences of smooth approximate initial data functions $(\rho_0^{\varepsilon,\delta,b}, u_0^{\varepsilon,\delta,b})$ and $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta})$ satisfying (3.5) and the properties:

$$
\begin{cases}
(\rho_0^{\varepsilon,\delta,b}, m_0^{\varepsilon,\delta,b})(r) \to (\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r) & \text{in } L^1_{\mathrm{loc}}([\delta,\infty); r^{N-1}\mathrm{d}r) \text{ as } b\to\infty,\\
(E_0^{\varepsilon,\delta,b}, E_1^{\varepsilon,\delta,b}) \to (E_0^{\varepsilon,\delta}, E_1^{\varepsilon,\delta}) & \text{as } b\to\infty,\\
E_2^{\varepsilon,\delta,b}+\tilde{E}_0^{\varepsilon,\delta,b}+\|u_{0r}^{\varepsilon,\delta,b}\|_{L^2} & \text{is uniform bounded in } b,
\end{cases}\tag{5.1}
$$

where

$$E_0^{\varepsilon,\delta} := \int_\delta^\infty \bar{\eta}^*(\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})\, r^{N-1}\mathrm{d}r < \infty, \tag{5.2}$$

$$E_1^{\varepsilon,\delta} := \varepsilon^2 \int_\delta^\infty \big(1 + 2\alpha\delta(\rho_0^{\varepsilon,\delta})^{\alpha-1} + \alpha^2\delta^2(\rho_0^{\varepsilon,\delta})^{2\alpha-2}\big)\big|\big(\sqrt{\rho_0^{\varepsilon,\delta}}\big)_r\big|^2\, r^{N-1}\mathrm{d}r < \infty. \tag{5.3}$$

From (3.7), (3.12), (4.4), (4.19), and (4.81), there exists a positive constant $\tilde{C} > 0$ that may depend on (ε, δ, T), but is independent of b, so that

$$0 < \tilde{C}^{-1} \leqq \rho^{\varepsilon,\delta,b}(t,r) \leqq \tilde{C}, \tag{5.4}$$

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big(\big\|(\rho^{\varepsilon,\delta,b} - \bar{\rho}, u^{\varepsilon,\delta,b})\big\|^2_{H^1([\delta,b])} + \big\|\rho_t^{\varepsilon,\delta,b}\big\|^2_{L^2([\delta,b])}\Big)(t)\\
&\quad + \int_0^T \big\|(u_t^{\varepsilon,\delta,b}, u_{rr}^{\varepsilon,\delta,b})\big\|^2_{L^2([\delta,b])}(t)\,\mathrm{d}t \leqq \tilde{C}.
\end{aligned} \tag{5.5}$$

We extend $\rho^{\varepsilon,\delta,b}(t,r)$ and $u^{\varepsilon,\delta,b}(t,r)$ to $[0,T]\times[\delta,\infty)$ by defining $\rho^{\varepsilon,\delta,b}(t,r) = \bar{\rho}$ and $u^{\varepsilon,\delta,b}(t,r) = 0$ for all $r \in [0,T]\times(b,\infty)$. Then it follows from (5.5) and the Aubin-Lions lemma that

$$(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b}) \qquad \text{is compact in } C([0,T]; L^p_{\mathrm{loc}}[\delta,\infty)) \text{ with } p \in [1,\infty).$$

More precisely, we have

Lemma 5.1. *There exist functions $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})(t,r)$ so that, as $b \to \infty$ (up to a subsequence),*

$$(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b}) \to (\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \quad \textit{strongly in } C([0,T]; L^p_{\mathrm{loc}}[\delta,\infty)) \textit{ for all } p \in [1,\infty).$$

In particular, as $b \to \infty$ (up to a subsequence),

$$(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b}) \to (\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \qquad \textit{almost everywhere } (t,r) \in [0,T]\times[\delta,\infty).$$

Using Lemma 5.1, it can immediately be proven that $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ is a weak solution of the initial-boundary value problem (IBVP) of the Navier–Stokes equations (3.1):

$$\begin{cases} (\rho,u)(0,r) = (\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta})(r) \qquad \text{for } r \in [\delta,\infty),\\ u|_{r=\delta} = 0 \qquad \text{for } t \geqq 0. \end{cases} \tag{5.6}$$

Moreover, it follows from (5.4)–(5.5) and the lower semicontinuity that

$$0 < \tilde{C}^{-1} \leqq \rho^{\varepsilon,\delta}(t,r) \leqq \tilde{C}, \tag{5.7}$$

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big(\big\|(\rho^{\varepsilon,\delta} - \bar{\rho}, u^{\varepsilon,\delta})\big\|^2_{H^1([\delta,\infty))} + \|\rho_t^{\varepsilon,\delta}\|^2_{L^2([\delta,\infty))}\Big)(t)\\
&\quad + \int_0^T \big\|(u_t^{\varepsilon,\delta}, u_{rr}^{\varepsilon,\delta})\big\|^2_{L^2([\delta,\infty))}(t)\,\mathrm{d}t \leqq \tilde{C}.
\end{aligned} \tag{5.8}$$

These facts yield that the weak solution $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ of (5.6) is indeed a strong solution. The uniqueness of this strong solution $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ is ensured by properties (5.7)–(5.8), the corresponding version of Lemmas 3.1–3.2 (*that is*, (5.10)–(5.11) below), and the basic L^2–energy estimate as in §3. This implies that the whole sequence $(\rho^{\varepsilon,\delta,b}, u^{\varepsilon,\delta,b})$ converges to $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ as $b \to \infty$.

Then it is direct to know that $(\rho^{\varepsilon,\delta}, \mathcal{M}^{\varepsilon,\delta})(t,\mathbf{x}) = (\rho^{\varepsilon,\delta}(t,r), m^{\varepsilon,\delta}(t,r)\,\frac{\mathbf{x}}{r})$ with $\rho^{\varepsilon,\delta}(t,\mathbf{x}) > 0$ is a strong solution of the initial-boundary problem of system (1.3) with (h, g) determined by (3.2) for $(t,\mathbf{x}) \in [0,\infty) \times \big(\mathbb{R}^N \backslash B_\delta(\mathbf{0})\big)$ with the following initial-boundary data:

$$\begin{cases}(\rho^{\varepsilon,\delta}, \mathcal{M}^{\varepsilon,\delta})(0,\mathbf{x}) = (\rho_0^{\varepsilon,\delta}(r), m_0^{\varepsilon,\delta}(r)\,\frac{\mathbf{x}}{r}),\\ \mathcal{M}^{\varepsilon,\delta}(t,\mathbf{x})|_{\mathbf{x}\in\partial B_\delta(\mathbf{0})} = 0.\end{cases} \tag{5.9}$$

From Lemma 5.1, (3.7), (3.12), (3.25), (4.3)–(4.4), (4.70), (5.1), Fatou's lemma, and the lower semicontinuity, we have

Proposition 5.2. *Under assumption* (5.1)*, for any fixed* (ε,δ)*, there exists a unique strong solution* $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ *of IBVP* (5.6)*. Moreover,* $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ *satisfies* (5.7) *and, for* $t \in (0, T]$,

$$\begin{aligned}&\int_\delta^\infty \Big(\frac{1}{2}\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2 + e(\rho^{\varepsilon,\delta},\bar\rho)\Big)(t,r)\,r^{N-1}\mathrm{d}r\\ &\quad+\varepsilon\int_0^T\int_\delta^\infty \Big(\rho^{\varepsilon,\delta}|u_r^{\varepsilon,\delta}|^2 + \rho^{\varepsilon,\delta}\frac{|u^{\varepsilon,\delta}|^2}{r^2}\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\ &\quad+c_N\varepsilon\delta\int_0^T\int_\delta^\infty \Big((\rho^{\varepsilon,\delta})^\alpha\big(|u_r^{\varepsilon,\delta}|^2 + \frac{|u^{\varepsilon,\delta}|^2}{r^2}\big)\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\ &\leqq E_0^{\varepsilon,\delta} \leqq C(E_0+1),\end{aligned} \tag{5.10}$$

$$\begin{aligned}&\varepsilon^2\int_\delta^\infty \Big(\big|(\sqrt{\rho^{\varepsilon,\delta}})_r\big|^2 + \delta(\rho^{\varepsilon,\delta})^{\alpha-2}|\rho_r^{\varepsilon,\delta}|^2 + \delta^2(\rho^{\varepsilon,\delta})^{2\alpha-3}|\rho_r^{\varepsilon,\delta}|^2\Big)(t,r)\,r^{N-1}\mathrm{d}r\\ &\quad+\varepsilon\int_0^T\int_\delta^\infty \Big(\big|\big((\rho^{\varepsilon,\delta})^{\frac{\gamma}{2}}\big)_r\big|^2 + \delta(\rho^{\varepsilon,\delta})^{\gamma+\alpha-3}|\rho_r^{\varepsilon,\delta}|^2\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\ &\leqq C(E_0+1),\end{aligned} \tag{5.11}$$

$$\int_0^T\int_d^D (\rho^{\varepsilon,\delta})^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t \leqq C(d,D,T,E_0), \tag{5.12}$$

$$\int_0^T\int_\delta^D \big(\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^3 + (\rho^{\varepsilon,\delta})^{\gamma+\theta}\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t \leqq C(D,T,E_0) \tag{5.13}$$

for any fixed $T > 0$ *and any compact subset* $[d, D]$ *of* (δ,∞)*, where* $c_N > 0$ *is some constant depending only on* N *determined in Lemma* 3.1.

5.2. Passage the Limit: $\delta \to 0+$

In this subsection, for fixed $\varepsilon > 0$, we consider the limit, $\delta \to 0+$, to obtain the weak solution of the Navier–Stokes equations. It follows from Lemma A.3 in the appendix that

$$\begin{cases} (\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r) \to (\rho_0^{\varepsilon}, m_0^{\varepsilon})(r) & \text{in } L^1_{\text{loc}}([0,\infty); r^{N-1}\mathrm{d}r) \text{ as } \delta \to 0+, \\ (E_0^{\varepsilon,\delta}, E_1^{\varepsilon,\delta}) \to (E_0^{\varepsilon}, E_1^{\varepsilon}) & \text{as } \delta \to 0. \end{cases} \tag{5.14}$$

To take the limit, we have to be careful since the weak solution may involve the vacuum state. We use similar compactness arguments as in [29,44] to consider the limit: $\delta \to 0+$. We first extend our solution $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ as the zero extension of $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ outside $[0,T]\times[\delta,\infty)$.

Lemma 5.3. *There exists a function $\rho^{\varepsilon}(t,r)$ such that, as $\delta \to 0+$ (up to a subsequence),*

$$(\rho^{\varepsilon,\delta}, \sqrt{\rho^{\varepsilon,\delta}}) \to (\rho^{\varepsilon}, \sqrt{\rho^{\varepsilon}}) \textit{ almost everywhere and strongly in } C(0,T; L^q_{\text{loc}}) \tag{5.15}$$

for any $q \in [1,\infty)$, where L^q_{loc} means $L^q(K)$ for any $K \Subset (0,\infty)$.

Proof. It follows from (5.10)–(5.11) that

$$\sqrt{\rho^{\varepsilon,\delta}} \in L^\infty(0,T; H^1_{\text{loc}}) \hookrightarrow L^\infty(0,T; L^\infty_{\text{loc}}) \qquad \text{uniformly.}$$

Notice that, for fixed $\varepsilon > 0$, the solution sequence $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ satisfies (3.1) for $(t,r) \in [0,\infty)\times[\delta,\infty)$. Using (5.10) and the mass equation $(3.1)_1$, we see that

$$\partial_t \sqrt{\rho^{\varepsilon,\delta}} = -(\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta})_r + \frac{1}{2}\sqrt{\rho^{\varepsilon,\delta}}u_r^{\varepsilon,\delta} - \frac{N-1}{2r}\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}$$

is uniformly bounded in $L^2(0,T; H^{-1}_{\text{loc}})$, which, using the Aubin-Lions lemma, implies that

$$\sqrt{\rho^{\varepsilon,\delta}} \qquad \text{is compact in } C(0,T; L^q_{\text{loc}}) \text{ for any } q \in [1,\infty).$$

Since $\sqrt{\rho^{\varepsilon,\delta}}$ and $\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}$ are uniformly bounded in $L^\infty(0,T; L^\infty_{\text{loc}})$ and $L^\infty(0,T; L^2_{\text{loc}})$ respectively, we see that

$$\rho^{\varepsilon,\delta}u^{\varepsilon,\delta} = \sqrt{\rho^{\varepsilon,\delta}}\big(\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}\big) \quad \text{is uniformly bounded in } L^\infty(0,T; L^2_{\text{loc}}). \tag{5.16}$$

Then it follows from the mass equation $(3.1)_1$ that

$$\partial_t \rho^{\varepsilon,\delta} = -(\rho^{\varepsilon,\delta}u^{\varepsilon,\delta})_r - \frac{N-1}{r}\rho^{\varepsilon,\delta}u^{\varepsilon,\delta} \quad \text{is uniformly bounded in } L^\infty(0,T; H^{-1}_{\text{loc}}).$$

Moreover, we obtain that

$$\rho_r^{\varepsilon,\delta} = 2\sqrt{\rho^{\varepsilon,\delta}}(\sqrt{\rho^{\varepsilon,\delta}})_r \quad \text{is uniformly bounded in } L^\infty(0,T; L^2_{\text{loc}}).$$

Then the Aubin-Lions lemma implies that

$$\rho^{\varepsilon,\delta} \qquad \text{is compact in } C(0,T; L^q_{\text{loc}}) \text{ with } q \in [1,\infty).$$

□

Corollary 5.4. *The pressure function sequence $p(\rho^{\varepsilon,\delta})$ is uniformly bounded in $L^\infty(0,T;L^q_{\mathrm{loc}})$ for all $q\in[1,\infty]$ and, as $\delta\to 0+$ (up to a subsequence),*

$$p(\rho^{\varepsilon,\delta})\to p(\rho^\varepsilon) \qquad \text{strongly in } L^q(0,T;L^q_{\mathrm{loc}}) \text{ for all } q\in[1,\infty). \tag{5.17}$$

Lemma 5.5. *As $\delta\to 0+$ (up to a subsequence), $m^{\varepsilon,\delta}$ converges strongly in $L^2(0,T;L^q_{\mathrm{loc}})$ to some function $m^\varepsilon(t,r)$ for all $q\in[1,\infty)$, which implies that*

$$m^{\varepsilon,\delta}(t,r)=(\rho^{\varepsilon,\delta}u^{\varepsilon,\delta})(t,r)\to m^\varepsilon(t,r) \qquad \text{almost everywhere in } [0,T]\times(0,\infty).$$

Proof. A direct calculation shows that

$$m_r^{\varepsilon,\delta}=2\big(\sqrt{\rho^{\varepsilon,\delta}}\big)_r\,\big(\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}\big)+\sqrt{\rho^{\varepsilon,\delta}}\,\big(\sqrt{\rho^{\varepsilon,\delta}}u_r^{\varepsilon,\delta}\big) \tag{5.18}$$

is uniformly bounded in $L^2(0,T;L^1_{\mathrm{loc}})$. Thus, it follows from (5.16)–(5.18) that

$$m^{\varepsilon,\delta} \qquad \text{is uniformly bounded in } L^2(0,T;W^{1,1}_{\mathrm{loc}}). \tag{5.19}$$

It follows from (5.10) and (5.17) that $\partial_r\big((\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta})^2\big)$, $\frac{N-1}{r}\big(\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}\big)^2$, and $\partial_r p(\rho^{\varepsilon,\delta})$ are uniformly bounded in $L^\infty(0,T;W^{-1,1}_{\mathrm{loc}})$, $L^\infty(0,T;L^1_{\mathrm{loc}})$, and $L^2(0,T;H^{-1}_{\mathrm{loc}})$, respectively.

From (5.10), we see that

$$\sqrt{(\rho^{\varepsilon,\delta})^\alpha}\big(\sqrt{\delta\,(\rho^{\varepsilon,\delta})^\alpha}(u_r^{\varepsilon,\delta}+\frac{N-1}{r}u^{\varepsilon,\delta})\big) \ \text{ and } \ \sqrt{\rho^{\varepsilon,\delta}}\big(\sqrt{\rho^{\varepsilon,\delta}}(u_r^{\varepsilon,\delta}+\frac{N-1}{r}u^{\varepsilon,\delta})\big)$$

are uniformly bounded in $L^2(0,T;L^2_{\mathrm{loc}})$.

