Second Derivatives Estimate of Suitable Solutions to the 3D Navier–Stokes Equations

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Abstract

We study the second spatial derivatives of suitable weak solutions to the incompressible Navier–Stokes equations in dimension three. We show that it is locally $L^{\frac{4}{3},q}$ for any $q > \frac{4}{3}$, which improves from the current result of $L^{\frac{4}{3},\infty}$. Similar improvements in Lorentz space are also obtained for higher derivatives of the vorticity for smooth solutions. We use a blow-up technique to obtain nonlinear bounds compatible with the scaling. The local study works on the vorticity equation and uses De Giorgi iteration. In this local study, we can obtain any regularity of the vorticity without any a priori knowledge of the pressure. The local-to-global step uses a recently constructed maximal function for transport equations.

1. Introduction

We study the three dimensional incompressible Navier–Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla P = \Delta u, \qquad \operatorname{div} u = 0. \tag{1}$$

Here $u : (0, T) \times \mathbb{R}^3 \to \mathbb{R}^3$ and $P : (0, T) \times \mathbb{R}^3 \to \mathbb{R}$ represent the velocity field and the pressure field of a fluid in $\mathbb{R}^3$, within a finite or infinite timespan of length T. Initial condition

$$u(0, \cdot) = u_0 \in L^2(\mathbb{R}^3)$$

is given by a divergence-free velocity profile u_0 of finite energy.

Leray [11] and Hopf [8] proved the existence of weak solutions for all time. They constructed solutions $u \in C_w(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^3))$ corresponding to each aforementioned initial value, and satisfying (1) in the sense of distribution. A weak solution is called a Leray-Hopf solution if it satisfies energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u\|_{L^2((0,t)\times\mathbb{R}^3)}^2 \leq \frac{1}{2}\|u_0\|_{L^2(\mathbb{R}^3)}^2$$

for every $t > 0$. Since Leray and Hopf much work has been developed in regard
to the uniqueness and regularity of weak solutions. Nonuniqueness of weak solu-
tions was proven very recently by Buckmaster and Vicol using convex integration
scheme [1]. However, the question of the uniqueness of Leray-Hopf solutions still
remains open. The uniqueness is related with the regularity of solutions by the
Ladyženskaya-Prodi-Serrin criteria [7,10,14,20,21]: if the velocity belongs to any
space interpolating $L_t^2 L_x^\infty$ and $L_t^\infty L_x^3$ then it is actually smooth, and hence unique.
The endpoint case $L_t^\infty L_x^3$ came much later by Iskauriaza, Serëgin and Shverak [9].
These spaces require $\frac{1}{6}$ higher spatial integrability than the energy space provides,
which is $\mathcal{E} = L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$.

   At the level of energy space, Scheffer began to study the partial regularity for
a class of Leray-Hopf solutions, called suitable weak solutions [16–19]. These
solutions exist globally and satisfy the following local energy inequality:

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + P\right)\right) + |\nabla u|^2 \leq \Delta \frac{|u|^2}{2}.$$

Scheffer showed the singular set, at which the solution is unbounded nearby, has
time-space Hausdorff dimension at most $\frac{5}{3}$. This result was later improved by
Caffarelli, Kohn and Nirenberg in [2] (see also [12,23]), where they showed the
1-dimensional Hausdorff measure of the singular set is zero. We will investigate the
regularity of suitable weak solutions. In the periodic setting, Constantin constructed
suitable weak solutions whose second derivatives have space-time integrability
$L^{\frac{4}{3}-\varepsilon}$ for any $\varepsilon > 0$, provided the initial vorticities are bounded measures [6].
This was improved by Lions to a slightly better space $L^{\frac{4}{3},\infty}$, a Lorentz space
which corresponds to weak $L^{\frac{4}{3}}$ space [13]. These estimates are extended to higher
derivatives of smooth solutions by one of the authors and Choi using blow-up
arguments: $L_{\text{loc}}^{p,\infty}$ space-time boundedness for $(-\Delta)^{\frac{\alpha}{2}} \nabla^n u$, where $p = \frac{4}{n+\alpha+1}$,
$n \geq 1, 0 \leq \alpha < 2$ [5,22]. They also constructed suitable weak solutions satisfying
these bounds for $n + \alpha < 3$.

   The aim of this paper is to improve these regularity results in Lorentz space.
The main result is the following. Note that the estimate does not rely on the size of
the pressure.

