Arch. Rational Mech. Anal. 242 (2021) 827–873
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-021-01694-9

Check for updates

The Random Heat Equation in Dimensions Three and Higher: The Homogenization Viewpoint

Alexander Dunlap, Yu Gu, Lenya Ryzhik & Ofer Zeitouni

Communicated by F. Otto

Abstract

We consider the stochastic heat equation $\partial_s u = \frac{1}{2}\Delta u + (\beta V(s,y) - \lambda)u$, with a smooth space-time-stationary Gaussian random field $V(s,y)$, in dimensions $d \geq 3$, with an initial condition $u(0,x) = u_0(\varepsilon x)$ and a suitably chosen $\lambda \in \mathbb{R}$. It is known that, for β small enough, the diffusively rescaled solution $u^\varepsilon(t,x) = u(\varepsilon^{-2}t, \varepsilon^{-1}x)$ converges weakly to a scalar multiple of the solution $\bar{u}(t,x)$ of the heat equation with an effective diffusivity a, and that fluctuations converge, also in a weak sense, to the solution of the Edwards-Wilkinson equation with an effective noise strength ν and the same effective diffusivity. In this paper, we derive a pointwise approximation $w^\varepsilon(t,x) = \bar{u}(t,x)\Psi^\varepsilon(t,x) + \varepsilon u_1^\varepsilon(t,x)$, where $\Psi^\varepsilon(t,x) = \Psi(t/\varepsilon^2, x/\varepsilon)$, Ψ is a solution of the SHE with constant initial conditions, and u_1^ε is an explicit corrector. We show that $\Psi(t,x)$ converges to a stationary process $\tilde{\Psi}(t,x)$ as $t \to \infty$, that $\mathbf{E}|u^\varepsilon(t,x) - w^\varepsilon(t,x)|^2$ converges pointwise to 0 as $\varepsilon \to 0$, and that $\varepsilon^{-d/2+1}(u^\varepsilon - w^\varepsilon)$ converges weakly to 0 for fixed t. As a consequence, we derive new representations of the diffusivity a and effective noise strength ν. Our approach uses a Markov chain in the space of trajectories introduced in [17], as well as tools from homogenization theory. The corrector $u_1^\varepsilon(t,x)$ is constructed using a seemingly new approximation scheme on a mesoscopic time scale.

1. Introduction

We consider the long-time and large-space behavior of the solutions $u(s,y)$ of the random heat equation with slowly varying initial conditions

$$\partial_s u = \frac{1}{2}\Delta u + (\beta V(s,y) - \lambda)u, \tag{1.1}$$

$$u(0,y) = u_0(\varepsilon y), \tag{1.2}$$

with $y \in \mathbb{R}^d$, $d \geq 3$. Here, u_0 is a smooth, compactly-supported initial condition, and the potential $V(s, y)$ is a smooth, isotropic, space-time-homogeneous, mean-zero Gaussian random field with a finite correlation length. These assumptions are stronger than we truly need, but we make them to avoid distracting from the focus of the paper. We assume that $V(s, y)$ has the form

$$V(s, y) = \int_{\mathbb{R}^{d+1}} \mu(s - s')\nu(y - y')\,\mathrm{d}W(s', y'),$$

where μ and ν are deterministic nonnegative functions of compact support, such that ν is isotropic,

$$\operatorname{supp}\mu \subset [0, 1], \qquad \operatorname{supp}\nu \subset \{y \in \mathbb{R}^d \mid |y| \leq 1/2\},$$

and $\mathrm{d}W$ is a space-time white noise. From this, we see that the covariance function is

$$R(s, y) := \mathbf{E}V(s + s', y + y')V(s', y') = \int_{\mathbb{R}} \mu(s + t)\mu(t)\,\mathrm{d}t \int_{\mathbb{R}^d} \nu(y + z)\nu(z)\,\mathrm{d}z. \tag{1.3}$$

The constant λ in (1.1) will be chosen – see Theorem 1.1 and (2.7)–(2.8) below – so that $\mathbf{E}u(t, x)$ does not grow exponentially as $t \to \infty$. The small parameter $\varepsilon \ll 1$ measures the ratio of the typical length scale of the initial condition to the correlation length of the random potential. As we are interested in the long-time behavior of u, we consider its macroscopic rescaling

$$u^\varepsilon(t, x) = u(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

which satisfies the rescaled problem

$$\partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \frac{1}{\varepsilon^2}\left(\beta V(\varepsilon^{-2}t, \varepsilon^{-1}x) - \lambda\right)u^\varepsilon \tag{1.4}$$

$$u^\varepsilon(0, x) = u_0(x). \tag{1.5}$$

Here and throughout the paper, we use s, y for the "microscopic" variables and $t = \varepsilon^2 s$, $x = \varepsilon^1 y$ for the rescaled "macroscopic" variables. It was shown in [17,20], and also in [22] at the level of the expectation, that there exists a $\beta_0 > 0$ so that, if $0 < \beta < \beta_0$, then there exists λ, depending on β, μ, ν, and $a, \overline{c} > 0$ so that, for any $t > 0$,

$$v^\varepsilon(t, \cdot) = \overline{c}u^\varepsilon(t, \cdot) \tag{1.6}$$

converges in probability and weakly in space as $\varepsilon \to 0$ to the solution $\overline{u}$ to the homogenized problem

$$\partial_t \overline{u} = \frac{1}{2}a\Delta\overline{u} \tag{1.7}$$

$$\overline{u}(0, x) = u_0(x), \tag{1.8}$$

with an effective diffusivity $a \neq 1$. It may come as a surprise that $\bar{c} \neq 1$ in general; see Remark 1.4 below. It was also shown that the fluctuations

$$\frac{1}{\varepsilon^{d/2-1}}\left(v^{\varepsilon}(t,\cdot) - \mathbf{E}v^{\varepsilon}(t,\cdot)\right) \tag{1.9}$$

converge in law and weakly in space as $\varepsilon \to 0$ to the solution $\mathscr{U}$ of the Edwards–Wilkinson equation

$$\partial_t \mathscr{U} = \frac{1}{2}a\Delta\mathscr{U} + \beta\nu\bar{u}\mathrm{d}W \tag{1.10}$$

$$\mathscr{U}(0,x) \equiv 0, \tag{1.11}$$

with an effective noise strength $\nu > 0$.

The results of [17,20] concern weak convergence, after integration against a macroscopic test function. We note that the restriction to dimension $d \geq 3$ is crucial: for $d = 2$ the behavior is different, as discussed in [4] and [5]. In this work, we seek to understand the microscopic behavior of the solutions, in the spirit of the classical random homogenization theory, and explain how the microscopic behavior leads to the macroscopic results of [17,20]. We are also interested in a more explicit interpretation of the macroscopic parameters: the renormalization constant λ, the effective diffusivity a in (1.7), the renormalization constant $\bar{c}$, and the effective noise strength ν in (1.10). In particular, we would like to connect these parameters to the classical objects of stochastic homogenization.

As is standard in PDE homogenization theory, we introduce fast variables and consider a formal asymptotic expansion for the solutions u^{ε} to (1.4)–(1.5) in the form

$$\begin{aligned} u^{\varepsilon}(t,x) = u^{(0)}(t,x,\varepsilon^{-2}t,\varepsilon^{-1}x) + \varepsilon u^{(1)}(t,x,\varepsilon^{-2}t,\varepsilon^{-1}x) \\ +\varepsilon^2 u^{(2)}(t,x,\varepsilon^{-2}t,\varepsilon^{-1}x) + \cdots . \end{aligned} \tag{1.12}$$

Two issues commonly arise in such expansions. First, it may be hard to prove, or even false, that the correctors exist as stationary random fields. Second, the correlations of the higher-order correctors may decay more slowly (in space) than those of lower-order correctors. Thus, after integration against a test function, all terms in the expansion may actually be of the same order, so including more correctors may not improve the expansion from the perspective of the weak approximation. We refer to [10,16] for a discussion of random fluctuations in elliptic homogenization, and [11,15] for a proof that stationary higher-order correctors exist in sufficiently high dimensions.

In the present case, it is easy to see that the leading order term in (1.12) should have the form

$$u^{(0)}(t,x,\varepsilon^{-2}t,\varepsilon^{-1}x) = \bar{\bar{u}}(t,x)\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x), \tag{1.13}$$

where Ψ is a solution to (1.1) and does not depend on the initial condition u_0 in (1.2), and $\bar{\bar{u}}$ is deterministic but depends on the initial condition u_0. We will see later that $\bar{\bar{u}} = \bar{u}$ with $\bar{u}$ taken to be the solution of the homogenized problem (1.7)–(1.8). In the context of the usual homogenization theory, one would like to think of

Ψ as being statistically stationary in space and time. In the context of the Cauchy problem (1.1)–(1.2), it turns out that better error bounds are achieved by letting Ψ solve the Cauchy problem with constant initial condition

$$\begin{aligned}\partial_s \Psi &= \frac{1}{2}\Delta\Psi + (\beta V - \lambda)\Psi \\ \Psi(0,\cdot) &\equiv 1.\end{aligned} \tag{1.14}$$

However, the intuition of a space-time-stationary Ψ is still justified, as we will see in Theorem 1.1 below that Ψ in fact converges to a space-time-stationary solution $\widetilde{\Psi}$ to (1.1).

As this paper was being written, we learned of the very interesting recent paper [6] (see also the subsequent [7]), which considers (in our notation) the pointwise error $(\Psi(s, y) - \widetilde{\Psi}(s, y))/\Psi(s, y)$ in the case where the random potential V is white in time, and shows that it is asymptotically Gaussian. This result is related but orthogonal to ours, and the proof techniques are quite different.

Existence of a Stationary Solution and the Leading-Order Term in the Expansion

The renormalization constant λ was understood in [17] as the unique value that keeps bounded the expectation of the solution to (1.1). Our first result refines this explanation by showing that, with this choice of λ, $\Psi(s,\cdot)$ in fact approaches a space-time-stationary solution, which we call $\widetilde{\Psi}$, as $s \to \infty$. As remarked above, this shows that it is reasonable to take $\Psi^\varepsilon(t, x) = \Psi(\varepsilon^{-2}t, \varepsilon^{-1}x)$ as a proxy for the stationary solution in the leading-order term for the asymptotic expansion (1.12). Note that neither Ψ nor its stationary limit $\widetilde{\Psi}$ depends on the initial condition u_0, so both are "universal" objects.

Theorem 1.1. *There is a $\beta_0 > 0$ so that for all $0 \le \beta < \beta_0$, there exists a $\lambda = \lambda(\beta) > 0$ and a space-time-stationary random function $\widetilde{\Psi} = \widetilde{\Psi}(s, y) > 0$ that solves*

$$\partial_s \widetilde{\Psi}(s, y) = \frac{1}{2}\Delta\widetilde{\Psi}(s, y) + (\beta V(s, y) - \lambda)\widetilde{\Psi}(s, y), \qquad s \in \mathbb{R}, y \in \mathbb{R}^d, \tag{1.15}$$

and there is a constant $C < \infty$ so that for any $y \in \mathbb{R}^d$ and $s > 0$, we have

$$\mathbf{E}|\Psi(s, y) - \widetilde{\Psi}(s, y)|^2 \le Cs^{-d/2+1}. \tag{1.16}$$

Throughout the paper, we will always assume that $\lambda = \lambda(\beta)$ is chosen as in the statement of Theorem 1.1. Theorem 1.1 can also be seen as an extension of [23, Theorem 2.1] to the colored-noise setting, even though that result was formulated in different terms. Some other relevant results in the literature are [8,25], which show the existence of stationary solutions and convergence along subsequences in weighted L^2 spaces, also in the white-noise setting.

The proof of Theorem 1.1 is similar in spirit to that of [23, Theorem 2.1], but uses the framework of [17] to deal with the necessary renormalization parameter λ. For the case of elliptic operators in divergence form, the existence of stationary correctors in high dimensions was studied in [1,12,14], and we refer the reader to the recent monograph [2] for a more complete list of references.

As an application of the existence of the stationary solution, we will show in Sect. 4 that the effective noise strength ν in (1.10), which has a complicated expression given in [17, (5.6)], has a more intuitive expression in terms of the stationary solution. Let

$$G_a(t,x) = (2\pi a t)^{-d/2} \exp\left\{-|x|^2/(2at)\right\} \tag{1.17}$$

be the heat kernel with diffusivity a, and note that there exists a constant c so that

$$\int_0^\infty \int_{\mathbb{R}^d} G_a(r,z) G_a(r,z+x)\,\mathrm{d}z\,\mathrm{d}r = \frac{c}{a|x|^{d-2}}. \tag{1.18}$$

Theorem 1.2. *For $0 \le \beta < \beta_0$, with λ taken as in Theorem 1.1, the effective noise strength ν in (1.10) has the expression*

$$\nu^2 = \frac{a \lim\limits_{\varepsilon\to 0} \int\int g(x)g(\tilde{x})\varepsilon^{-(d-2)}\,\mathrm{Cov}\left(\widetilde{\Psi}(0,\varepsilon^{-1}x), \widetilde{\Psi}(0,\varepsilon^{-1}\tilde{x})\right)\,\mathrm{d}x\,\mathrm{d}\tilde{x}}{c\beta^2 \mathrm{e}^{2\alpha_\infty} \int\int g(x)g(\tilde{x})|x-\tilde{x}|^{-(d-2)}\,\mathrm{d}x\,\mathrm{d}\tilde{x}} \tag{1.19}$$

for any test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. The deterministic constant α_∞ is defined in (2.8) below.

Theorem 1.2 should be read as a weak formulation of the asymptotics

$$\mathrm{Cov}(\widetilde{\Psi}(0,0), \widetilde{\Psi}(0,y)) \sim \frac{c\beta^2\nu^2\mathrm{e}^{2\alpha_\infty}}{a|y|^{d-2}}, \qquad |y| \gg 1.$$

In this sense, the effective noise strength in the Edwards–Wilkinson equation (1.10) is directly related to the decay of the covariance of the stationary solution. On the other hand, in Corollary 3.2, we provide an expression for the covariance term in (1.19) in terms of the Markov chain introduced in [17] and reviewed in Sect. 2 below.

Returning to the expansion (1.12), the leading order term in (1.13) is justified by the following microscopic convergence result:

Theorem 1.3. *For $0 \le \beta < \beta_0$, with λ taken as in Theorem 1.1, set $\Psi^\varepsilon(t,x) = \Psi(\varepsilon^{-2}t, \varepsilon^{-1}x)$. If $u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, then for all $t \ge 0$ and $x \in \mathbf{R}^d$ we have*

$$\lim_{\varepsilon\to 0} \mathbf{E}|u^\varepsilon(t,x) - \overline{u}(t,x)\Psi^\varepsilon(t,x)|^2 = 0. \tag{1.20}$$

Remark 1.4. We can now explain the non-divergent renormalization constant $\overline{c}$ in (1.6). The function $\Psi(s,\cdot)$ approaches a stationary solution $\widetilde{\Psi}$ as $s \to \infty$, that is, on a "microscopically large" time scale. However, even though $\Psi(0,\cdot) \equiv 1$, it is not necessarily the case that $\mathbf{E}\widetilde{\Psi}(s,\cdot) \equiv 1$. (This *would* be the case by the property of the Itô integral if V were white in time.) Thus we need to divide by the factor of $\overline{c} = \mathbf{E}\widetilde{\Psi}(s,\cdot)$ to see convergence to the effective diffusion problem (1.7)–(1.8) with initial condition u_0 rather than $\overline{c}u_0$.

A Higher-Order Approximation

In order to obtain higher-order corrections in the asymptotic expansion, if we plug (1.12) into (1.4) and group terms by powers of ε, we obtain the following equations for u_1 and u_2:

$$\partial_s u_1(t,x,s,y) = \frac{1}{2}\Delta_y u_1(t,x,s,y) + (\beta V(s,y) - \lambda)u_1(t,x,s,y) + \nabla_y \Psi(s,y)\cdot\nabla_x \overline{u}(t,x), \tag{1.21}$$

and

$$\begin{aligned}\partial_s u_2(t,x,s,y) = {} & \frac{1}{2}\Delta_y u_2(t,x,s,y) + (\beta V(s,y) - \lambda)u_2(t,x,s,y) \\ & + \nabla_y \cdot \nabla_x u_1(t,x,s,y) \\ & + \frac{1}{2}(1-a)\Psi(s,y)\Delta_x \overline{u}(t,x).\end{aligned} \tag{1.22}$$

As we will show in Sect. 5, the effective diffusivity a can be recovered from a formal solvability condition for (1.22) to have a solution u_2 that is stationary in the fast variables s and y, which is a rather standard situation in homogenization theory. However, here, as stationary correctors are not expected to exist in low dimensions, justifying this expression requires a construction of approximate correctors and passage to a large-time limit, similar to the “large box” limit in elliptic homogenization theory. In particular, Theorem 5.1 below shows how to evaluate the effective diffusivity in terms of objects familiar from the theory of homogenization.

Our last result concerns the connection between the local expansion (1.12) and the weak approximation of the solution. As we have mentioned, typically, the leading-order terms in such expansions in stochastic homogenization only provide local approximations, while a control of the weak error (after integration against a test function) requires extra terms. This is partly because the higher the order of the corrector, the slower the decay of its covariance function, leading to the accumulation of errors from terms of all orders. We circumvent this issue in a way reminiscent of the “straight-line” approximation of trajectories on a mesoscopic time scale that is “long but not too long”, such as is used for models of particles in random velocity fields or subject to random forces in [18,19].

If we look at (1.21) for each macroscopic $t > 0$ and $x \in \mathbb{R}^d$ fixed, as an evolution problem in s, we would have a “complete separation of scales” factorization

$$u_1(t,x,s,y) = \sum_{k=1}^{d} \zeta^{(k)}(s,y)\frac{\partial \overline{u}(t,x)}{\partial x_k}, \tag{1.23}$$

where $\zeta^{(k)}$ solves the microscopic problem

$$\partial_s \zeta^{(k)} = \frac{1}{2}\Delta\zeta^{(k)} + (\beta V(s,y) - \lambda)\zeta^{(k)} + \frac{\partial\Psi}{\partial y_k}. \tag{1.24}$$

Instead of using (1.24) directly, we consider mesoscopic time intervals in s of size $\varepsilon^{-\gamma}$, with $\gamma \in (0,2)$. To be precise, for each $j \geq 1$, let $\theta_j^{(k)} = \theta_j^{(k)}(s,y)$, $1 \leq k \leq d$, be the solution to

$$\begin{aligned} \partial_s \theta_j^{(k)} &= \frac{1}{2}\Delta_y \theta_j^{(k)} + (\beta V - \lambda)\theta_j^{(k)} + \frac{\partial \Psi}{\partial y_k}, \qquad s > \varepsilon^{-\gamma}(j-1), \\ \theta_j^{(k)}(\varepsilon^{-\gamma}(j-1), \cdot) &= 0. \end{aligned} \tag{1.25}$$

Then, define $u_{1;j} = u_{1;j}(s,y)$ to be the solution to

$$\begin{aligned} \partial_s u_{1;j} &= \frac{1}{2}\Delta u_{1;j} + (\beta V - \lambda) u_{1;j}, \qquad s > \varepsilon^{-\gamma} j \\ u_{1;j}(\varepsilon^{-\gamma} j, y) &= \sum_{k=1}^{d} \theta_j^{(k)}(\varepsilon^{-\gamma} j, y) \frac{\partial \overline{u}}{\partial x_k}(\varepsilon^{2-\gamma} j, \varepsilon y), \end{aligned} \tag{1.26}$$

and finally put

$$u_1^\varepsilon(t,x) = \sum_{j=1}^{\lfloor \varepsilon^{\gamma-2} t \rfloor} u_{1;j}(\varepsilon^{-2}t, \varepsilon^{-1}x) + \theta_{\lfloor \varepsilon^{\gamma-2}t \rfloor + 1}(\varepsilon^{-2}t, \varepsilon^{-1}x) \cdot \nabla \overline{u}(t,x). \tag{1.27}$$

This is similar to putting $s = \varepsilon^{-2}t$, $y = \varepsilon^{-1}x$ in the formal PDE (1.21), except that rather than multiplying the forcing by the "current" value of $\nabla \overline{u}$, we multiply it by an "out-of-date" value of $\nabla \overline{u}$ that is only updated to the correct current value of $\nabla \overline{u}$ at times of the form $\varepsilon^{-\gamma} j$, $j \in \mathbb{N}$. With this definition of u_1^ε, we have a weak convergence theorem for the fluctuations. Recall that $\Psi^\varepsilon(t,x) = \Psi(\varepsilon^{-2}t, \varepsilon^{-1}x)$ with Ψ solving (1.14).

