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The Boltzmann Equation for Uniform Shear Flow

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Abstract

The uniform shear flow for rarefied gas is governed by the time-dependent spatially homogeneous Boltzmann equation with a linear shear force. The main feature of such flow is that the temperature may increase in time due to the shearing motion that induces viscous heat, and the system strays far from equilibrium. For Maxwell molecules, we establish the unique existence, regularity, shear-rate-dependent structure and non-negativity of self-similar profiles for any small shear rate. The non-negativity is justified through the large time asymptotic stability even in spatially inhomogeneous perturbation framework, and the exponential rates of convergence are also obtained with the size proportional to the second order shear rate. This analysis supports the numerical result that the self-similar profile admits an algebraic high-velocity tail that is the key difficulty to overcome in the proof.

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1. Intoduction

1.1. Brief Background

In this paper we are concerned with the *uniform shear flow* (USF for short) described by the Boltzmann equation in the specific case of Maxwell molecules for which particles interacts via the exact inverse power law repulsive potential $U(r) = r^{-4}$ (cf. [15]). For the USF of the rarefied gas, the flow velocity behaves as $u^{\text{sh}} = (\alpha x_2, 0, 0)$ in space, namely, the velocity component in x_1-direction is linear along the x_2-direction for a constant shear rate $\alpha > 0$. The shearing motion and the induced viscous heating drive the system to depart from equilibrium. Thus, the energy and hence the temperature monotonically increase in time. It then becomes interesting to determine the global existence of such USF as well as its large time behavior. It turns out that for the Maxwell molecules, the existence can be transferred to look for self-similar profiles by taking into account the growth of temperature. Moreover, the self-similar profile is determined by non-Maxwellian solutions of a stationary problem on the Boltzmann equation with the shear force and the velocity relaxation term whose balance leads to the conservation of energy. The shear strength affects how far the self-similar profile is from the Maxwellian equilibrium and a perturbation approach in α is expected to give the existence of solutions for any small shear rate. In general, the self-similar profile is anisotropic in velocity variables due to shearing motion. The main feature of the self-similar profile verified numerically by Monte Carlo simulations (cf. [22]) is that it has the polynomial large-velocity tail that will induce the key difficulty in studying the topic.

We remark that the solutions to the Boltzmann equation for the USF are also called *homoenergetic solutions*; these were introduced by Galkin [21] and Truesdell [36]. Moreover, as pointed out by Truesdell and Muncaster [37], due to no boundary confinement, the USF is different from the planar Couette flow for a rarefied gas between two parallel infinite plates moving relative to each other with opposite velocities, cf. [22, Chapter 5], [31, Chapter 4] and [35, Chapter 4], for instance. We will study the latter topic accounting for boundary effects in another work.

1.2. Boltzmann Equation for USF

Mathematically, the USF is governed by the spatially homogeneous Boltzmann equation

$$\partial_t F - \alpha v_2 \partial_{v_1} F = Q(F, F). \tag{1.1}$$

Here the unknown $F = F(t, v) \geqq 0$ stands for the velocity distribution function of gas particles with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geqq 0$, and the constant $\alpha > 0$ denotes the shear rate as mentioned before. The Boltzmann collision operator $Q(\cdot, \cdot)$ is bilinear taking the non-symmetric form of

$$Q(F_1, F_2)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\cos\theta)[F_1(v'_*)F_2(v') - F_1(v_*)F_2(v)] \,\mathrm{d}\omega \mathrm{d}v_*, \tag{1.2}$$

where the velocity pairs (v_*, v) and (v'_*, v') satisfy the relation

$$v'_* = v_* - [(v_* - v) \cdot \omega]\omega, \quad v' = v + [(v_* - v) \cdot \omega]\omega, \tag{1.3}$$

denoting the ω-representation according to conservations of momentum and energy of two particles before and after the collision

$$v_* + v = v'_* + v', \quad |v_*|^2 + |v|^2 = |v'_*|^2 + |v'|^2.$$

Through this paper, we assume that the collision kernel $B_0(\cos\theta)$ with $\cos\theta = (v - v_*)\cdot\omega/|v - v_*|$ is independent of the relative speed $|v - v_*|$ for the Maxwell molecule model and satisfies the Grad's angular cutoff assumption

$$0 \leqq B_0(\cos\theta) \leqq C|\cos\theta| \tag{1.4}$$

for a generic constant $C > 0$.

1.3. Moment Equations and Self-similar Formulation

Provided that $F(t, v)$ decays in large velocity fast enough, we multiply (1.1) by the Boltzmann collision invariants and take integration in velocity so as to obtain

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} F\,\mathrm{d}v = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} v_1 F\,\mathrm{d}v + \alpha\int_{\mathbb{R}^3} v_2 F\,\mathrm{d}v = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} v_i F\,\mathrm{d}v = 0, \quad i = 2, 3, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 F\,\mathrm{d}v + \alpha\int_{\mathbb{R}^3} v_1 v_2 F\,\mathrm{d}v = 0 \end{cases} \tag{1.5}$$

for any $t \geqq 0$. In light of this, without loss of generality, we may assume that the solution $F(t, v)$ to (1.1) satisfies

$$\int_{\mathbb{R}^3} F(t, v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}^3} v_i F(t, v)\,\mathrm{d}v = 0, \; i = 1, 2, 3, \quad \forall t \geqq 0. \tag{1.6}$$

The last identity of (1.5) implies that the macroscopic energy of $F(t, v)$ can change in time due to the appearance of shear force. Physically the shearing motion should induce the viscous heat into the system so that the energy indeed increases in time. Moreover, it will be justified later that the heat flux $\int v_1 v_2 F\,\mathrm{d}v$ turns out to be strictly negative in large time for any small $\alpha > 0$.

From [22, Chapter 2] as well as [28, Section 5.1], for the Maxwell molecule model, a specific solution $F(t, v)$ can be self-similar of the form

$$F(t, v) = e^{-3\beta t} G\left(\frac{v}{e^{\beta t}}\right) \tag{1.7}$$

for a suitable constant β, where the self-similar stationary profile $G = G(v)$ satisfies

$$-\beta\nabla_v\cdot(vG) - \alpha v_2 \partial_{v_1} G = Q(G, G). \tag{1.8}$$

To find a solution to (1.8), it is natural to require that $G(v)$ also satisfies the same conservation laws (1.6) as for $F(t, v)$ and in addition G has a fixed positive energy, namely, without loss of generality,

$$\int_{\mathbb{R}^3} |v|^2 G(v)\,\mathrm{d}v = 3. \tag{1.9}$$

Therefore, from the solvability of the stationary equation (1.8)

$$\int_{\mathbb{R}^3} [1, v, |v|^2]\{-\beta\nabla_v \cdot (vG) - \alpha v_2 \partial_{v_1} G\}\, dv = 0,$$

the condition energy law (1.9) is equivalent to require that

$$\beta = -\alpha \frac{\int_{\mathbb{R}^3} v_1 v_2 G\, dv}{\int_{\mathbb{R}^3} |v|^2 G\, dv} = -\frac{\alpha}{3}\int_{\mathbb{R}^3} v_1 v_2 G\, dv. \tag{1.10}$$

Plugging this back to (1.8) gives

$$\frac{1}{3}\int_{\mathbb{R}^3} v_1 v_2 G\, dv\, \nabla_v \cdot (vG) - v_2 \partial_{v_1} G = \frac{1}{\alpha} Q(G, G). \tag{1.11}$$

The above equation is a crucial formulation for studying the existence of $G(v)$ via the Hilbert's perturbation approach in the small parameter $\alpha > 0$. In particular, β is no longer regarded as an unknown constant, but replaced by a nonlocal integral term. Note that $\alpha > 0$ plays the same role as the Knudsen number. We would emphasize that the current work is only focused on the small shear rate regime with unit Knudsen number, but it is still possible to make use of (1.11) to discuss the situation of the large shear rate for small Knudsen number in the hydrodynamic regime.

1.4. Main Results

With the preparations above, we are ready to state the main results of the paper regarding the existence and dynamical stability of the self-similar profile $G(v)$. It should be pointed out that the existence (obtained also in [28]) and the uniqueness, non-negativity and stability (as well as the analysis of the moments and the exponential rate of convergence) of self-similar profiles for the USF Boltzmann equation have been proved in [11] in the class of measures, for small values of the shear parameter. In particular, the approach used in [28] is based on the fixed point argument on the integral form of the problem over a set of non-negative Radon measures, while [11] gives a different proof by means of the Fourier transform method taking the full advantage of the Bobylev formula in the case of Maxwell molecules. Instead, in this paper, we consider the smooth solutions via the perturbative approach in α and obtain the C^∞ regularity and dependence on α up to the second order.

Specifically, our first result of the paper is concerned with the existence of smooth self-similar profiles for the stationary Boltzmann problem (1.11) under the assumption on smallness of shear rate $\alpha > 0$. To this end, we define the global Maxwellian

$$\mu = (2\pi)^{-3/2} \exp(-|v|^2/2), \tag{1.12}$$

and introduce the velocity weight function $w_l = w_l(v) := (1 + |v|^2)^l$ with $l \in \mathbb{R}$.

Theorem 1.1. *There is* $l_0 > 0$ *such that for any* $l \geqq l_0$*, there is* $\alpha_0 = \alpha_0(l) > 0$ *depending on* l *such that for any* $\alpha \in (0, \alpha_0)$*, the stationary Boltzmann equation* (1.11) *admits a unique smooth solution* $G = G(v) \in C^\infty(\mathbb{R}^3)$ *satisfying*

$$\int_{\mathbb{R}^3} [1, v, |v|^2] G(v)\, \mathrm{d}v = [1, 0, 3], \tag{1.13}$$

and

$$\left\| w_l \nabla_v^k \left[G - \left(1 - \frac{\alpha}{2b_0} v_1 v_2 \right) \mu \right] \right\|_{L^\infty} \leqq C_{k,l} \alpha^2 \tag{1.14}$$

for any integer $k \geqq 0$*, where* $C_{k,l}$ *is a constant independent of* α*, and* b_0 *is a positive constant defined by*

$$b_0 = 3\pi \int_{-1}^{1} B_0(z) z^2 (1 - z^2)\, dz. \tag{1.15}$$

Remark 1.1. Here are a few remarks in order on Theorem 1.1.

(a) The estimate (1.14) implies that as $\alpha \to 0$, the self-similar profile $G(v)$ behaves as

$$G(v) = \mu - \frac{\alpha}{2b_0} v_1 v_2 \mu + O(\alpha^2), \tag{1.16}$$

where μ is uniquely determined by conservation laws (1.13), and correspondingly by (1.10), as $\alpha \to 0$, the constant β behaves as

$$\beta = \frac{\alpha^2}{6b_0} + O(\alpha^3). \tag{1.17}$$

Thus, Theorem 1.1 not only gives the existence of smooth solutions $G(v)$, but also provides the α-dependent structure of G. Note that beyond the expansion (1.16) up to the first order, it is possible to further obtain the coefficient velocity functions of the second and third orders of α by iteration, see [22, (2.126) and (2.127), page 88]. Moreover, (1.17) implies that β is strictly positive and hence by (1.9), the energy of the self-similar solution (1.7), given by

$$\int_{\mathbb{R}^3} |v|^2 F(t, v)\, \mathrm{d}v = 3e^{2\beta t},$$

indeed tends to infinity exponentially in time.

(b) In general, from (1.14), $G(v)$ has to be anisotropic in v due to the shearing motion, and any l-th order velocity moments of $G(v)$ are finite as long as the shear rate $\alpha > 0$ is small enough. Due to the dependence of α_0 on l, in particular, one can choose $\alpha_0(l) \sim \frac{1}{l}$ for any $l > 0$ large enough from the later proof, it is impossible to obtain a positive shear rate α_0 such that (1.14) holds true uniformly in any $l > 0$, particularly allowing $l \to \infty$. In fact, as discussed in [22, Chapter 2.1, page 57], for any value of the shear rate α, all the velocity moments of order k with $k \geqq k_c(\alpha)$ are divergent. As α increases, the threshold order $k_c(\alpha)$ decreases but it always holds that $k_c(\alpha) > 2$. This property exactly features that $G(v)$ may admit the polynomial large-velocity tail as confirmed numerically by Monte Carlo simulations. It is also left open to obtain a sharp estimate on $k_c(\alpha)$ as well as the existence of $G(v)$ whenever $\alpha > 0$ becomes larger and larger.

(c) The constant $b_0 > 0$, describing the magnitude of collisions, is obviously finite under the angular cutoff assumption (1.4), and it has been assumed to be independent of $\alpha > 0$, meaning that the shear rate need to be small enough compared to the strength of collisions. It is interesting to further study the property of self-similar profiles in case when collisions are strong or weak enough corresponding to the hydrodynamic limit or the free molecule limit, respectively. Furthermore, since b_0 can be well-defined even in the case of the angular non-cutoff by (1.15), it is also interesting to extend the current result to the non-cutoff situation that is certainly more challenging than the current consideration due to the necessary L^∞ estimates on solutions.

Moreover, we are concerned with the global existence and large time behavior of solutions to the original USF equation (1.1) supplemented with a suitable initial data, namely, in terms of the self-similar reformulation (1.7) with the value of β obtained from Theorem 1.1, it is natural to further study whether or not it is holds true that

$$e^{3\beta t}F(t, e^{\beta t}v) \to G(v) \tag{1.18}$$

in a certain sense as time goes to infinity whenever they are close to each other initially, where $G(v)$ is the self-similar profile obtained in Theorem 1.1 and the constant β is defined by (1.10) in terms of $G(v)$. As a byproduct, a direct consequence of such large time asymptotic stability is the non-negativity of $G(v)$.

To treat the issue, instead of directly starting with the spatially homogeneous Boltzmann equation (1.1) for the USF, we turn to the spatially inhomogeneous setting for a more general purpose. In fact, let the rarefied gas flow be contained in an infinite channel $\mathbb{T}_{x_1} \times \mathbb{R}_{x_2}$ and uniform in x_3-direction, then the governing Boltzmann equation takes the form of

$$\partial_t \widetilde{F} + w_1\partial_{x_1}\widetilde{F} + w_2\partial_{x_2}\widetilde{F} = Q(\widetilde{F}, \widetilde{F}) \tag{1.19}$$

for the spatially inhomogeneous velocity distribution function $\widetilde{F} = \widetilde{F}(t, x_1, x_2, w)$ with $t \geqq 0$, $x_1 \in \mathbb{T}$, $x_2 \in \mathbb{R}$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$. We remark that when the Knudsen number is involved, under suitable scalings, the formal fluid limit of (1.19) gives the incompressible Euler or Navier-Stokes equations in the 2D domain $\mathbb{T}_{x_1} \times \mathbb{R}_{x_2}$. As mentioned at the end of Section 1.5 later on, for those fluid equations, there have been extensive studies of asymptotic stability of the planar Couette flow $(\alpha x_2, 0)$ with an arbitrary $\alpha > 0$. Thus, it would be interesting to explore the large time behavior of solutions to (1.19) in connection with the uniform shear flow in the self-similar framework. We will leave the study of such 2D problem with shear flows for the future.

Instead, to obtain the non-negativity of $G(v)$, we simply look for a one-dimensional solution of the specific form $\widetilde{F} = F(t, x_1, w_1 - \alpha x_2, w_2, w_3)$ to the 2D problem

(1.19). Then, one can see that F satisfies the following Boltzmann equation in a one-dimensional periodic box:

$$\partial_t F + v_1\partial_x F - \alpha v_2 \partial_{v_1} F = Q(F,F), \quad t>0, x\in\mathbb{T}, v\in\mathbb{R}^3, \tag{1.20}$$

supplemented with initial data

$$F(0,x,v) = F_0(x,v),\ x\in\mathbb{T},\ v\in\mathbb{R}^3. \tag{1.21}$$

Here for brevity of presentation we have used x to denote the first component of space variables. The second result of the paper is related to the large time asymptotics of solutions to the spatially inhomogeneous problem (1.20) and (1.21).

Theorem 1.2. *Let $G(v)$ be the self-similar profile obtained in Theorem 1.1 and the constant β be defined in (1.10). There are constants $\lambda>0$ and $C>0$ such that if $F_0(x,v)\geqq 0$ and*

$$\sum_{0\leqq\gamma_0\leqq 2}\left\| w_l\mu^{-\frac{1}{2}}\partial_x^{\gamma_0}\left[F_0(x,v)-G(v)\right]\right\|_{L^\infty}\leqq\alpha, \tag{1.22}$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{R}^3}[F_0(x,v)-G]\,\mathrm{d}v\mathrm{d}x=0, \quad \int_{\mathbb{T}}\int_{\mathbb{R}^3} vF_0(x,v)\,\mathrm{d}v\mathrm{d}x=0, \tag{1.23}$$

then the Cauchy problem (1.20) and (1.21) admits a unique solution $F(t,x,v)\geqq 0$ satisfying the following estimates:

$$\begin{aligned}&\left\| w_l(v)\left[e^{3\beta t}F(t,x,e^{\beta t}v)-G(v)\right]\right\|_{L^\infty}\\ &\leqq Ce^{-\lambda\beta t}\sum_{0\leqq\gamma_0\leqq 2}\left\| w_l\mu^{-\frac{1}{2}}\partial_x^{\gamma_0}\left[F_0(x,v)-G(v)\right]\right\|_{L^\infty},\end{aligned} \tag{1.24}$$

and

$$\begin{aligned}&\sum_{1\leqq\gamma_0\leqq 2}\left\| w_l(v)e^{3\beta t}\partial_x^{\gamma_0}F(t,x,e^{\beta t}v)\right\|_{L^\infty}\\ &\leqq Ce^{-\lambda t}\sum_{1\leqq\gamma_0\leqq 2}\left\| w_l(v)\mu^{-\frac{1}{2}}\partial_x^{\gamma_0}F_0(x,v)\right\|_{L^\infty},\end{aligned} \tag{1.25}$$

for any $t\geqq 0$.

Remark 1.2. We give a few remarks on Theorem 1.2 as follows:

(a) Whenever F is spatially homogeneous, as a direct consequence of Theorem 1.2, the large time asymptotics (1.18) for solutions to the USF (1.1) towards the self-similar profile G is also justified in the velocity weighted L^∞ setting. In particular, from (1.24) one has

$$\|w_l[e^{3\beta t}F(t,e^{\beta t}\cdot)-G]\|_{L^\infty}\leqq Ce^{-\lambda\beta t}\|w_l\mu^{-\frac{1}{2}}(F_0-G)\|_{L^\infty} \tag{1.26}$$

for any $t\geqq 0$.

(b) Estimate (1.24) or (1.26) implies that the rate of convergence is exponential with the size proportional to $\beta \sim \alpha^2$. Such property features the shearing motion for small $\alpha > 0$. In fact, when $\alpha = 0$, the large time behavior of solutions to (1.1) is the global Maxwellian equilibrium uniquely determined by initial data $F_0(v)$ through all the conservative fluid quantities, and the convergence rate is exponential with the size given by the spectral gap of the linearized Boltzmann operator.

For $\alpha > 0$, it is not necessary to impose that F_0 has the same energy as G except the mass and momentum conservation (1.23), because in the self-similar setting the energy of the rescaled distribution function is dissipative with the magnitude of dissipation rate proportional to β due to the linear relaxation effect arising from the term $-\beta\nabla_v \cdot (vF)$. Precisely, let $f(t, x, v) = e^{3\beta t} F(t, x, e^{\beta t} v)$, then it follows from (1.20) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \mathrm{d}x \int \frac{1}{2}|v|^2 (f - G)\,\mathrm{d}v + \beta \int_{\mathbb{T}} \mathrm{d}x \int |v|^2 (f - G)\,\mathrm{d}v \\
+ \alpha \int_{\mathbb{T}} \mathrm{d}x \int v_1 v_2 (f - G)\,\mathrm{d}v = 0. \qquad (1.27)
\end{aligned}
$$

This identity implies that the size of the exponential convergence rate in (1.24) or (1.26) is optimal; we also will explain this point in more detail in Section 1.6 later.

(c) Estimate (1.25) implies that the convergence rate of the higher order spatial derivatives is much faster than the one of the zero order, since the size of convergence is independent of the shear rate α. This indicates that the collision of particles dominates the long time asymptotics of the energy for the higher order spatial derivatives.

(d) The smallness assumption (1.22) on initial data implies that the initial perturbation has to admit an additional large velocity decay as $\mu^{1/2}$. This restriction is essentially due to the perturbation method of the proof. It is interesting to remove such restriction using an alternative approach, for instance, in [23].

1.5. Literature

In what follows, we mention some known results on the self-similar solutions to the Boltzmann equation in case of the Maxwell molecule model. When $\alpha = 0$, namely, there is no shear effect, the mathematical study of the problem was initiated by Bobylev and Cercignani [7–9]. Since the energy remains conservative, the self-similar profile exists only when it has infinite second order moments. The dynamical stability of such infinite energy self-similar profile was proved by Morimoto, Yang and Zhao [33] in the angular non-cutoff case; see also the previous investigation Cannone and Karch [13,14] on this topic.

When $\alpha \neq 0$, the global-in-time existence of solutions to the Boltzmann equation (1.1) for the USF was first established by Cercignani [16–18]. The group invariant property in the higher dimensional case was discussed in Bobylev, Caraffini and Spiga [10]. Recently, in series of significant progress by James, Nota and Velázquez [28–30], the existence of homoenergetic mild solutions as non-negative Radon measures was studied in a systematic way for a large class of initial data, where the admissible macroscopic shear velocity $u^{\text{sh}} = L(t)x$ with $L(t) := A(I + tA)^{-1}$ for a constant matrix A is characterized and the asymptotics of homoenergetic solutions that do not have self-similar profiles is also conjectured in certain situations. An interesting work by Matthies and Theil [32] also showed that the self-similar profile does not have the same exponential large-velocity tail as the global Maxwellian. Applying the Fourier transform method that is a fundamental analysis tool in the Boltzmann theory introduced by Bobylev [5,6], a recent progress Bobylev, Nota and Velázquez [11] proved the self-similar asymptotics of solutions in large time for the Boltzmann equation with a general deformation of the form

$$\partial_t F - \nabla_v \cdot (AvF) = Q(F, F)$$

under a smallness condition on the matrix A, and they also showed that the self-similar profile can have the finite polynomial moments of higher order as long as the norm of A is smaller. It seems that [11] is the only known result on the large time asymptotics to the self-similar profile in weak topology.

In the end, we remark that there have been extensive studies of stability of shear flow in an infinite channel domain $\mathbb{T}_{x_1} \times \mathbb{R}_{x_2}$ in the context of fluid dynamic equations, cf. [34], in particular, we mention great contributions [2–4] recently made by Bedrossian together with his collaborators. In fact, in comparison with (1.20) under consideration, it would be more interesting to study the large time behavior of solutions to the original Boltzmann equation (1.19) in the 2D domain $\mathbb{T} \times \mathbb{R}$ in order to gain further understandings of the stability issue similar to those aforementioned works on fluid equations by taking the limit of either small or large Knudsen number.

1.6. Strategy of the Proof

The main ideas and techniques used in the paper are outlined as follows:

- First of all, in the framework of perturbation, there is a severe velocity-growth term in the form of $v_1 v_2 G$ which is caused by the shearing motion. Specifically, to solve (1.11), setting the perturbation as $G = \mu + \alpha\mu^{1/2}(G_1 + G_R)$ where G_1 as in (3.8) is used to remove the zero-order inhomogeneous term, the remainder G_R satisfies an equation of the form

$$\cdots + \frac{\alpha}{2} v_1 v_2 G_R + LG_R = \cdots ,$$

  see (3.9). Here, one can see that $\frac{\alpha}{2} v_1 v_2 G_R$ becomes a trouble term to control in the basic energy estimate in term of the dissipation of the linearized self-adjoint operator L.

To overcome the difficulty, we borrow the idea given by Caflisch [12], where the solution is split into two parts: one includes the exponential weight while the other does not, namely, we set

$$\mu^{\frac{1}{2}} G_R = G_{R,1} + \mu^{\frac{1}{2}} G_{R,2}.$$

The key point here is that we put the bad terms mentioned above into the one without exponential weight, so as to eliminate the velocity growth. Roughly $G_{R,1}$ and $G_{R,2}$ satisfy the coupling equations of the form

$$\cdots - \alpha v_2 \partial_{v_1} G_{R,1} + \frac{\alpha}{2} v_1 v_2 \mu^{\frac{1}{2}} G_{R,2} + \nu_0 G_R = \chi_M \mathcal{K} G_{R,1} + \cdots ,$$
$$\cdots - \alpha v_2 \partial_{v_1} G_{R,2} + L G_{R,2} = (1-\chi_M)\mu^{-\frac{1}{2}} \mathcal{K} G_{R,1} + \cdots ,$$

after ignoring the high order or nonlinear terms, where χ_M is a velocity cutoff function defined in (2.6), and other notations on $\mathcal{K}$ and so on are introduced in Section 2. We may refer to (3.11) and (3.12) for the full coupling system.
We should point out that as confirmed by the numerical result, one may only expect the first part $G_{R,1}$ to decay polynomially in large velocity. To understand this issue mathematically, we consider the equation of the form

$$-\alpha v_2 \partial_{v_1} G_{R,1} + \nu_0 G_{R,1} = \cdots ,$$

where $\nu_0 > 0$ is the constant collision frequency corresponding to μ in case of the Maxwell molecules. Multiplying the above equation with the polynomial weight $w_l = (1+|v|^2)^l$ gives

$$-\alpha v_2 \partial_{v_1}(w_l G_{R,1}) + \left(\nu_0 + 2\alpha l \frac{v_1 v_2}{1+|v|^2}\right) w_l G_{R,1} = \cdots . \tag{1.28}$$

Therefore, given $l > 0$ large, we need to require $0 < \alpha < \alpha_0(l) \sim \frac{1}{l}$ that is small enough such that

$$\inf_v \left(\nu_0 + 2\alpha l \frac{v_1 v_2}{1+|v|^2}\right) \geqq \frac{1}{2}\nu_0$$

holds true and hence $w_l G_{R,1}$ can be shown to be bounded in all v in terms of (1.28).