Theorem 1. *Suppose we have a smooth solution u to the Navier–Stokes equations
in $(0, T) \times \mathbb{R}^3$ for some $0 < T \leq \infty$ with smooth divergence free initial data
$u_0 \in L^2$. Then for any integer $n \geq 0$, for any real number $q > 1$, the vorticity
$\omega = \operatorname{curl} u$ satisfies*

$$\left\| |\nabla^n \omega|^{\frac{4}{n+2}} \mathbf{1}_{\{|\nabla^n \omega|^{\frac{4}{n+2}} > C_n t^{-2}\}} \right\|_{L^{1,q}((0,T)\times\mathbb{R}^3)} \leq C_{q,n} \|u_0\|_{L^2}^2 \tag{2}$$

*for some constant C_n depending on n and $C_{q,n}$ depending only on q and n, uniform
in T. The above estimate (2) also holds for suitable weak solutions with only L^2
divergence free initial data in the case $n = 1$.*

   This theorem gives the following improvement on the second derivatives:

Corollary 1. *Let u be a suitable weak solution in* $(0, \infty) \times \mathbb{R}^3$ *with initial data* $u_0 \in L^2$. *Then for any* $q > \frac{4}{3}$, $K \subset\subset (0, \infty) \times \mathbb{R}^3$, *there exists a constant* $C_{q,K}$ *depending on q and K such that the following holds*

$$\|\nabla^2 u\|_{L^{\frac{4}{3}, q}(K)} \leq C_{q,K} \left(\|u_0\|_{L^2}^{\frac{3}{2}} + 1 \right).$$

Let us explain the main ideas of the proof. Similar as previous work on higher derivatives, the proof is also based on blow-up techniques. In particular, we blow up the equation along a trajectory, using the scaling symmetry and the Galilean invariance of the Navier–Stokes equations. That is, if we fix an initial time t_0 and move the frame of reference along some $X(t)$, and zoom in into ε scale, then it is easy to verify that $\tilde{u}(s, y)$ and $\tilde{P}(s, y)$, defined by

$$\frac{1}{\varepsilon} \tilde{u} \left(\frac{t - t_0}{\varepsilon^2}, \frac{x - X(t)}{\varepsilon} \right) := u(t, x) - \dot{X}(t)$$

$$\frac{1}{\varepsilon^2} \tilde{P} \left(\frac{t - t_0}{\varepsilon^2}, \frac{x - X(t)}{\varepsilon} \right) := P(t, x) + x \cdot \ddot{X}(t), \tag{3}$$

also satisfy the Navier–Stokes equation

$$\partial_s \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{P} = \Delta \tilde{u}, \quad \operatorname{div} \tilde{u} = 0.$$

We develop the following local theorem for $\tilde{u}$ and $\tilde{P}$. Note that it needs nothing from the pressure. Denote $B_r \subset \mathbb{R}^3$ to be a ball centered at the origin with radius r, and $Q_r = (-r^2, 0) \times B_r \subset \mathbb{R}^4$ to be a space-time cylinder.

Theorem 2. (Local Theorem) *There exists a universal constant* $\eta_1 > 0$, *such that for any suitable weak solution u to the Navier–Stokes equations in* $(-4, 0) \times \mathbb{R}^3$ *satisfying*

$$\int_{B_1} u(t, x)\phi(x)\mathrm{d}x = 0 \quad \text{a.e.} t \in (-4, 0), \tag{4}$$

$$\|\nabla u\|_{L_t^{p_1} L_x^{q_1}(Q_2)} + \|\omega\|_{L_t^{p_2} L_x^{q_2}(Q_2)} \leq \eta_1, \tag{5}$$

where $\phi \in C_c^\infty(B_1)$ *is a non-negative function with* $\int \phi = 1$, $\omega = \operatorname{curl} u$ *is the vorticity,* $\frac{4}{3} \leq p_1 \leq \infty$, $1 \leq p_2 \leq \infty$, $1 \leq q_1, q_2 < 3$ *satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} < 1, \quad \frac{1}{q_1} + \frac{1}{q_2} \leq \frac{7}{6},$$