Since

$$\begin{aligned}&\big(\mu(\rho^{\varepsilon,\delta})+\lambda(\rho^{\varepsilon,\delta})\big)\big(u_r^{\varepsilon,\delta}+\frac{N-1}{r}u^{\varepsilon,\delta}\big)\\&=\Big(\alpha\delta\sqrt{(\rho^{\varepsilon,\delta})^\alpha}+\sqrt{(\rho^{\varepsilon,\delta})^{2-\alpha}}\Big)\Big(\sqrt{(\rho^{\varepsilon,\delta})^\alpha}u_r^{\varepsilon,\delta}+\frac{N-1}{r}\sqrt{(\rho^{\varepsilon,\delta})^\alpha}u^{\varepsilon,\delta}\Big),\end{aligned}$$

we conclude that

$$\partial_r\Big(\big(\mu(\rho^{\varepsilon,\delta})+\lambda(\rho^{\varepsilon,\delta})\big)\big(u_r^{\varepsilon,\delta}+\frac{N-1}{r}u^{\varepsilon,\delta}\big)\Big)$$

is uniformly bouneded in $L^2(0,T;H^{-1}_{\mathrm{loc}})$. Also, it follows from (5.10)–(5.11) that

$$\frac{N-1}{r}\partial_r\mu(\rho^{\varepsilon,\delta})u^{\varepsilon,\delta}=\frac{2(N-1)}{r}\big((\sqrt{\rho^{\varepsilon,\delta}})_r+\alpha\delta(\rho^{\varepsilon,\delta})^{\alpha-\frac32}\rho_r^{\varepsilon,\delta}\big)\big(\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}\big)$$

is uniformly bounded in $L^2(0,T;L^1_{\mathrm{loc}})$. Then we conclude that

$$\partial_t m^{\varepsilon,\delta} \qquad \text{is uniformly bounded in } L^2(0,T;W^{-2,\frac43}_{\mathrm{loc}}),$$

which, with (5.19) and the Aubin-Lions lemma, implies that

$$m^{\varepsilon,\delta} \qquad \text{is compact in } L^2(0,T;L^p_{\mathrm{loc}}) \text{ for all } p\in[1,\infty).$$

□

Lemma 5.6. *$m^\varepsilon(t,r)=0$ almost everywhere on $\{(t,r)\,:\,\rho^\varepsilon(t,r)=0\}$. Furthermore, there exists a function $u^\varepsilon(t,r)$ so that $m^\varepsilon(t,r)=\rho^\varepsilon(t,r)u^\varepsilon(t,r)$ almost everywhere, $u^\varepsilon(t,r)=0$ almost everywhere on $\{(t,r)\,:\,\rho^\varepsilon(t,r)=0\}$, and*

$$\begin{aligned} &m^{\varepsilon,\delta}\to m^\varepsilon && \textit{strongly in } L^2(0,T;L^p_{\mathrm{loc}}) \textit{ for } p\in[1,\infty),\\ &\frac{m^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}\to\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}=\sqrt{\rho^\varepsilon}u^\varepsilon && \textit{strongly in } L^2(0,T;L^2_{\mathrm{loc}}). \end{aligned}$$

Proof. Since $\frac{m^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}r^{\frac{N-1}{2}}$ is uniformly bounded in $L^\infty(0,T;L^2)$, then Fatou's lemma implies

$$\begin{aligned} &\int_0^T\int_0^\infty \liminf_{\delta\to0+}\frac{|m^{\varepsilon,\delta}(t,r)|^2}{\rho^{\varepsilon,\delta}(t,r)}\,r^{N-1}\mathrm{d}r\mathrm{d}t\\ &\leqq \lim_{\delta\to0+}\int_0^T\int_0^\infty\frac{|m^{\varepsilon,\delta}(t,r)|^2}{\rho^{\varepsilon,\delta}(t,r)}\,r^{N-1}\mathrm{d}r\mathrm{d}t<\infty. \end{aligned}$$

Thus, $m^\varepsilon(t,r)=0$ almost everywhere on $\{(t,r)\,:\,\rho^\varepsilon(t,r)=0\}$. Then, if the limit velocity $u^\varepsilon(t,r)$ is defined by setting $u^\varepsilon(t,r):=\frac{m^\varepsilon(t,r)}{\rho^\varepsilon(t,r)}$ almost everywhere on $\{(t,r)\,:\,\rho^\varepsilon(t,r)\neq0\}$ and $u^\varepsilon(t,r)=0$ almost everywhere on $\{(t,r)\,:\,\rho^\varepsilon(t,r)=0\}$, we have

$$\begin{aligned} &m^\varepsilon(t,r)=\rho^\varepsilon(t,r)u^\varepsilon(t,r)\quad\text{almost everywhere},\\ &\int_0^T\int_0^\infty\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2r^{N-1}\,\mathrm{d}r\mathrm{d}t=\int_0^T\int_0^\infty\rho^\varepsilon|u^\varepsilon|^2r^{N-1}\mathrm{d}r\mathrm{d}t<\infty. \end{aligned}$$

Moreover, it follows from (5.13) and Fatou's lemma that, for $[d,D]\Subset(0,\infty)$,

$$\int_0^T\int_d^D\rho^\varepsilon|u^\varepsilon|^3\,\mathrm{d}r\mathrm{d}t\leqq\lim_{\delta\to0+}\int_0^T\int_d^D\frac{|m^{\varepsilon,\delta}|^3}{(\rho^{\varepsilon,\delta})^2}\,\mathrm{d}r\mathrm{d}t\leqq C(d,D,T,E_0)<\infty. \tag{5.20}$$

Next, since $m^{\varepsilon,\delta}$ and $\rho^{\varepsilon,\delta}$ converge almost everywhere, it is direct to know that sequence $\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}=\frac{m^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}$ converges almost everywhere to $\sqrt{\rho^\varepsilon}u^\varepsilon=\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}$ on $\{(t,r)\,:\,\rho^\varepsilon(t,r)\neq0\}$. Moreover, for any given positive constant $R>0$, it follows from Lemmas 5.3 and 5.6 that

$$\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}I_{|u^{\varepsilon,\delta}|\leqq R}\to\sqrt{\rho^\varepsilon}u^\varepsilon I_{|u^\varepsilon|\leqq R}\qquad\text{almost everywhere} \tag{5.21}$$

For $R\geqq1$, we cut the L^2-norm as follows:

$$\begin{aligned} &\int_0^T\int_d^D\big|\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}-\sqrt{\rho^\varepsilon}u^\varepsilon\big|^2\,\mathrm{d}r\mathrm{d}t\\ &\leqq\int_0^T\int_d^D\big|\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}I_{|u^{\varepsilon,\delta}|\leqq R}-\sqrt{\rho^\varepsilon}u^\varepsilon I_{|u^\varepsilon|\leqq R}\big|^2\,\mathrm{d}r\mathrm{d}t \end{aligned}$$

$$+2\int_0^T\int_d^D \left|\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}I_{|u^{\varepsilon,\delta}|\geqq R}\right|^2 \mathrm{d}r\mathrm{d}t$$
$$+2\int_0^T\int_d^D \left|\sqrt{\rho^{\varepsilon}}u^{\varepsilon}I_{|u^{\varepsilon}|\geqq R}\right|^2 \mathrm{d}r\mathrm{d}t. \tag{5.22}$$

It is direct to know that $\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}I_{|u^{\varepsilon,\delta}|\leqq R}$ is uniformly bounded in $L^\infty(0,T;L^p_{\mathrm{loc}})$ for all $p\in[1,\infty)$. Then it follows from (5.21) that

$$\int_0^T\int_d^D \left|\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}I_{|u^{\varepsilon,\delta}|\leqq R}-\sqrt{\rho^{\varepsilon}}u^{\varepsilon}I_{|u^{\varepsilon}|\leqq R}\right|^2 \mathrm{d}r\mathrm{d}t\to 0 \qquad \text{as } \delta\to 0+. \tag{5.23}$$

Using (5.20), we have

$$\int_0^T\int_d^D \Big(\left|\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}I_{|u^{\varepsilon,\delta}|\geqq R}\right|^2+\left|\sqrt{\rho^{\varepsilon}}u^{\varepsilon}I_{|u^{\varepsilon}|\geqq R}\right|^2\Big)\mathrm{d}r\mathrm{d}t$$
$$\leqq \frac{1}{R}\int_0^T\int_d^D \big(\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^3+\rho^{\varepsilon}|u^{\varepsilon}|^3\big)\mathrm{d}r\mathrm{d}t\leqq C(d,D,T,E_0)R^{-1}. \tag{5.24}$$

Substituting (5.23)–(5.24) into (5.22) leads to

$$\lim_{\delta\to 0+}\int_0^T\int_d^D \left|\sqrt{\rho^{\varepsilon,\delta}}u^{\varepsilon,\delta}-\sqrt{\rho^{\varepsilon}}u^{\varepsilon}\right|^2\mathrm{d}r\mathrm{d}t\leqq C(d,D,T,E_0)R^{-1}\quad \text{for all } R>0.$$

Then the lemma follows by taking $R\to\infty$. □

Let $(\rho^\varepsilon, m^\varepsilon)$ be the limit obtained above. By using Fatou's lemma and the lower semicontinuity and Proposition 5.2, it is direct to obtain

Proposition 5.7. *Under assumption (5.14), for any fixed ε and $T>0$, the limit functions $(\rho^\varepsilon, m^\varepsilon)=(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)$ satisfy*

$$\rho^\varepsilon(t,r)\geqq 0 \quad \textit{almost everywhere}, \tag{5.25}$$

$$u^\varepsilon(t,r)=0,\ \Big(\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big)(t,r)=\sqrt{\rho^\varepsilon}(t,r)u^\varepsilon(t,r)=0$$
$$\textit{almost everywhere on } \{(t,r)\,:\,\rho^\varepsilon(t,r)=0\}, \tag{5.26}$$

$$\int_0^\infty \Big(\frac12\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2+e(\rho^\varepsilon,\bar\rho)\Big)(t,r)\,r^{N-1}\mathrm{d}r+\varepsilon\int_{\mathbb{R}^2_+}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2(s,r)\,r^{N-3}\mathrm{d}r\mathrm{d}s$$
$$\leqq E_0^\varepsilon\leqq E_0+1 \quad \textit{for } t\geqq 0, \tag{5.27}$$

$$\varepsilon^2\int_0^\infty \Big|\big(\sqrt{\rho^\varepsilon(t,r)}\big)_r\Big|^2 r^{N-1}\mathrm{d}r+\varepsilon\int_{\mathbb{R}^2_+}\Big|\big((\rho^\varepsilon(s,r))^{\frac{\gamma}{2}}\big)_r\Big|^2 r^{N-1}\mathrm{d}r\mathrm{d}s$$
$$\leqq C(E_0+1) \quad \textit{for } t\geqq 0, \tag{5.28}$$

$$\int_0^T\int_d^D (\rho^\varepsilon)^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t\leqq C(d,D,T,E_0), \tag{5.29}$$

$$\int_0^T\int_0^D \big(\rho^\varepsilon|u^\varepsilon|^3+(\rho^\varepsilon)^{\gamma+\theta}\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\leqq C(D,T,E_0), \tag{5.30}$$

where $[d,D]\Subset(0,\infty)$.

We now show that

$$(\rho^\varepsilon, \mathcal{M}^\varepsilon)(t,\mathbf{x}) = (\rho^\varepsilon(t,r), m^\varepsilon(t,r)\frac{\mathbf{x}}{r}) \tag{5.31}$$

is a weak solution of the Cauchy problem (1.3) and (2.6) in $\mathbb{R}^N$ in the sense of Definition 2.3.

Lemma 5.8. *Let $0 \leqq t_1 < t_2 \leqq T$, and let $\zeta(t,\mathbf{x}) \in C^1([0,T]\times\mathbb{R}^N)$ be any smooth function with compact support. Then*

$$\begin{aligned}&\int_{\mathbb{R}^N} \rho^\varepsilon(t_2,\mathbf{x})\zeta(t_2,\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\quad= \int_{\mathbb{R}^N} \rho^\varepsilon(t_1,\mathbf{x})\zeta(t_1,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \big(\rho^\varepsilon\zeta_t + \mathcal{M}^\varepsilon\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned} \tag{5.32}$$

Proof. Notice that $(\rho^{\varepsilon,\delta}, \mathcal{M}^{\varepsilon,\delta})$ is a strong solution of (1.3) and (5.9) over $[0,\infty)\times\big(\mathbb{R}^N\setminus B_\delta(\mathbf{0})\big)$. It follows from $(1.3)_1$ and a direct calculation that

$$\begin{aligned}0 &= \int_{t_1}^{t_2}\int_{\mathbb{R}^N\setminus B_\delta(\mathbf{0})} \big((\rho^{\varepsilon,\delta})_t + \mathrm{div}\mathcal{M}^{\varepsilon,\delta}\big)\zeta(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &= \int_{\mathbb{R}^N\setminus B_\delta(\mathbf{0})} \rho^{\varepsilon,\delta}\zeta\, d\mathbf{x}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^N\setminus B_\delta(\mathbf{0})} \big(\rho^{\varepsilon,\delta}\zeta_t + \mathcal{M}^{\varepsilon,\delta}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &= \int_{\mathbb{R}^N} \rho^{\varepsilon,\delta}\zeta\,\mathrm{d}\mathbf{x}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \big(\rho^{\varepsilon,\delta}\zeta_t + \mathcal{M}^{\varepsilon,\delta}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned} \tag{5.33}$$

where we have used the fact that $(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta})$ is extended by zero in $[0,T]\times[0,\delta)$.

Notice that, for $i=1,2$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N} \big(\rho^{\varepsilon,\delta}-\rho^\varepsilon\big)(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big|\\ &\leqq \Big|\int_{\mathbb{R}^N\setminus B_\sigma(\mathbf{0})} \big(\rho^{\varepsilon,\delta}-\rho^\varepsilon\big)(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big|\\ &\quad + \Big|\int_{B_\sigma(\mathbf{0})} \big(\rho^{\varepsilon,\delta}-\rho^\varepsilon\big)(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big|.\end{aligned} \tag{5.34}$$

Denote

$$\phi(t,r) := \int_{\partial B_1(\mathbf{0})} \zeta(t,r\omega)\,\mathrm{d}\omega \in C_0^1([0,T]\times[0,\infty)). \tag{5.35}$$

Then, with (5.15), for any fixed $\sigma>0$, we have

$$\begin{aligned}&\lim_{\delta\to0+}\Big|\int_{\mathbb{R}^N\setminus B_\sigma(\mathbf{0})} \big(\rho^{\varepsilon,\delta}-\rho^\varepsilon\big)(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big|\\ &\quad= \omega_N\lim_{\delta\to0+}\Big|\int_\sigma^\infty \big(\rho^{\varepsilon,\delta}-\rho^\varepsilon\big)(t_i,r)\phi(t_i,r)\,r^{N-1}\mathrm{d}r\Big| = 0.\end{aligned} \tag{5.36}$$

Using (5.10) and (5.27), we obtain

$$\begin{aligned}
&\Big|\int_{B_\sigma(\mathbf{0})}(\rho^{\varepsilon,\delta}-\rho^{\varepsilon})(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big|\\
&\leqq C\|\zeta\|_{L^\infty}\Big\{\int_0^\sigma\big((\rho^{\varepsilon,\delta})^\gamma+(\rho^\varepsilon)^\gamma\big)\,r^{N-1}\mathrm{d}r\Big\}^{\frac{1}{\gamma}}\sigma^{N(1-\frac{1}{\gamma})}\\
&\leqq C(E_0)\|\zeta\|_{L^\infty}\sigma^{N(1-\frac{1}{\gamma})}\to 0 \qquad \text{as } \sigma\to 0,
\end{aligned} \tag{5.37}$$

which, along with (5.34) and (5.36), yields

$$\lim_{\delta\to 0+}\int_{\mathbb{R}^N}\rho^{\varepsilon,\delta}(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^N}\rho^{\varepsilon}(t_i,\mathbf{x})\zeta(t_i,\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \text{for } i=1,2. \tag{5.38}$$

From (5.35), a direct calculation shows

$$\phi_r=\int_{\partial B_1(\mathbf{0})}\omega\cdot\nabla\zeta(t,r\omega)\,\mathrm{d}\omega \tag{5.39}$$

which, with (5.15) and Lemma 5.6, implies

$$\begin{aligned}
&\lim_{\delta\to 0+}\int_{t_1}^{t_2}\int_{\mathbb{R}^N\setminus B_\sigma(\mathbf{0})}\big(\rho^{\varepsilon,\delta}\zeta_t+\mathcal{M}^{\varepsilon,\delta}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&=\omega_N\lim_{\delta\to 0+}\int_{t_1}^{t_2}\int_\sigma^\infty\big(\rho^{\varepsilon,\delta}\phi_t+m^{\varepsilon,\delta}\phi_r\big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&=\omega_N\int_{t_1}^{t_2}\int_\sigma^\infty\big(\rho^{\varepsilon}\phi_t+m^{\varepsilon}\phi_r\big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^N\setminus B_\sigma(\mathbf{0})}\big(\rho^{\varepsilon}\zeta_t+\mathcal{M}^{\varepsilon}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned} \tag{5.40}$$