- Although the Caflisch's decomposition provides us the great advantage above, it also prevents us from deducing the L^2 estimates of the solution, particularly for the first part of the decomposition, due to the decay-loss of the operator $\mathcal{K}$ as in (2.4). To treat the difficulty, we resort to the L^∞-L^2 method developed recently by Guo [25]; see also [19,20,27]. One of the key points when applying this approach is the decay of the operator K for large velocity. At the current stage, it is quite hard to achieve any decay rates of $\mathcal{K}$. Fortunately, motivated by Arkeryd, Esposito and Pulvirenti [1], we justify the crucial estimates for such $\mathcal{K}$ with the algebraic velocity weight. More precisely, we find out the following "decay" rate for the large power of the velocity weight

$$\sup_{|v|\geqq M} w_l |\nabla_v^k \mathcal{K} f| \leqq \frac{C}{l} \sum_{0\leqq k' \leqq k} \|w_l \nabla_v^{k'} f\|_{L^\infty},$$

with $k \geqq 1$ and $M \sim l^2$, where we refer to Proposition 2.1 for more details. Thus, we can treat $\chi_M \mathcal{K} G_{R,1}$ as a source term. We remark that such estimate holds true for the Maxwell molecules only, as seen from the derivation of (2.13) in the proof of Proposition 2.1 later on.

- Based on the above observations, we treat the steady problem for the existence of $G(v)$ as follows. We design an approximation procedure (3.14) to obtain the approximation solution sequence with conservation of mass. Moreover, motivated by [19], we also introduce the σ-parametrized problem (3.19) with a parameter $0 \leqq \sigma \leqq 1$ to take care of the linear nonlocal terms $\mathcal{K}\mathcal{G}_1$ and $\mathcal{K}\mathcal{G}_2$. The method of characteristics can be directly applied to obtain the explicit solution in case of $\sigma = 0$ and then an iteration argument is employed to extend the solvability from $\sigma = 0$ to $\sigma = 1$. Once uniform estimates for the linear inhomogeneous problem are obtained through Lemma 3.2, we can therefore apply them to construct the approximation solutions. Passing the limit $\varepsilon \to 0$, we then establish the existence of $G(v)$.
- In addition, the L^2 estimate for the second part of the decomposition is also difficult to obtain due to the inhomogeneous structure of the splitting equation.
  To deal with this difficulty, for the steady case, the conservation laws of solutions are essentially used, so that both the first order correction and the remainder of the steady solution are microscopic, then the L^2 estimate can be directly obtained by the energy estimate.
  As to the unsteady case, since the energy is no longer conserved, the argument for the steady problem is invalid. In fact, in the time-dependent situation, the zero order dissipation of the temperature is captured by exploring the structure of the macroscopic equations which contains the weak damping generated by the shear flow. More specifically, inspired by the Guo's energy method, we have to turn to the macro thirteen moments equations (5.3), (5.4), (5.5) and (5.6) to obtain the dissipation of the macro components a, $\mathbf{b}$ and c, cf. (5.1), in terms of the micro dissipation. Indeed, for the dissipation of the macro component c, we have found the cancellation property

$$(\partial_x c, \int_0^x d_{12}\, \mathrm{d}y) + (d_{12}, c) = 0,$$

  so that one can derive from (5.3), (5.4) and (5.5) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|c\|^2 + (b_1, \frac{1}{3}\alpha e^{-\beta t}\int_0^x d_{12}\, \mathrm{d}y)\right\} + 2\beta\|c\|^2 \leqq \cdots .$$

  We may refer to (5.20) and the desired estimate (5.24) for more details. The above energy estimate in perturbation framework is consistent with the energy identity (1.27) mentioned in Remark 1.2. Furthermore, it is a usual way to derive from the thirteen-moments equations the dissipation of derivatives of a, $\mathbf{b}$ and c, see (5.28), and the zero-order dissipation of a and $\mathbf{b}$ then follows from the Poincaré inequality. The desired last estimate (5.7) is a consequence of the suitable combination of those obtained dissipation estimates.

1.7. Organization of the Paper

The rest structure of this paper is arranged as follows. In Section 2 we give a key estimate for the operator $\mathcal{K}$ which shows smallness of $\chi_M w_l \mathcal{K}$ for large enough M and l. The existence of the self-similar stationary profile $G(v)$ for (1.11) is constructed in Section 3. In Section 4, we turn to the unsteady problem of the spatially inhomogeneous Boltzmann equation (1.20) and (1.21) and establish the local-in-time existence of solutions. In Section 5 we are devoted to showing the global existence of solutions and large time asymptotic behavior for the Cauchy problem (1.20) and (1.21). Finally, in the appendix Section 6, we list some known basic estimates on the linearized operator L as well as the nonlinear operators Γ and Q, and also present an explicit formula of $L(v_i v_j \mu^{1/2})$ in the case of the Maxwell molecule model.

1.8. Notations

We now list some notations used in the paper.

- Throughout this paper, C denotes some generic positive (generally large) constant and λ denotes some generic positive (generally small) constants, where C and λ may take different values in different places. $D \lesssim E$ means that there is a generic constant $C > 0$ such that $D \leqq CE$. $D \sim E$ means $D \lesssim E$ and $E \lesssim D$.
- We denote $\|\cdot\|$ the $L^2(\mathbb{T}\times\mathbb{R}^3)$−norm or the $L^2(\mathbb{T})$−norm or $L^2(\mathbb{R}^3)$−norm. Sometimes, we use $\|\cdot\|_{L^\infty}$ to denote either the $L^\infty(\mathbb{T}\times\mathbb{R}^3)$−norm or the $L^\infty(\mathbb{R}^3)$−norm. Moreover, $(\cdot,\cdot)$ denotes the L^2 inner product in $\mathbb{T}\times\mathbb{R}^3$ with the L^2 norm $\|\cdot\|$ and $\langle\cdot\rangle$ denotes the L^2 inner product in $\mathbb{R}^3_v$.

2. Large Velocity Decay of $\mathcal{K}$

Let us first give some notations to be used through the paper. The linearized collision operator L and nonlinear collision operator Γ are respecively defined by

$$Lg = -\mu^{-1/2}\left\{Q(\mu,\sqrt{\mu}g) + Q(\sqrt{\mu}g,\mu)\right\}, \tag{2.1}$$

and

$$\begin{aligned}\Gamma(f,g) &= \mu^{-1/2}Q(\sqrt{\mu}f,\sqrt{\mu}g)\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0\mu^{1/2}(v_*)[f(v_*')g(v') - f(v_*)g(v)]\,\mathrm{d}\omega\mathrm{d}v_*.\end{aligned}$$

Note that $Lf = \nu f - Kf$ with

$$\begin{aligned}\nu &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\cos\theta)\mu(v_*)\,\mathrm{d}\omega\mathrm{d}v_* = \nu_0,\\ Kf &= \mu^{-\frac12}\left\{Q(\mu^{\frac12}f,\mu) + Q_{\mathrm{gain}}(\mu,\mu^{\frac12}f)\right\},\end{aligned} \tag{2.2}$$

where Q_{gain} denotes the positive part of Q in (1.2). The kernel of L, denoted as $\ker L$, is a five-dimensional space spanned by $\{1, v, |v|^2-3\}\sqrt{\mu} := \{\phi_i\}_{i=1}^5$. We further define a projection from L^2 to $\ker(L)$ by

$$\mathbf{P}_0 g = \left\{a_g + \mathbf{b}_g\cdot v + (|v|^2-3)c_g\right\}\sqrt{\mu}$$

for $g \in L^2$, and correspondingly denote the operator $\mathbf{P}_1$ by $\mathbf{P}_1 g = g - \mathbf{P}_0 g$, which is orthogonal to $\mathbf{P}_0$.

It is also convenient to define

$$\mathcal{L} f = -\{Q(f, \mu) + Q(\mu, f)\} = \nu f - \mathcal{K} f,$$

with

$$\nu f = \nu_0 f, \quad \mathcal{K} f = Q(f, \mu) + Q_{\text{gain}}(\mu, f) = \sqrt{\mu} K\left(\frac{f}{\sqrt{\mu}}\right), \tag{2.3}$$

according to (2.2). Note that we have

$$\mathcal{K} f = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\cos\theta)(f'_*\mu' - f_*\mu + \mu'_* f')\,\mathrm{d}\omega \mathrm{d}v_*. \tag{2.4}$$

The main goal of this section is to present a crucial estimate on $\mathcal{K}$ meaning that the weighted velocity derivatives of $\mathcal{K}$ are small for large velocities as long as the power of the polynomial velocity weight is large enough. Such a property plays a vital role in the proof of the next sections.

Proposition 2.1. *Let $\mathcal{K}$ be given by (2.4), then for any positive integer $k \geqq 1$, there is $C > 0$ such that for any arbitrarily large $l > 0$, there is $M = M(l) > 0$ such that it holds that*

$$\sup_{|v| \geqq M} w_l |\nabla_v^k \mathcal{K} f| \leqq \frac{C}{l} \sum_{0 \leqq k' \leqq k} \|w_l \nabla_v^{k'} f\|_{L^\infty}. \tag{2.5}$$

In particular, one can choose $M = l^2$.

Proof. Fix an integer $k \geqq 1$, and take $l > 0$ that can be arbitrarily large. Let $M > 0$ be large to be suitably chosen in terms of l in the later proof. We define $\chi_M(v)$ to be a non-negative smooth cutoff function such that

$$\chi_M(v) = \begin{cases} 1, & |v| \geqq M + 1, \\ 0, & |v| \leqq M. \end{cases} \tag{2.6}$$

In light of (2.4), we have

$$w_l \chi_M \nabla_v^k \mathcal{K} f \overset{\text{def}}{=} \mathcal{I}_1 + \mathcal{I}_2,$$

with

$$\mathcal{I}_1 = -w_l \chi_M \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 f(v_*) \nabla_v^k \mu(v)\,\mathrm{d}\omega \mathrm{d}v_*,$$

and

$$\mathcal{I}_2 = w_l \chi_M \sum_{k_1 \leqq k} C_k^{k_1} \left\{ \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \nabla_v^{k_1} f(v'_*) \nabla_v^{k-k_1} \mu(v')\,\mathrm{d}\omega \mathrm{d}v_*\right.$$

$$+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \nabla_v^{k_1} f(v')\nabla_v^{k-k_1}\mu(v'_*)\,\mathrm{d}\omega \mathrm{d}v_*\Big\}.$$

We now compute $\mathcal{I}_1$ and $\mathcal{I}_2$. For $\mathcal{I}_1$, one directly has

$$\mathcal{I}_1 \leqq C w_l \chi_M \nabla_v^k \mu(v)\|w_l f\|_{L^\infty}\int_{\mathbb{R}^3} w_l^{-1}\mathrm{d}v \leqq Ce^{-\frac{M^2}{16}}\|w_l f\|_{L^\infty}, \tag{2.7}$$

thanks to the assumption that $M \gg 1$ and $l > \frac{3}{2}$, for instance.

For $\mathcal{I}_2$, we first rewrite it as

$$\mathcal{I}_2 = w_l \chi_M \sum_{k_1 \leqq k} C_k^{k_1}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0^* \nabla_v^{k_1} f(v')\nabla_v^{k-k_1}\mu(v'_*)\,\mathrm{d}\omega \mathrm{d}v_*,$$

where $B_0^* = \frac{1}{2}(B_0(\cos\theta) + B_0(\sin\theta))$. As it is shown in [1, (3.2), pp.397], we now resort to the Carleman's representation, i.e.

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0^* \nabla_v^{k_1} f(v')\nabla_v^{k-k_1}\mu(v'_*)\,\mathrm{d}\omega \mathrm{d}v_* \\ &\quad = \int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)}{|v-v'_*|^2}\int_{E(v,v'_*)} \nabla_v^{k_1} f(v') B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_*,\end{aligned}$$

where

$$E(v, v'_*) = \{v' \big| (v-v')\cdot(v-v'_*) = 0,\ |v-v'| \leqq |v-v'_*|\} \subset \mathbb{R}^2,$$

and $\Pi_{v'}$ is the Lebesgue measure on the hyperplane $E(v, v'_*)$. Next, we define

$$\chi_1 = \chi_1(\xi) = \begin{cases} 1, & |\xi| < \frac{|v|}{\sqrt{2}}, \\ 0, & \text{otherwise},\end{cases}$$

and $\chi_0 = 1 - \chi_1$. We then decompose $\mathcal{I}_2$ into

$$\begin{aligned}\mathcal{I}_2 &= w_l \chi_M \sum_{k_1\leqq k} C_k^{k_1}\int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)\chi_0(v'_*)}{|v-v'_*|^2}\int_{E(v,v'_*)}\nabla_v^{k_1} f(v')\chi_0(v')B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_* \\ &\quad + w_l \chi_M \sum_{k_1\leqq k} C_k^{k_1}\int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)\chi_1(v'_*)}{|v-v'_*|^2}\int_{E(v,v'_*)}\nabla_v^{k_1} f(v')\chi_0(v')B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_* \\ &\quad + w_l \chi_M \sum_{k_1\leqq k} C_k^{k_1}\int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)\chi_0(v'_*)}{|v-v'_*|^2}\int_{E(v,v'_*)}\nabla_v^{k_1} f(v')\chi_1(v')B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_* \\ &\stackrel{\text{def}}{=} \sum_{n=1}^{3}\mathcal{I}_{2,n}.\end{aligned}$$

Note that the term simultaneously involving $\mu\chi_1$ and $f\chi_1$ has vanished due to the fact that $|v'|^2+|v'_*|^2=|v|^2+|v_*|^2$. We now turn to estimate $\mathcal{I}_{2,n}(1\leqq n\leqq 3)$ term by term. First of all, a direct computation gives

$$\nabla_v^{k-k_1}\mu(v'_*)\leqq C\mu^{\frac{1}{2}}(v'_*),\ w_l\chi_{|v|\geqq M}\mu^{\frac{1}{4}}(v'_*)\chi_0(v'_*)\leqq Ce^{-\frac{M^2}{32}}. \tag{2.8}$$

Moreover, standard calculation yields

$$\int_{\mathbb{R}^3}\frac{\mu^{\frac{1}{4}}(v'_*)}{|v-v'_*|^2}\,\mathrm{d}v'_*\leqq C\langle v\rangle^{-2}. \tag{2.9}$$

By using (2.8) and (2.9), one sees that, for $l>\frac{3}{2}$,

$$\begin{aligned}\mathcal{I}_{2,1},\ \mathcal{I}_{2,3}&\leqq Ce^{-\frac{M^2}{32}}\langle v\rangle^{-2}\sum_{k_1\leqq k}\|w_l\nabla_v^{k_1}f\|_{L^\infty}\int_{\mathbb{R}^2}w_{-l}(v')\,\mathrm{d}\Pi v'\\&\leqq Ce^{-\frac{M^2}{32}}\sum_{k_1\leqq k}\|w_l\nabla_v^{k_1}f\|_{L^\infty}.\end{aligned} \tag{2.10}$$

It remains now to estimate the delicate term $\mathcal{I}_{2,2}$ where the smallness is hard to be obtained. As [1, Proposition 3.1, pp.397], we introduce the two cutoff functions

$$\chi_\delta(v'_*)=\begin{cases}1, & |v'_*|<\delta|v|,\\ 0, & \text{otherwise},\end{cases}\qquad \chi_\eta(v_*)=\begin{cases}1, & |v_*|<\eta|v|,\\ 0, & \text{otherwise},\end{cases}$$

where $0<\delta<\eta<1$. Then we spilt $\mathcal{I}_{2,2}$ as

$$\begin{aligned}\mathcal{I}_{2,2}=&\,w_l\chi_M\sum_{k_1\leqq k}C_k^{k_1}\int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)\chi_1(v'_*)(1-\chi_\delta(v'_*))}{|v-v'_*|^2}\\&\int_{E(v,v'_*)}\nabla_v^{k_1}f(v')\chi_0(v')B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_*\\&+w_l\chi_M\sum_{k_1\leqq k}C_k^{k_1}\int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)\chi_1(v'_*)\chi_\delta(v'_*)}{|v-v'_*|^2}\\&\int_{E(v,v'_*)}\nabla_v^{k_1}f(v')\chi_0(v')\chi_\eta(v_*)B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_*\\&+w_l\chi_M\sum_{k_1\leqq k}C_k^{k_1}\int_{\mathbb{R}^3}\frac{\nabla_v^{k-k_1}\mu(v'_*)\chi_1(v'_*)\chi_\delta(v'_*)}{|v-v'_*|^2}\\&\times\int_{E(v,v'_*)}\nabla_v^{k_1}f(v')\chi_0(v')(1-\chi_\eta(v_*))B_0^*\,\mathrm{d}\Pi_{v'}\mathrm{d}v'_*\\\stackrel{\text{def}}{=}&\sum_{n=1}^{3}\mathcal{I}_{2,2}^n.\end{aligned}$$

Performing similar calculations as to those for obtaining (2.10), one has

$$\mathcal{I}_{2,2}^1 \leqq Ce^{-\frac{\delta^2 M^2}{16}} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty} \int_{\mathbb{R}^2} w_{-l}(v')\, \mathrm{d}\Pi v'$$
$$\leqq Ce^{-\frac{\delta^2 M^2}{16}} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty}. \tag{2.11}$$

For $\mathcal{I}_{2,2}^2$, we first have that if $|v'_*| < \delta|v|$ and $|v_*| < \eta|v|$, then

$$|v' - v| = |v_* - v'_*| \leqq (\eta + \delta)|v|, \quad |v'| = |v|^2 + |v_*|^2 - |v'_*|^2 \geqq (1 - \delta^2)|v|^2,$$

which further implies that the measure of $E(v, v'_*)$ is bounded as

$$|E(v, v'_*)| \leqq \pi(\eta + \delta)^2 |v|^2 \leqq 4\pi\eta^2 |v|^2,$$

and it holds true that

$$(1 + |v'|^2)^{-l} \leqq (1 + (1 - \delta^2)|v|^2)^{-l} \leqq (1 + |v|^2)^{-l}(1 - \delta^2)^{-l}.$$

Consequently, applying (2.8) and (2.9) again, we obtain

$$\mathcal{I}_{2,2}^2 \leqq C w_l \chi_M \sum_{k_1 \leqq k} \int_{\mathbb{R}^3} \frac{\mu^{\frac{1}{2}}(v'_*)}{|v - v'_*|^2} \int_{E(v, v'_*)} \nabla_v^{k_1} f(v') \chi_0(v') \chi_\eta(v_*)\, \mathrm{d}\Pi_{v'} \mathrm{d}v'_*$$
$$\leqq C w_l(v) \chi_M \langle v \rangle^{-2} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty} \eta^2 |v|^2 (1 + |v|^2)^{-l}(1 - \delta^2)^{-l}$$
$$\leqq C\eta^2 (1 - \delta^2)^{-l} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty}. \tag{2.12}$$

We are now in a position to compute the last term $\mathcal{I}_{2,2}^3$. Since for the case of $\mathcal{I}_{2,2}^3$, we have $|v'_*| < \delta|v|$ and $|v_*| \geqq \eta|v|$, then it follows that

$$|v'|^2 = |v|^2 + |v_*|^2 - |v'_*|^2 \geqq |v|^2 + \eta^2|v|^2 - \delta^2|v|^2 = (1 + \eta^2 - \delta^2)|v|^2,$$

which implies

$$\mathcal{I}_{2,2}^3 \leqq C w_l \chi_M \langle v \rangle^{-2} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty} \int_{|v|\sqrt{1+\eta^2-\delta^2}}^{+\infty} \frac{r}{(1 + r^2)^l}\, dr$$
$$\leqq C \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty} \chi_M \langle v \rangle^{-2} w_l(v) \frac{1}{l - 1} \Big(1 + (1 + \eta^2 - \delta^2)|v|^2\Big)^{-l+1}$$
$$\leqq \frac{C}{l} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty}, \tag{2.13}$$

where the last inequality holds due to the fact that $1 + \eta^2 - \delta^2 \geqq 1$ and $l \gg 1$.

Therefore, putting (2.11), (2.12) and (2.13) together, we arrive at

$$\mathcal{I}_{2,2} \leqq C\left\{e^{-\frac{\delta^2 M^2}{32}} + \eta^2(1-\delta^2)^{-l} + \frac{1}{l}\right\} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty}. \tag{2.14}$$

Furthermore, if one chooses $\delta = \frac{1}{l}$, $\eta = \frac{1}{\sqrt{l}}$ and $M = l^2 \gg 1$, then

$$(1-\delta^2)^{-l} < e,\ e^{-\frac{\delta^2 M^2}{32}} \leqq e^{-\frac{M}{32}}. \tag{2.15}$$

As a consequence, (2.14) and (2.15) give rise to

$$\mathcal{I}_{2,2} \leqq \left\{\frac{1}{M} + \frac{2}{l}\right\} \sum_{k_1 \leqq k} \|w_l \nabla_v^{k_1} f\|_{L^\infty}. \tag{2.16}$$

Finally, the desired estimate (2.5) follows from (2.7), (2.10) and (2.16). This ends the proof of Proposition 2.1. □

3. Steady Problem

This section is devoted to studying the steady problem

$$-\beta \nabla_v \cdot (vG) - \alpha v_2 \partial_{v_1} G = Q(G, G) \tag{3.1}$$

with

$$\beta = -\frac{\alpha}{3} \int_{\mathbb{R}^3} v_1 v_2 G \, \mathrm{d}v, \tag{3.2}$$

where the solution $G(v)$ is required to satisfy

$$\int_{\mathbb{R}^3} G \, \mathrm{d}v = 1, \quad \int_{\mathbb{R}^3} v_i G \, \mathrm{d}v = 0, i = 1, 2, 3, \quad \int_{\mathbb{R}^3} |v|^2 G \, \mathrm{d}v = 3, \tag{3.3}$$

which is equivalent to the fact that G has the same fluid quantities as μ in (1.12) for any $\alpha > 0$. Note that through the paper we have omitted the dependence of G on the parameter α.

