Arch. Rational Mech. Anal. 242 (2021) 1571–1627
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-021-01711-x

# *Curved Fronts of Bistable Reaction-Diffusion Equations in Spatially Periodic Media*

Hongjun Guo, Wan-Tong Li, Rongsong Liu & Zhi-Cheng Wang

*Communicated by* P. Rabinowitz

**Abstract**

In this paper, curved fronts are constructed for spatially periodic bistable reaction-diffusion equations under the a priori assumption that there exist pulsating fronts in every direction. Some sufficient and some necessary conditions of the existence of curved fronts are given. Furthermore, the curved front is proved to be unique and stable. Finally, a curved front with varying interfaces is also constructed. Despite the effect of the spatial heterogeneity, the result shows the existence of curved fronts for spatially periodic bistable reaction-diffusion equations which is known for the homogeneous case.

## 1. Introduction

In this paper, we consider spatially periodic reaction-diffusion equations of the type

$$u_t - \Delta u = f(x, y, u), \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^2, \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $\Delta = \partial_{xx} + \partial_{yy}$ denotes the Laplace operator with respect to the space variables $(x, y) \in \mathbb{R}^2$. The reaction term $f(x, y, u)$ is assumed to be periodic in $(x, y)$ and bistable in $u$. More precisely, we assume throughout this paper that

Guo and Liu are partially supported by by NSF Grant 1826801. Guo is also partially supported by the fundamental research funds for the central universities and the National Natural Science Foundation of China under Grant 12101456. Li is supported by the National Natural Science Foundation of China under Grants 11731005 and 11671180. Wang is supported by the National Natural Science Foundation of China under Grant 12071193.

(F1) $f(x, y, u)$ is continuous, of class $C^\alpha$ in $(x, y)$ uniformly in $u \in [0, 1]$ with $\alpha \in (0, 1)$, of class $C^2$ in $u$ uniformly in $(x, y) \in \mathbb{R}^2$ with $f_u(x, y, u)$ and $f_{uu}(x, y, u)$ being Lipschitz continuous in $u \in \mathbb{R}$;

(F2) $f(x, y, u)$ is $L$-periodic with respect to $(x, y)$ where $L = (L_1, L_2) \in \mathbb{R}^2$, that is, $f(x + k_1 L_1, y + k_2 L_2, u) = f(x, y, u)$ for any $k_1, k_2 \in \mathbb{Z}$;

(F3) for every $(x, y) \in \mathbb{R}^2$, 0 and 1 are stable zeroes of $f(x, y, \cdot)$, that is,

$$f(x, y, 0) = f(x, y, 1) = 0,$$

and there exist $\lambda > 0$ and $\sigma \in (0, 1/2)$ such that

$$\begin{aligned} &-f_u(x, y, u) \geq \lambda \text{ for all} \\ &\quad (x, y, u) \in \mathbb{R}^2 \times [0, \sigma] \text{ and } (x, y, u) \in \mathbb{R}^2 \times [1 - \sigma, 1]. \end{aligned}$$

A typical example of $f(x, y, u)$ is the cubic nonlinearity

$$f(x, y, u) = u(1 - u)(u - \theta_{x,y}),$$

where $\theta_{x,y} \in (0, 1)$ is a $L$-periodic function. The Eq. (1.1) is a special generalization of the famous Allen–Cahn equation [1]. For mathematical convenience, we extend $f(x, y, u)$ out of the interval $u \in [0, 1]$ such that

$$-f_u(x, y, u) \geq \lambda \text{ for all } (x, y, u) \in \mathbb{R}^2 \times (-\infty, \sigma] \text{ and } (x, y, u) \in \mathbb{R}^2 \times [1 - \sigma, +\infty). \tag{1.2}$$

Then, $f(x, y, u)$ is globally Lipschitz continuous in $u \in \mathbb{R}$.

Before proceeding further, we first recall some well-known results in the homogeneous case, that is,

$$u_t - \Delta u = f(u), \ (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.3}$$

where $f$ is of bistable type, that is, $f(0) = f(1) = f(\theta)$, $f < 0$ on $(0, \theta)$ and $f > 0$ on $(\theta, 1)$, for some $\theta \in (0, 1)$. For one-dimensional space, it follows from celebrated results due to Fife and McLeod [13] that (1.3) admits a unique (up to shifts) traveling front $\phi(x - c_f t)$ satisfying

$$0 < \phi < 1, \ \phi(-\infty) = 1 \text{ and } \phi(+\infty) = 0.$$

Moreover, the speed $c_f$ has the sign of $\int_0^1 f(u)du$ and the front is globally and exponentially stable. A trivial extension of the traveling front to higher dimensional spaces is the planar front $\phi(x \cdot e - c_f t)$ where $e \in \mathbb{S}^{N-1}$ denotes the propagation direction. Notice that every level set of a planar front is a plane. In addition to planar fronts, more types of fronts are also known to exist in high dimensional spaces, such as $V$-shaped fronts, conical shaped fronts and pyramidal fronts, see Hamel et. al. [19], Ninomiya and Taniguchi [21] and Taniguchi [24,25]. All these fronts are transition fronts connecting 0 and 1 defined by Berestycki and Hamel [3]. The notions of transition fronts generalize the standard notions of traveling fronts. Roughly speaking, transition fronts connecting 0 and 1 are those entire solutions

$u(t,x)$ for which there is a set $\Gamma_t$ (which is called interface and can be picked as a level set of entire solutions) splitting the space into two parts $\Omega_t^{\pm}$ satisfying

$$\begin{cases} u(t,x) \to 1 \ \text{ as } d(x,\Gamma_t) \to +\infty \text{ for } x \in \Omega_t^{+} \text{ uniformly in } t \in \mathbb{R}, \\ u(t,x) \to 0 \ \text{ as } d(x,\Gamma_t) \to +\infty \text{ for } x \in \Omega_t^{-} \text{ uniformly in } t \in \mathbb{R}, \end{cases} \tag{1.4}$$

For more conditions on $\Gamma_t$ and $\Omega_t^{\pm}$, we refer to [3]. For above fronts, their interfaces between 0 and 1 can be given by their level sets and different shapes of interfaces actually show some structures of the solutions. One can roughly imagine a global appearance of such solutions in the framework of transition fronts by noticing that the solutions are approaching to 1 and 0 on one side and the other of the interfaces, respectively.

As far as a spatially periodic bistable reaction-diffusion equation considered, the situation is more complicated than the homogenous case. Because of the effect of hetereogeneities, there may even not exist transition fronts connecting states 0 and 1, see Zlatoš [33]. However, what we are concerned with in this paper is the existence of curved fronts when there exist some fronts in every direction, that is, pulsating fronts. We now introduce the notion of pulsating front by referring to [2,23,28–30].

**Definition 1.1.** Denote a periodic cell by $\mathbb{T}^2 = [0, L_1] \times [0, L_2]$. A pair $(U_e, c_e)$ with $U_e : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$ and $c_e \in \mathbb{R}$ is said to be a pulsating front of (1.1) with effective speed $c_e$ in the direction $e \in \mathbb{S}$ connecting 0 and 1 if the two following conditions are satisfied:

(i) For every $\xi \in \mathbb{R}$, the profile $U_e(\xi, x, y)$ is $L$-periodic in $(x, y)$ and satisfies

$$\lim_{\xi\to+\infty} U_e(\xi,x,y) = 0, \ \lim_{\xi\to-\infty} U_e(\xi,x,y) = 1, \ \text{ uniformly for } \ (x,y) \in \mathbb{T}^2.$$

(ii) The map $u(t,x,y) := U_e((x,y)\cdot e - c_e t, x, y)$ is an entire (classical) solution of the parabolic Eq. (1.1).

We now recall some existence results of pulsating fronts for the general reaction-diffusion equation in spatially periodic media

$$u_t = \sum_i (a(x)u_{x_i})_{x_i} + \sum_i b_i(x)u_{x_i} + f(x,u), \ t \in \mathbb{R}, \ x \in \mathbb{R}^N. \tag{1.5}$$

For one dimensional case of (1.5) when $f(x,u) = g(x)f(u)$, Nolen and Ryzhik [22] proved the existence of pulsating fronts with nonzero speed by provided with some restrictions for $g$ and $f$. Moreover, Ducrot, Giletti and Matano [9] also got some existence results of pulsating fronts with a positive speed, if the solutions of (1.5) with some compactly supported initial conditions can converge locally uniformly to 1 as $t \to +\infty$. Still for one-dimensional case, Ding, Hamel and Zhao [7] applied the implicit function theorem and abstract results of Fang and Zhao [12] to get the existence of pulsating fronts for small period and large period. For higher dimensions, when the diffusivity matrix $a$ is close to identity and $f$ is independent of $x$, the existence of pulsating fronts is obtained by Xin [28–30] through refined perturbation arguments. Ducrot [8] also got some existence results of fronts connecting 0 and 1 in every direction for slowly varying medium and rapidly varying medium (that is, $d << 1$ and $d >> 1$ respectively when the reaction term is $f(dx, u)$), in which the fronts are either moving pulsating waves or standing transition waves. Although such existence results are known, there may not exist pulsating fronts in general. Zlatoš [33] constructed a periodic pure bistable reaction such that there is no pulsating fronts of (1.1). We also refer to [7,31,32] for some nonexistence results.

In this work, we aim to construct curved fronts by using some pulsating fronts with nonzero speeds. Therefore, we need to assume a priori that

(H1) $\int_{\mathbb{T}^2\times[0,1]} f(x, y, u)dxdydu \neq 0$,
(H2) for every unit vector $e \in \mathbb{R}^2$, the Eq. (1.1) admits a pulsating front $U_e((x, y)\cdot e - c_e t, x, y)$ with $c_e \neq 0$.

From the results of Ducrot [8] and Guo [15], one knows that if (H1), (H2) hold, the propagation speed $c_e$ of the pulsating front in every direction has the sign of $\int_{\mathbb{T}^2\times[0,1]} f(x, y, u)dxdydu$. We assume, without loss of generality, that

$$\int_{\mathbb{T}^2\times[0,1]} f(x, y, u)dxdydu > 0, \tag{1.6}$$

which implies $c_e > 0$ for all $e \in \mathbb{S}$. Otherwise, one can replace $u$, $f$, $U_e(\xi, x, y)$ by $\tilde{u} = 1 - u$, $g(x, y, u) = -f(x, y, 1 - u)$, $\tilde{U}_e(\xi, x, y) = 1 - U_e(-\xi, x, y)$ and consider the new pulsating front $\tilde{U}_e$ with speed $-c_e$. From [3] and [15], the speed $c_e$ and the profile $U_e$ of the pulsating front are unique up to shifts in time for any direction $e$. We fix the pulsating front in every direction $e$ by

$$U_e(0, 0, 0) = \frac{1}{2}.$$

From [15], we also know that $\partial_\xi U_e < 0$, the family $\{c_e\}_{e\in\mathbb{S}}$ is uniformly bounded with respect to $e$ and the minimum and maximum of $c_e$ can be reached with the following inequality:

$$0 < \min_{e\in\mathbb{S}} c_e \leq \max_{e\in\mathbb{S}} c_e < +\infty.$$

In the whole paper, we always assume that (F1)–(F3), (H1)–(H2) and (1.6) hold and we do not repeat it in the sequel. We now focus on construction of curved fronts by some pulsating fronts. To the best of our knowledge, few results of the existence of curved fronts are known for bistable reaction-diffusion in spatially periodic media. However, one can refer to [10,11] for the existence of curved fronts of monostable and combustion reaction-diffusion equations with a periodic shear flow and refer to [4] for a space-time periodic monostable reaction-advection-diffusion equation. Although the pulsating front $U_e((x, y)\cdot e - c_e t, x, y)$ is not exactly planar, every level set is still bounded with a plane. Thus, the pulsating front is also called almost-planar in the framework of transition fronts (see [17]). We try to apply the ideas of Ninomiya and Taniguchi [21] which they used for homogeneous bistable case, to construct the curved fronts. But, since the profiles $U_e$ and speeds $c_e$ of pulsating fronts are different in general with respect to the direction $e$, we have to update their ideas.

We then claim our results. Let $\alpha \in (0, \pi)$. Then, by Assumption (H2), there exists a pulsating front in the direction $(\cos\alpha, \sin\alpha)$, that is,

$$U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y).$$

For any $\alpha, \beta \in (0, \pi)$, define

$$\begin{aligned} &U^-_{\alpha\beta}(t, x, y) \\ &:= \max\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y), U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)\}, \end{aligned} \tag{1.7}$$

which is a subsolution of (1.1). Our first result shows the existence of a curved front which converges to pulsating fronts along its asymptotic lines under some conditions on angles $\alpha$ and $\beta$. The curved front is actually a transition front connecting 0 and 1 whose interfaces can be chosen as a V-shaped curve.

**Theorem 1.2.** *For any $\theta \in (0, \pi)$, let $g(\theta) = c_\theta / \sin\theta$. For any $0 < \alpha < \beta < \pi$ such that*

$$\frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta} := c_{\alpha\beta} > \frac{c_\theta}{\sin\theta} \text{ for any } \theta \in (\alpha, \beta),\ g'(\alpha) < 0 \text{ and } g'(\beta) > 0, \tag{1.8}$$

*there exists an entire solution $V(t, x, y)$ of (1.1) such that $V_t(t, x, y) > 0$ for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ and*

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \left| V(t, x, y) - U^-_{\alpha\beta}(t, x, y) \right| = 0. \tag{1.9}$$

**Remark 1.3.** In [15], Guo has shown that $c_e$ is differentiable with respect to $e \in \mathbb{S}$ and hence $c_\theta = c_{(\cos\theta, \sin\theta)}$ is differentiable with respect to $\theta$. Obviously, $g(\theta)$ is then differentiable with respect to $\theta \in (0, \pi)$. Recently, Ding and Giletti [6] have shown that the set of admissible speeds $c_e$ is rather large and it is conjectured that $c_e$ could be any continuous sign-unchanging function. It means that conditions $g'(\alpha) < 0$ and $g'(\beta) > 0$ could be easily satisfied. We will also show that conditions $g'(\alpha) < 0$ and $g'(\beta) > 0$ are not empty later. It seems that in Theorem 1.2, conditions $g'(\alpha) < 0$ and $g'(\beta) > 0$ can not be removed by our methods. These conditions are actually true for homogeneous unbalanced bistable case with the reaction term having positive integration from 0 to 1 ($\alpha$ has to be smaller than $\pi/2$ in this case by symmetry and $\beta = \pi - \alpha$), but false for homogeneous balanced bistable case. Moreover, the V-shaped front exists in homogeneous unbalanced bistable case, but does not exist in homogeneous balanced bistable case, see [18]. Nevertheless, for the balanced case, there exist some fronts whose level sets have an exponential shape for 2-dimensional space and a paraboloidal shape for $N$-dimensional space with $N \geq 3$, see [5,26,27].

**Remark 1.4.** One can easily check that the curved front $V(t,x,y)$ in Theorem 1.2 is a transition front connecting 0 and 1 (see [17] for the definition) with sets

$$\begin{aligned}\Gamma_t &:= \{x \le 0,\, y \in \mathbb{R};\, x\cos\alpha + y\sin\alpha - c_\alpha t\}\\ &\cup\{x > 0,\, y \in \mathbb{R};\, x\cos\beta + y\sin\beta - c_\beta t\},\\ \Omega_t^+ &:= \{x \le 0,\, y \in \mathbb{R};\, x\cos\alpha + y\sin\alpha - c_\alpha t < 0\}\\ &\cup\{x > 0,\, y \in \mathbb{R};\, x\cos\beta + y\sin\beta - c_\beta t < 0\},\end{aligned}$$

and

$$\begin{aligned}\Omega_t^- &:= \{x \le 0,\, y \in \mathbb{R};\, x\cos\alpha + y\sin\alpha - c_\alpha t > 0\}\\ &\cup\{x > 0,\, y \in \mathbb{R};\, x\cos\beta + y\sin\beta - c_\beta t > 0\}.\end{aligned}$$

Notice that for any fixed $t$, $\Gamma_t$ is a connected polyline since $c_\alpha/\sin\alpha = c_\beta/\sin\beta$ and the shape of $\Gamma_t$ is invariant with respect to $t$. Moreover, by the definition of the global mean speed [17], the curved front $V(t,x,y)$ has a global mean speed equal to $\min\{c_\alpha, c_\beta\}$, in the sense that

$$\frac{d(\Gamma_t,\Gamma_s)}{|t-s|} \to \min\{c_\alpha, c_\beta\}, \text{ as } \quad |t-s| \to +\infty.$$

Here, the distance $d(A,B)$ between two subsets $A$ and $B$ of $\mathbb{R}^2$, is defined by the smallest geodesic distance between pairs of points in $A$ and $B$. Another definition of the distance $\tilde{d}$ like

$$\tilde{d}(A,B) = \min\Big(\sup\{d(x,B); x \in A\}, \sup\{d(y,A); y \in B\}\Big),$$

could be used. Then, there holds that $d(A,B) \le \tilde{d}(A,B)$ and the global mean speed is equal to $\max\{c_\alpha, c_\beta\}$, in the sense that

$$\frac{\tilde{d}(\Gamma_t,\Gamma_s)}{|t-s|} \to \max\{c_\alpha, c_\beta\}, \text{ as } \quad |t-s| \to +\infty.$$

This is different with the homogeneous case, in which the global mean speeds under these two definitions are the same, see [17] and see [16] for the underlying domains being exterior domains and domains with multiple branches.

We then show that the condition (1.8) is not empty, that is, it is satisfied when $\alpha$ close to 0 and $\beta$ close to $\pi$, see Fig. 2.25.

**Corollary 1.5.** *There exist $0 < \alpha_1 < \beta_1 < \pi$ such that for any $\alpha \in (0,\alpha_1)$, there is $\beta \in (\beta_1,\pi)$ such that (1.8) holds for such $\alpha$, $\beta$ and there exists an entire solution $V(t,x,y)$ of (1.1) satisfying (1.9).*

Indeed, one can rotate the coordinate such that $y$-axis points to any direction. Although the periodicity is not preserved by rotation, the same proofs of Theorem 1.2 and Corollary 1.5 can be applied. Therefore, Corollary 1.5 implies that for any two pulsating fronts whose propagation directions are close to reversed with each other, one can use them to construct a curved front.

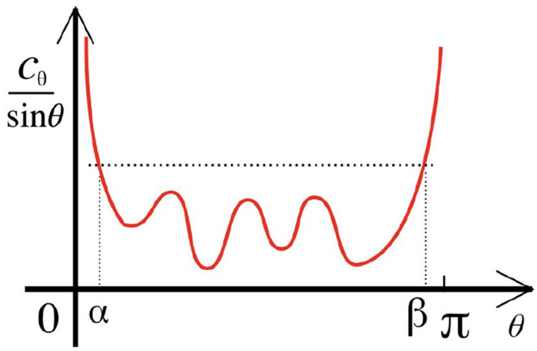


**Fig. 1.** An example of $\alpha$ and $\beta$ satisfying (1.8)

**Corollary 1.6.** *There exist* $0 < \rho < 1$ *such that for any directions* $e_1$, $e_2$ *with* $-1 < e_1 \cdot e_2 < -1 + \rho$, *there exist a direction* $e_0$ *such that*

$$\frac{c_{e_1}}{e_1 \cdot e_0} = \frac{c_{e_2}}{e_2 \cdot e_0} := c_{e_1 e_2} \tag{1.10}$$

*and there is an entire solution* $V(t, x, y)$ *of* (1.1) *satisfying*

$$\lim_{R \to +\infty} \sup_{((x,y) - c_{e_1 e_2} t e_0)^2 > R^2} \left| V(t, x, y) - U^-_{e_1 e_2}(t, x, y) \right| = 0, \tag{1.11}$$

*where*

$$U^-_{e_1 e_2}(t, x, y) := \max\{U_{e_1}((x, y) \cdot e_1 - c_{e_1} t, x, y), U_{e_2}((x, y) \cdot e_2 - c_{e_2} t, x, y)\}.$$

By Theorem 1.2, one knows that (1.8) is a sufficient condition for the existence of $V(t, x, y)$ satisfying (1.9). However, we cannot show that (1.8) is necessary, but can show that (1.8) without $g'(\alpha) < 0$ and $g'(\beta) > 0$ is necessary.

**Theorem 1.7.** *If there are two angles* $\alpha$ *and* $\beta$ *of* $(0, \pi)$ *and a constant* $c_{\alpha\beta} > 0$ *such that there exists an entire solution* $V(t, x, y)$ *of* (1.1) *satisfying* (1.9), *then it holds that*

$$c_{\alpha\beta} = \frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta} > \frac{c_\theta}{\sin\theta} \text{ for any } \quad \theta \in (\alpha, \beta).$$

Now, we show the uniqueness and the stability of the curved front $V(t, x, y)$.

**Theorem 1.8.** *For any fixed* $0 < \alpha < \beta < \pi$ *satisfying*

$$\frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta} := c_{\alpha\beta},$$

*the entire solution* $V(t, x, y)$ *of* (1.1) *satisfying* (1.9) *is unique; that is, if there is an entire solution* $V^*(t, x, y)$ *satisfying* (1.9), *then* $V^*(t, x, y) \equiv V(t, x, y)$.

**Theorem 1.9.** *Let $\alpha$ and $\beta$ be fixed angles satisfying* (1.8) *and $V(t,x,y)$ be the entire solution of* (1.1) *satisfying* (1.9). *Let $0 \le u_0(x,y) \le 1$ be an initial value satisfying*

$$\lim_{R\to+\infty} \sup_{x^2+y^2>R^2} \left| u_0(x,y) - U^-_{\alpha\beta}(0,x,y) \right| = 0. \tag{1.12}$$

*Then, the solution $u(t,x,y)$ of* (1.1) *for $t>0$ with $u(0,x,y)=u_0(x,y)$ satisfies*

$$\lim_{t\to+\infty} \|u(t,x,y) - V(t,x,y)\|_{L^\infty(\mathbb{R}^2)} = 0.$$

Next, we construct a transition front connecting 0 and 1 with varying interfaces. Such a kind of transition front is known in homogeneous case by [17], in which the solution is orthogonal symmetric with respect to $y$-axis and behaves as three planar fronts as $t \to -\infty$. However, in our case, this transition front can not be symmetric in general.

**Theorem 1.10.** *Let $\alpha$ and $\beta$ be fixed angles satisfying* (1.8) *and let $V_{\alpha\beta}(t,x,y)$ be the entire solution of* (1.1) *satisfying* (1.9). *Denote $e_\alpha = (\cos\alpha, \sin\alpha)$ and $e_\beta = (\cos\beta, \sin\beta)$. Assume that there exist another angle $\theta \in (\alpha,\beta)$ and a direction $e_\theta = (\cos\theta, \sin\theta)$ such that*

*(i) for $e_\alpha$ and $e_\theta$, there is a direction $e_{\alpha\theta}$ such that* (1.10) *holds for $e_1 = e_\alpha$, $e_2 = e_\theta$ and $e_0 = e_{\alpha\theta}$, it holds $h'(\alpha) < 0$ where $h(s) = c_s/(e_{\alpha\theta}\cdot(\cos s, \sin s))$ for $0 < s < \theta$ and there is an entire solution $V_{\alpha\theta}(t,x,y)$ satisfying* (1.11).

*(ii) for $e_\beta$ and $e_\theta$, there is a direction $e_{\beta\theta}$ such that* (1.10) *holds for $e_1 = e_\beta$, $e_2 = e_\theta$ and $e_0 = e_{\beta\theta}$, it holds $h'(\beta) > 0$ where $h(s) = c_s/(e_{\beta\theta}\cdot(\cos s, \sin s))$ for $\theta < s < \pi$ and $e_0 = e_{\alpha\theta}$ and there is an entire solution $V_{\beta\theta}(t,x,y)$ satisfying* (1.11).

*Then, there exists an entire solution $u(t,x,y)$ of* (1.1) *such that*

$$u(t,x,y) \to \begin{cases} V_{\alpha\theta}(t,x,y), \text{ uniformly in the half plane } \{(x,y)\in\mathbb{R}^2; x<0\}, \\ V_{\beta\theta}(t,x,y)\}, \text{ uniformly in the half plane } \{(x,y)\in\mathbb{R}^2; x>0\}, \end{cases} \quad \text{as } t\to-\infty. \tag{1.13}$$

*and*

$$u(t,x) \to V_{\alpha\beta}(t,x,y), \text{ as } \ t\to+\infty \ \text{ uniformly in } \mathbb{R}^2. \tag{1.14}$$

The convergence in above theorem is in the sense of $L^\infty$ norm.

**Remark 1.11.** From the proof of Theorem 1.10, one can easily check that the entire solution $u(t,x,y)$ is a transition front connecting 0 and 1 with the interfaces

$$\begin{aligned} \Gamma_t := & \left\{ x \le \frac{c_\alpha \sin\theta - c_\theta \sin\alpha}{\sin(\theta-\alpha)} t, y\in\mathbb{R}; x\cos\alpha + y\sin\alpha - c_\alpha t = 0 \right\} \\ & \cup \left\{ \frac{c_\alpha \sin\theta - c_\theta \sin\alpha}{\sin(\theta-\alpha)} t < x \right. \\ & \left. \le \frac{c_\beta \sin\theta - c_\theta \sin\beta}{\sin(\theta-\beta)} t, y\in\mathbb{R}; x\cos\theta + y\sin\theta - c_\theta t = 0 \right\} \end{aligned}$$

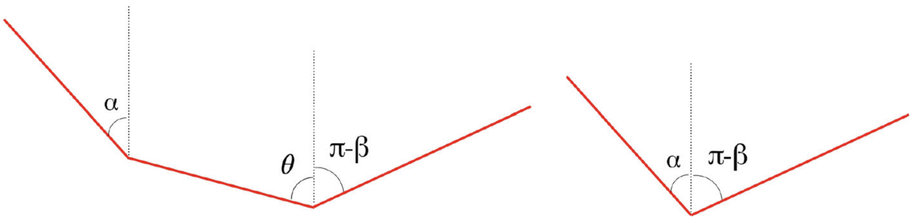


**Fig. 2.** Left: interface when $t << -1$; Right: interface when $t >> 1$

$$\cup \left\{ x > \frac{c_\beta \sin\theta - c_\theta \sin\beta}{\sin(\theta-\beta)} t,\ y \in \mathbb{R};\right.$$
$$\left. x\cos\beta + y\sin\beta - c_\beta t = 0 \right\}, \quad \text{for} \quad t \le 0,$$

and

$$\Gamma_t := \{x \le 0,\ y \in \mathbb{R};\ x\cos\alpha + y\sin\alpha - c_\alpha t = 0\}$$
$$\cup\{x > 0,\ y \in \mathbb{R};\ x\cos\beta + y\sin\beta - c_\beta t = 0\},\ \text{for} \quad t > 0,$$

see Fig. 2.