then for any integer $n \geq 0$, *we have*

$$\|\nabla^n \omega\|_{L^\infty(Q_{8-n-2})} \leq C_n$$

for some constant C_n *depending only on n.*

Let us illustrate the ideas of how to go from this local theorem towards the main result. We want to choose a "pivot quantity", blow up near a point, and use this quantity to control $\nabla^n \omega$. When we patch the local results together, we will obtain a nonlinear bound with the same scaling as the pivot quantity, so we want the pivot quantity to have the best possible scaling. The ideal pivot quantities would be $\int |\nabla u|^2 \mathrm{d}x\mathrm{d}t$ and $\int |\nabla^2 P|\mathrm{d}x\mathrm{d}t$. $\int |u|^{\frac{10}{3}}\mathrm{d}x\mathrm{d}t$ has a worse scaling and should not be used. However, we still need to control the flux in the local theorem, so we want to take out the mean velocity and control u by ∇u using Poincaré's inequality.

In order to take out the mean velocity, we choose $X(t)$ to be the trajectory of the mollified flow so that (4) can be realized. Notice that a cylinder Q_r in the local (s, y) coordinate will be transformed into a "skewed cylinder" growing along $X(t)$ in the global (t, x) coordinate. One of the authors recently constructed a maximal function $\mathcal{M}_\mathcal{Q}$ associated with these cylinders [24], which serves as a bridge between the local theorem and the global result, and is one of the main reasons for the improvement in this paper. The idea is, if locally the vorticity gradient can be controlled in L^∞ by the integral of something in the skewed cylinder, and the integral in a skewed cylinder can be controlled by the maximal function $\mathcal{M}_\mathcal{Q}$, then vorticity gradient is pointwise bounded by the maximal function.

If one uses $\int |\nabla u|^2 \mathrm{d}x\mathrm{d}t$ and $\int |\nabla^2 P|\mathrm{d}x\mathrm{d}t$ as the pivot quantity, then unfortunately the best possible outcome would just be an $L^{1,\infty}$ bound, as obtained in [24]. The reason is, the maximal function is bounded on L^p for $p > 1$, but for $p = 1$ it is only bounded from L^1 to $L^{1,\infty}$. Unfortunately $|\nabla u|^2$ and $|\nabla^2 P|$ are both L^1 quantities, so $\mathcal{M}_\mathcal{Q} \left(|\nabla u|^2 + |\nabla^2 P| \right)$ is only $L^{1,\infty}$. We need two things to improve from $L^{1,\infty}$: replace $\int |\nabla u|^2$ by $\int |\nabla u|^p$, and drop the pressure $\nabla^2 P$.

Suppose we could use $(\int |\nabla u|^p \mathrm{d}x\mathrm{d}t)^{\frac{2}{p}}$ as the pivot quantity for some $p < 2$, then we can majorize it by $\mathcal{M}_\mathcal{Q} (|\nabla u|^p)^{\frac{2}{p}} \in L^1$, since $\frac{2}{p} > 1$ and $\mathcal{M}_\mathcal{Q}$ is bounded in $L^{\frac{2}{p}}$. However, this poses significant difficulties in the local theorem. The nonlinear term $u \cdot \nabla u$ is quadratic, and if we only have a subquadratic integrability to begin with, we cannot treat this quadratic transport term as a source term because it is not integrable. Observe that what we lack is the temporal integrability rather than the spatial one: if p is slightly smaller than two, then $u \cdot \nabla u$ is still $L^{\frac{3}{2}-}$ in space, but L^{1-} in time. To overcome this difficulty, we write $u \cdot \nabla u$ as $\omega \times u$ up to a gradient term, and put $L_t^{2-} L_x^{6-}$ on u and $L_t^{2+} L_x^{2-}$ on ω. We compensate the lower integrability term by pairing with a higher integrability term to make $\omega \times u$ integrable. $L_t^{2+} L_x^{2-}$ of ω can be interpolated between $L_t^{2-} L_x^{2-}$ and $L_t^\infty L_x^1$, while the latter is controlled by $L_{t,x}^2$ of ∇u. Since $L_t^{2+} L_x^{2-}$ is closer to $L_t^{2-} L_x^{2-}$ than to $L_t^\infty L_x^1$, the pivot quantity that we use is actually $\delta^{-\nu}\|\nabla u\|_{L^p}^2 + \delta\|\nabla u\|_{L^2}^2$ for ν close to 0. By using more subquadratic integrability and a tiny bit of the quadratic one, we can complete the task by interpolation. That is why we obtain $L^{1,q}$ in the end: it interpolates L^1 bound from $\|\nabla u\|_{L^p}$ and $L^{1,\infty}$ bound from $\|\nabla u\|_{L^2}$. Unfortunately we still miss the endpoint L^1.