Since one expects $G \to \mu$ as $\alpha \to 0$, to look for the solution, let us first set

$$G = \mu + \alpha\sqrt{\mu}\,\{G_1 + G_R\}, \tag{3.4}$$

with $\mathbf{P}_0 G_1 = 0$ and $\mathbf{P}_0 G_R = 0$ such that (3.3) holds true, i.e.

$$\begin{cases} \int_{\mathbb{R}^3} G_1 \sqrt{\mu} \, \mathrm{d}v = \int_{\mathbb{R}^3} G_R \sqrt{\mu} \, \mathrm{d}v = 0, \\ \int_{\mathbb{R}^3} G_1 v_i \sqrt{\mu} \, \mathrm{d}v = \int_{\mathbb{R}^3} G_R v_i \sqrt{\mu} \, \mathrm{d}v = 0,\ i = 1, 2, 3, \\ \int_{\mathbb{R}^3} G_1 |v|^2 \sqrt{\mu} \, \mathrm{d}v = \int_{\mathbb{R}^3} G_R |v|^2 \sqrt{\mu} \, \mathrm{d}v = 0, \end{cases} \tag{3.5}$$

where G_1 accounts for the first order correction and G_R denotes the higher order remainder. We now turn to determine G_1 and derive the equation for the remainder G_R. In fact, plugging (3.4) into (3.2) gives

$$\beta = -\frac{\alpha}{3}\int_{\mathbb{R}^3} v_1 v_2 G\,\mathrm{d}v = -\frac{\alpha^2}{3}\int_{\mathbb{R}^3} v_1 v_2 \mu^{1/2}(G_1+G_R)\,\mathrm{d}v, \tag{3.6}$$

which implies that β is at least the second order of α. Therefore, substituting (3.4) into (3.1), one can write

$$\begin{aligned}
&-\frac{\beta}{\alpha}\mu^{-\frac{1}{2}}\nabla_v\cdot(v\mu) - \beta\mu^{-\frac{1}{2}}\nabla_v\cdot(v\mu^{\frac{1}{2}}(G_1+G_R))\\
&+v_1v_2\mu^{\frac{1}{2}} - \alpha\mu^{-\frac{1}{2}}v_2\partial_{v_1}(\mu^{\frac{1}{2}}(G_1+G_R))\\
&+LG_1+LG_R = \alpha\Gamma(G_1,G_1)\\
&+\alpha\{\Gamma(G_R,G_1)+\Gamma(G_1,G_R)\}+\alpha\Gamma(G_R,G_R).
\end{aligned} \tag{3.7}$$

To remove the zero order term from (3.7), we set

$$G_1 = -L^{-1}(v_1v_2\mu^{\frac{1}{2}}),$$

where we have noticed that $v_1v_2\mu^{\frac{1}{2}} \in (\ker L)^{\perp}$ so that G_1 is well-defined and $G_1 \in (\ker L)^{\perp}$ is purely microscopic, satisfying (3.5). Moreover, it follows from Lemma 6.6 that

$$G_1 = -\frac{1}{2b_0}v_1v_2\mu^{\frac{1}{2}} \tag{3.8}$$

with the constant $b_0 > 0$ defined in (1.15). Then, (3.7) is further reduced to

$$\begin{aligned}
&\beta\mu^{-\frac{1}{2}}\nabla_v\cdot(v\mu^{\frac{1}{2}}G_R) - \alpha\mu^{-\frac{1}{2}}v_2\partial_{v_1}(\mu^{\frac{1}{2}}G_R) + LG_R\\
&\quad= \frac{\beta}{\alpha}\mu^{-\frac{1}{2}}\nabla_v\cdot(v\mu) + \beta\mu^{-\frac{1}{2}}\nabla_v\cdot(v\mu^{\frac{1}{2}}G_1) + \alpha\mu^{-\frac{1}{2}}v_2\partial_{v_1}(\mu^{\frac{1}{2}}G_1)\\
&\qquad+\alpha\Gamma(G_1,G_1)+\alpha\{\Gamma(G_R,G_1)+\Gamma(G_1,G_R)\}+\alpha\Gamma(G_R,G_R),
\end{aligned} \tag{3.9}$$

and in light of (3.8), β in (3.6) is given as

$$\beta = \beta^0 - \frac{\alpha^2}{3}\int_{\mathbb{R}^3} v_1v_2\mu^{1/2}G_R\,\mathrm{d}v, \tag{3.10}$$

where for later use we have denoted

$$\beta^0 = -\frac{\alpha^2}{3}\int_{\mathbb{R}^3} v_1v_2\mu^{1/2}G_1\,\mathrm{d}v = \frac{\alpha^2}{6b_0} > 0.$$

To solve (3.9) on G_R, it is necessary to use the decomposition

$$\sqrt{\mu}G_R = G_{R,1} + \sqrt{\mu}G_{R,2},$$

where $G_{R,1}$ and $G_{R,2}$ are supposed to satisfy

$$-\beta\nabla_v\cdot(vG_{R,1}) - \alpha v_2\partial_{v_1}G_{R,1} + \frac{\beta}{2}|v|^2\mu^{\frac{1}{2}}G_{R,2}$$

$$
\begin{aligned}
&+\alpha\frac{v_1v_2}{2}\mu^{\frac12}G_{R,2}+\nu_0G_R-\chi_M\mathcal{K}G_{R,1}\\
&=\frac{\beta}{\alpha}\nabla_v\cdot(v\mu)+\beta\nabla_v\cdot(v\mu^{\frac12}G_1)+\alpha v_2\partial_{v_1}(\mu^{\frac12}G_1)+\alpha Q(\mu^{\frac12}G_1,\mu^{\frac12}G_1)\\
&\quad+\alpha\{Q(\mu^{\frac12}G_R,\mu^{\frac12}G_1)+Q(\mu^{\frac12}G_1,\mu^{\frac12}G_R)\}+\alpha Q(\mu^{\frac12}G_R,\mu^{\frac12}G_R),
\end{aligned}
\tag{3.11}
$$

and

$$
-\beta\nabla_v\cdot(vG_{R,2})-\alpha v_2\partial_{v_1}G_{R,2}+LG_{R,2}-(1-\chi_M)\mu^{-\frac12}\mathcal{K}G_{R,1}=0, \tag{3.12}
$$

respectively. Here, we recall that ν_0 and $\mathcal{K}$ are defined in (2.3). Moreover, in order for G_R to satisfy (3.5), we require that

$$
\begin{cases}
\displaystyle\int_{\mathbb{R}^3}G_{R,1}\,\mathrm{d}v+\int_{\mathbb{R}^3}\sqrt{\mu}G_{R,2}\,\mathrm{d}v=0,\\
\displaystyle\int_{\mathbb{R}^3}v_iG_{R,1}\,\mathrm{d}v+\int_{\mathbb{R}^3}v_i\sqrt{\mu}G_{R,2}\,\mathrm{d}v=0,\ i=1,2,3,\\
\displaystyle\int_{\mathbb{R}^3}|v|^2G_{R,1}\,\mathrm{d}v+\int_{\mathbb{R}^3}|v|^2\sqrt{\mu}G_{R,2}\,\mathrm{d}v=0.
\end{cases}
\tag{3.13}
$$

The existence of (3.11) and (3.12) under conditions (3.13) will be established through the approximation solution sequence by iteratively solving the equations

$$
\begin{cases}
\varepsilon G_{R,1}^{n+1}-\beta^n\nabla_v\cdot(vG_{R,1}^{n+1})-\alpha v_2\partial_{v_1}G_{R,1}^{n+1}+\nu_0G_{R,1}^{n+1}-\chi_M\mathcal{K}G_{R,1}^{n+1}\\
\quad+\dfrac{\beta^n}{2}|v|^2\mu^{\frac12}G_{R,2}^{n+1}+\alpha\dfrac{v_1v_2}{2}\mu^{\frac12}G_{R,2}^{n+1}-\dfrac{\beta^{n+1}-\frac{\alpha^2}{6b_0}}{\alpha}\nabla_v\cdot(v\mu)\\
\quad=\beta^n\nabla_v\cdot(v\mu^{\frac12}G_1)+\dfrac{\alpha}{6b_0}\nabla_v\cdot(v\mu)+\alpha v_2\partial_{v_1}(\mu^{\frac12}G_1)\\
\quad+\alpha Q(\mu^{\frac12}G_1,\mu^{\frac12}G_1)\\
\qquad+\alpha\{Q(\mu^{\frac12}G_R^n,\mu^{\frac12}G_1)+Q(\mu^{\frac12}G_1,\mu^{\frac12}G_R^n)\}+\alpha Q(\mu^{\frac12}G_R^n,\mu^{\frac12}G_R^n),\\
\varepsilon G_{R,2}^{n+1}-\beta^n\nabla_v\cdot(vG_{R,2}^{n+1})-\alpha v_2\partial_{v_1}G_{R,2}^{n+1}+LG_{R,2}^{n+1}\\
\quad-(1-\chi_M)\mu^{-\frac12}\mathcal{K}G_{R,1}^{n+1}=0,
\end{cases}
\tag{3.14}
$$

for a small parameter $\varepsilon>0$, where we have denoted

$$
\begin{aligned}
\mu^{\frac12}G_R^n&=G_{R,1}^n+\mu^{\frac12}G_{R,2}^n,\\
\beta^n&=\beta^0-\frac{\alpha^2}{3}\int_{\mathbb{R}^3}v_1v_2(G_{R,1}^n+\mu^{\frac12}G_{R,2}^n)\,\mathrm{d}v,\quad n\geqq 0,
\end{aligned}
\tag{3.15}
$$

the constant β^0 is defined in (3.10), and initially for $n=0$ we set

$$
G_{R,1}^0=G_{R,2}^0=0.
$$

Indeed, whenever $[G_{R,1}^n,G_{R,2}^n]$ is given, one has to solve the linear inhomogeneous system (3.14) for $[G_{R,1}^{n+1},G_{R,2}^{n+1}]$ as β^n and G_1 are also given. Thus, the approximation solution sequence $\{[G_{R,1}^n,G_{R,2}^n]\}_{n=1}^\infty$ would be expected to be well-defined.

For brevity we have omitted the explicit dependence of $\{[G^n_{R,1}, G^n_{R,2}]\}_{n=1}^\infty$ on ε. Note that we put the penalty terms $\varepsilon G^{n+1}_{R,i}(i = 1, 2)$ on the left hand side of (3.14) so as to guarantee the mass conservation in (3.5). In addition, since it holds that

$$\left\langle \frac{\alpha}{6b_0}\nabla_v\cdot(v\mu) + \alpha v_2\partial_{v_1}(\mu^{\frac{1}{2}}G_1), |v|^2\mu^{\frac{1}{2}}\right\rangle = 0,$$

and

$$\left\langle \frac{\beta^{n+1} - \frac{\alpha^2}{6b_0}}{\alpha}\nabla_v\cdot(v\mu) + \alpha v_2\partial_{v_1}G^{n+1}_{R,1}, |v|^2\right\rangle + \langle \alpha v_2\partial_{v_1}G^{n+1}_{R,2}, |v|^2\mu^{\frac{1}{2}}\rangle = 0,$$

one sees that

$$\langle G^{n+1}_{R,1}, [1, v_i, |v|^2]\rangle + \langle G^{n+1}_{R,2}, [1, v_i, |v|^2]\mu^{\frac{1}{2}}\rangle = 0,\ i = 1, 2, 3, \tag{3.16}$$

for any $\varepsilon > 0$.

We first show that in an appropriate function space there exists a solution $[G_{R,1}, G_{R,2}]$ satisfying

$$\langle G_{R,1}, [1, v_i, |v|^2]\rangle + \langle G_{R,2}, [1, v_i, |v|^2]\mu^{\frac{1}{2}}\rangle = 0,\ i = 1, 2, 3 \tag{3.17}$$

to the coupled linear inhomogeneous system corresponding to (3.14). To do so, in terms of (3.14), let us first define the following linear operator parameterized by $\sigma \in [0, 1]$ (cf. [19]):

$$\mathscr{L}_\sigma[\mathcal{G}_1, \mathcal{G}_2] = [\mathscr{L}^1_\sigma, \mathscr{L}^2_\sigma][\mathcal{G}_1, \mathcal{G}_2],$$

where

$$\begin{cases} \mathscr{L}^1_\sigma[\mathcal{G}_1, \mathcal{G}_2] = \varepsilon\mathcal{G}_1 - \beta'\nabla_v\cdot(v\mathcal{G}_1) - \alpha v_2\partial_{v_1}\mathcal{G}_1 + \nu_0\mathcal{G}_1 - \sigma\chi_M\mathcal{K}\mathcal{G}_1 \\ \qquad\qquad + \dfrac{\beta'}{2}|v|^2\sqrt{\mu}\mathcal{G}_2 + \alpha\dfrac{v_1v_2}{2}\sqrt{\mu}\mathcal{G}_2 - \dfrac{\beta''(\mathcal{G})}{\alpha}\nabla_v\cdot(v\mu), \\ \mathscr{L}^2_\sigma[\mathcal{G}_1, \mathcal{G}_2] = \varepsilon\mathcal{G}_2 - \beta'\nabla_v\cdot(v\mathcal{G}_2) - \alpha v_2\partial_{v_1}\mathcal{G}_2 \\ \qquad\qquad + \nu_0\mathcal{G}_2 - \sigma K\mathcal{G}_2 - \sigma(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1. \end{cases}$$

Here K is defined as (2.2), β' is a given constant satisfying $\beta' \sim \alpha^2$, and

$$\beta''(\mathcal{G}) = -\frac{\alpha^2}{3}\int_{\mathbb{R}^3} v_1v_2(\mathcal{G}_1 + \mu^{\frac{1}{2}}\mathcal{G}_2)\,\mathrm{d}v. \tag{3.18}$$

Then we consider the solvability of the general coupled linear system

$$\begin{cases} \mathscr{L}^1_\sigma[\mathcal{G}_1, \mathcal{G}_2] = \mathcal{F}_1, \\ \mathscr{L}^2_\sigma[\mathcal{G}_1, \mathcal{G}_2] = \mathcal{F}_2, \end{cases} \tag{3.19}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given sources satisfying

$$\begin{cases} \langle \mathcal{F}_1, [1, v_i, |v|^2]\rangle + \langle \mathcal{F}_2, [1, v_i, |v|^2]\mu^{\frac{1}{2}}\rangle = 0,\ i = 1, 2, 3, \\ \|w_l\nabla^k_v\mathcal{F}_1\|_{L^\infty} + \|w_l\nabla^k_v\mathcal{F}_2\|_{L^\infty} < +\infty,\ \text{for any } k \geqq 0. \end{cases} \tag{3.20}$$

In what follows, we look for solutions to the system (3.19) in the Banach space

$$\mathbf{X}_{\alpha,m}=\left\{[\mathcal{G}_1,\mathcal{G}_2]\in W^{m,\infty}(\mathbb{R}^3_v)\Big|\sum_{0\leqq k\leqq m}\|w_l\nabla_v^k[\mathcal{G}_1,\mathcal{G}_2]\|_{L^\infty}<+\infty,\right.$$
$$\left.\langle\mathcal{G}_1,[1,v_i,|v|^2]\rangle+\langle\mathcal{G}_2,[1,v_i,|v|^2]\mu^{\frac12}\rangle=0,\ i=1,2,3\right\},$$

associated with the norm

$$\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\mathbf{X}_{\alpha,m}}=\sum_{0\leqq k\leqq m}\left\{\|w_l\nabla_v^k\mathcal{G}_1\|_{L^\infty}+\|w_l\nabla_v^k\mathcal{G}_2\|_{L^\infty}\right\}.$$

Let us now deduce the *a priori* estimate for the parameterized linear system (3.19).

Lemma 3.1. (a priori estimate) *Let* $[\mathcal{G}_1,\mathcal{G}_2]\in\mathbf{X}_{\alpha,m}$ *with* $\alpha>0$ *and* $m\geqq 0$ *be a solution to* (3.19) *with* $\varepsilon>0$ *suitably small,* $\sigma\in[0,1]$ *and* $[\mathcal{F}_1,\mathcal{F}_2]$ *satisfying* (3.20)*. There is* $l_0>0$ *such that for any* $l\geqq l_0$ *arbitrarily large, there are* $\alpha_0=\alpha_0(l)>0$ *and large* $M=M(l)>0$ *such that for any* $0<\alpha<\alpha_0$*, the solution* $[\mathcal{G}_1,\mathcal{G}_2]:=\mathscr{L}_\sigma^{-1}[\mathcal{F}_1,\mathcal{F}_2]$ *satisfies the following estimate*

$$\begin{aligned}\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\mathbf{X}_{\alpha,m}}&=\|\mathscr{L}_\sigma^{-1}[\mathcal{F}_1,\mathcal{F}_2]\|_{\mathbf{X}_{\alpha,m}}\\&\leqq C_{\mathscr{L}}\sum_{0\leqq k\leqq m}\left\{\|w_l\nabla_v^k\mathcal{F}_1\|_{L^\infty}+\|w_l\nabla_v^k\mathcal{F}_2\|_{L^\infty}\right\},\end{aligned}\tag{3.21}$$

where $\mathscr{L}_\sigma^{-1}$ *denotes the solution operator for the problem* (3.19) *and the constant* $C_{\mathscr{L}}>0$ *is independent of* σ*,* ε *and* α*.*

Proof. The proof is divided into two steps.
Step 1. L^∞ estimates. Taking $0\leqq k\leqq m$ and l, we set $H_{1,k}=w_l\nabla_v^k\mathcal{G}_1$ and $H_{2,k}=w_l\nabla_v^k\mathcal{G}_2$. Then, $\mathbf{H}_k=[H_{1,k},H_{2,k}]$ satisfies the following equations:

$$\begin{aligned}&\varepsilon H_{1,k}-\beta'\nabla_v\cdot(vH_{1,k})+2l\beta'\frac{|v|^2}{1+|v|^2}H_{1,k}-\alpha v_2\partial_{v_1}H_{1,k}\\&\quad+2l\alpha\frac{v_2v_1}{1+|v|^2}H_{1,k}+\nu_0H_{1,k}\\&\quad-\sigma\chi_Mw_l\mathcal{K}\left(\frac{H_{1,k}}{w_l}\right)-w_l\frac{\beta''(\frac{\mathbf{H}_0}{w_l})}{\alpha}\nabla_v^k\nabla_v\cdot(v\mu)\\&\quad=\mathbf{1}_{|\gamma'|=1}w_l\beta'C_\gamma^{\gamma'}\nabla_v\cdot(\partial^{\gamma'}v\partial_v^{\gamma-\gamma'}\mathcal{G}_1)+\mathbf{1}_{\gamma'=(0,1,0)}\alpha C_\gamma^{\gamma'}w_l\partial_{v_1}\partial_v^{\gamma-\gamma'}\mathcal{G}_1\\&\qquad-\frac{\beta'}{2}w_l\sum_{\gamma'\leqq\gamma}C_\gamma^{\gamma'}\partial_v^{\gamma'}(|v|^2\mu^{\frac12})\partial_v^{\gamma-\gamma'}\mathcal{G}_2-\frac{\alpha}{2}\sum_{\gamma'\leqq\gamma}w_lC_\gamma^{\gamma'}\partial_v^{\gamma'}\left(v_1v_2\sqrt{\mu}\right)\partial_v^{\gamma-\gamma'}\mathcal{G}_2\\&\qquad+\sigma\sum_{0<\gamma'\leqq\gamma}C_\gamma^{\gamma'}w_l(\partial_v^{\gamma'}(\chi_M\mathcal{K}))\left(\partial_v^{\gamma-\gamma'}\mathcal{G}_1\right)+w_l\nabla_v^k\mathcal{F}_1,\end{aligned}\tag{3.22}$$

and

$$
\begin{aligned}
&\varepsilon H_{2,k} - \beta' \nabla_v \cdot (v H_{2,k}) + 2l\beta' \frac{|v|^2}{1+|v|^2} H_{2,k} - \alpha v_2 \partial_{v_1} H_{2,k} + 2l\alpha \frac{v_2 v_1}{1+|v|^2} H_{2,k} \\
&\quad + \nu_0 H_{2,k} - \sigma w_l K\left(\frac{H_{2,k}}{w_l}\right) \\
&\quad = \mathbf{1}_{|\gamma'|=1} w_l \beta' C_\gamma^{\gamma'} \nabla_v \cdot (\partial^{\gamma'} v \partial_v^{\gamma-\gamma'} \mathcal{G}_2) + \mathbf{1}_{\gamma'=(0,1,0)} \alpha C_\gamma^{\gamma'} w_l \partial_{v_1} \partial_v^{\gamma-\gamma'} \mathcal{G}_2 \\
&\qquad + \sigma w_l \sum_{0<\gamma'\leqq\gamma} C_\gamma^{\gamma'} (\partial^{\gamma'} K)\left(\partial_v^{\gamma-\gamma'} \mathcal{G}_2\right) \\
&\qquad + \sigma \sum_{\gamma'\leqq\gamma} C_\gamma^{\gamma'} w_l \partial_v^{\gamma'} ((1-\chi_M)\mu^{-\frac{1}{2}} \mathcal{K})\left(\partial_v^{\gamma-\gamma'} \mathcal{G}_1\right) + w_l \nabla_v^k \mathcal{F}_2,
\end{aligned} \tag{3.23}
$$

where $\mathbf{H}_0 \overset{\text{def}}{=} [H_1, H_2] = [H_{1,0}, H_{2,0}] = w_l[\mathcal{G}_1, \mathcal{G}_2]$. The method of characteristics will be employed to construct the existence of solutions to (3.22) and (3.23) in L^∞ space (cf. [20]). To do so, we first introduce a uniform parameter t, and regard $H_{i,k}(v) = H_{i,k}(t, v)(i = 1, 2)$, then define the characteristic line $[s, V(s; t, v)]$ for both the equations (3.22) and (3.23) going through (s, v) such that

$$
\begin{cases}
\frac{dV_1}{ds} = -\beta' V_1(s;t,v) - \alpha V_2(s;t,v), \\
\frac{dV_i}{ds} = -\beta' V_i(s;t,v),\ i = 2,3, \\
V(t;t,v) = v,
\end{cases} \tag{3.24}
$$

which is equivalent to

$$
\begin{aligned}
V_1(s;t,v) &= e^{\beta'(t-s)}(v_1 + \alpha v_2(t-s)), \\
V_i(s;t,v) &= e^{\beta'(t-s)} v_i, \quad i = 2,3.
\end{aligned}
$$

Integrating along the backward trajectory (3.24), one can write the solutions of (3.22) and (3.23) as the mild form of

$$
\begin{aligned}
H_{1,k} &= e^{-\int_0^t \mathcal{A}^\varepsilon(\tau, V(\tau))\mathrm{d}\tau} H_{1,k}(V(0)) \\
&\quad + \sigma \int_0^t e^{-\int_s^t \mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau} \left\{\chi_M w_l \mathcal{K}\left(\frac{H_{1,k}}{w_l}\right)\right\}(V(s))\,\mathrm{d}s \\
&\quad - \int_0^t e^{-\int_s^t \mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau} \left\{w_l \frac{\beta''(\frac{\mathbf{H}_0}{w_l})}{\alpha} \nabla_v^k \nabla_v \cdot (v\mu)\right\}(V(s))\,\mathrm{d}s \\
&\quad + \int_0^t e^{-\int_s^t \mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau} \Big\{\mathbf{1}_{|\gamma'|=1} w_l \beta' C_\gamma^{\gamma'} \nabla_v \cdot (\partial^{\gamma'} v \partial_v^{\gamma-\gamma'} \mathcal{G}_1) \\
&\quad + \mathbf{1}_{\gamma'=(0,1,0)} \alpha C_\gamma^{\gamma'} w_l \partial_{v_1} \partial_v^{\gamma-\gamma'} \mathcal{G}_1\Big\}(V(s))\,\mathrm{d}s \\
&\quad - \int_0^t e^{-\int_s^t \mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau} \Big\{\frac{\beta'}{2} w_l \sum_{\gamma'\leqq\gamma} C_\gamma^{\gamma'} \partial_v^{\gamma'} (|v|^2 \mu^{\frac{1}{2}}) \partial_v^{\gamma-\gamma'} \mathcal{G}_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha}{2}\sum_{\gamma'\leqq\gamma} w_l C_\gamma^{\gamma'}\partial_v^{\gamma'}\left(v_1v_2\sqrt{\mu}\right)\partial_v^{\gamma-\gamma'}\mathcal{G}_2\Big\}(V(s))\,\mathrm{d}s\\
&+\sigma\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\\
&\left\{\mathbf{1}_{|\gamma|\geqq1}\sum_{0<\gamma'\leqq\gamma}C_\gamma^{\gamma'}w_l(\partial_v^{\gamma'}(\chi_M\mathcal{K}))\left(\partial_v^{\gamma-\gamma'}\mathcal{G}_1\right)\right\}(V(s))\,\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\left(w_l\nabla_v^k\mathcal{F}_1\right)(V(s))\,\mathrm{d}s\overset{\text{def}}{=}\sum_{i=1}^{7}\mathcal{I}_i,
\end{aligned}\tag{3.25}
$$

and

$$
\begin{aligned}
H_{2,k}&=e^{-\int_0^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}H_{2,k}(V(0))\\
&+\sigma\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\left[w_lK\left(\frac{H_{2,k}}{w_l}\right)\right](V(s))\,\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\Big\{\mathbf{1}_{|\gamma'|=1}w_l\beta'C_\gamma^{\gamma'}\nabla_v\cdot(\partial^{\gamma'}v\partial_v^{\gamma-\gamma'}\mathcal{G}_2)\\
&+\mathbf{1}_{\gamma'=(0,1,0)}\alpha C_\gamma^{\gamma'}w_l\partial_{v_1}\partial_v^{\gamma-\gamma'}\mathcal{G}_2\Big\}(V(s))\,\mathrm{d}s\\
&+\sigma\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\\
&\left\{\mathbf{1}_{|\gamma|\geqq1}w_l\sum_{0<\gamma'\leqq\gamma}C_\gamma^{\gamma'}(\partial^{\gamma'}K)\left(\partial_v^{\gamma-\gamma'}\mathcal{G}_2\right)\right\}(V(s))\,\mathrm{d}s\\
&+\sigma\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\\
&\left\{w_l\sum_{\gamma'\leqq\gamma}C_\gamma^{\gamma'}\partial_v^{\gamma'}((1-\chi_M)\mu^{-\frac12}\mathcal{K})\left(\partial_v^{\gamma-\gamma'}\mathcal{G}_1\right)\right\}(V(s))\,\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t\mathcal{A}^\varepsilon(\tau,V(\tau))\mathrm{d}\tau}\left(w_l\nabla_v^k\mathcal{F}_2\right)(V(s))\,\mathrm{d}s\overset{\text{def}}{=}\sum_{i=8}^{13}\mathcal{I}_i,
\end{aligned}\tag{3.26}
$$

where

$$
\mathcal{A}^\varepsilon(\tau,V(\tau))=\nu_0+\varepsilon-3\beta'+2l\beta'\frac{|V(\tau)|^2}{1+|V(\tau)|^2}+2l\alpha\frac{V_2(\tau)V_1(\tau)}{1+|V(\tau)|^2}\geqq\nu_0/2,\tag{3.27}
$$

provided that $\varepsilon>0$, $\beta'\sim\alpha^2$, $l\beta'\sim l\alpha^2$, and $l\alpha$ is suitably small. In what follows, we will estimate $\mathcal{I}_i(1\leqq i\leqq 13)$ term by term.