Theorem 1.5. *Suppose that $0 \leq \beta < \beta_0$ and take λ as in Theorem 1.1. Let $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Let $\gamma \in (0,2)$ and define u_1^ε as in (1.27). For any $\zeta < (1-\gamma/2) \vee (\gamma - 1)$ and any $t > 0$, there exists a $C > 0$ (also depending on $\|u_0\|_{\mathcal{C}^3(\mathbb{R}^d)}$) so that*

$$\mathbf{E}\left(\varepsilon^{-d/2+1} \int g(x)[u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x) - \varepsilon u_1^\varepsilon(t,x)]\,\mathrm{d}x\right)^2 \leq C\varepsilon^{2\zeta}. \tag{1.28}$$

The optimal bound in Theorem 1.5 is achieved when $\gamma = 4/3$, in which case ζ is required to be less than $1/3$.

We note that it would be hopeless to get a convergence-of-fluctuations result like Theorem 1.5, even with an error of size $\varepsilon^{d/2-1}$ as in (1.9), using only the first term of the expansion (1.12) as in Theorem 1.3. This is because at that scale, [17] gives different central limit theorem statements for u and for $\Psi \overline{u}$: the rescaled and renormalized fluctuations of u converge to a solution of the SPDE

$$\partial_t \mathscr{U} = \frac{1}{2} a \Delta \mathscr{U} + \beta \nu \overline{u} \dot{W}, \tag{1.29}$$

while the rescaled and renormalized fluctuations of Ψ converge to a solution of the SPDE

$$\partial_t \psi = \frac{1}{2} a \Delta \psi + \beta \nu \dot{W}, \tag{1.30}$$

and so the rescaled and renormalized fluctuations of $\Psi \overline{u}$ converge to a solution of the SPDE

$$\begin{aligned} \partial_t (\psi \overline{u}) &= \frac{1}{2} a \overline{u} \Delta \psi + \beta \nu \overline{u} \dot{W} + \frac{1}{2} a \psi \Delta \overline{u} \\ &= \frac{1}{2} a \Delta (\psi \overline{u}) - a \nabla \psi \cdot \nabla \overline{u} + \beta \nu \overline{u} \dot{W}. \end{aligned} \tag{1.31}$$

The limiting SPDEs (1.29) and (1.31) are not the same, so an extra correction, besides the first term $\Psi^\varepsilon(t, x)\overline{u}$ of the homogenization expansion, is needed. This phenomenon is not new in the study of random fluctuations in homogenization, and has been discussed e.g. in [10,16].

The definitions (1.25)–(1.26) sit midway between two natural ways of interpreting the formal problem (1.21). On one hand, (1.21), for fixed x and t, can be solved as in (1.23)–(1.24). However, defining the corrector u_1 by (1.23), with initial condition 0, and then evaluating at time $s = \varepsilon^{-2} t$ does not seem to yield a good convergence result, because $\nabla_x \overline{u}(\tau, x)$ is not constant on the time scale from $\tau = 0$ to $\tau = \varepsilon^2 s = t$. On the other hand, (1.21) could also be solved by plugging $t = \varepsilon^2 s$, $x = \varepsilon y$ into (1.21), yielding the PDE

$$\begin{aligned} \partial_s u_1(s, y) &= \frac{1}{2} \Delta_y u_1(s, y) + (\beta V(s, y) - \lambda) u_1(s, y) \\ &\quad + \nabla_y \Psi(s, y) \cdot \nabla_x \overline{u}(\varepsilon^2 s, \varepsilon y). \end{aligned} \tag{1.32}$$

However, using a solution to (1.32) with initial condition 0 also fails to yield a result along the lines of Theorem 1.5. This is because the Feynman–Kac formula that arises from the solution to (1.32) involves the behavior of the Markov chain of [17] on microscopically short time scales, while the limits appear to arise from the averaged behavior of the Markov chain on long time scales. The delay in multiplying by $\nabla_x \overline{u}$ introduced by only updating its value at mesoscopic intervals allows the short-time fluctuations to be averaged out, leaving only the averaged behavior of the Markov chain, which allows us to deduce the limiting behavior.

Proof Strategies and the Organization of the Paper

Although our study is in part motivated by the goal of understanding results in the vein of [17] from the perspective of PDE theory and stochastic homogenization, our proofs remain probabilistic, relying entirely on the Feynman–Kac formula. In particular, we extensively use a certain Markov chain, introduced in [17], representing the tilting of the measure on Brownian paths induced by the time-correlations of the random potential V. Because this Markov chain is somewhat technical, we start the paper by explaining how it appears via the Feynman-Kac representation of the solution, and provide the definition and properties of the Markov chain in Sect.

2, including a few properties which were not needed in [17], and then complete the proof of Theorem 1.1 in Sect. 3.

The next two sections of the paper are devoted to the parameters a and ν obtained in [17]. In Sect. 4, we prove Theorem 1.2 regarding the effective noise strength ν, showing that it is directly related to the spatial decay of correlations of the stationary solution. In Sect. 5, we first show how the effective diffusivity can be recovered from the formal asymptotic expansion (1.12); see expression (5.7) below. However, as the correctors that appear in the asymptotic expansion may not be stationary (especially in lower dimensions), this formula does not necessarily make sense directly. Instead, we devise an approximation procedure via a sequence of problems on long but finite time intervals and then pass to the limit. This is the content of Theorem 5.1. Finally, in the last two sections we establish our convergence results for the formal asymptotic expansion: the strong convergence (Theorem 1.3) in Sect. 6, and the weak convergence (Theorem 1.5) in Sect. 7.

As we have emphasized above, the Markov chain introduced in [17] plays a key technical role in the analysis throughout the paper. However, the key observation of [17] is that on microscopically long time scales, the Markov chain mixes exponentially fast so that its partial sum essentially behaves like a Brownian motion. Our results still hold when the noise V is taken to be white rather than colored in time. In that case the Brownian motion is not tilted by the environment, and the Markov chain is just its i.i.d. Gaussian increments, even on microscopically short time scales. Thus, the reader may find it helpful on first reading to ignore the time correlations and pretend that the Markov chain is in fact a sequence of i.i.d. Gaussian random increments, which eliminates the need for most of the technicalities introduced in Sect. 2. The analysis of [17] constructing the Markov chain is orthogonal to the new applications of this chain in the present paper.

2. The Tilted Brownian Motion and the Markov Chain

All of the proofs in this paper rely heavily on a Markov chain introduced in [17] representing a tilted Wiener measure arising in the Feynman–Kac representation of solutions to the stochastic heat equation. In order to recall this Feynman–Kac representation, we first introduce some notation. By $\mathbb{E}_B^y$ we denote expectation with respect to the probability measure in which $B = (B^1, \ldots, B^d)$ is a standard d-dimensional Brownian motion with $B_0 = y$, which we will always assume to be two-sided (i.e. running both forward and backward from time 0) since this will be convenient in some formulas. We use $\mathbf{E}$ for expectation with respect to the randomness in V, and use $\mathbb{E}$, with various adornments, for expectation with respect to auxiliary Brownian motions or Markov chains used in some way in the Feynman-Kac formula. Also, whenever we denote an expectation with a letter "E," we will use the letter "P" with the same font and adornments to represent the corresponding probability measure. For any $s \in \mathbb{R}$ and $\mathfrak{A} \subset \mathbb{R}$, we set

$$\mathscr{V}_{s;\mathfrak{A}}[B] = \int_{\mathfrak{A}} V(s-\tau, B_\tau)\,\mathrm{d}\tau. \tag{2.1}$$

We will often use the shorthand $\mathscr{V}_s = \mathscr{V}_{s;[0,s]}$. Thus, for example, the solution to (1.14) can be expressed in the Feynman–Kac representation

$$\Psi(s, y) = \mathbb{E}_B^y \exp\{\beta \mathscr{V}_s[B] - \lambda s\}. \tag{2.2}$$

There are, of course, also Feynman–Kac formulas for solutions to the other equations in the introduction, which we will write as they are needed.

2.1. The Tilted Brownian Motion

In computing moments of $\Psi(s, y)$, due to the Gaussianity of $\mathscr{V}_{\mathfrak{A}}[B]$, it becomes necessary to evaluate the covariances of the latter. Recall the definition (1.3) of the covariance kernel R of the noise. We define, for any pair of sets $\mathfrak{A}, \widetilde{\mathfrak{A}} \subset \mathbb{R}$, the quantity

$$\mathscr{R}_{\mathfrak{A},\widetilde{\mathfrak{A}}}[B, \widetilde{B}] = \mathbf{E}\left(\mathscr{V}_{s;\mathfrak{A}}[B]\mathscr{V}_{s;\widetilde{\mathfrak{A}}}[\widetilde{B}]\right) = \int_{\widetilde{\mathfrak{A}}}\int_{\widetilde{\mathfrak{A}}} R(\tau - \tilde{\tau}, B_\tau - \widetilde{B}_{\tilde{\tau}})\,\mathrm{d}\tau\,\mathrm{d}\tilde{\tau}, \tag{2.3}$$

which is independent of the choice of s due to the stationarity of V, and use the abbreviations

$$\mathscr{R}_{s,\tilde{s}} = \mathscr{R}_{[0,s],[0,\tilde{s}]}, \quad \mathscr{R}_{\mathfrak{A}} = \mathscr{R}_{\mathfrak{A},\mathfrak{A}}, \quad \mathscr{R}_s = \mathscr{R}_{s,s}.$$

We will also abbreviate $\mathscr{R}_\bullet[B] = \mathscr{R}_\bullet[B, B]$, where the $\bullet$ can be replaced by any allowable subscript for $\mathscr{R}$, so that, for example,

$$\mathscr{R}_{s,\tilde{s}}[B] = \mathscr{R}_{s,\tilde{s}}[B, B] = \mathscr{R}_{[0,s],[0,\tilde{s}]}[B, B].$$

As an example of the use of this notation, we have by Fubini's theorem and the formula for the expectation of the integral of a Gaussian that

$$\mathbf{E}\Psi(s, y) = \mathbb{E}_B^y \mathbf{E} \exp\{\beta \mathscr{V}_s[B] - \lambda s\} = \mathbb{E}_B^y \exp\left\{\frac{\beta^2}{2}\mathscr{R}_s[B] - \lambda s\right\}. \tag{2.4}$$

Similarly, we can compute

$$\begin{aligned}\mathbf{E}\Psi(s, y)\Psi(\tilde{s}, \tilde{y}) &= \mathbb{E}_B^y \mathbb{E}_{\tilde{B}}^{\tilde{y}} \mathbf{E} \exp\left\{\beta \mathscr{V}_s[B] + \beta \mathscr{V}_{\tilde{s}}^{\tilde{y}}[\widetilde{B}] - \lambda(s + \tilde{s})\right\} \\ &= \mathbb{E}_B^y \mathbb{E}_{\tilde{B}}^{\tilde{y}} \exp\left\{\left(\frac{\beta^2}{2}\mathscr{R}_s[B] - \lambda s\right) + \beta^2 \mathscr{R}_{s,\tilde{s}}[B, \widetilde{B}] \right.\\ &\quad \left. + \left(\frac{\beta^2}{2}\mathscr{R}_{\tilde{s}}[\widetilde{B}] - \lambda \tilde{s}\right)\right\}.\end{aligned} \tag{2.5}$$

We recognize the first and third terms in the last exponential from the exponential in (2.4). This motivates the definition of the tilted path measure $\widehat{\mathbb{P}}_{B;\bullet}^y$ by

$$\widehat{\mathbb{E}}^{y}_{B;\bullet}\mathscr{F}[B] = \frac{1}{Z_\bullet}\mathbb{E}^{y}_{B}\left[\mathscr{F}[B]\exp\left\{\frac{1}{2}\beta^2\mathscr{R}_\bullet[B]\right\}\right],$$
$$Z_\bullet = \mathbb{E}^{y}_{B}\exp\left\{\frac{1}{2}\beta^2\mathscr{R}_\bullet[B]\right\} \tag{2.6}$$

for any measurable functional $\mathscr{F}$ on the space $\mathcal{C}([0,\infty);\mathbb{R}^d)$, where $\bullet$ can be taken to be any of the allowable subscripts for $\mathscr{R}$. We also define $\widehat{\mathbb{P}}^{y,\tilde{y}}_{B,\widetilde{B};\bullet} = \widehat{\mathbb{P}}^{y}_{B;\bullet}\otimes\widehat{\mathbb{P}}^{\tilde{y}}_{\widetilde{B};\bullet}$ and denote by $\widehat{\mathbb{E}}^{y,\tilde{y}}_{B,\widetilde{B};\bullet}$ the corresponding expectation. Finally, we define

$$\alpha_s = \log Z_s - \lambda s, \tag{2.7}$$

and note that, according to [17, Lemma A.1] and its proof, there exists a unique $\lambda = \lambda(\beta)$ so that

$$|\alpha_s - \alpha_\infty| \le C\mathrm{e}^{-cs} \tag{2.8}$$

for some $\alpha_\infty \in (0,\infty)$, $c > 0$, and $C < \infty$. This is where the constant λ comes from, and we fix it for the rest of the paper. This definition of λ should be interpreted in terms of (2.4): λ is chosen so that $\mathbf{E}\Psi(s,y)$ remains of order $O(1)$ as $s\to+\infty$. Equivalently, it is the exponential rate of growth of the unrenormalized, that is, with $\lambda = 0$, multiplicative stochastic heat equation with noise strength β. We note that a consequence of Theorem 1.1 is that $\mathrm{e}^{\alpha_\infty} = \mathbf{E}\widetilde{\Psi}(s,y) = \overline{c}$, where $\overline{c}$ is as in (1.6). Another consequence is that

$$\lambda = \frac{\beta\mathbf{E}\widetilde{\Psi}(t,x)V(t,x)}{\mathbf{E}\widetilde{\Psi}(t,x)}. \tag{2.9}$$

This allows λ to be recovered directly from the law of the stationary solution. The problem (1.15) already depends on λ, so we cannot use this expression as a definition of λ. However, if as in (1.16) we approximate $\tilde{\Psi}$ by Ψ evaluated at a large time, then the right side of the resulting version of (2.9) does not depend on the choice of λ in (1.1). Thus we could *define*

$$\lambda = \frac{\lim\limits_{t\to\infty}\beta\mathbf{E}\Psi(t,x)V(t,x)}{\lim\limits_{t\to\infty}\mathbf{E}\Psi(t,x)}. \tag{2.10}$$

An example of the utility of this tilted measure is that it lets us rewrite (2.5) by

$$\mathbf{E}\Psi(s,y)\Psi(\tilde{s},\tilde{y}) = \mathrm{e}^{\alpha_s+\alpha_{\tilde{s}}}\widehat{\mathbb{E}}^{y,\tilde{y}}_{B,\widetilde{B};s,\tilde{s}}\exp\left\{\beta^2\mathscr{R}_{s,\tilde{s}}[B,\widetilde{B}]\right\}. \tag{2.11}$$

In light of (2.8), the factor $\mathrm{e}^{\alpha_s+\alpha_{\tilde{s}}}$ should be thought of, for ("microscopically") large $s,\tilde{s}$, as essentially a constant. This expression is analogous to the computation in [23, Lemma 3.1], to which it indeed reduces if V is taken to be white in time rather than colored as it is in our setting. Indeed, in the white-in-time case, the kernel $R(s,y)$ becomes a delta mass in s at $s=0$, and thus the quantity $\mathscr{R}_s[B]$ becomes the constant λs (the Itô–Stratonovich correction), not depending on B, so also $\alpha_\infty = 0$. In particular, in the white-in-time case the tilting (2.6) becomes trivial: we use the tilting to account for the time-correlations of the noise. Then from (3.2) we recover exactly the first display in the proof of [23, Lemma 3.1].

2.2. The Markov chain

A key point of [17] is that a Brownian motion tilted according to (2.6) can be approximately represented by a Markov chain. Since $R(s, y)$ is supported on times $s \in [-1, 1]$, the functional $\mathscr{R}_\bullet[B]$ that appears in (2.6) only involves interactions between the values of B at times of distance at most 2 from each other. Thus, if we "chunk" the Brownian motion into segments of length 1, the tilting only takes into account the interactions between each segment and the immediate preceding and succeeding segments. One can then represent the tilted Brownian motion as a Markov chain on the chunks, with the caveat that another, ultimately small, tilting is needed to account for the edge effects at time T.

It is shown in [17] that the Markov chain satisfies the *Doeblin condition*, which is to say that the transition measures uniformly majorize a (small) multiple of the stationary measure. This condition is an elementary tool in the theory of Markov chains; see e.g. [21] for an introduction. Therefore, at every step of the chain corresponding to a length-1 chunk of the Brownian motion, there is a probability bounded away from zero that the next step of the chain can be considered to be sampled from the stationary distribution. Conditional on this event occurring at a particular step, the chain is then at its stationary distribution. Therefore, the chain converges to its stationary distribution exponentially quickly.

We state these ideas precisely in the following theorem, which summarizes several results and discussions in [17]. We let $\Xi_T = \{\omega \in \mathcal{C}([0, T]) \mid \omega(0) = 0\}$, and, given $W_i \in \Xi_{T_i}$, we define $[W_1, \ldots, W_k] \in \Xi_{\sum_i T_i}$ by concatenating the increments, as in [17, (4.2)].

Theorem 2.1. *([17]) Let $T > 1$ and $N = \lfloor T \rfloor - 1$. There is a Markov chain $w_0, w_1, \ldots, w_N, w_{N+1}$, with $w_0 \in \Xi_{T-[T]}$ and $w_j \in \Xi_1$ for $1 \le j \le N+1$, which has the following properties.*

1. *(Time-homogeneity.) The transition probability measure*

$$\widehat{\pi}(w_j, \cdot) = \mathrm{Law}(w_{j+1} \mid w_j)$$

does not depend on j for $j = 1, \ldots, N-1$.

2. *(Relationship with the tilted Brownian motion.) There is a bounded, measurable, even functional $\mathscr{G} : \Xi_1 \to \mathbf{R}$ such that,if we put $W = [w_0, \ldots, w_{N+1}] \in \Xi_T$, and let $\widetilde{\mathbb{E}}_W$ denote expectation with respect to the measure in which W is obtained from the Markov chain, then we have, for any bounded continuous function $\mathscr{F}$ on Ξ_T, that*

$$\widehat{\mathbb{E}}_{B;T} \mathscr{F}[B] = \widetilde{\mathbb{E}}_W[\mathscr{F}[W]\mathscr{G}[w_N]]. \tag{2.12}$$

3. *(Doeblin condition.) There is a sequence of i.i.d. Bernoulli random variables η_j^W, $j = 1, 2, \ldots$, with success probability not depending on T, so that*

$$\mathrm{Law}(w_j \mid \eta_j^W = 1, \{w_i : i < j\}) = \overline{\pi},$$

where $\overline{\pi}$ is the invariant measure of $\widehat{\pi}$.

Theorem 2.1 summarizes several results of [17]. The Markov chain (w_k) is constructed in [17, Sect. 4.1]. Equation (2.12) is [17, (4.25)], where we use the notation $\mathscr{G}$ instead of $\mathscr{G}_\varepsilon$ because the functional in fact does not depend on the ε of [17] (which is the same as the ε in the present paper, but is playing no role in the present discussion). The functional $\mathscr{G}$ represents the additional tilting to account for edge effects at time T. This additional tilting should be thought of as an error term and in our arguments we will always strive to show that it does not play an important role; the reader who pretends that $\mathscr{G} \equiv 1$ will not miss the thrust of the arguments in the paper. The Doeblin condition is established in [17, (4.18)], as explained in the discussion surrounding [17, (4.27)].

We note again that Theorem 2.1 is trivial in the case when V is white in time: then the Markov chain is simply given by the independent increments of the Brownian motion, and is always at its stationary distribution.