Finally, we give an example showing that Theorem 1.10 is not empty.

**Corollary 1.12.** *Assume that $e_*$ is the direction such that the family of speeds $\{c_e\}_{e\in\mathbb{S}}$ reaches its minimum, that is, $c_{e_*} = \min_{e\in\mathbb{S}}\{c_e\}$. Then, there exist $e_1$ and $e_2$ close to $e_*$ such that (1.10) holds for $e_0 = e_*$ and there is an entire solution $V_{e_1e_2}(t,x,y)$ of (1.1) satisfying (1.11). Moreover, there exist a direction $e_3$ close to $-e_*$ and a direction $e_{**}$ such that there is an entire solution $u(t,x,y)$ of (1.1) such that*

$$u(t,x,y) \to \begin{cases} V_{e_1e_2}(t,x,y), & \textit{uniformly in the half plane} \quad \{(x,y)\in\mathbb{R}^2;\ (x,y)\cdot e_{**} < 0\}, \\ V_{e_2e_3}(t,x,y)\}, & \textit{uniformly in the half plane} \quad \{(x,y)\in\mathbb{R}^2;\ (x,y)\cdot e_{**} > 0\}, \end{cases}$$

*as $t \to -\infty$ and*

$$u(t,x) \to V_{e_1e_3}(t,x,y),\ \textit{as} \quad t\to+\infty \quad \textit{uniformly in} \quad \mathbb{R}^2.$$

rest of this paper as organized as follows: in Section 2, we first prove the existence of the curved front, that is, Theorem 1.2. Then, we give some examples showing that Theorem 1.2 is not empty. We also show a necessary condition for the existence of the curved front in this section. Section 3 is devoted to the proof of the uniqueness and stability of the curved front in Theorem 1.2. In Section 4, we construct a curved front with varying interfaces and give an example.

## 2. Existence of Curved Fronts

This section is devoted to the construction of a curved front satisfying Theorem 1.2. We will need some properties of the pulsating front, especially the differentiability of the profile $U_e$ and the speed $c_e$ with respect to the direction $e$.

*2.1. Preliminaries*

We will use the hyperbolic function $\mathrm{sech}(x)$ frequently in the sequel. Thus, we recall some known properties of it which can be checked easily.

**Lemma 2.1.** *It holds that*

$$|\,sech'(x)|,\ |\,sech''(x)| \le\ sech(x),\ for\quad x \in \mathbb{R},$$

*and there is a positive constant $p$ such that*

$$sech'(x) > 0\,for\quad x \le -p,\quad sech'(x) < 0\,for \\ x \ge p\quad and\ \ sech''(x) > 0\,for\quad |x| \ge p.$$

Then, we need a smooth V-shaped curve with $y = -x\cot\alpha$ and $y = -x\cot\beta$ being its asymptotic lines.

**Lemma 2.2.** *For any $0 < \alpha < \beta < \pi$, there is a smooth function $\psi(x)$ for $x \in \mathbb{R}$ with $y = -x\cot\alpha$ and $y = -x\cot\beta$ being its asymptotic lines and there are positive constants $k_1$, $k_2$ and $K_1$ such that*

$$\begin{cases} \psi''(x) > 0, & for\ all\quad x \in \mathbb{R} \\ -\cot\alpha < \psi'(x) < -\cot\beta, & for\ all\quad x \in \mathbb{R} \\ k_1\ sech(x) \le \psi'(x) + \cot\alpha \le K_1\ sech(x), & for\quad x < 0, \\ k_2\ sech(x) \le -\cot\beta - \psi'(x) \le K_1\ sech(x), & for\quad x \ge 0, \\ \max(|\psi''(x)|, |\psi'''(x)|) \le K_1\ sech(x), & for\ all\quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

**Proof.** Let $0 < \alpha < \beta < \pi$. Since $\alpha < \beta$, there are two positive constants $a$, $b$ and a smooth function $\varphi(x)$ such that

$$\varphi(x) = \begin{cases} -x\cot\alpha, & x \le -a \\ -x\cot\beta, & x \ge b. \end{cases} \text{ and } \varphi''(x) > 0 \text{ for } \quad -a < x < b.$$

An example of such a function is that one can take an incircle of the straight lines $y = -x\cot\alpha$ and $y = -x\cot\beta$ with tangent points $(-a, a\cot\alpha)$ and $(b, -b\cot\beta)$ and $\varphi(x)$ is made of the line $y = -x\cot\alpha$ for $x \le -a$, the arc of the incircle between $-a$ and $b$, and the line $y = -x\cot\beta$ for $x \ge b$. One can mollify $\varphi(x)$ at $(-a, a\cot\alpha)$ and $(b, -b\cot\beta)$ such that $\varphi(x) \in C^\infty(\mathbb{R})$, see Fig. 3. Define a smooth function $\psi(x)$ as follows:

$$\psi(x) := \varphi(x) + \rho\,\mathrm{sech}(x).$$

Here $\rho > 0$ is a constant. Since $\mathrm{sech}''(x)$ is bounded and by Lemma 2.1, one can make $\rho$ small enough and $a$, $b$ sufficiently large such that

$$\psi''(x) > 0 \text{ for all } \quad x \in \mathbb{R}.$$

Moreover, one can easily check that $\psi(x)$ satisfies all properties in (2.1). This completes the proof.

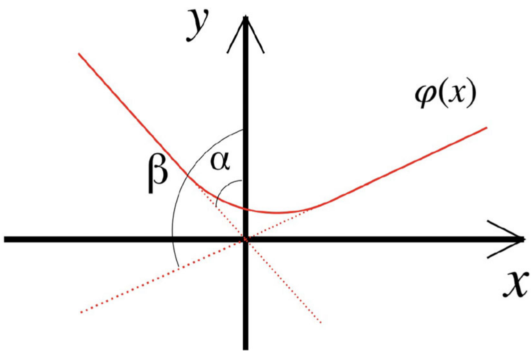


**Fig. 3.** The function $\varphi(x)$

We now recall some properties of the pulsating front $U_e((x, y) \cdot e - c_e t, x, y)$. One can substitute the form $U_e((x, y) \cdot e - c_e t, x, y)$ into (1.1) and get that $(U_e(\xi, x, y), c_e)$ satisfies the semi-linear elliptic degenerate equation

$$\begin{aligned} &c_e \partial_\xi U_e + \partial_{\xi\xi} U_e + 2\nabla_{x,y} \partial_\xi U_e \cdot e + \Delta_{x,y} U_e \\ &+ f(x, y, U_e) = 0, \quad \text{for all} \quad (\xi, x, y) \in \mathbb{R} \times \mathbb{T}^2. \end{aligned} \tag{2.2}$$

From [15, Lemma 2.1], we have

**Lemma 2.3.** *For any pulsating front $(U_e(\xi, x, y), c_e)$ with $c_e > 0$, there exist $\mu_1 > 0$, $\mu_2 > 0$, $C_1 > 0$ and $C_2 > 0$ independent of e such that*

$$\begin{aligned} 0 < U_e(\xi, x, y) \leq C_1 e^{-\mu_1 \xi} \ \ &for \ \ \xi > 0, \ (x, y) \in \mathbb{T}^2 \\ 0 < 1 - U_e(\xi, x, y) \leq C_2 e^{\mu_2 \xi} \ \ &for \ \ \xi \leq 0, \ (x, y) \in \mathbb{T}^2. \end{aligned}$$

Then, by standard parabolic estimates applied to $u(t, x, y) = U_e((x, y) \cdot e - c_e t, x, y)$, one can get that $|\nabla_{x,y} u_t|$, $|u_{tt}|$, $|u_t| \leq C u(t+1, x, y)$ for some constant $C > 0$ and $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$. Notice that $u_t(t, x, y) = -c_e \partial_\xi U_e((x, y) \cdot e - c_e t, x, y)$ with $c_e > 0$. Then, by Lemma 2.3, we have the following lemma:

**Lemma 2.4.** *For any pulsating front $(U_e(\xi, x, y), c_e)$ with $c_e > 0$, there exist $\mu_3 > 0$ and $C_3 > 0$ independent of e such that*

$$\begin{aligned} &|\partial_\xi U_e(\xi, x, y)|, \ |\partial_{\xi\xi} U_e(\xi, x, y)|, \ |\nabla_{x,y} \partial_\xi U_e(\xi, x, y)| \\ &\leq C_3 e^{-\mu_3 |\xi|} \ \ for \ \ \xi \in \mathbb{R}, \ (x, y) \in \mathbb{T}^2. \end{aligned}$$

We also need the following properties:

**Lemma 2.5.** *For any $C > 0$, there is $0 < \delta < 1/2$ independent of e such that*

$$\delta \leq U_e(\xi, x, y) \leq 1 - \delta, \ for \quad -C \leq \xi \leq C \ and \quad (x, y) \in \mathbb{T}^2, \tag{2.3}$$

*and there is $r > 0$ independent of e such that*

$$-\partial_\xi U_e(\xi, x, y) \geq r \ for \ for \quad -C \leq \xi \leq C \quad and \quad (x, y) \in \mathbb{T}^2. \tag{2.4}$$

**Proof.** Let $u(t,x,y) = U_e((x,y)\cdot e - c_e t, x, y)$. One can easily check that $u(t,x,y)$ is a transition front connecting 0 and 1 with set $\{(t,x,y)\in\mathbb{R}\times\mathbb{R}^2; (x,y)\cdot e - c_e t = 0\}$ being its interfaces. Then, by [3, Theorem 1.2], one immediately has that there is $0<\delta<1/2$ such that

$$\delta \le u(t,x,y) \le 1-\delta, \text{ for } \quad -C \le (x,y)\cdot e - c_e t \le C.$$

By continuity of $U_e$ with respect to $e$ (see [15]), one has that $\delta$ can be independent of $e$.

The following proof for (2.4) can be simplified for the pulsating front $U_e$. However, we do it in a general way in purpose that such idea can be used to prove that the curved front which we construct later has similar properties. Notice that $u_t(t,x,y)>0$ satisfies

$$(u_t)_t - \Delta u_t - f_u(x,y,u)u_t = 0, \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2.$$

Assume that there is a sequence $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}}$ of $\mathbb{R}\times\mathbb{R}^2$ such that $-C\le (x_n,y_n)\cdot e - c_e t_n \le C$ and $u_t(t_n,x_n,y_n)\to 0$ as $n\to+\infty$. Since $f(x,y,u)$ is periodic in $(x,y)$, there is $(x',y')\in\mathbb{R}^2$ such that $f(x+x_n, y+y_n, u)\to f(x+x', y+y', u)$ as $n\to+\infty$. Let $u_n(t,x,y) = u(t+t_n, x+x_n, y+y_n)$ and $v_n(t,x,y) = u_t(t+t_n, x+x_n, y+y_n)$. By standard parabolic estimates, $u_n(t,x,y)$ converges to a solution $u_\infty(t,x,y)$ of

$$u_t - \Delta u - f(x+x', y+y', u_\infty) = 0, \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2,$$

and $v_n(t,x,y)$ converges to a solution $v_\infty(t,x,y)$ of

$$v_t - \Delta v - f_u(x+x', y+y', u_\infty)v = 0, \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2.$$

Moreover, $v_\infty(t,x,y)$ satisfies $v_\infty(t,x,y)\ge 0$ and $v_\infty(0,0,0)=0$. By the maximum principle, $v_\infty(t,x,y)\equiv 0$. Since $U_e(\xi,x,y)\to 1$ as $\xi\to-\infty$, there is $R>0$ large enough such that

$$u(t,x,y)\ge 1-\sigma \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \text{ such that } \quad (x,y)\cdot e - c_e t \le -R,$$

where $\sigma$ is defined in (F3). Take $(x_*,y_*)\in\mathbb{R}^2$ such that $(x_*,y_*)\cdot e < -R-C$. Then, $v_\infty(t,x,y)\equiv 0$ implies that $u_t(t+t_n, x+x_*+x_n, y+y_*+y_n)\to 0$ as $n\to+\infty$ locally uniformly in $\mathbb{R}\times\mathbb{R}^2$. Notice that $(x_*+x_n, y_*+y_n)\cdot e - c_e t_n \le -R$ and hence, $u(t_n, x_*+x_n, y_*+y_n)\ge 1-\sigma$. Also notice that 1 is the only equilibrium of (1.1) over $1-\sigma$ from (F3) and (1.2). It further implies that $u(t+t_n, x+x_*+x_n, y+y_*+y_n)\to 1$ locally uniformly in $\mathbb{R}\times\mathbb{R}^2$. Since $(x_*,y_*)$ is fixed and $-C\le (x_n,y_n)\cdot e - c_e t_n \le C$, it reaches a contradiction with (2.3). This completes the proof.

It follows from [15, Theorem 1.5] that $U_e$ and $c_e$ are differentiable with respect to $e$. Remember that $U_e$ are normalized by $U_e(0,0,0)=1/2$ for all $e\in\mathbb{S}$. For any $b\in\mathbb{R}^2\setminus\{0\}$, define

$$U_b = U_{\frac{b}{|b|}} \text{ and } c_b = c_{\frac{b}{|b|}}. \tag{2.5}$$

Define Banach spaces as follows:

$$\begin{aligned}L^2(\mathbb{R}\times\mathbb{T}^2) = \{&u\in L^2_{loc}(\mathbb{R}\times\mathbb{R}^2);\ u(\xi, x+k_1L_1, y+k_2L_2)\\ &= u(\xi,x,y)\ \text{a.e. in}\quad \mathbb{R}\times\mathbb{R}^2\\ &\text{for any}\quad k_1,k_2\in\mathbb{Z},\ \text{and}\quad u\in L^2(\mathbb{R}\times K)\\ &\quad\text{for any bounded set}\quad K\subset\mathbb{R}^2\},\\ H^1(\mathbb{R}\times\mathbb{T}^2) = \{&u\in H^1_{loc}(\mathbb{R}\times\mathbb{R}^2);\ u(\xi, x+k_1L_1, y+k_2L_2)\\ &= u(\xi,x,y)\ \text{a.e. in}\ \mathbb{R}\times\mathbb{R}^2,\\ &\text{for any}\quad k_1,k_2\in\mathbb{Z},\ \text{and}\quad u\in H^1(\mathbb{R}\times K)\\ &\quad\text{for any bounded set}\quad K\subset\mathbb{R}^2\},\end{aligned}$$

and

$$\begin{aligned}H^2(\mathbb{R}\times\mathbb{T}^2) = \{&u\in H^2_{loc}(\mathbb{R}\times\mathbb{R}^2);\ u(\xi, x+k_1L_1, y+k_2L_2)\\ &= u(\xi,x,y)\ \text{a.e. in}\quad \mathbb{R}\times\mathbb{R}^2,\\ &\text{for any}\quad k_1,k_2\in\mathbb{Z},\ \text{and}\ u\in H^2(\mathbb{R}\times K)\\ &\quad\text{for any bounded set}\quad K\subset\mathbb{R}^2\},\end{aligned}$$

and define their norms as

$$\begin{aligned}\|u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} &= \left(\int_{\mathbb{R}}\int_{\mathbb{T}^2}|u|^2dxdyd\xi\right)^{1/2},\\ &\|u\|_{H^1(\mathbb{R}\times\mathbb{T}^2)} = \|u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \|\partial_\xi u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\\ &+\|\partial_x u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \|\partial_y u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)},\end{aligned}$$

and

$$\begin{aligned}\|u\|_{H^2(\mathbb{R}\times\mathbb{T}^2)} =& \|u\|_{H^1(\mathbb{R}\times\mathbb{T}^2)} + \|\partial_{\xi\xi}u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\\ &+ \|\partial_\xi\partial_x u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \|\partial_\xi\partial_y u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\\ &+ \|\partial_{xx}u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \|\partial_x\partial_y u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \|\partial_{yy}u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}.\end{aligned}$$

**Lemma 2.6.** *Let $U_b$ and $c_b$ be defined in (2.5). Then, $U_b$ and $c_b$ are doubly continuously Fréchet differentiable at any $b\in\mathbb{R}^N\setminus\{0\}$, that is, there exist linear operators $(U_b', c_b'): \mathbb{R}^2\to L^2(\mathbb{R}\times\mathbb{T}^2)\times\mathbb{R}$ and $(U_b'', c_b''): \mathbb{R}^2\times\mathbb{R}^2\to L^2(\mathbb{R}\times\mathbb{T}^2)\times\mathbb{R}$ such that for any $h$, $\rho\in\mathbb{R}^2$, $(U_{b+h}, c_{b+h}) - (U_b, c_b) = (U_b', c_b')\cdot h + o(|h|)$, $(U_{b+\rho}'\cdot h, c_{b+\rho}'\cdot h) - (U_b'\cdot h, c_b'\cdot h) = (U_b''\cdot h, c_b''\cdot h)\cdot\rho + o(|\rho|)$ as $|h|, |\rho|\to 0$.*

Let us denote the Fréchet derivatives up to second order of $U_e$ and $c_e$ with respect to $e$ by $U_e'$, $U_e''$, $c_e'$ and $c_e''$. The Fréchet derivatives are all bounded in the sense that

$$\|U_e'\| = \sup_{h\in\mathbb{R}^N}\frac{\|U_e'\cdot h\|_{L^\infty(\mathbb{R}\times\mathbb{T}^N)}}{|h|} < +\infty,$$

$$\|U_e''\| = \sup_{(h,\rho)\in\mathbb{R}^N\times\mathbb{R}^N} \frac{\|(U_e''\cdot h)\cdot\rho\|_{L^\infty(\mathbb{R}\times\mathbb{T}^N)}}{|h||\rho|} < +\infty,$$

and

$$\|c_e'\| = \sup_{h\in\mathbb{R}^N} \frac{|c_e'\cdot h|}{|h|} < +\infty,\ \ \|c_e''\| = \sup_{(h,\rho)\in\mathbb{R}^N\times\mathbb{R}^N} \frac{|(c_e''\cdot h)\cdot\rho|}{|h||\rho|} < +\infty.$$

The boundedness of $c_e'$ and $c_e''$ can be easily followed. Let $h\in\mathbb{R}^N$ with $|h|=1$. One can also easily get that $\|U_e'\cdot h\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}$ is uniformly bounded for any $h\in\mathbb{R}^N$ with $|h|=1$. By differentiating (2.2), it follows that

$$\begin{aligned} &c_e'\cdot h\partial_\xi U_e + c_e\partial_\xi U_e'\cdot h + \partial_{\xi\xi}U_e'\cdot h + 2\nabla_{x,y}\partial U_e\cdot(h-(h\cdot e)e) \\ &\quad +2\nabla_{x,y}\partial_\xi U_e'\cdot h + \Delta_{x,y}U_e'\cdot h + f_u(x,y,U_e)U_e'\cdot h = 0. \end{aligned} \tag{2.6}$$

By rewriting (2.6) in its weak form in the variables $(t,x,y)$ (namely $\xi=(x,y)\cdot e - c_e t$), it follows from parabolic regularity theory and bootstrap arguments that $U_e'\cdot h$ is a bounded classical solution of (2.6) and the $L^\infty$ bound of $U_e'\cdot h$ is uniform for $h\in\mathbb{R}^N$ with $|h|=1$. Thus, $U_e'$ is bounded in the above sense. Similar arguments can be applied to $U_e''$. We also know from [15] that for any $h\in\mathbb{R}^2$, $\rho\in\mathbb{R}^2$, $U_e'\cdot h$ and $(U_e''\cdot h)\cdot\rho$ are differentiable with respect to $\xi$, $x$ and $y$ up to second order and these derivatives are bounded too. We then need the following properties of $U_e'$:

**Lemma 2.7.** *For any $e\in\mathbb{S}$, there exist $\mu_4>0$ and $C_4>0$ independent of $e$ such that*

$$\begin{aligned} &|(U_e'\cdot h)(\xi,x,y)|,\ |(\partial_\xi U_e'\cdot h)(\xi,x,y)| \\ &\quad \le C_4 e^{-\mu_4|\xi|}|h|,\ \textit{for any}\ \ h\in\mathbb{R}^2, \xi\in\mathbb{R}\ \ \textit{and}\ \ (x,y)\in\mathbb{T}^2. \end{aligned}$$

**Proof.** Take a smooth nonincreasing function $p(\xi)$ such that

$$p(\xi)=1 \text{ for }\ \ \xi\le 0\ \ \text{ and } p(\xi)=e^{-r\xi} \text{ for }\ \ \xi\ge b$$

for some positive constants $r$ and $b$. Here, one can make $r$ and $b$ to be small and large enough respectively such that

$$r<\min\{\mu_1,\mu_2,\mu_3\}, \tag{2.7}$$

and

$$c_e\left|\frac{p'(\xi)}{p(\xi)}\right| + \left|\frac{p''(\xi)}{p(\xi)}\right| \le \frac{\lambda}{2} \text{ for all }\ \ \xi\in\mathbb{R} \text{ and } e\in\mathbb{S}, \tag{2.8}$$

where $\lambda>0$ is defined in (F3).

For every direction $e$, we define a function $V_e(\xi,x,y)$ by

$$V_e(\xi,x,y) := p^{-1}(\xi)U_e(\xi,x,y),\ \text{for }\ \ \xi\in\mathbb{R}\ \ \text{and}\ \ (x,y)\in\mathbb{T}^2.$$

By Lemmas 2.3, 2.4 and (2.7), one has

$$V_e(-\infty,x,y)=1 \text{ and } V_e(+\infty,x,y)=0,\ \text{uniformly for }\ \ (x,y)\in\mathbb{T}^2 \text{ and }\ \ e\in\mathbb{S},$$

$V_e(\xi, x, y) \in L^2(\mathbb{R}^+ \times \mathbb{T}^2)$, $1 - V_e(\xi, x, y) \in L^2(\mathbb{R}^- \times \mathbb{T}^2)$ and all derivatives of $V_e$ up to second order are in $L^2(\mathbb{R} \times \mathbb{T}^2)$. Since $U_e(\xi, x, y)$ satisfies (2.2), one can get that $V_e(\xi, x, y)$ satisfies

$$c_e\partial_\xi V_e + \partial_{\xi\xi} V_e + 2\nabla_{x,y}\partial_\xi V_e \cdot e + \Delta_{x,y} V_e + \frac{2p'}{p}\partial_\xi V_e + \frac{2p'}{p}\nabla_{x,y} V_e \cdot e$$
$$+ \frac{1}{p} f(x, y, pV_e) + \left(c_e\frac{p'}{p} + \frac{p''}{p}\right)V_e = 0, \text{ for } \quad (\xi, x, y) \in \mathbb{R} \times \mathbb{T}^2.$$

From (F3) and (2.8), there is $C > 0$ such that

$$\begin{cases} \frac{1}{p} f(x, y, pV_e) + \left(c_e\frac{p'}{p} + \frac{p''}{p}\right)V_e \le -\frac{\lambda}{2}V_e, & \text{for} \quad (x, y) \in \mathbb{T}^2 \text{ and } \xi \ge C, \\ \frac{1}{p} f(x, y, pV_e) + \left(c_e\frac{p'}{p} + \frac{p''}{p}\right)V_e \ge \frac{\lambda}{2}(1 - V_e), & \text{for} \quad (x, y) \in \mathbb{T}^2 \text{ and } \xi \le -C. \end{cases} \tag{2.9}$$

For any $e \in \mathbb{S}$, define a linear operator

$$M_e(v) := c_e\partial_\xi v + \partial_{\xi\xi} v + 2\nabla_{x,y}\partial_\xi v \cdot e + \Delta_{x,y} v + \frac{2p'}{p}\partial_\xi v + \frac{2p'}{p}\nabla_{x,y} v \cdot e - \beta v,$$

where $\beta > 0$ is a fixed real number and

$$v \in D := \{v \in H^1(\mathbb{R} \times \mathbb{T}^N);\ \partial_{\xi\xi} v + 2\nabla_y\partial_\xi v \cdot e + \Delta_y v \in L^2(\mathbb{R} \times \mathbb{T}^N)\}.$$

The space $D$ is endowed with the norm $\|v\|_D = \|v\|_{H^1(\mathbb{R}\times\mathbb{T}^N)} + \|\partial_{\xi\xi} v + 2\nabla_y\partial_\xi v \cdot e + \Delta_y v\|_{L^2(\mathbb{R}\times\mathbb{T}^N)}$. Then, by the similar proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3 in [7] (one can trivially extend the proofs to the high dimensional space), one knows that $M_e$ satisfies all the properties in Lemma 2.7 of [15], such as invertibility and boundedness. For any $e \in \mathbb{S}$, we then define

$$H_e(v) := c_e\partial_\xi v + \partial_{\xi\xi} v + 2\nabla_{x,y}\partial_\xi v \cdot e + \Delta_{x,y} v + \frac{2p'}{p}\partial_\xi v + \frac{2p'}{p}\nabla_{x,y} v \cdot e$$
$$+ f_u(y, pV_e)v + \left(c_e\frac{p'}{p} + \frac{p''}{p}\right)v, \quad v \in D.$$

Notice that $H_e(v) = \widetilde{H}_e(pv)/p$ with $0 < p(\xi) \le 1$, where

$$\widetilde{H}_e(v) := c_e\partial_\xi v + \partial_{\xi\xi} v + 2\nabla_y\partial_\xi v \cdot e + \Delta_y v + f_u(y, U_e)v, \quad v \in D.$$

By Lemma 4.1 in [7], one knows that the operator $\widetilde{H}_e$ and its adjoint operator $\widetilde{H}_e^*$ have algebraically simple eigenvalue 0 and the kernel of $\widetilde{H}_e$ is generated by $\partial_\xi U_e$. Therefore, the operator $H_e$ and its adjoint operator $H_e^*$ also have algebraically simple eigenvalue 0 and the kernel of $H_e$ is generated by $p^{-1}\partial_\xi U_e$. Moreover, the property that the range of $H_e$ is closed in $L^2(\mathbb{R}) \times \mathbb{T}^2$ can be proved in the same line of the proof of [7, Lemma 4.1] by using (2.9).