The second task is more subtle and technical. Without any information on the pressure, we don't have any control on the nonlocal effect. However, the role of the pressure is not important at the vorticity level; if we take the curl of the Navier–

Stokes equation, the pressure will disappear and we are left with the vorticity equation involving only local quantities:

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega. \tag{6}$$

Inspired by Chamorro, Lemarié-Rieusset and Mayoufi [4], we introduce a new velocity variable $v = -\operatorname{curl} \varphi^\sharp \Delta^{-1} \varphi \omega$ using only local information of vorticity (φ and $\varphi^\sharp$ are spatial cut-off functions), and this helps us to prove the local theorem. This is another main reason for the improvement in this paper. Consequently, the bounds we obtain in the end are on the vorticity ω rather than on the velocity u.

    This paper is organized as follow: in the preliminary Section 2 we introduce the analysis tools to the reader. We show how to rigorously derive the main results from the local theorem in Section 3, and then deal with technicalities of the local theorem in the later sections. The proof of the local theorem consists of three parts. Section 4 introduces the new variables v, and shows the smallness of v in the energy space. Then we use De Giorgi iteration argument in Section 5 to prove boundedness of v. Finally, we inductively bound ω and all its higher derivatives in Section 6.

2. Preliminary

    In this section, we introduce a few tools that we are going to use in the paper, including the maximal function, Lorentz space, and Helmholtz decomposition.

2.1. Maximal Function Associated with Skewed Cylinders

    This is recently developed for incompressible flows in [24]. We quote useful results here without proof.

    Suppose $u \in L^p(0, T; \dot{W}^{1,p}(\mathbb{R}^3; \mathbb{R}^3))$ is a vector field in $\mathbb{R}^3$. Fix $\phi \in C_c^\infty(B_1)$ to be a nonnegative function with $\int \phi = 1$ through out the paper. For $\varepsilon > 0$ define $\phi_\varepsilon(x) = \varepsilon^{-3} \phi(-x/\varepsilon)$, and let $u_\varepsilon(t, \cdot) = u(t, \cdot) * \phi_\varepsilon$ be the mollified velocity. For a fixed (t, x) we let $X(s)$ solve the following initial value problem:

$$\begin{cases} \dot{X}(s) = u_\varepsilon(s, X(s)), \\ X(t) = x. \end{cases}$$

The skewed parabolic cylinder $Q_\varepsilon(t, x)$ is then defined to be

$$Q_\varepsilon(t, x) := \left\{ (t + \varepsilon^2 s, X(t) + \varepsilon y) : -9 \le s \le 0, \, y \in B_3 \right\}. \tag{7}$$

We use $\mathcal{M}$ to denote the spatial Hardy-Littlewood maximal function, which is defined by

$$\mathcal{M}(f)(t, x) = \sup_{r > 0} \fint_{B_r(x)} |f(t, y)| \mathrm{d}y.$$

Then we construct the space-time maximal function adapted to the flow.

Theorem 3. (Q-Maximal Function) *There exists a universal constant η_0 such that the following is true. We say $Q_\varepsilon(t, x)$ is admissible if $Q_\varepsilon(t, x) \subset (0, T) \times \mathbb{R}^3$ and*

$$\varepsilon^2 \fint_{Q_\varepsilon(t,x)} \mathcal{M}(|\nabla u|) \mathrm{d}x \mathrm{d}t \leq \eta_0. \tag{8}$$

Define the maximal function

$$\mathcal{M}_Q(f)(t, x) := \sup_{\varepsilon > 0} \left\{ \fint_{Q_\varepsilon(t,x)} |f(s, y)| \mathrm{d}s \mathrm{d}y : Q_\varepsilon(t, x) \text{ is admissible} \right\}.$$

If u is divergence free and $\mathcal{M}(|\nabla u|) \in L^q$ for some $1 \leq q \leq \infty$, then $\mathcal{M}_Q$ is bounded from $L^1((0, T) \times \mathbb{R}^3)$ to $L^{1,\infty}((0, T) \times \mathbb{R}^3)$ and from $L^p((0, T) \times \mathbb{R}^3)$ to itself for any $p > 1$ with norm depending on p.