Similar to that in (5.37), we also have

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_{B_\sigma(\mathbf{0})}(\rho^{\varepsilon,\delta}-\rho^{\varepsilon})\zeta_t\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|\leqq C(T,E_0)\|\zeta_t\|_{L^\infty}\,\sigma^{N(1-\frac{1}{\gamma})},\\
&\Big|\int_{t_1}^{t_2}\int_{B_\sigma(\mathbf{0})}\big(\mathcal{M}^{\varepsilon,\delta}-\mathcal{M}^{\varepsilon}\big)\cdot\nabla\zeta\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|\\
&\leqq C\|\nabla\zeta\|_{L^\infty}\Big\{\int_{t_1}^{t_2}\int_0^\sigma\Big(\frac{|m^{\varepsilon,\delta}|^2}{\rho^{\varepsilon,\delta}}+\frac{|m^\varepsilon|^2}{\rho^\varepsilon}\Big)(t,r)r^{N-1}\,\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}}\\
&\qquad\times\Big\{\int_{t_1}^{t_2}\int_0^\sigma\big(\rho^{\varepsilon,\delta}+\rho^\varepsilon\big)(t,r)r^{N-1}\,\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}}\\
&\leqq C(T,E_0)\|\nabla\zeta\|_{L^\infty}\sigma^{\frac{N}{2}(1-\frac{1}{\gamma})},
\end{aligned}$$

which, with (5.40), yields

$$\lim_{\delta\to 0+}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\big(\rho^{\varepsilon,\delta}\zeta_t+\mathcal{M}^{\varepsilon,\delta}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\big(\rho^{\varepsilon}\zeta_t+\mathcal{M}^{\varepsilon}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t. \tag{5.41}$$

Combining (5.38) and (5.41) with (5.33), we conclude (5.32). □

Lemma 5.9. *Let* $\psi(t,\mathbf{x}) \in \big(C_0^2([0,\infty)\times\mathbb{R}^N)\big)^N$ *be any smooth function with* supp $\psi \Subset [0,T)\times\mathbb{R}^N$ *for some fixed* $T\in(0,\infty)$. *Then*

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+}\Big\{\mathcal{M}^\varepsilon\cdot\partial_t\psi+\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\big)\psi+p(\rho^\varepsilon)\,\mathrm{div}\,\psi\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\qquad+\int_{\mathbb{R}^N}\mathcal{M}_0^\varepsilon(\mathbf{x})\cdot\psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&=-\varepsilon\int_{\mathbb{R}^{N+1}_+}\Big\{\frac{1}{2}\mathcal{M}^\varepsilon\cdot\big(\Delta\psi+\nabla\mathrm{div}\,\psi\big)+\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\big(\nabla\sqrt{\rho^\varepsilon}\cdot\nabla\big)\psi\\
&\qquad\qquad+\nabla\sqrt{\rho^\varepsilon}\cdot\big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\big)\psi\Big\}\mathrm{d}\mathbf{x}\mathrm{d}t \qquad (5.42)\\
&=\sqrt{\varepsilon}\int_{\mathbb{R}^{N+1}_+}\sqrt{\rho^\varepsilon}\Big\{V^\varepsilon\frac{\mathbf{x}\otimes\mathbf{x}}{r^2}+\frac{\sqrt{\varepsilon}}{r}\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big(I_{N\times N}-\frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big)\Big\}:\nabla\psi\,\mathrm{d}\mathbf{x}\mathrm{d}t, \qquad (5.43)
\end{aligned}$$

where $V^\varepsilon(t,r)\in L^2(0,T;L^2(\mathbb{R}^N))$ *is a function such that*

$$\int_0^T\int_{\mathbb{R}^N}|V^\varepsilon(t,\mathbf{x})|^2\mathrm{d}\mathbf{x}\mathrm{d}t\leqq CE_0$$

for some $C>0$, *independent of* $T>0$.

Proof. Let $\psi=(\psi_1,\dots,\psi_N)\in\big(C_0^2([0,\infty)\times\mathbb{R}^N)\big)^N$ be a smooth function with supp $\psi\Subset[0,T)\times\mathbb{R}^N$. For any given $\sigma\in(0,1]$, let $\chi_\sigma(r)\in C^\infty(\mathbb{R})$ be a cut-off function satisfying that

$$\begin{aligned}
&\chi_\sigma(r)=0\ \text{ for } r\leqq\sigma,\quad \chi_\sigma(r)=1\ \text{ for } r\geqq 2\sigma,\\
&|\chi_\sigma'(r)|\leqq\frac{C}{\sigma},\quad |\chi_\sigma''(r)|\leqq\frac{C}{\sigma^2}.
\end{aligned} \qquad (5.44)$$

Denote $\Psi_\sigma(t,\mathbf{x}):=\psi(t,\mathbf{x})\chi_\sigma(|\mathbf{x}|)$.

Taking δ small enough so that $\delta\leqq\sigma$, then it follows from $(1.3)_2$ and integration by parts that

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+}\Big\{\mathcal{M}^{\varepsilon,\delta}\cdot\partial_t\Psi_\sigma+\frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}\cdot\big(\frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}\cdot\nabla\big)\Psi_\sigma+p(\rho^{\varepsilon,\delta})\,\mathrm{div}\,\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\qquad+\int_{\mathbb{R}^N}\mathcal{M}_0^{\varepsilon,\delta}(\mathbf{x})\cdot\Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&=:J_1^{\varepsilon,\delta}+J_2^{\varepsilon,\delta}, \qquad (5.45)
\end{aligned}$$

where

$$J_1^{\varepsilon,\delta}:=\delta\varepsilon\int_{\mathbb{R}^{N+1}_+}(\rho^{\varepsilon,\delta})^\alpha\Big\{D(\frac{\mathcal{M}^{\varepsilon,\delta}}{\rho^{\varepsilon,\delta}}):\nabla\Psi_\sigma+(\alpha-1)\mathrm{div}\big(\frac{\mathcal{M}^{\varepsilon,\delta}}{\rho^{\varepsilon,\delta}}\big)\,\mathrm{div}\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t, \qquad (5.46)$$

$$
\begin{aligned}
J_2^{\varepsilon,\delta} := & -\varepsilon \int_{\mathbb{R}_+^{N+1}} \Big\{ \frac{1}{2}\mathcal{M}^{\varepsilon,\delta}\cdot\big(\Delta\Psi_\sigma + \nabla\mathrm{div}\Psi_\sigma\big) + \frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}\cdot\big(\nabla\sqrt{\rho^{\varepsilon,\delta}}\cdot\nabla\big)\Psi_\sigma \\
& \qquad + \nabla\sqrt{\rho^{\varepsilon,\delta}}\cdot\big(\frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}\cdot\nabla\big)\Psi_\sigma \Big\}\mathrm{d}\mathbf{x}\mathrm{d}t \\
= & \ \varepsilon\int_{\mathbb{R}_+^{N+1}} \sqrt{\rho^{\varepsilon,\delta}}\sqrt{\rho^{\varepsilon,\delta}} D\big(\frac{\mathcal{M}^{\varepsilon,\delta}}{\rho^{\varepsilon,\delta}}\big) : \nabla\Psi_\sigma \, \mathrm{d}\mathbf{x}\mathrm{d}t. \qquad (5.47)
\end{aligned}
$$

A direct calculation leads to

$$
\partial_i\big(\frac{\mathcal{M}_j^{\varepsilon,\delta}}{\rho^{\varepsilon,\delta}}\big) = u_r^{\varepsilon,\delta}\frac{x_i x_j}{r^2} + \frac{u^{\varepsilon,\delta}}{r}\big(\delta_{ij} - \frac{x_i x_j}{r^2}\big). \qquad (5.48)
$$

Using (5.10), there exists a function $V^\varepsilon(t,r)$ so that

$$
\sqrt{\varepsilon}\sqrt{\rho^{\varepsilon,\delta}} D\big(\frac{\mathcal{M}_j^{\varepsilon,\delta}}{\rho^{\varepsilon,\delta}}\big) \rightharpoonup V^\varepsilon\frac{\mathbf{x}\otimes\mathbf{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r}\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big(I_{N\times N} - \frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big) \qquad (5.49)
$$

in $L^2(0,T;(L^2(\mathbb{R}^N\backslash B_\sigma(\mathbf{0})))^{N\times N})$ as $\delta\to 0+$ for any given $\sigma>0$. Moreover, we have

$$
\int_0^T\int_{\mathbb{R}^N} \big|V^\varepsilon\big|^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \leqq CE_0. \qquad (5.50)
$$

It follows from (5.10) and (5.48) that

$$
\begin{aligned}
|J_1^{\varepsilon,\delta}| \leqq & \ C(\|\psi\|_{C^1}, \mathrm{supp}\,\psi, \sigma)\sqrt{\varepsilon\delta}\Big\{\int_{\mathrm{supp}\,\Psi_\sigma} (\rho^{\varepsilon,\delta})^\alpha\, r^{N-1}\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}} \\
& \times\Big\{\delta\varepsilon\int_{\mathrm{supp}\,\Psi_\sigma}(\rho^{\varepsilon,\delta})^\alpha\Big(|u_r^{\varepsilon,\delta}|^2 + \frac{|u^{\varepsilon,\delta}|^2}{r^2}\Big) r^{N-1}\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}} \\
\leqq & \ C(\|\psi\|_{C^1}, \mathrm{supp}\,\psi, \sigma, E_0)\sqrt{\varepsilon\delta}\to 0 \qquad \text{as } \delta\to 0+. \qquad (5.51)
\end{aligned}
$$

Denote

$$
\phi_{1\sigma}(t,r) := \int_{\partial B_1(\mathbf{0})} \big(\omega\cdot(\Delta\Psi_\sigma)(t,r\omega) + \omega\cdot(\nabla\mathrm{div}\Psi_\sigma)(t,r\omega)\big)\,\mathrm{d}\omega.
$$

Then it is clear that $\phi_{1\sigma}\in C_0^2([0,T]\times(0,\infty))$. Thus, using Lemma 5.6, we have

$$
\begin{aligned}
& \int_{\mathbb{R}_+^{N+1}} \mathcal{M}^{\varepsilon,\delta}\cdot\big(\Delta\Psi_\sigma + \nabla\mathrm{div}\Psi_\sigma\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
& = \omega_N\int_{\mathbb{R}_+^2} m^{\varepsilon,\delta}\phi_{1\sigma}\, r^{N-1}\,\mathrm{d}r\mathrm{d}t \to \omega_N\int_{\mathbb{R}_+^2} m^\varepsilon\phi_{1\sigma}\, r^{N-1}\,\mathrm{d}r\mathrm{d}t \\
& = \int_{\mathbb{R}_+^{N+1}} \mathcal{M}^\varepsilon\cdot\big(\Delta\Psi_\sigma + \nabla\mathrm{div}\Psi_\sigma\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t \qquad \text{as } \delta\to 0. \qquad (5.52)
\end{aligned}
$$

Similarly, using Lemmas 5.3 and 5.6, we can prove

$$\int_{\mathbb{R}^2_+} \Big\{\frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}} \cdot \big(\nabla\sqrt{\rho^{\varepsilon,\delta}} \cdot \nabla\big) + \nabla\sqrt{\rho^{\varepsilon,\delta}} \cdot \big(\frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}} \cdot \nabla\big)\Big\}\Psi_\sigma \, \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$\to \int_{\mathbb{R}^2_+} \Big\{\frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \big(\nabla\sqrt{\rho^{\varepsilon}} \cdot \nabla\big) + \nabla\sqrt{\rho^{\varepsilon}} \cdot \big(\frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \nabla\big)\Big\}\Psi_\sigma \, \mathrm{d}\mathbf{x}\mathrm{d}t \;\text{ as } \delta \to 0. \quad (5.53)$$

Combining (5.49) with (5.52)–(5.53), we obtain that, as $\delta \to 0$,

$$\begin{aligned} J_2^{\varepsilon,\delta} \to &-\varepsilon \int_{\mathbb{R}^{N+1}_+} \Big\{\frac{1}{2}\mathcal{M}^{\varepsilon} \cdot \big(\Delta\Psi_\sigma + \nabla \mathrm{div}\Psi_\sigma\big) + \frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \big(\nabla\sqrt{\rho^{\varepsilon}} \cdot \nabla\big)\Psi_\sigma \\ &\qquad + \nabla\sqrt{\rho^{\varepsilon}} \cdot \big(\frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \nabla\big)\Psi_\sigma\Big\}\mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \sqrt{\varepsilon}\int_{\mathbb{R}^{N+1}_+} \sqrt{\rho^{\varepsilon}}\Big\{V^{\varepsilon}\frac{\mathbf{x}\otimes\mathbf{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r}\frac{m^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}}\big(I_{N\times N} - \frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big)\Big\} : \nabla\Psi_\sigma \, \mathrm{d}\mathbf{x}\mathrm{d}t. \end{aligned} \quad (5.54)$$

Also, by similar arguments as in (5.52), applying Lemma 5.3, Corollary 5.4, and Lemma 5.6, we have

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} \Big\{\mathcal{M}^{\varepsilon,\delta} \cdot \partial_t\Psi_\sigma + \frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}} \cdot \big(\frac{\mathcal{M}^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}} \cdot \nabla\big)\Psi_\sigma + p(\rho^{\varepsilon,\delta})\,\mathrm{div}\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^N} \mathcal{M}_0^{\varepsilon,\delta}(\mathbf{x}) \cdot \Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\to \int_{\mathbb{R}^{N+1}_+} \Big\{\mathcal{M}^{\varepsilon} \cdot \partial_t\Psi_\sigma + \frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \big(\frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \nabla\big)\Psi_\sigma + p(\rho^{\varepsilon})\mathrm{div}\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &\qquad + \int_{\mathbb{R}^N} \mathcal{M}_0^{\varepsilon}(\mathbf{x}) \cdot \Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x} \end{aligned}$$

as $\delta \to 0$, which, with (5.54), yields

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} \Big\{\mathcal{M}^{\varepsilon} \cdot \partial_t\Psi_\sigma + \frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \big(\frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \nabla\big)\Psi_\sigma + p(\rho^{\varepsilon})\,\mathrm{div}\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^N} \mathcal{M}_0^{\varepsilon}(\mathbf{x}) \cdot \Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= -\varepsilon \int_{\mathbb{R}^{N+1}_+} \Big\{\frac{1}{2}\mathcal{M}^{\varepsilon} \cdot \big(\Delta\Psi_\sigma + \nabla\mathrm{div}\Psi_\sigma\big) + \frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \big(\nabla\sqrt{\rho^{\varepsilon}} \cdot \nabla\big)\Psi_\sigma \\ &\qquad + \nabla\sqrt{\rho^{\varepsilon}} \cdot \big(\frac{\mathcal{M}^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}} \cdot \nabla\big)\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \sqrt{\varepsilon}\int_{\mathbb{R}^{N+1}_+} \sqrt{\rho^{\varepsilon}}\Big\{V^{\varepsilon}\frac{\mathbf{x}\otimes\mathbf{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r}\frac{m^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}}\big(I_{N\times N} - \frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big)\Big\} : \nabla\Psi_\sigma \, \mathrm{d}\mathbf{x}\mathrm{d}t. \end{aligned} \quad (5.55)$$

Next, we consider the limit, $\sigma \to 0$, in (5.55). We first define

$$\begin{aligned}\varphi(t,r) :&= \int_{\partial B_1(\mathbf{0})} \omega\cdot\psi(t,r\omega)\,\mathrm{d}\omega\\&=\frac{1}{r^{N-1}}\int_{\partial B_r(\mathbf{0})}\omega\cdot\psi(t,\mathbf{y})\,\mathrm{d}S_{\mathbf{y}}\\&=\frac{1}{r^{N-1}}\int_{B_r(\mathbf{0})}\operatorname{div}\psi(t,\mathbf{y})\,\mathrm{d}\mathbf{y},\end{aligned}\tag{5.56}$$

which implies

$$|\varphi(t,r)| \leqq C(\|\psi\|_{C^1})r;\tag{5.57}$$

also see [34,53]. Using (5.56), Lebesgue's dominated convergence theorem, and Proposition 5.7, we have

$$\begin{aligned}&\lim_{\sigma\to 0}\left\{\int_{\mathbb{R}^{N+1}_+}\mathcal{M}^\varepsilon\cdot\partial_t\Psi_\sigma\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}\mathcal{M}^\varepsilon_0(\mathbf{x})\cdot\Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\right\}\\&=\omega_N\lim_{\sigma\to 0}\left\{\int_{\mathbb{R}^2_+}m^\varepsilon\,\partial_t\varphi\,\chi_\sigma(r)\,r^{N-1}\mathrm{d}r\mathrm{d}t+\int_0^\infty m^\varepsilon_0(r)\varphi(0,r)\chi_\sigma(r)\,r^{N-1}\mathrm{d}r\right\}\\&=\omega_N\int_{\mathbb{R}^2_+}m^\varepsilon\,\partial_t\varphi\,r^{N-1}\mathrm{d}r\mathrm{d}t+\omega_N\int_0^\infty m^\varepsilon_0(r)\varphi(0,r)\,r^{N-1}\mathrm{d}r\\&=\int_{\mathbb{R}^{N+1}_+}\mathcal{M}^\varepsilon\cdot\partial_t\psi\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}\mathcal{M}^\varepsilon_0(\mathbf{x})\cdot\psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{5.58}$$