Since the parameter t here is arbitrary, we may take t sufficiently large such that

$$e^{-\int_0^t \mathcal{A}^\varepsilon(\tau,V(\tau))\,\mathrm{d}\tau} \leqq e^{-\frac{\nu_0 t}{2}} \leqq \frac{1}{8},$$

from which one sees that

$$\mathcal{I}_1 \leqq \frac{1}{8}\|H_{1,k}\|_{L^\infty},\ \mathcal{I}_8 \leqq \frac{1}{8}\|H_{2,k}\|_{L^\infty}.$$

Next, Proposition 2.1 and (3.27) give that

$$\mathcal{I}_2 \leqq \frac{C}{l}\|H_{1,k}\|_{L^\infty}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\,\mathrm{d}s \leqq \frac{C}{l}\|H_{1,k}\|_{L^\infty}.$$

In view of (3.18), it follows that for $l > 5/2$,

$$\begin{aligned}
\mathcal{I}_3 &\leqq C\alpha\int_{\mathbb{R}^3}|v_1v_2(\mathcal{G}_1+\mu^{\frac{1}{2}}\mathcal{G}_2)|\,\mathrm{d}v\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\,\mathrm{d}s\\
&\leqq C\alpha\|w_l\mathcal{G}_1\|_{L^\infty}\int_{\mathbb{R}^3}|v_1v_2w_l^{-1}|\,\mathrm{d}v + C\alpha\|w_l\mathcal{G}_2\|_{L^\infty}\int_{\mathbb{R}^3}|\mu^{\frac{1}{2}}w_l^{-1}|\,\mathrm{d}v\\
&\leqq C\alpha\|H_{1,0}\|_{L^\infty} + C\alpha\|H_{2,0}\|_{L^\infty}.
\end{aligned}$$

For $\mathcal{I}_4$, noticing that $\beta' \sim \alpha^2$, we have

$$\begin{aligned}
\mathcal{I}_4 &\leqq C\beta'\|w_l\nabla_v\cdot(\partial^{\gamma'}v\partial_v^{\gamma-\gamma'}\mathcal{G}_1)\|_{L^\infty}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\,\mathrm{d}s\\
&\quad + C\alpha\|w_l\partial_{v_1}\partial_v^{\gamma-\gamma'}\mathcal{G}_1\|_{L^\infty}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\,\mathrm{d}s\\
&\leqq C\alpha\sum_{k'\leqq k}\|H_{1,k'}\|_{L^\infty}.
\end{aligned}$$

Similarly, it holds that

$$\mathcal{I}_5,\ \mathcal{I}_{10} \leqq C\alpha\sum_{k'\leqq k}\|H_{2,k'}\|_{L^\infty}.$$

For $\mathcal{I}_6$, we first rewrite $\partial_v^{\gamma'}(\chi_M\mathcal{K})(\partial_v^{\gamma-\gamma'}\mathcal{G}_1)$ as

$$\begin{aligned}
&\partial_v^{\gamma'}(\chi_M\mathcal{K})(\partial_v^{\gamma-\gamma'}\mathcal{G}_1)\\
&\quad = \sum_{\gamma''\leqq\gamma'} C_{\gamma'}^{\gamma''}\partial_v^{\gamma'-\gamma''}\chi_M\partial_v^{\gamma''}\mathcal{K}(\partial_v^{\gamma-\gamma'}\mathcal{G}_1)\\
&\quad = \sum_{\gamma''\leqq\gamma'} C_{\gamma'}^{\gamma''}\partial_v^{\gamma'-\gamma''}\chi_M\left\{Q(\partial_v^{\gamma''}\mu,\partial_v^{\gamma-\gamma'}\mathcal{G}_1)+Q(\partial_v^{\gamma-\gamma'}\mathcal{G}_1,\partial_v^{\gamma''}\mu)\right\}.
\end{aligned}$$

Then one sees that

$$\mathcal{I}_6\mathbf{1}_{k\geqq 1} \leqq C\left\|w_l\{Q(\partial_v^{\gamma''}\mu,\partial_v^{\gamma-\gamma'}\mathcal{G}_1)+Q(\partial_v^{\gamma-\gamma'}\mathcal{G}_1,\partial_v^{\gamma''}\mu)\|\right\|_{L^\infty}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\,\mathrm{d}s$$

$$\leqq C \sum_{k'<k} \|H_{1,k'}\|_{L^\infty},$$

according to Lemma 6.4. And likewise, we also have

$$\mathcal{I}_{12} \leqq C \sum_{k' \leqq k} \|H_{1,k'}\|_{L^\infty}.$$

Next, by using Lemma 6.3, one gets

$$\begin{aligned} \mathcal{I}_{11}\mathbf{1}_{k\geqq 1} \leqq C \Big\{ &\Big\| w_1 \Gamma_{\text{gain}}(\partial^{\gamma'}(\sqrt{\mu}), \partial_v^{\gamma-\gamma'}\mathcal{G}_2) \Big\|_{L^\infty} \\ &+ \Big\| w_1 \Gamma(\partial_v^{\gamma-\gamma'}\mathcal{G}_2, \partial^{\gamma'}(\sqrt{\mu})) \Big\|_{L^\infty} \Big\} \int_0^t e^{-\frac{\nu_0}{2}(t-s)}\, \mathrm{d}s \\ \leqq C &\sum_{k'<k} \|H_{2,k'}\|_{L^\infty}. \end{aligned}$$

For $\mathcal{I}_7$ and $\mathcal{I}_{13}$, one directly has

$$\mathcal{I}_7 \leqq C\|w_l \nabla_v^k \mathcal{F}_1\|_{L^\infty},\ \mathcal{I}_{13} \leqq C\|w_l \nabla_v^k \mathcal{F}_2\|_{L^\infty}.$$

Finally, for the delicate term $\mathcal{I}_9$, we divide our computations into the following three cases.
Case 1. $|V| \geqq M$ with M suitably large. From Lemma 6.2, it follows that

$$\int \mathbf{k}_w(V, v_*)\, \mathrm{d}v_* \leqq \frac{C}{(1+|V|)} \leqq \frac{C}{M}.$$

Using this, it follows that

$$\mathcal{I}_9 \leqq \sup_{0\leqq s\leqq t} \int_{\mathbb{R}^3} \mathbf{k}_w(V, v_*)\, \mathrm{d}v_* \|H_{2,k}\|_{L^\infty} \leqq \frac{C}{M}\|H_{2,k}\|_{L^\infty}. \tag{3.28}$$

Case 2. $|V| \leqq M$ and $|v_*| \geqq 2M$. In this situation, we have $|V - v_*| \geqq M$, then

$$\mathbf{k}_w(V, v_*) \leqq Ce^{-\frac{\varepsilon M^2}{8}} \mathbf{k}_w(V, v_*) e^{\frac{\varepsilon|V-v_*|^2}{8}}.$$

By virtue of Lemma 6.2, one sees that $\int \mathbf{k}_w(V, v_*) e^{\frac{\varepsilon|V-v_*|^2}{8}}\, \mathrm{d}v_*$ is still bounded. At this stage, we have by a similar argument as for obtaining (3.28) that

$$\mathcal{I}_9 \leqq Ce^{-\frac{\varepsilon M^2}{8}} \|H_{2,k}\|_{L^\infty}.$$

To obtain the final bound for $\mathcal{I}_9$, we are now in a position to handle the last case.
Case 3. $|V| \leqq M$, $|v_*| \leqq 2M$. In this case, our strategy is to convert the bound in L^∞-norm to the one in L^2-norm which will be established later on. To do so, for any large $M > 0$, we choose a number $p = p(M)$ to define

$$\mathbf{k}_{w,p}(V, v_*) \equiv \mathbf{1}_{|V-v_*|\geqq \frac{1}{p}, |v_*|\leqq p} \mathbf{k}_w(V, v_*), \tag{3.29}$$

such that $\sup_V \int_{\mathbf{R}^3} |\mathbf{k}_{w,p}(V, v_*) - \mathbf{k}_w(V, v_*)| \, dv_* \leqq \frac{1}{M}$. One then has

$$\begin{aligned}\mathcal{I}_9 &\leqq C \sup_s \int_{|v_*| \leqq 2M} \mathbf{k}_{w,p}(V, v_*) |\nabla_v^k \mathcal{G}_2(v_*)| dv_* + \frac{1}{M} \|H_{2,k}\|_{L^\infty} \\ &\leqq C(p) \sup_s \|\nabla_v^k \mathcal{G}_2\| + \frac{1}{M} \|H_{2,k}\|_{L^\infty},\end{aligned}$$

according to Hölder's inequality and the fact that $\int_{\mathbb{R}^3} \mathbf{k}^2_{w,p}(V, v_*) dv_* < \infty$.
Therefore, it follows that for any large $M > 0$,

$$\mathcal{I}_9 \leqq C \left(e^{-\frac{\varepsilon M^2}{8}} + \frac{1}{M} \right) \|H_{2,k}\|_{L^\infty} + C \sup_s \|\nabla_v^k \mathcal{G}_2\|. \tag{3.30}$$

Combining all the estimates above together, we now conclude to have

$$\begin{cases} \|H_{1,k}\|_{L^\infty} \leqq \left(\frac{1}{8} + \frac{C}{l} + C\alpha \right) \|H_{1,k}\|_{L^\infty} + C\alpha \|H_{1,0}\|_{L^\infty} \\ \qquad + C\alpha \sum_{k' \leqq k} \|H_{2,k'}\|_{L^\infty} + \mathbf{1}_{k \geqq 1} C \sum_{k' < k} \|H_{1,k'}\|_{L^\infty} + C \|w_l \nabla_v^k \mathcal{F}_1\|_{L^\infty}, \\ \|H_{2,k}\|_{L^\infty} \leqq \left(\frac{1}{8} + \frac{C}{M} + C\alpha \right) \|H_{2,k}\|_{L^\infty} + \mathbf{1}_{k \geqq 1} C \sum_{k' < k} \|H_{2,k'}\|_{L^\infty} \\ \qquad + C \sum_{k' \leqq k} \|H_{1,k'}\|_{L^\infty} + C \|\nabla_v^k \mathcal{G}_2\| + C \|w_l \nabla_v^k \mathcal{F}_2\|_{L^\infty}. \end{cases} \tag{3.31}$$

It should be pointed out that the constant C in (3.31) is independent of σ and ε.
Step 2. L^2 estimates. We now deduce the L^2 estimate on $\mathcal{G}_2$ which is necessary due to (3.31). Let us start from the macroscopic part of $(\mathcal{G}_1, \mathcal{G}_2)$. Recalling the definition of $\mathbf{P}_0$, at this stage, we may write

$$\mathbf{P}_0 \mathcal{G}_2 = (a_2 + \mathbf{b}_2 \cdot v + c_2(|v|^2 - 3))\sqrt{\mu},$$

and define the projection $\bar{\mathbf{P}}_0$, from L^2 to $\ker(\mathcal{L})$, as

$$\bar{\mathbf{P}}_0 \mathcal{G}_1 = (a_1 + \mathbf{b}_1 \cdot v + c_1(|v|^2 - 3))\mu,$$

and because $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha,m}$, it also follows that

$$a_1 + a_2 = 0, \ \mathbf{b}_1 + \mathbf{b}_2 = 0, \ c_1 + c_2 = 0. \tag{3.32}$$

The following significant observation will be used in the later deductions:

$$\nu_0 f - \sigma K f = (1 - \sigma)\nu_0 f + \sigma L f, \ \nu_0 f - \sigma \mathcal{K} f = (1 - \sigma)\nu_0 f + \sigma \mathcal{L} f. \tag{3.33}$$

By applying (3.33) and (3.32), for any $k \geqq 0$, we get from $\langle \nabla_v^k \mathbf{P}_1 (3.19)_2, \nabla_v^k \mathbf{P}_1 \mathcal{G}_2 \rangle$ and Lemma 6.1 that

$$\frac{1}{2} \min\{\nu_0, \delta_0\} \|\nabla_v^k \mathbf{P}_1 \mathcal{G}_2\| - C \mathbf{1}_{k \geqq 1} \|\mathbf{P}_1 \mathcal{G}_2\|$$

$$\leqq C\left|\left\langle \nabla_v^k \mathbf{P}_1[(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1], \nabla_v^k \mathbf{P}_1\mathcal{G}_2\right\rangle\right|^{\frac{1}{2}} + C\|\nabla_v^k \mathcal{F}_2\| + C\alpha|[a_1, \mathbf{b}_1, c_1]|, \tag{3.34}$$

and using Lemma 6.4, one also has

$$\begin{aligned}
&\left|\left\langle \nabla_v^k \mathbf{P}_1[(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1], \nabla_v^k \mathbf{P}_1\mathcal{G}_2\right\rangle\right| \\
&\leqq \eta\|\nabla_v^k \mathbf{P}_1\mathcal{G}_2\|^2 + C_\eta \left\|\nabla_v^k \mathbf{P}_1[(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1]\right\|^2 \\
&\leqq \eta\|\nabla_v^k \mathbf{P}_1\mathcal{G}_2\|^2 + C_\eta \left\|\nabla_v^k [(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1]\right\|^2 \\
&\quad + C_\eta \left\|\nabla_v^k \mathbf{P}_0[(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1]\right\|^2 \\
&\leqq \eta\|\nabla_v^k \mathbf{P}_1\mathcal{G}_2\|^2 + C_\eta \sum_{k'\leqq k} \|w_l \nabla_v^{k'} \mathcal{G}_1\|_{L^\infty}^2.
\end{aligned} \tag{3.35}$$

For $l > 5/2$, it follows that

$$|[a_1, \mathbf{b}_1, c_1]| \leqq C\|w_l \mathcal{G}_1\|_{L^\infty}. \tag{3.36}$$

Now, (3.34), (3.35) and (3.36) give rise to

$$\sum_{0\leqq k\leqq m} \|\nabla_v^k \mathbf{P}_1\mathcal{G}_2\| + |[a_1, \mathbf{b}_1, c_1]| \leqq C \sum_{0\leqq k'\leqq k} \|w_l \nabla_v^{k'} \mathcal{G}_1\|_{L^\infty} + C\sum_{0\leqq k\leqq m} \|w_l \nabla_v^k \mathcal{F}_2\|_{L^\infty}, \tag{3.37}$$

for $l > 5/2$, where $C > 0$ is independent of ε.

Consequently, taking the linear combination of (3.31) and (3.37) for $0 \leqq k \leqq m$ and adjusting constants, we arrive at

$$\sum_{0\leqq k\leqq m} \left\{\|H_{1,k}\|_{L^\infty} + \|H_{2,k}\|_{L^\infty}\right\} \leqq C \sum_{0\leqq k\leqq m} \|w_l \nabla_v^k [\mathcal{F}_1, \mathcal{F}_2]\|_{L^\infty}.$$

This shows the desired estimate (3.21) and ends the proof of Lemma 3.1. □

With Lemma 3.1 in hand, we now turn to prove the existence of solutions to (3.19) in L^∞ framework by the contraction mapping method. We employ the continuity technique in the parameter σ developed in [19].

Lemma 3.2. *Under the same assumption of Lemma 3.1, there exists a unique solution* $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha,m}$ *to* (3.19) *with* $\sigma = 1$ *satisfying*

$$\sum_{0\leqq k\leqq m} \left\{\|w_l \nabla_v^k \mathcal{G}_1\|_{L^\infty} + \|w_l \nabla_v^k \mathcal{G}_2\|_{L^\infty}\right\} \leqq C \sum_{0\leqq k\leqq m} \left\{\|w_l \nabla_v^k \mathcal{F}_1\|_{L^\infty} + \|w_l \nabla_v^k \mathcal{F}_2\|_{L^\infty}\right\}. \tag{3.38}$$

Proof. As in Lemma 3.1, we denote $\mathscr{L}_\sigma^{-1}$ to be the solution operator for the problem (3.19) and (3.20). Recalling $\mathbf{H}_0 = [H_{1,0}, H_{2,0}] = w_l[\mathcal{G}_1, \mathcal{G}_2]$, if $\sigma = 0$, then (3.22) and (3.23) with $k = 0$ can be reduced to

$$\begin{aligned}&\varepsilon H_1 - \beta'\nabla_v\cdot(vH_1) + 2l\beta'\frac{|v|^2}{1+|v|^2}H_1 - \alpha v_2\partial_{v_1}H_1 + 2l\alpha\frac{v_2v_1}{1+|v|^2}H_1 + \nu_0 H_1\\&\quad+\frac{\beta'}{2}|v|^2\sqrt{\mu}H_2 + \alpha\frac{v_1v_2}{2}\sqrt{\mu}H_2 - w_l\frac{\beta''(\frac{\mathbf{H}_0}{w_l})}{\alpha}\nabla_v\cdot(v\mu) = w_l\mathcal{F}_1,\end{aligned}$$

and

$$\begin{aligned}&\varepsilon H_2 - \beta'\nabla_v\cdot(vH_2) + 2l\beta'\frac{|v|^2}{1+|v|^2}H_2 - \alpha v_2\partial_{v_1}H_2 + 2l\alpha\frac{v_2v_1}{1+|v|^2}H_2 + \nu_0 H_2\\&\quad= w_l\mathcal{F}_2,\end{aligned}$$

respectively. Then, in this case of $\sigma = 0$, the existence of L^∞-solutions can be easily proved by the characteristic method and the contraction mapping theorem, since there is no trouble term involving K or $\mathcal{K}$. That is, one can directly show that

$$\|\mathscr{L}_0^{-1}[\mathcal{F}_1, \mathcal{F}_2]\|_{\mathbf{X}_{\alpha,m}} \leqq C_{\mathscr{L}}\|[\mathcal{F}_1, \mathcal{F}_2]\|_{\mathbf{X}_{\alpha,m}}. \tag{3.39}$$

We now define an operator

$$\mathcal{T}_\sigma[\mathcal{G}_1, \mathcal{G}_2] = \mathscr{L}_0^{-1}\Big[\sigma\chi_M\mathcal{K}\mathcal{G}_1 + \mathcal{F}_1,\ \sigma(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1 + \sigma K\mathcal{G}_2 + \mathcal{F}_2\Big].$$

Moreover, since $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha,m}$, one also has

$$\langle\mathcal{K}\mathcal{G}_1, [1, v_i, |v|^2]\rangle + \langle K\mathcal{G}_2, [1, v_i, |v|^2]\mu^{\frac{1}{2}}\rangle = 0,\ 1 \leqq i \leqq 3,$$

according to (3.33), which further implies

$$\langle\mathcal{T}_\sigma[\mathcal{G}_1, \mathcal{G}_2], [\phi, \phi\mu^{\frac{1}{2}}]\rangle = 0,$$

for any $\varepsilon > 0$, where ϕ denotes 1, v_i $(i = 1, 2, 3)$, and $|v|^2$. Then (3.39) yields

$$\begin{aligned}&\|\mathcal{T}_\sigma[\mathcal{G}_1, \mathcal{G}_2] - \mathcal{T}_\sigma[\mathcal{G}_1', \mathcal{G}_2']\|_{\mathbf{X}_{\alpha,m}}\\&\quad= \Big\|\mathscr{L}_0^{-1}\Big[\sigma\chi_M\mathcal{K}\left(\mathcal{G}_1 - \mathcal{G}_1'\right), \sigma(1-\chi_M)w_l\mu^{-\frac{1}{2}}\mathcal{K}\left(\mathcal{G}_1 - \mathcal{G}_1'\right)\\&\qquad+\sigma w_l K\left(\mathcal{G}_2 - \mathcal{G}_2'\right)\Big]\Big\|_{\mathbf{X}_{\alpha,m}}\\&\quad= \sigma\Big\|\mathscr{L}_0^{-1}\Big[w_l\chi_M\mathcal{K}\left(\mathcal{G}_1 - \mathcal{G}_1'\right), (1-\chi_M)w_l\mu^{-\frac{1}{2}}\mathcal{K}\left(\mathcal{G}_1 - \mathcal{G}_1'\right)\\&\qquad+ w_l K\left(\mathcal{G}_2 - \mathcal{G}_2'\right)\Big]\Big\|_{\mathbf{X}_{\alpha,m}}\\&\quad\leqq \sigma C_{\mathscr{L}}\|[\mathcal{G}_1 - \mathcal{G}_1', \mathcal{G}_2 - \mathcal{G}_2']\|_{\mathbf{X}_{\alpha,m}} \leqq \frac{1}{2}\|[\mathcal{G}_1 - \mathcal{G}_1', \mathcal{G}_2 - \mathcal{G}_2']\|_{\mathbf{X}_{\alpha,m}},\end{aligned} \tag{3.40}$$

provided that $\sigma \in [0, \sigma_*]$ with $0 < \sigma_* \leqq \min\{\frac{1}{2C_{\mathscr{L}}}\}$.

Thus, we obtain a unique fixed point $[\mathcal{G}_1, \mathcal{G}_2]$ in $\mathbf{X}_{\alpha,m}$ such that

$$\mathcal{T}_\sigma[\mathcal{G}_1, \mathcal{G}_2] = [\mathcal{G}_1, \mathcal{G}_2],$$

which is equivalent to

$$\mathscr{L}_0[\mathcal{G}_1, \mathcal{G}_2] = \Big[\sigma\chi_M\mathcal{K}\mathcal{G}_1 + \mathcal{F}_1, \sigma(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1 + \sigma K\mathcal{G}_2 + \mathcal{F}_2\Big].$$

Therefore $[\mathcal{G}_1, \mathcal{G}_2]$ is a unique solution to the system

$$\mathscr{L}_\sigma[\mathcal{G}_1, \mathcal{G}_2] = [\mathcal{F}_1, \mathcal{F}_2], \ \sigma \in [0, \sigma_*].$$

Next, we define

$$\mathcal{T}_{\sigma_*+\sigma} = \mathscr{L}_{\sigma_*}^{-1}\Big[\sigma\chi_M\mathcal{K}\mathcal{G}_1 + \mathcal{F}_1, \sigma(1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\mathcal{G}_1 + \sigma K\mathcal{G}_2 + \mathcal{F}_2\Big].$$

Since the constant $C_{\mathscr{L}}$ in (3.21) is uniform in $\sigma \in [0, 1]$, one can further verify that $\mathcal{T}_{\sigma_*+\sigma}$ with $\sigma \in [0, \sigma_*]$ is also a contraction mapping on $\mathbf{X}_{\alpha,m}$ by using a similar argument as for obtaining (3.40). Namely, we have shown the existence of $\mathscr{L}_{2\sigma_*}^{-1}$ on $\mathbf{X}_{\alpha,m}$ and (3.21) holds true for $\sigma = 2\sigma_*$. Hence, step by step, one can see that $\mathscr{L}_1^{-1}$ exists in case $\sigma = 1$ and (3.38) also follows simultaneously. This completes the proof of Lemma 3.2. □

Once Lemma 3.2 has been obtained, we can now turn to complete the

Proof of Theorem 1.1. We prove the existence of $W^{m,\infty}$ solution to the coupled system (3.11) and (3.12) under the condition (3.17).

According to Lemma 3.2, there indeed exists a unique solution $[G_{R,1}^{n+1}, G_{R,2}^{n+1}]$ to the system (3.14) satisfying (3.16), provided that $[G_{R,1}^n, G_{R,2}^n] \in \mathbf{X}_{\alpha,m}$ for any $m \geqq 0$. We now show that the solution sequence $\{[G_{R,1}^n, G_{R,2}^n]\}_{n=0}^\infty$ is a Cauchy sequence in $\mathbf{X}_{\alpha,m-1}$ with $m \geqq 1$, hence it is convergent and the limit is the unique solution of the following system

$$\begin{aligned}
&\varepsilon G_{R,1} - \beta\nabla_v\cdot(vG_{R,1}) - \alpha v_2\partial_{v_1}G_{R,1} + \frac{\beta}{2}|v|^2\sqrt{\mu}G_{R,2} \\
&\quad + \alpha\frac{v_1v_2}{2}\sqrt{\mu}G_{R,2} + \nu_0 G_{R,1} - \chi_M\mathcal{K}G_{R,1} \\
&\quad = \frac{\beta}{\alpha}\nabla_v\cdot(v\mu) + \beta\nabla_v\cdot(v\mu^{\frac{1}{2}}G_1) + \alpha v_2\partial_{v_1}(\mu^{\frac{1}{2}}G_1) + \alpha Q(\mu^{\frac{1}{2}}G_1, \mu^{\frac{1}{2}}G_1) \\
&\qquad + \alpha\{Q(\mu^{\frac{1}{2}}G_R, \mu^{\frac{1}{2}}G_1) + Q(\mu^{\frac{1}{2}}G_1, \mu^{\frac{1}{2}}G_R)\} + \alpha Q(\mu^{\frac{1}{2}}G_R, \mu^{\frac{1}{2}}G_R),
\end{aligned} \tag{3.41}$$

and

$$\varepsilon G_{R,2} - \beta\nabla_v\cdot(vG_{R,2}) - \alpha v_2\partial_{v_1}G_{R,2} + LG_{R,2} - (1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}G_{R,1} = 0. \tag{3.42}$$

To do this, we first denote

$$\tilde{\beta}^{n+1} = \beta^{n+1} - \beta^n = -\frac{\alpha^2}{3}\int v_1v_2\mu^{1/2}\tilde{G}_R^{n+1}\,\mathrm{d}v,$$

with β^n given by (3.15), and set

$$\mu^{1/2}\tilde{G}_R^{n+1} = \tilde{G}_{R,1}^{n+1} + \mu^{1/2}\tilde{G}_{R,2}^{n+1},$$

with

$$[\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}] = [G_{R,1}^{n+1} - G_{R,1}^{n}, G_{R,2}^{n+1} - G_{R,2}^{n}].$$

Then $\tilde{G}_{R,1}^{n+1}$ and $\tilde{G}_{R,2}^{n+1}$ satisfy the equations

$$\begin{cases} \varepsilon\tilde{G}_{R,1}^{n+1} - \beta^n\nabla_v\cdot(v\tilde{G}_{R,1}^{n+1}) - \alpha v_2\partial_{v_1}\tilde{G}_{R,1}^{n+1} + \nu_0\tilde{G}_{R,1}^{n+1} - \chi_M\mathcal{K}\tilde{G}_{R,1}^{n+1} \\ \quad + \dfrac{\beta^n}{2}|v|^2\sqrt{\mu}\tilde{G}_{R,2}^{n+1} + \alpha\dfrac{v_1v_2}{2}\sqrt{\mu}\tilde{G}_{R,2}^{n+1} - \dfrac{\tilde{\beta}^{n+1}}{\alpha}\nabla_v\cdot(v\mu) \\ \quad = \tilde{\beta}^n\nabla_v\cdot(vG_{R,1}^n) + \tilde{\beta}^n\nabla_v\cdot(v\mu^{\frac{1}{2}}G_1) \\ \quad + \alpha\{Q(\mu^{\frac{1}{2}}\tilde{G}_R^n, \mu^{\frac{1}{2}}G_1) + Q(\mu^{\frac{1}{2}}G_1, \mu^{\frac{1}{2}}\tilde{G}_R^n)\} \\ \quad\quad + \alpha\left\{Q(\mu^{\frac{1}{2}}\tilde{G}_R^n, \mu^{\frac{1}{2}}\tilde{G}^n) + Q(\mu^{\frac{1}{2}}\tilde{G}_R^n, \mu^{\frac{1}{2}}G_R^n) + Q(\mu^{\frac{1}{2}}G_R^n, \mu^{\frac{1}{2}}\tilde{G}_R^n)\right\} \\ \quad \overset{\text{def}}{=} \mathcal{M}(\tilde{G}_R^n, \tilde{G}_R^n), \\ \varepsilon\tilde{G}_{R,2}^{n+1} - \beta^n\nabla_v\cdot(v\tilde{G}_{R,2}^{n+1}) - \alpha v_2\partial_{v_1}\tilde{G}_{R,2}^{n+1} \\ \quad + L\tilde{G}_{R,2}^{n+1} - (1-\chi_M)\mu^{-\frac{1}{2}}\mathcal{K}\tilde{G}_{R,1}^{n+1} = 0, \end{cases} \tag{3.43}$$

with

$$\langle\tilde{G}_{R,1}^n, [1, v_i, |v|^2]\rangle + \langle\tilde{G}_{R,2}^n, [1, v_i, |v|^2]\mu^{\frac{1}{2}}\rangle = 0, \ i = 1, 2, 3.$$