An important consequence of the weak type $(1, 1)$ bound of the Hardy-Littlewood maximal function is the Lebesgue differentiation theorem in $\mathbb{R}^n$. Similarly, we can use the Q-maximal function to prove the Q-Lebesgue differentiation theorem.

Theorem 4. (Q-Lebesgue Differentiation Theorem) *Let $f \in L^1_{\mathrm{loc}}((0, T) \times \mathbb{R}^3)$. Then for almost every $(t, x) \in (0, T) \times \mathbb{R}^3$,*

$$\lim_{\varepsilon \to 0} \fint_{Q_\varepsilon(t,x)} |f(s, y) - f(t, x)| \mathrm{d}s \mathrm{d}y = 0.$$

In this case we say (t, x) is a Q-Lebesgue point of f.

2.2. Lorentz Space

Let (X, μ) be a measure space. Recall that for a measurable function f, its decreasing rearrangement is defined as

$$f^*(\lambda) := \inf \{ \alpha > 0 : \mu(\{|f| > \alpha\}) < \lambda \}, \qquad \lambda \geq 0.$$

For $0 < p < \infty$, $0 < q \leq \infty$, Lorentz space $L^{p,q}(X)$ is defined as the set of functions f for which

$$\|f\|_{L^{p,q}(X)} := \|\lambda^{\frac{1}{p}} f^*\|_{L^q(\frac{\mathrm{d}\lambda}{\lambda})} = \|\lambda^{\frac{1}{p} - \frac{1}{q}} f^*(\lambda)\|_{L^q} < \infty.$$

Now we introduce the interpolation lemma for Lorentz spaces.

Lemma 1. (Interpolation of Lorentz Spaces) *Let $\nu > 0$ be a fixed positive number. Assume $f_0 \in L^{p_0, q_0}$, $f_1 \in L^{p_1, q_1}$, where $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$. If f is a measurable function satisfying*

$$2|f| \leq \delta f_0 + \delta^{-\nu} f_1 \qquad \text{for all } \delta > 0, \tag{9}$$

then $f \in L^{p,q}$, where

$$\frac{1}{p} = \frac{\nu}{1 + \nu} \frac{1}{p_0} + \frac{1}{1 + \nu} \frac{1}{p_1}, \qquad \frac{1}{q} = \frac{\nu}{1 + \nu} \frac{1}{q_0} + \frac{1}{1 + \nu} \frac{1}{q_1}.$$

Proof. It is easy to check from the definition of decreasing rearrangement that if $h \le f + g$, then $h^*(2\lambda) \le (f + g)^*(2\lambda) \le f^*(\lambda) + g^*(\lambda)$. Thus (9) implies

$$2|f^*(2\lambda)| \le \delta f_0^*(\lambda) + \delta^{-\nu} f_1^*(\lambda), \qquad \text{for all } \lambda \ge 0, \delta > 0.$$

Set $\theta = \frac{1}{1+\nu}, \delta = f_0^*(\lambda)^{-\theta} f_1^*(\lambda)^\theta$, then

$$2|f^*(2\lambda)| \le f_0^*(\lambda)^{-\theta} f_1^*(\lambda)^\theta f_0^*(\lambda) + f_0^*(\lambda)^{\nu\theta} f_1^*(\lambda)^{-\nu\theta} f_1^*(\lambda)$$
$$= 2 f_0^*(\lambda)^{1-\theta} f_1^*(\lambda)^\theta.$$

Therefore,

$$\begin{aligned}
\|f\|_{L^{p,q}} &= \|\lambda^{\frac{1}{p}-\frac{1}{q}} f^*(\lambda)\|_{L^q} = C\|\lambda^{\frac{1}{p}-\frac{1}{q}} f^*(2\lambda)\|_{L^q} \\
&\le C\|\lambda^{\frac{1-\theta}{p_0}-\frac{1-\theta}{q_0}} f_0^*(\lambda)^{1-\theta} \cdot \lambda^{\frac{\theta}{p_1}-\frac{\theta}{q_1}} f_1^*(\lambda)^\theta\|_{L^q} \\
&\le C\|\lambda^{\frac{1-\theta}{p_0}-\frac{1-\theta}{q_0}} f_0^*(\lambda)^{1-\theta}\|_{L^{\frac{q_0}{1-\theta}}} \|\lambda^{\frac{\theta}{p_1}-\frac{\theta}{q_1}} f_1^*(\lambda)^\theta\|_{L^{\frac{q_1}{\theta}}} \\
&= C\|\lambda^{\frac{1}{p_0}-\frac{1}{q_0}} f_0^*(\lambda)\|_{L^{q_0}}^{1-\theta} \|\lambda^{\frac{1}{p_1}-\frac{1}{q_1}} f_1^*(\lambda)\|_{L^{q_1}}^\theta \\
&= C\|f_0\|_{L^{p_0,q_0}}^{1-\theta} \|f_1\|_{L^{p_1,q_1}}^\theta,
\end{aligned}$$