Using (5.57) and Proposition 5.7, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^{N+1}_+}\Big(\frac{|m^\varepsilon|^2}{\rho^\varepsilon}+p(\rho^\varepsilon)\Big)\Big(\psi\cdot\frac{\mathbf{x}}{r}\Big)\chi'_\sigma(r)\,\mathrm{d}\mathbf{x}\mathrm{d}t\right|\\&\leqq C\int_0^T\int_\sigma^{2\sigma}\Big(\frac{|m^\varepsilon|^2}{\rho^\varepsilon}+p(\rho^\varepsilon)\Big)|\varphi(t,r)\chi'_\sigma(r)|\,r^{N-1}\mathrm{d}r\mathrm{d}t\\&\leqq C\int_0^T\int_\sigma^{2\sigma}\Big(\frac{|m^\varepsilon|^2}{\rho^\varepsilon}+p(\rho^\varepsilon)\Big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\to 0\quad\text{as }\sigma\to 0,\end{aligned}\tag{5.59}$$

$$\begin{aligned}&\left|\varepsilon\int_{\mathbb{R}^{N+1}_+}\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}(\sqrt{\rho^\varepsilon})_r\Big(\psi\cdot\frac{\mathbf{x}}{r}\Big)\chi'_\sigma(r)\,\mathrm{d}\mathbf{x}\mathrm{d}t\right|\\&\leqq C\varepsilon\int_0^T\int_\sigma^{2\sigma}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}(\sqrt{\rho^\varepsilon})_r\Big|\,|\varphi(t,r)\chi'_\sigma(r)|\,r^{N-1}\mathrm{d}r\mathrm{d}t\\&\leqq C\int_0^T\int_\sigma^{2\sigma}\Big(\frac{|m^\varepsilon|^2}{\rho^\varepsilon}+\varepsilon^2\big|(\sqrt{\rho^\varepsilon})_r\big|^2\Big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\to 0\quad\text{as }\sigma\to 0,\end{aligned}\tag{5.60}$$

$$\left|\int_{\mathbb{R}^{N+1}_+}\chi'_\sigma(r)\sqrt{\rho^\varepsilon}\Big\{V^\varepsilon\frac{\mathbf{x}\otimes\mathbf{x}}{r^2}+\frac{\sqrt{\varepsilon}}{r}\frac{\mu^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big(I_{N\times N}-\frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\Big)\Big\}:\Big(\psi\otimes\frac{\mathbf{x}}{r}\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}t\right|$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}_+^{N+1}} \chi_\sigma'(r) \sqrt{\rho^\varepsilon} V^\varepsilon \big(\psi \cdot \frac{\mathbf{x}}{r}\big)\, \mathrm{d}\mathbf{x}\mathrm{d}t \right| \\
&\leqq C \left| \int_0^T \int_\sigma^{2\sigma} \sqrt{\rho^\varepsilon} V^\varepsilon\, r^{N-1} \mathrm{d}r\mathrm{d}t \right| \to 0 \qquad \text{as } \sigma \to 0. \qquad (5.61)
\end{aligned}
$$

Using (5.59)–(5.61), Lebesgue's dominated convergence theorem, and Proposition 5.7, we obtain

$$
\begin{aligned}
&\lim_{\sigma\to 0} \int_{\mathbb{R}_+^{N+1}} \Big\{ \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \nabla\big) \Psi_\sigma + p(\rho^\varepsilon)\, \mathrm{div} \Psi_\sigma \Big\}\, \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad = \int_{\mathbb{R}_+^{N+1}} \Big\{ \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \nabla\big) \psi + p(\rho^\varepsilon)\, \mathrm{div} \psi \Big\}\, \mathrm{d}\mathbf{x}\mathrm{d}t, \qquad (5.62) \\
&\lim_{\sigma\to 0} \int_{\mathbb{R}_+^{N+1}} \Big\{ \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \big(\nabla\sqrt{\rho^\varepsilon} \cdot \nabla\big) \Psi_\sigma + \big(\nabla\sqrt{\rho^\varepsilon}\big) \cdot \big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \nabla\big) \Psi_\sigma \Big\}\, \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad = \int_{\mathbb{R}_+^{N+1}} \Big\{ \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \big(\nabla\sqrt{\rho^\varepsilon} \cdot \nabla\big) \psi + \big(\nabla\sqrt{\rho^\varepsilon}\big) \cdot \big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \cdot \nabla\big) \psi \Big\}\, \mathrm{d}\mathbf{x}\mathrm{d}t, \qquad (5.63) \\
&\lim_{\sigma\to 0} \int_{\mathbb{R}_+^{N+1}} \sqrt{\rho^\varepsilon} \Big\{ V^\varepsilon \frac{\mathbf{x}\otimes\mathbf{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r} \frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}} \big(I_{N\times N} - \frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big) \Big\} : \nabla \Psi_\sigma\, \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad = \int_{\mathbb{R}_+^{N+1}} \sqrt{\rho^\varepsilon} \Big\{ V^\varepsilon \frac{\mathbf{x}\otimes\mathbf{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r} \frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}} \big(I_{N\times N} - \frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big) \Big\} : \nabla \psi\, \mathrm{d}\mathbf{x}\mathrm{d}t. \qquad (5.64)
\end{aligned}
$$

We notice that

$$
\begin{aligned}
\Delta(\Psi_\sigma)_i &= \chi_\sigma(r)\Delta\psi_i + 2\nabla\psi_i \cdot \nabla\chi_\sigma(r) + \psi_i \Delta\chi_\sigma(r), \\
\partial_i \mathrm{div}\Psi_\sigma &= \chi_\sigma(r)\, \partial_i \mathrm{div}\psi + \mathrm{div}\psi\, \partial_i \chi_\sigma(r) + \partial_i \psi \cdot \nabla\chi_\sigma(r) \\
&\qquad + \frac{x_i}{r} \chi_\sigma''(r)\, \psi \cdot \frac{\mathbf{x}}{r} + \chi_\sigma'(r) \psi \cdot \big(\frac{\nabla x_i}{r} - \frac{x_i}{r^2}\frac{\mathbf{x}}{r}\big).
\end{aligned} \qquad (5.65)
$$

It follows from (5.57) and Proposition 5.7 that

$$
\begin{aligned}
&\Big| \sum_{i=1}^N \varepsilon \int_{\mathbb{R}_+^{N+1}} m^\varepsilon \frac{x_i}{r} \Big\{ 2\nabla\psi_i \cdot \nabla\chi_\sigma + \psi_i \Delta\chi_\sigma + \mathrm{div}\psi\, \partial_i \chi_\sigma(r) + \partial_i \psi \cdot \nabla\chi_\sigma(r) \\
&\qquad + \frac{x_i}{r} \chi_\sigma''(r) \big(\psi \cdot \frac{\mathbf{x}}{r}\big) + \chi_\sigma'(r) \big(\psi \cdot \frac{\nabla x_i}{r} - \big(\psi \cdot \frac{\mathbf{x}}{r}\big) \frac{x_i}{r^2}\big) \Big\}\, \mathrm{d}\mathbf{x}\mathrm{d}t \Big| \\
&\leqq C(\|\psi\|_{C^1}) \int_0^T \int_\sigma^{2\sigma} \varepsilon |m^\varepsilon| \Big(|\chi_\sigma'(r)| + \frac{1}{r} \varphi(r) |\chi_\sigma'(r)| + \varphi(r) |\chi_\sigma''(r)| \Big)\, r^{N-1} \mathrm{d}r\mathrm{d}t \\
&\leqq C(\|\psi\|_{C^1}) \int_0^T \int_\sigma^{2\sigma} \varepsilon |m^\varepsilon|\, r^{N-2} \mathrm{d}r\mathrm{d}t \\
&\leqq C(\|\psi\|_{C^1}) \Big\{ \int_0^T \int_\sigma^{2\sigma} \rho^\varepsilon\, r^{N-1} \mathrm{d}r\mathrm{d}t \Big\}^{\frac12} \Big\{ \varepsilon \int_0^T \int_\sigma^{2\sigma} \frac{|m^\varepsilon|^2}{\rho^\varepsilon}\, r^{N-3} \mathrm{d}r\mathrm{d}t \Big\}^{\frac12} \\
&\to 0 \qquad \text{as } \sigma \to 0. \qquad (5.66)
\end{aligned}
$$

Using (5.65)–(5.66), Lebesgue's dominated convergence theorem, and Proposition 5.7, we have

$$\lim_{\sigma\to 0} \varepsilon \int_{\mathbb{R}_+^{N+1}} \mathcal{M}^\varepsilon \cdot \big(\Delta\Psi_\sigma + \nabla \mathrm{div}\Psi_\sigma\big)\, \mathrm{d}\mathbf{x}\mathrm{d}t \\ = \varepsilon \int_{\mathbb{R}_+^{N+1}} \mathcal{M}^\varepsilon \cdot \big(\Delta\psi + \nabla \mathrm{div}\psi\big)\, \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{5.67}$$

Substituting (5.58), (5.62)–(5.64), and (5.67) into (5.55), we conclude (5.42)–(5.43). □

Remark 5.10. It is not so clear to show that the right-hand side terms of (5.42) vanish as $\varepsilon \to 0$ by direct arguments. However, we can prove the vanishing of these terms by using (5.43), which is the main reason why the form of (5.43) is important to us.

We also need the H^{-1}_{loc}-compactness of weak entropy dissipation measures of $(\rho^\varepsilon, m^\varepsilon)$.

Lemma 5.11 (H^{-1}_{loc}-compactness). *Let (η, q) be a weak entropy pair defined in (2.7) for any smooth compact supported function $\psi(s)$ on $\mathbb{R}$. Then, for $\varepsilon \in (0,1]$,*

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon) \qquad \textit{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+). \tag{5.68}$$

Proof. To obtain (5.68), we have to be careful since $(\rho^\varepsilon, \mathcal{M}^\varepsilon)$ is a weak solution of the Navier–Stokes equations (1.3). In fact, we first have to study the equation for $\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon)$ in the distributional sense, which is much complicated than that in [15,17]. We divide the proof into five steps.

1. Since

$$\eta(\rho, m) = \rho \int_{-1}^{1} \psi(u + \rho^\theta s)[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s, \\ q(\rho, m) = \rho \int_{-1}^{1} (u + \theta\rho^\theta s)\psi(u + \rho^\theta s)[1-s^2]_+^{\mathfrak{b}}\, \mathrm{d}s,$$

it follows from [15, Lemma 2.1] that

$$|\eta(\rho,m)| + |q(\rho,m)| \leqq C_\psi \rho \qquad \text{for } \gamma \in (1,3], \tag{5.69}$$

$$|\eta(\rho,m)| \leqq C_\psi \rho, \quad |q(\rho,m)| \leqq C_\psi\big(\rho + \rho^{1+\theta}\big) \quad \text{for } \gamma \in (3,\infty), \tag{5.70}$$

$$|\partial_\rho \eta(\rho,m)| \leqq C_\psi\big(1+\rho^\theta\big), \quad |\partial_m \eta(\rho,m)| \leqq C_\psi. \tag{5.71}$$

Moreover, if $\partial_m \eta(\rho, \rho u)$ is regarded as a function of (ρ, u), then

$$|\partial_{m\rho}\eta| \leqq C_\psi \rho^{\theta-1}, \qquad |\partial_{mu}\eta| \leqq C_\psi. \tag{5.72}$$

2. Denote $(\eta^{\varepsilon,\delta}, q^{\varepsilon,\delta}) := (\eta, q)(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta})$ and $(\eta^\varepsilon, q^\varepsilon) := (\eta, q)(\rho^\varepsilon, m^\varepsilon)$ for simplicity. Multiply $(3.1)_1$ by $\eta_\rho^{\varepsilon,\delta}$, $(3.1)_2$ by $\eta_m^{\varepsilon,\delta}$, and add them together to obtain

$$
\begin{aligned}
&\partial_t \eta^{\varepsilon,\delta} + \partial_r q^{\varepsilon,\delta} \\
&= -\frac{N-1}{r} m^{\varepsilon,\delta}\big(\eta_\rho^{\varepsilon,\delta} + u^{\varepsilon,\delta}\eta_m^{\varepsilon,\delta}\big) \\
&\quad + \varepsilon \partial_m \eta^{\varepsilon,\delta}\Big\{\big(\rho^{\varepsilon,\delta}(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta})\big)_r - \frac{N-1}{r}\rho_r^{\varepsilon,\delta}u^{\varepsilon,\delta}\Big\} \\
&\quad + \varepsilon \partial_m \eta^{\varepsilon,\delta}\Big\{\alpha\delta\big((\rho^{\varepsilon,\delta})^\alpha (u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta})\big)_r - \delta\frac{N-1}{r}\big((\rho^{\varepsilon,\delta})^\alpha\big)_r u^{\varepsilon,\delta}\Big\}.
\end{aligned} \tag{5.73}
$$

Let $\phi(t,r) \in C_0^\infty(\mathbb{R}_+^2)$, and let $\delta \ll 1$ so that $\operatorname{supp}(\phi(t,\cdot)) \Subset (\delta,\infty)$. Multiplying (5.73) by ϕ and integrating by parts, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+^2} \big(\partial_t \eta^{\varepsilon,\delta} + \partial_r q^{\varepsilon,\delta}\big)\phi \, \mathrm{d}r\mathrm{d}t \\
&= -\int_{\mathbb{R}_+^2} \frac{N-1}{r} m^{\varepsilon,\delta}\big(\eta_\rho^{\varepsilon,\delta} + u^{\varepsilon,\delta}\eta_m^{\varepsilon,\delta}\big)\phi \, \mathrm{d}r\mathrm{d}t \\
&\quad - \varepsilon \int_{\mathbb{R}_+^2} \rho^{\varepsilon,\delta}(\partial_m \eta^{\varepsilon,\delta})_r \, \big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\big)\phi \, \mathrm{d}r\mathrm{d}t \\
&\quad - \varepsilon \int_{\mathbb{R}_+^2} \rho^{\varepsilon,\delta}\partial_m \eta^{\varepsilon,\delta} \, \big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\big)\phi_r \, \mathrm{d}r\mathrm{d}t \\
&\quad - \varepsilon \int_{\mathbb{R}_+^2} \partial_m \eta^{\varepsilon,\delta}\frac{N-1}{r}\rho_r^{\varepsilon,\delta}u^{\varepsilon,\delta}\phi \, \mathrm{d}r\mathrm{d}t \\
&\quad - \alpha\varepsilon\delta \int_{\mathbb{R}_+^2} (\rho^{\varepsilon,\delta})^\alpha (\partial_m \eta^{\varepsilon,\delta})_r \, \big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\big)\phi \, \mathrm{d}r\mathrm{d}t \\
&\quad - \alpha\varepsilon\delta \int_{\mathbb{R}_+^2} \Big((\rho^{\varepsilon,\delta})^\alpha \partial_m \eta^{\varepsilon,\delta} \, \big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\big)\phi_r \\
&\qquad + \partial_m \eta^{\varepsilon,\delta}\frac{N-1}{r}(\rho^{\varepsilon,\delta})^{\alpha-1}\rho_r^{\varepsilon,\delta}u^{\varepsilon,\delta}\phi\Big) \, \mathrm{d}r\mathrm{d}t \\
&:= \sum_{j=1}^{6} I_j^{\varepsilon,\delta}.
\end{aligned} \tag{5.74}
$$

3. It is direct to see that

$$
\eta^{\varepsilon,\delta} \to \eta^\varepsilon \qquad \text{almost everywhere in } \{(t,r)\,:\,\rho^\varepsilon(t,r) \neq 0\} \text{ as } \delta \to 0+. \tag{5.75}
$$