Our goal next is to prove

$$\|[\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}]\|_{\mathbf{X}_{\alpha,m-1}} \leqq C\alpha\|[\tilde{G}_{R,1}^{n}, \tilde{G}_{R,2}^{n}]\|_{\mathbf{X}_{\alpha,m-1}}, \tag{3.44}$$

under the condition that

$$\|[G_{R,1}^n, G_{R,2}^n]\|_{\mathbf{X}_{\alpha,m}} < C_{\alpha,m}, \tag{3.45}$$

where the constant $C_{\alpha,m}$ is finite independent of n. In fact, on the one hand, thanks to Lemma 3.1, it follows that

$$\|[\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}]\|_{\mathbf{X}_{\alpha,m-1}} \leqq C\sum_{0\leqq k\leqq m-1}\|w_l\nabla_x^k\mathcal{M}(\tilde{G}_R^n, \tilde{G}_R^n)\|_\infty,$$

where $\mathcal{M}(\tilde{G}_R^n, \tilde{G}_R^n)$ is given in (3.43). On the other hand, we get from Lemma 6.4 that

$$\begin{aligned} \sum_{0\leqq k\leqq m-1}\|w_l\nabla_x^k\mathcal{M}(\tilde{G}_R^n, \tilde{G}_R^n)\|_\infty &\leqq C\alpha\sum_{0\leqq k\leqq m-1}\left\{\|w_l\nabla_x^k\tilde{G}_{R,1}^n\|_{L^\infty}^2 + \|w_l\nabla_x^k\tilde{G}_{R,2}^n\|_{L^\infty}^2\right\} \\ &\quad + C\alpha\sum_{0\leqq k\leqq m-1}\|w_l\nabla_x^k\tilde{G}_R^n\|_{L^\infty}\sum_{0\leqq k\leqq m}\|w_l\nabla_x^kG_R^n\|_{L^\infty}, \end{aligned}$$

which is further bounded by

$$C\alpha \sum_{0\leqq k\leqq m-1} \left\{ \|w_l \nabla_v^k \tilde{G}_{R,1}^n\|_{L^\infty} + \|w_l \nabla_v^k \tilde{G}_{R,2}^n\|_{L^\infty} \right\},$$

due to (3.45). Thus, (3.44) is valid, in other words, $\{[G_{R,1}^n, G_{R,2}^n]\}_{n=0}^\infty$ is a Cauchy sequence in $\mathbf{X}_{\alpha,m-1}$ for $\alpha > 0$ suitably small. Hence,

$$[G_{R,1}^n, G_{R,2}^n] \to [G_{R,1}^\varepsilon, G_{R,2}^\varepsilon]$$

strongly in $\mathbf{X}_{\alpha,m-1}$ as $n \to +\infty$, and

$$\beta^n \to \beta^\varepsilon = \frac{\alpha^2}{6b_0} - \frac{\alpha^2}{3} \int_{\mathbb{R}^3} v_1 v_2 (G_{R,1}^\varepsilon + \mu^{1/2} G_{R,2}^\varepsilon)\, \mathrm{d}v. \tag{3.46}$$

And the limit $[G_{R,1}^\varepsilon, G_{R,2}^\varepsilon]$ is a unique solution to (3.41) and (3.42). Furthermore, it can be directly shown that $[G_{R,1}^\varepsilon, G_{R,2}^\varepsilon]$ enjoys the estimate

$$\|[G_{R,1}^\varepsilon, G_{R,2}^\varepsilon]\|_{\mathbf{X}_{\alpha,m}} \leqq C\alpha. \tag{3.47}$$

Furthermore, taking the limit $\varepsilon \to 0$, we may repeat the same procedure as for letting $n \to \infty$, so that the limit function $[G_{R,1}, G_{R,2}] \in \mathbf{X}_{\alpha,m}$ is the unique solution of (3.11) and (3.12) and enjoys the same bound as (3.47).

Moreover, for any $\varepsilon > 0$, it follows that

$$\begin{aligned}
&\varepsilon\langle G_{R,1}^\varepsilon, 1\rangle + \varepsilon\langle G_{R,2}^\varepsilon, \mu^{\frac{1}{2}}\rangle = 0,\\
&(\varepsilon + \beta^\varepsilon)\langle G_{R,1}^\varepsilon, v_1\rangle + (\varepsilon + \beta^\varepsilon)\langle G_{R,2}^\varepsilon, v_1\mu^{\frac{1}{2}}\rangle + \alpha\langle G_{R,1}^\varepsilon, v_2\rangle\\
&+ \alpha\langle G_{R,2}^\varepsilon, v_2\mu^{\frac{1}{2}}\rangle = 0,\\
&(\varepsilon + \beta^\varepsilon)\langle G_{R,1}^\varepsilon, v_i\rangle + (\varepsilon + \beta^\varepsilon)\langle G_{R,2}^\varepsilon, v_i\mu^{\frac{1}{2}}\rangle = 0,\ i = 2, 3,
\end{aligned}$$

and

$$(\varepsilon + \beta^\varepsilon)\langle G_{R,1}^\varepsilon, |v|^2\rangle + (\varepsilon + \beta^\varepsilon)\langle G_{R,2}^\varepsilon, |v|^2\mu^{\frac{1}{2}}\rangle = 0,$$

consequently,

$$\langle G_{R,1}^\varepsilon, [1, v_i, |v|^2]\rangle + \langle G_{R,2}^\varepsilon, [1, v_i, |v|^2]\mu^{\frac{1}{2}}\rangle = 0,\ i = 1, 2, 3, \tag{3.48}$$

since $\beta^\varepsilon > 0$ due to (3.47) and (3.46) and α can be suitably small. Taking $\varepsilon \to 0$ in (3.48) gives rise to (3.3). This finishes the proof of Theorem 1.1. □

4. Local Existence

In the previous section, via the perturbation approach we have proved the existence of the self-similar profile $G(v)$ governed by the steady problem (1.11) whenever the shear rate α is suitably small. However, the non-negativity of $G(v)$ is still unknown. Thus, we will study the time-dependent problem and obtain the time-asymptotic stability of $G(v)$ under initial small perturbations, which in turn can give the non-negativity of $G(v)$. To be more general, we reformulate the problem in the spatially inhomogeneous setting where the one-dimensional transport only along the v_1-direction is allowed. We remark that the justification of dynamical stability in the 2D framework as for the original problem (1.19) should be more challenging and is hence left for the future study.

The goal of this section is to first establish the existence of the unique local-in-time solution to the Cauchy problem (1.20) and (1.21) in the one-dimensional spatially inhomogeneous setting. The proof of the global existence as well as the large time behavior of solutions will be given in the next section. In light of (1.7), we let

$$F(t,x,v)=e^{-3\beta t}f(t,x,\frac{v}{e^{\beta t}})\overset{\text{def}}{=}e^{-3\beta t}f(t,x,\xi),$$

then the Cauchy problem (1.20) and (1.21) is converted to

$$\begin{cases}\partial_t f+e^{\beta t}\xi_1\partial_x f-\beta\nabla_\xi\cdot(\xi f)-\alpha\xi_2\partial_{\xi_1}f=Q(f,f),\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3,\\ f(0,x,\xi)=F_0(x,\xi),\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3.\end{cases}\tag{4.1}$$

We notice that as mentioned in (1.18), $F(t,x,v)$ is expected to behave like $e^{-3\beta t}G(e^{-\beta t}v)$ in large time and hence $f(t,x,\xi)$ should tend asymptotically towards the self-similar profile $G(\xi)$, where the self-similar profile G is determined in Theorem 1.1. Now, setting $\tilde f(t,x,\xi)=f(t,x,\xi)-G(\xi)$ as the perturbation, one can see that $\tilde f=\tilde f(t,x,\xi)$ satisfies

$$\begin{cases}\partial_t\tilde f+e^{\beta t}\xi_1\partial_x\tilde f-\beta\nabla_\xi\cdot(\xi\tilde f)-\alpha\xi_2\partial_{\xi_1}\tilde f\\ \quad=Q(\tilde f,\tilde f)+Q(\tilde f,G)+Q(G,\tilde f),\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3,\\ \tilde f(0,x,\xi)=F_0(x,\xi)-G(\xi),\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3.\end{cases}$$

Defining next $\tilde f(t,x,\xi)=\sqrt{\mu}\tilde g$ and recalling $G=\mu+\sqrt{\mu}\{\alpha G_1+\alpha G_R\}$, we have

$$\begin{cases}\partial_t\tilde g+e^{\beta t}\xi_1\partial_x\tilde g-\beta\nabla_\xi\cdot(\xi\tilde g)+\dfrac{\beta}{2}|\xi|^2\tilde g-\alpha\xi_2\partial_{\xi_1}\tilde g+\dfrac{\alpha}{2}\xi_1\xi_2\tilde g+L\tilde g\\ \quad=\Gamma(\tilde g,\tilde g)+\Gamma(\tilde g,\alpha G_1+\alpha G_R)+\Gamma(\alpha G_1+\alpha G_R,\tilde g),\quad t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3,\\ \tilde g(0,x,\xi)=\tilde g_0=\dfrac{F_0(x,\xi)-G(\xi)}{\sqrt{\mu}},\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3.\end{cases}\tag{4.2}$$

To solve (4.2), since there is a strong growth term $\frac{\alpha}{2}\xi_1\xi_2\tilde g$ in (4.2), it is more convenient to consider the decomposition $\sqrt{\mu}\tilde g=g_1+\sqrt{\mu}g_2$, where g_1 and g_2 satisfy

$$\begin{cases} \partial_t g_1 + e^{\beta t}\xi_1\partial_x g_1 - \beta\nabla_\xi\cdot(\xi g_1) - \alpha\xi_2\partial_{\xi_1}g_1 + \nu_0 g_1 \\ \quad = \chi_M\mathcal{K}g_1 - \frac{\beta}{2}|\xi|^2\mu^{\frac{1}{2}}g_2 - \frac{\alpha}{2}\mu^{\frac{1}{2}}\xi_1\xi_2 g_2 \\ \quad + \tilde{H}(g_1, g_2),\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ g_1(0,x,\xi) = 0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \partial_t g_2 + e^{\beta t}\xi_1\partial_x g_2 - \beta\nabla_\xi\cdot(\xi g_2) - \alpha\xi_2\partial_{\xi_1}g_2 + Lg_2 \\ \quad = \mu^{-1/2}(1-\chi_M)\mathcal{K}g_1,\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ g_2(0,x,\xi) = \frac{F_0(x,\xi)-G(\xi)}{\sqrt{\mu}} \overset{\text{def}}{=} \tilde{g}_0(x,\xi),\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{4.4}$$

respectively. Here

$$\tilde{H}(g_1,g_2) = Q(g_1+\mu^{\frac{1}{2}}g_2, g_1+\mu^{\frac{1}{2}}g_2) + Q(g_1+\mu^{\frac{1}{2}}g_2, \mu^{\frac{1}{2}}(\alpha G_1+\alpha G_R) \\ + Q(\mu^{\frac{1}{2}}(\alpha G_1+\alpha G_R), g_1+\mu^{\frac{1}{2}}g_2).$$

We shall look for solutions of (4.3) and (4.4) in the following function space

$$\bar{\mathbf{Y}}_{\alpha,T} = \Big\{[\mathcal{G}_1,\mathcal{G}_2]\in L^\infty(0,T;W^{2,\infty}_x L^\infty_v)\Big| \\ \sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T}\left\{\|w_l\partial_x^{\gamma_0}\mathcal{G}_1(t)\|_{L^\infty} + \alpha\|w_l\partial_x^{\gamma_0}\mathcal{G}_2(t)\|_{L^\infty}\right\} < +\infty, \\ \langle\mathcal{G}_1,[1,v_i]\rangle + \langle\mathcal{G}_2,[1,v_i]\mu^{\frac{1}{2}}\rangle = 0,\ i=1,2,3\Big\},$$

supplemented with the norm

$$\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\bar{\mathbf{Y}}_{\alpha,T}} = \sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T}\left\{\|w_l\partial_x^{\gamma_0}\mathcal{G}_1(t)\|_{L^\infty} + \alpha\|w_l\partial_x^{\gamma_0}\mathcal{G}_2(t)\|_{L^\infty}\right\}.$$

Theorem 4.1. (Local existence) *Under the conditions listed in Theorem 1.2, there exits $T_*>0$ which may depend on α such that the coupling systems (4.3) and (4.4) admit a unique local in time solution $[g_1(t,x,\xi), g_2(t,x,\xi)]$ satisfying*

$$\|[g_1,g_2]\|_{\mathbf{Y}_{\alpha,T_*}} \leqq 2\alpha.$$

Proof. Our proof is based on the Duhamel's principle and contraction mapping method. We first consider the following approximation equations

$$\begin{cases} \partial_t g_1 + e^{\beta t}\xi_1\partial_x g_1 - \beta\nabla_\xi\cdot(\xi g_1) - \alpha\xi_2\partial_{\xi_1}g_1 + \nu_0 g_1 \\ \quad = \chi_M\mathcal{K}g_1' - \frac{\beta}{2}|\xi|^2\mu^{\frac{1}{2}}g_2' \\ \quad - \frac{\alpha}{2}\mu^{\frac{1}{2}}\xi_1\xi_2 g_2' + \tilde{H}(g_1', g_2'),\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ g_1(0,x,\xi) = 0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \partial_t g_2 + e^{\beta t}\xi_1\partial_x g_2 - \beta\nabla_\xi\cdot(\xi g_2) - \alpha\xi_2\partial_{\xi_1} g_2 + \nu_0 g_2 \\ \quad = K g_2' + \mu^{-\frac{1}{2}}(1-\chi_M)\mathcal{K} g_1',\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ g_2(0,x,\xi) = \dfrac{F_0(x,\xi)-G(\xi)}{\sqrt{\mu}} \overset{\text{def}}{=} \tilde{g}_0(x,\xi),\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3. \end{cases} \tag{4.6}$$

Let $[g_1, g_2]$ be a solution of the pair of (4.5) and (4.6) with $[g_1', g_2']$ being given. Then the nonlinear operator $\mathcal{N}$ is formally defined as

$$\mathcal{N}([g_1', g_2']) = [g_1, g_2].$$

Our aim is to prove that there exists a sufficiently small $T_* > 0$ such that $\mathcal{N}[g_1', g_2']$ has a unique fixed point in some Banach space by adopting the contraction mapping method. In fact, since

$$\sum_{\gamma_0\leqq 2} \|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty} \leqq \alpha,$$

we can define the Banach space

$$\begin{aligned} \mathbf{Y}_{\alpha,T} = \Big\{ &(\mathcal{G}_1,\mathcal{G}_2)\in L^\infty(0,T; W_x^{2,\infty}L_v^\infty) \Big| \\ &\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T}\left\{\|w_l\partial_x^{\gamma_0}\mathcal{G}_1(t)\|_{L^\infty} + \alpha\|w_l\partial_x^{\gamma_0}\mathcal{G}_2(t)\|_{L^\infty}\right\} \leqq 2\alpha, \\ &\mathcal{G}_1(0)=0,\ \mathcal{G}_2(0)=\tilde{g}_0 \Big\}, \end{aligned}$$

associated with the norm

$$\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\mathbf{Y}_{\alpha,T}} = \sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T}\left\{\|w_l\partial_x^{\gamma_0}\mathcal{G}_1(t)\|_{L^\infty} + \alpha\|w_l\partial_x^{\gamma_0}\mathcal{G}_2(t)\|_{L^\infty}\right\}.$$

We now show that

$$\mathcal{N}: \mathbf{Y}_{\alpha,T} \to \mathbf{Y}_{\alpha,T},$$

is well-defined and $\mathcal{N}$ is a contraction mapping for some $T>0$.

Let us denote $h_1 = w_l\partial_x^{\gamma_0} g_1$ and $h_2 = w_l\partial_x^{\gamma_0} g_2$ with $\gamma_0 \leqq 2$, then $[h_1, h_2]$ satisfies

$$\begin{cases} \partial_t h_1 + e^{\beta t}\xi_1\partial_x h_1 - \beta\nabla_\xi\cdot(\xi h_1) - \alpha\xi_2\partial_{\xi_1}h_1 + 2l\beta\dfrac{|\xi|^2}{1+|\xi|^2}h_1 \\ \quad + 2l\alpha\dfrac{\xi_2\xi_1}{1+|\xi|^2}h_1 + \nu_0 h_1 \\ \quad = \chi_M w_l\mathcal{K}\left(\dfrac{h_1'}{w_l}\right) - \dfrac{\beta}{2}|\xi|^2\mu^{\frac{1}{2}}h_2' - \dfrac{\alpha}{2}\mu^{\frac{1}{2}}\xi_1\xi_2 h_2' \\ \qquad + w_l\partial_x^{\gamma_0}\tilde{H}(g_1',g_2'),\quad t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ h_1(0,x,\xi)=0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{4.7}$$

and

$$\begin{cases} \partial_t h_2 + e^{\beta t}\xi_1\partial_x h_2 - \beta\nabla_\xi\cdot(\xi h_2) - \alpha\xi_2\partial_{\xi_1}h_2 + 2l\beta\frac{|\xi|^2}{1+|\xi|^2}h_2 \\ \quad + 2l\alpha\frac{\xi_2\xi_1}{1+|\xi|^2}h_2 + \nu_0 h_2 - w_l K\left(\frac{h_2'}{w_l}\right) \\ \quad = w_l\mu^{-1/2}(1-\chi_M)\mathcal{K}\left(\frac{h_1'}{w_l}\right), t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ h_2(0,x,\xi) = w_l\partial_x^{\gamma_0}\tilde{g}_0(x,\xi),\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{4.8}$$

where $h_i' = w_l\partial_x^{\gamma_0}g_i'(i=1,2)$.

Next, we define the characteristic line $[s, X(s;t,x,\xi), V(s;t,x,\xi)]$ for equations (4.7) and (4.8) passing through (t,x,ξ) such that

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}s} = e^{\beta s}V_1(s;t,x,\xi), \\ \frac{\mathrm{d}V_1}{\mathrm{d}s} = -\beta V_1(s;t,x,\xi) - \alpha V_2(s;t,x,\xi), \\ \frac{\mathrm{d}V_i}{\mathrm{d}s} = -\beta V_i(s;t,x,\xi),\ i=2,3, \\ X(t;t,x,\xi)=x,\ V(t;t,x,\xi)=\xi, \end{cases}$$

which is equivalent to

$$\begin{cases} X(s;t,x,\xi) = e^{\beta(t-s)}\Big(x-(t-s)\xi_1 - \frac{1}{2}\alpha(t-s)^2\xi_2\Big), \\ V_1(s;t,x,\xi) = e^{\beta(t-s)}(\xi_1+\alpha\xi_2(t-s)), \\ V_i(s;t,x,\xi) = e^{\beta(t-s)}\xi_i.\ i=2,3. \end{cases} \tag{4.9}$$

Using this, as (3.25) and (3.26), we can write the solution of (4.7) as

$$[h_1,h_2] = \mathcal{Q}(g_1',g_2') = [\mathcal{Q}_1(g_1',g_2'), \mathcal{Q}_2(g_1',g_2')],$$

with

$$\begin{aligned} \mathcal{Q}_1(g_1',g_2') &= \int_0^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left\{\chi_M w_l\mathcal{K}\left(\frac{h_1'}{w_l}\right)\right\}(V(s))\,\mathrm{d}s \\ &\quad + \frac{\beta}{2}\int_0^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}|V(s)|^2\sqrt{\mu}(V(s))h_2'(V(s))\,\mathrm{d}s \\ &\quad + \alpha\int_0^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\frac{V_1(s)V_2(s)}{2}\sqrt{\mu}(V(s))h_2'(V(s))\,\mathrm{d}s \\ &\quad + \int_0^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left(w_l\tilde{H}\right)(V(s))\,\mathrm{d}s, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \mathcal{Q}_2(g_1',g_2') &= e^{-\int_0^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}(w_l\partial_x^{\gamma_0}\tilde{g}_0)(X(0),V(0)) \\ &\quad + \int_0^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left\{(1-\chi_M)\mu^{-\frac{1}{2}}w_l\mathcal{K}\left(\frac{h_1'}{w_l}\right)\right\}(V(s))\,\mathrm{d}s \end{aligned}$$

$$+\int_0^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left[w_l K\left(\frac{h_2'}{w_l}\right)\right](V(s))\,\mathrm{d}s, \tag{4.11}$$

where $\mathcal{A}=\mathcal{A}^\varepsilon-\varepsilon$.

Let $[g_1',g_2']\in\mathbf{Y}_{\alpha,T_*}$. In light of (3.27), taking L^∞ estimates of $\mathcal{Q}[g_1',g_2']$ and applying Lemmas 6.2, 6.4 and 2.1, one directly has

$$\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\|\mathcal{Q}_1[g_1',g_2']\|_{L^\infty}\leqq\left(\frac{C}{l}+C\alpha\right)T_*\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\|h_1'\|_{L^\infty}+C\alpha T_*\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\|h_2'\|_{L^\infty}\leqq\frac{\alpha}{2}, \tag{4.12}$$

and

$$\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\alpha\|\mathcal{Q}_2[g_1',g_2']\|_{L^\infty}\leqq\alpha+CT_*\alpha\sum_{\gamma_0\leqq 2}\left\{\sup_{0\leqq t\leqq T_*}\|h_1'\|_{L^\infty}+\sup_{0\leqq t\leqq T_*}\|h_2'\|_{L^\infty}\right\}\leqq\frac{3\alpha}{2}, \tag{4.13}$$

provided that $T_*>0$ is suitably small. And similarly, for $[g_1',g_2']\in\mathbf{Y}_{\alpha,T_*}$ and $[g_1'',g_2'']\in\mathbf{Y}_{\alpha,T_*}$, it follows that

$$\begin{aligned}&\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\|\mathcal{Q}_1[g_1',g_2']-\mathcal{Q}_1[g_1'',g_2'']\|_{L^\infty}\\&\leqq\left(\frac{C}{l}+C\alpha\right)T_*\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\|h_1'-h_1''\|_{L^\infty}\\&\quad+C\alpha T_*\sum_{\gamma_0\leqq 2}\sup_{0\leqq t\leqq T_*}\|h_2'-h_2''\|_{L^\infty}\\&\leqq\frac{1}{4}\sum_{\gamma_0\leqq 2}\left\{\sup_{0\leqq t\leqq T_*}\|h_1'-h_1''\|_{L^\infty}+\alpha\sup_{0\leqq t\leqq T_*}\|h_2'-h_2''\|_{L^\infty}\right\},\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\sum_{\gamma_0\leqq 2}\alpha\sup_{0\leqq t\leqq T_*}\|\mathcal{Q}_2[g_1',g_2']-\mathcal{Q}_2[g_1'',g_2'']\|_{L^\infty}\\&\leqq CT_*\alpha\sum_{\gamma_0\leqq 2}\left\{\sup_{0\leqq t\leqq T_*}\|h_1'-h_1''\|_{L^\infty}+\sup_{0\leqq t\leqq T_*}\|h_2'-h_2''\|_{L^\infty}\right\}\\&\leqq\frac{1}{4}\sum_{\gamma_0\leqq 2}\left\{\sup_{0\leqq t\leqq T_*}\|h_1'-h_1''\|_{L^\infty}+\alpha\sup_{0\leqq t\leqq T_*}\|h_2'-h_2''\|_{L^\infty}\right\},\end{aligned} \tag{4.15}$$

for $T_* > 0$ small enough. Here, the following type of estimates have been also used:

$$
\begin{aligned}
&\|w_l[Q(\mu^{\frac{1}{2}}g_2', \mu^{\frac{1}{2}}g_2') - Q(\mu^{\frac{1}{2}}g_2'', \mu^{\frac{1}{2}}g_2'')]\|_{L^\infty} \\
&\quad \leqq \|w_l Q(\mu^{\frac{1}{2}}(g_2' - g_2''), \mu^{\frac{1}{2}}(g_2' - g_2''))\|_{L^\infty} + \|w_l Q(\mu^{\frac{1}{2}}(g_2' - g_2''), \mu^{\frac{1}{2}}g_2'')\|_{L^\infty} \\
&\qquad + \|w_l Q(\mu^{\frac{1}{2}}g_2'', \mu^{\frac{1}{2}}(g_2' - g_2''))\|_{L^\infty} \\
&\quad \leqq C\left\{\|w_l[g_2' - g_2'']\|_{L^\infty}^2 + 2\|w_l g_2''\|_{L^\infty}\|w_l[g_2' - g_2'']\|_{L^\infty}\right\}.
\end{aligned}
$$

Consequently, (4.14) and (4.15) lead to

$$
\|\mathcal{Q}[g_1', g_2'] - \mathcal{Q}[g_1'', g_2'']\|_{\mathbf{Y}_{\alpha,T_*}} \leqq \frac{1}{2}\|[g_1', g_2'] - [g_1'', g_2'']\|_{\mathbf{Y}_{\alpha,T_*}}.
$$

This together with (4.12) and (4.13) imply that there exists $T_* > 0$ such that $\mathcal{N}$ is a contraction mapping on $\mathbf{Y}_{\alpha,T_*}$. Hence, there exists a unique $[g_1, g_2] \in \mathbf{Y}_{\alpha,T_*}$ such that

$$
[g_1, g_2] = \mathcal{N}(g_1, g_2).
$$

This completes the proof of Theorem 4.1. $\square$

5. Convergence to the Steady State

Following the previous section regarding the local existence, the goal of this section is to establish the global existence of the Cauchy problem (1.20) and (1.21). More precisely, we shall construct a unique global-in-time solution around the self-similar profile, and also prove its large time asymptotic behavior with the exponential rate of convergence.