where $C = 2^{\frac{1}{p}}$. □

We would also like to mention that Riesz transform is bounded on Lorentz space. The proof can be found in [3]. See [15] for general Lorentz spaces.

Lemma 2. *For $1 < p < \infty$, $1 \le q \le \infty$, $R_{ij} = \partial_i \partial_j \Delta^{-1}$ is a bounded linear operator from $L^{p,q}(\mathbb{R}^n)$ to itself. As a spatial operator, it is also bounded in time-space from $L^{p,q}((0, T) \times \mathbb{R}^n)$ to itself.*

2.3. Helmholtz Decomposition

First, recall two vector calculus identities:

$$\nabla(u \cdot v) = (u \cdot \nabla)v + (v \cdot \nabla)u + u \times \text{curl } v + v \times \text{curl } u, \tag{10}$$
$$\text{curl}(u \times v) = u \, \text{div } v - v \, \text{div } u + (v \cdot \nabla)u - (u \cdot \nabla)v. \tag{11}$$

For operators A and B, denote $[A, B] = AB - BA$ to be their commutator. Define $\mathbb{P}_{\text{curl}} = -\text{curl curl } \Delta^{-1}$ and $\mathbb{P}_\nabla = \nabla\Delta^{-1} \text{div} = \text{Id} - \mathbb{P}_{\text{curl}}$ to be the Helmholtz decomposition. Then we compute the following commutators:

$$[\varphi, \text{curl}]u = -\nabla\varphi \times u, \tag{12}$$
$$[\varphi, \Delta]u = -2\nabla\varphi \cdot \nabla u - (\Delta\varphi)u = -2 \, \text{div}(\nabla\varphi \otimes u) + (\Delta\varphi)u, \tag{13}$$
$$[\varphi, \Delta^{-1}]u = \Delta^{-1}\left\{2\nabla\varphi \cdot \nabla\Delta^{-1}u + (\Delta\varphi)\Delta^{-1}u\right\}, \tag{14}$$
$$\begin{aligned}
[\varphi, \mathbb{P}_{\text{curl}}]u = {}& \nabla\varphi \times \text{curl }\Delta^{-1}u + \nabla\varphi \, \text{div }\Delta^{-1}u - \Delta^{-1}u\Delta\varphi \\
& + (\Delta^{-1}u \cdot \nabla)\nabla\varphi - (\nabla\varphi \cdot \nabla)\Delta^{-1}u \\
& + \mathbb{P}_{\text{curl}}\left\{2\nabla\varphi \cdot \nabla\Delta^{-1}u + (\Delta\varphi)\Delta^{-1}u\right\}.
\end{aligned} \tag{15}$$

The first two are straightforward. The third uses

$$[\varphi, \Delta^{-1}] = -\Delta^{-1}[\varphi, \Delta]\Delta^{-1},$$

and the last one is because

$$[\varphi, \mathbb{P}_{\mathrm{curl}}] = [\varphi, -\operatorname{curl}\operatorname{curl}\Delta^{-1}]$$
$$= -[\varphi, \operatorname{curl}]\operatorname{curl}\Delta^{-1} - \operatorname{curl}[\varphi, \operatorname{curl}]\Delta^{-1} - \operatorname{curl}\operatorname{curl}[\varphi, \Delta^{-1}],$$
$$[\varphi, \mathbb{P}_{\mathrm{curl}}]u = \nabla\varphi \times \operatorname{curl}\Delta^{-1}u + \operatorname{curl}(\nabla\varphi \times \Delta^{-1}u)$$
$$- \operatorname{curl}\operatorname{curl}\Delta^{-1}\left\{2\nabla\varphi \cdot \nabla\Delta^{-1}u + (\Delta\varphi)\Delta^{-1}u\right\},$$

so we can expand $\operatorname{curl}(\nabla\varphi \times \Delta^{-1}u)$ by (11).