In $\{(t,r)\,:\,\rho^\varepsilon(t,r) = 0\}$,

$$
|\eta^{\varepsilon,\delta}| \leqq C_\psi \rho^{\varepsilon,\delta} \to 0 = \eta^\varepsilon \qquad \text{as } \delta \to 0+. \tag{5.76}
$$

Thus, combining (5.75) with (5.76), we have

$$
\eta^{\varepsilon,\delta} \to \eta^\varepsilon \qquad \text{almost everywhere as } \delta \to 0+. \tag{5.77}
$$

Similarly, we have

$$
q^{\varepsilon,\delta} \to q^\varepsilon \qquad \text{almost everywhere as } \delta \to 0+. \tag{5.78}
$$

Let $K \Subset (0, \infty)$ be any compact subset. For $\gamma \in (1, 3]$, it follows from (5.12) and (5.69) that

$$\int_0^T \int_K \big(|\eta^{\varepsilon,\delta}| + |q^{\varepsilon,\delta}|\big)^{\gamma+1} \, \mathrm{d}r\mathrm{d}t \leqq C_\psi \int_0^T \int_K |\rho^{\varepsilon,\delta}|^{\gamma+1} \, \mathrm{d}r\mathrm{d}t \leqq C_\psi(K, T, E_0). \tag{5.79}$$

For $\gamma \in (3, \infty)$, it follows from (5.13) and (5.70) that

$$\int_0^T \int_K \big(|\eta^{\varepsilon,\delta}| + |q^{\varepsilon,\delta}|\big)^{\frac{\gamma+\theta}{1+\theta}} \, \mathrm{d}r\mathrm{d}t \leqq C_\psi \int_0^T \int_K \big(|\rho^{\varepsilon,\delta}|^{\frac{\gamma+\theta}{1+\theta}} + |\rho^{\varepsilon,\delta}|^{\gamma+\theta}\big) \, \mathrm{d}r\mathrm{d}t \leqq C_\psi(K, T, E_0). \tag{5.80}$$

We take $p_1 = \gamma + 1 > 2$ when $\gamma \in (1, 3]$, and $p_1 = \frac{\gamma+\theta}{1+\theta} > 2$ when $\gamma \in (3, \infty)$. Then it follows from (5.79)–(5.80) that

$$(\eta^{\varepsilon,\delta}, q^{\varepsilon,\delta}) \qquad \text{is uniformly bounded in } L^{p_1}_{\mathrm{loc}}(\mathbb{R}^2_+), \tag{5.81}$$

which, with (5.77)–(5.78), implies that, up to a subsequence,

$$(\eta^{\varepsilon,\delta}, q^{\varepsilon,\delta}) \to (\eta^{\varepsilon}, q^{\varepsilon}) \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^2_+) \text{ as } \delta \to 0+.$$

Thus, for any $\phi \in C^1_0(\mathbb{R}^2_+)$, we see that, as $\delta \to 0+$ (up to a subsequence),

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \big(\partial_t \eta^{\varepsilon,\delta} + \partial_r q^{\varepsilon,\delta}\big)\phi \, \mathrm{d}r\mathrm{d}t = -\int_{\mathbb{R}^2_+} \big(\eta^{\varepsilon,\delta}\partial_t\phi + q^{\varepsilon,\delta}\partial_r\phi\big) \, \mathrm{d}r\mathrm{d}t \\ &\to -\int_{\mathbb{R}^2_+} \big(\eta^{\varepsilon}\partial_t\phi + q^{\varepsilon}\partial_r\phi\big) \, \mathrm{d}r\mathrm{d}t = \int_{\mathbb{R}^2_+} \big(\partial_t \eta^{\varepsilon} + \partial_r q^{\varepsilon}\big)\phi \, \mathrm{d}r\mathrm{d}t. \end{aligned} \tag{5.82}$$

Furthermore, $(\eta^\varepsilon, q^\varepsilon)$ is uniformly bounded in $L^{p_1}_{\mathrm{loc}}(\mathbb{R}^2_+)$ for some $p_1 > 2$, which implies that

$$\partial_t \eta^\varepsilon + \partial_r q^\varepsilon \qquad \text{is uniformly bounded in } \varepsilon > 0 \text{ in } W^{-1,p_1}_{\mathrm{loc}}(\mathbb{R}^2_+). \tag{5.83}$$

4. Now we estimate the terms on the right-hand side of (5.74). For $I_1^{\varepsilon,\delta}$, a direct calculation shows that $|\eta_\rho + u\eta_m| \leqq C_\psi\big(1 + \rho^\theta\big)$, which, together Lemma 5.6 and similar arguments as to those in (5.75)–(5.77), leads to

$$\frac{N-1}{r} m^{\varepsilon,\delta}\big(\eta^{\varepsilon,\delta}_\rho + u^{\varepsilon,\delta}\eta^{\varepsilon,\delta}_m\big) \to \frac{N-1}{r} m^{\varepsilon}\big(\eta^{\varepsilon}_\rho + u^{\varepsilon}\eta^{\varepsilon}_m\big) \qquad \text{almost everywhere as } \delta \to 0+. \tag{5.84}$$

Then it follows from (5.12)–(5.13) that

$$\begin{aligned} &\int_0^T \int_K \Big|\frac{N-1}{r} m^{\varepsilon,\delta}\big(\eta^{\varepsilon,\delta}_\rho + u^{\varepsilon,\delta}\eta^{\varepsilon,\delta}_m\big)\Big|^{\frac{7}{6}} \mathrm{d}r\mathrm{d}t \\ &\leqq C(K) \int_0^T \int_K \big(\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2 + \rho^{\varepsilon,\delta} + (\rho^{\varepsilon,\delta})^\gamma\big)^{\frac{7}{6}} \, \mathrm{d}r\mathrm{d}t \end{aligned}$$

$$
\leqq \begin{cases} C(K)\Big(1+\int_0^T\int_K \rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^3\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{7}{9}}\Big(\int_0^T\int_K \big(1+|\rho^{\varepsilon,\delta}|^{\gamma+1}\big)\mathrm{d}r\mathrm{d}t\Big)^{\frac{2}{9}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \gamma\in(1,3], \\ C(K)\Big(1+\int_0^T\int_K \rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^3\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{7}{9}}\Big(\int_0^T\int_K \big(1+|\rho^{\varepsilon,\delta}|^{\gamma+\theta}\big)\mathrm{d}r\mathrm{d}t\Big)^{\frac{2}{9}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \gamma\in(3,\infty) \end{cases}
$$

$$
\leqq C(K,T,E_0). \tag{5.85}
$$

Using (5.84)–(5.85), we have

$$
I_1^{\varepsilon,\delta} \to -\int_{\mathbb{R}^2_+} \frac{N-1}{r} m^\varepsilon\big(\eta^\varepsilon_\rho + u^\varepsilon \eta^\varepsilon_m\big)\phi\,\mathrm{d}r\mathrm{d}t \ \text{ as } \delta\to 0+ \ \text{ (up to a subsequence)}, \tag{5.86}
$$

$$
\int_0^T\int_K \Big|\frac{N-1}{r} m^\varepsilon\big(\eta^\varepsilon_\rho + u^\varepsilon\eta^\varepsilon_m\big)\Big|^{\frac{7}{6}}\mathrm{d}r\mathrm{d}t \leqq C(K,T,E_0). \tag{5.87}
$$

For $I_2^{\varepsilon,\delta}$, $I_4^{\varepsilon,\delta}$, and $I_5^{\varepsilon,\delta}$, it follows from (5.10)–(5.11) and (5.71)–(5.72) that

$$
\begin{aligned}
&\int_0^T\int_K \Big|\varepsilon\rho^{\varepsilon,\delta}(\partial_m\eta^{\varepsilon,\delta})_r\,\Big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}\frac{m^{\varepsilon,\delta}}{\rho^{\varepsilon,\delta}}\Big)\Big|\,\mathrm{d}r\mathrm{d}t \\
&\leqq C_\psi(K)\int_0^T\int_K \big(\varepsilon\rho^{\varepsilon,\delta}|u_r^{\varepsilon,\delta}|^2 + \varepsilon(\rho^{\varepsilon,\delta})^{\gamma-2}|\rho_r^{\varepsilon,\delta}|^2 + \rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2\big)\,\mathrm{d}r\mathrm{d}t \\
&\leqq C_\psi(K,T,E_0), \\
&\int_0^T\int_K \Big|\varepsilon\frac{N-1}{r}\partial_m\eta^{\varepsilon,\delta}\rho_r^{\varepsilon,\delta}u^{\varepsilon,\delta}\Big|\,\mathrm{d}r\mathrm{d}t \\
&\leqq C_\psi(K)\Big(\varepsilon^2\int_0^T\int_K \frac{|\rho_r^{\varepsilon,\delta}|^2}{\rho^{\varepsilon,\delta}}\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^T\int_K \rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\leqq C_\psi(K,T,E_0), \\
&\int_0^T\int_K \Big|\varepsilon\delta(\rho^{\varepsilon,\delta})^\alpha(\partial_m\eta^{\varepsilon,\delta})_r\,\Big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\Big)\Big|\,\mathrm{d}r\mathrm{d}t \\
&\leqq C_\psi(K)\int_0^T\int_K \varepsilon\delta(\rho^{\varepsilon,\delta})^\alpha\big(|u_r^{\varepsilon,\delta}|^2 + (\rho^{\varepsilon,\delta})^{\gamma-3}|\rho_r^{\varepsilon,\delta}|^2 + |u^{\varepsilon,\delta}|^2\big)\,\mathrm{d}r\mathrm{d}t \\
&\leqq C_\psi(K,T,E_0).
\end{aligned}
$$

Thus, there exist local bounded Radon measures μ_1^ε, μ_2^ε, and μ_3^ε on $\mathbb{R}^2_+$ so that, as $\delta\to 0+$ (up to a subsequence),

$$
\begin{aligned}
&-\varepsilon\rho^{\varepsilon,\delta}(\partial_m\eta^{\varepsilon,\delta})_r\,\Big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\Big) \rightharpoonup \mu_1^\varepsilon, \\
&-\varepsilon\partial_m\eta^{\varepsilon,\delta}\frac{N-1}{r}\rho_r^{\varepsilon,\delta}u^{\varepsilon,\delta} \rightharpoonup \mu_2^\varepsilon, \\
&-\alpha\varepsilon\delta(\rho^{\varepsilon,\delta})^\alpha(\partial_m\eta^{\varepsilon,\delta})_r\,\Big(u_r^{\varepsilon,\delta} + \frac{N-1}{r}u^{\varepsilon,\delta}\Big) \rightharpoonup \mu_3^\varepsilon.
\end{aligned}
$$

In addition,

$$\mu_i^\varepsilon((0,T)\times V) \leqq C_\psi(K,T,E_0) \qquad \text{for } i=1,2,3, \tag{5.88}$$

for each open subset $V \subset K$. Then we have

$$I_2^{\varepsilon,\delta}+I_4^{\varepsilon,\delta}+I_5^{\varepsilon,\delta} \to \langle \mu_1^\varepsilon+\mu_2^\varepsilon+\mu_3^\varepsilon,\, \phi\rangle \quad \text{as } \delta\to 0+ \ \text{(up to a subsequence)}. \tag{5.89}$$

For $I_3^{\varepsilon,\delta}$, we notice from (5.10) that

$$\begin{aligned}
&\int_0^T\int_K \Big|\sqrt{\varepsilon}\rho^{\varepsilon,\delta}\partial_m\eta^{\varepsilon,\delta}\,\big(u_r^{\varepsilon,\delta}+\frac{N-1}{r}u^{\varepsilon,\delta}\big)\Big|^{\frac{4}{3}}\mathrm{d}r\mathrm{d}t\\
&\leqq C_\psi(K)\int_0^T\int_K \Big|\sqrt{\varepsilon}\rho^{\varepsilon,\delta}(|u_r^{\varepsilon,\delta}|+|u^{\varepsilon,\delta}|)\Big|^{\frac{4}{3}}\mathrm{d}r\mathrm{d}t\\
&\leqq C_\psi(K)\Big(\varepsilon\int_0^T\int_K \big(\rho^{\varepsilon,\delta}|u_r^{\varepsilon,\delta}|^2+\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2\big)\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{2}{3}}\Big(\int_0^T\int_K |\rho^{\varepsilon,\delta}|^2\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{3}}\\
&\leqq C_\psi(K,T,E_0).
\end{aligned}$$

Then there exists a function f^ε such that, as $\delta \to 0+$ (up to a subsequence),

$$\sqrt{\varepsilon}\rho^{\varepsilon,\delta}\partial_m\eta^{\varepsilon,\delta}\,\big(u_r^{\varepsilon,\delta}+\frac{N-1}{r}u^{\varepsilon,\delta}\big) \rightharpoonup f^\varepsilon \quad \text{weakly in } L^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}^2_+), \tag{5.90}$$

$$\int_0^T\int_K |f^\varepsilon|^{\frac{4}{3}}\,\mathrm{d}r\mathrm{d}t \leqq C_\psi(K,T,E_0). \tag{5.91}$$

It follows from (5.90) that, as $\delta \to 0+$ (up to a subsequence),

$$I_3^{\varepsilon,\delta} \to \sqrt{\varepsilon}\int_0^T\int_K f^\varepsilon\phi_r\,\mathrm{d}r\mathrm{d}t. \tag{5.92}$$

For $I_6^{\varepsilon,\delta}$, it follows from (5.10)–(5.11) and (5.71) that

$$\begin{aligned}
|I_6^{\varepsilon,\delta}| &\leqq C_\psi(\mathrm{supp}\,\phi)\varepsilon\delta\int_{\mathbb{R}^2_+}\Big((\rho^{\varepsilon,\delta})^\alpha\big(|u_r^{\varepsilon,\delta}|+|u^{\varepsilon,\delta}|\big)\phi_r+\big|(\rho^{\varepsilon,\delta})^{\alpha-1}\rho_r^{\varepsilon,\delta}u^{\varepsilon,\delta}\phi\big|\Big)\,\mathrm{d}r\mathrm{d}t\\
&\leqq C_\psi(\mathrm{supp}\,\phi)\varepsilon\delta\Big(\int_{\mathbb{R}^2_+}\big(\rho^{\varepsilon,\delta}|u_r^{\varepsilon,\delta}|^2+\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2\big)|\phi_r|\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\qquad\times\Big(\int_{\mathbb{R}^2_+}(\rho^{\varepsilon,\delta}+1)\,|\phi_r|\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\quad+C_\psi(\mathrm{supp}\,\phi)\sqrt{\delta}\Big(\varepsilon^2\delta\int_{\mathbb{R}^2_+}(\rho^{\varepsilon,\delta})^{\alpha-2}|\rho_r^{\varepsilon,\delta}|^2\,|\phi|\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\qquad\times\Big(\int_{\mathbb{R}^2_+}(\rho^{\varepsilon,\delta})^\alpha|u^{\varepsilon,\delta}|^2\,|\phi|\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\leqq C_\psi(\mathrm{supp}\,\phi,\|\phi\|_{C^1},T,E_0)\sqrt{\delta}\\
&\qquad\times\Big(\sqrt{\varepsilon}+\big(\int_{\mathbb{R}^2_+}\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^3\,|\phi|\,\mathrm{d}r\mathrm{d}t\big)^{\frac{1}{3}}\big(\int_{\mathbb{R}^2_+}|\rho^{\varepsilon,\delta}|^{3(\alpha-\frac{2}{3})}\,|\phi|\,\mathrm{d}r\mathrm{d}t\big)^{\frac{1}{6}}\Big)\\
&\leqq C_\psi(\mathrm{supp}\,\phi,\|\phi\|_{C^1},T,E_0)\big(\sqrt{\varepsilon}+1\big)\sqrt{\delta}\to 0 \qquad \text{as } \delta\to 0+,
\end{aligned} \tag{5.93}$$

where we have used $\alpha = \frac{2N-1}{2N} \in [\frac{3}{4}, 1)$ for $N \geqq 2$.