As in the previous sections, we denote the macroscopic part of $\tilde{g}$ by

$$
\mathbf{P}_0\tilde{g} = \{a + \mathbf{b}\cdot\xi + c(|\xi|^2 - 3)\}\sqrt{\mu}. \tag{5.1}
$$

By taking the velocity moments

$$
\mu^{\frac{1}{2}}, \xi_j\mu^{\frac{1}{2}}, \frac{1}{6}(|\xi|^2 - 3)\mu^{\frac{1}{2}}, A_{ij} = \left(\xi_i\xi_j - \frac{\delta_{ij}}{3}|\xi|^2\right)\mu^{\frac{1}{2}}, B_i = \frac{1}{10}(|\xi|^2 - 5)\xi_i\mu^{\frac{1}{2}}
$$

with $1 \leqq i, j \leqq 3$ for the equation

$$
\begin{cases}
\partial_t\tilde{g} + e^{\beta t}\xi_1\partial_x\tilde{g} - \beta\nabla_\xi\cdot(\xi\tilde{g}) + \dfrac{\beta}{2}|\xi|^2\tilde{g} - \alpha\xi_2\partial_{\xi_1}\tilde{g} + \dfrac{\alpha}{2}\xi_1\xi_2\tilde{g} + L\tilde{g} \\
\quad = \underbrace{\Gamma(\tilde{g}, \tilde{g}) + \Gamma(\tilde{g}, \alpha G_1 + \alpha G_R) + \Gamma(\alpha G_1 + \alpha G_R, \tilde{g})}_{\tilde{\mathcal{F}}}, \\
\quad t > 0,\ x \in \mathbb{T},\ \xi \in \mathbb{R}^3, \\
\tilde{g}(0, x, \xi) = \tilde{g}_0 = \dfrac{F_0(x, \xi) - G(\xi)}{\sqrt{\mu}},\ x \in \mathbb{T},\ \xi \in \mathbb{R}^3,
\end{cases} \tag{5.2}
$$

one sees that the coefficient functions $[a, \mathbf{b}, c] = [a, \mathbf{b}, c](t, x)$ satisfy the fluid-type system

$$\begin{cases} \partial_t a + e^{\beta t}\partial_x b_1 = 0, \\ \partial_t b_1 + \beta b_1 + e^{\beta t}\partial_x(a+2c) + \alpha b_2 + e^{\beta t}\partial_x \int_{\mathbb{R}^3} \xi_1^2 \sqrt{\mu}\mathbf{P}_1\tilde{g}\,\mathrm{d}\xi = 0, \\ \partial_t b_i + \beta b_i + e^{\beta t}\partial_x \langle A_{1i}, \mathbf{P}_1\tilde{g}\rangle = 0, \ i = 2, 3, \end{cases} \tag{5.3}$$

$$\begin{aligned} &\partial_t c + \beta a + 2\beta c + \frac{e^{\beta t}}{3}\partial_x b_1 + \frac{e^{\beta t}}{6}\partial_x \int_{\mathbb{R}^3} \xi_1(|\xi|^2 - 3)\sqrt{\mu}\mathbf{P}_1\tilde{g}\,\mathrm{d}\xi \\ &\quad + \frac{\alpha}{3}\langle A_{12}, \mathbf{P}_1\tilde{g}\rangle = 0, \end{aligned} \tag{5.4}$$

$$\begin{cases} \partial_t \langle A_{11}, \mathbf{P}_1\tilde{g}\rangle + \frac{4e^{\beta t}}{3}\partial_x b_1 + e^{\beta t}\partial_x\langle \xi_1 A_{11}, \mathbf{P}_1\tilde{g}\rangle + 2\beta\langle A_{11}, \mathbf{P}_1\tilde{g}\rangle \\ \quad + \frac{4\alpha}{3}\langle A_{12}, \mathbf{P}_1\tilde{g}\rangle + \langle L\tilde{g}, A_{11}\rangle = \langle \tilde{\mathcal{F}}, A_{11}\rangle, \\ \partial_t \langle A_{12}, \mathbf{P}_1\tilde{g}\rangle + e^{\beta t}\partial_x b_2 + e^{\beta t}\partial_x\langle \xi_1 A_{12}, \mathbf{P}_1\tilde{g}\rangle + 2\beta\langle A_{12}, \mathbf{P}_1\tilde{g}\rangle \\ \quad + \alpha(a+2c) + \alpha\langle A_{22}, \mathbf{P}_1\tilde{g}\rangle + \langle L\tilde{g}, A_{12}\rangle = \langle \tilde{\mathcal{F}}, A_{12}\rangle, \\ \partial_t \langle A_{13}, \mathbf{P}_1\tilde{g}\rangle + e^{\beta t}\partial_x b_3 + e^{\beta t}\partial_x\langle \xi_1 A_{13}, \mathbf{P}_1\tilde{g}\rangle + 2\beta\langle A_{13}, \mathbf{P}_1\tilde{g}\rangle \\ \quad + \alpha\langle A_{23}, \mathbf{P}_1\tilde{g}\rangle + \langle L\tilde{g}, A_{13}\rangle = \langle \tilde{\mathcal{F}}, A_{13}\rangle, \end{cases} \tag{5.5}$$

and

$$\begin{aligned} &\partial_t \langle B_1, \mathbf{P}_1\tilde{g}\rangle + e^{\beta t}\partial_x c + e^{\beta t}\partial_x\langle \xi_1 B_1, \mathbf{P}_1\tilde{g}\rangle + \frac{\beta}{5}b_1 + \frac{\beta}{10}\langle(|\xi|^2-3)\xi_1\sqrt{\mu}, \mathbf{P}_1\tilde{g}\rangle \\ &+ \frac{\alpha}{5}b_2 + \frac{\alpha}{5}\int_{\mathbb{R}^3} \xi_1^2\xi_2\sqrt{\mu}\mathbf{P}_1\tilde{g}\,\mathrm{d}\xi + \alpha\langle B_2, \mathbf{P}_1\tilde{g}\rangle + \langle L\tilde{g}, B_1\rangle = \langle \tilde{\mathcal{F}}, B_1\rangle, \end{aligned} \tag{5.6}$$

respectively. Here and in the sequel, we have denoted $[b_1, b_2, b_3] = \mathbf{b}$. From (5.3) and the initial condition (1.23), it follows that

$$\int_{\mathbb{T}} a\,\mathrm{d}x = \int_{\mathbb{T}} b_i\,\mathrm{d}x = 0, \ 1 \leqq i \leqq 3.$$

We are now in a position to complete the

Proof of Theorem 1.2. The global existence of (5.2) follows from the standard continuation argument based on the local existence which has been established in Section 4 and the *a priori* estimate. In what follows, we intend to obtain the *a priori* estimate

$$\begin{aligned} &\sup_{0\leqq t\leqq T} e^{\lambda\beta_{\gamma_0} t}\left\{\|w_l\partial_x^{\gamma_0} g_1(t)\|_{L^\infty} + \alpha_{\gamma_0}\|w_l\partial_x^{\gamma_0} g_2(t)\|_{L^\infty}\right\} \\ &\quad \leqq C\sum_{\gamma_0\leqq 2}\alpha_{\gamma_0}\|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty}, \ \gamma_0 \leqq 2, \end{aligned} \tag{5.7}$$

under the *a priori* assumption that $[g_1, g_2]$ is a unique solution to the coupled system (4.3) and (4.4) and satisfies

$$\sup_{0\leqq t\leqq T} e^{\lambda\beta_{\gamma_0}t}\left\{\|w_l\partial_x^{\gamma_0}g_1(t)\|_{L^\infty}+\alpha_{\gamma_0}\|w_l\partial_x^{\gamma_0}g_2(t)\|_{L^\infty}\right\}\leqq \alpha\alpha_{\gamma_0},\ \gamma_0\leqq 2. \quad (5.8)$$

Here,

$$0<\lambda<\frac{1}{4}\min\{1,\lambda_0\} \quad (5.9)$$

with λ_0 being determined as (5.28). Moreover, α_{γ_0} and β_{γ_0} are defined as

$$\alpha_{\gamma_0}=\begin{cases}\alpha, & \gamma_0=0,\\ 1, & \gamma_0=1,2,\end{cases}\quad \beta_{\gamma_0}=\begin{cases}\beta, & \gamma_0=0,\\ 1, & \gamma_0=1,2.\end{cases}$$

Step 1. L^∞ estimates. Recalling (4.10) and (4.11), one has

$$e^{\lambda\beta_{\gamma_0}t}|h_1|\leqq\sum_{i=1}^{3}\mathcal{J}_i,\quad e^{\lambda\beta_{\gamma_0}t}|h_2|\leqq\sum_{i=4}^{6}\mathcal{J}_i, \quad (5.10)$$

with

$$\begin{aligned}
\mathcal{J}_1&=e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\int_s^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left\{\chi_M w_l\mathcal{K}\left(\frac{h_1}{w_l}\right)\right\}(V(s))\,\mathrm{d}s,\\
\mathcal{J}_2&=\frac{\beta}{2}e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\int_s^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}|V(s)|^2\sqrt{\mu}(V(s))h_2(V(s))\,\mathrm{d}s\\
&\quad+\alpha e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\int_s^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\frac{V_1(s)V_2(s)}{2}\sqrt{\mu}(V(s))h_2(V(s))\,\mathrm{d}s,\\
\mathcal{J}_3&=e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\int_s^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left(w_l\tilde{H}\right)(V(s))\,\mathrm{d}s,\\
\mathcal{J}_4&=e^{\lambda\beta_{\gamma_0}t}e^{-\int_0^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}(w_l\partial_x^{\gamma_0}\tilde{g}_0)(X(0),V(0)),\\
\mathcal{J}_5&=e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\int_s^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left\{(1-\chi_M)\mu^{-\frac{1}{2}}w_l\mathcal{K}\left(\frac{h_1}{w_l}\right)\right\}(V(s))\,\mathrm{d}s,
\end{aligned}$$

and

$$\mathcal{J}_6=e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\int_s^t\mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau}\left[w_l K\left(\frac{h_2}{w_l}\right)\right](V(s))\,\mathrm{d}s.$$

In what follows, we will compute each $\mathcal{J}_i$, $1\leqq i\leqq 6$, separately. Since $\beta>0$ is sufficiently small and λ satisfies (5.9), in view of (3.27), (5.8), Lemma 6.4 and Proposition 2.1, one directly has

$$\begin{aligned}
\mathcal{J}_1&\leqq\frac{C}{l}\sum_{0\leqq s\leqq t}e^{\lambda\beta_{\gamma_0}s}\|h_1(s)\|_{L^\infty}e^{\lambda\beta_{\gamma_0}t}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}e^{\lambda\beta_{\gamma_0}s}\,\mathrm{d}s\\
&\leqq\frac{C}{l}\sum_{0\leqq s\leqq t}e^{\lambda\beta_{\gamma_0}s}\|h_1(s)\|_{L^\infty},
\end{aligned}$$

$$\mathcal{J}_2 \leqq C\alpha \sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|h_2(s)\|_{L^\infty},$$

$$\mathcal{J}_3 \leqq C\alpha \sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\left(\|h_1(s)\|_{L^\infty} + \|h_2(s)\|_{L^\infty}\right),$$

$$\mathcal{J}_4 \leqq e^{\lambda\beta_{\gamma_0}t-\frac{\nu_0}{2}t}\|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty} \leqq \|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty},$$

$$\mathcal{J}_5 \leqq C \sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|h_1(s)\|_{L^\infty}.$$

For the delicate term $\mathcal{J}_6$, we first split it as

$$\mathcal{J}_6 = e^{\lambda\beta_{\gamma_0}t}\left\{\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right\} ds\ e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\mathrm{d}\tau}$$
$$\left[w_l K\left(\frac{h_2}{w_l}\right)\right](V(s)) \overset{\text{def}}{=} \mathcal{J}_{6,1} + \mathcal{J}_{6,2},$$

where $\varepsilon > 0$ is small enough. Then, for $\mathcal{J}_{6,2}$, by applying (3.27) and Lemma 6.3, one has

$$|\mathcal{J}_{6,2}| \leqq C \sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|h_2(s)\|_{L^\infty}\int_{t-\varepsilon}^t e^{\lambda\beta_{\gamma_0}t}e^{-\frac{\nu_0}{2}(t-s)}e^{-\lambda\beta_{\gamma_0}s}\,\mathrm{d}s$$
$$\leqq C\varepsilon \sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|h_2(s)\|_{L^\infty}.$$

However, $\mathcal{J}_{6,1}$ needs more attentions. We rewrite $\mathcal{J}_{6,1}$ as

$$\mathcal{J}_{6,1} = e^{\lambda\beta_{\gamma_0}t}\int_0^{t-\varepsilon} ds\ e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,\mathrm{d}\tau} w_l(V(s))$$
$$\int_{\mathbb{R}^3}\mathbf{k}(V(s),\xi_*)\frac{h_2(s,X(s),\xi_*)}{w_l(\xi_*)}\,\mathrm{d}\xi_*.$$

Then as for obtaining (3.30), we divide the computations in the following three cases.

Case 1. $|V(s)| \geqq M$ with M suitably large. In view of Lemma 6.2, it follows that

$$\int \mathbf{k}_w(V,\xi_*)\,\mathrm{d}\xi_* \leqq \frac{C}{(1+|V|)} \leqq \frac{C}{M},$$

which implies

$$\mathcal{J}_{6,1} \leqq \sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|h_2(s)\|_{L^\infty}\int_0^{t-\varepsilon} ds\ e^{\lambda\beta_{\gamma_0}t}e^{-\frac{\nu_0}{2}(t-s)}e^{-\lambda\beta_{\gamma_0}s}$$
$$\int_{\mathbb{R}^3}\mathbf{k}_w(V(s),\xi_*)\,\mathrm{d}\xi_*$$
$$\leqq \frac{C}{M}\sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|h_2(s)\|_{L^\infty}\int_0^{t-\varepsilon} e^{-\frac{\nu_0}{4}(t-s)}\,\mathrm{d}s$$

$$\leqq \frac{C}{M} \sum_{0 \leqq s \leqq t} e^{\lambda \beta_{\gamma_0} s} \|h_2(s)\|_{L^\infty}. \tag{5.11}$$

Case 2. $|V(s)| \leqq M$ and $|\xi_*| \geqq 2M$. In this situation one has $|V(s) - \xi_*| \geqq M$, so it holds that

$$\mathbf{k}_w(V, \xi_*) \leqq Ce^{-\frac{\varepsilon M^2}{16}} \mathbf{k}_w(V, v_*) e^{\frac{\varepsilon |V-\xi_*|^2}{16}},$$

where one sees that the integral $\int \mathbf{k}_w(V, \xi_*) e^{\frac{\varepsilon |V-\xi_*|^2}{16}} \, \mathrm{d}\xi_*$ is further bounded according to Lemma 6.2. Thus as for obtaining (5.11), one has

$$\mathcal{J}_{6,1} \leqq Ce^{-\frac{\varepsilon M^2}{16}} \sum_{0 \leqq s \leqq t} e^{\lambda \beta_{\gamma_0} s} \|h_2(s)\|_{L^\infty}.$$

Case 3. $|V| \leqq M$, $|\xi_*| \leqq 2M$. In this bad case, one possible way is to convert the bound in L^∞-norm to the one in L^2-norm by an iteration approach. As in the proof of Lemma 3.1 (Case 3), we compute $\mathcal{J}_{6,1}$ as

$$\begin{aligned}
\mathcal{J}_{6,1} \leqq e^{\lambda \beta_{\gamma_0} t} & \int_0^{t-\varepsilon} ds \, e^{-\int_s^t \mathcal{A}(\tau, V(\tau)) \, \mathrm{d}\tau} \\
& \int_{|\xi_*| \leqq 2M} \mathbf{k}_{w,p}(V(s), \xi_*) |h_2(s, X(s), \xi_*)| \, \mathrm{d}\xi_* \\
& + \frac{1}{M} \sum_{0 \leqq s \leqq t} e^{\lambda \beta_{\gamma_0} s} \|h_2(s)\|_{L^\infty},
\end{aligned}$$

where $\mathbf{k}_{w,p}$ is given by (3.29). Next, by plugging the above estimates for $\mathcal{J}_4$, $\mathcal{J}_5$ and $\mathcal{J}_6$ into the second inequality of (5.10), one has

$$\begin{aligned}
e^{\lambda \beta_{\gamma_0} t} |h_2| \leqq C \|w_l \partial_x^{\gamma_0} \tilde{g}_0\|_{L^\infty} & + C \sum_{0 \leqq s \leqq t} e^{\lambda \beta_{\gamma_0} s} \|w_l h_1(s)\|_{L^\infty} \\
& + Ce^{\lambda \beta_{\gamma_0} t} \int_0^{t-\varepsilon} ds \, \mathbf{1}_{|V(s)| \leqq M} e^{-\int_s^t \mathcal{A}(\tau, V(\tau)) \mathrm{d}\tau} e^{-\lambda \beta_{\gamma_0} s} \\
& \times \int_{|\xi_*| \leqq 2M} \mathbf{k}_{w,p}(V(s), v_*) e^{\lambda \beta_{\gamma_0} s} |h_2(s, X(s), \xi_*)| \, \mathrm{d}\xi_*.
\end{aligned}$$

Substituting it again, we get

$$\begin{aligned}
e^{\lambda \beta_{\gamma_0} t} |h_2| \leqq Ce^{\lambda \beta_{\gamma_0} t} & \int_0^{t-\varepsilon} ds \, \mathbf{1}_{|V| \leqq M} e^{-\int_s^t \mathcal{A}(\tau, V(\tau)) \mathrm{d}\tau} e^{-\lambda \beta_{\gamma_0} s} \\
& \int_{|\xi_*| \leqq 2M} \mathbf{k}_{w,p}(V(s), \xi_*) e^{\lambda \beta_{\gamma_0} s} \\
& \times \int_0^{s-\varepsilon} ds_1 \, \mathbf{1}_{|V(s_1)| \leqq M} e^{-\int_{s_1}^s \mathcal{A}(\tau, V(\tau)) \mathrm{d}\tau} e^{-\lambda \beta_{\gamma_0} s_1} \\
& \times \int_{|\xi_*'| \leqq 2M} \mathbf{k}_{w,p}(\xi_*, \xi_*') e^{\lambda \beta_{\gamma_0} s_1} |h_2(s_1, X(s_1), \xi_*')| \, \mathrm{d}\xi_* \mathrm{d}\xi_*'
\end{aligned}$$

$$
+ C\|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty} + C\sup_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|w_l h_1(s)\|_{L^\infty}. \tag{5.12}
$$

On the other hand, thanks to (4.9), it follows that

$$
\begin{aligned}
X(s_1) &= X(s_1; s, X(s), \xi_*)\\
&= e^{\beta(s-s_1)}\left(X(s;t,x,\xi) - (s-s_1)\xi_{*1} - \frac{1}{2}\alpha(s-s_1)^2\xi_{*2}\right)\\
&= e^{\beta(s-s_1)}\left(e^{\beta(t-s)}\left(x-(t-s)\xi_1 - \frac{1}{2}\alpha(t-s)^2\xi_2\right)\right.\\
&\quad\left. -(s-s_1)\xi_{*1} - \frac{1}{2}\alpha(s-s_1)^2\xi_{*2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
V_1(s_1) &= V_1(s_1; s, X(s), \xi_*) = e^{\beta(s-s_1)}\left(\xi_{*1} + \alpha\xi_{*2}(t-s)\right),\\
V_i(s_1) &= V_i(s_1; s, X(s), \xi_*) = e^{\beta(s-s_1)}\xi_{*i}, \quad i = 2, 3.
\end{aligned}
$$

Therefore, for $s - s_1 \geqq \varepsilon$, one has

$$
\left|\frac{\partial\xi_{*1}}{\partial X(s_1)}\right| = \frac{e^{-\beta(s-s_1)}}{s-s_1} \leqq \varepsilon^{-1}e^{-\beta(s-s_1)}.
$$

Let $y = X(s_1)$, then it holds that

$$
\begin{aligned}
&\left|e^{\beta(s-s_1)}X(s;t,x,\xi) - \frac{1}{2}\alpha e^{\beta(s-s_1)}(s-s_1)^2\xi_{*2} - y\right|\\
&\quad\leqq e^{\beta(s-s_1)}|s-s_1||\xi_{*1}| \leqq 2Me^{\beta(s-s_1)}|s-s_1|,
\end{aligned}
$$

where we have used the fact that $|\xi_*| \leqq 2M$ in order to further estimate the integral term on the right hand side of (5.12). Consequently, if $\gamma_0 = 0$, we can bound the integral term on the right hand side of (5.12) as

$$
\begin{aligned}
&Ce^{\lambda\beta t}\int_0^t ds\int_0^{s-\varepsilon} ds_1 e^{-\frac{\nu_0}{2}(t-s)}e^{-\frac{\nu_0}{2}(s-s_1)}e^{-\lambda\beta s_1}\\
&\quad\times\int_{|\xi_*'|\leqq 2M}\int_{|\xi_{*2}|^2+|\xi_{*3}|^2\leqq 4M^2}\left(\int_{|\xi_{*1}|\leqq 2M}|e^{\lambda\beta s_1}h_2(s_1,y,\xi_*')|^2\mathrm{d}\xi_{*_1}\right)^{\frac{1}{2}}\mathrm{d}\xi_{*_2}\mathrm{d}\xi_{*_3}\mathrm{d}\xi_*'\\
&\leqq Ce^{\lambda\beta t}\int_0^t ds\int_0^{s-\varepsilon} ds_1 e^{-\frac{\nu_0}{2}(t-s)}e^{-\frac{\nu_0}{2}(s-s_1)}e^{-\lambda\beta s_1}\frac{e^{-\frac{\beta}{2}(s-s_1)}}{(s-s_1)^{\frac{1}{2}}}\\
&\quad\times\int_{|\xi_*'|\leqq 2M}\left(\int_{\Omega'}|e^{\lambda\beta s_1}g_2(s_1,y,\xi_*')|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}\xi_*'\\
&\leqq Ce^{\lambda\beta t}\int_0^t ds\int_0^{s-\varepsilon} ds_1 e^{-\frac{\nu_0}{2}(t-s)}e^{-\frac{\nu_0}{2}(s-s_1)}e^{-\lambda\beta s_1}\frac{e^{-\frac{\beta}{2}(s-s_1)}}{(s-s_1)^{\frac{1}{2}}}\\
&\quad\times\left(M^{\frac{1}{2}}e^{\frac{\beta}{2}(s-s_1)}|s-s_1|^{\frac{1}{2}}+1\right)\left(\int_{\mathbb{R}^3}\int_{\mathbb{T}}|e^{\lambda\beta s_1}g_2(s_1,y,\xi_*')|^2\,\mathrm{d}y\mathrm{d}\xi_*'\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leqq C \sup_{0\leqq s\leqq t} e^{\lambda\beta s}\|g_2(s)\|,$$

where we have denoted

$$\Omega' = \left\{ y \middle| |e^{\beta(s-s_1)}X(s;t,x,\xi) - \frac{1}{2}\alpha e^{\beta(s-s_1)}(s-s_1)^2\xi_{*2} - y| \leqq 2Me^{\beta(s-s_1)}|s-s_1| \right\}.$$

While for $\gamma_0 = 1, 2$, because $\left|\frac{\partial X(s_1)}{\partial x}\right| = e^{\beta(t-s_1)}$, we can also bound the integral term on the right hand side of (5.12) as

$$\begin{aligned}
& Ce^{\lambda t}\int_0^t ds \int_0^{s-\varepsilon} ds_1\, e^{-\frac{\nu_0}{2}(t-s)}e^{-\frac{\nu_0}{2}(s-s_1)}e^{-\lambda s_1} \\
& \int_{|\xi_*'|\leqq 2M}\left(\int_{|\xi_{*1}|\leqq 2M}|e^{\lambda s_1}h_2(s_1,y,\xi_*')|^2\,\mathrm{d}\xi_{*1}\right)^{\frac{1}{2}}\mathrm{d}\xi_*' \\
& \leqq Ce^{\lambda t}\int_0^t ds\int_0^{s-\varepsilon} ds_1 e^{-\frac{\nu_0}{2}(t-s)}e^{-\frac{\nu_0}{2}(s-s_1)}e^{-\lambda s_1}\frac{e^{-\frac{\beta}{2}(s-s_1)}}{(s-s_1)^{\frac{1}{2}}}e^{\gamma_0\beta(t-s_1)} \\
& \quad\times\int_{|\xi_*'|\leqq 2M}\left(\int_{\Omega'}|e^{\lambda s_1}\partial_y^{\gamma_0}g_2(s_1,y,\xi_*')|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}\xi_*' \\
& \leqq Ce^{\lambda t}\int_0^t ds\int_0^{s-\varepsilon} ds_1\, e^{-\frac{\nu_0}{2}(t-s)}e^{-\frac{\nu_0}{2}(s-s_1)}e^{-\lambda s_1}\frac{e^{-\frac{\beta}{2}(s-s_1)}}{(s-s_1)^{\frac{1}{2}}}e^{\gamma_0\beta(t-s_1)} \\
& \quad\times\left(M^{\frac{1}{2}}e^{\frac{\beta}{2}(s-s_1)}|s-s_1|^{\frac{1}{2}}+1\right)\left(\int_{\mathbb{R}^3}\int_{\mathbb{T}}|e^{\lambda s_1}\partial_y^{\gamma_0}g_2(s_1,y,\xi_*')|^2\,\mathrm{d}y\mathrm{d}\xi_*'\right)^{\frac{1}{2}} \\
& \leqq C\sup_{0\leqq s\leqq t} e^{\lambda s}\|\partial_y^{\gamma_0}g_2(s)\|,
\end{aligned}$$

where we notice that $0 < \beta \sim \alpha^2 \ll \nu_0$.