We will use the notation

$$\widetilde{\mathbb{E}}^y_W \mathscr{F}[W] = \widetilde{\mathbb{E}}_W \mathscr{F}[y + W]. \tag{2.13}$$

Define the stopping times $\sigma_0^W = 0$, $\sigma_n^W = \min\{t \geq \sigma_{n-1}^W \mid \eta_t^W = 1\}$ and put, for $n \geq 0$,

$$\mathbf{W}_n^W = W_{\sigma_{n+1}^W} - W_{\sigma_n^W}. \tag{2.14}$$

This is the construction in [17, (4.27)]. The following lemma summarizes some results of [17] about these stopping times:

Lemma 2.2. *The family $\{\mathbf{W}_n^W\}_{n\geq 0}$ is a collection of independent, statistically isotropic random variables with exponential tails. Moreover, the elements of $\{\mathbf{W}_n^W\}_{n\geq 1}$ are identically distributed.*

Proof. The fact that $\{\mathbf{W}_n^W\}_{n\geq 0}$ is independent, and that the elements of $\{\mathbf{W}_n^W\}_{n\geq 1}$ are identically distributed, is an immediate consequence of the Doeblin condition and the time-homogeneity of the Markov chain. Isotropy follows from the isotropy of the construction. Exponential tails were established in [17, Lemma A.2]. □

The construction leading to (2.14) can be applied to pairs of paths as well, as explained at the end of [17, Sect. 4.1]. Given two independent copies W, $\widetilde{W}$ of the Markov chain, define

$$\eta_j^{W,\widetilde{W}} = \eta_j^W \eta_j^{\widetilde{W}},$$

and the stopping times

$$\sigma_n^{W,\widetilde{W}} = \begin{cases} 0 & n = 0; \\ \min\{t \geq \sigma_{n-1} \,:\, \eta_t^{W,\widetilde{W}} = 1\} & n \geq 1. \end{cases} \tag{2.15}$$

Then put

$$\mathbf{W}_n^{W,\widetilde{W}} = W_{\sigma_{n+1}^{W,\widetilde{W}}} - W_{\sigma_n^{W,\widetilde{W}}}, \qquad \widetilde{\mathbf{W}}_n^{W,\widetilde{W}} = \widetilde{W}_{\sigma_{n+1}^{W,\widetilde{W}}} - \widetilde{W}_{\sigma_n^{W,\widetilde{W}}}.$$

Analogously to (2.13), we use the notation $\widetilde{\mathbb{P}}^{y,\tilde{y}}_{W,\widetilde{W}} = \widetilde{\mathbb{P}}^y_W \otimes \widetilde{\mathbb{P}}^{\tilde{y}}_{\widetilde{W}}$. We have the following corollary of Lemma 2.2:

Corollary 2.3. *The family* $\{\mathbf{W}_n^{W,\widetilde{W}}\}_{n\geq 0} \cup \{\widetilde{\mathbf{W}}_n^{W,\widetilde{W}}\}_{n\geq 0}$ *is a collection of independent isotropic random variables with exponential tails.*[1] *Moreover, the elements of* $\{\mathbf{W}_n^{W,\widetilde{W}}\}_{n\geq 1} \cup \{\widetilde{\mathbf{W}}_n^{W,\widetilde{W}}\}_{n\geq 1}$ *are identically distributed.*

Now let us set

$$\kappa_1 = \mathbb{P}(\eta_j^W = 1), \qquad \kappa_2 = \mathbb{P}(\eta_j^{W,\widetilde{W}} = 1) = \kappa_1^2. \tag{2.16}$$

The next proposition gives an expression for the effective diffusivity a in (1.7) in terms of the Markov chain.

Proposition 2.4. *([17, Proposition 4.1]) There is a diagonal* $d \times d$ *matrix*

$$\mathbf{a} = aI_{d\times d} = \kappa_1 \widetilde{\mathbb{E}}_W[\mathbf{W}_n^w (\mathbf{W}_n^W)^t] \tag{2.17}$$

so that for any $t > 0$*, as* $\varepsilon \to 0$*, the process* $\{\varepsilon W_{\varepsilon^2 \tau}\}_{0\leq\tau\leq t}$ *(under the measure* $\widetilde{\mathbb{P}}_W$*) converges in distribution in* $\mathcal{C}([0,t])$ *to a Brownian motion with covariance matrix* $\mathbf{a}$.

Two Brownian motions in $d \geq 3$ will almost surely spend at most a finite amount of time within distance 1 of each other. The fact that this is also true for the Markov chains $W, \widetilde{W}$ is expressed in the next two propositions, and will play a crucial role in the sequel.

Proposition 2.5. *([17, Corollary 4.4]) There is a* $\beta_0 > 0$ *and a deterministic constant* $C < \infty$ *so that if* $0 \leq \beta < \beta_0$ *then for any* $s \geq 0$*,* $y, \tilde{y} \in \mathbb{R}^d$*, we have*

$$\widetilde{\mathbb{E}}_{W,\widetilde{W}}^{y,\tilde{y}}\left[\exp\left\{\beta^2 \mathscr{R}_{[s,\infty)}[W,\widetilde{W}]\right\} \mid \mathcal{F}_s\right] \leq C$$

with probability 1*, where* $\mathcal{F}_s$ *is the* σ*-algebra generated by the paths* $W, \widetilde{W}$ *on the time interval* $[0,s]$.

We will require a slightly stronger version of Proposition 2.5, which can be proved similarly.

Proposition 2.6. *There is a* $\beta_0 > 0$ *and a deterministic constant* $C < \infty$ *so that if* $0 \leq \beta < \beta_0$ *then for all* $r, \tilde{r} > 0$*, we have*

$$\widetilde{\mathbb{E}}_{W,\widetilde{W}}^{y,\tilde{y}}\left[\exp\left\{\beta^2 \mathscr{R}_\infty[W,\widetilde{W}]\right\} \mid \mathcal{F}_{r,\tilde{r}}\right] \leq C$$

with probability 1*, where* $\mathcal{F}_{r,\tilde{r}}$ *is the* σ*-algebra generated by the path* W *on* $[0,r]$ *and the path* $\widetilde{W}$ *on* $[0,\tilde{r}]$.

We also need some estimates from [17] on various error terms.

Lemma 2.7. *([17, (4.30)]) There is a constant* C *so that*

$$\widetilde{\mathbb{E}}_W|\varepsilon W_{\varepsilon^{-2}t_2} - \varepsilon W_{\varepsilon^{-2}t_1}|^2 \leq C(t_2 - t_1). \tag{2.18}$$

[1] We use the standard terminology that a random variable X has exponential tails if there are constants $C, c > 0$ such that $\mathbf{P}(|X| > x) \leq Ce^{-cx}$ for all $x > 0$.

Lemma 2.8. *([17, Lemma A.3]) For any $\chi > 0$, there are constants $0 < c, C < \infty$ so that if, for each T, $\mathscr{F}_T : \Xi_T \to \mathbf{R}$ is a bounded functional on Ξ_T, and $\{S_n\}$, $\{T_n\}$ are sequences of real numbers such that $S_n, T_n, S_n - T_n \to +\infty$, then*

$$\left|\widetilde{\mathbb{E}}_W \mathscr{F}_{T_n}[W|_{[0,T_n]}] - \widetilde{\mathbb{E}}_W \mathscr{F}_{T_n}[W|_{[0,T_n]}]\mathscr{G}(w_{S_n})\right| \\ \le C\left(\widetilde{\mathbb{E}}_W\left(\mathscr{F}_{T_n}[W|_{[0,T_n]}]\right)^{\chi}\right)^{1/\chi} \exp\{-c(T_n \wedge (S_n - T_n))\}.$$

Here, $\mathscr{G}$ is as in Theorem 2.1. The rate of convergence is not stated explicitly in [17, Lemma A.3], but it comes from the proof there.

Lemma 2.9. *([17, Lemma A.2]) We have constants $0 < c, C < \infty$ so that*

$$\widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}\left[\max_{r,\tilde{r}\in[\sigma_n,\sigma_{n+2}]}\left(\left|W_r - W_{\sigma_n^{W,\widetilde{W}}}\right| + \left|\widetilde{W}_{\tilde{r}} - \widetilde{W}_{\sigma_n^{W,\widetilde{W}}}\right|\right) > a\right] \le C\mathrm{e}^{-ca}.$$

2.3. Estimates on Path Intersections

These preliminaries having been completed, we now prove a fact that will be essential for us: that two independent copies of the Markov chain, started at distance of order ε^{-1} from each other, pass within distance 1 of each other with probability ε^{d-2}. This is the same situation as for the standard Brownian motion. Explicitly, we prove the following (which does not in fact require the assumption that β is small):

Proposition 2.10. *There is a constant C so that*

$$\widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}\left[\inf_{\substack{r,\tilde{r}>0\\|r-\tilde{r}|\le 1}} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] \le \frac{C}{|x-\tilde{x}|^{d-2}}.$$

In order to prove Proposition 2.10, we first prove it just at regeneration times. For the rest of this section, to economize on notation we put $\sigma_n := \sigma_n^{W,\widetilde{W}}$ (defined in (2.15)).

Lemma 2.11. *For all $A > 0$, we have*

$$\widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}\left[\inf_{n\ge 0}\left|W_{\sigma_n} - \widetilde{W}_{\sigma_n}\right| \le A\right] \le \frac{A^{d-2}}{|x-\tilde{x}|^{d-2}}.$$

Proof. Let

$$X_n = W_{\sigma_n} - \widetilde{W}_{\sigma_n},$$

let $\mathcal{H}_n$ be the σ-algebra generated by $X_1, \dots, X_n$, and set

$$q(z) = \frac{1}{(|z| \vee A)^{d-2}}.$$

For any $z \in \mathbb{R}^d$ with $|z| \geq A$ and $M > 0$, if we let $\mathrm{d}S$ denote the surface measure on $\{|\tilde{z} - z| = M\}$, then we have

$$\fint_{|\tilde{z}-z|=M} q(\tilde{z})\,\mathrm{d}S(\tilde{z}) \leq \fint_{|\tilde{z}-z|=M} \frac{1}{|\tilde{z}|^{d-2}}\,\mathrm{d}S(\tilde{z}) \leq \frac{1}{|z|^{d-2}} = q(z) \tag{2.19}$$

by the mean value inequality for superharmonic functions, as $z \mapsto |z|^{-d+2}$ is superharmonic. Here, the notation $\fint$ means that we normalize the surface measure to have total mass 1. Let ω be the smallest n so that $|X_n| \leq A$, or ∞ if $|X_n| > A$ for all n. Note that ω is a stopping time with respect to the filtration $\{\mathcal{H}_n\}$. Also, the distribution of $X_n - X_{n-1}$ is isotropic and independent of $\mathcal{H}_{n-1}$ for each $n \geq 1$ by Corollary 2.3. Therefore, we have, whenever $n - 1 < \omega$,

$$\begin{aligned}\widetilde{\mathbb{E}}_{W,\widetilde{W}}[q(X_n) \mid \mathcal{H}_{n-1}] &= \int_{\mathbb{R}^d} q(z)\,\mathrm{d}\widetilde{\mathbb{P}}_{W,\widetilde{W}}(X_n = z \mid \mathcal{H}_{n-1}) \\ &= \int_{\mathbb{R}} \fint_{|z-X_{n-1}|=M} q(z)\,\mathrm{d}S(z)\,\mathrm{d}\widetilde{\mathbb{P}}_{W,\widetilde{W}}(|X_n - X_{n-1}| = M) \\ &\leq \int_{\mathbb{R}} q(X_{n-1})\,\mathrm{d}\widetilde{\mathbb{P}}_{W,\widetilde{W}}(|X_n - X_{n-1}| = M) = q(X_{n-1}),\end{aligned}$$

where the last inequality is by (2.19). Thus, the sequence $(q(X_{n\wedge\omega}))_n$ is a supermartingale. By the optional stopping theorem, for any n we have

$$\frac{1}{|x-\tilde{x}|^{d-2}} = q(X_0) \geq \widetilde{\mathbb{E}}_{W,\widetilde{W}} q(X_{n\wedge\omega}) \geq \frac{1}{A^{d-2}} \widetilde{\mathbb{P}}_{W,\widetilde{W}}(\omega \leq n).$$

Therefore, we have

$$\widetilde{\mathbb{P}}_{W,\widetilde{W}}(\omega < \infty) \leq \frac{A^{d-2}}{|x-\tilde{x}|^{d-2}}$$

by Fatou's lemma. □

Proof of Proposition 2.10. Let

$$B_n = \max_{r,\tilde{r}\in[\sigma_n,\sigma_{n+2}]} \left(\left|W_r - W_{\sigma_n^{W,\widetilde{W}}}\right| + \left|\widetilde{W}_{\tilde{r}} - \widetilde{W}_{\sigma_n^{W,\widetilde{W}}}\right|\right)$$

and

$$\omega_M = \inf\left\{n \geq 0 : |W_{\sigma_n} - \widetilde{W}_{\sigma_n}| \leq 2^M\right\}$$

We have

$$\begin{aligned}&\left\{\inf_{|r-\tilde{r}|\leq 1} |W_r - \widetilde{W}_{\tilde{r}}| \leq 1\right\} \\ &\quad\subseteq \bigcup_{M=0}^{\infty}\bigcup_{n=0}^{\infty}\left(\left\{|W_{\sigma_n} - \widetilde{W}_{\sigma_n}| \leq 2^M\right\} \cap \left\{B_n \geq 2^{M-1} - 1\right\}\right)\end{aligned}$$

$$\subseteq \bigcup_{M=0}^{\infty} \left[\{\omega_M < \infty\} \cap \left(\bigcup_{n=\omega_M}^{\infty} \left(\left\{ |W_{\sigma_n} - \widetilde{W}_{\sigma_n}| \le 2^M \cap \left\{ B_n \ge 2^{M-1} - 1 \right\} \right\} \right) \right) \right]. \tag{2.20}$$

Therefore, we can estimate, abbreviating $\mathbb{P} = \widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}$ and letting the constant C change from line to line,

$$\begin{aligned}
&\mathbb{P}\left[\inf_{|r-\tilde{r}|\le 1} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] \\
&\le \sum_{M,\ell=0}^{\infty} \mathbb{P}(\omega_M = \ell) \sum_{n=\ell}^{\infty} \mathbb{P}\left[|W_{\sigma_n} - \widetilde{W}_{\sigma_n}| \le 2^M \mid \omega_M = \ell\right] \mathbb{P}\left[B_n \ge 2^{M-1} - 1\right] \\
&\le C \sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)} \sum_{\ell=0}^{\infty} \mathbb{P}(\omega_M = \ell) \sum_{n=\ell}^{\infty} \frac{2^{Md}}{(n-\ell+1)^{d/2}} \\
&= C \sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)+CMd} \mathbb{P}(\omega_M < \infty) \\
&\le C \sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)+CMd} \cdot \frac{2^{(d-2)M}}{|x-\tilde{x}|^{d-2}} \le \frac{C}{|x-\tilde{x}|^{d-2}},
\end{aligned}$$

where the first inequality is by (2.20), the second is by Lemma 2.9 and a local central limit theorem ([24] as applied in [17, (4.36)]) and the third is by Lemma 2.11. □

We also need a slightly different version of the bound in Proposition 2.10:

Proposition 2.12. *There is a constant C so that*

$$\widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}\left[\inf_{\substack{r,\tilde{r}>s \\ |r-\tilde{r}|\le 1}} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] \le C s^{-d/2+1}.$$

Proof. Recall the definition (2.16) of κ_2 and put $n_0 = \frac{s}{2\kappa_2}$. Again we abbreviate $\mathbb{P} = \widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}$ and let constants change from line to line. We can estimate

$$\mathbb{P}\left[\inf_{\substack{r,\tilde{r}>s \\ |r-\tilde{r}|\le 1}} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] \le \mathbb{P}\left[\inf_{\substack{r,\tilde{r}>\sigma_{n_0} \\ |r-\tilde{r}|\le 1}} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] + \mathbb{P}(\sigma_{n_0} \ge s).$$

A simple large-deviations estimate for geometric random variables yields

$$\mathbb{P}(\sigma_{n_0} \ge s) \le C\mathrm{e}^{-cn_0} \le Cs^{1-d/2},$$

so it suffices to show that

$$\mathbb{P}\left[\inf_{\substack{r,\tilde{r}>\sigma_{n_0} \\ |r-\tilde{r}|\le 1}} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] \le C n_0^{1-d/2}.$$

Define

$$B_k = \max_{r,\tilde{r}\in[\sigma_k,\sigma_{k+2}]} \left(|W_r - W_{\sigma_k}| + |\widetilde{W}_{\tilde{r}} - \widetilde{W}_{\sigma_k}|\right),$$

so we have

$$\mathbb{P}\left[\inf_{\substack{r,\tilde{r}>\sigma_{n_0}\\ |r-\tilde{r}|\le 1}} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right] \le \sum_{M=0}^{\infty}\sum_{k=n_0}^{\infty} \mathbb{P}\left[|W_{\sigma_k} - \widetilde{W}_{\sigma_k}| \le 2^M\right]$$
$$\cdot \mathbb{P}[B_k \ge 2^{M-1} - 1]$$
$$\le C\sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)} \sum_{k=n_0}^{\infty} \frac{2^{Md}}{k^{d/2}} = Cn_0^{1-d/2} \sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)+CMd} \le Cn_0^{1-d/2},$$

where the second inequality again uses the local limit theorem of [24]. □

3. The Stationary Solution

The strategy of the proof of Theorem 1.1 is typical for the construction of a stationary solution to a PDE: we consider the Cauchy problem with initial data given at time $s = -S$, and pass to the limit $S \to +\infty$. This lets us obtain a global-in-time solution to the problem that satisfies appropriate uniform bounds, provided that the Lyapunov exponent $\lambda = \lambda(\beta)$ is chosen appropriately. Let $\Psi(s, y; S)$ be the solution to

$$\cdot\partial_s \Psi(s, y; S) = \frac{1}{2}\Delta\Psi(s, y; S) + (\beta V(s, y) - \lambda)\Psi(s, y; S), \qquad s > -S; \tag{3.1}$$
$$\Psi(-S, y; S) = 1.$$

The heart of the proof of Theorem 1.1 is the following proposition:

Proposition 3.1. *If β is sufficiently small, then there exists $\lambda = \lambda(\beta)$ and a constant $C < \infty$ so that, with this choice of λ in (3.1), for any $0 \le S_1 \le S_2$, we have*

$$\mathbf{E}(\Psi(0, y; S_2) - \Psi(0, y; S_1))^2 \le CS_1^{-d/2+1}.$$

Before we prove Proposition 3.1, we show how it implies Theorem 1.1.

Proof. (Proof of Theorem 1.1.) For a positive weight $w \in L^1(\mathbb{R}^d)$, consider the weighted space $L^2_w(\mathbb{R}^d)$, with the inner product

$$\langle f, g\rangle_{L^2_w(\mathbb{R}^d)} = \int f(y)\overline{g(y)}w(y)\,\mathrm{d}y.$$

By Proposition 3.1 and the stationarity of V in time, we have

$$\begin{aligned}
&\mathbf{E}\|\Psi(s,\cdot;S_1)-\Psi(s,\cdot;S_2)\|^2_{L^2_w(\mathbb{R}^d)}\\
&\quad=\int \mathbf{E}|\Psi(0,y;s+S_1)-\Psi(0,y;s+S_2)|^2 w(y)\,\mathrm{d}y\\
&\quad\le C(s+S_1)^{-d/2+1}\|w\|_{L^1(\mathbb{R}^d)},
\end{aligned}\tag{3.2}$$

and the right-hand side converges to 0 as $S_1, S_2 \to \infty$, locally uniformly in s. Hence, the family $\Psi(s,y;S)$ converges in $L^2(\Omega; L^2_w(\mathbb{R}^d))$, locally uniformly in s, to a limit $\widetilde{\Psi}$. (Here Ω denotes the probability space on which V is defined.) The stationarity of $\widetilde{\Psi}$ is standard. The convergence of Ψ to $\widetilde{\Psi}$ locally in $L^2(\Omega; L^2_w(\mathbb{R}^d))$ implies that $\widetilde{\Psi}$ satisfies (1.15) in a weak sense almost surely, hence in a strong sense almost surely by standard parabolic regularity.