5. Taking $\delta \to 0+$ (up to a subsequence) on both sides of (5.74), then it follows from (5.82), (5.86), (5.89), and (5.92)–(5.93) that

$$\partial_t \eta^\varepsilon + \partial_r q^\varepsilon = -\frac{N-1}{r} m^\varepsilon \big(\eta_\rho^\varepsilon + u^\varepsilon \eta_m^\varepsilon\big) + \mu_1^\varepsilon + \mu_2^\varepsilon + \mu_3^\varepsilon - \sqrt{\varepsilon} f_r^\varepsilon \tag{5.94}$$

in the sense of distributions. From (5.87)–(5.88), we see that

$$-\frac{N-1}{r} m^\varepsilon \big(\eta_\rho^\varepsilon + u^\varepsilon \eta_m^\varepsilon\big) + \mu_1^\varepsilon + \mu_2^\varepsilon + \mu_3^\varepsilon \tag{5.95}$$

are bounded uniformly in $\varepsilon > 0$ as Radon measures. From (5.91), we have

$$\sqrt{\varepsilon} f_r^\varepsilon \to 0 \qquad \text{in } W_{\mathrm{loc}}^{-1,\frac{4}{3}}(\mathbb{R}_+^2) \text{ as } \varepsilon \to 0+. \tag{5.96}$$

Thus, it follows from (5.95)–(5.96) that

$$\partial_t \eta^\varepsilon + \partial_r q^\varepsilon \text{ is confined in a compact subset of } W_{\mathrm{loc}}^{-1,p_2}(\mathbb{R}_+^2) \text{ for some } p_2 \in (1,2). \tag{5.97}$$

The interpolation compactness theorem (*cf.* [13,22]) indicates that, for $p_2 > 1$, $p_1 \in (p_2, \infty]$, and $p_0 \in [p_2, p_1)$,

$$\begin{aligned}&(\text{compact set of } W_{\mathrm{loc}}^{-1,p_2}(\mathbb{R}_+^2)) \cap (\text{bounded set of } W_{\mathrm{loc}}^{-1,p_1}(\mathbb{R}_+^2))\\ &\subset (\text{compact set of } W_{\mathrm{loc}}^{-1,p_0}(\mathbb{R}_+^2)),\end{aligned}$$

which is a generalization of Murat's lemma in [47,57]. Combining this interpolation compactness theorem for $1 < p_2 < 2$, $p_1 > 2$, and $p_0 = 2$ with the facts in (5.83) and (5.97), we conclude (5.68). □

Combining Proposition 5.7 with Lemmas 5.8–5.9 and 5.11, we have

Theorem 5.12. *Let* $(\rho_0^\varepsilon, m_0^\varepsilon)$ *be the initial data satisfying* (2.11)–(2.14). *For each* $\varepsilon > 0$, *there exists a spherical symmetry weak solution*

$$(\rho^\varepsilon, \mathcal{M}^\varepsilon)(t, \mathbf{x}) := (\rho^\varepsilon(t,r), m^\varepsilon(t,r)\frac{\mathbf{x}}{r})$$

of the compressible Navier–Stokes equations (1.3) *in the sense of Definition* 2.3. *Moreover,* $(\rho^\varepsilon, m^\varepsilon)(t,r) = (\rho^\varepsilon(t,r), \rho^\varepsilon(t,r)u^\varepsilon(t,r))$, *with* $u^\varepsilon(t,r) := \frac{m^\varepsilon(t,r)}{\rho^\varepsilon(t,r)}$ *almost everywhere on* $\{(t,r) : \rho^\varepsilon(t,r) \neq 0\}$ *and* $u^\varepsilon(t,r) := 0$ *almost everywhere on* $\{(t,r) : \rho^\varepsilon(t,r) = 0 \text{ or } r = 0\}$, *satisfies the following bounds*:

$$\begin{aligned}&\rho^\varepsilon(t,r) \geqq 0 \quad \textit{almost everywhere},\\ &\Big(\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big)(t,r) = \sqrt{\rho^\varepsilon(t,r)}u^\varepsilon(t,r) = 0 \quad \textit{almost everywhere on } \{(t,r) : \rho^\varepsilon(t,r) = 0\},\\ &\int_0^\infty \Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2 + e(\rho^\varepsilon, \bar{\rho})\Big)(t,r)\, r^{N-1} \mathrm{d}r + \varepsilon \int_{\mathbb{R}_+^2} \Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2(s,r)\, r^{N-3} \mathrm{d}r\mathrm{d}s\\ &\qquad \leqq E_0^\varepsilon \leqq E_0 + 1 \ \textit{for } t > 0,\end{aligned} \tag{5.98}$$

$$\varepsilon^2 \int_0^\infty \big|\big(\sqrt{\rho^\varepsilon(t,r)}\big)_r\big|^2 r^{N-1}\mathrm{d}r + \varepsilon \int_{\mathbb{R}^2_+} \big|\big((\rho^\varepsilon(s,r))^{\frac{\gamma}{2}}\big)_r\big|^2 r^{N-1}\mathrm{d}r\mathrm{d}s$$
$$\leqq C(E_0+1) \qquad \textit{for } t>0,$$
$$\int_0^T \int_d^D (\rho^\varepsilon)^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t \leqq C(d,D,T,E_0), \tag{5.99}$$
$$\int_0^T \int_0^D \big(\rho^\varepsilon|u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta}\big)(t,r)\, r^{N-1}\mathrm{d}r\mathrm{d}t \leqq C(D,T,E_0) \tag{5.100}$$

for any fixed $T>0$ *and any compact subset* $[d,D]\Subset(0,\infty)$.

Let (η,q) *be an entropy pair defined in* (2.7) *for a smooth compact supported function* $\psi(s)$ *on* $\mathbb{R}$. *Then, for* $\varepsilon\in(0,1]$,

$$\partial_t\eta(\rho^\varepsilon,m^\varepsilon)+\partial_r q(\rho^\varepsilon,m^\varepsilon) \qquad \textit{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+).$$

6. Proof of the Main Theorems

In this section, we give a complete proof of Main Theorem II: Theorem 2.4, which leads to Main Theorem I: Theorem 2.2, as indicated in Remark 2.5.

The uniform estimates and compactness properties obtained in Theorem 5.12 imply that the weak solutions $(\rho^\varepsilon,m^\varepsilon)$ of the Navier–Stokes equations (1.7) satisfy the compensated compactness framework in Chen-Perepelitsa [15]. Then the compactness theorem established in [15] for the case $\gamma>1$ (also see LeFloch-Westdickenberg [37] for $\gamma\in(1,5/3]$) implies that there exist functions $(\rho,m)(t,r)$ such that

$$(\rho^\varepsilon,m^\varepsilon)\to(\rho,m) \qquad \textit{almost everywhere}\ (t,r)\in\mathbb{R}^2_+ \ \text{as } \varepsilon\to0+ \ \text{(up to a subsequence)}.$$

By similar arguments as to those in the proof of Lemma 5.6, we find that $m(t,r)=0$ almost everywhere on $\{(t,r)\,:\,\rho(t,r)=0\}$. We can define the limit velocity $u(t,r)$ by setting $u(t,r):=\frac{m(t,r)}{\rho(t,r)}$ almost everywhere on $\{(t,r)\,:\,\rho(t,r)\neq0\}$ and $u(t,r):=0$ almost everywhere on $\{(t,r)\,:\,\rho(t,r)=0 \text{ or } r=0\}$. Then we have

$$m(t,r)=\rho(t,r)u(t,r).$$

We can also define $(\frac{m}{\sqrt{\rho}})(t,r):=\sqrt{\rho(t,r)}u(t,r)$, which is 0 almost everywhere on the vacuum states $\{(t,r)\,:\,\rho(t,r)=0\}$. Moreover, we obtain that, as $\varepsilon\to0+$,

$$\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\equiv\sqrt{\rho^\varepsilon}u^\varepsilon\to\frac{m}{\sqrt{\rho}}\equiv\sqrt{\rho}u \ \ \text{strongly in } \ L^2([0,T]\times[0,D],r^{N-1}\mathrm{d}r\mathrm{d}t). \tag{6.1}$$

Notice that $|m|^{\frac{3(\gamma+1)}{\gamma+3}}\leqq C\big(\frac{|m|^3}{\rho^2}+\rho^{\gamma+1}\big)$, which, along with (5.99)–(5.100), implies

$$(\rho^\varepsilon,m^\varepsilon)\to(\rho,m) \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2_+)\times L^q_{\mathrm{loc}}(\mathbb{R}^2_+) \text{ as } \varepsilon\to0+ \tag{6.2}$$

for $p \in [1, \gamma+1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$, where $L^q_{\mathrm{loc}}(\mathbb{R}^2_+)$ represents $L^q([0,T]\times K)$ for any $T > 0$ and $K \Subset (0, \infty)$.

From the same estimates, we also obtain the convergence of the relative mechanical energy as $\varepsilon \to 0+$:

$$\bar{\eta}^*(\rho^\varepsilon, m^\varepsilon) \to \bar{\eta}^*(\rho, m) \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2_+).$$

Since $\bar{\eta}^*(\rho, m)$ is a convex function, by passing the limit in (5.98), we have

$$\int_{t_1}^{t_2}\int_0^\infty \bar{\eta}^*(\rho, m)(t,r)\, r^{N-1}\mathrm{d}r\mathrm{d}t \leqq (t_2 - t_1)\int_0^\infty \bar{\eta}^*(\rho_0, m_0)(r)\, r^{N-1}\mathrm{d}r,$$

which implies

$$\int_0^\infty \bar{\eta}^*(\rho, m)(t,r)\, r^{N-1}\mathrm{d}r \leqq \int_0^\infty \bar{\eta}^*(\rho_0, m_0)(r)\, r^{N-1}\mathrm{d}r \quad \text{for } \textit{almost everywhere } t \geqq 0. \tag{6.3}$$

This indicates that there is no concentration formed in the density $\rho(t, r)$ at origin $r = 0$.

Define

$$(\rho, \mathcal{M})(t, \mathbf{x}) := (\rho(t,r), m(t,r)\frac{\mathbf{x}}{r}) = (\rho(t,r), \rho(t,r)u(t,r)\frac{\mathbf{x}}{r}). \tag{6.4}$$

From (6.3), we know that $\frac{\mathcal{M}}{\sqrt{\rho}} = \sqrt{\rho}u\,\frac{\mathbf{x}}{r}$ is well-defined and in L^2 for almost everywhere $t > 0$. We now prove that $(\rho, \mathcal{M})(t, \mathbf{x})$ is a weak solution of the Cauchy problem for the Euler equations (1.1) in $\mathbb{R}^N$.

Let $\zeta(t, \mathbf{x}) \in C^1_0([0,\infty)\times\mathbb{R}^N)$ be a smooth, compactly supported function. Then it follows from (5.32) that

$$\int_{\mathbb{R}^{N+1}_+} \big(\rho^\varepsilon\zeta_t + \mathcal{M}^\varepsilon\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathbb{R}^N}\rho_0^\varepsilon(\mathbf{x})\zeta(0,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0. \tag{6.5}$$

Let $\phi(t, r)$ be the corresponding function defined in (5.35). Using (6.2) and similar arguments as in the proof of Lemma 5.8, we obtain that, for any fixed $\sigma > 0$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\int_0^\infty\int_{\mathbb{R}^N\setminus B_\sigma(\mathbf{0})}\big(\rho^\varepsilon\zeta_t + \mathcal{M}^\varepsilon\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\quad = \omega_N \lim_{\varepsilon\to 0+}\int_0^\infty\int_\sigma^\infty\big(\rho^\varepsilon\phi_t + m^\varepsilon\phi_r\big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&\quad = \omega_N\int_0^\infty\int_\sigma^\infty\big(\rho\phi_t + m\phi_r\big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&\quad = \int_0^\infty\int_{\mathbb{R}^N\setminus B_\sigma(\mathbf{0})}\big(\rho\zeta_t + \mathcal{M}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned} \tag{6.6}$$

Using (6.3) and by similar arguments as to those in (5.37), we have

$$\Big|\int_0^\infty\int_{B_\sigma(\mathbf{0})}(\rho^\varepsilon - \rho)\zeta_t\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|$$

$$
\begin{aligned}
&\leqq C(\|\zeta\|_{C^1}, \operatorname{supp}\zeta)\Big\{\int_0^\infty\int_0^\sigma \big((\rho^\varepsilon)^\gamma+\rho^\gamma\big)\,|\phi_t|\,r^{N-1}\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{\gamma}}\sigma^{N(1-\frac{1}{\gamma})}\\
&\leqq C(\|\zeta\|_{C^1}, \operatorname{supp}\zeta, E_0)\,\sigma^{N(1-\frac{1}{\gamma})}\to 0 \qquad \text{as } \sigma\to 0, \qquad (6.7)
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_0^\infty\int_{B_\sigma(\mathbf{0})}(\mathcal{M}^\varepsilon-\mathcal{M})\cdot\nabla\zeta\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|\\
&\leqq C\Big\{\int_0^\infty\int_0^\sigma\Big(\frac{|m^\varepsilon|^2}{\rho^\varepsilon}+\frac{m^2}{\rho}\Big)(t,r)\,|\phi_r|\,r^{N-1}\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}}\\
&\quad\times\Big\{\int_0^\infty\int_0^\sigma\big(\rho^\varepsilon+\rho\big)(t,r)\,|\phi_r|\,r^{N-1}\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}}\\
&\leqq C(\|\zeta\|_{C^1}, \operatorname{supp}\zeta, E_0)\sigma^{\frac{N}{2}(1-\frac{1}{\gamma})}\to 0 \qquad \text{as } \sigma\to 0, \qquad (6.8)
\end{aligned}
$$

which, with (6.6)–(6.8), implies

$$
\lim_{\delta\to 0+}\int_{\mathbb{R}^{N+1}_+}\big(\rho^\varepsilon\zeta_t+\mathcal{M}^\varepsilon\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t=\int_{\mathbb{R}^{N+1}_+}\big(\rho\zeta_t+\mathcal{M}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t. \qquad (6.9)
$$

Letting $\varepsilon\to 0+$ in (6.5) and using (6.9), we conclude that $(\rho,\mathcal{M})$ satisfies (2.4).

Next we consider the momentum equations. Let $\psi=(\psi_1,\ldots,\psi_N)\in\big(C_0^2(\mathbb{R}\times\mathbb{R}^N)\big)^N$ be a smooth function with compact support, and let $\chi_\sigma(r)\in C^\infty(\mathbb{R})$ be a cut-off function satisfying (5.44). Without loss of generality, we assume that $\operatorname{supp}\psi\subset[-T,T]\times B_D(\mathbf{0})$. Denote $\Psi_\sigma=\psi\chi_\sigma$. Then we have

$$
\begin{aligned}
&\Big|\varepsilon\int_{\mathbb{R}^{N+1}_+}\Big\{\frac{1}{2}\mathcal{M}^\varepsilon\cdot\big(\Delta\Psi_\sigma+\nabla\mathrm{div}\Psi_\sigma\big)+\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\big(\nabla\sqrt{\rho^\varepsilon}\cdot\nabla\big)\Psi_\sigma\\
&\qquad\qquad+(\nabla\sqrt{\rho^\varepsilon})\cdot\big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\big)\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|\\
&=\Big|\sqrt{\varepsilon}\int_{\mathbb{R}^{N+1}_+}\sqrt{\rho^\varepsilon}\Big\{V^\varepsilon\frac{\mathbf{x}\otimes\mathbf{x}}{r^2}+\frac{\sqrt{\varepsilon}}{r}\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big(I_{N\times N}-\frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big)\Big\}:\nabla\Psi_\sigma\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|\\
&\leqq C\Big\{\int_{\mathbb{R}^{N+1}_+}|V^\varepsilon|^2\,\mathrm{d}\mathbf{x}+\varepsilon\int_{\mathbb{R}^2_+}\frac{|m^\varepsilon|^2}{\rho^\varepsilon}\,r^{N-3}\mathrm{d}r\mathrm{d}t\Big\}^{\frac{1}{2}}\Big\{\varepsilon\int_{\mathbb{R}^{N+1}_+}\rho^\varepsilon|\nabla\Psi_\sigma|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big\}^{\frac{1}{2}}\\
&\leqq C(\sigma,D,T,E_0)\sqrt{\varepsilon}\to 0 \qquad \text{as } \varepsilon\to 0. \qquad (6.10)
\end{aligned}
$$

Using (6.1) and (6.10), and passing the limit: $\varepsilon\to 0+$ (up to a subsequence) in (5.55), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+}\Big\{\mathcal{M}\cdot\partial_t\Psi_\sigma+\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\big(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla\big)\Psi_\sigma+p(\rho)\,\mathrm{div}\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\quad+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x}=0. \qquad (6.11)
\end{aligned}
$$

Notice that, for any $T>0$ and $D>0$,

$$
\int_0^T\int_0^D\big(\frac{m^2}{\rho}+\rho^\gamma\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\leqq C(D,T,E_0), \qquad (6.12)
$$

which, with similar arguments as to those in (5.58), leads to

$$\lim_{\sigma\to 0}\Big\{\int_{\mathbb{R}^{N+1}_+}\mathcal{M}\cdot\partial_t\Psi_\sigma\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big\}$$
$$=\int_{\mathbb{R}^{N+1}_+}\mathcal{M}\cdot\partial_t\psi\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{6.13}$$

Using (5.56)–(5.57) and (6.12), we have

$$\Big|\int_{\mathbb{R}^{N+1}_+}\big(\frac{m^2}{\rho}+p(\rho)\big)\big(\psi\cdot\frac{\mathbf{x}}{r}\big)\chi'_\sigma(r)\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big|$$
$$\leqq C\int_0^\infty\int_\sigma^{2\sigma}\big(\frac{m^2}{\rho}+p(\rho)\big)\varphi(t,r)|\chi'_\sigma(r)|\,r^{N-1}\mathrm{d}r\mathrm{d}t$$
$$\leqq C\int_0^T\int_\sigma^{2\sigma}\big(\frac{m^2}{\rho}+p(\rho)\big)\,r^{N-1}\mathrm{d}r\mathrm{d}t\to 0\qquad\text{as }\sigma\to 0,$$

which, with (6.12) and the Lebesgue dominated convergence theorem, leads to

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^{N+1}_+}\Big\{\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\big(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla\big)\Psi_\sigma+p(\rho)\,\mathrm{div}\Psi_\sigma\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t$$
$$=\int_{\mathbb{R}^{N+1}_+}\Big\{\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\big(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla\big)\psi+p(\rho)\,\mathrm{div}\psi\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t. \tag{6.14}$$

Substituting (6.13)–(6.14) into (6.11), we conclude that $(\rho,\mathcal{M})$ satisfies (2.5).