Now, for any $e \in \mathbb{S}$, $v \in H^2(\mathbb{R} \times \mathbb{T}^2)$, $\vartheta \in \mathbb{R}$ and $\eta \in \mathbb{R}^2$, define

$$K_e(v, \vartheta, \eta) = \vartheta\partial_\xi(V_e + v) + 2\nabla_y\partial_\xi(V_e + v) \cdot \eta + \frac{2p'}{p}\nabla_{x,y}(V_e + v) \cdot \eta$$
$$+ \frac{1}{p} f(y, p(V_e + v)) - \frac{1}{p} f(y, V_e) + \left(c_e\frac{p'}{p} + \frac{p''}{p} + \beta\right)v,$$

and

$$G_e(v,\vartheta,\eta) := \left(v + M_e^{-1}(K_e(v,\vartheta,\eta)), \int_{\mathbb{R}^+\times\mathbb{T}^N}\left[(V_e(\xi,y)+v(\xi,y))^2 - U_e^2(\xi,y)\right]dyd\xi\right).$$

By following the proof of [15, Lemma 2.10], one can get that for every $e \in \mathbb{S}$, the function $G_e : H^2(\mathbb{R}\times\mathbb{T}^N)\times\mathbb{R}\times\mathbb{R}^N \to D\times\mathbb{R}$ is continuous and it is continuously Fréchet differentiable with respect to $(v,\vartheta)$ and doubly continuously Fréchet differentiable with respect to $\eta$. For any $e \in \mathbb{S}^{N-1}$ and $(\tilde{v},\tilde{\vartheta}) \in D\times\mathbb{R}$, define

$$\begin{aligned} &Q_e(\tilde{v},\tilde{\vartheta}) \\ &= \left(\tilde{v} + M_e^{-1}(\tilde{\vartheta}\partial_\xi V_e + f_u(y,U_e)\tilde{v} + \left(c_e\frac{p'}{p} + \frac{p''}{p} + \beta\right)\tilde{v}), 2\int_{\mathbb{R}^+\times\mathbb{T}^N} V_e(\xi,y)\tilde{v}(\xi,y)dyd\xi\right), \end{aligned}$$

which has the same form as $\partial_{(v,\vartheta)}G_e(0,0,0)$. By the properties of $H_e$ and the same line of the proofs of [7, Lemma 3.3] and [15, Lemma 2.11], one can get that $Q_e$ satisfies all properties in [15, Lemma 2.11], such as invertibility and boundedness.

As soon as we have all these properties of these operators, we can follow the same proof of [15, Theorem 1.5] to get that $V_b(\xi,x,y) = p^{-1}(\xi)U_b(\xi,x,y)$ is doubly Fréchet differentiable at any $b \in \mathbb{R}^2\setminus\{0\}$. Moreover, $\|V_e'\|$ is bounded for any $e \in \mathbb{S}$.

Thus, by the definition of Fréchet differentiation, we have

$$(U_e'\cdot h)(\cdot,\cdot,\cdot) = p(\xi)(V_e'\cdot h)(\cdot,\cdot,\cdot), \text{ for any } \quad e\in\mathbb{S} \quad \text{and} \quad h\in\mathbb{R}^2.$$

Therefore, there exists a positive constant $C_4$ such that

$$|(U_e'\cdot h)(\xi,x,y)| \le p(\xi)\|V_e'\||h| \le C_4e^{-r\xi}|h| \text{ for } \quad \xi\ge 0, (x,y)\in\mathbb{T}^2 \text{ and } h\in\mathbb{R}^2. \tag{2.10}$$

By applying similar arguments to the other side, that is, $\xi < 0$, one can also get that there are positive constants $C_5$ and $\mu_5$ such that

$$|(U_e'\cdot h)(\xi,x,y)| \le C_5e^{\mu_5\xi}|h| \text{ for } \quad \xi<0, (x,y)\in\mathbb{T}^2 \text{ and } \quad h\in\mathbb{R}^2. \tag{2.11}$$

Lastly, we differentiate (2.2) at $e$ on $h \in \mathbb{R}^2$ and get that

$$\begin{aligned} &(c_e'\cdot h)\partial_\xi U_e + c_e\partial_\xi(U_e'\cdot h) + \partial_{\xi\xi}(U_e'\cdot h) + 2\nabla_y\partial_\xi U_e\cdot(h-(e\cdot h)e) \\ &\quad + 2\nabla_{x,y}\partial_\xi(U_e'\cdot h)\cdot e + \Delta_{x,y}(U_e'\cdot h) + f_u(x,y,U_e)(U_e'\cdot h) = 0. \end{aligned}$$

By changing variables $\xi = (x,y)\cdot e - c_et$, one has that $u(t,x) := (U_e'\cdot h)((x,y)\cdot e - c_et, x, y)$ satisfies a parabolic equation

$$u_t - \Delta u = f_u(x,y,U_e)u + (c_e'\cdot h)\partial_\xi U_e + 2\nabla_{x,y}\partial_\xi U_e\cdot(h-(e\cdot h)e).$$

By parabolic estimates, Lemma 2.4 and (2.10)-(2.11), one can get that there are positive constants $C_6$ and $\mu_6$ such that

$$|u_t(t,x,y)| \le C_6e^{-\mu_6|(x,y)\cdot e - c_et|}|h|,$$

that is,

$$|(\partial_\xi U_e'\cdot h)(\xi,x,y)| \le C_6e^{-\mu_6|\xi|}|h| \text{ for any } \quad h\in\mathbb{R}^2, \xi\in\mathbb{R} \text{ and } (x,y)\in\mathbb{T}^2.$$

This completes the proof.

### 2.2. *Proof of Theorem 1.2*

Take any two angles $\alpha, \beta$ of $(0, \pi)$ such that (1.8) holds. Let $\psi(x)$ be a smooth function satisfying Lemma 2.2 for $\alpha$ and $\beta$. Take a constant $\varrho$ to be determined later. For every point $(x, y)$ on the curve $y = \psi(\varrho x)/\varrho$, there is a unit normal

$$e(x) = (e_1(x), e_2(x)) = \Big( -\frac{\psi'(\varrho x)}{\sqrt{\psi'^2(\varrho x)+1}}, \frac{1}{\sqrt{\psi'^2(\varrho x)+1}} \Big). \tag{2.12}$$

By Lemma 2.2, every component of $e(x)$ is differentiable with respect to $x$ and

$$e(x) \to (\cos\alpha, \sin\alpha) \text{ as } \quad x \to -\infty \quad \text{ and } e(x) \to (\cos\beta, \sin\beta) \text{ as } \quad x \to +\infty;$$

its derivatives can be denoted by

$$e'(x) = (e_1'(x), e_2'(x)) = \Big( -\frac{\varrho\psi''(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{3}{2}}}, -\frac{\varrho\psi'(\varrho x)\psi''(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{3}{2}}} \Big),$$

and

$$\begin{aligned} e''(x) = (e_1''(x), e_2''(x)) =\Big( &-\frac{\varrho^2\psi'''(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{3}{2}}} + \frac{3\varrho^2\psi'(\varrho x)\psi''^2(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{5}{2}}}, \\ &-\frac{\varrho^2\psi''^2(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{3}{2}}} - \frac{\varrho^2\psi'(\varrho x)\psi'''(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{3}{2}}} + \frac{3\varrho^2\psi'^2(\varrho x)\psi''^2(\varrho x)}{(\psi'^2(\varrho x)+1)^{\frac{5}{2}}} \Big). \end{aligned}$$

Therefore, by Lemma 2.2, there exist $K_2 > 0$ and $K_3 > 0$ such that

$$|e'(x)| \le \varrho K_2 \operatorname{sech}(\varrho x) \text{ and } |e''(x)| \le \varrho^2 K_3 \operatorname{sech}(\varrho x) \text{ for all } \quad x \in \mathbb{R}. \tag{2.13}$$

Remember that $U_{\alpha\beta}^-(t, x, y)$ defined by (2.8) is a subsolution of (1.1). Now, take a positive constant $\varepsilon$ and we define

$$U^+(t, x, y) = U_{e(x)}(\xi(t, x, y), x, y) + \varepsilon \operatorname{sech}(\varrho x), \tag{2.14}$$

where

$$\xi(t, x, y) = \frac{y - c_{\alpha\beta}t - \psi(\varrho x)/\varrho}{\sqrt{\psi'^2(\varrho x)+1}}, \tag{2.15}$$

and $c_{\alpha\beta}$ is defined by (1.8). We prove that $U^+(t, x, y)$ is a supersolution of (1.1) for small $\varepsilon$ and $\varrho$.

**Lemma 2.8.** *There exist $\varepsilon_0 > 0$ and $\varrho(\varepsilon_0) > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$ and $0 < \varrho \le \varrho(\varepsilon_0)$, the function $U^+(t, x, y)$ is a supersolution of (1.1) with $U_t^+ > 0$. Moreover, this satisfies*

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha,\beta}t)^2>R^2} \Big| U^+(t, x, y) - U_{\alpha\beta}^-(t, x, y) \Big| \le 2\varepsilon, \tag{2.16}$$

*and*

$$U^+(t, x, y) \ge U_{\alpha\beta}^-(t, x, y), \text{ for all } \quad t \in \mathbb{R} \quad \text{and} \quad (x, y) \in \mathbb{R}^2. \tag{2.17}$$

**Proof.** We divide the proof into three steps.

*Step 1: $U^+$ is a supersolution.* We will pick $\varepsilon_0 > 0$ and $\varrho(\varepsilon)$ such that Lemma 2.8 holds. Assume that

$$\varepsilon_0 \leq \frac{\sigma}{2},$$

where $\sigma > 0$ is defined in (F3). More restrictions on $\varepsilon_0$ will be given later. One can compute that

$$\begin{aligned}
LU^+ :&= U_t^+ - \Delta_{x,y}U^+ - f(x, y, U^+) \\
&= \partial_\xi U_{e(x)}\xi_t - \partial_{\xi\xi}U_{e(x)}(\xi_x^2 + \xi_y^2) - 2\nabla_{x,y}\partial_\xi U_{e(x)} \cdot (\xi_x, \xi_y) \\
&\quad - \Delta_{x,y}U_{e(x)} - \partial_\xi U_{e(x)}\xi_{xx} \\
&\quad - U''_{e(x)} \cdot e'(x) \cdot e'(x) - U'_{e(x)} \cdot e''(x) - 2\partial_\xi U'_{e(x)} \cdot e'(x)\xi_x \\
&\quad - 2\partial_x U'_{e(x)} \cdot e'(x) \\
&\quad - \varepsilon\varrho^2 \mathrm{sech}''(\varrho x) - f(x, y, U^+),
\end{aligned}$$

where $\partial_\xi U_{e(x)}$, $\partial_{\xi\xi}U_{e(x)}$, $\nabla_{x,y}\partial_\xi U_{e(x)}$, $\Delta_{x,y}U_{e(x)}$, $U''_{e(x)} \cdot e'(x) \cdot e'(x)$, $U'_{e(x)} \cdot e''(x)$, $\partial_\xi U'(e(x)) \cdot e'(x)$, $\partial_x U'_{e(x)} \cdot e'(x)$ are taking values at $(\xi(t, x, y), x, y)$ and $U^+$, $\xi_t$, $\xi_x$, $\xi_y$ are taking values at $(t, x, y)$. By (2.15), it follows from a direct computation that

$$\begin{aligned}
\xi_t &= -\frac{c_{\alpha\beta}}{\sqrt{\psi'^2(\varrho x) + 1}}, \\
\xi_x &= -\frac{\varrho\psi'(\varrho x)\psi''(\varrho x)}{\psi'^2(\varrho x) + 1}\xi - \frac{\psi'(\varrho x)}{\sqrt{\psi'^2(\varrho x) + 1}}, \\
\xi_y &= \frac{1}{\sqrt{\psi'^2(\varrho x) + 1}}, \\
\xi_{xx} &= -\frac{\varrho^2\psi'(\varrho x)\psi'''(\varrho x)}{\psi'^2(\varrho x) + 1}\xi + \frac{\varrho^2\psi''^2(\varrho x)(2\psi'^2(\varrho x) - 1)}{(\psi'^2(\varrho x) + 1)^2}\xi \\
&\quad + \frac{\varrho(\psi'^2(\varrho x) - 1)\psi''(\varrho x)}{(\psi'^2(\varrho x) + 1)^{\frac{3}{2}}}, \\
\xi_x^2 + \xi_y^2 - 1 &= \left(\frac{\varrho\psi'(\varrho x)\psi''(\varrho x)}{\psi'^2(\varrho x) + 1}\right)^2\xi^2 + 2\frac{\varrho\psi'^2(\varrho x)\psi''(\varrho x)}{(\psi'^2(\varrho x) + 1)^{\frac{3}{2}}}\xi.
\end{aligned} \tag{2.18}$$

By noticing that $\xi_y = e_2(x)$ and by (2.2), one has

$$\begin{aligned}
LU^+ &= (c_{e(x)} + \xi_t)\partial_\xi U_{e(x)} - \partial_{\xi\xi}U_{e(x)}(\xi_x^2 + \xi_y^2 - 1) \\
&\quad - 2\partial_x\partial_\xi U_{e(x)}(\xi_x - e_1(x)) - \partial_\xi U_{e(x)}\xi_{xx} \\
&\quad - U''_{e(x)} \cdot e'(x) \cdot e'(x) - U'_{e(x)} \cdot e''(x) - 2\partial_\xi U'_{e(x)} \cdot e'(x)\xi_x \\
&\quad - 2\partial_x U'_{e(x)} \cdot e'(x) \\
&\quad - \varepsilon\varrho^2 \mathrm{sech}''(\varrho x) + f(x, y, U_{e(x)}) - f(x, y, U^+),
\end{aligned} \tag{2.19}$$

where $\partial_\xi U_{e(x)}$, $\partial_{\xi\xi} U_{e(x)}$, $\partial_x\partial_\xi U_{e(x)}$, $U''_{e(x)}\cdot e'(x)\cdot e'(x)$, $U'_{e(x)}\cdot e''(x)$, $\partial_\xi U'(e(x))\cdot e'(x)$, $\partial_x U'_{e(x)}\cdot e'(x)$, $U_{e(x)}$ are taking values at $(\xi(t,x,y),x,y)$ and $U^+$, $\xi_t$, $\xi_x$, $\xi_y$ are taking values at $(t,x,y)$. By Lemma 2.4, one has that $|\partial_{\xi\xi} U_{e(x)}\xi^2|$, $|\partial_{\xi\xi} U_{e(x)}\xi|$, $|\partial_x\partial_\xi U_{e(x)}\xi|$ and $|\partial_\xi U_{e(x)}\xi|$ are uniformly bounded for $\xi\in\mathbb{R}$, $(x,y)\in\mathbb{R}^2$. Then, by Lemmas 2.2 and (2.18), there is $C_5>0$ such that

$$|\partial_{\xi\xi} U_{e(x)}(\xi_x^2+\xi_y^2-1)|+2|\partial_x\partial_\xi U_{e(x)}(\xi_x-e_1(x))|+|\partial_\xi U_{e(x)}\xi_{xx}|\le C_5\varrho\mathrm{sech}(\varrho x). \tag{2.20}$$

Since $\|U'_e\|$, $\|U''_e\|$, $\|\partial_\xi U'_e\|$, $\|\partial_x U'_e\|$ are bounded and by Lemma 2.7, (2.13), there is $C_6>0$ such that

$$\begin{aligned}&|U''_{e(x)}\cdot e'(x)\cdot e'(x)|+|U'_{e(x)}\cdot e''(x)|+2|\partial_\xi U'_{e(x)}\cdot e'(x)\xi_x|\\&+2|\partial_x U'_{e(x)}\cdot e'(x)|\le C_6\varrho\mathrm{sech}(\varrho x).\end{aligned} \tag{2.21}$$

We make the following claim:

**Claim 2.9.** *There is $C_7>0$ such that*

$$-\xi_t-c_{e(x)}=\frac{c_{\alpha\beta}}{\sqrt{\psi'^2(\varrho x)+1}}-c_{e(x)}\ge C_7 sech(\varrho x)>0. \tag{2.22}$$

We postpone the proof of this claim after the proof of this lemma.

Then, it follows from (2.19), (2.20), (2.21), (2.22), Lemma 2.1 and $\partial_\xi U_e<0$ that

$$\begin{aligned}LU^+\ge&-\partial_\xi U_{e(x)}C_7\mathrm{sech}(\varrho x)-(C_5+C_6)\varrho\mathrm{sech}(\varrho x)-2\varepsilon\varrho^2\mathrm{sech}(\varrho x)\\&+f(x,y,U_{e(x)})-f(x,y,U^+).\end{aligned} \tag{2.23}$$

By Lemma 2.3, there is $C>0$ such that

$$0<U_e(\xi,x,y)\le\frac{\sigma}{2}\ \text{for}\quad \xi\ge C\ \text{and}\quad 0<1-U_e(\xi,x,y)\le\frac{\sigma}{2}\ \text{for}\quad \xi\le -C, \tag{2.24}$$

uniformly for $(x,y)\in\mathbb{T}^2$ and $e\in\mathbb{S}$. Then, for $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$ such that $\xi(t,x,y)\ge C$ and $\xi(t,x,y)\le -C$ respectively, one has that $U^+(t,x,y)\le\sigma/2+\varepsilon\le\sigma$ and $U^+(t,x,y)\ge 1-\sigma/2$ respectively since $\varepsilon\le\varepsilon_0\le\sigma/2$ and hence, it follows from (1.2) that

$$f(x,y,U_{e(x)})-f(x,y,U^+)\ge\lambda\varepsilon\mathrm{sech}(\varrho x). \tag{2.25}$$

Since $\partial_\xi U_e<0$ and by (2.23), (2.25), one has that

$$LU^+\ge\Big(-(C_5+C_6)\varrho-2\varepsilon\varrho^2+\lambda\varepsilon\Big)\mathrm{sech}(\varrho x)\ge 0,$$

by taking $0<\varrho\le\varrho(\varepsilon)$ where $\varrho(\varepsilon)>0$ is small enough such that

$$-(C_5+C_6)\varrho-2\varepsilon\varrho^2+\lambda\varepsilon>0,\ \text{for all}\quad 0<\varrho\le\varrho(\varepsilon). \tag{2.26}$$

Finally, for $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$ such that $-C\le\xi(t,x,y)\le C$, it follows from Lemma 2.5 that there is $k>0$ such that

$$-\partial_\xi U_e(\xi,x,y)\ge k\ \text{for all}\quad e\in\mathbb{S}. \tag{2.27}$$

Notice that

$$f(x, y, U_{e(x)}) - f(x, y, U^+) \geq -M\varepsilon \text{sech}(\varrho x), \tag{2.28}$$

where $M := \max_{(x,y,u)\in\mathbb{T}^2\times\mathbb{R}} |f_u(x, y, u)|$. Thus, it follows from (2.23), (2.26), (2.27) and (2.28) that

$$\begin{aligned} LU^+ \geq &\Big(kC_7 - (C_5 + C_6)\varrho - 2\varepsilon\varrho^2 - M\varepsilon\Big) \\ \text{sech}(\varrho x) \geq &\Big(kC_7 - (\lambda + M)\varepsilon\Big)\text{sech}(\varrho x) \geq 0, \end{aligned}$$

by taking $\varepsilon_0 = \min\{\sigma/2, kC_7/(\lambda + M)\}$ and $0 < \varepsilon \leq \varepsilon_0$.

Therefore, $LU^+ \geq 0$ for all $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$. By the comparison principle, $U^+(t, x, y)$ is a supersolution of (1.1). The property $U_t^+ > 0$ comes from $\partial_\xi U_e < 0$ and $c_{\alpha\beta} > 0$.

*Step 2: the proof of* (2.16). Since $e(x) \to (\cos\alpha, \sin\alpha)$ as $x \to -\infty$ and by the definition of $U_e'$, there is $R_1 > 0$ such that

$$\begin{aligned} &|U_{e(x)}(\xi(t, x, y), x, y) - U_\alpha(\xi(t, x, y), x, y)| \\ &\quad \leq \|U_\alpha'\||e(x) - (\cos\alpha, \sin\alpha)| + o(|e(x) - (\cos\alpha, \sin\alpha)|) \\ &\quad \leq \frac{\varepsilon}{4}, \quad \text{for} \quad x \leq -R_1 \quad \text{and} \quad t \in \mathbb{R}, y \in \mathbb{R}. \end{aligned} \tag{2.29}$$

Notice that $1/\sqrt{\psi'^2(\varrho x) + 1} \to \sin\alpha$ as $x \to -\infty$ and $c_{\alpha\beta}\sin\alpha = c_\alpha$. Then, by Lemma 2.2, one has that

$$\xi(t, x, y) \to x\cos\alpha + y\sin\alpha - c_\alpha t, \text{ as } \quad x \to -\infty \quad \text{for any } t \in \mathbb{R} \text{ and } y \in \mathbb{R}.$$

Thus, there is $R_2 > 0$ such that

$$\begin{aligned} &|U_\alpha(\xi(t, x, y), x, y) - U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)| \\ &\quad \leq \frac{\varepsilon}{4}, \text{ for } \quad x \leq -R_2 \text{ and } t \in \mathbb{R}, y \in \mathbb{R}. \end{aligned}$$

By the definition of $U^+(t, x, y)$ and together with (2.29), it follows that

$$\begin{aligned} &|U^+(t, x, y) - U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)| \\ &\quad \leq \frac{3}{2}\varepsilon, \text{ for } \quad x \leq -\max\{R_1, R_2\} \quad \text{and} \quad t \in \mathbb{R}, y \in \mathbb{R}. \end{aligned} \tag{2.30}$$

Similarly, one can prove that there is $R_3 > 0$ such that

$$\begin{aligned} &|U^+(t, x, y) - U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)| \\ &\quad \leq \frac{3}{2}\varepsilon, \text{ for } \quad x \geq R_3 \quad \text{and } t \in \mathbb{R}, y \in \mathbb{R}. \end{aligned} \tag{2.31}$$

Now, for $-\max\{R_1, R_2\} \leq x \leq R_3$, we know that $\psi(\varrho x)$ and $\psi'(\varrho x)$ are bounded. Then, as $y - c_{\alpha\beta}t \to +\infty$, one has that

$$\begin{aligned} &\xi(t, x, y) \to +\infty \text{ and } x\cos\alpha + y\sin\alpha - c_\alpha t \to +\infty, \\ &\text{for} \quad -\max\{R_1, R_2\} \leq x \leq R_3. \end{aligned}$$

Thus, there is $R_4 > 0$ such that

$$0 < U_{e(x)}(\xi(t,x,y),x,y) \le \frac{\varepsilon}{2},$$

and

$$0 < U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y),\ U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y) \le \frac{\varepsilon}{2},$$

for $-\max\{R_1, R_2\} \le x \le R_3$ and $y - c_{\alpha\beta}t \ge R_4$. Hence,

$$|U^+(t,x,y) - U^-_{\alpha\beta}(t,x,y)| \le 2\varepsilon, \tag{2.32}$$

for $-\max\{R_1, R_2\} \le x \le R_3$ and $y - c_{\alpha\beta}t \ge R_4$. Similarly, since $U_{e(x)}(-\infty,x,y) = U_\alpha(-\infty,x,y) = 1$ uniformly for $(x,y) \in \mathbb{T}^2$, there is $R_5$ such that

$$|U^+(t,x,y) - U^-_{\alpha\beta}(t,x,y)| \le 2\varepsilon, \tag{2.33}$$

for $-\max\{R_1, R_2\} \le x \le R_3$ and $y - c_{\alpha\beta}t \le -R_5$.

On the other hand, since $U_e(-\infty,x,y) = 1$ and $U_e(+\infty,x,y) = 0$ for any $(x,y) \in \mathbb{T}^2$ and $e \in \mathbb{S}$, it follows that there is $C_\varepsilon > 0$ such that

$$0 < U_e(\xi,x,y) \le \varepsilon/4 \text{ for } \ \xi \ge C_\varepsilon \ \ \text{and} (x,y) \in \mathbb{T}^2,$$

and

$$1 - \varepsilon/4 \le U_e(\xi,x,y) < 1 \text{ for } \ \xi \le -C_\varepsilon \ \ \text{and} \ \ (x,y) \in \mathbb{T}^2.$$

This then means that

$$\begin{aligned} &U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y) \\ &\quad \ge U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y), \text{ for } \ (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \\ &\quad \text{such that } \ x\cos\alpha + y\sin\alpha - c_\alpha t \le -C_\varepsilon \\ &\quad \text{and } \ \ x\cos\beta + y\sin\beta - c_\beta t \ge C_\varepsilon. \end{aligned} \tag{2.34}$$

For any fixed $r \in \mathbb{R}$ and any point $(t,x,y) \in \mathbb{R}\times\mathbb{R}^2$ such that $x\cos\alpha + y\sin\alpha - c_\alpha t = r$, one has that

$$\begin{aligned} x\cos\beta + y\sin\beta - c_\beta t = x\frac{\sin(\alpha-\beta)}{\sin\alpha} + \frac{\sin\beta}{\sin\alpha} r \to +\infty, \\ \text{as } \ \ x \to -\infty \ \ \text{uniformly for } \ r \ge -C_\varepsilon, \end{aligned}$$

since $-\pi < \alpha - \beta < 0$ and $c_\alpha/\sin\alpha = c_\beta/\sin\beta$. It implies that $U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y) \to 0$ as $x \to -\infty$ uniformly for $(t,x,y) \in \mathbb{R}\times\mathbb{R}^2$ such that $x\cos\alpha + y\sin\alpha - c_\alpha t = r \ge -C_\varepsilon$. While, by Lemma 2.5, there is $\varepsilon' > 0$ such that $U_\alpha(r,x,y) \ge \varepsilon'$ for $-C_\varepsilon \le r \le C_\varepsilon$. Thus, there is $R_6 > 0$ such that

$$\begin{aligned} &U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y) \\ &\quad \ge U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y), \text{ for } \ (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \\ &\quad \text{such that } \ x \le -R_6 \text{ and } \ \ -C_\varepsilon \\ &\quad \le x\cos\alpha + y\sin\alpha - c_\alpha t \le C_\varepsilon. \end{aligned} \tag{2.35}$$

and

$$U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y) \le \frac{\varepsilon}{4}, \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \quad \text{such that} \quad x\le -R_6$$
$$\text{and} \quad x\cos\alpha + y\sin\alpha - c_\alpha t \ge C_\varepsilon.$$

It follows that

$$\begin{aligned} &U^-_{\alpha\beta}(t,x,y)\\ &\quad = \max\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y), U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)\} \le \frac{\varepsilon}{4},\\ &\quad \text{for} \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \quad \text{such that} \quad x\le -R_6 \quad \text{and} x\cos\alpha + y\sin\alpha - c_\alpha t \ge C_\varepsilon. \end{aligned} \tag{2.36}$$

For any point $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$ such that $x\cos\beta + y\sin\beta - c_\beta t = r$, one has that

$$\cos\alpha + y\sin\alpha - c_\alpha t = x\frac{\sin(\beta-\alpha)}{\sin\beta} + \frac{\sin\alpha}{\sin\beta} r \to -\infty,$$
$$\text{as} \quad x\to-\infty \quad \text{uniformly for} \quad r\le C_\varepsilon.$$

This implies that $U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)\to 1$ as $x\to-\infty$ uniformly for $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$ such that $x\cos\beta + y\sin\beta - c_\beta t = r \le C_\varepsilon$. While, by Lemma 2.5, there is $\varepsilon'' > 0$ such that $U_\beta(r,x,y)\le 1-\varepsilon''$ for $-C_\varepsilon \le r\le C_\varepsilon$. Thus, even if it means increasing $R_6$, one can get that

$$\begin{aligned} &U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)\\ &\quad \ge U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y), \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\\ &\quad \text{such that} \quad x\le -R_6 \quad \text{and} \quad -C_\varepsilon \le x\cos\beta + y\sin\beta - c_\beta t\le C_\varepsilon. \end{aligned} \tag{2.37}$$

and

$$U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y) \ge 1-\frac{\varepsilon}{4}, \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \text{such that} \quad x\le -R_6$$
$$\text{and } x\cos\beta + y\sin\beta - c_\beta t \le -C_\varepsilon.$$

It follows that

$$\begin{aligned} &U^-_{\alpha\beta}(t,x,y) \ge 1-\frac{\varepsilon}{4}, \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\\ &\quad \text{such that} \quad x\le -R_6 \quad \text{and} \quad x\cos\beta + y\sin\beta - c_\beta t\le -C_\varepsilon. \end{aligned} \tag{2.38}$$

By (2.34)-(2.38), one gets that

$$\begin{aligned} U^-_{\alpha\beta}(t,x,y) = &U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y), \text{ for}\\ &(t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \quad \text{such that} \quad x\le -R_6,\\ &x\cos\alpha + y\sin\alpha - c_\alpha t \le C_\varepsilon \quad \text{and} \quad x\cos\beta + y\sin\beta - c_\beta t \ge -C_\varepsilon. \end{aligned}$$

and

$$|U^-_{\alpha\beta}(t,x,y) - U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)| \le \frac{\varepsilon}{4},$$

for $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ such that $x \leq -R_6$, $x\cos\alpha + y\sin\alpha - c_\alpha t \geq C_\varepsilon$ and $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ such that $x \leq -R_6$, $x\cos\beta + y\sin\beta - c_\beta t \geq -C_\varepsilon$. Above arguments also imply that

$$U^-_{\alpha\beta}(t, x, y) - U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y) \to 0,$$
$$\text{as} \quad x \to -\infty \quad \text{uniformly for} t \in \mathbb{R} \quad \text{and} \quad y \in \mathbb{R}.$$

Similar proof can deduce that

$$U^-_{\alpha\beta}(t, x, y) - U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y) \to 0, \text{ as } \quad x \to +\infty \quad \text{uniformly for} \quad t \in \mathbb{R} \quad \text{and} \quad y \in \mathbb{R}.$$

Combined with (2.30), it follows that

$$|U^+(t, x, y) - U^-_{\alpha\beta}(t, x, y)| \leq 2\varepsilon, \text{ for } x \leq -\max\{R_1, R_2, R_6\} \quad \text{and} \quad t \in \mathbb{R}, y \in \mathbb{R}. \tag{2.39}$$

Similarly, there is $R_7 > 0$ such that

$$|U^+(t, x, y) - U^-_{\alpha\beta}(t, x, y)| \leq 2\varepsilon, \text{ for } \quad x \geq \max\{R_3, R_7\} \quad \text{and} t \in \mathbb{R}, y \in \mathbb{R}. \tag{2.40}$$

By (2.32), (2.33), (2.39) and (2.40), we have our conclusion (2.16).