By plugging the above estimates into (5.10), we then conclude

$$e^{\lambda\beta_{\gamma_0}t}|h_1| \leqq C\alpha\sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|w_l h_2(s)\|_{L^\infty}, \tag{5.13}$$

and

$$\begin{aligned}
e^{\lambda\beta_{\gamma_0}t}|h_2| &\leqq C\|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty} + C\sum_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|w_l h_1(s)\|_{L^\infty} \\
&\quad + C\sup_{0\leqq s\leqq t} e^{\lambda\beta_{\gamma_0}s}\|\partial_x^{\gamma_0}g_2(s)\|.
\end{aligned} \tag{5.14}$$

Step 2. L^2 estimates. Recall $\sqrt{\mu}\tilde{g} = g_1 + \sqrt{\mu}g_2$. We now denote

$$d_{ij} = \langle A_{ij}, \mathbf{P}_1\tilde{g}\rangle = \langle A_{ij}\mu^{-\frac{1}{2}}, \bar{\mathbf{P}}_1 g_1\rangle + \langle A_{ij}, \mathbf{P}_1 g_2\rangle,$$

and we also use the notations

$$\mathbf{P}_1\tilde{g} = \bar{\mathbf{P}}_1 g_1 + \mathbf{P}_1 g_2,\ \ \mathbf{P}_0\tilde{g} = \bar{\mathbf{P}}_0 g_1 + \mathbf{P}_0 g_2,$$

with

$$\bar{\mathbf{P}}_1 g_1 = g_1 - \bar{\mathbf{P}}_0 g_1,\ \text{ and }\ \bar{\mathbf{P}}_0 g_1 = (a_1 + \mathbf{b}_1\cdot\xi + c_1(|\xi|^2-3))\mu(\xi).$$

We now clarify the relation for $\mathbf{P}_0 g_2$, $\mathbf{P}_1 g_2$, $\mathbf{P}_0\tilde{g}$ and $\mathbf{P}_1\tilde{g}$. Noticing $\sqrt{\mu}\tilde{g} = g_1 + \sqrt{\mu} g_2$, one sees that

$$\mathbf{P}_0 g_2 = \mathbf{P}_0\tilde{g} - \mathbf{P}_0\left(\frac{g_1}{\sqrt{\mu}}\right).$$

Therefore it holds that

$$\|\mathbf{P}_0 g_2\| \leqq \|\mathbf{P}_0\widetilde{g}\| + \left\|\mathbf{P}_0\left(\frac{g_1}{\sqrt{\mu}}\right)\right\| \leqq \|\mathbf{P}_0\widetilde{g}\| + C\|w_l g_1\|_{L^\infty},\ \text{ for } l > \frac{5}{2}, \tag{5.15}$$

and in particular,

$$\begin{aligned} \|[a_2,\mathbf{b}_2]\| = \|[a-a_1,\mathbf{b}-\mathbf{b}_1]\| &\leqq \|\partial_x[a,\mathbf{b}]\| + C\|w_l g_1\|_{L^\infty},\\ \|c_2\| &\leqq \|c\| + C\|w_l g_1\|_{L^\infty},\ \text{ for } l > \frac{5}{2}. \end{aligned} \tag{5.16}$$

Likewise, one obtains that

$$\|\langle\mathbf{P}_1\tilde{g}, |\xi|^3\mu^{\frac{1}{2}}\rangle\| \leqq \|\mathbf{P}_1 g_2\| + C\|w_l g_1\|_{L^\infty},\ \text{ for } l > 3. \tag{5.17}$$

We multiply (5.3) by $\frac{\alpha e^{-\beta t}}{6}\int_0^x d_{12}\,\mathrm{d}y$ and (5.4) by c, respectively, add them together and then take integration of the resulting equation in $x\in\mathbb{T}$. Further using integration by parts, one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(b_1, \frac{\alpha e^{-\beta t}}{6}\int_0^x d_{12}\,\mathrm{d}y\right) - \left(b_1, \frac{\alpha e^{-\beta t}}{6}\int_0^x \partial_t d_{12}\,\mathrm{d}y\right) + \beta\left(b_1, \frac{\alpha e^{-\beta t}}{6}\int_0^x d_{12}\,\mathrm{d}y\right)\\ &\quad + \beta\left(b_1, \frac{\alpha e^{-\beta t}}{6}\int_0^x d_{12}\,\mathrm{d}y\right) + \frac{\alpha}{6}\left(\partial_x a, \int_0^x d_{12}\,\mathrm{d}y\right) + \frac{\alpha^2}{6}\left(b_2, \int_0^x d_{12}\,\mathrm{d}y\right)\\ &\quad - \frac{\alpha}{6}\left(\int_{\mathbb{R}^3}\xi_1^2\sqrt{\mu}\mathbf{P}_1\tilde{g}\,\mathrm{d}\xi, d_{12}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c\|^2 + \beta(a,c) + 2\beta\|c\|^2 - \frac{e^{\beta t}}{3}(b_1,\partial_x c)\\ &\quad - \frac{e^{\beta t}}{6}\left(\int_{\mathbb{R}^3}\xi_1(|\xi|^2-3)\sqrt{\mu}\mathbf{P}_1\tilde{g}\,\mathrm{d}\xi, \partial_x c\right) = 0, \end{aligned} \tag{5.18}$$

where we have used the cancellation

$$\left(\partial_x c, \int_0^x d_{12}\,\mathrm{d}y\right) + (d_{12}, c) = 0,$$

and we also recall that $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbb{T}_x)$. On the other hand, using the second equation of (5.5) and Lemma 6.6, we have

$$-\left(b_1, \frac{\alpha e^{-\beta t}}{6}\int_0^x \partial_t d_{12}\,\mathrm{d}y\right)$$

$$
\begin{aligned}
&= \frac{2\alpha}{9}(b_1, b_2) + \frac{\alpha}{6}(b_1, \langle \xi_1 A_{12}, \mathbf{P}_1 \tilde{g} \rangle) + \frac{\beta\alpha e^{-\beta t}}{3}\left(b_1, \int_0^x d_{12}\,\mathrm{d}y\right) \\
&\quad + \frac{\alpha^2 e^{-\beta t}}{6}\left(b_1, \int_0^x (a+2c)\,\mathrm{d}y\right) + \frac{\alpha^2 e^{-\beta t}}{6}\left(b_1, \int_0^x \langle A_{22}, \mathbf{P}_1 \tilde{g}\rangle\,\mathrm{d}y\right) \\
&\quad + \frac{\alpha b_0 e^{-\beta t}}{3}\left(b_1, \int_0^x d_{12}\,\mathrm{d}y\right) - \frac{\alpha e^{-\beta t}}{6}\left(b_1, \int_0^x \langle \tilde{\mathcal{F}}, A_{12}\rangle\,\mathrm{d}y\right). \qquad (5.19)
\end{aligned}
$$

As a consequence, (5.18), (5.19) and (5.17) lead us to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|c\|^2 + \left(b_1, \frac{\alpha e^{-\beta t}}{3}\int_0^x d_{12}\,\mathrm{d}y\right)\right\} + 2\beta\|c\|^2 \\
&\leqq \eta_0 \|\langle B_1, \mathbf{P}_1\tilde{g}\rangle\|^2 + C\alpha\|\langle \xi_1 A_{12}, \mathbf{P}_1 \tilde{g}\rangle\|^2 \\
&\quad + \alpha\|d_{22}\|^2 + \alpha\|d_{11}\|^2 + \alpha\|d_{12}\|^2 \\
&\quad + \frac{C}{\eta_0} e^{2\beta t}\|\partial_x[a, \mathbf{b}, c]\|^2 + C\alpha\|\langle \tilde{\mathcal{F}}, A_{12}\rangle\|^2 \\
&\leqq (\eta_0 + \alpha)(\|\mathbf{P}_1 g_2\|^2 + \|w_l g_1\|_{L^\infty}^2) \\
&\quad + \frac{C}{\eta_0} e^{2\beta t}\|\partial_x[a, \mathbf{b}, c]\|^2 + C\alpha^2\|\langle \tilde{\mathcal{F}}, A_{12}\rangle\|^2, \qquad (5.20)
\end{aligned}
$$

where $\eta_0 > 0$ is an arbitrary constant and we also have used the Poincaré inequality $\|u - \int_{\mathbb{T}} u \mathrm{d}x\| \leqq C\|\partial_x u\|$.

We next compute carefully the last term on the right hand side of (5.20). First of all, recalling the definition for $\tilde{\mathcal{F}}$, one has

$$
\begin{aligned}
\|\langle \tilde{\mathcal{F}}, A_{12}\rangle\|^2 &\lesssim \int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} |Q(\mu^{\frac12} g_2, \mu^{\frac12} g_2)||\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} |Q(g_1, g_1)||\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} |Q(g_1, \mu^{\frac12} g_2)||\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} |Q(\mu^{\frac12} g_2, g_1)||\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\quad + \alpha^2\int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} Q(g_1, \mu^{\frac12}(G_1+G_R) + Q(\mu^{\frac12}(G_1+G_R), g_1)\xi_1\xi_2\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\quad + \alpha^2\int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} Q(\mu^{\frac12} g_2, \mu^{\frac12}(G_1+G_R) + Q(\mu^{\frac12}(G_1+G_R), \mu^{\frac12} g_2)\xi_1\xi_2\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\overset{\text{def}}{=} \sum_{i=1}^{6} \mathcal{H}_i. \qquad (5.21)
\end{aligned}
$$

We then compute $\mathcal{H}_i (1 \leqq i \leqq 6)$ term by term. For $\mathcal{H}_1$, thanks to Lemma 6.4 and the *a priori* assumption (5.8), it follows that

$$
\begin{aligned}
\mathcal{H}_1 &\lesssim \|w_l Q(\mu^{\frac12} g_2, \mu^{\frac12} g_2)\|_{L^\infty}^2 \int_{\mathbb{T}}\left(\int_{\mathbb{R}^3} w_{-l}(\xi)|\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x \\
&\lesssim \|w_l \mu^{\frac12} g_2\|_{L^\infty}^4 \lesssim \alpha^2\|w_l g_2\|_{L^\infty}^2,
\end{aligned}
$$

where we have let $l > 3$ such that the integral $\int_{\mathbb{R}^3} w_{-l}(\xi)|\xi_1\xi_2|\,\mathrm{d}\xi$ is convergent. Similarly, it holds that

$$\mathcal{H}_2 \lesssim \|w_l Q(g_1, g_1)\|_{L^\infty}^2 \int_{\mathbb{T}} \left(\int_{\mathbb{R}^3} w_{-l}|\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x \lesssim \|w_l g_1\|_{L^\infty}^4 \lesssim \alpha^4 \|w_l g_1\|_{L^\infty}^2,$$

and

$$\mathcal{H}_3, \mathcal{H}_4 \lesssim \left(\|w_l Q\left(g_1, \mu^{\frac{1}{2}} g_2\right)\|_{L^\infty}^2 + \|w_l Q(\mu^{\frac{1}{2}} g_2, g_1)\|_{L^\infty}^2\right) \int_{\mathbb{T}} \left(\int_{\mathbb{R}^3} w_{-l}|\xi_1\xi_2|\,\mathrm{d}\xi\right)^2 \mathrm{d}x$$
$$\lesssim \|w_l g_1\|_{L^\infty}^2 \|w_l g_2\|_{L^\infty}^2 \lesssim \alpha^2 \|w_l g_1\|_{L^\infty}^2.$$

Next, applying Lemma 6.4 and Theorem 1.1, one directly has

$$\mathcal{H}_5, \mathcal{H}_6 \lesssim \alpha^2 \|w_l g_1\|_{L^\infty}^2 + \alpha^2 \|w_l g_2\|_{L^\infty}^2.$$

Putting now the above estimates for $\mathcal{H}_i\,(1 \leqq i \leqq 6)$ into (5.20), one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|c\|^2 + (b_1, \frac{\alpha e^{-\beta t}}{3}\int_0^x d_{12}\,\mathrm{d}y)\right\} + 2\beta\|c\|^2 \\
&\quad \leqq C(\eta_0 + \alpha^2)\|\mathbf{P}_1 g_2\|^2 + C\alpha^4 \|w_l g_2\|_{L^\infty}^2 \\
&\qquad + C(\alpha^2 + \eta_0)\|w_l g_1\|_{L^\infty}^2 + \frac{C}{\eta_0} e^{2\beta t}\|\partial_x[a, \mathbf{b}, c]\|^2,
\end{aligned} \tag{5.22}$$

where the Poincaré inequality has been also used. Thus (5.22) further implies that for $0 < \lambda \leqq \frac{1}{4}$,

$$\begin{aligned}
&\sup_{0 \leqq s \leqq t} e^{2\lambda\beta s}\|c(s)\|^2 \\
&\quad \leqq \|c(0)\|^2 + \alpha\|[b_1, d_{12}](0)\|^2 + \alpha \sup_{0 \leqq s \leqq t} e^{2\lambda\beta s}\|[b_1, d_{12}]\|^2 \\
&\qquad + C(\eta_0 + \alpha^2) e^{2\lambda\beta t} \int_0^t e^{-2\beta(t-s)} e^{-2\lambda\beta s} e^{2\lambda\beta s} \|\mathbf{P}_1 g_2\|^2(s)\,\mathrm{d}s \\
&\qquad + C(\alpha^2 + \eta_0) e^{2\lambda\beta t} \int_0^t e^{-2\beta(t-s)} e^{-2\lambda\beta s} e^{2\lambda\beta s} \|w_l g_1\|_{L^\infty}^2\,\mathrm{d}s \\
&\qquad + C(\alpha^4 + \alpha^2) e^{2\lambda\beta t} \int_0^t e^{-2\beta(t-s)} e^{-2\lambda\beta s} e^{2\lambda\beta s} \|w_l g_2\|_{L^\infty}^2\,\mathrm{d}s \\
&\qquad + \frac{C}{\eta_0} e^{2\lambda\beta t} \int_0^t e^{-2\beta(t-s)} e^{4\beta s} e^{-2\lambda s} e^{2\lambda s} \|\partial_x[a, \mathbf{b}, c]\|^2\,\mathrm{d}s \\
&\quad \leqq \|c(0)\|^2 + \alpha\|[b_1, d_{12}](0)\|^2 + \alpha \sup_{0 \leqq s \leqq t} e^{2\lambda\beta s}\|[b_1, d_{12}]\|^2 \\
&\qquad + \frac{C(\alpha^2 + \eta_0)}{\beta} \sup_{0 \leqq s \leqq t} e^{2\lambda\beta s}\|\mathbf{P}_1 g_2(s)\|^2 \\
&\qquad + \frac{C(\alpha^2 + \eta_0)}{\beta} \sup_{0 \leqq s \leqq t} e^{2\lambda\beta s}\|w_l g_1(s)\|_{L^\infty}^2
\end{aligned}$$

$$+\frac{C\alpha^4}{\beta}\sup_{0\leqq s\leqq t}e^{2\lambda\beta s}\|w_l g_2(s)\|_{L^\infty}^2+\frac{C}{\eta_0}\sup_{0\leqq s\leqq t}e^{2\lambda s}\|\partial_x[a,\mathbf{b},c]\|^2.$$

Here we have used the following estimate

$$\begin{aligned}&e^{2\lambda\beta t}\int_0^t e^{-2\beta(t-s)}e^{4\beta s}e^{-2\lambda s}e^{-2\lambda s}\|\partial_x[a,\mathbf{b},c]\|^2\,\mathrm{d}s\\&\quad\leqq\sup_{0\leqq s\leqq t}e^{2\lambda s}\|\partial_x[a,\mathbf{b},c]\|^2\int_0^t e^{-2\beta(t-s)}e^{2\lambda\beta t}e^{4\beta s}e^{-2\lambda s}\,\mathrm{d}s\\&\quad\leqq C\sup_{0\leqq s\leqq t}e^{2\lambda s}\|\partial_x[a,\mathbf{b},c]\|^2,\end{aligned}\tag{5.23}$$

due to the fact that $0<\beta\sim\alpha^2\ll 1$. Consequently, it follows that

$$\begin{aligned}&\alpha\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|c(s)\|\\&\quad\leqq\alpha\|c(0)\|^2+\alpha^{\frac{3}{2}}\|[b_1,d_{12}](0)\|+\alpha^{\frac{3}{2}}\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|[b_1,d_{12}]\|\\&\qquad+C(\alpha^2+\eta_0)^{\frac{1}{2}}\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|\mathbf{P}_1 g_2(s)\|+C(\alpha^2+\eta_0)^{\frac{1}{2}}\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|w_l g_1(s)\|_{L^\infty}\\&\qquad+C\alpha^2\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|w_l g_2(s)\|_{L^\infty}+\frac{C\alpha}{\sqrt{\eta_0}}\sup_{0\leqq s\leqq t}e^{\lambda s}\|\partial_x[a,\mathbf{b},c]\|.\end{aligned}\tag{5.24}$$

On the other hand, taking the inner product of the first equation of (4.4) and $\mathbf{P}_1 g_2$ with respect to (x,v) over $\mathbb{T}\times\mathbb{R}^3$, we also have by Lemma 6.1, (5.15) and (5.16) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{P}_1 g_2\|^2+\frac{\delta_0}{2}\|\mathbf{P}_1 g_2\|^2&\leqq C\|w_l g_1\|_{L^\infty}^2+Ce^{2\beta t}\|\partial_x\mathbf{P}_0 g_2\|^2+C\alpha^2\|\mathbf{P}_0 g_2\|^2\\&\leqq C\|w_l g_1\|_{L^\infty}^2+Ce^{2\beta t}\|\partial_x\mathbf{P}_0 g_2\|^2+C\alpha^2\|c\|^2,\end{aligned}$$

which further yields

$$\begin{aligned}\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|\mathbf{P}_1 g_2(s)\|\leqq\|\mathbf{P}_1 g_2(0)\|&+C\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|w_l g_1(s)\|_{L^\infty}\\&+C\sup_{0\leqq s\leqq t}e^{\lambda s}\|\partial_x\mathbf{P}_0 g_2\|\\&+C\alpha\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|c(s)\|.\end{aligned}\tag{5.25}$$

Therefore, putting (5.24) and (5.25) together, we have that for $\alpha^2=\eta_0$,

$$\begin{aligned}&\alpha\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|c(s)\|+\alpha\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|\mathbf{P}_1 g_2(s)\|\\&\quad\leqq\alpha\|c(0)\|^2+\alpha^{\frac{3}{2}}\|[b_1,d_{12}](0)\|+C\alpha\|\mathbf{P}_1\tilde{g}_0\|+C(\alpha+\alpha^2)\sup_{0\leqq s\leqq t}e^{\lambda\beta s}\|w_l g_1(s)\|_{L^\infty}\end{aligned}$$

$$+ C\alpha^2 \sup_{0\leqq s\leqq t} e^{\lambda\beta s}\|w_l g_2(s)\|_{L^\infty} + C\alpha \sup_{0\leqq s\leqq t} e^{\lambda s}\|w_l\partial_x g_1(s)\|_{L^\infty}$$
$$+ C \sup_{0\leqq s\leqq t} e^{\lambda s}\|\partial_x[a, \mathbf{b}, c]\|. \tag{5.26}$$

Let us now turn to deduce the higher order energy estimate. For this, we claim that

$$\sum_{1\leqq\gamma_0\leqq 2}\sup_{0\leqq s\leqq t} e^{\lambda s}\|\partial_x^{\gamma_0} g_2\|(s) \leqq \sum_{1\leqq\gamma_0\leqq 2}\|\partial_x^{\gamma_0}\tilde{g}_0\| + C\sum_{1\leqq\gamma_0\leqq 2}\sup_{0\leqq s\leqq t} e^{\lambda s}\|w_l\partial_x^{\gamma_0} g_1(s)\|_{L^\infty}$$
$$+ C\alpha \sup_{0\leqq s\leqq t} e^{\lambda s}\|w_l\partial_x g_2(s)\|_{L^\infty}. \tag{5.27}$$

Indeed, for some $\lambda_0 > 0$, the inner products

$$(\partial_x(5.3)_2, e^{\beta t}\partial_x^2 a), (\partial_x(5.6), e^{\beta t}\partial_x^2 c) and (\partial_x(5.5)_i, e^{\beta t}\partial_x^2 b_i)$$

together with (5.3) and (5.4) give rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{int} + \lambda_0\sum_{1\leqq\gamma_0\leqq 2} e^{2\beta t}\|\partial_x^{\gamma_0}[a, \mathbf{b}, c]\|^2 \lesssim \sum_{1\leqq\gamma_0\leqq 2} e^{2\beta t}\|\langle\varsigma_i, \partial_x^{\gamma_0}\mathbf{P}_1\tilde{g}\rangle\|^2$$
$$+ \|\langle L\partial_x\tilde{g}, \varsigma_i\rangle\|^2 + \|\langle\partial_x\tilde{\mathcal{F}}, \varsigma_i\rangle\|^2, \tag{5.28}$$

where we have set

$$\mathcal{E}_{int} = \sum_{i=1}^{3}(\partial_x\langle A_{1i}, \mathbf{P}_1\tilde{g}\rangle, e^{\beta t}\partial_x^2 b_i) + (\partial_x\langle B_1, \mathbf{P}_1\tilde{g}\rangle, e^{\beta t}\partial_x^2 c) + \kappa_1(\partial_x b_1, e^{\beta t}\partial_x^2 a),$$

and the velocity moments ς_i in those inner products on the right-hand side of (5.28) denote all A_{ij}, B_i, $\xi_1^2\xi_2\mu^{\frac{1}{2}}$ and so on appearing in (5.3), (5.4), (5.5) and (5.6). Moreover, the Poincaré inequality

$$\|\partial_x[a, \mathbf{b}, c]\| \leqq C\|\partial_x^2[a, \mathbf{b}, c]\|$$

has been used here. Furthermore, performing similar calculations to those used to estimate $\|\langle\tilde{\mathcal{F}}, A_{12}\rangle\|^2$ in (5.21) before, one has

$$\|\langle\partial_x\tilde{\mathcal{F}}, \varsigma_i\rangle\|^2 \lesssim \alpha^2\left(\|w_l\partial_x g_2\|_{L^\infty}^2 + \|w_l\partial_x g_1\|_{L^\infty}^2\right). \tag{5.29}$$

Lemma 6.3 and Lemma 6.4 with $l > 3$ imply that

$$\|\langle L\partial_x\tilde{g}, \varsigma_i\rangle\| \lesssim \|w_l\partial_x g_1\|_{L^\infty} + \|\partial_x\mathbf{P}_1 g_2\|. \tag{5.30}$$

Consequently, plugging (5.29) and (5.30) into (5.28) and employing (5.17), we arrive at

$$e^{-2\beta t}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{int} + \lambda_0\sum_{1\leqq\gamma_0\leqq 2}\|\partial_x^{\gamma_0}[a, \mathbf{b}, c]\|^2$$

$$\leqq C\sum_{1\leqq\gamma_0\leqq 2}\|\partial_x^{\gamma_0}\mathbf{P}_1 g_2\|^2 + C\sum_{1\leqq\gamma_0\leqq 2}\|w_l\partial_x^{\gamma_0} g_1\|_{L^\infty}^2 + C\alpha^2\|w_l\partial_x g_2\|_{L^\infty}^2. \tag{5.31}$$

On the other hand, energy estimate on (4.4) leads us to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{\gamma_0} g_2\|^2 + \lambda\|\partial_x^{\gamma_0}\mathbf{P}_1 g_2\|^2 \leqq C_{\eta_1}\|\partial_x^{\gamma_0} g_1\|^2 + (C\alpha^2+\eta_1)\|\partial_x^{\gamma_0}\mathbf{P}_0 g_2\|^2, \tag{5.32}$$

for $\gamma_0 = 1, 2$, where η_1 is positive and suitably small.

Combing (5.31) and (5.32) together, one has that for $\kappa_2 > 0$ and suitably small,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{1\leqq\gamma_0\leqq 2}\|\partial_x^{\gamma_0} g_2\|^2 + \kappa_2 e^{-2\beta t}\mathcal{E}_{int}\Big\} + \sum_{1\leqq\gamma_0\leqq 2}\lambda_0\left\{\|\partial_x^{\gamma_0}\mathbf{P}_1 g_2\|^2 + \|\partial_x^{\gamma_0}[a,\mathbf{b},c]\|^2\right\}$$
$$\leqq C\sum_{1\leqq\gamma_0\leqq 2}\|w_l\partial_x^{\gamma_0} g_1\|_{L^\infty}^2 + C\alpha^2\|w_l\partial_x g_2\|_{L^\infty}^2.$$

From the above energy inequality, we further obtain that for $0 < \lambda \leqq \frac{\lambda_0}{4}$,

$$\sup_{0\leqq s\leqq t} e^{2\lambda s}\sum_{1\leqq\gamma_0\leqq 2}\|\partial_x^{\gamma_0} g_2(s)\|^2$$
$$\leqq \sum_{1\leqq\gamma_0\leqq 2}\|\partial_x^{\gamma_0} g_2(0)\|^2 + C\sum_{1\leqq\gamma_0\leqq 2} e^{2\lambda t}\int_0^t e^{-\lambda_0(t-s)}\|w_l\partial_x^{\gamma_0} g_1(s)\|_{L^\infty}^2\,\mathrm{d}s$$
$$+ C\alpha^2 e^{2\lambda t}\int_0^t e^{-\lambda_0(t-s)}\|w_l\partial_x g_2(s)\|_{L^\infty}^2 ds. \tag{5.33}$$

Therefore (5.27) follows from (5.33) and an estimate similar to that for (5.23).