To prove the convergence claimed in (1.16), we use an argument similar to the above. In particular, we note that the solution $\Psi(s,y)$ to (1.14) is stationary in y, as is $\widetilde{\Psi}(s,y)$, so for any fixed $y \in \mathbb{R}^d$ we have

$$\begin{aligned}
\mathbf{E}|\Psi(s,y)-\widetilde{\Psi}(s,y)|^2\int w(y')\,\mathrm{d}y' &= \int \mathbf{E}|\Psi(s,y')-\widetilde{\Psi}(s,y')|^2 w(y')\,\mathrm{d}y'\\
&= \int \mathbf{E}|\Psi(0,y';s)-\widetilde{\Psi}(0,y')|^2 w(y')\,\mathrm{d}y,
\end{aligned}\tag{3.3}$$

and the right-hand side is bounded by a constant times $s^{-d/2+1}$ as $s \to \infty$ by the definition of $\widetilde{\Psi}$. (In the second equality of (3.3), we used the time-stationarity of $(V, \widetilde{\Psi})$.) □

We also record the covariance kernel of the stationary solution.

Corollary 3.2. *We have*

$$\mathbf{E}[\widetilde{\Psi}(s,y)\widetilde{\Psi}(s,\tilde{y})] = \mathrm{e}^{2\alpha_\infty}\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}} \exp\left\{\beta^2\mathscr{R}_\infty[W,\widetilde{W}]\right\}.$$

In the remainder of this section, we set about proving Proposition 3.1. The proof will rely on the Feynman–Kac formula. We recall the Feynman–Kac formula for $\Psi(s,y;S)$, which comes from (2.2) by a simple time-change:

$$\Psi(s,y;S) = \mathbb{E}^y_B \exp\left\{\beta\mathscr{V}_{s;s+S}[B] - \lambda(s+S)\right\}.\tag{3.4}$$

We first note that spatial stationarity allow us to take $y=0$, and then the same computation that leads to (3.2) gives

$$
\begin{aligned}
&\mathbf{E}(\Psi(0,0;S_2)-\Psi(0,0;S_1))^2\\
&\quad= e^{2\alpha_{S_2}}\widehat{\mathbb{E}}^{0,0}_{B,\widetilde{B};S_2}\exp\left\{\beta^2\mathscr{R}_{S_2}[B,\widetilde{B}]\right\}\\
&\qquad-2e^{\alpha_{S_2}+\alpha_{S_1}}\widehat{\mathbb{E}}^{0,0}_{B,\widetilde{B};S_2,S_1}\exp\left\{\beta^2\mathscr{R}_{S_2,S_1}[B,\widetilde{B}]\right\}\\
&\qquad+e^{2\alpha_{S_1}}\widehat{\mathbb{E}}^{0,0}_{B,\widetilde{B};S_2,S_1}\exp\left\{\beta^2\mathscr{R}_{S_1}[B,\widetilde{B}]\right\}.
\end{aligned}
\tag{3.5}
$$

Let us now explain intuitively why the right-hand side of this expression should be small. First, we recall that α_s has a limit as $s\to\infty$ by (2.8). Second, as we have observed in Sect. 2.3, in dimension $d\geq 3$ two Brownian motions will almost surely spend at most a finite amount of time within distance 1 of each other. In fact, the amount of time they spend within distance 1 of each other has (some but not all) exponential moments. Only such times contribute to $\mathscr{R}_\bullet[B,\widetilde{B}]$. The thrust of Sect. 2 above was that the tilted Brownian motion, on large scales, again looks like a Brownian motion. This makes it plausible that, under the tilted measure, the exponential moments of $\mathscr{R}_{S_1}[B,\widetilde{B}]$, $\mathscr{R}_{S_2,S_1}[B,\widetilde{B}]$, and $\mathscr{R}_{S_2}[B,\widetilde{B}]$ are all close to each other, making the right-hand side of (3.5) small as $S_1,S_2\to\infty$.

In the rest of this section, we make this reasoning precise. We emphasize that the computation that we will do still has content in the case when V is white in time; in this case the tilting has no effect and B and $\widetilde{B}$ are simply Brownian motions. In that case, the approximations from Sect. 2 are unnecessary and the previous paragraph is essentially a proof. Nonetheless, the reader may find it helpful on first reading to pretend that B and $\widetilde{B}$ are Brownian motions. (In this case the computation is very similar to that of [23].)

Our first lemma is the workhorse of the argument. It makes the above intuition, which is standard for the Brownian motion, precise for the case of the Markov chain.

Lemma 3.3. *There exists a constant $C<\infty$ so that for all β sufficiently small, the following holds. If $1\leq s\leq s'\leq\tilde{s}\leq\tilde{s}'$, then*

$$
\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}}\left|\exp\left\{\beta^2\mathscr{R}_{\tilde{s},\tilde{s}'}[W,\widetilde{W}]\right\}-\exp\left\{\beta^2\mathscr{R}_{s,s'}[W,\widetilde{W}]\right\}\right|\leq C(s-1)^{1-d/2}.
$$

Proof. We have

$$
\begin{aligned}
&\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}}\left|\exp\left\{\beta^2\mathscr{R}_{\tilde{s},\tilde{s}'}[W,\widetilde{W}\right\}-\exp\left\{\beta^2\mathscr{R}_{s,s'}[W,\widetilde{W}]\right\}\right|\\
&\quad\leq\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}}\left|\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde{W}]\right\}-\exp\left\{\beta^2\mathscr{R}_{s}[W,\widetilde{W}]\right\}\right|\\
&\quad\leq\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}}\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde{W}]\right\}\mathbf{1}\{\mathscr{R}_\infty[W,\widetilde{W}]\neq\mathscr{R}_S[W,\widetilde{W}]\}\\
&\quad\leq\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}}\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde{W}]\right\}\mathbf{1}\{(\exists r,\tilde{r}\geq s-1)\,|r-\tilde{r}|\\
&\quad\leq 2\text{ and }|W_r-\widetilde{W}_{\tilde{r}}|\leq 1\}.
\end{aligned}
$$

On the event that $\{\mathscr{R}_\infty[W,\widetilde{W}]\neq\mathscr{R}_s[W,\widetilde{W}]\}$, let $\tau<\tilde{\tau}$ be the first pair of times after $s-1$ such that $|\tau-\tilde{\tau}|\leq 2$ and $|W_\tau-\widetilde{W}_{\tilde{\tau}}|\leq 1$. Then we have

$$
\begin{aligned}
&\widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}} \left| \exp\left\{\beta^2 \mathscr{R}_{\tilde{s},\tilde{s}'}[W,\widetilde{W}\right\} - \exp\left\{\beta^2 \mathscr{R}_{s,s'}[W,\widetilde{W}]\right\}\right| \\
&\quad \le \int_{s-1}^{\infty} \int_{r}^{r+2} \widetilde{\mathbb{E}}^{y,\tilde{y}}_{W,\widetilde{W}} \left[\exp\left\{\beta^2 \mathscr{R}_{\infty}[W,\widetilde{W}]\right\} \mid \tau = r, \tilde{\tau} = \tilde{r}\right] \\
&\qquad \cdot \mathrm{d}\widetilde{\mathbb{P}}^{y,\tilde{y}}_{W,\widetilde{W}}(\tau = r, \tilde{\tau} = \tilde{r}) \\
&\quad \le C \widetilde{\mathbb{P}}^{y,\tilde{y}}_{W,\widetilde{W}} \left((\exists r, \tilde{r} \ge s-1)\, |r - \tilde{r}| \le 2 \text{ and } |W_r - \widetilde{W}_{\tilde{r}}| \le 1\right) \le C(s-1)^{1-d/2},
\end{aligned}
$$

where the second inequality is by Proposition 2.6 and the last is by Proposition 2.12. □

Now we can prove Proposition 3.1. The proof combines Lemma 3.3 with various error bounds from Sect. 2.

Proof of Proposition 3.1. We first re-write (3.5) in terms of the Markov chain using (2.12):

$$
\begin{aligned}
\mathbf{E}(\Psi(0,0;S_2) - \Psi(0,0;S_1))^2 &= \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{2\alpha_{S_2}} \exp\left\{\beta^2 \mathscr{R}_{S_2}[W,\widetilde{W}]\right\} \mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \\
&\quad \mathscr{G}[\tilde{w}_{\lfloor S_2 \rfloor - 1}] \\
&\quad - 2\mathrm{e}^{\alpha_{S_2} + \alpha_{S_1}} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\} \mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}] \\
&\quad + \mathrm{e}^{2\alpha_{S_1} + \alpha_{S_1}} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{S_1}[W,\widetilde{W}]\right\} \mathscr{G}[w_{\lfloor S_1 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}]. \qquad (3.6)
\end{aligned}
$$

For any $S_1 \le S_2$ we can decompose

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{\alpha_{S_2} + \alpha_{S_1}} \exp\left\{\beta^2 \mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\} \mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}] \\
&\quad \cdot = \mathrm{e}^{2\alpha_\infty} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{\frac{9}{10}S_2, \frac{9}{10}S_1}[W,\widetilde{W}]\right\} \\
&\qquad + \mathrm{e}^{2\alpha_\infty} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{\frac{9}{10}S_2, \frac{9}{10}S_1}[W,\widetilde{W}]\right\} \left(\mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}] - 1\right) \\
&\qquad + \mathrm{e}^{2\alpha_\infty} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left(\exp\left\{\beta^2 \mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\} - \exp\left\{\beta^2 \mathscr{R}_{\frac{9}{10}S_2, \frac{9}{10}S_1}[W,\widetilde{W}]\right\}\right) \\
&\qquad \cdot \mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}] \\
&\qquad + \left(\mathrm{e}^{\alpha_{S_2} + \alpha_{S_1}} - \mathrm{e}^{2\alpha_\infty}\right) \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\} \\
&\qquad \cdot \mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}]. \qquad (3.7)
\end{aligned}
$$

Now (2.8), Lemma 3.3, and Proposition 2.6 allow us to control the last term of (3.7):

$$
\begin{aligned}
&\lim_{S_1,S_2 \to \infty} S_2^{d/2-1} \left(\mathrm{e}^{\alpha_{S_2} + \alpha_{S_1}} - \mathrm{e}^{2\alpha_\infty}\right) \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\} \\
&\quad \cdot \mathscr{G}[w_{\lfloor S_2 \rfloor - 1}] \mathscr{G}[\tilde{w}_{\lfloor S_1 \rfloor - 1}] = 0.
\end{aligned}
$$

Lemma 3.3 also allows us to bound the third term of (3.7):

$$\cdot S_1^{\frac{d}{2}-1}\left|\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left(\exp\left\{\beta^2\mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\}-\exp\left\{\beta^2\mathscr{R}_{\frac{9}{10}S_2,\frac{9}{10}S_1}[W,\widetilde{W}]\right\}\right)\right.$$
$$\left.\cdot\mathscr{G}[w_{\lfloor S_2\rfloor-1}]\mathscr{G}[\tilde{w}_{\lfloor S_1\rfloor-1}]\right|$$
$$\le \|\mathscr{G}\|_\infty^2(S_1\wedge S_2)^{d/2-1}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left(\exp\left\{\beta^2\mathscr{R}_{S_2,S_1}[W,\widetilde{W}]\right\}\right.$$
$$\left.-\exp\left\{\beta^2\mathscr{R}_{\frac{9}{10}S_2,\frac{9}{10}S_1}[W,\widetilde{W}]\right\}\right)\le C,$$

for a constant C independent of S_1 and S_2. For the second term of (3.7), we can use Lemma 2.8 to get

$$\limsup_{S_1,S_2\to\infty} S_1^{d/2-1}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\exp\left\{\beta^2\mathscr{R}_{\frac{9}{10}S_2,\frac{9}{10}S_1}[W,\widetilde{W}]\right\}\left(\mathscr{G}[w_{\lfloor S_2\rfloor-1}]\mathscr{G}[\tilde{w}_{\lfloor S_1\rfloor-1}]-1\right)=0.$$

Finally, we have that

$$S_1^{d/2-1}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left[\exp\left\{\beta^2\mathscr{R}_{\frac{9}{10}S_2}[W,\widetilde{W}]\right\}-2\exp\left\{\beta^2\mathscr{R}_{\frac{9}{10}S_2,\frac{9}{10}S_1}[W,\widetilde{W}]\right\}\right.$$
$$\left.+\exp\left\{\beta^2\mathscr{R}_{\frac{9}{10}S_1}[W,\widetilde{W}]\right\}\right]$$

is bounded above independently of S_1 and S_2, also by Lemma 3.3. Substituting (3.7) into (3.6), and then applying the last four bounds, we see that

$$\mathbf{E}(\Psi(0,y;S_2)-\Psi(0,y;S_1))^2\le CS_1^{-d/2+1},$$

as claimed. □

4. The Effective Noise Strength

In this section, we explain how the effective noise strength parameter ν in (1.10) arises from the stationary solution $\widetilde{\Psi}$ and prove Theorem 1.2.

Lemma 4.1. *If β is sufficiently small and $g\in\mathcal{C}_c^\infty(\mathbb{R}^d)$, then we have*

$$\lim_{t\to\infty}\left|\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x)\,\mathrm{d}x\right)\right.$$
$$\left.-\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\widetilde{\Psi}(0,\varepsilon^{-1}x)\,\mathrm{d}x\right)\right|=0,$$

uniformly in $\varepsilon>0$.

Proof. We have

$$\left|\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x)\,\mathrm{d}x\right)-\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\widetilde{\Psi}(0,\varepsilon^{-1}x)\,\mathrm{d}x\right)\right|$$
$$\le\varepsilon^{-d+2}\|g\|_{L^1(\mathbb{R}^d)}\int|g(x)|\mathbf{E}\left|\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x)-\widetilde{\Psi}(\varepsilon^{-2}t,\varepsilon^{-1}x)\right|^2\,\mathrm{d}x$$
$$\le C\varepsilon^{-d+2}\|g\|_{L^1(\mathbb{R}^d)}^2(\varepsilon^{-2}t)^{-d/2+1}\le C\|g\|_{L^1(\mathbb{R}^d)}^2t^{-d/2+1},$$

where the first inequality is by the time-stationarity of $\widetilde{\Psi}$ and Jensen's inequality and the second is by (1.16). □

We recall from [17, Lemmas 3.1, 3.2 and 3.3] that

$$\lim_{\varepsilon\to 0} \operatorname{Var}\left(\frac{e^{-\alpha_{t/\varepsilon^2}}}{\varepsilon^{d/2-1}}\int g(x)\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x)\,dx\right) = \operatorname{Var}\left(\int g(x)\psi(t,x)\,dx\right), \tag{4.1}$$

where ψ is the solution to the Edwards-Wilkinson stochastic partial differential equation

$$\begin{aligned} \partial_t\psi &= \frac{1}{2}a\Delta\psi + \beta\nu\dot{W}, \qquad t>0, x\in\mathbb{R}^d;\\ \psi(0,x) &= 0, \end{aligned} \tag{4.2}$$

which is simply (1.10) with $\overline{u}\equiv 1$.

Lemma 4.2. *We have*

$$\lim_{t\to\infty} \operatorname{Var}\left(\int g(x)\psi(t,x)\,dx\right) = \beta^2\nu^2\int_0^\infty\int |\overline{g}(r,x)|^2\,dx\,dr, \tag{4.3}$$

where $\overline{g}$ is the solution of

$$\begin{aligned} \partial_t\overline{g}(t,x) &= \frac{1}{2}a\Delta\overline{g}(t,x), \qquad t>0, x\in\mathbb{R}^d;\\ \cdot\overline{g}(0,x) &= g(x). \end{aligned}$$

Proof. As in [17, (3.16)], we have

$$\operatorname{Var}\left(\int g(x)\psi(t,x)\,dx\right) = \beta^2\nu^2\int_0^t\int|\overline{g}(t-r,x)|^2\,dx\,dr = \beta^2\nu^2 \cdot\int_0^t\int|\overline{g}(r,x)|^2\,dx\,dr,$$

and then the result follows by taking $t\to\infty$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $\delta>0$. By Lemmas 4.1 and 4.2, we can choose t large enough, independently of ε, so that

$$\left|\operatorname{Var}\left(\int g(x)\psi(t,x)\,dx\right) - \beta^2\nu^2\int_0^\infty\int|\overline{g}(r,x)|^2\,dx\,dr\right| < \delta/3$$

and

$$\left|\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x)\,dx\right) - \operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\widetilde{\Psi}(\varepsilon^{-2}t,\varepsilon^{-1}x)\,dx\right)\right| < \delta/3.$$

Then by (4.1) we can choose ε so small that

$$\left|\operatorname{Var}\left(\varepsilon^{-\alpha_{t/\varepsilon^2}}\varepsilon^{-d/2+1}\int g(x)\Psi(\varepsilon^{-2}t,\varepsilon^{-1}x)\,\mathrm{d}x\right)-\operatorname{Var}\left(\int g(x)\psi(t,x)\,\mathrm{d}x\right)\right|<\delta/3.$$

Using the triangle inequality on the last three expressions, and recalling (2.8), we obtain

$$\lim_{\varepsilon\to 0}\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\widetilde{\Psi}(0,\varepsilon^{-1}x)\,\mathrm{d}x\right)=\mathrm{e}^{2\alpha_\infty}\beta^2\nu^2\int_0^\infty\int|\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r. \tag{4.4}$$

The left-hand side of (4.4) is equal to

$$\lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\varepsilon^{-d+2}\operatorname{Cov}\left(\widetilde{\Psi}(0,\varepsilon^{-1}x),\widetilde{\Psi}(0,\varepsilon^{-1}\tilde{x})\right)\,\mathrm{d}x\,\mathrm{d}\tilde{x},$$

while the right-hand side of (4.4) is equal to

$$\begin{aligned}\mathrm{e}^{2\alpha_\infty}\beta^2\nu^2&\int\int\left(\int_0^\infty\int G_a(r,z-x)G_a(r,z-\tilde{x})\,\mathrm{d}z\,\mathrm{d}r\right)g(x)g(\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x}\\&=\mathrm{e}^{2\alpha_\infty}\beta^2\nu^2ca^{-1}\int\int|x-\tilde{x}|^{-d+2}g(x)g(\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x},\end{aligned}$$

where G_a and c are defined as in (1.17)–(1.18). Therefore, we have

$$\nu^2=\frac{a\lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\varepsilon^{-d+2}\operatorname{Cov}\left(\widetilde{\Psi}(0,\varepsilon^{-1}x),\widetilde{\Psi}(0,\varepsilon^{-1}\tilde{x})\right)\,\mathrm{d}x\,\mathrm{d}\tilde{x}}{c\mathrm{e}^{2\alpha_\infty}\beta^2\int\int g(x)g(\tilde{x})|x-\tilde{x}|^{-d+2}\,\mathrm{d}x\,\mathrm{d}\tilde{x}},$$

which is (1.19). □

5. The Effective Diffusivity

In this section we explain how to relate the effective diffusivity a to the asymptotic expansion (1.12).