*Step 3: the proof of* (2.17). We only have to prove that $U^+(t, x, y) \geq U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t)$ and $U^+(t, x, y) \geq U_\beta(x\cos\beta + y\sin\beta - c_\beta t)$ for all $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$.

Since $U_e(-\infty, x, y) = 1$ and $U_e(+\infty, x, y) = 0$ for any $(x, y) \in \mathbb{T}^2$ and $e \in \mathbb{S}$, there is $C > 0$ such that

$$0 < U_e(\xi, x, y) \leq \sigma \text{ for } \quad \xi \geq C \quad \text{and} \quad (x, y) \in \mathbb{T}^2,$$

and

$$1 - \sigma \leq U_e(\xi, x, y) < 1 \text{ for } \quad \xi \leq -C \quad \text{and} (x, y) \in \mathbb{T}^2,$$

where $\sigma$ is defined in (F3). By (2.16) and letting $\varepsilon \leq \sigma/4$, there is $R > 0$ such that

$$U^+(t, x, y) \leq \sigma, \text{ for } \quad (t, x, y) \in \Omega^+_R$$
$$\text{and } U^+(t, x, y) \geq 1 - \sigma, \text{ for } \quad (t, x, y) \in \Omega^-_R,$$

where

$$\Omega^+_R := \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2; x \leq 0 \text{ and } x\cos\alpha + y\sin\alpha - c_\alpha t \geq c_\alpha R\} \cup \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2; x > 0 \text{ and } x\cos\beta + y\sin\beta - c_\beta t \geq c_\beta R\},$$

and

$$\Omega^-_R := \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2; x \leq 0 \text{ and } x\cos\alpha + y\sin\alpha - c_\alpha t \leq -c_\alpha R\} \cup \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2; x > 0 \text{ and } x\cos\beta + y\sin\beta - c_\beta t \leq -c_\beta R\}.$$

Notice that for any $t$, the boundaries of $\Omega_t^+$ and $\Omega_t^-$ are connected polylines since $c_\alpha/\sin\alpha = c_\beta/\sin\beta$. By Lemma 2.5 and the definition of $U^+(t,x,y)$, there is $0<\sigma'\leq\sigma$ such that

$$\sigma' \leq U^+(t,x,y) \leq 1-\sigma', \text{ for } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\setminus(\Omega_R^+\cup\Omega_R^-),$$

and

$$\sigma' \leq U_e(\xi,x,y) \leq 1-\sigma' \text{ for } \quad -C\leq\xi\leq C, (x,y)\in\mathbb{T}^2 \quad \text{and any} \quad e\in\mathbb{S}.$$

For any $\tau\in\mathbb{R}$, let $u_\tau(t,x,y)=U_\alpha(x\cos\alpha+y\sin\alpha-c_\alpha t+\tau)$. Let

$$\omega_\tau^+ := \{(t,x,y)\in\mathbb{R}\times\mathbb{R}^2; x\cos\alpha+y\sin\alpha-c_\alpha t+\tau\geq C\},$$

and

$$\omega_\tau^- := \{(t,x,y)\in\mathbb{R}\times\mathbb{R}^2; x\cos\alpha+y\sin\alpha-c_\alpha t+\tau\leq -C\}.$$

Notice that since $\alpha<\beta$, one has that

$$\{(t,x,y)\in\mathbb{R}\times\mathbb{R}^2; x\cos\alpha+y\sin\alpha-c_\alpha t\leq -c_\alpha R\}\subset\Omega_R^-,$$

and

$$\Omega_R^+\subset\{(t,x,y)\in\mathbb{R}\times\mathbb{R}^2; x\cos\alpha+y\sin\alpha-c_\alpha t\geq c_\alpha R\}.$$

Thus,

$$\mathbb{R}\times\mathbb{R}^2\setminus(\omega_\tau^+\cup\omega_\tau^-)\subset\Omega^-_{(\tau-C)/c_\alpha} \text{ and } \mathbb{R}\times\mathbb{R}^2\setminus(\Omega_R^+\cup\Omega_R^-)\subset\omega^+_{C+c_\alpha R}.$$

Then, by (2.16), $U_e(-\infty,x,y)=1$ and $U_e(+\infty,x,y)=0$, there is $\tau_1\geq c_\alpha R+C$ large enough such that for any $\tau\geq\tau_1$,

$$U^+(t,x,y)\geq 1-\sigma'\geq u_\tau(t,x,y), \text{ for all } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\setminus(\omega_\tau^+\cup\omega_\tau^-),$$

and

$$u_\tau(t,x,y)\leq\sigma'\leq U^+(t,x,y), \text{ for all } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\setminus(\Omega_R^+\cup\Omega_R^-).$$

Moreover, since $\tau\geq\tau_1\geq c_\alpha R+C$, one has that

$$U^+(t,x,y)\geq 1-\sigma\geq\sigma\geq u_\tau(t,x,y), \text{ for all } \quad (t,x,y)\in\omega_\tau^+\cap\Omega_R^-.$$

Thus, it follows that

$$\begin{aligned} u_\tau(t,x,y)\leq U^+(t,x,y), \text{ for any } \quad \tau\geq\tau_1 \\ \text{and all } \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\setminus(\omega_\tau^-\cup\Omega_R^+). \end{aligned} \tag{2.41}$$

Also notice that

$$u_\tau(t,x,y),\ U^+(t,x,y)\geq 1-\sigma \text{ in } \omega_\tau^- \text{and } u_\tau(t,x,y),\ U^+(t,x,y)\leq\sigma \text{ in } \Omega_R^+,$$

and $f(x,y,u)$ is nonincreasing in $u\in(-\infty,\sigma]$ and $u\in[1-\sigma,+\infty)$ for any $(x,y)\in\mathbb{T}^2$ by (1.2). By following similar proof as the proof of [3, Lemma 4.2] which mainly applied the sliding method and the linear parabolic estimates, one can get that

$$U^+(t,x,y) \geq u_\tau(t,x,y), \text{ in } \omega_\tau^- \text{ and } \Omega_R^+.$$

Combined with (2.41), one has that

$$U^+(t,x,y) \geq u_\tau(t,x,y), \text{ for any } \tau \geq \tau_1 \text{ and all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2.$$

Now, we decrease $\tau$. Define

$$\tau_* = \inf\{\tau \in \mathbb{R}; U^+(t,x,y) \geq u_\tau(t,x,y) \text{ for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2\}.$$

From above arguments, one knows that $\tau_* < +\infty$. Since $U^+(t,x,y) \to U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)$ as $x \to -\infty$, $U_\alpha(\xi,x,y)$ is decreasing in $\xi$ and by the definition of $u_\tau(t,x,y)$, one also knows that $\tau_* \geq 0$. Assume that $\tau_* > 0$. If

$$\inf\{U^+(t,x,y) - u_{\tau_*}(t,x,y); (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)\} > 0,$$

then there is $\eta > 0$ such that

$$U^+(t,x,y) \geq u_{\tau_*-\eta}(t,x,y), \text{ for } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*-\eta}^- \cup \Omega_R^+).$$

Then, one can apply the above arguments again and get that $U^+(t,x,y) \geq u_{\tau_*-\eta}(t,x,y)$ for all $(t,x,y) \in \mathbb{R}\times\mathbb{R}^2$ which contradicts the definition of $\tau_*$. Thus,

$$\inf\{U^+(t,x,y) - u_{\tau_*}(t,x,y); (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)\} = 0.$$

Since $\alpha < \beta$, there is a sequence $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)$ such that

$$-C - \tau_* \leq x_n\cos\alpha + y_n\sin\alpha - c_\alpha t_n \leq c_\alpha R,$$

and

$$U^+(t_n,x_n,y_n) - u_{\tau_*}(t_n,x_n,y_n) \to 0, \text{ as } n \to +\infty.$$

Then, there is $\xi_* \in \mathbb{R}$ such that $x_n\cos\alpha + y_n\sin\alpha - c_\alpha t_n \to \xi_*$ as $n \to +\infty$. Since $U^+(t,x,y) \to U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)$ as $x \to -\infty$, $U^+(t,x,y) \to U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)$ as $x \to +\infty$ with $\alpha < \beta$ and $\tau_* > 0$, one has that $x_n$ is bounded and there is $x_* \in \mathbb{R}$ such that $x_n \to x_*$ as $n \to +\infty$. Again by $U^+(t,x,y) \to U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)$ as $x \to -\infty$ and by (2.30), there is $R' > 0$ such that

$$\begin{aligned}&|U^+(t,x,y) - U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)| \\ &\leq \varepsilon \text{ for } x \leq -R' \text{ and } t \in \mathbb{R}, y \in \mathbb{R}.\end{aligned} \tag{2.42}$$

Let $v(t,x,y) = U^+(t,x,y) - u_{\tau_*}(t,x,y)$. Then, $v(t,x,y) \geq 0$ and

$$\begin{aligned}&v(t,x,y) > 0 \text{ for any } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \\ &\quad \text{such that } x \leq -R, x\cos\alpha + y\sin\alpha - c_\alpha t = \xi_*,\end{aligned} \tag{2.43}$$

by (2.42), $\tau_* > 0$ and taking $\varepsilon$ sufficiently small. Since $U^+(t,x,y)$ is a supersolution and $u_{\tau_*}(t,x,y)$ is a solution of (1.1), we have that $v(t,x,y)$ satisfies

$$v_t - \Delta v \geq -b(x,y)v, \text{ for } (t,x,y) \in \mathbb{R} \times \mathbb{R}^2,$$

where $b(x,y)$ is bounded. Since $v(t_n,x_n,y_n) \to 0$ and by the linear parabolic estimates and $x_n$ is bounded, one gets that

$$v(t_n - 1, x_n - R', y_n + \frac{R'\cos\alpha - c_\alpha}{\sin\alpha}) \to 0 \text{ as } n \to +\infty,$$

which contradicts (2.43). Thus, $\tau_* = 0$ and $U^+(t,x,y) \geq U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$.

Similarly one can prove that $U^+(t,x,y) \geq U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$. In conclusion, $U^+(t,x,y) \geq U^-_{\alpha\beta}(t,x,y)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$.

*Proof of Claim 2.9.* Notice that

$$-\frac{\psi'(\rho x)}{\sqrt{\psi'^2(\varrho x) + 1}} = e_1(x) \text{ and } \frac{1}{\sqrt{\psi'^2(\varrho x) + 1}} = e_2(x).$$

Let $\hat{\theta}(x) = \arccos e_1(x)$. By Lemma 2.2, one can get that $\alpha < \hat{\theta}(x) < \beta$ for all $x \in \mathbb{R}$ and $\hat{\theta}(-\infty) = \alpha$, $\hat{\theta}(+\infty) = \beta$. Then, $e(x) = (\cos\hat{\theta}, \sin\hat{\theta})$ and

$$\frac{c_{e(x)}}{e_2(x)} = \frac{c_{\hat{\theta}}}{\sin\hat{\theta}}.$$

Thus,

$$\frac{c_{\alpha\beta}}{\sqrt{\psi'^2(\varrho x) + 1}} - c_{e(x)} = \sin\hat{\theta}\left(c_{\alpha\beta} - \frac{c_{\hat{\theta}}}{\sin\hat{\theta}}\right).$$

Since $c_{\alpha\beta} > c_\theta/\sin\theta$ for any $\theta \in (\alpha,\beta)$ and $0 < \min\{\sin\alpha, \sin\beta\} \leq \sin\hat{\theta} \leq 1$, one only has to prove that

$$\begin{aligned} &c_{\alpha\beta} - \frac{c_{\hat{\theta}}}{\sin\hat{\theta}} \\ &\geq C_7 \operatorname{sech}(\varrho x), \text{ for some positive constant } C_7 \text{ and when } |x| \text{ is large.} \end{aligned} \tag{2.44}$$

We only consider when $x < 0$ and similar arguments can be applied for $x > 0$. Define

$$g(\theta) = \frac{c_\theta}{\sin\theta}, \text{ for } \theta \in (0,\pi).$$

Obviously, $g(\theta)$ is a $C^2$ function since $c_e$ is doubly differentiable with respect to $e$. By (1.8), one has that $g'(\alpha) < 0$. Since $\hat{\theta}(x) \to \alpha$ as $x \to -\infty$, it then follows that

$$\frac{c_\alpha}{\sin\alpha} - \frac{c_{\hat{\theta}}}{\sin\hat{\theta}} = g'(\alpha)(\alpha - \hat{\theta}(x)) + o(|\alpha - \hat{\theta}(x)|), \text{ for } x \text{ negative enough.} \tag{2.45}$$

Moreover, by (2.1), one has that

$$\hat{\theta}(x)-\alpha=\int_{-\infty}^{x}\hat{\theta}'(s)ds=-\int_{-\infty}^{x}\frac{e_1'(s)}{\sqrt{1-e_1^2(s)}}ds=\int_{\infty}^{x}\frac{\varrho\psi''(\varrho s)}{\psi'^2(\varrho s)+1}ds$$
$$\geq\frac{1}{|\psi'|_{L^\infty}^2+1}(\psi'(\varrho x)+\cot\alpha)\geq\frac{k_1}{|\psi'|_{L^\infty}^2+1}\operatorname{sech}(\varrho x).$$

One then can conclude (2.44) from (2.45) for $x$ negative enough.

Now, we are ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $u_n(t,x)$ be the solution of (1.1) for $t\geq -n$ with initial data

$$u_n(-n,x,y)=U_{\alpha\beta}^{-}(-n,x,y).$$

By Lemma 2.8, one can get from the comparison principle that

$$U_{\alpha\beta}^{-}(t,x,y)\leq u_n(t,x,y)\leq U^{+}(t,x,y),\ \text{for}\quad t\geq -n\quad\text{and}\quad (x,y)\in\mathbb{R}^2. \tag{2.46}$$

Since $U_{\alpha\beta}^{-}(t,x,y)$ is a subsolution, the sequence $u_n(t,x,y)$ is increasing in $n$. Letting $n\to+\infty$ and by parabolic estimates, the sequence $u_n(t,x,y)$ converges to an entire solution $V(t,x,y)$ of (1.1). By (2.46), one has that

$$U_{\alpha\beta}^{-}(t,x,y)\leq V(t,x,y)\leq U^{+}(t,x,y),\ \text{for}\quad t\in\mathbb{R}\quad\text{and}\quad (x,y)\in\mathbb{R}^2.$$

Then, it follows from Lemma 2.8 that (1.9) holds.

By $U_{\alpha\beta}^{-}(t,x,y)$ is increasing in $t$ and the maximum principle, one has that $(u_n)_t(t,x,y)>0$ for all $t\in(-n,+\infty)$ and $(x,y)\in\mathbb{R}^2$. By letting $n\to+\infty$ and the strong maximum principle, one concludes that $u_t(t,x,y)>0$ for all $t\in\mathbb{R}$ and $(x,y)\in\mathbb{R}^2$. This completes the proof.

### 2.3. *Proofs of Corollaries 1.5, 1.6 and Theorem 1.7*

We then give some examples to show that Theorem 1.2 is not empty, that is, Corollaries 1.5, 1.6.

*Proof of Corollary 1.5.* Notice that $c_\theta$ and $c_\theta'$ are uniformly bounded for $\theta\in[0,\pi]$. Let $g(\theta):=c_\theta/\sin\theta$. Then,

$$g'(\theta)=\frac{c_\theta'\cdot(-\sin\theta,\cos\theta)}{\sin\theta}-\frac{c_\theta\cos\theta}{\sin^2\theta}.$$

Obviously, there are constants $0<\alpha_1<\beta_1<\pi$ such that $g'(\theta)<0$ for $\theta\in(0,\alpha_1)$ and $g'(\theta)>0$ $\theta\in(\beta_1,\pi)$ since $c_e'$ is bounded for any $e\in\mathbb{S}$ and $\sin\theta\to 0$ as $\theta\to 0$ or $\pi$. One can also notice that $g(\theta)\to+\infty$ as $\theta\to 0$ or $\theta\to\pi$. By continuity, one can take any $\alpha\in(0,\alpha_1)$ and there is $\beta\in(\beta_1,\pi)$ such that $g(\alpha)=g(\beta)$ and $g(\theta)<g(\alpha)=g(\beta)$ for all $\theta\in(\alpha,\beta)$.

Then, the conclusion of Corollary 1.5 follows from Theorem 1.2.

*Proof of Corollary 1.6.* Take two directions $e_1 = (\cos\theta_1, \sin\theta_1)$ and $e_2 = (\cos\theta_2, \sin\theta_2)$ where $\theta_1, \theta_2 \in (0, 2\pi)$. Assume without loss of generality that $\theta_2 > \theta_1$. Rotate the coordinate by changing variables as

$$\begin{cases} X = x\cos\theta + y\sin\theta, \\ Y = -x\sin\theta + y\cos\theta, \end{cases}$$

where $\theta$ varies from $\theta_2 - \pi/2$ to $\theta_1 + \pi/2$. Assume without loss of generality that $\theta_2 - \theta_1 < \pi$. Otherwise, if $\theta_2 - \theta_1 > \pi$, we can take $\theta$ varying from $\theta_2$ to $2\pi + \theta_1$. Then, under the new coordinate, directions $e_1$ and $e_2$ become $(\cos(\theta_1 + \pi/2 - \theta), \sin(\theta_1 + \pi/2 - \theta))$ and $(\cos(\theta_2 + \pi/2 - \theta), \sin(\theta_2 + \pi/2 - \theta))$ where $0 < \theta_1 + \pi/2 - \theta < \theta_2 + \pi/2 - \theta < \pi$. Since $\sin\theta$ is increasing in $[0, \pi/2]$ and decreasing in $[\pi/2, \pi]$, one has that

$$\frac{c_{e_1}}{\sin(\theta_1 + \pi/2 - \theta)} \text{ is increasing from } \frac{c_{e_1}}{\sin(\theta_1 - \theta_2 + \pi)} \text{ to } +\infty \text{ as} \theta \text{ varies from } \theta_2 - \pi/2 \text{ to} \theta_1 + \pi/2,$$

and

$$\frac{c_{e_2}}{\sin(\theta_2 + \pi/2 - \theta)} \text{ is decreasing from } +\infty \text{ to } \frac{c_{e_2}}{\sin(\theta_2 - \theta_1)} \text{ as } \theta \text{ varies from } \theta_2 - \pi/2 \text{ to } \theta_1 + \pi/2.$$

By continuity and for any $0 < \theta_2 - \theta_1 < \pi$, there is $\theta^* \in (\theta_2 - \pi/2, \theta_1 + \pi/2)$ such that

$$\frac{c_{e_1}}{\sin(\theta_1 + \pi/2 - \theta^*)} = \frac{c_{e_2}}{\sin(\theta_2 + \pi/2 - \theta^*)}.$$

On the other hand, by the proof of Corollary 1.5, there is $0 < \alpha_1 < \pi$ small enough such that for $0 < \pi - (\theta_2 - \theta_1) < \alpha_1$, it holds

$$\frac{c_{e_1}}{\sin(\theta_1 + \pi/2 - \theta^*)} = \frac{c_{e_2}}{\sin(\theta_2 + \pi/2 - \theta^*)} > \frac{c_\theta}{\sin(\theta - \theta^*)} \text{ for } \theta_1 + \pi/2 < \theta < \theta_2 + \pi/2.$$

Now, under the new coordinate $(X, Y) = (x\cos\theta^* + y\sin\theta^*, -x\sin\theta^* + y\cos\theta^*)$, one can construct a curve $Y = \psi(X)$ with $x\cos\theta_1 + y\sin\theta_1 = 0$ and $x\cos\theta_2 + y\sin\theta_2 = 0$ (the half parts such that $Y \geq 0$) being its asymptotic lines and define normals $e(X)$ for the curve $Y = \psi(\varrho X)/\varrho$. Then, define a function

$$U^+(t, X, Y) = U_{e(X)}\Big(\frac{Y - c_{e_1e_2}t - \psi(\varrho X)/\varrho}{\sqrt{\psi'^2(\varrho X) + 1}}, x, y\Big) + \varepsilon \operatorname{sech}(\varrho X).$$

By following similar arguments of Lemma 2.8, Theorem 1.2 and Corollary 1.5, one can prove that $U^+(t, X, Y)$ is a supersolution and there is an entire solution $V(t, x, y)$ of (1.1) satisfying (1.11) for all $\alpha_1$ small enough. By taking $\rho = \cos(\pi - \alpha_1) - 1$ and $e_0 = (\cos\theta^*, \sin\theta^*)$, the conclusion of Corollary 1.6 immediately follows.

Now, we show that condition (1.8) without $g'(\alpha) < 0$ and $g'(\beta) > 0$ is necessary for the existence of the curved front in Theorem 1.2.

*Proof of Theorem1.7.* We first prove that

$$\frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta}. \tag{2.47}$$

Assume by contradiction that $c_\alpha/\sin\alpha \neq c_\beta/\sin\beta$. Take a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$. Then, for the sequence

$$(x_n, y_n) = \Big(\frac{(c_\alpha \sin\beta - c_\beta \sin\alpha)t_n}{\sin(\beta-\alpha)}, \frac{(c_\alpha \cos\beta - c_\beta \cos\alpha)t_n}{\sin(\alpha-\beta)}\Big),$$

one has that $x_n^2 + (y_n - c_{\alpha\beta}t_n)^2 \to +\infty$ as $n \to +\infty$ for any $c_{\alpha\beta} \in \mathbb{R}$ since $c_\alpha/\sin\alpha \neq c_\beta/\sin\beta$. Notice that for any $n$, there are $k_n^1, k_n^2 \in \mathbb{Z}$ and $x_n', y_n' \in [0, L_2)$ such that $x_n = k_n^1 L_1 + x_n'$ and $y_n = k_n^2 L_2 + y_n'$. Moreover, up to extract subsequences of $x_n$ and $y_n$, there are $x_*' \in [0, L_1]$ and $y_*' \in [0, L_2]$ such that $x_n' \to x_*'$ and $y_n' \to y_*'$ as $n \to +\infty$. Since $f(x, y, \cdot)$ is $L$-periodic in $(x, y)$, one has $f(x + x_n, y + y_n, \cdot) \to f(x + x_*', y + y_*', \cdot)$ as $n \to +\infty$. Let $v_n(t, x, y) = V(t+t_n, x+x_n, y+y_n)$. By standard parabolic estimates, $v_n(t, x, y)$, up to extract of a subsequence, converges to a solution $v_\infty(t, x, y)$ of

$$v_t - \Delta v = f(x + x_*', y + y_*', v), \text{ for } \ t \in \mathbb{R} \ \text{ and } \ (x, y) \in \mathbb{R}^2. \tag{2.48}$$

By definitions of $x_n$ and $y_n$, one can also have that

$$U_{\alpha\beta}^-(t + t_n, x + x_n, y + y_n) \to \hat{U}_{\alpha\beta}^-(t, x, y), \text{ as } \quad n \to$$
$$+\infty \quad \text{uniformly in} \quad \mathbb{R} \times \mathbb{R}^2,$$

where

$$\hat{U}_{\alpha\beta}^-(t, x, y) := \max\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x + x_*', y + y_*'),$$
$$U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x + x_*', y + y_*')\}.$$

Moreover, by (1.9) and $x_n^2 + (y_n - c_{\alpha\beta}t_n)^2 \to +\infty$ as $n \to +\infty$, one gets that

$$v_n(t, x, y) \to \hat{U}_{\alpha\beta}^-(t, x, y) \text{ as } \quad n \to +\infty \quad \text{locally uniformly in} \mathbb{R} \times \mathbb{R}^2.$$

It implies that $v_\infty(t, x, y) = \hat{U}_{\alpha\beta}^-(t, x, y)$ which is impossible since $\hat{U}_{\alpha\beta}^-(t, x, y)$ is not a solution of (2.48). Therefore, (2.47) holds.