Step 3. Combination. We are now in position to obtain our final estimates (5.7). To do so, for $\gamma_0 = 0$, we get from the summation of (5.13), $\alpha \times$ (5.14) and (5.26) that

$$e^{\lambda\beta t}|w_l g_1| + \alpha e^{\lambda\beta t}|w_l g_2| \leqq C\alpha\|w_l\tilde{g}_0\|_{L^\infty} + \alpha\|[c,d_{12}](0)\| + \alpha\|\mathbf{P}_1\tilde{g}_0\|$$
$$+ C\sup_{0\leqq s\leqq t} e^{\lambda s}\|w_l\partial_x g_1(s)\|_{L^\infty} + C\sup_{0\leqq s\leqq t} e^{\lambda s}\|\partial_x[a,\mathbf{b},c]\|. \tag{5.34}$$

As to $\gamma_0 = 1, 2$, we set $\kappa_3 > 0$ sufficiently small, take the summation of (5.13) and $\kappa_3 \times$ (5.14), and plug (5.27) into the resultant inequality, so as to obtain

$$\sum_{1\leqq\gamma_0\leqq 2} e^{\lambda t}\left\{|w_l\partial_x^{\gamma_0} g_1| + |w_l\partial_x^{\gamma_0} g_2|\right\} \leqq C\sum_{1\leqq\gamma_0\leqq 2}\left\{\|\partial_x^{\gamma_0}\tilde{g}_0\| + \|w_l\partial_x^{\gamma_0}\tilde{g}_0\|_{L^\infty}\right\}. \tag{5.35}$$

On the other hand, it follows that

$$\|\mathbf{P}_1 g_2\| \leqq C\|w_l g_2\|_{L^\infty},\ \text{for } l > \frac{3}{2}, \tag{5.36}$$

and for $l > \frac{5}{2}$, one has

$$\begin{aligned} \|\partial_x[a, \mathbf{b}, c]\| &\leqq C\|\partial_x \mathbf{P}_0 \tilde{g}\| \leqq C\|\partial_x g_2\| + C\|w_l \partial_x g_1\|_{L^\infty} \\ &\leqq C\|w_l \partial_x g_2\|_{L^\infty} + C\|w_l \partial_x g_1\|_{L^\infty}, \end{aligned} \tag{5.37}$$

and

$$\|[c, b_1, d_{12}](0)\| \leqq C\|w_l g_2\|_{L^\infty} + C\|w_l g_1(0)\|_{L^\infty} \leqq C\|w_l g_2\|_{L^\infty}. \tag{5.38}$$

Finally, putting (5.34), (5.35), (5.36), (5.37) and (5.38) together and adjusting constants, we have

$$\sum_{\gamma_0 \leqq 2} e^{\lambda \beta_{\gamma_0} t} \left\{ |w_l \partial_x^{\gamma_0} g_1| + \alpha^{\gamma_0} |w_l \partial_x^{\gamma_0} g_2| \right\} \leqq C \sum_{\gamma_0 \leqq 2} \alpha^{\gamma_0} \|w_l \partial_x^{\gamma_0} \tilde{g}_0\|_{L^\infty}.$$

Thus (5.7) is valid.

Step 4. Non-negativity. We now turn to prove that the unique global solution constructed above is non-negative, i.e. $e^{3\beta t} F(t, x, e^{\beta t}\xi) = G(\xi) + \tilde{f}(t, x, \xi) \geqq 0$ under the condition that $F_0(x, \xi) = G + \tilde{f}(0, x, \xi) \geqq 0$, which also indicates the non-negativity of the self-similar solution $G(v)$ obtained in Theorem 1.1 due to the large time asymptotic behavior (1.24). To do so, let us start from the following linearized equation of (4.1) in Section 4

$$\begin{cases} \partial_t f + e^{\beta t} \xi_1 \partial_x f - \beta \nabla_\xi \cdot (\xi f) - \alpha \xi_2 \partial_{\xi_1} f + f \mathcal{V}(f') \\ \quad = Q_+(f', f'), \ t > 0, \ x \in \mathbb{T}, \ \xi \in \mathbb{R}^3, \\ f(0, x, \xi) = F_0(x, \xi), \ x \in \mathbb{T}, \ \xi \in \mathbb{R}^3, \end{cases} \tag{5.39}$$

where

$$\mathcal{V}(f') = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\cos\theta) f'(\xi_*) \, d\omega d\xi_*.$$

One can see that if $F_0(x, \xi) \geqq 0$ and $f'(t, x, \xi) \geqq 0$, then any solution of (5.39) should be non-negative. Denote $f = G + \tilde{f}$ and $f' = G + \tilde{f}'$, and decompose $\tilde{f}$ and $\tilde{f}'$ as

$$\tilde{f} = f_1 + \sqrt{\mu} f_2, \quad \tilde{f}' = f_1' + \sqrt{\mu} f_2'.$$

We now verify that there exists a unique solution in the form of $G + f_1 + \sqrt{\mu} f_2$ to (5.39) under the condition that $[f_1', f_2']$ belongs to the function space

$$\begin{aligned} \mathbf{W}_{\alpha,T} = \Big\{ &(\mathcal{G}_1, \mathcal{G}_2) \in L^\infty(0, T; L^\infty(\mathbb{T} \times \mathbb{R}^3)) \Big| \\ &\sup_{0 \leqq t \leqq T} \{\|w_l \mathcal{G}_1(t)\|_{L^\infty} + \alpha \|w_l \mathcal{G}_2(t)\|_{L^\infty}\} \leqq 2\alpha, \\ &\mathcal{G}_1(0) = 0, \ \mathcal{G}_2(0) = \tilde{g}_0, \ G + \mathcal{G}_1 + \mu^{\frac{1}{2}} \mathcal{G}_2 \geqq 0 \Big\}. \end{aligned}$$

We now consider the coupled equations for f_1 and f_2

$$\begin{cases} \partial_t f_1 + e^{\beta t}\xi_1\partial_x f_1 - \beta\nabla_\xi\cdot(\xi f_1) - \alpha\xi_2\partial_{\xi_1} f_1 + \frac{\beta}{2}|\xi|^2\mu^{\frac{1}{2}} f_2 \\ + \frac{\alpha}{2}\mu^{\frac{1}{2}}\xi_1\xi_2 f_2 + \nu_0 f_1 \\ \quad + (f_1+\mu^{\frac{1}{2}} f_2)\mathcal{V}(\mu^{\frac{1}{2}}(\alpha G_1+\alpha G_R)) + (f_1+\mu^{\frac{1}{2}} f_2)\mathcal{V}(f_1'+\mu^{\frac{1}{2}} f_2') \\ \quad = \underbrace{\chi_M\mathcal{K} f_1' + \mathcal{F}_3(f_1', f_2')}_{\mathcal{F}_4},\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ f_1(0,x,\xi) = 0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{5.40}$$

and

$$\begin{cases} \partial_t f_2 + e^{\beta t}\xi_1\partial_x f_2 - \beta\nabla_\xi\cdot(\xi f_2) - \alpha\xi_2\partial_{\xi_1} f_2 + \nu_0 f_2 \\ \quad = \underbrace{K f_2' + \mu^{-1/2}(1-\chi_M)\mathcal{K} f_1'}_{\mathcal{F}_5},\ t>0,\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \\ f_2(0,x,\xi) = \frac{F_0(x,\xi)-G(\xi)}{\sqrt{\mu}} \overset{\text{def}}{=} \tilde{g}_0(x,\xi),\ x\in\mathbb{T},\ \xi\in\mathbb{R}^3, \end{cases} \tag{5.41}$$

where

$$\mathcal{F}_3(f_1', f_2') = -\mu^{\frac{1}{2}}(\alpha G_1+\alpha G_R)\mathcal{V}(f_1'+\mu^{\frac{1}{2}} f_2') + Q_+(f_1'+\mu^{\frac{1}{2}} f_2', f_1'+\mu^{\frac{1}{2}} f_2').$$

Let $[f_1, f_2]$ be a solution of the pair of (5.40) and (5.41) with $[f_1', f_2'] \in \mathbf{W}_{\alpha,T}$. Then the nonlinear operator $\mathcal{W}$ is formally defined as

$$\mathcal{W}([f_1', f_2']) = [\mathcal{W}_1, \mathcal{W}_2]([f_1', f_2']) = [f_1, f_2].$$

We next show that $\mathcal{W}$ is a contraction mapping on $\mathbf{W}_{\alpha,T}$. To do this, let us first rewrite the solution of (5.40) and (5.41) as

$$\begin{aligned} w_l[f_1, \alpha f_2] &= e^{-\int_0^t \mathcal{M}(\tau,V(\tau))\,\mathrm{d}\tau}[0, \alpha w_l \tilde{g}_0(X(0), V(0))] \\ &\quad + \int_0^t e^{-\int_s^t \mathcal{M}(\tau,V(\tau))\,\mathrm{d}\tau} w_l[\mathcal{F}_4, \alpha\mathcal{F}_5]\,\mathrm{d}s, \end{aligned} \tag{5.42}$$

where $\mathcal{M}$ is a 2×2 matrix given by

$$\begin{bmatrix} M_{11} & M_{12} \\ 0 & \alpha M_{22} \end{bmatrix},$$

with

$$\begin{aligned} M_{11} &= \nu_0 - 3\beta + 2l\beta\frac{|V(\tau)|^2}{1+|V(\tau)|^2} + 2l\alpha\frac{V_2(\tau)V_1(\tau)}{1+|V(\tau)|^2} \\ &\quad + \mathcal{V}(\mu^{\frac{1}{2}}(\alpha G_1+\alpha G_R)) + \mathcal{V}(f_1'+\mu^{\frac{1}{2}} f_2') \geqq \nu_0/2, \end{aligned} \tag{5.43}$$

$$M_{12} = \frac{\beta |V(\tau)|^2}{2} \mu^{\frac{1}{2}} + \frac{\alpha}{2} \mu^{\frac{1}{2}} V_2(\tau) V_1(\tau) + \mu^{\frac{1}{2}} \mathcal{V}(\mu^{\frac{1}{2}}(\alpha G_1 + \alpha G_R))$$
$$+ \mu^{\frac{1}{2}} \mathcal{V}(f_1' + \mu^{\frac{1}{2}} f_2'),$$
$$M_{22} = \nu_0 - 3\beta + 2l\beta \frac{|V(\tau)|^2}{1 + |V(\tau)|^2} + 2l\alpha \frac{V_2(\tau) V_1(\tau)}{1 + |V(\tau)|^2} \geqq \nu_0/2, \tag{5.44}$$

and moreover, $X(s)$ and $V(s)$ is defined as (4.9).

Given $[f_1', f_2'] \in \mathbf{W}_{\alpha,T}$, we show that also $[f_1, f_2] \in \mathbf{W}_{\alpha,T}$. Since $[f_1', f_2'] \in \mathbf{W}_{\alpha,T}$, one sees that $|M_{12}| \leqq C$. With this, (5.43) and (5.44), we have by utilizing Lemma 6.4 that

$$|w_l[f_1, \alpha f_2]| \leqq e^{CT}\alpha + Te^{CT} \|w_l[\mathcal{F}_4, \alpha\mathcal{F}_5]\|_{L^\infty}$$
$$\leqq e^{CT}\alpha + Te^{CT} \left\{ \|w_l f_1'\|_{L^\infty} + \|w_l f_1'\|_{L^\infty}^2 + \alpha \|w_l f_2'\|_{L^\infty} + \|w_l f_2'\|_{L^\infty}^2 \right\},$$

which is further bounded by 2α, provided that $0 < T \leqq T_{**}$ with T_{**} sufficiently small. Thus $\mathcal{W}([f_1', f_2']) \in \mathbf{W}_{\alpha,T_{**}}$. Note that $G + f_1 + \mu^{\frac{1}{2}} f_2 \geqq 0$ follows from (5.39) and $f' \geqq 0$. It remains now to verify that $\mathcal{W}$ is a contraction. In fact, given $[f_1', f_2'], [f_1'', f_2''] \in \mathbf{W}_{\alpha,T_{**}}$, from (5.42), it follows that

$$|w_l \mathcal{W}_1([f_1', f_2']) - w_l \mathcal{W}_1([f_1'', f_2''])| + \alpha |w_l \mathcal{W}_2([f_1', f_2']) - w_l \mathcal{W}_2([f_1'', f_2''])|$$
$$\leqq \int_0^t e^{-\int_s^t \mathcal{M}(\tau, V(\tau))\mathrm{d}\tau} w_l \left| [\mathcal{F}_4, \alpha\mathcal{F}_5](f_1', f_2') - [\mathcal{F}_4, \alpha\mathcal{F}_5](f_1'' f_2'') \right| ds$$
$$\leqq T_{**} e^{CT_{**}} \left\{ \|w_l(f_1' - f_1'')\|_{L^\infty} + \alpha \|w_l(f_2' - f_2'')\|_{L^\infty} + \|w_l(f_2' - f_2'')\|_{L^\infty}^2 \right\}$$
$$\leqq \frac{1}{2} \left\{ \|w_l(f_1' - f_1'')\|_{L^\infty} + \alpha \|w_l(f_2' - f_2'')\|_{L^\infty} \right\},$$

provided that $T_{**} > 0$ is sufficiently small. Therefore, there exists a unique function $[f_1, f_2] \in \mathbf{W}_{\alpha,T_{**}}$ such that $[f_1, f_2] = \mathcal{W}([f_1, f_2])$, namely (4.1) admits a unique non-negative solution $f = G + f_1 + \mu^{\frac{1}{2}} f_2$ with $[f_1, f_2] \in \mathbf{W}_{\alpha,T_{**}}$ for $T_{**} > 0$ small enough. Finally, since we have obtained the uniform bound (5.7) in the previous steps of this section, one can extend the existing time interval of the above non-negative solution to an arbitrary time $t > 0$. Thus, the proof of Theorem 1.2 is completed. □

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6. Appendix

In this section, we collect some known basic estimates which have been used in the previous sections. The following lemma can be found in [24, Lemmas 3.2, 3.3, pp.638-639], where the more general hard sphere case is proved:

Lemma 6.1. *In the Maxwell molecular case, there is a constant* $\delta_0 > 0$ *such that*

$$\langle Lf, f\rangle = \langle L\mathbf{P}_1 f, \mathbf{P}_1 f\rangle \geqq \delta_0 \|\mathbf{P}_1 f\|^2.$$

Moreover, for $\gamma > 0$ *and* $l \geqq 0$*,*

$$\langle w_l^2 \partial_v^\gamma Lf, \partial_v^\gamma f\rangle \geqq \delta_0 \|w_l \partial_v^\gamma f\|^2 - C\|f\|^2.$$

The following lemma is concerned with the integral operator K given by (2.2), and its proof in case of the hard sphere model has been given by [25, Lemma 3, pp.727].

Lemma 6.2. *Let* K *be defined as* (2.2)*, then it holds that*

$$Kf(v) = \int_{\mathbb{R}^3} \mathbf{k}(v, v_*) f(v_*)\, \mathrm{d}v_*$$

with

$$|\mathbf{k}(v, v_*)| \leqq C\{1 + |v - v_*|^{-2}\} e^{-\frac{1}{8}|v-v_*|^2 - \frac{1}{8}\frac{\left||v|^2 - |v_*|^2\right|^2}{|v-v_*|^2}}.$$

Moreover, let $\mathbf{k}_w(v, v_*) = w_l(v)\mathbf{k}(v, v_*)w_{-l}(v_*)$ *with* $l \geqq 0$*, then it also holds that*

$$\int_{\mathbb{R}^3} \mathbf{k}_w(v, v_*) e^{\frac{\varepsilon|v-v_*|^2}{8}} \mathrm{d}v_* \leqq \frac{C}{1 + |v|},$$

for $\varepsilon \geqq 0$ *small enough.*

For the velocity weighted derivative estimates on the nonlinear operator Γ, one has

Lemma 6.3. *In the Maxwell molecular case, it holds that*

$$\|w_l \partial_v^\gamma \Gamma(f, g)\|_{L_v^2} \leqq C \sum_{\gamma_1 \leqq \gamma} \|w_l \partial_v^{\gamma_1} f\|_{L_v^2} \|w_l \partial_v^{\gamma-\gamma_1} g\|_{L_v^2}, \tag{6.1}$$

and

$$\|w_l \partial_v^\gamma \Gamma(f, g)\|_{L^\infty} \leqq C \sum_{\gamma_1 \leqq \gamma} \|w_l \partial_v^{\gamma_1} f\|_{L^\infty} \|w_l \partial_v^{\gamma-\gamma_1} g\|_{L^\infty}, \tag{6.2}$$

for any multiple index γ *and any* $l \geqq 0$*.*

Proof. The proof of (6.1) and (6.2) is similar as that of [26, Lemma 2.3, pp.1111] and [25, Lemma 5, pp.730], respectively. Thus we omit the details for brevity. □

The following Lemma on the velocity weighted derivative estimates for the original Boltzmann equation Q can be verified by using the parallel argument as obtaining [1, Proposition 3.1, pp.397] where the hard potential case and the case $|\gamma| = 0$ were proved.

Lemma 6.4. *In the Maxwell molecular case, for $l > \frac{3}{2}$ and $|\gamma| \geqq 0$, it holds that*

$$\|w_l \partial_v^\gamma Q(F_1, F_2)\|_{L^\infty} \leqq C \sum_{\gamma_1 \leqq \gamma} \|w_l \partial_v^{\gamma-\gamma_1} F_1\|_{L^\infty} \|w_l \partial_v^{\gamma_1} F_2\|_{L^\infty}.$$

We now give the following two useful results concerning the second momentum invariant property of the linearized operator L in the case of Maxwell molecules. The first one is due to [28, Proposition 4.10, pp.804].

Lemma 6.5. *Let $W_{ij}(v)$ be quadratic functions in the form of $W_{ij}(v) = v_i v_j (1 \leqq i, j \leqq 3)$ and define*

$$T_{ij} = \frac{1}{2} \int_{\mathbb{S}^2} \mathrm{d}\omega\, B_0(\cos\theta) \left[W_{i,j}(v') + W_{i,j}(v'_*) - W_{i,j}(v) - W_{i,j}(v_*)\right], \quad (6.3)$$

where (v, v_*) and (v', v'_*) satisfies (1.3). Then it holds that

$$T_{ij} = -b_0 \left[(v - v_*)_i (v - v_*)_j - \frac{\delta_{ij}}{3} |v - v_*|^2\right], \quad (6.4)$$

with b_0 given in (1.15).

Based on the above nice lemma, we can obtain

Lemma 6.6. *Let L be defined as (2.1), then it holds that for all $1 \leqq i, j \leqq 3$,*

$$L(v_i v_j \mu^{1/2}) = 2b_0 \left(v_i v_j - \frac{\delta_{ij}}{3}|v|^2\right) \mu^{1/2}. \quad (6.5)$$

Proof. For $f = \mu^{1/2} W$ with a general function $W = W(v)$, one has

$$Lf = -\mu^{1/2} \int \mu_*\, \mathrm{d}v_* \int \mathrm{d}\omega\, B_0(\cos\theta)[W' + W'_* - W - W_*].$$

In particular, letting $W = W_{ij}(v) = v_i v_j$ and applying Lemma 6.5, we have

$$L(\mu^{1/2} W_{ij}) = -2\mu^{1/2} \int \mu_* T_{ij}\, \mathrm{d}v_*, \quad (6.6)$$

where T_{ij} is given by (6.3). Plugging (6.4) into (6.6), one sees that (6.5) is valid. This completes the proof of Lemma 6.6. □

References

1. Arkeryd, L., Esposito, R., Pulvirenti, M.: The Boltzmann equation for weakly inhomogeneous data. *Comm. Math. Phys.* **111**(3), 393–407, 1987
2. Bedrossian, J., Masmoudi, N.: Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.* **122**, 195–300, 2015
3. Bedrossian, J., Masmoudi, N., Vicol, V.: Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow. *Arch. Ration. Mech. Anal.* **219**, 1087–1159, 2016
4. Bedrossian, J., Germain, P., Masmoudi, N.: On the stability threshold for the 3D Couette flow in Sobolev regularity. *Ann. Math.* **185**, 541–608, 2017
5. Bobylev, A.V.: The method of the Fourier transform in the theory of the Boltzmann equation for Maxwell molecules (Russian). *Dokl. Akad. Nauk. SSSR* **225**, 1041–1044, 1975. *Soviet Phys. Dokl.* **20** (1976), 820–822 the Boltzmann equation for Maxwell molecules. (Russian) *Dokl. Akad. Nauk. SSSR* **225** (1975), 1041–1044,
6. Bobylev, A.V.: The theory of the nonlinear spatially uniform Boltzmann equation for Maxwellian molecules. *Sov. Scient. Rev. C* **7**, 111–233, 1988
7. Bobylev, A.V., Cercignani, C.: Exact eternal solutions of the Boltzmann equation. *J. Stat. Phys.* **106**(5–6), 1019–1039, 2002
8. Bobylev, A.V., Cercignani, C.: Self-similar solutions of the Boltzmann equation and their applications. *J. Stat. Phys.* **106**(5–6), 1039–1071, 2002
9. Bobylev, A.V., Cercignani, C.: Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions. *J. Stat. Phys.* **1–2**, 335–375, 2003
10. Bobylev, A.V., Caraffini, G.L., Spiga, G.: On group invariant solutions of the Boltzmann equation. *Journal Math. Phys.* **37**, 2787–2795, 1996
11. Bobylev, A., Nota, A., Velázquez, J.J.L.: Self-similar asymptotics for a modified Maxwell-Boltzmann equation in systems subject to deformations. *Comm. Math. Phys.* **380**(1), 409–448, 2020
12. Caflisch, R.: The Boltzmann equation with a soft potential, II. Nonlinear, spatially-periodic. *Comm. Math. Phys.* **74**(2), 97–109, 1980
13. Cannone, M., Karch, G.: Infinite energy solutions to the homogeneous Boltzmann equation. *Comm. Pure Appl. Math.* **63**, 747–778, 2010
14. Cannone, M., Karch, G.: On self-similar solutions to the homogeneous Boltzmann equation. *Kinet. Relat. Models* **6**(4), 801–808, 2013
15. Cercignani, C.: *The Boltzmann Equation and Its Applications. Applied Mathematical Sciences*, vol. 67. Springer, New York, 1988
16. Cercignani, C.: Existence of homoenergetic affine flows for the Boltzmann equation. *Arch. Rat. Mech. Anal.* **105**(4), 377–387, 1989
17. Cercignani, C.: Shear Flow of a Granular Material. *J. Stat. Phys.* **102**(5), 1407–1415, 2001
18. Cercignani, C.: The Boltzmann equation approach to the shear flow of a granular material. *Philosophical Trans. Royal Society.* **360**, 437–451, 2002
19. Duan, R.-J., Huang, F.-M., Wang, Y., Zhang, Z.: Effects of soft interaction and non-isothermal boundary upon long-time dynamics of rarefied gas. *Arch. Ration. Mech. Anal.* **234**(2), 925–1006, 2019
20. Esposito, R., Guo, Y., Kim, C., Marra, R.: Non-isothermal boundary in the Boltzmann theory and Fourier law. *Comm. Math. Phys.* **323**(1), 177–239, 2013
21. Galkin, V.S.: On a class of solutions of Grad's moment equation. *PMM* **22**(3), 386–389, 1958. (Russian version *PMM* **20** (1956), 445–446
22. Garzó, V., Santos, A.: Kinetic Theory of Gases in Shear Flows. *Nonlinear transport. Fundamental Theories of Physics* **131**, 2003. Kluwer Academic Publishers, Dordrecht
23. Gualdani, M.P., Mischler, S., Mouhot, C.: Factorization of non-symmetric operators and exponential H-theorem. *Mém. Soc. Math. Fr. (N.S.)* **153**, 2017. 137 pp

24. Guo, Y.: Boltzmann diffusive limit beyond the Navier-Stokes approximation. *Comm. Pure. Appl. Math.* **55**(9), 0626–0687, 2006
25. Guo, Y.: Decay and continuity of the Boltzmann equation in bounded domains. *Arch. Ration. Mech. Anal.* **197**(3), 713–809, 2010
26. Guo, Y.: The Vlasov-Poisson-Boltzmann system near Maxwellians. *Comm. Pure Appl. Math.* **55**, 1104–1135, 2002
27. Guo, Y., Liu, S.-Q.: The Boltzmann equation with weakly inhomogeneous data in bounded domain. *J. Funct. Anal.* **272**(5), 2038–2057, 2017
28. James, R.D., Nota, A., Velázquez, J.J.L.: Self-similar profiles for homoenergetic solutions of the Boltzmann equation: particle velocity distribution and entropy. *Arch. Rat. Mech. Anal.* **231**(2), 787–843, 2019
29. James, R.D., Nota, A., Velázquez, J.J.L.: Long-time asymptotics for homoenergetic solutions of the Boltzmann equation: collision-dominated case. *J. Nonlinear Sci.* **29**(5), 1943–1973, 2019
30. James, R.D., Nota, A., Velázquez, J.J.L.: Long-time asymptotics for homoenergetic solutions of the Boltzmann equation: hyperbolic-dominated case. *Nonlinearity* **33**(8), 3781–3815, 2020
31. Kogan, M.N.: *Rarefied Gas Dynamics*. Plenum Press, New York, 1969
32. Matthies, K., Theil, F.: Rescaled objective solutions of Fokker-Planck and Boltzmann equations. *SIAM Journal on Mathematical Analysis* **51**(2), 1321–1348, 2019
33. Morimoto, Y., Yang, T., Zhao, H.-J.: Convergence to self-similar solutions for the homogeneous Boltzmann equation. *J. Eur. Math. Soc.* **19**(8), 2241–2267, 2017
34. Schmid, P.J., Henningson, D.S.: Stability and Transition in Shear Flows. *Applied Mathematical Sciences*, vol. 142. Springer, New York, 2001
35. Sone, Y.: *Molecular Gas Dynamics: Theory, Techniques, and Applications*. Birkhauser, Boston, 2007
36. Truesdell, C.: On the pressures and flux of energy in a gas according to Maxwell's kinetic theory II. *J. Rat. Mech. Anal.* **5**, 55–128, 1956
37. Truesdell, C., Muncaster, R.G.: *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas*. Academic Press, New York, 1980

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