5.1. The Solvability Condition

We first explain how the effective diffusivity a can be recovered formally from the homogenization correctors for (1.12). We define these correctors now, and for the moment we disregard the question of their existence. We start with the equations (1.21)–(1.22) for the terms u_1 and u_2 in the formal asymptotic expansion (1.12) for u^ε. We will replace Ψ on the right-hand side of these equations by the stationary solution $\widetilde{\Psi}$, so our formal starting point is

$$\begin{aligned}\partial_s u_1(t,x,s,y)=&\frac{1}{2}\Delta_y u_1(t,x,s,y)+(\beta V(s,y)-\lambda)u_1(t,x,s,y)\\&+\nabla_y\widetilde{\Psi}(s,y)\cdot\nabla_x\overline{u}(t,x)\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\partial_s u_2(t,x,s,y) = \frac{1}{2}\Delta_y u_2(t,x,s,y) + (\beta V(s,y) - \lambda)u_2(t,x,s,y)\\ + \nabla_y \cdot \nabla_x u_1(t,x,s,y)\\ + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\Delta_x \overline{u}(t,x).\end{aligned} \tag{5.2}$$

We can now formally decompose the solution to (5.1) as

$$u_1(t,x,s,y) = \tilde{\boldsymbol{\omega}}(s,y) \cdot \nabla_x \overline{u}(t,x), \tag{5.3}$$

where $\tilde{\boldsymbol{\omega}}(s,y) = (\tilde{\omega}^{(1)}(s,y),\ldots,\tilde{\omega}^{(d)}(s,y))$ is a space-time-stationary solution to

$$\partial_s \tilde{\omega}^{(k)} = \frac{1}{2}\Delta_y \tilde{\omega}^{(k)} + (\beta V - \lambda)\tilde{\omega}^{(k)} + \frac{\partial \widetilde{\Psi}}{\partial y_k}. \tag{5.4}$$

We note that, unlike the random heat equation (1.15), the forced equation (5.4) may not have stationary solutions in all $d \geq 3$. Nevertheless, the formal computation will give us an idea of how the effective diffusivity can be approximated. By Theorem 1.1, applied with time reversed (or equivalently to the random heat equation with potential $V(-s,y)$), we also have a stationary solution $\widetilde{\Phi}$ to the equation

$$-\partial_s \widetilde{\Phi} = \frac{1}{2}\Delta \widetilde{\Phi} + (\beta V - \lambda)\widetilde{\Phi}. \tag{5.5}$$

Multiplying (5.2) by Φ and using (5.3) and (5.5) gives

$$\begin{aligned}\partial_s(\widetilde{\Phi}(s,y)u_2(t,x,s,y)) = \frac{1}{2}\widetilde{\Phi}(s,y)\Delta_y u_2(t,x,s,y) - \frac{1}{2}u_2(t,x,s,y)\Delta\widetilde{\Phi}(s,y)\\ + \widetilde{\Phi}(s,y)\,\mathrm{tr}(\nabla_y \tilde{\boldsymbol{\omega}}(s,y) \cdot \mathrm{Hess}\,\overline{u}(t,x))\\ + \frac{1}{2}(1-a)\widetilde{\Phi}(s,y)\widetilde{\Psi}(s,y)\Delta_x \overline{u}(t,x).\end{aligned} \tag{5.6}$$

The assumed stationarity of u_2 in s and the stationarity of $\widetilde{\Phi}$ in s imply that the expectation of the left-hand side is 0. Stationarity of u_2 in y, on the other hand, implies that

$$\mathbf{E}\left[\widetilde{\Phi}(s,y)\Delta_y u_2(t,x,s,y) - u_2(t,x,s,y)\Delta\widetilde{\Phi}(s,y)\right] = 0.$$

Therefore, taking the expectation of (5.6) yields

$$\mathbf{E}\widetilde{\Phi}(s,y)\left[\mathrm{tr}(\nabla_y \tilde{\boldsymbol{\omega}}(s,y) \cdot \mathrm{Hess}\,\overline{u}(t,x)) + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\Delta_x \overline{u}(t,x)\right] = 0.$$

Due to the assumption of isotropy, we have

$$\mathbf{E}\widetilde{\Phi}\nabla_y \tilde{\boldsymbol{\omega}} = \frac{1}{d}\,\mathrm{tr}\left(\mathbf{E}\widetilde{\Phi}\nabla_y \tilde{\boldsymbol{\omega}}\right) I_{d\times d} = \frac{1}{d}\mathbf{E}\widetilde{\Phi}(\nabla_y \cdot \tilde{\boldsymbol{\omega}}) I_{d\times d},$$

and thus

$$
\begin{aligned}
0 &= \mathbf{E}\widetilde{\Phi}(s,y)\left[\operatorname{tr}(\nabla_y\tilde{\boldsymbol{\omega}}(s,y)\cdot\operatorname{Hess}\overline{u}(t,x)) + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\Delta_x\overline{u}(t,x)\right]\\
&= \mathbf{E}\widetilde{\Phi}(s,y)\left[\frac{1}{d}\nabla_y\cdot\tilde{\boldsymbol{\omega}}(s,y) + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\right]\Delta_x\overline{u}(t,x),
\end{aligned}
$$

leading to

$$
a = 1 + \frac{2}{d}\frac{\mathbf{E}[\widetilde{\Phi}(s,y)\nabla_y\cdot\tilde{\boldsymbol{\omega}}(s,y)]}{\mathbf{E}[\widetilde{\Phi}(s,y)\widetilde{\Psi}(s,y)]}. \tag{5.7}
$$

As we have not proved that a stationary corrector $\tilde{\boldsymbol{\omega}}$ actually exists, the expression (5.7) is purely formal. In the next section, we will explain how we can use an approximate version of $\tilde{\boldsymbol{\omega}}$ to write a rigorous version of the computation leading to (5.7).

5.2. An Approximation of the Effective Diffusivity

In this section, we will show how approximate correctors can be used in the right-hand side of (5.7) to provide a good approximation of the effective diffusivity. Instead of trying to build a stationary solution to the corrector equation (5.4), we take $S > 0$ and consider the the solution $\boldsymbol{\omega}(s,y;S)$ of the Cauchy problem for (5.4), with $\widetilde{\Psi}(s,y)$ replaced by $\Psi(s,y;S)$ (defined in (3.1)):

$$
\begin{aligned}
&\partial_s\omega^{(k)}(s,y) = \frac{1}{2}\Delta_y\omega^{(k)}(s,y) + (\beta V(s,y)-\lambda)\omega^{(k)}(s,y) + \frac{\partial\Psi(s,y;S)}{\partial y_k},\\
&\qquad\qquad s > -S,\ k=1,\ldots,d;\\
&\boldsymbol{\omega}(-S,\cdot;S)\equiv 0.
\end{aligned}
$$

The solution is given by the Feynman–Kac formula

$$
\boldsymbol{\omega}(s,y;S) = \mathbb{E}_B^y\left[\int_0^{s+S}\exp\left\{\beta\mathscr{V}_{s;[0,r]}[B]-\lambda r\right\}\nabla\Psi(s-r,B_r;S)\,\mathrm{d}r\right]. \tag{5.8}
$$

We also define, similarly to the definition (3.1)/(3.4) of $\Psi(s,y;S)$, the

$$
\Phi(s,y;T) = \mathbb{E}_B^y\exp\left\{\beta\mathscr{V}_{s;[s-T,0]}[B]-\lambda(T-s)\right\},\qquad s<T, \tag{5.9}
$$

which solves (5.5) with terminal condition

$$
\Phi(T,y;T) = 1.
$$

Recall that $\mathscr{V}$ was defined in (2.1), so, in particular, we have

$$
\begin{aligned}
\mathscr{V}_{s;[0,r]}[B] &= \int_0^r V(s-\tau,B_\tau)\,\mathrm{d}\tau;\\
\mathscr{V}_{s;[s-T,0]} &= \int_{s-T}^0 V(s-\tau,B_\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Note that in the second expression we are evaluating B at negative times, interpreting it as a two-sided Brownian motion. Now we define an approximate version of (5.7).

$$a_{S,T}(s,y) = 1 + \frac{2}{d}\frac{\mathbf{E}[\Phi(s,y;T)\nabla_y \cdot \boldsymbol{\omega}(s,y;S)]}{\mathbf{E}[\Phi(s,y;T)\Psi(s,y;S)]}. \tag{5.10}$$

The next theorem, which is the main result of this section, shows that the "large S, T" limit of (5.10) agrees with the effective diffusivity from (2.17) (established in [17]).

Theorem 5.1. *Let a be the effective diffusivity defined by (2.17). Then we have, for each $s \in \mathbb{R}$ and $y \in \mathbb{R}^d$,*

$$\lim_{\substack{S\to\infty \\ T\to\infty}} a_{S,T}(s,y) = a.$$

We note that if a stationary $\tilde{\boldsymbol{\omega}}$ given by

$$\tilde{\boldsymbol{\omega}}(s,y) = \lim_{S\to\infty} \boldsymbol{\omega}(s,y;S)$$

exists, then Theorem 5.1 verifies the formal expression (5.7). Such large-scale approximations of the effective diffusivity have been used in the different context of elliptic homogenization theory; see [13].

Without loss of generality, we will take $s = 0$ and $y = 0$ in the proof of Theorem 5.1. In the course of the proof, we will denote by $H(x)$ the standard Heaviside function $H(x) = \mathbf{1}\{x \geq 0\}$ and also use its regularization

$$H_\gamma(x) = \begin{cases} 0 & x \leq 0; \\ \gamma^{-1}x & 0 \leq x \leq \gamma; \\ 1 & x \geq \gamma, \end{cases}$$

as well as $J(x) = xH(x)$. While several of the following lemmas are written using this regularization, the statement of Theorem 5.1 does not depend on the regularization. (Ultimately we take $\gamma \to 0$.) We begin with a Feynman–Kac formula for the numerator on the right-hand side of Theorem 5.1.

Lemma 5.2. *We have*

$$\begin{aligned}
&\mathbf{E}\left[\Phi(0,0;T)(\nabla_y \cdot \boldsymbol{\omega})(0,0;S)\right] \\
&\quad = \nabla_\eta|_{\eta=0} \cdot \nabla_\xi|_{\xi=0} \mathbf{E}\mathbb{E}_B^0 \exp\left\{\beta \int_{-T}^{S} V(-\tau, B_\tau + H(\tau)\eta + J(\tau)\xi)\,\mathrm{d}\tau \right. \\
&\qquad\qquad \left. -\lambda(T+S)\right\} \\
&\quad = \nabla_\eta|_{\eta=0} \cdot \nabla_\xi|_{\xi=0} \mathbb{E}_B^0 \exp\left\{\frac{1}{2}\beta^2 \mathscr{R}_{[-T,S]}[B + H\eta + J\xi] - \lambda(T+S)\right\}.
\end{aligned} \tag{5.11}$$

Proof. From (5.8) and (3.4), we have

$$\begin{aligned}\boldsymbol{\omega}(0,y;S) &= \mathbb{E}_B^y \int_0^S \exp\left\{\beta \int_0^r V(-\tau,B_\tau)\,\mathrm{d}\tau - \lambda r\right\} \nabla\Psi(-r,B_r;S)\,\mathrm{d}r \\ &= \nabla_\xi|_{\xi=0}\mathbb{E}_B^y \int_0^S \exp\left\{\beta \int_0^S V(-\tau,B_\tau + H(\tau-r)\xi)\,\mathrm{d}\tau - \lambda S\right\}\mathrm{d}r.\end{aligned} \tag{5.12}$$

One can check by explicit differentiation of both expressions that the right-hand side of (5.12) can be re-written as

$$\boldsymbol{\omega}(0,y;S) = \nabla_\xi|_{\xi=0}\mathbb{E}_B^y \exp\left\{\beta\int_0^S V(-\tau,B_\tau+\tau\xi)\,\mathrm{d}\tau - \lambda S\right\}. \tag{5.13}$$

Taking the divergence and setting $y=0$, we can write

$$(\nabla_y\cdot\boldsymbol{\omega})(0,0;S) = \nabla_\eta|_{\eta=0}\cdot\nabla_\xi|_{\xi=0}\mathbb{E}_B^0 \exp\left\{\beta\int_0^S V(-\tau,B_\tau+\eta+\tau\xi)\,\mathrm{d}\tau - \lambda S\right\}.$$

Multiplying by (5.9) gives

$$\begin{aligned}&\Phi(0,0;T)(\nabla_y\cdot\boldsymbol{\omega})(0,0;S) \\ &\quad= \nabla_\eta|_{\eta=0}\cdot\nabla_\xi|_{\xi=0}\mathbb{E}_B^0 \exp\left\{\beta\int_{-T}^S V(-\tau,B_\tau+H(\tau)\eta\right. \\ &\qquad\qquad \left. +J(\tau)\xi)\,\mathrm{d}\tau - \lambda(T+S)\right\}.\end{aligned}$$

(Now we are evaluating B at both positive and negative times.) Taking the expectation yields the first equality in (5.11). The second inequality then arises from evaluating the expectation. □

It will be useful to write a regularized version of (5.11), which will later allow us to use the Girsanov formula.

Corollary 5.3. *We have*

$$\begin{aligned}&\mathbf{E}\left[\Phi(0,0;T)(\nabla_y\cdot\boldsymbol{\omega})(0,0;S)\right] \\ &\quad= \lim_{\gamma\downarrow 0}\nabla_\eta|_{\eta=0}\cdot\nabla_\xi|_{\xi=0}\mathbf{E}\mathbb{E}_B^0\exp\left\{\int_{-T}^S V(-\tau,B_\tau+H_\gamma(\tau)\eta+J(\tau)\xi)\right. \\ &\qquad \left.\mathrm{d}\tau - \lambda(T+S)\right\} \\ &\quad= \nabla_\eta|_{\eta=0}\cdot\nabla_\xi|_{\xi=0}\mathbb{E}_B^0\exp\left\{\frac{1}{2}\beta^2\mathscr{R}_{[-T,S]}[B+H_\gamma\eta+J\xi] - \lambda(T+S)\right\}.\end{aligned} \tag{5.14}$$

Proof. Similarly to (5.11), the second equality of (5.14) is a simple computation, so it suffices to prove that the first expression is equal to the third. We write out all of the gradients in the third expression. Define $\delta f(\tau,\tilde\tau) = f(\tau) - f(\tilde\tau)$. For all $\gamma \geq 0$ we have

$$\nabla_\eta|_{\eta=0}\cdot\nabla_\xi|_{\xi=0}\mathrm{e}^{-\lambda(T+S)}\mathbb{E}_B^y\exp\left\{\beta^2\mathscr{R}_{[-T,S]}[B+H_\gamma\eta+J\xi]\right\}$$

$$= \beta^2 e^{-\lambda(T+S)} \mathbb{E}_B^y (g_{1;\gamma}[B] + g_{2;\gamma}[B] \cdot g_{3;\gamma}[B]) \cdot \exp\left\{\beta^2 \mathscr{R}_{[-T,S]}[B]\right\}, \tag{5.15}$$

where we define

$$g_{1;\gamma}[B] = \iint_{[-2,2]^2} \delta H_\gamma(\tau, \tilde{\tau}) \delta J(\tau, \tilde{\tau}) \Delta R(\tau - \tilde{\tau}, \delta B(\tau, \tilde{\tau}))\, d\tau\, d\tilde{\tau},$$

$$g_{2;\gamma}[B] = \iint_{[-S,-T]^2} \delta J(\tau, \tilde{\tau}) \nabla R(\tau - \tilde{\tau}, \delta B(\tau, \tilde{\tau}))\, d\tau\, d\tilde{\tau},$$

$$g_{3;\gamma}[B] = \iint_{[-2,2]^2} \delta H_\gamma(\tau, \tilde{\tau}) \nabla R(\tau - \tilde{\tau}, \delta B(\tau, \tilde{\tau}))\, d\tau\, d\tilde{\tau}.$$

Here we have used the fact that $R(s, y) = 0$ whenever $s \neq [-1, 1]$. Then the bounded convergence theorem implies the right-hand side of (5.15) is continuous in γ, so

$$\nabla_\eta|_{\eta=0} \cdot \nabla_\xi|_{\xi=0} \lim_{\gamma \downarrow 0} e^{-\lambda(T+S)} \mathbb{E}_B^y \exp\left\{\beta^2 \mathscr{R}_{[-T,S]}[B + H_\gamma \eta + J\xi]\right\}$$
$$= \nabla_\eta|_{\eta=0} \cdot \nabla_\xi|_{\xi=0} e^{-\lambda(T+S)} \mathbb{E}_B^y \exp\left\{\beta^2 \mathscr{R}_{[-T,S]}[B + H\eta + J\xi]\right\},$$

and the result follows from Lemma 5.11. □

Lemma 5.4. *We have*

$$a_{S,T}(0,0) = 1 + 2 \lim_{\gamma \downarrow 0} \widehat{\mathbb{E}}^0_{B;[-T,S]} \left(\frac{1}{\gamma d} B_S \cdot B_\gamma - 1\right). \tag{5.16}$$

Proof. To address the numerator of (5.10), continue from (5.14) and use the Girsanov formula, writing

$$\nabla_\eta|_{\eta=0} \cdot \nabla_\xi|_{\xi=0} \mathbb{E}_B^0 \exp\left\{\beta \int_{-T}^S V(-\tau, B_\tau + H_\gamma(\tau)\eta + J(\tau)\xi)\, d\tau - \lambda(T+S)\right\}$$
$$= \nabla_\eta|_{\eta=0} \cdot \nabla_\xi|_{\xi=0} \mathbb{E}_B^0 \exp\left\{\beta \int_{-T}^S V(-\tau, B_\tau)\, d\tau - \right.$$
$$\left. -\lambda(T+S) + \frac{1}{\gamma} B_\gamma \cdot \eta - \frac{1}{2\gamma}|\eta|^2 - \xi \cdot \eta + B_S \cdot \xi - \frac{1}{2}|\xi|^2 S\right\}$$
$$= \mathbb{E}_B^0 \left(\frac{1}{\gamma} B_S \cdot B_\gamma - d\right) \exp\left\{\beta \int_{-T}^S V(-\tau, B_\tau)\, d\tau - \lambda(T+S)\right\}.$$

Passing to the limit as $\gamma \downarrow 0$ and taking expectations shows that

$$\mathbf{E}[\Phi(0,0;T)(\nabla \cdot \boldsymbol{\omega})(0,0;S)] = e^{-\lambda(T+S)} \lim_{\gamma \downarrow 0} \mathbb{E}_B^0 \left(\gamma^{-1} B_S \cdot B_\gamma - d\right) \cdot \exp\left\{\beta^2 \mathscr{R}_{[-T,S]}[B]\right\}. \tag{5.17}$$

For the denominator of (5.10), we write

$$\Phi(0,0;T)\Psi(0,0;S) = \mathbb{E}_B^0 \exp\left\{\beta \mathscr{V}_{0;[-T,S]}[B] - \lambda(T+S)\right\}$$

(where again we use the interpretation of B as a two-sided Brownian motion), so

$$\mathbf{E}\Phi(0,0;T)\Phi(0,0;S) = \mathrm{e}^{-\lambda(T+S)}\mathbb{E}_B^0 \exp\left\{\beta^2 \mathscr{R}_{[-T,S]}[B]\right\}. \tag{5.18}$$

Dividing (5.17) by (5.18) yields (5.16). □

Lemma 5.5. *We have*

$$\lim_{\gamma\downarrow 0} \frac{1}{\gamma d} \widehat{\mathbb{E}}^0_{B;[-T,S]}|B_\gamma|^2 = 1,$$

uniformly in S and T.

Proof. We have

$$\begin{aligned}\widehat{\mathbb{E}}^0_{B;[-T,S]}|B_\gamma|^2 - \mathbb{E}_B^0|B_\gamma|^2 &= \mathbb{E}_B^0|B_\gamma|^2\left(\frac{1}{Z_{[-T,S]}}\exp\left\{\frac{1}{2}\beta^2\mathscr{R}_{[-T,S]}[B]\right\} - 1\right)\\ &= \frac{1}{Z_{[-T,S]}}\mathbb{E}_B^0|B_\gamma|^2\left(\exp\left\{\frac{1}{2}\beta^2\mathscr{R}_{[-T,S]}[B]\right\}\right.\\ &\qquad \left. - \exp\left\{\frac{1}{2}\beta^2\mathscr{R}_{[-T,S]}[\widetilde{B}]\right\}\right),\end{aligned}$$

where $\widetilde{B}$ is a Brownian motion whose increments on $[-T,0]$ and $[\gamma,S]$ are identical to those of B and whose increments on $[0,\gamma]$ are independent of those of B. (Thus the second equality is because $\mathscr{R}_{[-T,S]}[\widetilde{B}]$ is independent of B_γ.) This means that

$$\begin{aligned}\left|\widehat{\mathbb{E}}^0_{B;[-T,S]}|B_\gamma|^2 - \mathbb{E}_B^0|B_\gamma|^2\right| &= \frac{1}{Z_{[-T,S]}}\mathbb{E}_B^0\left(\exp\left\{\beta^2\mathscr{R}_{[-T,0]}[B]\right\} + \exp\left\{\beta^2\mathscr{R}_{[\gamma,S]}[B]\right\}\right)\\ &\quad \times \mathbb{E}_B^0|B_\gamma|^2\left|\exp\left\{2\beta^2\int_{-1}^{\gamma}\int_{\tau\vee 0}^{1} R(\tau-\tilde{\tau}, B_\tau - B_{\tilde{\tau}})\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau\right\}\right.\\ &\quad \left. - \exp\left\{2\beta^2\int_{-1}^{\gamma}\int_{\tau\vee 0}^{1} R(\tau-\tilde{\tau}, \widetilde{B}_\tau - \widetilde{B}_{\tilde{\tau}})\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau\right\}\right|\\ &\le C(\mathbb{E}_B^0|B_\gamma|^4)^{1/2}(\mathbb{E}_B^0(\exp\{4\beta^2\max_{0\le s\le\gamma}|B_s - \widetilde{B}_s|\}-1)^2)^{1/2} \le C\gamma^2,\end{aligned}$$

where C is a constant that may depend on β and R. Since $\mathbb{E}_B^0|B_\gamma|^2 = \gamma d$, this proves the lemma. □

Corollary 5.6. *We have*

$$a_{S,T}(0,0) = \lim_{\gamma\downarrow 0} a_{S,T;\gamma},$$

where

$$a_{S,T;\gamma} = 1 + \frac{2}{d\gamma}\widehat{\mathbb{E}}^0_{B;[-T,S]}(B_S - B_\gamma)\cdot B_\gamma. \tag{5.19}$$

Proof. This is a simple consequence of Lemma 5.4 and Lemma 5.5. □

Lemma 5.7. *The limit*

$$\lim_{\substack{T\to\infty\\ S\to\infty}} a_{S,T}(0,0) \tag{5.20}$$

exists.