Then, we prove that

$$c_{\alpha\beta} = \frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta}. \tag{2.49}$$

Assume by contradiction that $c_{\alpha\beta} \neq c_\alpha/\sin\alpha$. Take a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = L_2 n \sin\alpha/c_\alpha \to +\infty$ and consider the sequence

$$(x_n, y_n) = (0, \frac{c_\alpha}{\sin\alpha}t_n).$$

Notice that $x_n^2+(y_n-c_{\alpha\beta}t_n)^2\to+\infty$ as $n\to+\infty$ since $c_{\alpha\beta}\neq c_\alpha/\sin\alpha$, $t_nc_\alpha/\sin\alpha=nL_2$ and $U^-_{\alpha\beta}(t+t_n,x+x_n,y+y_n)=U^-_{\alpha\beta}(t,x,y)$. Then, one can make the similar arguments as above to get a contradiction. Thus, (2.49) holds.

At last, we prove that

$$\frac{c_\theta}{\sin\theta}<c_{\alpha\beta}=\frac{c_\alpha}{\sin\alpha}=\frac{c_\beta}{\sin\beta}\ \text{for any}\quad \theta\in(\alpha,\beta).$$

Assume by contradiction that there is $\theta\in(\alpha,\beta)$ such that $c_\theta/\sin\theta\geq c_{\alpha\beta}$. Then, two cases may occur: (i) $c_\theta/\sin\theta>c_{\alpha\beta}$; (ii) $c_\theta/\sin\theta=c_{\alpha\beta}$.

For case (i), take $t=0$ and by (1.9), for any $\varepsilon>0$, there is $R_\varepsilon>0$ such that

$$\sup_{|(x,y)|>R_\varepsilon}\left|V(0,x,y)-U^-_{\alpha\beta}(0,x,y)\right|\leq\varepsilon. \tag{2.50}$$

We claim that

**Claim 2.10.** *There exist constants $\tau\in\mathbb{R}$ and $\delta>0$ such that*

$$V(t,x,y)\geq U_\theta(x\cos\theta+y\sin\theta-c_\theta t+\tau,x,y)-\delta e^{-\delta t},\ \textit{for}\quad t\geq0\quad \textit{and} x\in\mathbb{R}^2.$$

In order to not lengthen the proof, we postpone the proof of this claim after the proof of Theorem 1.7. Take a sequences $(t_n)_{n\in\mathbb{N}}$ such that $t_n\to+\infty$ as $n\to+\infty$ and $y_n=c_{\alpha\beta}t_n+R$ where $R$ is a constant. Then, since $U_e(+\infty,x,y)=0$ for all $e\in\mathbb{S}$ and $(x,y)\in\mathbb{T}^2$, one can take $R$ large enough such that

$$\begin{aligned}U^-_{\alpha\beta}(t_n,0,y_n)&=\max\{U_\alpha(y_n\sin\alpha-c_\alpha t_n,0,y_n),U_\beta(y_n\sin\beta-c_\beta t_n,0,y_n)\}\\&=\max\{U_\alpha(R\sin\alpha,0,y_n),U_\beta(R\sin\beta,0,y_n)\}\leq\frac{1}{4}.\end{aligned}$$

By (1.9) and even if it means increasing $R$, one has that

$$V(t_n,0,y_n)\leq U^-_{\alpha\beta}(t_n,0,y_n)+\frac{1}{4}\leq\frac{1}{2}\ \text{for all}\quad n. \tag{2.51}$$

However, since $c_\theta/\sin\theta>c_{\alpha\beta}$ and hence,

$$y_n\sin\theta-c_\theta t_n=(c_{\alpha\beta}\sin\theta-c_\theta)t_n+R\sin\theta\to-\infty,\ \text{as}\quad n\to+\infty,$$

it follows from Claim 2.10 that

$$V(t_n,0,y_n)\geq U_\theta(y_n\sin\theta-c_\theta t_n+\tau,0,y_n)-\delta e^{-\delta t_n}\to1\ \text{as}\quad n\to+\infty,$$

which contradicts (2.51). Case (i) is ruled out.

Now we consider case (ii). Since $U_e(-\infty,x,y)=1$ and $U_e(+\infty,x,y)=0$ for any $(x,y)\in\mathbb{T}^2$ and $e\in\mathbb{S}$, there is $C>0$ such that

$$0<U_e(\xi,x,y)\leq\sigma\ \text{for}\quad \xi\geq C\quad \text{and}(x,y)\in\mathbb{T}^2,$$

and

$$1-\sigma\leq U_e(\xi,x,y)<1\ \text{for}\quad \xi\leq -C\quad \text{and}(x,y)\in\mathbb{T}^2,$$

where $\sigma$ is defined in (F3). By (1.9), there is $R > 0$ such that

$$V(t,x,y) \le \sigma, \text{ for } (t,x,y) \in \Omega_R^+ \\ \text{and } V(t,x,y) \ge 1-\sigma, \text{ for } (t,x,y) \in \Omega_R^-,$$

where

$$\Omega_R^+ := \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; x \le 0 \text{ and } x\cos\alpha + y\sin\alpha - c_\alpha t \ge c_\alpha R\} \\ \cup \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; \\ x > 0 \text{ and } x\cos\beta + y\sin\beta - c_\beta t \ge c_\beta R\},$$

and

$$\Omega_R^- := \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; x \le 0 \text{ and } x\cos\alpha + y\sin\alpha - c_\alpha t \le -c_\alpha R\} \\ \cup \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; \\ x > 0 \text{ and } x\cos\beta + y\sin\beta - c_\beta t \le -c_\beta R\}.$$

By a similar proof as of Lemma 2.5, there is $0 < \sigma' \le \sigma$ such that

$$\sigma' \le V(t,x,y) \le 1-\sigma', \text{ for } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\Omega_R^+ \cup \Omega_R^-),$$

and

$$\sigma' \le U_e(\xi,x,y) \le 1-\sigma' \text{ for } -C \le \xi \le C, (x,y) \in \mathbb{T}^2 \text{ and any } e \in \mathbb{S}.$$

For any $\tau \in \mathbb{R}$, let $u_\tau(t,x,y) = U_\theta(x\cos\theta + y\sin\theta - c_\theta t + \tau, x, y)$. Let

$$\omega_\tau^+ := \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; x\cos\theta + y\sin\theta - c_\theta t + \tau \ge C\},$$

and

$$\omega_\tau^- := \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; x\cos\theta + y\sin\theta - c_\theta t + \tau \le -C\}.$$

Since $\alpha < \theta < \beta$ and $c_\theta/\sin\theta = c_\alpha/\sin\alpha = c_\beta/\sin\beta$, one can easily check that

$$\mathbb{R}\times\mathbb{R}^2 \setminus (\omega_\tau^+ \cup \omega_\tau^-) \subset \Omega^-_{(C-\tau)/c_\alpha} \text{ and } \mathbb{R}\times\mathbb{R}^2 \setminus (\Omega_R^+ \cup \Omega_R^-) \subset \omega^+_{C+c_\alpha R}.$$

Then, by (1.9), $U_e(-\infty,x,y) = 1$ and $U_e(+\infty,x,y) = 0$, there is $\tau_1 \ge c_\alpha R + C$ large enough such that for any $\tau \ge \tau_1$,

$$V(t,x,y) \ge 1-\sigma' \ge u_\tau(t,x,y), \text{ for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_\tau^+ \cup \omega_\tau^-),$$

and

$$u_\tau(t,x,y) \le \sigma' \le V(t,x,y), \text{ for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\Omega_R^+ \cup \Omega_R^-).$$

Moreover, since $\tau \ge \tau_1 \ge c_\alpha R + C$, one has that

$$V(t,x,y) \ge 1-\sigma \ge \sigma \ge u_\tau(t,x,y), \text{ for all } (t,x,y) \in \omega_\tau^+ \cap \Omega_R^-.$$

Thus, it follows that

$$u_\tau(t,x,y) \le V(t,x,y), \text{ for any } \quad \tau \ge \tau_1 \quad \text{and all} \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2\setminus(\omega_\tau^- \cup \Omega_R^+). \tag{2.52}$$

Also notice that

$$u_\tau(t,x,y),\ V(t,x,y) \ge 1-\sigma \text{ in } \quad \omega_\tau^- \quad \text{and } u_\tau(t,x,y),\ V(t,x,y) \le \sigma \text{ in } \quad \Omega_R^+,$$

and $f(x,y,u)$ is nonincreasing in $u \in (-\infty,\sigma]$ and $u \in [1-\sigma,+\infty)$ for any $(x,y) \in \mathbb{T}^2$ by (1.2). By following similar proof as the proof of [3, Lemma 4.2] which mainly applied the sliding method and the linear parabolic estimates, one can get that

$$V(t,x,y) \ge u_\tau(t,x,y), \text{ in } \quad \omega_\tau^- \quad \text{and} \quad \Omega_R^+.$$

Combined with (2.52), one has that

$$V(t,x,y) \ge u_\tau(t,x,y), \text{ for any } \quad \tau \ge \tau_1 \quad \text{and all} \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2.$$

Let

$$\tau_* = \inf\{\tau \in \mathbb{R};\, u_\tau(t,x,y) \le V(t,x,y) \text{ for all } \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2\}.$$

By above arguments, one knows that $\tau_* < +\infty$. On the other hand, for any fixed $(t,x,y)$, $u_\tau(t,x,y) = U_\theta(x\cos\theta + y\sin\theta - c_\theta t + \tau, x, y) \to 1$ as $\tau \to -\infty$ and $V(t,x,y) < 1$ by the maximum principle. By the definition of $\tau_*$, one also has that $\tau_* > -\infty$. Thus, $|\tau_*|$ is bounded. If

$$\inf\{V(t,x,y) - u_{\tau_*};\, (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)\} > 0,$$

there is $\eta > 0$ such that

$$V(t,x,y) \ge u_{\tau_*-\eta}(t,x,y), \text{ for } \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+).$$

Then, one can follow the above arguments again to get that

$$u_{\tau_*-\eta}(t,x,y) \le V(t,x,y), \text{ for } \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2,$$

which contradicts the definition of $\tau_*$. Thus,

$$\inf\{V(t,x,y) - u_{\tau_*};\, (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)\} = 0.$$

Since $V(t,x,y) \ge \sigma'$ in $\mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)$ and $u_{\tau_*}(t,x,y) = U_\theta(x\cos\theta + y\sin\theta - c_\theta t + \tau_*, x, y) \to 0$ as $x\cos\theta + y\sin\theta - c_\theta t \to +\infty$, there is $R_1 > 0$ and there is a sequence $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{\tau_*}^- \cup \Omega_R^+)$ such that

$$-C - \tau_* \le x_n\cos\theta + y_n\sin\theta - c_\theta t_n \le R_1 \tag{2.53}$$

and

$$V(t_n,x_n,y_n) - U_\theta(x_n\cos\theta + y_n\sin\theta - c_\theta t_n + \tau_*, x_n, y_n) \to 0, \text{ as } \quad n \to +\infty. \tag{2.54}$$

Notice that $x_n$ is bounded. Otherwise, if $x_n \to -\infty$ as $n \to +\infty$, then it follows from (2.53) and $\theta > \alpha$ that

$$\begin{aligned} x_n \cos\alpha + y_n \sin\alpha - c_\alpha t_n &= x_n \cos\alpha + \sin\alpha\Big(y_n - \frac{c_\alpha}{\sin\alpha}t_n\Big) \\ &= x_n \cos\alpha + \sin\alpha\Big(y_n - \frac{c_\theta}{\sin\theta}t_n\Big) \\ &= \Big(\cos\alpha - \frac{\sin\alpha\cos\theta}{\sin\theta}\Big)x_n \\ &\quad + \frac{\sin\alpha}{\sin\theta}\Big(\cos\theta x_n + \sin\theta y_n - c_\theta t_n\Big) \\ &\to -\infty, \qquad \text{as} \quad n \to +\infty, \end{aligned}$$

and $x_n^2 + (y_n - c_{\alpha\beta}t_n)^2 \to +\infty$ as $n \to +\infty$. It implies that $V(t_n, x_n, y_n) \to U^-_{\alpha\beta}(t_n, x_n, y_n) \to 1$ as $n \to +\infty$ which contradicts $u_{\tau_*}(t, x, y) \leq 1 - \sigma'$ in $\mathbb{R} \times \mathbb{R}^2 \setminus \omega^-_{\tau_*}$ and (2.54). Similarly, it is not possible that $x_n \to +\infty$ as $n \to +\infty$. Thus, there is $x_* \in \mathbb{R}$ such that $x_n \to x_*$ as $n \to +\infty$. Let $w(t, x, y) = V(t, x, y) - u_{\tau_*}(t, x, y)$. Then, by (2.54), $w(t_n, x_n, y_n) \to 0$ as $n \to +\infty$. Consider the point $(t_n - 1, x_n - R', y_n - c_\theta/\sin\theta + R'\cos\theta/\sin\theta)$ for some constant $R'$. Notice that by (2.53),

$$\begin{aligned} &(x_n - R')\cos\theta + (y_n - c_\theta/\sin\theta + R'\cos\theta/\sin\theta)\sin\theta \\ &\quad -c_\theta(t_n - 1) \in [-C - \tau_*, R_1], \ \text{for any} \quad n, \end{aligned}$$

and

$$\begin{aligned} &(x_n - R')\cos\alpha + (y_n - c_\theta/\sin\theta + R'\cos\theta/\sin\theta)\sin\alpha \\ &\quad -c_\alpha(t_n - 1) \to -\infty, \ \text{as} \quad R' \to +\infty, \end{aligned}$$

for any $n$. By taking $R'$ large enough, one can let

$$V(t_n - 1, x_n - R', y_n - c_\theta + R'\cos\theta/\sin\theta) \geq 1 - \frac{\sigma'}{2}, \ \text{for any} \quad n.$$

Then, by noticing that $(t_n - 1, x_n - R, y_n - c_\theta + R\cos\theta/\sin\theta)$ satisfies (2.53) and hence $u_{\tau_*}(t_n - 1, x_n - R, y_n - c_\theta + R\cos\theta/\sin\theta) \leq 1 - \sigma'$, one has that

$$w(t_n - 1, x_n - R, y_n - c_\theta + R\cos\theta/\sin\theta) \geq \frac{\sigma'}{2} > 0. \tag{2.55}$$

However, since $V(t, x, y)$ and $u_{\tau_*}(t, x, y)$ are solutions of (1.1), we have that $w(t, x, y)$ satisfies

$$w_t - \Delta w \geq -b(x, y)w, \ \text{for} \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^2,$$

where $b(x, y)$ is bounded. By the linear parabolic estimates, one can get that

$$w(t_n - 1, x_n - R, y_n - c_\theta + R\cos\theta/\sin\theta) \to 0, \ \text{as} \quad n \to +\infty,$$

which contradicts (2.55). Therefore, case (ii) is ruled out.

In conclusion, $c_\theta/\sin\theta < c_{\alpha\beta}$ for any $\theta \in (\alpha, \beta)$.

We finish this section by proving Claim 2.10.

*Proof of Claim 2.10.* Take $\delta > 0$ such that

$$\delta \leq \min\left\{\frac{\sigma}{2}, \lambda\right\},$$

where $\sigma$ and $\lambda$ are defined in (F3). Since $U_\theta(-\infty, x, y) = 1$ and $U_\theta(+\infty, x, y) = 0$ for $(x, y) \in \mathbb{T}^2$, there is $C > 0$ such that

$$\begin{cases} 0 < U_\theta(\xi, x, y) \leq \delta, & \text{for} \quad \xi \geq C \quad \text{and} \quad (x, y) \in \mathbb{T}^2 \\ 1 - \delta \leq U_\theta(\xi, x, y) < 1, & \text{for} \quad \xi \leq -C \quad \text{and} (x, y) \in \mathbb{T}^2. \end{cases}$$

From Lemma 2.5, there is $k > 0$ such that $-\partial_\xi U_\theta(\xi, x, y) \geq k$ for $-C \leq \xi \leq C$ and $(x, y) \in \mathbb{T}^2$. Take $\omega > 0$ such that

$$k\omega \geq \delta + M, \tag{2.56}$$

where $M = \max_{(x,y,u)\in\mathbb{T}^2\times\mathbb{R}} |f_u(x, y, u)|$. It follows from (2.50) and the definition of $U^-_{\alpha\beta}$ that there is $R_\delta > 0$ such that

$$V(0, x, y) \geq 1 - \delta, \text{ for } \quad (x, y) \in \Omega$$

where

$$\begin{aligned} \Omega := &\{x \leq 0, y \in \mathbb{R}; x \cos\alpha + y \sin\alpha \leq -R_\delta\} \\ &\cup\{x \geq 0, y \in \mathbb{R}; x \cos\beta + y \sin\beta \leq -R_\delta\}. \end{aligned}$$

Define

$$v^-(t, x, y) = U_\theta(\xi^-(t, x, y), x, y) - \delta e^{-\delta t},$$

where

$$\xi^-(t, x, y) = x\cos\theta + y\sin\theta - c_\theta t - \omega e^{-\delta t} + \omega + \hat{R}_\delta + C,$$

and $\hat{R}_\delta = R_\delta \sin\theta \max\{1/\sin\alpha, 1/\sin\beta\}$. We prove that $v^-(t, x, y)$ is a subsolution of the problem satisfied by $V(t, x, y)$ for $t \geq 0$ and $(x, y) \in \mathbb{R}^2$.

Firstly, we check the initial data. Since $\alpha < \theta < \beta$, one has that

$$\{(x, y) \in \mathbb{R}^2; \xi^-(0, x, y) \leq C\} \subset \Omega.$$

Then,

$$v^-(0, x, y) \leq 1 - \delta \leq V(0, x, y), \text{ for } \quad (x, y) \in \mathbb{R}^2 \quad \text{such that} \quad \xi^-(0, x, y) \leq C.$$

For $(x, y) \in \mathbb{R}^2$ such that $\xi(0, x, y) \geq C$, one has that

$$v^-(0, x, y) \leq \delta - \delta = 0 \leq V(0, x, y).$$

Thus, $v^-(0, x, y) \leq V(0, x, y)$ for all $(x, y) \in \mathbb{R}^2$.

We then check that

$$Nv := v^-_t - \Delta v^- - f(x, y, v^-) \leq 0, \text{ for } \quad t \geq 0 \quad \text{and} (x, y) \in \mathbb{R}^2.$$

By some computation and (2.2), one has that

$$N v = \omega\delta e^{-\delta t}\partial_\xi U_\theta + \delta^2 e^{-\delta t} + f(x, y, U_\theta) - f(x, y, v^-). \tag{2.57}$$

For $t \geq 0$ and $(x, y) \in \mathbb{R}^2$ such that $\xi(t, x, y) \geq C$, one has that $0 < U_\theta(\xi(t, x, y), x, y) \leq \delta$ and hence $v^-(t, x, y) \leq 2\delta \leq \sigma$. Thus, by (1.2), it follows that

$$f(x, y, U_\theta) - f(x, y, v^-) \leq -\lambda\delta e^{-\delta t}. \tag{2.58}$$

Since $\partial_\xi U_\theta < 0$, it follows from (2.57) and (2.58) that

$$N v \leq \delta^2 e^{-\delta t} - \lambda\delta e^{-\delta t} \leq 0.$$

Similarly, one can prove that $N v \leq 0$ for $t \geq 0$ and $(x, y) \in \mathbb{R}^2$ such that $\xi(t, x, y) \leq -C$. Finally, for $t \geq 0$ and $(x, y) \in \mathbb{R}^2$ such that $-C \leq \xi(t, x, y) \leq C$, one has that $-\partial_\xi U_\theta(\xi(t, x, y), x, y) \geq k$ and

$$f(x, y, U_\theta) - f(x, y, v^-) \leq M\delta e^{-\delta t}, \tag{2.59}$$

where $M = \max_{(x,y,u)\in\mathbb{T}^2\times\mathbb{R}} |f_u(x, y, u)|$. Then, it follows from (2.56), (2.57) and (2.59) that

$$N v \leq -k\omega\delta e^{-\delta t} + \delta^2 e^{-\delta t} + M\delta e^{-\delta t} \leq 0.$$

By the comparison principle, one gets that

$$V(t, x, y) \geq v^-(t, x, y), \text{ for } \quad t \geq 0 \quad \text{and} \quad x \in \mathbb{R}^2.$$

Then, the conclusion of Claim 2.10 follows immediately.

## 3. Uniqueness and Stability of the Curved Front

This section is devoted to the proofs of uniqueness and stability of the curved front in Theorem 1.2, that is, Theorems 1.8 and 1.9.

### *3.1. Proof of Theorem 1.8*

The idea of the proof of the uniqueness is inspired by Berestycki and Hamel [3] who proved that for any two almost-planar fronts $u_1(t, x, y)$ and $u_2(t, x, y)$, there is $T \in \mathbb{R}$ such that either $u_1(t+T, x, y) > u_2(t, x, y)$ or $u_1(t+T, x, y) = u_2(t, x, y)$.

*Proof of Theorem 1.8.* Assume that there is another curved front $V^*(t, x, y)$ satisfying (1.9). By (1.9), there is $R > 0$ large enough such that

$$V(t, x, y),\ V^*(t, x, y) \leq \sigma \text{ for } \quad (t, x, y) \in \omega_t^+$$
$$\text{and } V(t, x, y),\ V^*(t, x, y) \geq 1 - \sigma \text{ for } \quad (t, x, y) \in \omega_t^-,$$

where $\sigma$ is defined in (F3),

$$\omega_t^+ := \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2; x \leq 0 \text{ and}$$

$$x\cos\alpha + y\sin\alpha - c_\alpha t \geq c_\alpha R\} \cup \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2;$$
$$x > 0 \text{ and } x\cos\beta + y\sin\beta - c_\beta t \geq c_\beta R\},$$

and

$$\omega_t^- := \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2; x \leq 0 \text{ and}$$
$$x\cos\alpha + y\sin\alpha - c_\alpha t \leq -c_\alpha R\} \cup \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2;$$
$$x > 0 \text{ and } x\cos\beta + y\sin\beta - c_\beta t \leq -c_\beta R\}.$$

Since $c_\alpha/\sin\alpha = c_\beta/\sin\beta$, one knows that $\omega_t^+$ and $\omega_t^-$ are connected. By a similar proof as of Lemma 2.5, there is $0 < \sigma' \leq \sigma$ such that

$$\sigma' \leq V(t,x,y),\ V^*(t,x,y) \leq 1-\sigma', \text{ in } \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_t^+ \cup \omega_t^-).$$

Then, by taking $\tau$ large enough, one has

$$V^*(t-\tau,x,y) \leq \sigma' \leq V(t,x,y), \text{ for } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_t^+ \cup \omega_t^-),$$

and

$$V(t,x,y) \geq 1-\sigma' \geq V^*(t-\tau,x,y), \text{ for } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_{t-\tau}^+ \cup \omega_{t-\tau}^-).$$

Since

$$V(t,x,y) \geq 1-\sigma \geq \sigma \geq V^*(t,x,y), \text{ for } (t,x,y) \in \omega_{t-\tau}^+ \cap \omega_t^-,$$

This means that

$$V^*(t-\tau,x,y) \leq V(t,x,y), \text{ in } \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_t^+ \cup \omega_{t-\tau}^-).$$

Since $f(x,y,u)$ is nonincreasing in $u \in (-\infty,\sigma]$ and $u \in [1-\sigma,+\infty)$ for $(x,y) \in \mathbb{T}^2$ and by the same line of the proof of [3, Lemma 4.2], one can get that

$$V^*(t-\tau,x,y) \leq V(t,x,y), \text{ for all } (t,x,y) \in \omega_{t-\tau}^- \text{ and } (t,x,y) \in \omega_t^+.$$

and hence,

$$V^*(t-\tau,x,y) \leq V(t,x,y), \text{ for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \text{ and } \tau \text{ large enough.}$$

Now, we decrease $\tau$ and let

$$\tau_* = \inf\{\tau \in \mathbb{R}; V^*(t-\tau,x,y) \leq V(t,x,y), \text{ for } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2\}.$$

Since both $V(t,x,y)$ and $V^*(t,x,y)$ satisfy (1.9), one knows that $\tau_* \geq 0$. Assume that $\tau_* > 0$. If

$$\inf\{V(t,x,y) - V^*(t-\tau_*,x,y); (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_t^+ \cup \omega_{t-\tau_*}^-)\} > 0,$$

then there is $\eta > 0$ such that

$$V^*(t-(\tau_*-\eta),x,y) \leq V(t,x,y), \text{ for } (t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \setminus (\omega_t^+ \cup \omega_{t-\tau_*}^-).$$

By applying above arguments again, one can get that

$$V^*(t-(\tau_*-\eta),x,y)\le V(t,x,y),\ \text{for}\quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2,$$

which contradicts the definition of $\tau_*$. Thus,

$$\inf\{V(t,x,y)-V^*(t-\tau_*,x,y);\,(t,x,y)\in\mathbb{R}\times\mathbb{R}^2\setminus(\omega_t^+\cup\omega_{t-\tau_*}^-)\}=0,$$

and there is a sequence $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}}$ such that

$$V(t_n,x_n,y_n)-V^*(t_n-\tau_*,x_n,y_n)\to 0,\ \text{as}\quad n\to+\infty.$$

Then, by following similar arguments as Step 3 of the proof of Lemma 2.8, one can get a contradiction. Thus, $\tau_*=0$.

Therefore,

$$V(t,x,y)\ge V^*(t,x,y),\ \text{for all}\quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2.$$

The same arguments can be applied by changing positions of $V(t,x,y)$ and $V^*(t,x,y)$, and then, we can get that

$$V^*(t,x,y)\ge V(t,x,y),\ \text{for all}\quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2.$$

In conclusion, $V^*(t,x,y)\equiv V(t,x,y)$.