By the Lebesgue theorem, we can weaken the assumption: $\psi\in C_0^2$ as $\psi\in C_0^1$. This completes the proof of Theorem 2.4. □

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Declaration

Conflicts of interest: The authors declare that they have no conflict of interest.

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Appendix A. Construction and Estimates of Approximate Initial Data

In this appendix, we construct the approximate initial data functions with desired estimates and regularity. From (1.5), we know that there exists a constant $R\gg 1$ so that

$$0<\frac{1}{2}\bar{\rho}\leqq\rho_0(r)\leqq\frac{3}{2}\bar{\rho}\qquad\text{for }r\geqq R. \tag{A.1}$$

We first cut-off the density function $\rho_0(r)$ as

$$\tilde{\rho}_0^\varepsilon(r) = \begin{cases} (\beta\varepsilon)^{\frac{1}{4}} & \text{if } \rho_0(r) \leqq (\beta\varepsilon)^{\frac{1}{4}}, \\ \rho_0(r) & \text{if } (\beta\varepsilon)^{\frac{1}{4}} \leqq \rho_0(r) \leqq (\beta\varepsilon)^{-\frac{1}{2}}, \\ (\beta\varepsilon)^{-\frac{1}{2}} & \text{if } \rho_0(r) \geqq (\beta\varepsilon)^{-\frac{1}{2}}, \end{cases} \tag{A.2}$$

where $\varepsilon \in (0, 1]$, and $0 < \beta \ll 1$ is a given small positive constant, which is used to ensure $(\beta\varepsilon)^{\frac{1}{4}} \ll (\beta\varepsilon)^{-\frac{1}{2}}$ for all $\varepsilon \in (0, 1]$. It is easy to check that

$$\tilde{\rho}_0^\varepsilon(r) \leqq \rho_0(r) + 1, \qquad \tilde{\rho}_0^\varepsilon(r) \to \rho_0(r) \text{ as } \varepsilon \to 0 \text{ almost everywhere } r \in \mathbb{R}_+. \tag{A.3}$$

To keep the L^p-properties of mollification, it is more convenient to smooth out the initial data in the original coordinate $\mathbb{R}^N$; so we do not distinguish between functions $(\rho_0, m_0)(r)$ and $(\rho_0, m_0)(\mathbf{x}) = (\rho_0, m_0)(|\mathbf{x}|)$ when no confusion arises. It follows from (2.2)–(2.3) that $\rho_0(\mathbf{x}) \in L^\gamma_{\mathrm{loc}}(\mathbb{R}^N)$. Using the convexity of $e(\rho, \bar{\rho})$, we have

$$0 \leqq e(\tilde{\rho}_0^\varepsilon(\mathbf{x}), \bar{\rho}) \leqq e(\rho_0(\mathbf{x}), \bar{\rho}). \tag{A.4}$$

Combining (2.2) with (A.3)–(A.4) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon \to 0+} \int_K \Big(\big|\tilde{\rho}_0^\varepsilon(\mathbf{x}) - \rho_0(\mathbf{x})\big|^\gamma + \Big|\sqrt{\tilde{\rho}_0^\varepsilon(\mathbf{x})} - \sqrt{\rho_0(\mathbf{x})}\Big|^{2\gamma} \Big) \, \mathrm{d}\mathbf{x} = 0 \tag{A.5}$$

for any $K \Subset \mathbb{R}^N$.
Since we need a better decay property for approximate initial data, we further cut-off the function $\tilde{\rho}_0^\varepsilon(\mathbf{x})$ at the far-field:

$$\hat{\rho}_0^\varepsilon(\mathbf{x}) = \begin{cases} \tilde{\rho}_0^\varepsilon(\mathbf{x}) & \text{if } |\mathbf{x}| \leqq (\beta\varepsilon)^{-\frac{1}{2N}}, \\ \bar{\rho} & \text{if } |\mathbf{x}| > (\beta\varepsilon)^{-\frac{1}{2N}}. \end{cases} \tag{A.6}$$

Here we further choose β small enough so that $|\mathbf{x}| \geqq (\beta\varepsilon)^{-\frac{1}{2N}} \geqq R + 2$ for all $\varepsilon \in (0, 1]$. It is clear that $\hat{\rho}_0^\varepsilon(\mathbf{x})$ is not a smooth function so that we need to mollify $\hat{\rho}_0^\varepsilon(\mathbf{x})$. Let $J(\mathbf{x})$ be the standard mollification function and $J_\sigma(\mathbf{x}) := \frac{1}{\sigma^N} J(\frac{\mathbf{x}}{\sigma})$ for $\sigma \in (0, 1)$. For later use, we take $\sigma = \varepsilon^{\frac{1}{4}}$ and define $\rho_0^\varepsilon(\mathbf{x})$ as

$$\rho_0^\varepsilon(\mathbf{x}) := \Big(\int_{\mathbb{R}^N} \sqrt{\hat{\rho}_0^\varepsilon(\mathbf{x} - \mathbf{y})} J_\sigma(\mathbf{y}) \, \mathrm{d}\mathbf{y} \Big)^2. \tag{A.7}$$

Then $\rho_0^\varepsilon(\mathbf{x})$ is still a spherically symmetric function, *that is*, $\rho_0^\varepsilon(\mathbf{x}) = \rho_0^\varepsilon(|\mathbf{x}|)$.

Lemma A.1. *For any given $\varepsilon \in (0, 1]$, $\rho_0^\varepsilon(\mathbf{x})$ defined in (A.7) is in $C^\infty(\mathbb{R}^N)$ with $(\beta\varepsilon)^{\frac{1}{4}} \leqq \rho_0^\varepsilon(\mathbf{x}) \leqq (\beta\varepsilon)^{-\frac{1}{2}}$ and satisfies*

$$\lim_{\varepsilon\to 0+}\Big(\|\rho_0^\varepsilon-\rho_0\|_{L^\gamma_{\mathrm{loc}}(\mathbb{R}^N)}+\Big|\int_{\mathbb{R}^N}\big(e(\rho_0^\varepsilon(\mathbf{x}),\bar\rho)-e(\rho_0(\mathbf{x}),\bar\rho)\big)\,\mathrm{d}\mathbf{x}\Big|\Big)=0, \quad \text{(A.8)}$$

$$\varepsilon^2\int_{\mathbb{R}^N}\big|\nabla_{\mathbf{x}}\sqrt{\rho_0^\varepsilon(\mathbf{x})}\big|^2\,\mathrm{d}\mathbf{x}\leqq C\sqrt{\varepsilon}, \quad \text{(A.9)}$$

$$\int_{\mathbb{R}^N}e(\rho_0^\varepsilon(\mathbf{x}),\bar\rho)(1+|\mathbf{x}|)^{N-1+\vartheta}\,\mathrm{d}\mathbf{x}\leqq CE_0\varepsilon^{-\frac{N-1+\vartheta}{2N}}, \quad \text{(A.10)}$$

where E_0 is defined in (2.12), and $\vartheta\in(0,1)$.

Proof. We divide the proof into four steps.

1. We first consider the first part of (A.8). A direct calculation shows

$$\begin{aligned}
&\big|\sqrt{\rho_0^\varepsilon(\mathbf{x})}-\sqrt{\rho_0(\mathbf{x})}\big|\\
&\leqq\Big|\int_{\mathbb{R}^N}\big(\sqrt{\hat\rho_0^\varepsilon(\mathbf{x}-\mathbf{y})}-\sqrt{\rho_0(\mathbf{x}-\mathbf{y})}\big)J_\sigma(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big|\\
&\quad+\Big|\int_{\mathbb{R}^N}\big(\sqrt{\rho_0(\mathbf{x}-\mathbf{y})}-\sqrt{\rho_0(\mathbf{x})}\big)J_\sigma(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big|.
\end{aligned} \quad \text{(A.11)}$$

For any given $M\gg 1$, it follows from (A.11) and Hölder's inequality that

$$\begin{aligned}
&\int_{|\mathbf{x}|\leqq M+1}\big|\sqrt{\rho_0^\varepsilon(\mathbf{x})}-\sqrt{\rho_0(\mathbf{x})}\big|^{2\gamma}\mathrm{d}\mathbf{x}\\
&\leqq C\int_{\mathbb{R}^N}J_\sigma(\mathbf{y})\int_{|\mathbf{x}|\leqq M+1}\Big(\big|\sqrt{\hat\rho_0^\varepsilon(\mathbf{x}-\mathbf{y})}-\sqrt{\rho_0(\mathbf{x}-\mathbf{y})}\big|^{2\gamma}\\
&\qquad\qquad+\big|\sqrt{\rho_0(\mathbf{x}-\mathbf{y})}-\sqrt{\rho_0(\mathbf{x})}\big|^{2\gamma}\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
&\leqq C\sup_{|\mathbf{y}|\leqq\varepsilon^{\frac14}}\big\|\sqrt{\rho_0(\cdot+\mathbf{y})}-\sqrt{\rho_0(\cdot)}\big\|_{L^{2\gamma}(\{|\mathbf{x}|\leqq M+1\})}\\
&\quad+C\big\|\sqrt{\tilde\rho_0^\varepsilon}-\sqrt{\rho_0}\big\|_{L^{2\gamma}(\{|\mathbf{x}|\leqq M+2\})}\to 0
\end{aligned} \quad \text{(A.12)}$$

as $\varepsilon\to 0+$, where we have used (A.5), $\sigma=\varepsilon^{\frac14}$, and $\hat\rho_0^\varepsilon(\mathbf{x})=\tilde\rho_0^\varepsilon(\mathbf{x})$ for $|\mathbf{x}|\leqq(\beta\varepsilon)^{-\frac{1}{2N}}$. Using (A.12), we immediately obtain

$$\int_{|\mathbf{x}|\leqq M+1}|\rho_0^\varepsilon(\mathbf{x})-\rho_0(\mathbf{x})|^\gamma\,\mathrm{d}\mathbf{x}\to 0 \qquad \text{as } \varepsilon\to 0. \quad \text{(A.13)}$$

2. We now consider the second part of (A.8). For any given $M\gg 1$, it follows from (A.13) that

$$\lim_{\varepsilon\to 0+}\int_{|\mathbf{x}|\leqq M+1}\big(e(\rho_0^\varepsilon(\mathbf{x}),\bar\rho)-e(\rho_0(\mathbf{x}),\bar\rho)\big)\,\mathrm{d}\mathbf{x}=0. \quad \text{(A.14)}$$

For $|\mathbf{x}|>M+1$ with $M\geqq R+1$, noting (A.1)–(A.2) and (A.6)–(A.7), we have

$$0<\frac12\bar\rho\leqq\rho_0^\varepsilon(\mathbf{x})\leqq\frac32\bar\rho. \quad \text{(A.15)}$$

It follows from (A.2) and (A.6) that $\left|\sqrt{\hat{\rho}_0^\varepsilon(\mathbf{x})}-\sqrt{\bar{\rho}}\right| \leqq \left|\sqrt{\tilde{\rho}_0^\varepsilon(\mathbf{x})}-\sqrt{\bar{\rho}}\right|$ for $\mathbf{x} \in \mathbb{R}^N$, which, with (A.15), yields

$$
\begin{aligned}
&\int_{|\mathbf{x}|>M+1} e(\rho_0^\varepsilon(\mathbf{x}), \bar{\rho})\,\mathrm{d}\mathbf{x} \\
&\quad \leqq C(\bar{\rho}) \int_{|\mathbf{x}|>M+1} \left|\sqrt{\rho_0^\varepsilon(\mathbf{x})}-\sqrt{\bar{\rho}}\right|^2 \mathrm{d}\mathbf{x} \\
&\quad \leqq C(\bar{\rho}) \int_{|\mathbf{x}|>M+1} \left|\int_{\mathbb{R}^N} \left(\sqrt{\hat{\rho}_0^\varepsilon(\mathbf{x}-\mathbf{y})}-\sqrt{\bar{\rho}}\right) J_\sigma(\mathbf{y})\mathrm{d}\mathbf{y}\right|^2 \mathrm{d}\mathbf{x} \\
&\quad \leqq C(\bar{\rho}) \int_{|\mathbf{x}|>M} \left|\sqrt{\hat{\rho}_0^\varepsilon(\mathbf{x})}-\sqrt{\bar{\rho}}\right|^2 \mathrm{d}\mathbf{x} \\
&\quad \leqq C(\bar{\rho}) \int_{|\mathbf{x}|>M} \left|\sqrt{\tilde{\rho}_0^\varepsilon(\mathbf{x})}-\sqrt{\bar{\rho}}\right|^2 \mathrm{d}\mathbf{x} \\
&\quad = C(\bar{\rho}) \int_{|\mathbf{x}|>M} \left|\sqrt{\rho_0(\mathbf{x})}-\sqrt{\bar{\rho}}\right|^2 \mathrm{d}\mathbf{x} \\
&\quad \leqq C(\bar{\rho}) \int_{|\mathbf{x}|>M} e(\rho_0(\mathbf{x}), \bar{\rho})\,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{A.16}
$$

For any given small $\varrho > 0$, there exists $M(\varrho) \gg 1$ such that

$$
\int_{|\mathbf{x}|>M(\varrho)} e(\rho_0(\mathbf{x}), \bar{\rho})\,\mathrm{d}\mathbf{x} \leqq \varrho. \tag{A.17}
$$

Using (A.14) and (A.16)–(A.17), we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} \left(e(\rho_0^\varepsilon(\mathbf{x}), \bar{\rho}) - e(\rho_0(\mathbf{x}), \bar{\rho})\right) \mathrm{d}\mathbf{x}\right| \\
&\quad \leqq \left|\int_{|\mathbf{x}|\leqq M(\varrho)+1} \left(e(\rho_0^\varepsilon(\mathbf{x}), \bar{\rho}) - e(\rho_0(\mathbf{x}), \bar{\rho})\right) \mathrm{d}\mathbf{x}\right| \\
&\qquad + C(\bar{\rho}) \int_{|\mathbf{x}|>M(\varrho)} e(\rho_0(\mathbf{x}), \bar{\rho})\,\mathrm{d}\mathbf{x} \\
&\quad \leqq C(\bar{\rho})\varrho,
\end{aligned}
$$

provided that $\varepsilon \ll 1$. Then (A.8) is proved.

3. Noting (A.6), we have

$$
\partial_{\mathbf{x}_i}\sqrt{\rho_0^\varepsilon(\mathbf{x})} = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \sqrt{\hat{\rho}_0^\varepsilon(\mathbf{x}-\mathbf{y})}\,\partial_{\mathbf{y}_i} J_\sigma(y)\,\mathrm{d}\mathbf{y} & \text{for } |\mathbf{x}| \leqq 1 + (\beta\varepsilon)^{-\frac{1}{2N}}, \\ 0 & \text{for } |\mathbf{x}| \geqq 1 + (\beta\varepsilon)^{-\frac{1}{2N}}, \end{cases}
$$

which, with (A.2) and (A.6), leads to

$$\varepsilon^2 \int_{\mathbb{R}^N} \left|\nabla_{\mathbf{x}}\sqrt{\rho_0^\varepsilon(\mathbf{x})}\right|^2 d\mathbf{x} = \frac{C\varepsilon^2}{\sigma^2} \int_{|\mathbf{x}|\leqq 1+(\beta\varepsilon)^{-\frac{1}{2N}}} \sup_{\mathbf{y}\in\mathbb{R}^N} \hat{\rho}_0^\varepsilon(\mathbf{y})\, d\mathbf{x}$$
$$\leqq \frac{C\varepsilon^2}{\sigma^2}(\beta\varepsilon)^{-1} \leqq C\varepsilon^{\frac{1}{2}},$$

where we have used $\sigma = \varepsilon^{\frac{1}{4}}$. Thus, (A.9) is proved.