Proof. We have, for any $\tau_1 < \tau_2 < \tau_3 < \tau_4 \le \tau_5$,

$$\begin{aligned}\widehat{\mathbb{E}}_{B;\tau_5}(B_{\tau_4} - B_{\tau_3})\cdot(B_{\tau_2} - B_{\tau_1}) &= \widehat{\mathbb{E}}_W(W_{\tau_4} - W_{\tau_5})\cdot(W_{\tau_2} - W_{\tau_1})\mathscr{G}(w_{\lfloor\tau_5\rfloor-1})\\ &= \widehat{\mathbb{E}}_W(W_{\tau_4\wedge\sigma} - W_{\tau_3\wedge\sigma})\cdot(W_{\tau_2} - W_{\tau_1}),\end{aligned} \tag{5.21}$$

where σ is the first regeneration time after τ_4 and the second equality comes from the fact that $\mathscr{G}$ is even and the increments of W after a regeneration time are isotropic. This makes it clear that there are constants $0 < c, C < \infty$ so that

$$\widehat{\mathbb{E}}_{B;\tau_5}(B_{\tau_4} - B_{\tau_3})\cdot(B_{\tau_2} - B_{\tau_1}) \le C\mathrm{e}^{-c(\tau_3-\tau_2)}, \tag{5.22}$$

since the increments of W have exponential tails and, conditional on there being a regeneration time in (τ_2, τ_3), the expectation of the right-hand side of (5.21) is 0. Then it follows from Corollary 5.6 that $a_{S,T}$ is Cauchy in S and also in T, and thus the limit (5.20) exists. □

Now we prove Theorem 5.1.

Proof of Theorem 5.1. We have, using (2.17), (5.22), and Lemma 2.8, that

$$\begin{aligned}a &= \lim_{U\to\infty}\frac{1}{dU}\widetilde{\mathbb{E}}_W(W_{3U} - W_0)\cdot(W_{2U} - W_U)\\ &= \lim_{U\to\infty}\frac{1}{dU}\widehat{\mathbb{E}}^0_{B;3U}(B_{3U} - B_0)\cdot(B_{2U} - B_U).\end{aligned} \tag{5.23}$$

Define

$$\tau_j^{(\gamma)} = (U + j\gamma)\wedge 2U$$

and note that

$$B_{2U} - B_U = \sum_{j=1}^{\lceil U/\gamma\rceil-1}(B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}).$$

Substituting this into (5.23) yields

$$\begin{aligned}a &= \lim_{U\to\infty}\frac{1}{dU}\lim_{\gamma\downarrow 0}\widehat{\mathbb{E}}^0_{B;3U}(B_{3U} - B_0)\cdot\sum_{j=0}^{\lceil U/\gamma\rceil-1}(B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}})\\ &= \lim_{U\to\infty}\frac{1}{dU}\lim_{\gamma\downarrow 0}\sum_{j=0}^{\lceil U/\gamma\rceil-1}\widehat{\mathbb{E}}^0_{B;3U}\Big((B_{3U} - B_{\tau_{j+1}^{(\gamma)}})\\ &\quad+(B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) + (B_{\tau_j^{(\gamma)}} - B_0)\Big)\cdot(B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}).\end{aligned}$$

Now by Lemma 5.5, we have

$$\lim_{U\to\infty}\frac{1}{dU}\lim_{\gamma\downarrow 0}\sum_{j=0}^{\lceil U/\gamma\rceil-1}\widehat{\mathbb{E}}^0_{B;3U}(B_{\tau^{(\gamma)}_{j+1}}-B_{\tau^{(\gamma)}_j})\cdot(B_{\tau^{(\gamma)}_{j+1}}-B_{\tau^{(\gamma)}_j})=1.$$

Moreover, we have by (5.19) that

$$\widehat{\mathbb{E}}^0_{B;3U}(B_{3U}-B_{\tau^{(\gamma)}_{j+1}})\cdot(B_{\tau^{(\gamma)}_{j+1}}-B_{\tau^{(\gamma)}_j})=\frac{\gamma d}{2}(a_{3U-\tau^{(\gamma)}_j,\tau^{(\gamma)}_j;\tau^{(\gamma)}_{j+1}-\tau^{(\gamma)}_j}-1);$$
$$\widehat{\mathbb{E}}^0_{B;3U}(B_{\tau^{(\gamma)}_j}-B_0)\cdot(B_{\tau^{(\gamma)}_{j+1}}-B_{\tau^{(\gamma)}_j})=\frac{\gamma d}{2}(a_{\tau^{(\gamma)}_{j+1},3U-\tau^{(\gamma)}_{j+1};\tau^{(\gamma)}_{j+1}-\tau^{(\gamma)}_j}-1).$$

Therefore,

$$\begin{aligned}a&=1+\lim_{U\to\infty}\frac{1}{dU}\lim_{\gamma\downarrow 0}\sum_{j=0}^{\lceil U/\gamma\rceil-1}\Bigg(\frac{\gamma d}{2}(a_{3U-\tau^{(\gamma)}_j,\tau^{(\gamma)}_j;\tau^{(\gamma)}_{j+1}-\tau^{(\gamma)}_j}-1)\\&\quad+\frac{\gamma d}{2}(a_{\tau^{(\gamma)}_{j+1},3U-\tau^{(\gamma)}_{j+1};\tau^{(\gamma)}_{j+1}-\tau^{(\gamma)}_j}-1)\Bigg)\\&=\lim_{U\to\infty}\frac{1}{U}\lim_{\gamma\downarrow 0}\frac{\gamma}{2}\sum_{j=0}^{\lceil U/\gamma\rceil-1}(a_{3U-\tau^{(\gamma)}_j,\tau^{(\gamma)}_j;\tau^{(\gamma)}_{j+1}-\tau^{(\gamma)}_j}+a_{\tau^{(\gamma)}_{j+1},3U-\tau^{(\gamma)}_{j+1};\tau^{(\gamma)}_{j+1}-\tau^{(\gamma)}_j})\\&=\lim_{\substack{T\to\infty\\S\to\infty}}a_{S,T}(0,0),\end{aligned}$$

where the last equality is by Lemma 5.7. □

6. Strong Convergence of the Leading Term

In this section, we prove Theorem 1.3: convergence of the leading term in the homogenization expansion (1.12). We begin by deriving an expression for the error in (1.20) using the Feynman–Kac formula. We will use the Fourier transform for the initial condition $u_0\in\mathcal{C}^\infty_c(\mathbb{R}^d)$, which we normalize as

$$\widehat{u_0}(\omega)=\int \mathrm{e}^{-\mathrm{i}\omega\cdot x}u_0(x)\frac{\mathrm{d}x}{(2\pi)^d},\qquad u_0(x)=\int \mathrm{e}^{\mathrm{i}\omega\cdot x}\widehat{u_0}(\omega)\,\mathrm{d}\omega.$$

In this section ω and $\tilde\omega$ denote Fourier variables; the function $\boldsymbol{\omega}$ from the previous section makes no appearance.

Proposition 6.1. *We have that*

$$\begin{aligned}\mathbf{E}|u^\varepsilon(t,x)-\Psi^\varepsilon(t,x)\overline{u}(t,x)|^2&=\mathrm{e}^{2\alpha_{\varepsilon^{-2}t}}\int\int \mathrm{e}^{\mathrm{i}(\omega+\tilde\omega)\cdot x}\widehat{u_0}(\omega)\widehat{u_0}(\tilde\omega)\\&\cdot\widehat{\mathbb{E}}_{B,\widetilde{B};\varepsilon^{-2}t}\mathscr{A}^\varepsilon_{t;\omega,\tilde\omega}[B,\widetilde{B}]\,\mathrm{d}\omega\,\mathrm{d}\tilde\omega,\end{aligned}\tag{6.1}$$

where

$$\mathscr{A}^\varepsilon_{t;\omega,\tilde\omega}[B,\widetilde{B}]=\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[B,\widetilde{B}]\right\}\mathscr{E}^\varepsilon_{t,\omega}[B]\mathscr{E}^\varepsilon_{t,\tilde\omega}[\widetilde{B}];\tag{6.2}$$

$$\mathscr{E}^\varepsilon_{t,\omega}[B]=\mathrm{e}^{\mathrm{i}\omega\cdot\varepsilon(B_{\varepsilon^{-2}t}-B_0)}-\mathrm{e}^{-\frac{1}{2}at|\omega|^2}.\tag{6.3}$$

Proof. We start with the Feynman–Kac formula for (1.4)–(1.5):

$$u^\varepsilon(t,x) = \mathbb{E}_B^{\varepsilon^{-1}x} \exp\left\{\beta \mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\} u_0(\varepsilon B_{\varepsilon^{-2}t}), \tag{6.4}$$

and note that

$$u_0(\varepsilon B_{\varepsilon^{-2}t}) = \int \mathrm{e}^{\mathrm{i}\omega\cdot\varepsilon B_{\varepsilon^{-2}t}} \widehat{u_0}(\omega)\,\mathrm{d}\omega, \qquad \overline{u}(t,x) = \int \mathrm{e}^{\mathrm{i}\omega\cdot x - \frac{1}{2}at|\omega|^2} \widehat{u_0}(\omega)\,\mathrm{d}\omega,$$

so, if $B_0 = \varepsilon^{-1}x$, then

$$u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) = \int \mathrm{e}^{\mathrm{i}\omega\cdot x} \mathscr{E}^\varepsilon_{t,\omega}[B] \widehat{u_0}(\omega)\,\mathrm{d}\omega. \tag{6.5}$$

The Feynman–Kac formula also shows that

$$\Psi^\varepsilon(t,x) = \mathbb{E}_B^{\varepsilon^{-1}x} \exp\left\{\beta \mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\}. \tag{6.6}$$

This is simply (6.4) with initial condition $u_0 \equiv 1$; we also saw the unrescaled version before in (2.2). Combining (6.4), (6.5), and (6.6) yields

$$u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x) = \mathbb{E}_B^{\varepsilon^{-1}x} \exp\left\{\beta \mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\} \int \mathrm{e}^{\mathrm{i}\omega\cdot x} \mathscr{E}^\varepsilon_{t,\omega}[B] \widehat{u_0}(\omega)\,\mathrm{d}\omega.$$

We finish the proof of the lemma by simply computing the second moment:

$$\begin{aligned}
\mathbf{E}(u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x))^2 &= \mathbf{E}\left(\mathbb{E}_B^{\varepsilon^{-1}x} \exp\left\{\beta \mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\}\right. \\
&\qquad\left.\cdot \int \mathrm{e}^{\mathrm{i}\omega\cdot x} \mathscr{E}^\varepsilon_{t,\omega}[B] \widehat{u_0}(\omega)\,\mathrm{d}\omega\right)^2 \\
&= \int\int \mathrm{e}^{\mathrm{i}(\omega+\tilde{\omega})\cdot x} \widehat{u_0}(\omega)\widehat{u_0}(\tilde{\omega}) \mathbb{E}_{B,\widetilde{B}}^{\varepsilon^{-1}x,\varepsilon^{-1}x} \mathbf{E} \exp\left\{\mathscr{V}_{\varepsilon^{-2}t}[B] + \mathscr{V}_{\varepsilon^{-2}t}[\widetilde{B}] - 2\lambda\varepsilon^{-2}t\right\} \\
&\qquad \cdot \mathscr{E}^\varepsilon_{t,\omega}[B] \mathscr{E}^\varepsilon_{t,\tilde{\omega}}[\widetilde{B}]\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega} \\
&= \mathrm{e}^{2\alpha_{\varepsilon^{-2}t}} \int\int \mathrm{e}^{\mathrm{i}(\omega+\tilde{\omega})\cdot x} \widehat{u_0}(\omega)\widehat{u_0}(\omega) \widehat{\mathbb{E}}_{B,\widetilde{B};\varepsilon^{-2}t} \mathscr{A}^\varepsilon_{t;\omega,\tilde{\omega}}[B,\widetilde{B}]\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega}.
\end{aligned}$$

□

To prove Theorem 1.3, we will bound the expression on the right-hand side of (6.1) using the techniques of [17] recalled in Sect. 2. On first reading, the reader may again wish to consider the case when V is white in time, so the tilting of the Markov chain can be ignored and B and $\widetilde{B}$ are simply Brownian motions. The key idea is that with high probability, the only contributions to $\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[B,\widetilde{B}]\right\}$ come from times close to 0, so the expectation of (6.2) "almost" splits into a product of the expectations of $\mathscr{E}^\varepsilon_{t,\omega}[B]$ and $\mathscr{E}^\varepsilon_{t,\omega}[\widetilde{B}]$. Since the Markov chain has effective diffusivity a, each of the latter expectations is approximately 0. (In the white-in-time case, $a = 1$, and each of the latter expectations is exactly 0.)

Our first lemma is that the correction $\mathscr{G}$ appearing in (2.12) does not matter.

Lemma 6.2. *We have*

$$\lim_{\varepsilon\to 0}\left|\widehat{\mathbb{E}}_{B,\tilde{B};\varepsilon^{-2}t}\mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[B,\tilde{B}]-\widetilde{\mathbb{E}}_{W,\widetilde{W}}\mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]\right|=0.$$

As this lemma is a technical point, we defer its proof to the end of this section. Now we note that, for $r,\tilde{r}\geq 0$, we have

$$\begin{aligned}\frac{\partial^2}{\partial r\partial\tilde{r}}\exp\left\{\beta^2\mathscr{R}_{r,\tilde{r}}[W,\widetilde{W}]\right\}&=\frac{\partial}{\partial r}\left[\left(\beta^2\int_0^r R(\tau-\tilde{r},W_\tau-\widetilde{W}_{\tilde{r}})\,\mathrm{d}\tau\right)\right.\\&\qquad\left.\cdot\exp\left\{\beta^2\mathscr{R}_{r,\tilde{r}}[W,\widetilde{W}]\right\}\right]\\&=\mathscr{Q}_{r,\tilde{r}}[W,\widetilde{W}]\exp\left\{\beta^2\mathscr{R}_{r,\tilde{r}}[W,\widetilde{W}]\right\},\end{aligned}\tag{6.7}$$

where

$$\begin{aligned}\mathscr{Q}_{r,\tilde{r}}[W,\widetilde{W}]&=\beta^2R(r-\tilde{r},W_r-\widetilde{W}_{\tilde{r}})\\&\quad+\beta^4\int_{[\tilde{r}-2,r]}R(\tau-\tilde{r},W_\tau-\widetilde{W}_{\tilde{r}})\,\mathrm{d}\tau\int_{[r-2,\tilde{r}]}R(r-\tilde{\tau},W_r-\widetilde{W}_{\tilde{\tau}})\,\mathrm{d}\tilde{\tau}.\end{aligned}\tag{6.8}$$

We note that, for each $r,\tilde{r}$,

$$\mathscr{Q}_{r,\tilde{r}}[W,\widetilde{W}]\geq 0\tag{6.9}$$

almost surely, since R was assumed nonnegative. Now if we define the shorthand

$$\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]=\mathscr{E}^{\varepsilon}_{t,\omega}[W]\mathscr{E}^{\varepsilon}_{t,\tilde{\omega}}[\widetilde{W}],$$

then we can write

$$\begin{aligned}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]&=\widetilde{\mathbb{E}}_{W,\widetilde{W}}\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}\\&=\int_0^{\varepsilon^{-2}t}\int_0^{\varepsilon^{-2}t}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]\mathscr{Q}_{r,\tilde{r}}[W,\widetilde{W}]\\&\qquad\cdot\exp\left\{\beta^2\mathscr{R}_{r,\tilde{r}}[W,\widetilde{W}]\right\}\,\mathrm{d}r\,\mathrm{d}\tilde{r}.\end{aligned}\tag{6.10}$$

The next lemma gives an estimate for the contribution to the integral (6.10) from each $r,\tilde{r}$. The key point is that, if B is a Brownian motion with diffusivity σ^2, then $\exp\left\{\mathrm{i}\omega\cdot B_t+\frac{1}{2}t\sigma^2|\omega|^2\right\}$ is a martingale. Since W is converging to a Brownian motion with diffusivity a, the contribution to the integrand in (6.10) from $\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]$ should be small except for the contribution from time interval $[0,r\vee\tilde{r}]$, on which the term $\exp\left\{\beta^2\mathscr{R}_{r,\tilde{r}}[W,\widetilde{W}]\right\}$ could have an effect. But for fixed $r,\tilde{r}$, this time interval is microscopic, and thus does not contribute in the limit.

Lemma 6.3. *For fixed $r,\tilde{r}\geq 0$, we have*

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]\mathscr{Q}_{r,\tilde{r}}[W,\widetilde{W}]\exp\left\{\beta^2\mathscr{R}_{r,\tilde{r}}[W,\widetilde{W}]\right\}=0.$$

Proof. In this proof we will treat r and $\tilde{r}$ as fixed, and suppress them from the notation of the objects we define. We abbreviate $\sigma_j = \sigma_j^{W,\widetilde{W}}$ from (2.15) and recall the definition (2.16) of κ_2. Let $j_0 \in \{j \geq 0 \mid \sigma_j \geq r \vee \tilde{r}\}$ and let the σ-algebra $\mathcal{F}_{j_0}$ be generated by the collection of random variables

$$\{\eta_n^{W,\tilde{W}} \mid n < \sigma_{j_0}\} \cup \{w_n \mid n < \sigma_{j_0}\} \cup \{\tilde{w}_n \mid n < \sigma_{j_0}\},$$

with notation as in Theorem 2.1. We note that the random variable

$$\mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\}$$

is $\mathcal{F}_{j_0}$-measurable. Therefore, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \widetilde{W}] \mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\} \\
&\quad = \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left(\widetilde{\mathbb{E}}_{W,\widetilde{W}} \left[\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \widetilde{W}] \mid \mathcal{F}_{j_0}\right] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\} \mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}]\right) \\
&\quad = \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left(\mathrm{e}^{\mathrm{i}\omega\cdot\varepsilon W_{j_0}} \widetilde{\mathbb{E}}_W \left[\mathrm{e}^{\mathrm{i}\omega\cdot\varepsilon(W_{\varepsilon^{-2}t} - W_{j_0})} \mid \mathcal{F}_{j_0}\right] - \mathrm{e}^{-\frac{1}{2}at|\omega|^2}\right) \cdot \\
&\qquad \cdot \left(\mathrm{e}^{\mathrm{i}\tilde{\omega}\cdot\varepsilon \widetilde{W}_{j_0}} \widetilde{\mathbb{E}}_W \left[\mathrm{e}^{\mathrm{i}\tilde{\omega}\cdot\varepsilon(\widetilde{W}_{\varepsilon^{-2}t} - \widetilde{W}_{j_0})} \mid \mathcal{F}_{j_0}\right] - \mathrm{e}^{-\frac{1}{2}at|\tilde{\omega}|^2}\right) \mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}] \\
&\qquad \cdot \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\}.
\end{aligned} \tag{6.11}$$

Observe that

$$\widetilde{\mathbb{E}}_W \left[\mathrm{e}^{\mathrm{i}\omega\cdot\varepsilon(W_{\varepsilon^{-2}t} - W_{j_0})} \mid \mathcal{F}_{j_0}\right] = \widetilde{\mathbb{E}}_W \left[\mathrm{e}^{\mathrm{i}\omega\cdot\varepsilon(W_{\varepsilon^{-2}t} - W_{j_0})} \mid j_0\right] \to \mathrm{e}^{-\frac{1}{2}at|\omega|^2}$$

almost surely as $\varepsilon \to 0$ by Proposition 2.4, and similarly for $\widetilde{\mathbb{E}}_{\widetilde{W}} \left[\mathrm{e}^{\mathrm{i}\tilde{\omega}\cdot\varepsilon(\widetilde{W}_{\varepsilon^{-2}t} - \widetilde{W}_{j_0})} \mid \mathcal{F}_{j_0}\right]$. In addition, we have

$$\mathrm{e}^{\mathrm{i}\tilde{\omega}\cdot\varepsilon \widetilde{W}_{j_0}} \to 1$$

almost surely as $\varepsilon \to 0$. The statement of the lemma then follows from the bounded convergence theorem applied to (6.11). □

Now we upgrade the pointwise convergence to convergence of the integral.