### *3.2. Stability of the Curved Front*

Take any $0<\alpha<\beta<\pi$ such that Theorem 1.2 holds. Since $g'(\alpha)<0$, one can take $\alpha_1\in(0,\alpha)$ such that

$$\frac{c_\theta}{\sin\theta}>\frac{c_\alpha}{\sin\alpha},\ \text{for}\quad \theta\in[\alpha_1,\alpha].$$

Similar as Lemma 2.2, there is a smooth function $\varphi_1(x)$ with $y=-x\cot\alpha$ and $y=-x\cot\alpha_1$ being its asymptotic lines and there are positive constant $k_3$, $k_4$ and $K_4$ such that

$$\begin{cases}\varphi_1''(x)<0, & \text{for all}\quad x\in\mathbb{R},\\ -\cot\alpha>\varphi_1'(x)>-\cot\alpha_1, & \text{for all}\quad x\in\mathbb{R},\\ k_1\mathrm{sech}(x)\le-\cot\alpha-\varphi_1'(x)\le K_4\mathrm{sech}(x), & \text{for}\quad x<0,\\ k_2\mathrm{sech}(x)\le\varphi_1'(x)+\cot\alpha_1\le K_4\mathrm{sech}(x), & \text{for}\quad x\ge 0,\\ \max(|\varphi_1''(x)|,|\varphi_1'''(x)|)\le K_4\mathrm{sech}(x), & \text{for all}\quad x\in\mathbb{R}.\end{cases}\tag{3.1}$$

Take a constant $\varrho$ which will be determined later. For every point $(x,y)$ on the curve $y=\varphi_1(\varrho x)/\varrho$, there is a unit normal

$$e(x)=(e_1(x),e_2(x))=\Big(-\frac{\varphi_1'(\varrho x)}{\sqrt{\varphi_1'^2(\varrho x)+1}},\frac{1}{\sqrt{\varphi_1'^2(\varrho x)+1}}\Big).$$

For $(x,y)\in\mathbb{R}^2$ and $t\in\mathbb{R}$, take a constant $\varepsilon$ and we define

$$U_1^-(t,x,y)=U_{e(x)}(\underline{\xi}(t,x,y),x,y)-\varepsilon\mathrm{sech}(\varrho x),\tag{3.2}$$

where

$$\underline{\xi}(t,x,y)=\frac{y-c_{\alpha\beta}t-\varphi_1(\varrho x)/\varrho}{\sqrt{\varphi_1'^2(\varrho x)+1}}.$$

**Lemma 3.1.** *There exist* $\varepsilon_0$ *and* $\varrho(\varepsilon_0)$ *such that for any* $0<\varepsilon\le\varepsilon_0$ *and* $0<\varrho\le\varrho(\varepsilon_0)$*, the function* $U_1^-(t,x,y)$ *is a subsolution of* (1.1)*. Moreover, this satisfies*

$$\lim_{R\to+\infty}\sup_{x<-R}\left|U_1^-(t,x,y)-U_\alpha(x\cos\alpha+y\sin\alpha-c_\alpha t,x,y)\right|\le 2\varepsilon, \tag{3.3}$$

*and*

$$U_1^-(t,x,y)\le U_\alpha(x\cos\alpha+y\sin\alpha-c_\alpha t,x,y),\ \textit{for all}\quad t\in\mathbb{R}\quad\textit{and}\quad (x,y)\in\mathbb{R}^2. \tag{3.4}$$

**Proof.** Assume that

$$\varepsilon_0\le\frac{\sigma}{2},$$

where $\sigma>0$ is defined in (F3). More restrictions on $\varepsilon_0$ will be given later. It follows from similar computation as Step 1 of the proof of Lemma 2.8 that

$$\begin{aligned}
NU_1^- :=&(U_1^-)_t-\Delta_{x,y}U_1^- - f(x,y,U_1^-)\\
=&\partial_\xi U_{e(x)}\underline{\xi}_t-\partial_{\xi\xi}U_{e(x)}(\underline{\xi}_x^2+\underline{\xi}_y^2)-2\nabla_{x,y}\partial_\xi U_{e(x)}\cdot(\underline{\xi}_x,\underline{\xi}_y)\\
&-\Delta_{x,y}U_{e(x)}-\partial_\xi U_{e(x)}\underline{\xi}_{xx}\\
&-U''_{e(x)}\cdot e'(x)\cdot e'(x)-U'_{e(x)}\cdot e''(x)-2\partial_\xi U'_{e(x)}\cdot e'(x)\underline{\xi}_x\\
&-2\partial_x U'_{e(x)}\cdot e'(x)-\varepsilon\varrho^2\mathrm{sech}''(\varrho x)-f(x,y,U_1^-)\\
=&(c_{e(x)}+\underline{\xi}_t)\partial_\xi U_{e(x)}-\partial_{\xi\xi}U_{e(x)}(\underline{\xi}_x^2+\underline{\xi}_y^2-1)\\
&-2\partial_x\partial_\xi U_{e(x)}(\underline{\xi}_x-e_1(x))-\partial_\xi U_{e(x)}\underline{\xi}_{xx}\\
&-U''_{e(x)}\cdot e'(x)\cdot e'(x)-U'_{e(x)}\cdot e''(x)\\
&-2\partial_\xi U'_{e(x)}\cdot e'(x)\underline{\xi}_x-2\partial_x U'_{e(x)}\cdot e'(x)\\
&-\varepsilon\varrho^2\mathrm{sech}''(\varrho x)+f(x,y,U_{e(x)})-f(x,y,U_1^-),
\end{aligned}$$

where $U_{e(x)}$, $\partial_\xi U_{e(x)}$, $\partial_{\xi\xi}U_{e(x)}$, $\nabla_{x,y}\partial_\xi U_{e(x)}$, $\Delta_{x,y}U_{e(x)}$, $U''_{e(x)}\cdot e'(x)\cdot e'(x)$, $U'_{e(x)}\cdot e''(x)$, $\partial_\xi U'(e(x))\cdot e'(x)$, $\partial_x U'_{e(x)}\cdot e'(x)$ are taking values at $(\underline{\xi}(t,x,y),x,y)$ and $U_1^-$, $\underline{\xi}_t$, $\underline{\xi}_x$, $\underline{\xi}_y$ are taking values at $(t,x,y)$. Similar as (2.20), (2.21) in the proof of Lemma 2.8, there are $C_5>0$ and $C_6>0$ such that

$$|\partial_{\xi\xi}U_{e(x)}(\underline{\xi}_x^2+\underline{\xi}_y^2-1)|+2|\partial_x\partial_\xi U_{e(x)}(\underline{\xi}_x-e_1(x))|+|\partial_\xi U_{e(x)}\underline{\xi}_{xx}|\le C_5\varrho\,\mathrm{sech}(\varrho x), \tag{3.5}$$

and

$$|U''_{e(x)}\cdot e'(x)\cdot e'(x)|+|U'_{e(x)}\cdot e''(x)|+2|\partial_\xi U'_{e(x)}\cdot e'(x)\underline{\xi}_x|$$

$$+2|\partial_x U'_{e(x)} \cdot e'(x)| \leq C_6 \varrho \operatorname{sech}(\varrho x). \tag{3.6}$$

By a similar proof as of Claim 2.9, we can easily get that

$$c_{e(x)} + \underline{\xi}_t > 0, \text{ for } \quad x \in \mathbb{R}.$$

and there is $C_7 > 0$ such that

$$c_{e(x)} + \underline{\xi}_t = c_{e(x)} - \frac{c_{\alpha\beta}}{\sqrt{\varphi_1'^2(\varrho x) + 1}} \geq C_7 \operatorname{sech}(\varrho x) > 0,$$

$$\text{for} \quad x \quad \text{being negative enough.}$$

Since $\varphi_1'(x)(\varrho x) \to -\cot\alpha_1, e(x) \to (\cos\alpha_1, \sin\alpha_1)$ as $x \to +\infty$ and $c_{\alpha_1}/\sin\alpha_1 > c_\alpha/\sin\alpha$, there is a constant $c > 0$ such that

$$c_{e(x)} + \underline{\xi}_t = c_{e(x)} - \frac{c_{\alpha\beta}}{\sqrt{\varphi_1'^2(\varrho x) + 1}} \geq c > 0, \text{ for all } \quad x \geq 0. \tag{3.7}$$

For $x < 0$, one can make similar arguments as in the proof of Lemma 2.8 to get that $NU_1^- \leq 0$. For $x \geq 0$, one can get from (3.5), (3.6), (3.7) and Lemma 2.1 that

$$\begin{aligned} NU_1^- &\leq c\partial_\xi U_{e(x)} + (C_5 + C_6)\varrho \operatorname{sech}(\varrho x) \\ &\quad + 2\varepsilon\varrho^2 \operatorname{sech}(\varrho x) + f(x, y, U_{e(x)}) - f(x, y, U_1^-). \end{aligned} \tag{3.8}$$

For $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ such that $\underline{\xi}(t, x, y) \geq C$ and $\underline{\xi}(t, x, y) \leq -C$ where $C$ is defined by (2.24), it follows from (F3) and $\varepsilon \leq \varepsilon_0 \leq \sigma/2$ that

$$f(x, y, U_{e(x)}) - f(x, y, U_1^-) \leq -\lambda\varepsilon \operatorname{sech}(\varrho x).$$

Since $\partial_\xi U_e < 0$, one has that

$$NU_1^- \leq \Big((C_5 + C_6)\varrho + 2\varepsilon\varrho^2 - \lambda\varepsilon\Big) \operatorname{sech}(\varrho x) \leq 0,$$

by taking $\varrho(\varepsilon) > 0$ small enough such that

$$(C_5 + C_6)\varrho + 2\varepsilon\varrho^2 - \lambda\varepsilon < 0, \tag{3.9}$$

and $0 < \varrho \leq \varrho(\varepsilon)$. Finally, for $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ such that $-C \leq \underline{\xi}(t, x, y) \leq C$, there is $k > 0$ such that

$$-\partial_\xi U_e(\underline{\xi}, x, y) \geq k \text{ for all } \quad e \in \mathbb{S}.$$

Notice that

$$f(x, y, U_{e(x)}) - f(x, y, U_1^-) \leq M\varepsilon \operatorname{sech}(\varrho x),$$

where $M := \max_{(x,y,u) \in \mathbb{T}^2 \times \mathbb{R}} |f_u(x, y, u)|$. Thus, it follows from (3.8) and (3.9) that

$$NU_1^- \leq -kc + \Big((C_5 + C_6)\varrho + 2\varepsilon\varrho^2 + M\varepsilon\Big)$$

$$\mathrm{sech}(\varrho x) \le -kc + (\lambda + M)\varepsilon \mathrm{sech}(\varrho x) \le 0,$$

by taking $\varepsilon_0 = \min\{\sigma/2, kc/(\lambda + M)\}$ and $0 < \varepsilon \le \varepsilon_0$.

By similar arguments as to those in Step 2 of the proof of Lemma 2.8, one gets that (3.3) holds. The inequality (3.4) can be gotten by comparing $U_1^-(t, x, y)$ with $U_\alpha(x \cos\alpha + y \sin\alpha - c_\alpha t, x, y)$ through similar arguments as in Step 3 of the proof of Lemma 2.8. This completes the proof.

Similarly, since $g'(\beta) > 0$, one can take $\beta_1 \in (\beta, \pi)$ such that

$$\frac{c_\beta}{\sin\beta} < \frac{c_\theta}{\sin\theta}, \text{ for all } \theta \in (\beta, \beta_1].$$

Similarly to as Lemma 2.2, there is a smooth function $\varphi_2(x)$ with $y = -x\cot\beta$ and $y = -x\cot\beta_1$ being its asymptotic lines and there are positive constant $k_5, k_6$ and $K_5$ such that

$$\begin{cases} \varphi_2''(x) < 0, & \text{for all } x \in \mathbb{R}, \\ -\cot\beta_1 > \psi_2'(x) > -\cot\beta, & \text{for all } x \in \mathbb{R}, \\ k_5 \mathrm{sech}(x) \le -\cot\beta_1 - \varphi_2'(x) \le K_5 \mathrm{sech}(x), & \text{for } x < 0, \\ k_6 \mathrm{sech}(x) \le \varphi_2'(x) + \cot\beta \le K_5 \mathrm{sech}(x), & \text{for } x \ge 0, \\ \max(|\varphi_2''(x)|, |\varphi_2'''(x)|) \le K_5 \mathrm{sech}(x), & \text{for all } x \in \mathbb{R}. \end{cases} \tag{3.10}$$

Take a constant $\varrho$ which will be determined later. For every point $(x, y)$ on the curve $y = \varphi_2(\varrho x)/\varrho$, there is a unit normal

$$e(x) = (e_1(x), e_2(x)) = \Big( -\frac{\varphi_2'(\varrho x)}{\sqrt{\varphi_2'^2(\varrho x) + 1}}, \frac{1}{\sqrt{\varphi_2'^2(\varrho x) + 1}} \Big).$$

For $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, take a constant $\varepsilon$ and we define

$$U_2^-(t, x, y) = U_{e(x)}(\underline{\xi}(t, x, y), x, y) - \varepsilon \mathrm{sech}(\varrho x), \tag{3.11}$$

where

$$\underline{\xi}(t, x, y) = \frac{y - c_{\alpha\beta} t - \varphi_2(\varrho x)/\varrho}{\sqrt{\varphi_2'^2(\varrho x) + 1}}.$$

Similarly to Lemma 3.1, we can prove the following lemma:

**Lemma 3.2.** *There exist $\varepsilon_0$ and $\varrho(\varepsilon_0)$ such that for any $0 < \varepsilon \le \varepsilon_0$ and $0 < \varrho \le \varrho(\varepsilon_0)$, the function $U_2^-(t, x, y)$ is a subsolution of (1.1). Moreover, this satisfies*

$$\lim_{R \to +\infty} \sup_{x > R} \Big| U_2^-(t, x, y) - U_\beta(x \cos\beta + y \sin\beta - c_\beta t, x, y) \Big| \le 2\varepsilon,$$

*and*

$$U_2^-(t, x, y) \le U_\beta(x \cos\beta + y \sin\beta - c_\beta t, x, y), \ \textit{for all} \ \ t \in \mathbb{R} \ \ \textit{and} \ \ (x, y) \in \mathbb{R}^2.$$

Then, we need the following sub and supersolutions for the Cauchy problems of (1.1):

**Lemma 3.3.** *For any function $u(t,x,y) \in C^{1,2}(\mathbb{R}\times\mathbb{R}^2)$, if it is a subsolution of (1.1) for $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$ with $u_t > 0$ and for any $0 < \sigma_1 < 1/2$ there is a positive constant $k$ such that*

$$u_t \geq k, \ \textit{for} \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2 \quad \textit{such that} \quad \sigma_1 \leq u(t,x,y) \leq 1-\sigma_1, \tag{3.12}$$

*then for any $0<\delta<\sigma/2$ where $\sigma$ is defined in (F3), there exist positive constants $\omega$ and $\lambda$ such that*

$$\underline{u}(t,x,y) = u(t+\omega\delta e^{-\lambda t} - \omega\delta, x, y) - \delta e^{-\lambda t},$$

*is a subsolution of (1.1) for $t \geq 0$ and $(x,y)\in\mathbb{R}^2$. Similarly, if $u(t,x,y)$ is a smooth supersolution satisfying (3.12), then for any $0<\delta<\sigma/2$, there exist positive constants $\omega$ and $\lambda$ such that*

$$\overline{u}(t,x,y) = u(t-\omega\delta e^{-\lambda t} + \omega\delta, x, y) + \delta e^{-\lambda t}$$

*is a supersolution of (1.1) for $t \geq 0$ and $(x,y)\in\mathbb{R}^2$.*

**Proof.** We only prove for the subsolution. Similar arguments can be applied for the supersolution. Take any $0<\delta<\sigma/2$ where $\sigma$ is defined in (F3). Let $k>0$ such that $u_t \geq k$ for $(t,x,y)\in\mathbb{R}\times\mathbb{R}^2$ such that $\sigma/2 \leq u \leq 1-\sigma/2$. Take $\omega > 0$ such that

$$k\omega \geq \frac{\lambda + M}{\lambda},$$

where $\lambda$ is defined in (F3) and $M := \max_{(x,y,u)\in\mathbb{T}^2\times\mathbb{R}} |f_u(x,y,u)|$.

We then check that

$$N\underline{u} := \underline{u}_t - \Delta_{x,y}\underline{u} - f(x,y,\underline{u}) \leq 0, \ \text{for} \quad t>0 \quad \text{and} \quad (x,y)\in\mathbb{R}^2.$$

By computation, one can get that

$$N\underline{u} = -\omega\delta\lambda e^{-\lambda t}u_t + \delta\lambda e^{-\lambda t} + f(x,y,u(t+\omega\delta e^{-\lambda t}-\omega\delta,x,y)) - f(x,y,\underline{u}).$$

For $t>0$ and $(x,y)\in\mathbb{R}^2$ such that $1-\sigma/2 \leq u(t+\omega\delta e^{-\lambda t}-\omega\delta,x,y) \leq 1$ and $0 \leq u(t+\omega\delta e^{-\lambda t}-\omega\delta,x,y) \leq \sigma/2$ respectively, one has that $\underline{u}(t,x,y) \geq 1-\sigma$ and $\underline{u}(t,x,y) \leq \sigma$ respectively. Then, by (1.2), it follows that

$$f(x,y,u(t+\omega\delta e^{-\lambda t}-\omega\delta,x,y)) - f(x,y,\underline{u}) \leq -\lambda\delta e^{-\lambda t}.$$

Thus, by $u_t > 0$, we have

$$N\underline{u} \leq \delta\lambda e^{-\lambda t} - \lambda\delta e^{-\lambda t} \leq 0.$$

For $t>0$ and $(x,y)\in\mathbb{R}^2$ such that $\delta/2 \leq u(t+\omega\delta e^{-\lambda t},x,y) \leq 1-\sigma/2$, one has that

$$N\underline{u} \leq -k\omega\delta\lambda e^{-\lambda t} + \delta\lambda e^{-\lambda t} + M\delta e^{-\lambda t} \leq 0,$$

by the definition of $\omega$.

This completes the proof.

Now, we are ready to prove the stability of the curved front of Theorem 1.2.

*Proof of Theorem 1.9.* Take any $\delta \in (0, \sigma/2]$. Take $\varepsilon_0 \le \delta/4$ and $\varrho(\varepsilon_0)$ such that Lemmas 2.8, 3.1 and 3.2 hold for any $\varepsilon \in (0, \varepsilon_0]$ and $\varrho \in (0, \varrho(\varepsilon_0)]$. Pick any $\varepsilon \in (0, \varepsilon_0]$. Let $U^+(t, x, y)$, $U_1^-(t, x, y)$ and $U_2^-(t, x, y)$ be defined by (2.14), (3.2) and (3.11) respectively. Let $U_{12}^-(t, x, y) = \max\{U_1^-(t, x, y), U_2^-(t, x, y)\}$. Then, by Lemmas 3.1, 3.2 and similar arguments as Step 2 of the proof of Lemma 2.8, one can get that

$$U_{12}^-(t, x, y) \le U_{\alpha\beta}^-(t, x, y) \text{ for all } \quad t \in \mathbb{R} \quad \text{and} \quad (x, y) \in \mathbb{R}^2, \tag{3.13}$$

and

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha,\beta}t)^2>R^2} \left| U_{12}^-(t, x, y) - U_{\alpha\beta}^-(t, x, y) \right| \le 2\varepsilon. \tag{3.14}$$

By (1.12), there is $R_\delta > 0$ such that

$$U_{\alpha\beta}^-(0, x, y) - \frac{\delta}{2} \le u_0(x, y) \le U_{\alpha\beta}^-(0, x, y) + \frac{\delta}{2}, \text{ for } \quad (x, y) \in \mathbb{R}^2$$
$$\text{such that } \quad x^2 + y^2 > R_\delta^2.$$

By the definition of $\psi(x)$ from Lemma 2.2, one has that

$$\xi(0, x, y) = \frac{y - \psi(\varrho x)/\varrho}{\sqrt{\psi'^2(\varrho x) + 1}} \to -\infty \text{ as } \quad \varrho \to 0 \quad \text{for } \quad x^2 + y^2 \le R_\delta^2,$$

which implies $U_{e(x)}(\xi(0, x, y), x, y) \to 1$ as $\varrho \to 0$ for $x^2 + y^2 \le R_\delta^2$, where $e(x)$ is defined by (2.12). Then, take $\varrho \in (0, \varrho(\varepsilon_0)]$ small enough such that

$$u_0(x, y) \le 1 \le U^+(0, x, y) + \delta \text{ for } \quad (x, y) \in \mathbb{R}^2 \quad \text{such that} \quad x^2 + y^2 \le R_\delta^2. \tag{3.15}$$

Similarly, since $\varphi_1(0) < 0$ and $\varphi_2(0) < 0$, one can take a $\varrho \in (0, \varrho(\varepsilon_0)]$ such that

$$u_0(x, y) \ge 0 \ge U_1^-(0, x, y) - \delta \text{ and } u_0(x, y) \ge 0 \ge U_2^-(0, x, y) - \delta, \tag{3.16}$$

for $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \le R_\delta^2$. Define

$$\begin{aligned} \underline{U}(t, x, y) = \max\{&U_1^-(t + \omega\delta e^{-\lambda t} - \omega\delta, x, y) \\ &-\delta e^{-\lambda t}, U_2^-(t + \omega\delta e^{-\lambda t} - \omega\delta, x, y) - \delta e^{-\lambda t}\}, \end{aligned}$$

and

$$\overline{U}(t, x, y) = U^+(t - \omega\delta e^{-\lambda t} + \omega\delta, x, y) + \delta e^{-\lambda t},$$

where $\omega$, $\delta$ and $\lambda$ are defined in Lemma 3.3. It follows from (2.17) and (3.13) that

$$\underline{U}(0, x, y) \le u_0(x, y) \le \overline{U}(0, x, y), \text{ for } \quad (x, y) \in \mathbb{R}^2 \quad \text{such that} \quad x^2 + y^2 > R_\delta^2.$$

Together with (3.15) and (3.16), one has that

$$\underline{U}(0, x, y) \le u_0(x, y) \le \overline{U}(0, x, y), \text{ for all } \quad (x, y) \in \mathbb{R}^2.$$

On the other hand, by Lemma 2.5, one knows that $U_1^-(t,x,y)$, $U_2^-(t,x,y)$ and $U^+(t,x,y)$ satisfy (3.12). By Lemma 3.3 and the comparison principle, one can get that

$$\underline{U}(t,x,y) \le u(t,x,y) \le \overline{U}(t,x,y), \text{ for } \quad t \ge 0 \quad \text{and} \quad (x,y) \in \mathbb{R}^2.$$

Take a sequence $t_n = L_2 n/c_{\alpha\beta}$ where $L_2$ is the period of $y$. Then, $t_n \to +\infty$ as $n \to +\infty$. By parabolic estimates, the sequence $u_n(t,x,y) := u(t+t_n, x, y+L_2 n)$ converges, locally uniformly in $\mathbb{R} \times \mathbb{R}^2$, to a solution $u_\infty(t,x,y)$ of (1.1). Since $U_1^-(t+t_n, x, y+L_2 n) = U_1^-(t,x,y)$, $U_2^-(t+t_n, x, y+L_2 n) = U_2^-(t,x,y)$ and $U^+(t+t_n, x, y+L_2 n) = U^+(t,x,y)$, one has that

$$\begin{aligned}
\max\{U_1^-(t+\omega\delta e^{-\lambda(t+t_n)} - \omega\delta, x, y) - \delta e^{-\lambda(t+t_n)},&\\
U_2^-(t+\omega\delta e^{-\lambda(t+t_n)} - \omega\delta, x, y) - \delta e^{-\lambda(t+t_n)}\}&\\
\le u_n(t,x,y) \le U^+(t-\omega\delta e^{-\lambda(t+t_n)}&\\
+ \omega\delta, x, y) + \delta e^{-\lambda(t+t_n)},&
\end{aligned} \tag{3.17}$$

and by passing to the limit $n \to +\infty$, $u_\infty(t,x,y)$ satisfies

$$\begin{aligned}
&\max\{U_1^-(t-\omega\delta, x, y), U_2^-(t-\omega\delta, x, y)\} \le u_\infty(t,x,y)\\
&\le U^+(t+\omega\delta, x, y), \text{ for } \quad (t,x,y) \in \mathbb{R} \times \mathbb{R}^2.
\end{aligned}$$

Let $u(t+t_0, x, y; u_0(x,y))$ denote the solution of the initial value problem

$$\begin{cases} u_t - \Delta u = f(x,y,u), \quad t > t_0, \ (x,y) \in \mathbb{R}^2, \\ u(t_0, x, y) = u_0(x,y), \quad t > t_0. \end{cases} \tag{3.18}$$

Then, by the comparison principle, one can get that

$$U_{\alpha\beta}^-(t+t_0+\omega\delta, x, y) \le u(t+t_0, x, y; U^+(t_0+\omega\delta, x, y)) \le U^+(t+t_0+\omega\delta, x, y),$$

for $t \ge t_0$ and $(x,y) \in \mathbb{R}^2$. By uniqueness of the curved front, one can easily prove that

$$\begin{aligned}
&u(t+t_0, x, y; U^+(t_0+\omega\delta, x, y)) - V(t+t_0+\omega\delta, x, y) \to 0 \text{ as}\\
&\quad t \to +\infty \quad \text{for} \quad (x,y) \in \mathbb{R}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&u(t+t_0, x, y; U_{12}^-(t_0-\omega\delta, x, y)) - V(t+t_0-\omega\delta, x, y) \to 0 \text{ as}\\
&\quad t \to +\infty \quad \text{for} \quad (x,y) \in \mathbb{R}^2.
\end{aligned}$$

Thus, for any fixed $t$ and any $t_0 < t$,

$$\begin{aligned}
&u(t,x,y; U_{12}^-(t_0-\omega\delta, x, y)) \le u_\infty(t,x,y)\\
&\le u(t,x,y; U^+(t_0+\omega\delta, x, y)), \text{ for } \quad (x,y) \in \mathbb{R}^2.
\end{aligned}$$

By passing to the limit $t_0 \to -\infty$, then one has that

$$V(t - \omega\delta, x, y) \le u_\infty(t, x, y) \le V(t + \omega\delta, x, y).$$

Since $\delta$ can be taken arbitrary small, we have that $u_\infty(t, x, y) \equiv V(t, x, y)$. Thus, for any $\eta > 0$, it follows from (1.9), (3.14), (3.17), Lemma 2.8 and taking $\delta$ small enough that there is $t_0 > 0$ large enough such that

$$|u(t_0, x, y) - V(t_0, x, y)| \le \eta, \text{ for all } (x, y) \in \mathbb{R}^2.$$

Then, by $V_t > 0$ and a similar proof as of Lemma 2.5, one knows that $V(t, x, y)$ satisfies (3.12). By Lemma 3.3 again and the comparison principle, one gets that

$$\begin{aligned} &V(t_0 + t + \omega\eta e^{-\lambda t} - \omega\eta, x, y) - \eta e^{-\lambda t} \le u(t_0 + t, x, y) \\ &\le V(t_0 - \omega\eta e^{-\lambda t} + \omega\eta, x, y) + \eta e^{-\lambda t}, \end{aligned}$$

for $t \ge 0$ and $(x, y) \in \mathbb{R}^2$. Then, since $\eta$ can be arbitrary small, one finally has that

$$u(t, x, y) \to V(t, x, y), \text{ as } t \to +\infty \text{uniformly in } \mathbb{R} \times \mathbb{R}^2.$$

This completes the proof.