4. We finally consider (A.10). Noting (A.6), we see that $\rho_0^\varepsilon(\mathbf{x}) = \bar{\rho}$ for all $|\mathbf{x}| \geqq 1 + (\beta\varepsilon)^{-\frac{1}{2N}}$, which, with (A.8), implies

$$\begin{aligned}
&\int_{\mathbb{R}^N} e(\rho_0^\varepsilon(\mathbf{x}), \bar{\rho})(1+|\mathbf{x}|)^{N-1+\vartheta} d\mathbf{x} \\
&= \int_{|\mathbf{x}|\leqq 1+(\beta\varepsilon)^{-\frac{1}{2N}}} e(\rho_0^\varepsilon(\mathbf{x}), \bar{\rho})(1+|\mathbf{x}|)^{N-1+\vartheta} d\mathbf{x} \\
&\leqq C\varepsilon^{-\frac{N-1+\vartheta}{2N}} \int_{|\mathbf{x}|\leqq 1+(\beta\varepsilon)^{-\frac{1}{2N}}} e(\rho_0^\varepsilon(\mathbf{x}), \bar{\rho})\, d\mathbf{x} \\
&\leqq C(E_0+1)\varepsilon^{-\frac{N-1+\vartheta}{2N}}.
\end{aligned}$$

Therefore, we have proved (A.10). □

Denote $\mathbf{I}_{[4\delta,\delta^{-1}]}(\mathbf{x})$ to be the characteristic function $\{\mathbf{x} \in \mathbb{R}^N : 4\delta \leqq |\mathbf{x}| \leqq \delta^{-1}\}$ with $0 < \delta \ll 1$. Now, for the approximation of the velocity, we define $u_0^\varepsilon(\mathbf{x})$ and $u_0^{\varepsilon,\delta}(\mathbf{x})$:

$$u_0^\varepsilon(\mathbf{x}) := \frac{1}{\sqrt{\rho_0^\varepsilon(\mathbf{x})}}\Big(\frac{m_0}{\sqrt{\rho_0}}\Big)(\mathbf{x}), \tag{A.18}$$

$$u_0^{\varepsilon,\delta}(\mathbf{x}) := \frac{1}{\sqrt{\rho_0^\varepsilon(\mathbf{x})}} \int_{\mathbb{R}^N} \Big(\frac{m_0}{\sqrt{\rho_0}}\mathbf{I}_{[4\delta,\delta^{-1}]}\Big)(\mathbf{x}-\mathbf{y}) J_\delta(\mathbf{y})\, d\mathbf{y}, \tag{A.19}$$

where $\rho_0^\varepsilon(\mathbf{x})$ is the approximate density function defined in Lemma A.1.

Lemma A.2. *The function $u_0^\varepsilon(\mathbf{x})$ defined in (A.18) satisfies*

$$\int_{\mathbb{R}^N} \rho_0^\varepsilon(\mathbf{x})|u_0^\varepsilon(\mathbf{x})|^2\, d\mathbf{x} \equiv \int_{\mathbb{R}^N} \frac{|m_0(\mathbf{x})|^2}{\rho_0(\mathbf{x})}\, d\mathbf{x} \qquad \textit{for any } \varepsilon \in (0,1], \tag{A.20}$$

$$\lim_{\varepsilon\to 0+} \|\rho_0^\varepsilon u_0^\varepsilon - m_0\|_{L^1_{\mathrm{loc}}(\mathbb{R}^N)} = 0. \tag{A.21}$$

The function $u_0^{\varepsilon,\delta}(\mathbf{x})$ defined in (A.19) is in $C_0^\infty(\mathbb{R}^N)$ and satisfies

$$\operatorname{supp} u_0^{\varepsilon,\delta} \subset \{\mathbf{x} \in \mathbb{R}^N : 2\delta \leqq |\mathbf{x}| \leqq 1+\delta^{-1}\}, \tag{A.22}$$

$$\lim_{\delta\to 0+} \int_{\mathbb{R}^N} \rho_0^\varepsilon(\mathbf{x})|u_0^{\varepsilon,\delta}(\mathbf{x})|^2\, d\mathbf{x} = \int_{\mathbb{R}^N} \rho_0^\varepsilon(\mathbf{x})|u_0^\varepsilon(\mathbf{x})|^2\, d\mathbf{x}, \tag{A.23}$$

$$\lim_{\delta\to 0+} \left\|\rho_0^\varepsilon u_0^{\varepsilon,\delta} - \rho_0^\varepsilon u_0^\varepsilon\right\|_{L^1_{\mathrm{loc}}(\mathbb{R}^N)} = 0, \tag{A.24}$$

$$\int_{\mathbb{R}^N} \rho_0^\varepsilon(\mathbf{x})|u_0^{\varepsilon,\delta}(\mathbf{x})|^2(|\mathbf{x}|+1)^{N-1+\vartheta}\,\mathrm{d}\mathbf{x} \leqq CE_0\delta^{-N+1-\vartheta}, \tag{A.25}$$

where E_0 *is defined in* (2.12).

Proof. (A.20) follows directly from (A.18). Using (A.12) and (A.18), we have

$$\begin{aligned}
&\int_{|\mathbf{x}|\leqq M} \big|(\rho_0^\varepsilon u_0^\varepsilon - m_0)(\mathbf{x})\big|\,\mathrm{d}\mathbf{x}\\
&= \int_{|\mathbf{x}|\leqq M} \big|(\sqrt{\rho_0^\varepsilon}-\sqrt{\rho_0})(\mathbf{x})(\frac{m_0}{\sqrt{\rho_0}})(\mathbf{x})\big|\,\mathrm{d}\mathbf{x}\\
&\leqq \Big(\int_{\mathbb{R}^N} \frac{|m_0(\mathbf{x})|^2}{\rho_0(\mathbf{x})}\,\mathrm{d}\mathbf{x}\Big)^{\frac12}\Big(\int_{|\mathbf{x}|\leqq M} \big|(\sqrt{\rho_0^\varepsilon}-\sqrt{\rho_0})(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}\Big)^{\frac12}\\
&\to 0 \qquad \text{as } \varepsilon\to 0
\end{aligned} \tag{A.26}$$

for any $M \gg 1$, which leads to (A.21).
From (A.19), it is clear that $u_0^{\varepsilon,\delta}(\mathbf{x}) \in C_0^\infty(\mathbb{R}^N)$ and $\operatorname{supp} u_0^{\varepsilon,\delta} \subset \{\mathbf{x}\in\mathbb{R}^N : 2\delta \leqq |\mathbf{x}| \leqq 1+\delta^{-1}\}$. For any given small constant $\varrho > 0$, there exist small $\epsilon = \epsilon(\varrho) > 0$ and large $M = M(\varrho) \gg 1$ such that

$$\int_{B_{2\epsilon}(\mathbf{0})\cup\{|\mathbf{x}|\geqq M(\varrho)\}} \frac{|m_0(\mathbf{x})|^2}{\rho_0(\mathbf{x})}\,\mathrm{d}\mathbf{x} \leqq \varrho. \tag{A.27}$$

Taking $\delta > 0$ small enough so that $\epsilon \geqq 6\delta$, then it follows from (A.19) that

$$\int_{\epsilon\leqq|\mathbf{x}|\leqq M+2} \Big|\Big(\sqrt{\rho_0^\varepsilon}u_0^{\varepsilon,\delta} - \frac{m_0}{\sqrt{\rho_0}}\Big)(\mathbf{x})\Big|^2\mathrm{d}\mathbf{x} \to 0 \qquad \text{as } \delta\to 0+. \tag{A.28}$$

Since $\epsilon \geqq 6\delta$, we have

$$\begin{aligned}
&\int_{B_\epsilon(\mathbf{0})\cup\{|\mathbf{x}|\geq M+1\}} \big|\sqrt{\rho_0^\varepsilon(\mathbf{x})}u_0^{\varepsilon,\delta}(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}\\
&\leqq \int_{B_\epsilon(\mathbf{0})\cup\{|\mathbf{x}|\geq M+1\}} \Big|\int_{\mathbb{R}^N} \big(\frac{m_0}{\sqrt{\rho_0}}\mathbf{I}_{[4\delta,\delta^{-1}]}\big)(\mathbf{x}-\mathbf{y})J_\delta(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big|^2\,\mathrm{d}\mathbf{x}\\
&\leqq \int_{B_{2\epsilon}(\mathbf{0})\cup\{|\mathbf{x}|\geq M\}} \frac{|m_0(\mathbf{x})|^2}{\rho_0(\mathbf{x})}\,\mathrm{d}\mathbf{x} \leqq \varrho.
\end{aligned} \tag{A.29}$$

It follows from (A.18) and (A.27)–(A.29) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big|(\sqrt{\rho_0^\varepsilon}u_0^{\varepsilon,\delta} - \sqrt{\rho_0^\varepsilon}u_0^\varepsilon)(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^N} \big|(\sqrt{\rho_0^\varepsilon}u_0^{\varepsilon,\delta} - \frac{m_0}{\sqrt{\rho_0}})(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}\\
&\leqq \int_{\epsilon\leqq|\mathbf{x}|\leqq M+2} \big|(\sqrt{\rho_0^\varepsilon}u_0^{\varepsilon,\delta} - \frac{m_0}{\sqrt{\rho_0}})(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}
\end{aligned}$$

$$+ C \int_{B_{2\epsilon}(\mathbf{0})\cup\{|\mathbf{x}|\geqq M\}} \frac{|m_0(\mathbf{x})|^2}{\rho_0(\mathbf{x})}\,\mathrm{d}\mathbf{x}$$
$$\to 0 \qquad \text{as } \delta \to 0+, \tag{A.30}$$

which leads to (A.23).
Using (A.30), we have

$$\begin{aligned}
&\int_{|\mathbf{x}|\leqq M} \big|(\rho_0^\varepsilon u_0^{\varepsilon,\delta} - \rho_0^\varepsilon u_0^\varepsilon)(\mathbf{x})\big|\,\mathrm{d}\mathbf{x}\\
&\leqq \Big(\int_{|\mathbf{x}|\leqq M} \rho_0^\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} \big|(\sqrt{\rho_0^\varepsilon} u_0^{\varepsilon,\delta} - \sqrt{\rho_0^\varepsilon} u_0^\varepsilon)(\mathbf{x})\big|^2 \mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}}\\
&\to 0 \qquad \text{as } \delta \to 0+,
\end{aligned}$$

which implies (A.24).
Finally, noting (A.23) and $u_0^{\varepsilon,\delta}(\mathbf{x}) = 0$ for $|\mathbf{x}| \geqq 1 + \delta^{-1}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \rho_0^\varepsilon(\mathbf{x})|u_0^{\varepsilon,\delta}(\mathbf{x})|^2(|\mathbf{x}|+1)^{N-1+\vartheta}\,\mathrm{d}\mathbf{x}\\
&\leqq \int_{|\mathbf{x}|\leqq 1+\delta^{-1}} \rho_0^\varepsilon(\mathbf{x})|u_0^{\varepsilon,\delta}(\mathbf{x})|^2(|\mathbf{x}|+1)^{N-1+\vartheta}\,\mathrm{d}\mathbf{x}\\
&\leqq C\delta^{-N+1-\vartheta} \int_{\mathbb{R}^N} \rho_0^\varepsilon(\mathbf{x})|u_0^{\varepsilon,\delta}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\\
&\leqq C\delta^{-N+1-\vartheta} \int_{\mathbb{R}^N} \frac{|m_0(\mathbf{x})|^2}{\rho_0(\mathbf{x})}\,\mathrm{d}\mathbf{x}\\
&\leqq C E_0 \delta^{-N+1-\vartheta},
\end{aligned}$$

which yields (A.25). □

With $\rho_0^\varepsilon(\mathbf{x})$, $u_0^\varepsilon(\mathbf{x})$, and $u_0^{\varepsilon,\delta}(\mathbf{x})$ defined above, we can construct the approximate initial data $(\rho_0^{\varepsilon,\delta,b}, m_0^{\varepsilon,\delta,b})(r) = (\rho_0^{\varepsilon,\delta,b}, \rho_0^{\varepsilon,\delta,b} u_0^{\varepsilon,\delta,b})(r)$ for (3.1) and (3.4), and $(\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r) = (\rho_0^{\varepsilon,\delta}, \rho_0^{\varepsilon,\delta} u_0^{\varepsilon,\delta})(r)$ for (5.6): For $b \geqq 1 + \delta^{-1}$, define

$$(\rho_0^{\varepsilon,\delta,b}, u_0^{\varepsilon,\delta,b})(r) := (\rho_0^\varepsilon(\mathbf{x}), u_0^{\varepsilon,\delta}(\mathbf{x}))\mathbf{I}_{[\delta,b]}(\mathbf{x}) \qquad \text{for } r = |\mathbf{x}| \in [\delta, b] \tag{A.31}$$

to be the initial data for IBVP (3.1) and (3.4). On the other hand, for IBVP (5.6), we define

$$(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}(r) := (\rho_0^\varepsilon(\mathbf{x}), u_0^{\varepsilon,\delta}(\mathbf{x}))\mathbf{I}_{[\delta,\infty)}(\mathbf{x}) \qquad \text{for } r = |\mathbf{x}| \in [\delta, \infty). \tag{A.32}$$

Then, combining Lemma A.1 with Lemma A.2, we obtain

Lemma A.3. *The following three results hold*:

(i) *As* $\varepsilon \to 0$,

$$\begin{aligned}
&(E_0^\varepsilon, E_1^\varepsilon) \to (E_0, 0),\\
&(\rho_0^\varepsilon, m_0^\varepsilon)(r) \to (\rho_0, m_0)(r) \quad \text{in } L^1_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r),
\end{aligned} \tag{A.33}$$

where E_0^ε, E_1^ε, *and* E_0 *are defined in* (2.12), (2.13), *and* (2.2), *respectively.*

(ii) *For any fixed* $\varepsilon \in (0, 1]$*, as* $\delta \to 0$*,*

$$
\begin{aligned}
&(E_0^{\varepsilon,\delta}, E_1^{\varepsilon,\delta}) \to (E_0^{\varepsilon}, E_1^{\varepsilon}),\\
&(\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r) \to (\rho_0^{\varepsilon}, m_0^{\varepsilon})(r) \quad in\ L^1_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r),
\end{aligned} \tag{A.34}
$$

where $E_0^{\varepsilon,\delta}$ *and* $E_1^{\varepsilon,\delta}$ *are defined in* (5.2)–(5.3).

(iii) *For any fixed* (ε, δ)*, as* $b \to \infty$*,*

$$
(E_0^{\varepsilon,\delta,b}, E_1^{\varepsilon,\delta,b}) \to (E_0^{\varepsilon,\delta}, E_1^{\varepsilon,\delta}), \tag{A.35}
$$

$$
(\rho_0^{\varepsilon,\delta,b}, m_0^{\varepsilon,\delta,b})(r) \to (\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r) \quad in\ L^1_{\mathrm{loc}}((\delta,\infty); r^{N-1}\mathrm{d}r), \tag{A.36}
$$

where $E_0^{\varepsilon,\delta,b}$, $E_1^{\varepsilon,\delta,b}$, $E_2^{\varepsilon,\delta,b}$*, and* $\tilde{E}_0^{\varepsilon,\delta,b}$ *are defined in Lemmas* 3.1–3.2 *and* (4.2)*. In addition, the upper bounds of* $E_0^{\varepsilon,\delta,b}$, $E_1^{\varepsilon,\delta,b}$, $E_2^{\varepsilon,\delta,b}$*, and* $\tilde{E}_0^{\varepsilon,\delta,b}$ *are independent of* b (*but may depend on* ε, δ)*, and*

$$
E_0^{\varepsilon,\delta,b} + E_1^{\varepsilon,\delta,b} \leqq C(E_0 + 1), \tag{A.37}
$$

$$
\begin{aligned}
\tilde{E}_0^{\varepsilon,\delta,b} &\leqq \int_\delta^b \bar{\eta}^*(\rho_0^{\varepsilon,\delta,b}, m_0^{\varepsilon,\delta,b}) r^{N-1}(1+r)^{N-1+\vartheta}\,\mathrm{d}r\\
&\leqq C E_0\big(\delta^{-N+1-\vartheta} + \varepsilon^{-\frac{N-1+\vartheta}{2N}}\big),
\end{aligned} \tag{A.38}
$$

for some $C > 0$ *independent of* (ε, δ, b)*, where* $\vartheta \in (0, 1)$ *is any fixed constant.*

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