Lemma 6.4. *We have*

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \widetilde{W}] = 0.$$

Proof. Using (6.9), we have

$$\begin{aligned}
&\left|\widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathscr{E}^{\varepsilon}_{t,\omega,\tilde{\omega}}[W, \widetilde{W}] \mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\}\right| \leq 4\widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}] \\
&\quad \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\}.
\end{aligned}$$

Using (6.7), we have that

$$\int_0^{\tilde{q}} \int_0^q \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathscr{Q}_{r,\tilde{r}}[W, \widetilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W, \widetilde{W}]\right\} \mathrm{d}r \, \mathrm{d}\tilde{r}$$

$$= \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{q,\tilde{q}}[W, \widetilde{W}]\right\} \le \widetilde{\mathbb{E}}_{W,\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_{\infty}[W, \widetilde{W}]\right\} < \infty,$$

where the last equality is by Proposition 2.5. The dominated convergence theorem applied to the integral (6.10), in light of the pointwise convergence established in Lemma 6.3, then implies the result. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Combining Lemmas 6.2 and Lemma 6.4, we see that the integrand in (6.1) converges pointwise to 0 as $\varepsilon \to 0$. On the other hand, by Proposition 2.5, as long as $\beta < \beta_0$, there is a constant C so that

$$\left|\widehat{\mathbb{E}}_{B,\tilde{B};\varepsilon^{-2}t} \mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[B, \widetilde{B}]\right| \le C$$

independently of $\varepsilon, \omega, \tilde{\omega}$. As $u_0 \in \mathcal{C}^{\infty}_{\mathrm{c}}(\mathbb{R}^d)$, the dominated convergence theorem and (2.8) imply that

$$\mathbf{E}|u^{\varepsilon}(t,x) - \Psi^{\varepsilon}(t,x)\overline{u}(t,x)|^2 \to 0$$

as $\varepsilon \to 0$. □

It remains to prove Lemma 6.2.

Proof of Lemma 6.2. We have

$$\begin{aligned}
\widehat{\mathbb{E}}_{B,\widetilde{B};\varepsilon^{-2}t} \mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[B, \widetilde{B}] &= \widehat{\mathbb{E}}_{B,\widetilde{B};\varepsilon^{-2}t} \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[B, \widetilde{B}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[B, \widetilde{B}]\right\} \\
&= \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathscr{G}[w_{\lfloor \varepsilon^{-2}t \rfloor -1}] \mathscr{G}[\tilde{w}_{\lfloor \varepsilon^{-2}t \rfloor -1}] \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \widetilde{W}] \\
&\quad \cdot \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W, \widetilde{W}]\right\}. \qquad (6.12)
\end{aligned}$$

Let $\gamma \in (0, 2)$ be arbitrary. Then

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{W,\widetilde{W}} \left| \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \widetilde{W}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W, \widetilde{W}]\right\} - \mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W, \widetilde{W}] \right. \\
&\qquad \left. \cdot \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W, \widetilde{W}]\right\} \right| \\
&\quad \le \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left| \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \widetilde{W}] - \mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W, \widetilde{W}] \right| \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W, \widetilde{W}]\right\}
\end{aligned}$$

$$+\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left|\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\widetilde{W}]\right|\left|\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}\right.$$
$$\left.-\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}\right\}\right|. \tag{6.13}$$

We begin by addressing the first term of (6.13). By (2.18), we have

$$\widetilde{\mathbb{E}}_{W}\left|\varepsilon W_{\varepsilon^{-2}t}-\varepsilon W_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}\right|^2\leq C\varepsilon^{\gamma},$$

which in particular means that

$$\lim_{\varepsilon\to 0}\left|\varepsilon W_{\varepsilon^{-2}t}-\varepsilon W_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}\right|=0 \tag{6.14}$$

in probability. The same statement of course holds for $\widetilde{W}$. We then have, using Hölder's inequality, that for $\delta>0$ sufficiently small there is a constant C_δ so that

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]-\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\widetilde{W}]\right|\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}$$
$$\leq C_\delta\lim_{\varepsilon\to 0}\left(\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]-\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\widetilde{W}]\right|^{1/\delta+1}\right)=0 \tag{6.15}$$

by Proposition 2.5 and the bounded convergence theorem in light of (6.14).

Finally, we consider the second term of (6.13), which is easier. Here, we have

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left|\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\widetilde{W}]\right|\left|\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}\right.$$
$$\left.-\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\widetilde{W}]\right\}\right|$$
$$\leq 4\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left|\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}-\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\widetilde{W}]\right\}\right|=0 \tag{6.16}$$

by the dominated convergence theorem, again in light of (2.18). Applying (6.15) and (6.16) to (6.13) implies that

$$\lim_{\varepsilon\to 0}\widehat{\mathbb{E}}_{W,\widetilde{W}}\left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\widetilde{W}]\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}-\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\widetilde{W}]\right.$$
$$\left.\cdot\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\widetilde{W}]\right\}\right|=0. \tag{6.17}$$

Combining (6.12), (6.17), and Lemma 2.8, and recalling that $\mathscr{G}$ is bounded from above and away from zero, completes the proof of the lemma. □

7. The Second Term of the Expansion

In this section we will prove Theorem 1.5. We first introduce some notation. Fix $\gamma \in (1,2)$ and $t > 0$; all constants in this section will depend on γ and t. We define a discrete set of times

$$r_k = \begin{cases} 0 & k = 0; \\ t - \varepsilon^{-\gamma}(\lfloor \varepsilon^{\gamma} t \rfloor - (k-1)) & k > 0, \end{cases} \tag{7.1}$$

and set

$$\mathscr{I}_t^{\varepsilon}[B] = \sum_{k=0}^{K_t^{\varepsilon}} (\varepsilon B_{r_{k+1}} - \varepsilon B_{r_k}) \cdot \nabla \overline{u}(t - \varepsilon^2 r_k, \varepsilon B_{r_k}), \tag{7.2}$$

with

$$K_t^{\varepsilon} = \lfloor \varepsilon^{\gamma-2} t \rfloor. \tag{7.3}$$

The next lemma gives a Feynman–Kac formula for the corrector u_1^{ε} defined in (1.27).

Lemma 7.1. *We have*

$$u_1^{\varepsilon}(t,x) = \frac{1}{\varepsilon} \mathbb{E}_B^{\varepsilon^{-1}x} \exp\left\{ \mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda \varepsilon^{-2} t \right\} \mathscr{I}_t^{\varepsilon}[B]. \tag{7.4}$$

Proof. The Feynman–Kac formula applied to (1.25), in the same way as (5.13), gives the following expression for the solution $\theta_j(s,y)$ to that equation:

$$\begin{aligned}
\theta_j(s,y) &= \mathbb{E}_B^y \int_0^{s-\varepsilon^{-\gamma}(j-1)} \exp\left\{ \int_0^r [\beta V(s-\tau, B_\tau) - \lambda] \, d\tau \right\} \nabla \Psi(s-r, B_r) \, dr \\
&= \mathbb{E}_B^y \nabla_\xi |_{\xi=0} \int_0^{s-\varepsilon^{-\gamma}(j-1)} \exp\left\{ \int_0^s [\beta V(s-\tau, B_\tau + H(\tau - r)\xi) - \lambda] \, d\tau \right\} dr \\
&= \nabla_\xi |_{\xi=0} \mathbb{E}_B^y \exp\left\{ \int_0^s [\beta V(s-\tau, B_\tau + (\tau \wedge (s - \varepsilon^{-\gamma}(j-1)))\xi) - \lambda] \, d\tau \right\},
\end{aligned}$$

where H is the Heaviside function. The Girsanov formula then yields

$$\begin{aligned}
\theta_j(s,y) &= \nabla_\xi |_{\xi=0} \mathbb{E}_B^y \exp\Big\{ \mathscr{V}_s[B] - \lambda s + (B_{s-\varepsilon^{-\gamma}(j-1)} - y) \cdot \xi \\
&\qquad - \frac{s - \varepsilon^{-\gamma}(j-1)}{2} |\xi|^2 \Big\} \\
&= \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}(j-1)} - y) \exp\{\mathscr{V}_s[B] - \lambda s\}.
\end{aligned}$$

Given this expression for θ_j, we can then write the Feynman–Kac formula for (1.26):

$$u_{1;j}(s,y) = \mathbb{E}_B^y \exp\left\{\int_0^{s-\varepsilon^{-\gamma}j} [\beta V(s-\tau, B_\tau) - \lambda]\,\mathrm{d}\tau\right\} \theta_j(\varepsilon^{-\gamma}j, B_{s-\varepsilon^{-\gamma}j})\cdot$$
$$\cdot \nabla\overline{u}(\varepsilon^{2-\gamma}j, \varepsilon B_{s-\varepsilon^{-\gamma}j})$$
$$= \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}(j-1)} - B_{s-\varepsilon^{-\gamma}j}) \cdot \nabla\overline{u}(\varepsilon^{2-\gamma}j, \varepsilon B_{s-\varepsilon^{-\gamma}j})$$
$$\cdot \exp\{\mathscr{V}_s[B] - \lambda s\}.$$

Finally, by (1.27) we have

$$u_1^\varepsilon(t,x) = u_1(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

where

$$u_1(s,y) = \sum_{j=1}^{\lfloor\varepsilon^\gamma s\rfloor} \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}(j-1)} - B_{s-\varepsilon^{-\gamma}j}) \cdot \nabla\overline{u}(\varepsilon^{2-\gamma}j, \varepsilon B_{s-\varepsilon^{-\gamma}j})$$
$$\cdot \exp\{\mathscr{V}_s[B] - \lambda s\}$$
$$+ \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}\lfloor\varepsilon^\gamma s\rfloor} - y) \exp\{\mathscr{V}_s[B] - \lambda s\} \cdot \nabla\overline{u}(\varepsilon^2 s, \varepsilon y)$$
$$= \mathbb{E}_B^y \exp\{\mathscr{V}_s[B] - \lambda s\} \sum_{k=0}^{\lfloor\varepsilon^\gamma s\rfloor} (B_{r_{k+1}} - B_{r_k}) \cdot \nabla\overline{u}(\varepsilon^2(s-r_k), \varepsilon B_{r_k}),$$

with r_k defined in (7.1); this yields (7.4). □

Next we consider the error term

$$q^\varepsilon(t,x) = u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x) - \varepsilon u_1^\varepsilon(t,x).$$

Combining (6.4), (6.6), and (7.4) gives the expression

$$q^\varepsilon(t,x) = \mathbb{E}_B^{\varepsilon^{-1}x}\left[u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[B]\right] \exp\left\{\mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\},$$

with expectation

$$\mathbf{E}q^\varepsilon(t,x) = \mathrm{e}^{\alpha_{\varepsilon^{-2}t}} \widehat{\mathbb{E}}_{B;\varepsilon^{-2}t}^{\varepsilon^{-1}x}[u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[B]].$$

Taking covariances, we obtain

$$\mathbf{E}q^\varepsilon(t,x)q^\varepsilon(t,\tilde{x}) - \mathbf{E}q^\varepsilon(t,x)\mathbf{E}q^\varepsilon(t,\tilde{x})$$
$$= \mathrm{e}^{2\alpha_{\varepsilon^{-2}t}} \widehat{\mathbb{E}}_{B,\widetilde{B};\varepsilon^{-2}t}^{\varepsilon^{-1}x,\varepsilon^{-1}x} (u_0(\varepsilon B_{\varepsilon^{-2}t})$$
$$- \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[B])(u_0(\varepsilon \widetilde{B}_{\varepsilon^{-2}t}) - \overline{u}(t,\tilde{x}) - \mathscr{I}_t^\varepsilon[\widetilde{B}])\cdot$$
$$\cdot \left(\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[B,\widetilde{B}]\right\} - 1\right)$$
$$= \mathrm{e}^{2\alpha_{\varepsilon^{-2}t}} \widetilde{\mathbb{E}}_{W,\widetilde{W}}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}} (u_0(\varepsilon W_{\varepsilon^{-2}t})$$
$$- \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[W])(u_0(\varepsilon \widetilde{W}_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[\widetilde{W}])\cdot$$
$$\cdot \left(\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\} - 1\right) \mathscr{G}[w_{\lfloor\varepsilon^{-2}t\rfloor-1}]\mathscr{G}[\tilde{w}_{\lfloor\varepsilon^{-2}t\rfloor-1}]. \tag{7.5}$$

In the last equality of (7.5) we used Theorem 2.1.

In line with the framework of Sect. 2, we will proceed to approximate the times r_k by nearby regeneration times of the Markov chain. Thus, we define

$$\sigma^W(k) = (\varepsilon^{-2}t) \wedge \min\{r \geq r_k \mid \eta_r^W = 1\}, \tag{7.6}$$

where η_r^W is as in Theorem 2.1. Before we begin our argument in earnest, we record bounds on the relevant error terms. Put

$$\begin{aligned} Y &= \max_{0\leq k\leq K_t^\varepsilon}(\sigma^W(k) - r_k), \qquad F(\tau) = \max_{r\in[0,\varepsilon^{-2}t-\tau]}|W_{r+\tau} - W_r|, \\ Z &= \varepsilon^{\gamma/2}F(\varepsilon^{-\gamma} + Y). \end{aligned}$$

Lemma 7.2. *We have constants* $0 < c, C < \infty$ *so that, for all* $\xi \geq 0$*, we have*

$$\widetilde{\mathbb{P}}_W(Y \geq C|\log\varepsilon| + \xi) \leq C\mathrm{e}^{-c\xi}, \tag{7.7}$$

$$\widetilde{\mathbb{P}}_W(F(Y) \geq C|\log\varepsilon| + \xi) \leq C\mathrm{e}^{-c\xi}, \tag{7.8}$$

and

$$\widetilde{\mathbb{P}}_W(Z \geq C|\log\varepsilon| + \xi) \leq C\mathrm{e}^{-c\xi}. \tag{7.9}$$

These bounds are simple consequences of the regeneration structure of the Markov chain described in Sect. 2 and of [17, Lemma A.1]. We begin our approximation procedure by replacing the deterministic times r_k in the definition (7.2) of $\mathscr{I}_{t,x}^\varepsilon$ by the regeneration time approximations.

Lemma 7.3. *Let*

$$\widetilde{\mathscr{I}}_t^\varepsilon[W] = \sum_{k=0}^{K_t^\varepsilon}(\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)}) \cdot \nabla\overline{u}(t - \varepsilon^2\sigma^W(k), \varepsilon W_{\sigma^W(k)}). \tag{7.10}$$

For any $1 \leq p < \infty$ *and any* $\zeta < \gamma - 1$ *there exists a constant* $C = C(p, \zeta, t, \|u_0\|_{\mathcal{C}^2(\mathbb{R}^d)}) < \infty$ *so that*

$$\left(\widetilde{\mathbb{E}}_W^x|\mathscr{I}_t^\varepsilon[W] - \widetilde{\mathscr{I}}_t^\varepsilon[W]|^p\right)^{1/p} \leq C\varepsilon^\zeta. \tag{7.11}$$

Proof. We have

$$\begin{aligned} \mathscr{I}_t^\varepsilon[W] - \widetilde{\mathscr{I}}_t^\varepsilon[W] = \sum_{k=0}^{K_t^\varepsilon}\Big[&(\varepsilon W_{r_{k+1}} - \varepsilon W_{r_k}) \cdot \nabla\overline{u}(t - \varepsilon^2 r_k, \varepsilon W_{r_k}) \\ &-(\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)}) \cdot \nabla\overline{u}(t - \varepsilon^2\sigma^W(k), \varepsilon W_{\sigma^W(k)})\Big], \end{aligned}$$

hence

$$|\mathscr{I}_t^\varepsilon[W] - \widetilde{\mathscr{I}}_t^\varepsilon[W]| \leq \sum_{k=0}^{K_t^\varepsilon} |(\varepsilon W_{r_k+1} - \varepsilon W_{r_k}) - (\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)})| \cdot$$
$$\cdot |\nabla \overline{u}(t - \varepsilon^2 r_k, \varepsilon W_{r_k})|$$
$$+ \sum_{k=0}^{K_t^\varepsilon} |\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)}| \cdot |\nabla \overline{u}(t - \varepsilon^2 r_k, \varepsilon W_{r_k})$$
$$- \nabla \overline{u}(t - \varepsilon^2 \sigma^W(k), \varepsilon W_{\sigma^W(k)})|. \tag{7.12}$$

We bound above the first term on the right-hand side by

$$|(\varepsilon W_{r_{k+1}} - \varepsilon W_{r_k}) - (\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)})| \cdot |\nabla \overline{u}(t - \varepsilon^2 r_k)|$$
$$\leq 2F(Y)\varepsilon \|\overline{u}\|_{\mathcal{C}^1(\mathbb{R}^d)}, \tag{7.13}$$

and the second by

$$|\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)}| \cdot |\nabla \overline{u}(t - \varepsilon^2 r_k, \varepsilon W_{r_k}) - \nabla \overline{u}(t - \varepsilon^2 \sigma^W(k), \varepsilon W_{\sigma^W(k)})|$$
$$\leq \varepsilon F(Y + \varepsilon^{-\gamma}) \|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)} (\varepsilon^2 Y + \varepsilon F(Y)) = \varepsilon^{1-\gamma/2} Z \|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)} (\varepsilon^2 Y + \varepsilon F(Y)). \tag{7.14}$$

Combining (7.12), (7.13), and (7.14), and recalling the definition (7.3) of K_t^ε, gives us

$$|\mathscr{I}_t^\varepsilon[W] - \widetilde{\mathscr{I}}_t^\varepsilon[W]| \leq \varepsilon^{\gamma-2} t \Big[2F(Y)\varepsilon \|\overline{u}\|_{\mathcal{C}^1(\mathbb{R}^d)} + \varepsilon^{1-\gamma/2} Z \|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)} (\varepsilon^2 Y$$
$$+ \varepsilon F(Y)) \Big],$$

which, in light of Lemma 7.2, implies (7.11). □

Lemma 7.4. *For any power* $1 \leq p < \infty$*, there exists a* $C = C(p, t, \zeta, \|u_0\|_{\mathcal{C}^3(\mathbb{R}^d)}) < \infty$ *so that*

$$\left(\widetilde{\mathbb{E}}_W^{\varepsilon^{-1}x} \left| u_0(\varepsilon W_{\varepsilon^{-2}t}) - \overline{u}(t, x) - \mathscr{I}_t^\varepsilon[W] \right|^p \right)^{1/p} \leq C\varepsilon^\zeta \tag{7.15}$$

for any $\zeta < 1 - \gamma/2$.