## 4. A Curved Front with Varying Interfaces

In this section, we construct a curved front with varying interfaces. It behaves as three pulsating fronts as $t \to -\infty$ and as two pulsating fronts as $t \to +\infty$. We can not apply the idea of Hamel [17] by considering a Neumann boundary problem in the half plane $x < 0$ since our problem is not orthogonal symmetric with respect to $y$-axis in general.

Let $\alpha$, $\beta$ and $\theta$ satisfy Theorem 1.10. We will need the following properties:

**Lemma 4.1.** *It holds that*

$$c_{\alpha\theta} e_{\alpha\theta} = \left( \frac{c_\alpha \sin\theta - c_\theta \sin\alpha}{\sin(\theta - \alpha)}, \frac{c_\alpha \cos\theta - c_\theta \cos\alpha}{\sin(\alpha - \theta)} \right) := (c_1, c_2),$$

*and*

$$c_{\beta\theta} e_{\beta\theta} = \left( \frac{c_\beta \sin\theta - c_\theta \sin\beta}{\sin(\theta - \beta)}, \frac{c_\beta \cos\theta - c_\theta \cos\beta}{\sin(\beta - \theta)} \right) := (\hat{c}_1, \hat{c}_2),$$

*with* $c_1 > 0$ *and* $\hat{c}_1 < 0$. *Moreover,*

$$\begin{aligned} &\frac{c_\alpha}{e_{\alpha\theta} \cdot (\cos\alpha, \sin\alpha)} = \frac{c_\theta}{e_{\alpha\theta} \cdot (\cos\theta, \sin\theta)} \\ &= c_{\alpha\theta} > \frac{c_{\theta_1}}{e_{\alpha\theta} \cdot (\cos\theta_1, \sin\theta_1)}, \textit{ for any } \theta_1 \in (\alpha, \theta), \end{aligned}$$

*and*

$$\begin{aligned} &\frac{c_\beta}{e_{\beta\theta} \cdot (\cos\beta, \sin\beta)} = \frac{c_\theta}{e_{\beta\theta} \cdot (\cos\theta, \sin\theta)} \\ &= c_{\beta\theta} > \frac{c_{\theta_2}}{e_{\beta\theta} \cdot (\cos\theta_2, \sin\theta_2)}, \textit{ for any } \theta_2 \in (\theta, \beta). \end{aligned}$$

**Proof.** Assume by contradiction that $c_{\alpha\theta}e_{\alpha\theta} \neq (c_1, c_2)$. Take a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$. Then, for the sequence

$$(x_n, y_n) = (c_1, c_2)t_n,$$

one has that $((x_n, y_n) - c_{\alpha\theta}e_{\alpha\theta}t_n)^2 \to +\infty$ as $n \to +\infty$ since $c_{\alpha\theta}e_{\alpha\theta} \neq (c_1, c_2)$. Notice that for any $n$, there are $k_n^1, k_n^2 \in \mathbb{Z}$ and $x_n' \in [0, L_1]$, $y_n' \in [0, L_2)$ such that $x_n = k_n^1 L_1 + x_n'$ and $y_n = k_n^2 L_2 + y_n'$. Moreover, up to extract subsequences of $x_n$ and $y_n$, there are $x_*' \in [0, L_1]$ and $y_*' \in [0, L_2]$ such that $x_n' \to x_*'$ and $y_n' \to y_*'$ as $n \to +\infty$. Since $f(x, y, \cdot)$ is $L$-periodic in $(x, y)$, one has $f(x + x_n, y + y_n, \cdot) \to f(x + x_*', y + y_*', \cdot)$ as $n \to +\infty$. Let $v_n(t, x, y) = V_{\alpha\theta}(t + t_n, x + x_n, y + y_n)$. By standard parabolic estimates, $v_n(t, x, y)$, up to extract of a subsequence, converges to a solution $v_\infty(t, x, y)$ of

$$v_t - \Delta v = f(x + x_*', y + y_*', v), \text{ for } \quad t \in \mathbb{R} \quad \text{and} \quad (x, y) \in \mathbb{R}^2. \tag{4.1}$$

By definitions of $x_n$, $y_n$, $c_1$ and $c_2$, one can also have that

$$U_{\alpha\theta}^-(t + t_n, x + x_n, y + y_n) \to \hat{U}_{\alpha\theta}^-(t, x, y), \text{ as } \quad n \to +\infty \quad \text{uniformly in} \quad \mathbb{R} \times \mathbb{R}^2,$$

where

$$\hat{U}_{\alpha\theta}^-(t, x, y) := \max\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x + x_*', y + y_*'),\\ U_\theta(x\cos\theta + y\sin\theta - c_\theta t, x + x_*', y + y_*')\}.$$

Moreover, since $V_{\alpha\theta}(t, x, y)$ satisfies

$$\lim_{R\to+\infty} \sup_{((x,y)-c_{e_1e_2}te_{\alpha\theta})^2>R^2} \left|V_{\alpha\theta}(t, x, y) - U_{\alpha\theta}^-(t, x, y)\right| = 0,$$

one then gets that

$$v_n(t, x, y) \to \hat{U}_{\alpha\theta}^-(t, x, y) \text{ as } \quad n \to +\infty \quad \text{locally uniformly in} \quad \mathbb{R}^2.$$

This implies that $v_\infty(t, x, y) = \hat{U}_{\alpha\theta}^-(t, x, y)$ which is impossible since $\hat{U}_{\alpha\theta}^-(t, x, y)$ is not a solution of (4.1). Thus, $c_{\alpha\theta}e_{\alpha\theta} = (c_1, c_2)$. Similarly, one can prove that $c_{\beta\theta}e_{\beta\theta} = (\hat{c}_1, \hat{c}_2)$.

The signs of $c_1$ and $\hat{c}_1$ can be easily gotten from the facts $\alpha < \theta < \beta$ and $c_\alpha/\sin\alpha = c_\beta/\sin\beta > c_\theta/\sin\theta$.

Notice that the speed of the pulsating front $U_{\theta_1}(x\cos\theta_1 + y\sin\theta_1 - c_{\theta_1}t, x, y)$ in direction $e_{\alpha\theta}$ can be denoted by

$$\frac{c_{\theta_1}}{e_{\alpha\theta}\cdot(\cos\theta_1, \sin\theta_1)}.$$

By similar arguments as to those of Theorem 1.7, one has that

$$\frac{c_\alpha}{e_{\alpha\theta}\cdot(\cos\alpha, \sin\alpha)} = \frac{c_\theta}{e_{\alpha\theta}\cdot(\cos\theta, \sin\theta)}\\ = c_{\alpha\theta} \text{ and } \frac{c_{\theta_1}}{e_{\alpha\theta}\cdot(\cos\theta_1, \sin\theta_1)} < c_{\alpha\theta}, \text{ for any } \quad \theta_1 \in (\alpha, \theta).$$

This completes the proof.

Let $\varphi_1(x)$ be a smooth function such that there exist $a_1 < 0 < b_1$ such that

$$\varphi_1(x) = -x\cot\alpha,\ \text{for}\quad x \le a_1, \varphi_1(x) = -x\cot\theta,$$
$$\text{for}\quad x \ge b_1 \quad \text{and}\ \varphi_1''(x) > 0 \text{ for}\quad x \in (a_1, b_1).$$

Let $\varphi_2(x)$ be a smooth function such that there exist $a_2 < 0 < b_2$ such that

$$\varphi_2(x) = -x\cot\theta,\ \text{for}\quad x \le a_2, \varphi_2(x) = -x\cot\beta,\ \text{for}\quad x \ge b_2 \text{and}\ \varphi_2''(x) > 0 \text{ for}\quad x \in (a_2, b_2).$$

Let

$$\psi_1(t,x) = \varphi_1(x - c_1 t) + \rho\mathrm{sech}(x - c_1 t) + \rho\mathrm{sech}(x - \hat{c}_1 t),$$

and

$$\psi_2(t,x) = \varphi_2(x - \hat{c}_1 t) + \rho\mathrm{sech}(x - c_1 t) + \rho\mathrm{sech}(x - \hat{c}_1 t).$$

By $c_1 > 0$, $\hat{c}_1 < 0$ and making $|a_1|$, $|a_2|$, $b_1, b_2$ large enough and $\rho$ small enough, one can let $(\psi_1)_{xx} > 0$ for $t$ negative enough and $x \le (c_1 + \hat{c}_1)t/2$ and $(\psi_2)_{xx} > 0$ for $t$ negative enough and $x \ge (c_1 + \hat{c}_1)t/2$. Let

$$\psi(t,x) = \begin{cases} \psi_1(t,x), & \text{for}\quad x \le (c_1 + \hat{c}_1)t/2, \\ \psi_2(t,x), & \text{for}\quad x > (c_1 + \hat{c}_1)t/2. \end{cases} \tag{4.2}$$

Take a constant $\varrho$ to be determined. For every point on the curve $y = \psi_1(\varrho t, \varrho x)$, there is a unit normal

$$e(t,x) = (e_1(t,x), e_2(t,x)) = \left( -\frac{(\psi_1)_x(\varrho t, \varrho x)}{\sqrt{(\psi_1)_x^2(\varrho t, \varrho x) + 1}}, \frac{1}{\sqrt{(\psi_1)_x^2(\varrho t, \varrho x) + 1}} \right).$$

For every point on the curve $y = \psi_2(\varrho t, \varrho x)$, there is a unit normal

$$\eta(t,x) = \left( -\frac{(\psi_2)_x(\varrho t, \varrho x)}{\sqrt{(\psi_2)_x^2(\varrho t, \varrho x) + 1}}, \frac{1}{\sqrt{(\psi_2)_x^2(\varrho t, \varrho x) + 1}} \right).$$

Take $\varepsilon > 0$ to be determined. For $t \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$, define

$$\widetilde{U}^+(t,x,y) := \begin{cases} U_{e(t,x)}\left(\xi_1(t,x,y), x, y\right) + \varepsilon\mathrm{sech}(\varrho(x - c_1 t)) \\ \qquad\qquad + \varepsilon\mathrm{sech}(\varrho(x - \hat{c}_1 t)),\ \text{for}\quad x \le \dfrac{c_1 + \hat{c}_1}{2} t, \\ U_{\eta(t,x)}\left(\xi_2(t,x,y), x, y\right) + \varepsilon\mathrm{sech}(\varrho(x - c_1 t)) \\ \qquad\qquad + \varepsilon\mathrm{sech}(\varrho(x - \hat{c}_1 t)),\ \text{for}\quad x > \dfrac{c_1 + \hat{c}_1}{2} t, \end{cases}$$

where

$$\xi_1(t,x,y) := \frac{y - c_2 t - \psi_1(\varrho t, \varrho x)/\varrho}{\sqrt{(\psi_1)_x^2(\varrho t, \varrho x) + 1}} \text{ and } \xi_2(t,x,y) := \frac{y - \hat{c}_2 t - \psi_2(\varrho t, \varrho x)/\varrho}{\sqrt{(\psi_2)_x^2(\varrho t, \varrho x) + 1}}.$$

By the definition of $\psi_1$, $\psi_2$, $c_1$, $c_2$, $\hat{c}_1$ and $\hat{c}_2$, one can easily check that around $x = (c_1 + \hat{c}_1)t/2$,

$$\begin{aligned} &U_{e(t,x)}\left(\xi_1(t,x,y),x,y\right) = U_\theta(x\cos\theta + y\sin\theta - c_\theta t, x, y) \\ &= U_{\eta(t,x)}\left(\xi_2(t,x,y),x,y\right), \end{aligned}$$

for $t$ negative enough. Thus, $\widetilde{U}^+(t,x,y)$ is smooth for $t$ negative enough and $(x,y) \in \mathbb{R}^2$.

**Lemma 4.2.** *There exist* $\varepsilon_0$ *and* $\varrho(\varepsilon_0)$ *such that for any* $0 < \varepsilon \le \varepsilon_0$ *and* $0 < \varrho \le \varrho(\varepsilon_0)$*, the function* $\widetilde{U}^+(t,x,y)$ *is a supersolution of* (1.1) *for t negative enough. Moreover, this satisfies*

$$\lim_{R\to+\infty} \sup_{x\le 0, ((x,y)-c_{\alpha\theta}e_{\alpha\theta}t)^2 > R^2} \left|\widetilde{U}^+(t,x,y) - U^-_{\alpha\theta}(t,x,y)\right| \le 2\varepsilon, \tag{4.3}$$

$$\lim_{R\to+\infty} \sup_{x> 0, ((x,y)-c_{\beta\theta}e_{\beta\theta}t)^2 > R^2} \left|\widetilde{U}^+(t,x,y) - U^-_{\theta\beta}(t,x,y)\right| \le 2\varepsilon, \tag{4.4}$$

*and*

$$\begin{aligned} \widetilde{U}^+(t,x,y) \ge \max\{&U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y), \\ &U_\theta(x\cos\theta + y\sin\theta - c_\theta t, x, y), \\ &U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)\}, \\ &\textit{for} \quad t\textit{negative enough and}(x,y) \in \mathbb{R}^2. \end{aligned} \tag{4.5}$$

**Proof.** We only prove for the part $x \le (c_1 + \hat{c}_1)t/2$. Take $0 < \varepsilon_0 \le \sigma/2$ and more restrictions on $\varepsilon_0$ will be given later. Change variables $X = x - c_1 t$ and $Y = y - c_2 t$. Then,

$$\psi_1(t,X) := \varphi_1(X) + \rho\mathrm{sech}(X) + \rho\mathrm{sech}(X + (c_1 - \hat{c}_1)t),$$

and

$$e(t,X) = (e_1(t,X), e_2(t,X)) = \left(-\frac{(\psi_1)_X(\varrho t, \varrho X)}{\sqrt{(\psi_1)_X^2(\varrho t, \varrho X) + 1}}, \frac{1}{\sqrt{(\psi_1)_X^2(\varrho t, \varrho X) + 1}}\right).$$

One can compute that

$$\begin{aligned} (\psi_1)_t(t,X) &= (c_1 - \hat{c}_1)\rho\mathrm{sech}'(X + (c_1 - \hat{c}_1)t), \\ (\psi_1)_X(t,X) &= \varphi_1'(X) + \rho\mathrm{sech}'(X) + \rho\mathrm{sech}'(X + \varrho(c_1 - \hat{c}_1)t), \\ (\psi_1)_{tX}(t,X) &= (c_1 - \hat{c}_1)\rho\mathrm{sech}'(X + (c_1 - \hat{c}_1)t), \\ (\psi_1)_{XX}(t,X) &= \varphi_1''(X) + \rho\mathrm{sech}''(X) + \rho\mathrm{sech}''(X + (c_1 - \hat{c}_1)t), \\ (\psi_1)_{XXX}(t,X) &= \varphi_1'''(X) + \rho\mathrm{sech}'''(X) + \rho\mathrm{sech}'''(X + (c_1 - \hat{c}_1)t). \end{aligned}$$

This means that there exists a positive constant $C$ such that the $L^\infty$ norms of above derivatives of $\psi_1(t,X)$ are bounded by $C(\mathrm{sech}(X) + \mathrm{sech}(X + (c_1 - \hat{c}_1)t))$. One can also compute that

$$e_t = \left(-\frac{\varrho(\psi_1)_{tX}}{((\psi_1)_X^2 + 1)^{3/2}}, -\frac{\varrho(\psi_1)_X(\psi_1)_{tX}}{((\psi_1)_X^2 + 1)^{3/2}}\right),$$

$$e_X = \Big( -\frac{\varrho(\psi_1)_{XX}}{((\psi_1)_X^2+1)^{\frac{3}{2}}}, -\frac{\varrho(\psi_1)_X(\psi_1)_{XX}}{((\psi_1)_X^2+1)^{\frac{3}{2}}}\Big),$$

and

$$\begin{aligned} e_{XX} =\Big( & -\frac{\varrho^2(\psi_1)_{XXX}}{((\psi_1)_X^2+1)^{\frac{3}{2}}} + \frac{3\varrho^2(\psi_1)_X(\psi_1)_{XX}^2}{((\psi_1)_X^2+1)^{\frac{5}{2}}}, -\frac{\varrho^2(\psi_1)_{XX}^2}{((\psi_1)_X^2+1)^{\frac{3}{2}}} \\ & -\frac{\varrho^2(\psi_1)_X(\psi_1)_{XXX}}{((\psi_1)_X^2+1)^{\frac{3}{2}}} + \frac{3\varrho^2(\psi_1)_X^2(\psi_1)_{XX}^2}{((\psi_1)_X^2+1)^{\frac{5}{2}}}\Big), \end{aligned}$$

where $(\psi_1)_X$, $(\psi_1)_{XX}$, $(\psi_1)_{XXX}$, $(\psi_1)_{tX}$ are taking values at $(\varrho t, \varrho X)$ in $e_t$, $e_X$, $e_{XX}$. Let

$$\begin{aligned} &\widetilde{U}^+(t,X,Y) = U_{e(t,X)}\,(\xi_1(t,X,Y), X+c_1t, Y+c_2t) + \varepsilon \operatorname{sech}(\varrho X) \\ &+\varepsilon \operatorname{sech}(\varrho X - \varrho(c_1-\hat{c}_1)t), \end{aligned}$$

where

$$\xi_1(t,X,Y) = \frac{Y - \psi_1(\varrho t, \varrho X)/\varrho}{\sqrt{(\psi_1)_x^2(\varrho t,\varrho X)+1}}.$$

We need to verify that

$$N\widetilde{U}^+ := \widetilde{U}^+_t - \Delta_{X,Y}\widetilde{U}^+ - c_1\widetilde{U}^+_X - c_2\widetilde{U}^+_Y - f(X+c_1t, Y+c_2t, \widetilde{U}^+) \geq 0,$$

for $t$ negative enough and $(x,y)\in\mathbb{R}^2$. By (2.2) and after some computation, one can get that

$$\begin{aligned} N\widetilde{U}^+ =& \partial_\xi U_{e(t,X)}((\xi_1)_t - c_1(\xi_1)_X - c_2(\xi_1)_Y + c_{e(t,X)}) + U'_{e(t,X)}\cdot e_t \\ & - U''_{e(t,X)}\cdot e_X\cdot e_X - U'_{e(t,X)}\cdot e_{XX} \\ & - 2\partial_\xi U'_{e(t,X)}\cdot e_X(\xi_1)_X - 2\partial_X U'_{e(t,X)}\cdot e_X \\ & - \partial_{\xi\xi}U_{e(t,X)}((\xi_1)_X^2 + (\xi_1)_Y^2 - 1) \\ & - 2\partial_\xi\partial_X U_{e(t,X)}((\xi_1)_X - e_1(t,X)) - 2\partial_\xi\partial_Y \\ & U_{e(t,X)}((\xi_1)_Y - e_2(t,X)) - \partial_\xi U_{e(t,X)}\xi_{XX} \\ & - c_1U'_{e(t,X)}\cdot e_X - \varepsilon\varrho^2\operatorname{sech}''(\varrho X) \\ & - \varepsilon\varrho^2\operatorname{sech}''(\varrho X - \varrho(c_1-\hat{c}_1)t) - c_1\varepsilon\varrho\operatorname{sech}'(\varrho X) \\ & - c_1\varepsilon\varrho\operatorname{sech}'(\varrho X - \varrho(c_1-\hat{c}_1)t) \\ & + f(X+c_1t, Y+c_2t, U_{e(t,X)}) - f(X+c_1t, Y+c_2t, \widetilde{U}^+), \end{aligned}$$

where $\partial_\xi U_{e(t,X)}$, $\partial_{\xi\xi}U_{e(t,X)}$, $\nabla_{X,Y}\partial_\xi U_{e(t,X)}$, $U'_{e(t,X)}\cdot e_t$, $U''_{e(t,X)}\cdot e_X\cdot e_X$, $U'_{e(t,X)}\cdot e_{XX}$, $\partial_\xi U'_{e(t,X)}\cdot e_X$, $\partial_X U'_{e(t,X)}\cdot e_X$, $U'_{e(t,X)}\cdot e_X$, $U_{e(t,X)}$ are taking values at

$(\xi_1(t,X,Y),X,Y)$ and $\widetilde{U}^+$, $(\xi_1)_t$, $(\xi_1)_X$, $(\xi_1)_Y$ are taking values at $(t,X,Y)$. Similarly to the as those formulas of (2.18), one can also compute that

$$\begin{aligned}
(\xi_1)_t &= -\frac{\varrho(\psi_1)_X(\psi_1)_{tX}}{(\psi_1)_X^2+1}\xi_1 - \frac{(\psi_1)_t}{\sqrt{(\psi_1)_X^2+1}},\\
(\xi_1)_X &= -\frac{\varrho(\psi_1)_X(\psi_1)_{XX}}{(\psi_1)_X^2+1}\xi_1 - \frac{(\psi_1)_X}{\sqrt{(\psi_1)_X^2+1}},\\
(\xi_1)_Y &= \frac{1}{\sqrt{(\psi_1)_X^2+1}},\\
(\xi_1)_{XX} &= -\frac{\varrho^2(\psi_1)_X(\psi_1)_{XXX}}{(\psi_1)_X^2+1}\xi_1 + \frac{\varrho^2(\psi_1)_{XX}^2(2(\psi_1)_X^2-1)}{((\psi_1)_X^2+1)^2}\xi_1\\
&+\frac{\varrho((\psi_1)_X^2-1)(\psi_1)_{XX}}{((\psi_1)_X^2+1)^{\frac{3}{2}}},\\
(\xi_1)_X^2+(\xi_1)_Y^2-1 &= \left(\frac{\varrho(\psi_1)_X^2(\psi_1)_{XX}}{(\psi_1)_X^2+1}\right)^2\xi_1^2 + 2\frac{\varrho(\psi_1)_X(\psi_1)_{XX}}{((\psi_1)_X^2+1)^{\frac{3}{2}}}\xi_1,
\end{aligned} \tag{4.6}$$

where $(\psi_1)_X$, $(\psi_1)_t$ $(\psi_1)_{XX}$, $(\psi_1)_{tX}$ are taking values at $(\varrho t,\varrho X)$. By Lemma 2.1, Lemma 2.4, Lemma 2.7, boundedness of $\|U_e'\|$, $\|U_e''\|$, $\|\partial_\xi U_e'\|$, $\|\partial_x U_e'\|$ and above formulas, there are constants $C_8>0$ and $C_9>0$ such that

$$\begin{aligned}
&|\partial_{\xi\xi}U_{e(t,X)}((\xi_1)_X^2+(\xi_1)_Y^2-1)| + 2|\partial_X\partial_\xi U_{e(t,X)}((\xi_1)_X-e_1(t,X))|\\
&+2|\partial_Y\partial_\xi U_{e(t,X)}((\xi_1)_y-e_2(t,X))|\\
&+|\partial_\xi U_{e(t,X)}(\xi_1)_{XX}| \le C_8\varrho(\mathrm{sech}(\varrho X)+\mathrm{sech}(\varrho X-\varrho(c_1-\hat{c}_1)t)),
\end{aligned}$$

and

$$\begin{aligned}
&|U'_{e(t,X)}\cdot e_t| + |U''_{e(t,X)}\cdot e_X\cdot e_X| + |U'_{e(t,X)}\cdot e_{XX}| + 2|\partial_\xi U'_{e(t,X)}\cdot e_X(\xi_1)_X|\\
&+2|\partial_x U'_{e(t,X)}\cdot e_X|\\
&+c_1|U'_{e(t,X)}\cdot e_X| \le C_9\varrho(\mathrm{sech}(\varrho X)+\mathrm{sech}(\varrho X-\varrho(c_1-\hat{c}_1)t)),
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
N\widetilde{U}^+ \ge&\partial_\xi U_{e(t,X)}((\xi_1)_t-c_1(\xi_1)_X-c_2(\xi_1)_Y+c_{e(t,X)}) - (C_8+C_9)\varrho(\mathrm{sech}(\varrho X)\\
&+\varrho\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) - (1+c_1)\varepsilon\varrho^2(\mathrm{sech}(\varrho X)\\
&+\varrho\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t))\\
&+f(X+c_1t,Y+c_2t,U_{e(t,x)}) - f(X+c_1t,Y+c_2t,\widetilde{U}^+)
\end{aligned}$$

We claim that

**Claim 4.3.** *There exist positive constants $C_{10}$ and $C_{11}$ such that*

$$c_1(\xi_1)_X + c_2(\xi_1)_Y - (\xi_1)_t - c_{e(t,X)} \ge -C_{10}\varrho(\,\mathit{sech}(\varrho X)$$

$$
\begin{aligned}
&+\ sech(\varrho X+\varrho(c_1-\hat{c}_1)t))|\xi_1| \\
&-C_{10}\varrho\ sech(\varrho X+\varrho(c_1-\hat{c}_1)t))+C_{11}(\ sech(\varrho X) \\
&+\ sech(\varrho X+\varrho(c_1-\hat{c}_1)t)).
\end{aligned}
$$

In order to not lengthen the proof, we postpone the proof of Claim 4.3 after the proof of this lemma.

For $\xi_1(t,X,Y)\geq C$ and $\xi_1(t,X,Y)\leq -C$ where $C$ is defined by (2.24), it follows from (1.2) that

$$
\begin{aligned}
&f(X+c_1t,Y+c_2t,U_{e(t,x)})-f(X+c_1t,Y+c_2t,\widetilde{U}^+) \\
&\geq \lambda\varepsilon(\mathrm{sech}(\varrho x)+\mathrm{sech}(\varrho x+\varrho(c_1-\hat{c}t)))
\end{aligned}
$$

Then, by $\partial_\xi U_e<0$, Lemma 2.4 and Claim 4.3, it follows that

$$
\begin{aligned}
N\widetilde{U}^+\geq &-B_1C_{10}\varrho(\mathrm{sech}(\varrho X)+\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) \\
&-B_2C_{10}\varrho\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) \\
&-\Big((C_8+C_9)\varrho+(1+c_1)\varepsilon\varrho^2\Big)(\mathrm{sech}(\varrho X)+\varrho\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) \\
&+\lambda\varepsilon(\mathrm{sech}(\varrho X)+\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}t)))\geq 0
\end{aligned}
$$

where $B_1=\sup_{e\in\mathbb{S}}\|\partial_\xi U_e\xi_1\|_{L^\infty}$ and $B_2=\sup_{e\in\mathbb{S}}\|\partial_\xi U_e\|_{L^\infty}$, by taking $0<\varrho\leq\varrho(\varepsilon)$ where $\varrho(\varepsilon)$ is small enough such that

$$
-B_1C_{10}\varrho-B_2C_{10}\varrho-\Big((C_8+C_9)\varrho+(1+c_1)\varepsilon\varrho^2\Big)+\lambda\varepsilon>0,\ \text{for all}\quad 0<\varrho\leq\varrho(\varepsilon). \tag{4.7}
$$

For $-C\leq\xi_1(t,x,y)\leq C$, there is $k>0$ such that $-\partial_\xi U_{e(t,x)}(\xi_1(t,x,y),x,y)\geq k$. Then, it follows from Claim 4.3 that

$$
\begin{aligned}
N\widetilde{U}^+\geq &kC_{11}(\mathrm{sech}(\varrho X)+\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) \\
&-B_1C_{10}\varrho(\mathrm{sech}(\varrho X)+\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) \\
&-B_2C_{10}\varrho\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t)) \\
&-\Big((C_8+C_9)\varrho+(1+c_1)\varepsilon\varrho^2\Big)(\mathrm{sech}(\varrho X) \\
&+\varrho\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t))-M\varepsilon(\mathrm{sech}(\varrho X) \\
&+\mathrm{sech}(\varrho X+\varrho(c_1-\hat{c}t)))\geq 0
\end{aligned}
$$

where $M=\max_{(x,y,u)\in\mathbb{R}^2\times\mathbb{R}}|f_u(x,y,u)|$, by (4.7), taking $0<\varepsilon\leq\varepsilon_0$ and $\varepsilon_0=\max\{\sigma/2,kC_{11}/(\lambda+M)\}$.

By the comparison principle, $\widetilde{U}^+(t,x,y)$ is a supersolution of (1.1).

By the definition of $\psi_1(x)$, $\psi_2(x)$ and Lemma 4.1, one has that

$$
\xi_1(t,X,Y)\to X\cos\alpha+Y\sin\alpha=x\cos\alpha+y\sin\alpha-c_\alpha t \text{ as } X\to-\infty,
$$

and

$$
\xi_1(t,X,Y)\to X\cos\theta+Y\sin\theta=x\cos\theta+y\sin\theta-c_\theta t \text{ as } X\to+\infty.
$$

Then, by similar arguments as in Step 2 of the proof of Lemma 2.8, one can get (4.3) and (4.4). The inequality (4.5) can be gotten by comparing $\widetilde{U}^+(t,x,y)$ with $U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y)$, $U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)$, $U_\theta(x\cos\theta + y\sin\theta - c_\theta t, x, y)$ respectively for $t$ negative enough through similar arguments as in Step 3 of the proof of Lemma 2.8. This completes the proof.

We then prove Claim 4.3.

*Proof of Claim 4.3.* From (4.6), one has that

$$\begin{aligned}c_1(\xi_1)_X + c_2(\xi_1)_Y - (\xi_1)_t - c_{e(t,X)} =& -c_1\frac{\varrho(\psi_1)_X(\psi_1)_{XX}}{(\psi_1)_X^2+1}\xi_1 - c_1\frac{(\psi_1)_X}{\sqrt{(\psi_1)_X^2+1}}\\&+c_2\frac{1}{\sqrt{(\psi_1)_X^2+1}}\\&+\frac{\varrho(\psi_1)_X(\psi_1)_{tX}}{(\psi_1)_X^2+1}\xi_1+\frac{(\psi_1)_t}{\sqrt{(\psi_1)_X^2+1}}-c_{e(t,X)}.\end{aligned}$$

Then, by Lemma 2.1 and the definition of $\psi_1$, there is $C_{10}>0$ such that

$$\begin{aligned}&\left|-c_1\frac{\varrho(\psi_1)_X(\psi_1)_{XX}}{(\psi_1)_X^2+1}\xi_1+\frac{\varrho(\psi_1)_X(\psi_1)_{tX}}{(\psi_1)_X^2+1}\xi_1\right|\\&\le C_{10}\varrho(\operatorname{sech}(\varrho X)+\operatorname{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t))|\xi_1|,\end{aligned}\tag{4.8}$$

and

$$\left|\frac{(\psi_1)_t}{\sqrt{(\psi_1)_X^2+1}}\right|\le C_{10}\varrho\operatorname{sech}(\varrho X+\varrho(c_1-\hat{c}_1)t).\tag{4.9}$$

Let $\theta(t,X)=\arccos(e_1(t,X))$. Then, $e(t,X)=(\cos\theta(t,X),\sin\theta(t,X))$. By the definition of $\psi_1(t,X)$, one has $\alpha<\theta(t,X)<\theta$. It follows from Lemma 4.1 that

$$\begin{aligned}-c_1\frac{(\psi_1)_X}{\sqrt{(\psi_1)_X^2+1}}+c_2\frac{1}{\sqrt{(\psi_1)_X^2+1}}-c_{e(t,X)}=&(c_1,c_2)(\cos\theta(t,X),\sin\theta(t,X))-c_{\theta(t,X)}\\=&c_{\alpha\theta}e_{\alpha\theta}(\cos\theta(t,X),\sin\theta(t,X))-c_{\theta(t,X)}>0.\end{aligned}\tag{4.10}$$

Notice that $c_e>0$ for all $e\in\mathbb{S}$. By Lemma 4.1, one has that

$$e_{\alpha\theta}\cdot(\cos\theta(t,X),\sin\theta(t,X))>0,\ \text{for all}\quad X\in\mathbb{R}.$$

Let

$$h(s)=\frac{c_s}{e_{\alpha\theta}\cdot(\cos s,\sin s)}.$$

Notice that $h(\alpha)=c_{\alpha\theta}$. Also notice that $e_1(t,X)\to\cos\alpha$ as $X\to-\infty$ and $\theta(t,X)\to\alpha$ as $X\to-\infty$ for $X$ being very negative. Then, one has that

$$\begin{aligned}&c_{\alpha\theta}e_{\alpha\theta}(\cos\theta(t,X),\sin\theta(t,X))-c_{\theta(t,X)}\\=&e_{\alpha\theta}\cdot(\cos\theta(t,X),\sin\theta(t,X))(h(\alpha)-h(\theta(t,X)))\\=&e_{\alpha\theta}\cdot(\cos\theta(t,X),\sin\theta(t,X))(h'(\alpha)(\alpha-\theta(t,X))+o(|\alpha-\theta(t,X)|))\end{aligned}\tag{4.11}$$

Remember that $h'(\alpha) < 0$ by the assumptions of Theorem 1.10. Moreover, by the formulas in the proof of Lemma 4.2, there is $C_{11} > 0$ such that

$$\begin{aligned}\theta(t,X)-\alpha=\int_{-\infty}^{X}\theta_X(t,s)ds&=\int_{-\infty}^{X}\frac{\varrho(\psi_1)_{XX}(\varrho t,\varrho s)}{(\psi_1)_X^2(\varrho t,\varrho s)+1}ds\\ &\geq\frac{1}{\|(\psi_1)_X\|_{L^\infty}^2+1}((\psi_1)_X(\varrho t,\varrho X)+\cot\alpha)\\ &\geq C_{11}(\operatorname{sech}(\varrho X)+\operatorname{sech}(\varrho X+\varrho(c_1-\hat{c}t))).\end{aligned}\tag{4.12}$$

By (4.8)-(4.12), we have our conclusion.

Now, we turn to prove Theorem 1.10.

*Proof of Theorem 1.10.* Let $u_n(t,x,y)$ be the solution of (1.1) for $t \geq -n$ with initial data

$$u_n(-n,x,y)=U^-_{\alpha\theta\beta}(-n,x,y),$$

where

$$\begin{aligned}U^-_{\alpha\theta\beta}(t,x,y)=\max\{&U_\alpha(x\cos\alpha+y\sin\alpha-c_\alpha t,x,y),\\ &U_\theta(x\cos\theta+y\sin\theta-c_\theta t,x,y),\\ &U_\beta(x\cos\beta+y\sin\beta-c_\beta t,x,y)\}.\end{aligned}$$

By Lemma 4.2, it follows from the comparison principle that

$$U^-_{\alpha\theta\beta}(t,x,y)\leq u_n(t,x,y)\leq \widetilde{U}^+(t,x,y),\ \text{for}\ \ -n\leq t\leq T\ \ \text{and}\ \ (x,y)\in\mathbb{R}^2,\tag{4.13}$$

wherer $T$ is a negative constant such that Lemma 4.2 holds for $-\infty < t \leq T$. Since $U^-_{\alpha\theta\beta}(t,x,y)$ is a subsolution, the sequence $u_n(t,x,y)$ is increasing in $n$. Letting $n \to +\infty$ and by parabolic estimates, the sequence $u_n(t,x,y)$ converges to an entire solution $u(t,x,y)$ of (1.1).

By (4.13), $u(t,x,y)$ satisfies

$$U^-_{\alpha\theta\beta}(t,x,y)\leq u(t,x,y)\leq \widetilde{U}^+(t,x,y),\ \text{for}\ \ t\leq T\ \ \text{and}\ (x,y)\in\mathbb{R}^2.\tag{4.14}$$

Moreover, by (4.3), (4.4) and since $\varepsilon$ can be arbitrary small, one can get that $u(t,x,y)$ satisfies

$$\lim_{R\to+\infty}\sup_{x\leq0,((x,y)-c_{\alpha\theta}e_{\alpha\theta}t)^2>R^2}\left|u(t,x,y)-U^-_{\alpha\theta}(t,x,y)\right|=0,\tag{4.15}$$

and

$$\lim_{R\to+\infty}\sup_{x>0,((x,y)-c_{\beta\theta}e_{\beta\theta}t)^2>R^2}\left|u(t,x,y)-U^-_{\beta\theta}(t,x,y)\right|=0\tag{4.16}$$

for $t$ negative enough. Now, we consider the half plane $H := \{(x,y)\in\mathbb{R}^2; x<0\}$. Take any sequence $\{t_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}$ such that $t_n \to -\infty$ as $n \to +\infty$. Notice that for any $n$, there are $k_n^1, k_n^2 \in \mathbb{Z}$ and $x_n' \in [0,L_1)$, $y_n' \in [0,L_2)$ such that $c_1t_n = k_n^1L_1+x_n'$ and $c_2t_n = k_n^2L_2+y_n'$. Moreover, up to extract subsequences of $c_1t_n$ and $c_2t_n$, there are $x'_* \in [0, L_1]$ and $y'_* \in [0, L_2]$ such that $x'_n \to x'_*$ and $y'_n \to y'_*$ as $n \to +\infty$. Let $v_n(t, x, y) = u(t + t_n, x + c_1 t_n, y + c_2 t_n)$ and $H_n = H - c_1 t_n$. Then, $H_n \to \mathbb{R}^2$ as $n \to +\infty$. Since $f(x, y, \cdot)$ is $L$-periodic in $(x, y)$, one has that $f(x + c_1 t_n, y + c_2 t_n, \cdot) \to f(x + x'_*, y + y'_*, \cdot)$ By parabolic estimates, $v_n(t, x, y)$, up to extract of a subsequence, converges to a solution $v_\infty(t, x, y)$ of

$$v_t - \Delta v = f(x + x'_*, y + y'_*, v), \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^2. \tag{4.17}$$

By the definitions of $c_1$ and $c_2$, one can easily check that

$$U^-_{\alpha\theta}(t+t_n, x+x_n, y+y_n) \to \hat{U}^-_{\alpha\theta}(t, x, y), \text{ as } \quad n \to +\infty \quad \text{uniformly in} \quad \mathbb{R}\times\mathbb{R}^2, \tag{4.18}$$

where

$$\hat{U}^-_{\alpha\theta}(t, x, y) := \max\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x + x'_*, y + y'_*),$$
$$U_\theta(x\cos\theta + y\sin\theta - c_\theta t, x + x'_*, y + y'_*)\}.$$

By (4.15), it follows that

$$\lim_{R\to+\infty} \sup_{((x,y)-c_{\alpha\theta}e_{\alpha\theta}t)^2 > R^2} \left| v_\infty(t, x, y) - \hat{U}^-_{\alpha\theta}(t, x, y) \right| = 0.$$

By the uniqueness of the curved front, one then has that $v_\infty(t, x, y) \equiv \hat{V}_{\alpha\theta}(t, x, y)$ where $\hat{V}_{\alpha\theta}(t, x, y)$ is the curved front of (4.17) satisfying

$$\lim_{R\to+\infty} \sup_{((x,y)-c_{\alpha\theta}e_{\alpha\theta}t)^2 > R^2} \left| \hat{V}_{\alpha\theta}(t, x, y) - \hat{U}^-_{\alpha\theta}(t, x, y) \right| = 0. \tag{4.19}$$

Thus, for any fixed $t$,

$$v_n(t, x, y) \to \hat{V}_{\alpha\theta}(t, x, y), \text{ as } \quad n \to +\infty \quad \text{locally uniformly in} \quad H_n.$$

By (4.15), (4.18) and (4.19), the above convergence is uniform in $\overline{H_n}$. Thus, for any fixed $t$,

$$u(t + t_n, x + c_1 t_n, y + c_2 t_n) \to \hat{V}_{\alpha\theta}(t, x, y), \text{ as } \quad n \to +\infty \quad \text{uniformly in} \quad \overline{H_n},$$

which implies

$$u(t + t_n, x, y) \to \hat{V}_{\alpha\theta}(t, x - c_1 t_n, y - c_2 t_n), \text{ as } \quad n \to +\infty \quad \text{uniformly in} \quad \overline{H}. \tag{4.20}$$

By the above arguments applied to $\hat{V}_{\alpha\theta}(t - t_n + t_0, x - c_1 t_n, y - c_2 t_n)$ for arbitrary $t_0 \in \mathbb{R}$, one can get that

$$\hat{V}_{\alpha\theta}(t - t_n + t_0, x - c_1 t_n, y - c_2 t_n) \to V_{\alpha\theta}(t + t_0, x, y),$$
$$\text{as} \quad n \to +\infty \quad \text{locally uniformly for} \quad t \in \mathbb{R}$$
$$\text{and uniformly for} \quad (x, y) \in \mathbb{R}^2.$$

Since $t_0$ is arbitrary, the above convergence is also uniform for $t \in \mathbb{R}$. Thus, by (4.20), one gets that

$$u(t, x, y) \to V_{\alpha\theta}(t, x, y), \text{ as } t \to -\infty \text{ uniformly in } \overline{H}.$$

Similarly, one can prove that $u(t,x,y) \to V_{\beta\theta}(t,x,y)$ as $t \to -\infty$ uniformly in $\mathbb{R}^2 \setminus H$.

On the other hand, for fixed $T < 0$ such that Lemma 4.2 holds, one can easily check that

$$\lim_{R\to+\infty} \sup_{x^2+y^2>R^2} \left| U^-_{\alpha\theta\beta}(T,x,y) - U^-_{\alpha\beta}(T,x,y) \right| = 0,$$

and

$$\lim_{R\to+\infty} \sup_{x^2+y^2>R^2} \left| \widetilde{U}^+(T,x,y) - U^-_{\alpha\beta}(T,x,y) \right| \le 2\varepsilon.$$

Since $\varepsilon$ can be arbitrary small and by (4.14), one has that

$$\lim_{R\to+\infty} \sup_{x^2+y^2>R^2} \left| u(T,x,y) - U^-_{\alpha\beta}(T,x,y) \right| = 0.$$

By stability of the curved front, that is, Theorem 1.9, one has that

$$u(t,x,y) \to V_{\alpha\beta}(t,x,y), \text{ as } \quad t \to +\infty \quad \text{uniformly in } \quad \mathbb{R}^2.$$

This completes the proof of Theorem 1.10.

Finally, we prove Corollary 1.12 which implies that Theorem 1.10 is not empty.

*Proof of Corollary 1.12.* Assume that $e_* = (0,1)$. Since $c_{e_*} = \min_{e\in\mathbb{S}}\{c_e\}$ and $c'_e$ is bounded, there exist $\alpha_1 \in (0,\pi/2)$ and $\beta_1 \in (\pi/2,\pi)$ such that

$$\frac{dc_\theta}{d\theta}\Big|_{\theta=\alpha} = c'_\alpha \cdot (-\sin\alpha, \cos\alpha) \le 0 \text{ for } \quad \alpha \in [\alpha_1, \frac{\pi}{2}],$$

and

$$\frac{dc_\theta}{d\theta}\Big|_{\theta=\beta} = c'_\beta \cdot (-\sin\beta, \cos\beta) \ge 0 \text{ for } \quad \beta \in [\frac{\pi}{2}, \beta_1].$$

Let $g(\theta) = c_\theta / \sin\theta$. Then,

$$g'(\theta) = \frac{c'_\theta \cdot (-\sin\theta, \cos\theta)}{\sin\theta} - \frac{c_\theta \cos\theta}{\sin^2\theta}.$$

One can make $\alpha_1$, $\beta_1$ close to $\pi/2$ such that

$$g'(\alpha) < 0 \text{ for all } \quad \alpha \in [\alpha_1, \frac{\pi}{2}) \quad \text{and} g'(\beta) > 0 \text{ for all } \quad \beta \in [\frac{\pi}{2}, \beta_1].$$

Thus, $g(\theta)$ is decreasing from $g(\alpha)$ to $g(\pi/2)$ as $\theta$ varying from $\alpha_1$ to $\pi/2$, and is increasing from $g(\pi/2)$ to $g(\beta)$ as $\theta$ varying from $\pi/2$ to $\beta_1$. By continuity, one can pick $\alpha \in [\alpha_1, \pi/2)$ and $\beta \in (\pi/2, \beta_1]$ such that

$$g(\alpha) = g(\beta), \; g'(\alpha) < 0 \text{ and } g'(\beta) > 0.$$

Let $e_1 = (\cos\alpha, \sin\alpha)$ and $e_2 = (\cos\beta, \sin\beta)$. By Theorem 1.2, there is a curved front $V_{e_1e_2}(t,x,y)$ of (1.1) satisfying (1.11) with $e_0 = e_*$.

By the same arguments of Corollary 1.6, one can rotate the coordinate such that $e_*$ can be any direction and for $e_1$, $e_2$ close to $e_*$ enough, there is a curved front $V_{e_1e_2}(t, x, y)$ of (1.1) satisfying (1.11).

Assume that $e_*$ is denoted by $(\cos\theta_*, \sin\theta_*)$ where $\theta_* \in (0, \pi/2)$ is small enough. Take $e_1$, $e_2$ close to $e_*$ such that there exists a curved front $V_{e_1e_2}(t, x, y)$ of (1.1). Let $e_1$ and $e_2$ be denoted by $(\cos\theta_1, \sin\theta_1)$ and $\cos\theta_2, \sin\theta_2$ respectively, where $\theta_1$ and $\theta_2$ are close to $\theta_*$. By Corollary 1.5 and since $\theta_*$ is small enough which means that $\theta_1$ is small enough, there is $\theta_3 \in (\pi/2, \pi)$ such that

$$\frac{c_{\theta_1}}{\sin\theta_1} = \frac{c_{\theta_3}}{\sin\theta_3} := c_{\theta_1\theta_3},$$

and there is a curved front $V_{\theta_1\theta_3}$ of (1.1) satisfying (1.9) with $\alpha = \theta_1$, $\beta = \theta_3$ and $c_{\alpha\beta} = c_{\theta_1\theta_3}$. On the other hand, since $\theta_1$ is small enough, this implies that $\theta_3$ is close to $\pi$ enough. Then, since $\theta_2$ is also small enough, one has that $\theta_3 - \theta_2$ is close to $\pi$ enough and hence, $(\cos\theta_2, \sin\theta_2) \cdot (\cos\theta_3, \sin\theta_3) = \cos(\theta_3 - \theta_2)$ is close to $-1$ enough. By Corollary 1.6, there is $e_{**}$ such that (1.10) holds for $e_1 = (\cos\theta_2, \sin\theta_2)$, $e_2 = (\cos\theta_3, \sin\theta_3)$, $e_0 = e_{**}$ and there is a curved front $V_{\theta_2\theta_3}$ of (1.1) satisfying (1.11).

Then, by Theorem 1.10, there is an entire solution $u(t, x, y)$ of (1.1) satisfying (1.13) and (1.14) with $\alpha = \theta_1$, $\theta = \theta_2$, $\beta = \theta_3$.

*Acknowledgements.* We thank the anonymous referee for offering many helpful suggestions on revision and mentioning some problems which we did not notice initially.



## References

1. Allen, S.; Cahn, J.W.: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta. Mettall.* **27**, 1084–1095, 1979
2. Berestycki, H.; Hamel, F.: Front propagation in periodic excitable media. *Commun. Pure Appl. Math.* **55**, 949–1032, 2002
3. Berestycki, H.; Hamel, F.: Generalized transition waves and their properties. *Commun. Pure Appl. Math.* **65**, 592–648, 2012
4. Bu, Z.-H.; Wang, Z.-C.: Curved fronts of monostable reaction-advection-diffusion equations in space-time periodic media. *Commun. Pure Appl. Anal.* **15**, 139–160, 2016
5. Chen, X.; Guo, J.-S.; Hamel, F.; Ninomiya, H.; Roquejoffre, J.-M.: Traveling waves with paraboloid like interfaces for balanced bistable dynamics. *Ann. Inst. H. Poincaré Non Linear Anal.* **24**, 369–393, 2007
6. Ding, W.; Giletti, T.: Admissible speeds in spatially periodic bistable reaction-diffusion equations, preprint. (arXiv:2006.05118)
7. Ding, W.; Hamel, F.; Zhao, X.: Bistable pulsating fronts for reaction-diffusion equations in a periodic habitat. *Indiana Univ. Math. J.* **66**, 1189–1265, 2017
8. Ducrot, A.: A multi-dimensional bistable nonlinear diffusion equation in a periodic medium. *Math. Ann.* **366**, 783–818, 2016

9. DUCROT, A.; GILETTI, T.; MATANO, H.: Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations. *Trans. Am. Math. Soc.* **366**, 5541–5566, 2014
10. EI SMAILY, M.: Curved fronts in a shear flow: case of combustion nonlinearities. *Nonlinearity* **31**, 5643–5663, 2018
11. EL SMAILY, M.; HAMEL, F.; HUANG, R.: Two-dimensional curved fronts in a periodic shear flow, Nonlinear. *Analysis* **74**, 6469–6486, 2011
12. FANG, J.; ZHAO, X.-Q.: Bistable traveling waves for monotone semiflows with applications. *J. Eur. Math. Soc.* **17**, 2243–2288, 2015
13. FIFE, P.C.; MCLEOD, J.B.: The approach of solutions of nonlinear diffusion equations to traveling front solutions. *Arch. Ration. Mech. Anal.* **65**, 335–361, 1977
14. FISHER, R.A.: The wave of advance of advantageous genes. *Ann. Eugenics* **7**, 335–369, 1937
15. GUO, H.: Propagating speeds of bistable transition fronts in spatially periodic media. *Calc. Var. Part. Diff. Equ.* **57**, 47, 2018
16. GUO, H.; HAMEL, F.; SHENG, W.-J.: On the mean speed of bistable transition fronts in unbounded domains. *J. Math. Pures Appl.* **136**, 92–157, 2020
17. HAMEL, F.: Bistable transition fronts in $\mathbb{R}^N$. *Adv. Math.* **289**, 279–344, 2016
18. HAMEL, F.; MONNEAU, R.: Solutions of semilinear elliptic equations in $\mathbb{R}^N$ with conical-shaped level sets. *Commun. Part. Diff. Equations* **25**, 769–819, 2000
19. HAMEL, F.; MONNEAU, R.; ROQUEJOFFRE, J.-M.: Existence and qualitative properties of multidimensional conical bistable fronts. *Disc. Cont. Dyn. Syst. A* **13**, 1069–1096, 2005
20. KOLMOGOROV, A.N.; PETROVSKII, I.G.; PISKUNOV, S.N.: Étude de l equation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. État Moscou Ser. Intern. A* **1**, 1–26, 1937
21. NINOMIYA, H.; TANIGUCHI, M.: Existence and global stability of traveling curved fronts in the Allen–Cahn equations. *J. Diff. Equ.* **213**, 204–233, 2005
22. NOLEN, J.; RYZHIK, L.: Traveling waves in a one-dimensional heterogeneous medium. *Ann. Inst. H. Poincaré Analyse Non Linéaire* **26**, 1021–1047, 2009
23. SHIGESADA, N.; KAWASAKI, K.; TERAMOTO, E.: Traveling periodic waves in heterogeneous environments. *Theor. Pop. Bio.* **30**, 143–160, 1986
24. TANIGUCHI, M.: Traveling fronts of pyramidal shapes in the Allen–Cahn equation. *SIAM J. Math. Anal.* **39**, 319–344, 2007
25. TANIGUCHI, M.: The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen–Cahn equations. *J. Diff. Equ.* **246**, 2103–2130, 2009
26. Taniguchi, M.: Axially asymmetric traveling fronts in balanced bistable reaction-diffusion equations. *Ann. Inst. H. Poincaré Analyse Non Linéaire*, forthcoming
27. TANIGUCHI, M.: Axisymmetric traveling fronts in balanced bistable reaction-diffusion equations. *Disc. Cont. Dyn. Syst. A* **40**, 3981–3995, 2020
28. XIN, X.: Existence and uniqueness of travelling waves in a reaction-diffusion equation with combustion nonlinearity. *Indiana Univ. Math. J.* **40**, 985–1008, 1991
29. XIN, X.: Existence and stability of travelling waves in periodic media governed by a bistable nonlinearity. *J. Dyn. Diff. Equ.* **3**, 541–573, 1991
30. XIN, J.X.: Existence of planar flame fronts in convective-diffusive periodic media. *Arch. Ration. Mech. Anal.* **121**, 205–233, 1992
31. XIN, J.X.: Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media. *J. Stat. Phys.* **73**, 893–926, 1993
32. XIN, J.X.; ZHU, J.: Quenching and propagation of bistable reaction-diffusion fronts in multidimensional periodic media. *Phys. D* **81**, 94–110, 1995
33. ZLATOŠ, A.: Existence and non-existence of transition fronts for bistable and ignition reactions. *Ann. Inst. H. Poincaré Analyse Non Linéaire* **34**, 1687–1705, 2017

Hongjun Guo
School of Mathematical Sciences, Institute for Advanced Study,
Tongji University,
Shanghai
China.
e-mail: hongjun.guo@etu.univ-amu.fr

and

Wan-Tong Li
School of Mathematics and Statistics,
Lanzhou University,
Lanzhou
China.

and

Rongsong Liu
Department of Mathematics and Statistics,
University of Wyoming,
Laramie
WY
USA.

and

Zhi-Cheng Wang
School of Mathematics and Statistics,
Lanzhou University,
Lanzhou
China.

(*Received June 5, 2020 / Accepted September 8, 2021*)
*Published online September 27, 2021*

*corrected publication 2021*