Proof. To ease the notation, in this proof we will abbreviate $\sigma = \sigma^W$. (Recall the definition (7.6).) We write the Taylor expansion

$$\overline{u}(t - \varepsilon^2 \sigma(k+1), \varepsilon W_{\sigma(k+1)}) - \overline{u}(t - \varepsilon^2 \sigma(k), \varepsilon W_{\sigma(k)})$$
$$= -\varepsilon^2 (\sigma(k+1) - \sigma(k)) \partial_t \overline{u}(t - \varepsilon^2 \sigma(k), \varepsilon W_{\sigma(k)}) + \varepsilon (W_{\sigma(k+1)} - W_{\sigma(k)}) \cdot$$
$$\cdot \nabla \overline{u}(t - \varepsilon^2 \sigma(k), \varepsilon W_{\sigma(k)}) \tag{7.16}$$
$$+ \frac{1}{2} \varepsilon^2 \mathrm{Q}\overline{u}(t - \varepsilon^2 \sigma(k), \varepsilon W_{\sigma(k)})(W_{\sigma(k+1)} - W_{\sigma(k)}) + \mathscr{Y}_k[W],$$

where $\mathrm{Q}\overline{u}(t, x)$ is the quadratic form associated to the Hessian of $\overline{u}$ at (t, x) (so $\mathrm{Q}\overline{u}(t, x)(V) = \mathrm{Hess}\,\overline{u}(t, x)(V, V)$) and $\mathscr{Y}_k[W]$ is the remainder term. By Taylor's theorem, we have

$$|\mathscr{Y}_k[W]| \le C\|\overline{u}\|_{\mathcal{C}^3(\mathbb{R}^d)}\left(\varepsilon^4|\sigma(k+1)-\sigma(k)|^2+\varepsilon^3|W_{\sigma(k+1)}-W_{\sigma(k)}|^3\right). \tag{7.17}$$

Note that the second term of the second line of (7.16) appears in the definition (7.10) of $\widetilde{\mathscr{I}}_t^\varepsilon$. Thus, we can telescope the left side of (7.16) to obtain

$$\widetilde{\mathscr{I}}_t^\varepsilon[W] = u_0(\varepsilon W_{\varepsilon^{-2}t}) - \overline{u}(t-\varepsilon^2\sigma(0), \varepsilon W_{\sigma(0)}) + \sum_{k=0}^{K_t^\varepsilon}\left(\varepsilon^2\mathscr{X}_k[W] + \mathscr{Y}_k[W]\right), \tag{7.18}$$

where

$$\begin{aligned}\mathscr{X}_k[W] &= (\sigma(k+1)-\sigma(k))\partial_t\overline{u}(t-\varepsilon^2\sigma(k), \varepsilon W_{\sigma(k)}) \\ &\quad -\frac{1}{2}\mathrm{Q}\overline{u}(t-\varepsilon^2\sigma(k), \varepsilon W_{\sigma(k)})(W_{\sigma(k+1)}-W_{\sigma(k)}) \\ &= (\sigma(k+1)-\sigma(k))\frac{1}{2}a\Delta\overline{u}(t-\varepsilon^2\sigma(k), \varepsilon W_{\sigma(k)}) \\ &\quad -\frac{1}{2}\mathrm{Q}\overline{u}(t-\varepsilon^2\sigma(k), \varepsilon W_{\sigma(k)})(W_{\sigma(k+1)}-W_{\sigma(k)}).\end{aligned}$$

We now deal with each piece of this expression in term.

The drift terms. We first define

$$\begin{aligned}\widetilde{\mathscr{X}}_k &= (\tilde\sigma(k+1)-\tilde\sigma(k))\frac{1}{2}a\Delta\overline{u}(t-\varepsilon^2\sigma(k), \varepsilon W_{\sigma(k)}) \\ &\quad -\frac{1}{2}\mathrm{Q}\overline{u}(t-\varepsilon^2\sigma(k), \varepsilon W_{\sigma(k)})(W_{\tilde\sigma(k+1)}-W_{\tilde\sigma(k)}),\end{aligned}$$

where $\tilde\sigma(k) = \min\{r \ge r_k \mid \eta_r^W = 1\}$ differs from $\sigma(k)$ by not being restricted to be less than $\varepsilon^{-2}t$. Using the relation (2.17) between the effective diffusivity a and the variance of the increments $W_{\sigma_{n+1}^W} - W_{\sigma_n^W}$, as well as the isotropy of W, we see that

$$\widetilde{\mathbb{E}}_W\widetilde{\mathscr{X}}_k[W] = 0 \tag{7.19}$$

for each k. We also note the simple bound

$$\begin{aligned}|\widetilde{\mathscr{X}}_k[W]| &\le a\|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)}(\varepsilon^{-\gamma}+Y) + \|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)}(F(\varepsilon^{-\gamma}+Y))^2 \\ &\le a\|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)}(\varepsilon^{-\gamma}+Y) + \|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)}\varepsilon^{-\gamma}Z^2.\end{aligned}$$

Therefore, by Lemma 7.2, we have

$$\widetilde{\mathbb{E}}_W|\widetilde{\mathscr{X}}_k[W]|^p \le C\varepsilon^{-p\xi} \tag{7.20}$$

for any $\xi > \gamma$. We further define

$$M_\ell = \sum_{k=0}^{\ell}\widetilde{\mathscr{X}}_k[W].$$

For each $\ell \geq 0$, define $\mathcal{G}_\ell$ to be the σ-algebra generated by $\{W_t \mid t \leq \sigma(\ell)\} \cup \{\eta_t^W \mid t \leq \sigma(\ell)\}$. Then, according to (7.19), $\{M_\ell\}$ is a martingale with respect to the filtration $\{\mathcal{G}_\ell\}$. An L^p-version of the Burkholder–Gundy inequality as in [9] (see also [3, Theorem 9]) implies that

$$\left(\widetilde{\mathbb{E}}_W |\varepsilon^2 M_{K_t^\varepsilon}|^p\right)^{1/p} \leq C\varepsilon^2 \left[(K_t^\varepsilon + 1)^{p/2-1} \sum_{k=0}^{K_t^\varepsilon} \widetilde{\mathbb{E}}_W |\widetilde{\mathscr{X}_k}[W]|^p\right]^{1/p} \leq C\varepsilon^\zeta \tag{7.21}$$

for any $\zeta < 1 - \gamma/2$, where in the second inequality we used (7.3) and (7.20).

On the other hand, we note that $\mathscr{X}_k[W] - \widetilde{\mathscr{X}_k}[W]$ can be nonzero for at most one k, so we have

$$\left|\sum_{k=0}^{K_t^\varepsilon} (\mathscr{X}_k[W] - \widetilde{\mathscr{X}_k}[W])\right| = \max_{k=0}^{K_t^\varepsilon} \left|\mathscr{X}_k[W] - \widetilde{\mathscr{X}_k}[W]\right| \leq C\|\overline{u}\|_{\mathcal{C}^2(\mathbb{R}^d)} \varepsilon^{-\gamma/2} Z,$$

so

$$\left(\widetilde{\mathbb{E}}_W \left|\varepsilon^2 \sum_{k=0}^{K_t^\varepsilon} (\mathscr{X}_k[W] - \widetilde{\mathscr{X}_k}[W])\right|^p\right)^{1/p} \leq C\varepsilon^\zeta \tag{7.22}$$

for any $\zeta < 2 - \gamma/2$.

The error term. By (7.17), we have a constant C so that

$$\begin{aligned}\left|\sum_{k=0}^{K_t^\varepsilon} \mathscr{Y}_j^\varepsilon[W]\right| &\leq C\|\overline{u}\|_{\mathcal{C}^3(\mathbb{R}^d)} \sum_{j\geq 0} \Big(\varepsilon^4 |\sigma(k+1) - \sigma(j)|^2 \\ &\quad + |\varepsilon W_{\sigma(k+1)} - \varepsilon W_{\sigma(k)}|^3\Big) \\ &\leq C\|\overline{u}\|_{\mathcal{C}^3(\mathbb{R}^d)} K_t^\varepsilon \left(\varepsilon^4 (\varepsilon^{-\gamma} + Y)^2 + |\varepsilon F(\varepsilon^{-\gamma} + Y)|^3\right) \\ &\leq C\|\overline{u}\|_{\mathcal{C}^3(\mathbb{R}^d)} \left(\varepsilon^{2-\gamma}(1 + \varepsilon^\gamma Y)^2 + \varepsilon^{1-\gamma/2} Z^3\right),\end{aligned}$$

so by Lemma 7.2 we have

$$\left(\widetilde{\mathbb{E}}_W \left|\sum_{k=0}^{K_t^\varepsilon} \mathscr{Y}_j^\varepsilon[W]\right|^p\right)^{1/p} \leq C\varepsilon^\zeta \tag{7.23}$$

for any $\zeta < 1 - \gamma/2$.

The initial term. Finally, we observe that

$$\begin{aligned}&\left(\widetilde{\mathbb{E}}_W \left|\overline{u}(t - \varepsilon^2 \sigma(0), \varepsilon W_{\sigma(0)}) - \overline{u}(t, x)\right|^p\right)^{1/p} \leq \|\overline{u}\|_{\mathcal{C}^1(\mathbb{R}^d)} \\ &\quad \left(\widetilde{\mathbb{E}}_W (\varepsilon^2 \sigma(0) + \varepsilon |W_{\sigma(0)}|)^p\right)^{1/p} \leq C\varepsilon^\zeta \end{aligned} \tag{7.24}$$

for any $\zeta < 1 - \gamma/2$.

Applying the bounds (7.21), (7.22), (7.23), and (7.24) to (7.18) gives us (7.15). □

Corollary 7.5. *For any* $1 \le p < \infty$ *and* $\zeta < (\gamma - 1) \wedge (1 - \gamma/2)$ *there exists* $C = C(p, t, \zeta, \|u_0\|_{\mathcal{C}^3(\mathbb{R}^d)})$ *so that*

$$\left(\widetilde{\mathbb{E}}_W^{\varepsilon^{-1}x} |u_0(\varepsilon W_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[W]|^p\right)^{1/p} \le C\varepsilon^\zeta.$$

Proof. This is a simple consequence of the L^p triangle inequality applied to the results of the last two lemmas. □

We will also need the following auxiliary lemma:

Lemma 7.6. *There is a* $\beta_0 > 0$ *so that if* $\chi > 1$, $\beta > 0$ *are such that* $\chi\beta^2 < \beta_0^2$, *then there is a constant* $C = C(\chi, \beta) < \infty$ *so that for any* $\varepsilon > 0$ *and* $x, \tilde{x} \in \mathbb{R}^2$ *we have*

$$\widetilde{\mathbb{E}}_{W,\widetilde{W}}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}} \left(\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\} - 1\right)^\chi \le C\left(\frac{\varepsilon}{|x-\tilde{x}|} \wedge 1\right)^{d-2}.$$

Proof. Since $\mathscr{R}_t[W,\widetilde{W}] \ge 0$, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{W,\widetilde{W}}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}} \left(\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\} - 1\right)^\chi \\
&\quad \le \widetilde{\mathbb{P}}_{W,\widetilde{W}}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}} \left(\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}] > 0\right) \\
&\qquad \times \sup_{r>0,\, W|_{[0,r]},\, \widetilde{W}|_{[0,r]}} \widetilde{\mathbb{E}}_{W,\widetilde{W}}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}} \left[\exp\left\{\chi\beta^2 \mathscr{R}_{[r,\varepsilon^{-2}t]}[W,\widetilde{W}]\right\}\right. \\
&\quad \left.\Big|\, W|_{[0,r]}, \widetilde{W}|_{[0,r]}\right] \\
&\quad \le C\left(\frac{\varepsilon}{|x-\tilde{x}|} \wedge 1\right)^{d-2}
\end{aligned}$$

by Proposition 2.10 and Proposition 2.5, as long as $\chi\beta^2$ is sufficiently small. □

Proposition 7.7. *For all* $\chi > 1$, $\zeta < (1 - \gamma/2) \wedge (\gamma - 1)$, *and* $t > 0$, *there exists a constant* $C = C(\chi, \zeta, t, \|u\|_{\mathcal{C}^3(\mathbb{R}^d)})$ *so that*

$$\left|\mathbf{E} q^\varepsilon(t,x) q^\varepsilon(t,\tilde{x}) - \mathbf{E} q^\varepsilon(t,x) \mathbf{E} q^\varepsilon(t,\tilde{x})\right| \le C|x-\tilde{x}|^{-\frac{d-2}{\chi}} \varepsilon^{2\zeta + \frac{d-2}{\chi}}.$$

Proof. Take $p \ge 1$ so that $1/\chi + 2/p = 1$. We go back to (7.5) and apply Hölder's inequality, as well as Corollary 7.5 and Lemma 7.6, to get the bound

$$\begin{aligned}
&\left|\mathbf{E} q^\varepsilon(t,x) q^\varepsilon(t,\tilde{x}) - \mathbf{E} q^\varepsilon(t,x) \mathbf{E} q^\varepsilon(t,\tilde{x})\right| \\
&\quad \le \|\mathscr{G}\|_\infty^2 \left(\sup_{x \in \mathbb{R}^d} \widetilde{\mathbb{E}}_W^{\varepsilon^{-1}x} |u_0(\varepsilon W_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[W]|^p\right)^{2/p}
\end{aligned}$$

$$\times \left(\widetilde{\mathbb{E}}_{W,\widetilde{W}}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}}\left(\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\widetilde{W}]\right\}-1\right)^{\chi}\right)^{1/\chi}$$
$$\leq C\varepsilon^{2\zeta}\left(\frac{\varepsilon}{|x-\tilde{x}|}\right)^{\frac{d-2}{\chi}}.$$

□

We are finally ready to prove Theorem 1.5.

Proof of Theorem 1.5. By Proposition 7.7, we have, for any $\zeta < (1-\gamma/2)\wedge(\gamma-1)$ and any $\chi > 1$, that

$$\begin{aligned}
&\varepsilon^{-(d-2)}\mathbf{E}\left(\int g(x)q^{\varepsilon}(t,x)\,\mathrm{d}x-\mathbf{E}\int g(x)q^{\varepsilon}(t,x)\,\mathrm{d}x\right)^2\\
&=\varepsilon^{-(d-2)}\int\int g(x)g(\tilde{x})\left[\mathbf{E}q^{\varepsilon}(t,x)q^{\varepsilon}(t,\tilde{x})-\mathbf{E}q^{\varepsilon}(t,x)\mathbf{E}q^{\varepsilon}(t,\tilde{x})\right]\mathrm{d}x\,\mathrm{d}\tilde{x}\\
&\leq C\varepsilon^{(d-2)(1/\chi-1)+2\zeta}\int\int g(x)g(\tilde{x})|x-\tilde{x}|^{-\frac{d-2}{\chi}}\,\mathrm{d}x\,\mathrm{d}\tilde{x}.
\end{aligned}$$

The integral in the last line is finite because g is smooth and compactly supported. Now by taking χ sufficiently close to 1 and reducing ζ slightly, we achieve (1.28). □

Acknowledgements. We would like to thank the anonymous referees for multiple helpful comments and suggestions which helped improve the presentation of the paper. AD was supported by an NSF Graduate Research Fellowship under grant DGE-1147470, YG by NSF grant DMS-1613301/1807748/1907928 and the Center for Nonlinear Analysis at CMU, LR by NSF grant DMS-1613603 and ONR grant N00014-17-1-2145, and OZ by an Israel Science Foundation grant and funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement number 692452). We would like to thank S. Chatterjee, F. Hernandez, G. Papanicolaou, and M. Perlman for helpful comments and discussions.

References

1. Armstrong, S., Kuusi, T., Mourrat, J.-C.: The additive structure of elliptic homogenization. *Invent. Math* **208**(3), 999–1154, 2017
2. Armstrong, S.; Kuusi, T.; Mourrat, J.-C.: Quantitative stochastic homogenization and large-scale regularity, vol. 352. Grundlehren Math. Wiss. Springer, Cham 2019
3. Burkholder, D.L.: Martingale transforms. *Ann. Math. Statist*. **37**, 1494–1504, 1966
4. Caravenna, F., Sun, R., Zygouras, N.: Universality in marginally relevant disordered systems. *Ann. Appl. Probab*. **27**(5), 3050–3112, 2017
5. Caravenna, F., Sun, R., Zygouras, N.: On the moments of the (2 + 1)-dimensional directed polymer and stochastic heat equation in the critical window. *Commun. Math. Phys*. **372**(2), 385–440, 2019

6. COMETS, F.; COSCO, C.; MUKHERJEE, C.: Fluctuation and Rate of Convergence for the Stochastic Heat Equation in Weak Disorder, arXiv:1807.03902v2.
7. COMETS, F.; COSCO, C.; MUKHERJEE, C.: Space-time fluctuation of the Kardar-Parisi-Zhang equation in $d \geq 3$ and the Gaussian free field, arXiv:1905.03200v4.
8. DAWSON, D.A., SALEHI, H.: Spatially homogeneous random evolutions. *J. Multivariate Anal*. **10**(2), 141–180, 1980
9. DHARMADHIKARI, S.W., FABIAN, V., JOGDEO, K.: Bounds on the moments of martingales. *Ann. Math. Statist*. **39**, 1719–1723, 1968
10. DUERINCKX, M., GLORIA, A., OTTO, F.: The structure of fluctuations in stochastic homogenization. *Commun. Math. Phys*. **377**(1), 259–306, 2020
11. DUERINCKX, M. ; OTTO, F.: Higher-order pathwise theory of fluctuations in stochastic homogenization. *Stoch. Partial Differ. Equ. Anal. Comput.*, 2019
12. GLORIA, A., OTTO, F.: An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab*. **39**(3), 779–856, 2011
13. GLORIA, A., OTTO, F.: An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab*. **22**(1), 1–28, 2012
14. GLORIA, A., OTTO, F.: Quantitative results on the corrector equation in stochastic homogenization. *J. Eur. Math. Soc. (JEMS)* **19**(11), 3489–3548, 2017
15. GU, Y.: High order correctors and two-scale expansions in stochastic homogenization. *Probab. Theory Related Fields* **169**(3–4), 1221–1259, 2017
16. GU, Y., MOURRAT, J.-C.: Scaling limit of fluctuations in stochastic homogenization. *Multiscale Model. Simul*. **14**(1), 452–481, 2016
17. GU, Y., RYZHIK, L., ZEITOUNI, O.: The Edwards-Wilkinson limit of the random heat equation in dimensions three and higher. *Comm. Math. Phys*. **363**(2), 351–388, 2018
18. KESTEN, H., PAPANICOLAOU, G.C.: A limit theorem for turbulent diffusion. *Commun. Math. Phys*. **65**(2), 97–128, 1979
19. KESTEN, H.; PAPANICOLAOU, G. C.: A limit theorem for stochastic acceleration. *Comm. Math. Phys.*, 78(1):19–63, 1980/81.
20. MAGNEN, J., UNTERBERGER, J.: The scaling limit of the KPZ equation in space dimension 3 and higher. *J. Stat. Phys*. **171**(4), 543–598, 2018
21. MEYN, S., TWEEDIE, R.L.: *Markov Chains and Stochastic Stability*, 2nd edn. Cambridge University Press, Cambridge 2009
22. MUKHERJEE, C.: Central limit theorem for Gibbs measures on path spaces including long range and singular interactions and homogenization of the stochastic heat equation, arXiv:1706.09345v3
23. MUKHERJEE, C.; SHAMOV, A.; ZEITOUNI, O.: Weak and strong disorder for the stochastic heat equation and continuous directed polymers in $d \geq 3$. *Electron. Commun. Probab.*, 21:Paper No. 61, 12, 2016
24. STONE, C.: A local limit theorem for nonlattice multi-dimensional distribution functions. *Ann. Math. Statist*. **36**, 546–551, 1965
25. TESSITORE, G. ;ZABCZYK, J.: Invariant measures for stochastic heat equations. *Probab. Math. Statist.*, 18(2, Acta Univ. Wratislav. No. 2111):271–287, 1998

Alexander Dunlap
Department of Mathematics,
Stanford University,
Stanford
CA
94305 USA.
e-mail: alexander.dunlap@cims.nyu.edu

and

Yu Gu
Present address: Department of Mathematics,
Carnegie Mellon University,
Pittsburgh
PA 15213 USA.
e-mail: yug2@andrew.cmu.edu

and

Lenya Ryzhik
Department of Mathematics,
Carnegie Mellon University,
Pittsburgh
PA
15213 USA.
e-mail: ryzhik@stanford.edu

and

Ofer Zeitouni
Department of Mathematics,
Weizmann Institute of Science,
POB 26,
Rehovot
76100 Israel.
e-mail: ofer.zeitouni@weizmann.ac.il

(*Received August 22, 2018 / Accepted July 8, 2021*)
Published online August 24, 2021