Lemma 3. $\partial_i[\varphi, \mathbb{P}_{\mathrm{curl}}]$ and $[\varphi, \mathbb{P}_{\mathrm{curl}}]\partial_i$ are both bounded linear operator from L^p to L^p for any $1 < p < \infty$, i.e.

$$\|\partial_i[\varphi, \mathbb{P}_{\mathrm{curl}}]u\|_{L^p} + \|[\varphi, \mathbb{P}_{\mathrm{curl}}]\partial_i u\|_{L^p} \leq C_{p,\varphi}\|u\|_{L^p}.$$

Proof. First, we observe that by Jacobi identity $[\varphi, \mathbb{P}_{\mathrm{curl}}]\partial_i$ and $\partial_i[\varphi, \mathbb{P}_{\mathrm{curl}}]$ differ by

$$[[\varphi, \mathbb{P}_{\mathrm{curl}}], \partial_i] = [\varphi, [\mathbb{P}_{\mathrm{curl}}, \partial_i]] - [\mathbb{P}_{\mathrm{curl}}, [\varphi, \partial_i]] = 0 - [\mathbb{P}_{\mathrm{curl}}, \partial_i\varphi],$$

which is bounded from L^p to L^p for any p, because both $\mathbb{P}_{\mathrm{curl}}$ and multiplication by $\partial_i\varphi$ are bounded from L^p to L^p, so we can complete the proof by duality. For $1 < p < 3$, set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$, from (15) we can see that

$$\|[\varphi, \mathbb{P}_{\mathrm{curl}}]\partial_i u\|_{L^p} \lesssim \|\nabla\Delta^{-1}\partial_i u\|_{L^p(\mathbb{R}^3)} + C_{p,\varphi}\|\Delta^{-1}\partial_i u\|_{L^p(\mathrm{supp}\,\varphi)}$$
$$\lesssim \|u\|_{L^p(\mathbb{R}^3)} + C_{p,\varphi}\|\partial_i\Delta^{-1}u\|_{L^{p^*}(\mathrm{supp}\,\varphi)}$$
$$\leq C\|u\|_{L^p(\mathbb{R}^3)}.$$

For $\frac{3}{2} < p < \infty$, set $1 - \frac{1}{p} = \frac{1}{q} = \frac{1}{q^*} + \frac{1}{3}$, then $1 < p, q, q^* < \infty$. Take any $u \in L^p(\mathbb{R}^3)$ and any vector field $v \in L^q(\mathbb{R}^3)$ to get

$$\int \partial_i[\varphi, \mathbb{P}_{\mathrm{curl}}]u \cdot v\,\mathrm{d}x = -\int [\varphi, \mathbb{P}_{\mathrm{curl}}]u \cdot \partial_i v\,\mathrm{d}x$$
$$= \int u \cdot [\varphi, \mathbb{P}_{\mathrm{curl}}]\partial_i v\,\mathrm{d}x$$
$$\leq \|u\|_{L^p(\mathbb{R}^3)}(\|v\|_{L^q(\mathbb{R}^3)} + \|\partial_i\Delta^{-1}v\|_{L^q(\mathrm{supp}\,\varphi)})$$
$$\leq \|u\|_{L^p(\mathbb{R}^3)}(\|v\|_{L^q(\mathbb{R}^3)} + C_{q,\varphi}\|\partial_i\Delta^{-1}v\|_{L^{q^*}(\mathrm{supp}\,\varphi)})$$
$$\leq C\|u\|_{L^p(\mathbb{R}^3)}\|v\|_{L^q(\mathbb{R}^3)}.$$

$\square$

Corollary 2. $\partial_i[\varphi, \mathbb{P}_\nabla]$ and $[\varphi, \mathbb{P}_\nabla]\partial_i$ are both bounded linear operator from L^p to L^p for any $1 < p < \infty$:

Proof. $\text{Id} = \mathbb{P}_\nabla + \mathbb{P}_{\text{curl}}$ commutes with φ, so $[\varphi, \mathbb{P}_\nabla] = -[\varphi, \mathbb{P}_{\text{curl}}]$.   □

Because of the smoothing effect of the Laplace potential, we have the following.

Lemma 4. *Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be supported away from some openset $\Omega \subset \mathbb{R}^3$, that is, $\text{dist}(\text{supp}\,\varphi, \Omega) = d > 0$. Then for any $f \in L_{\text{loc}}^1(\mathbb{R}^3)$, $k > 0$,*

$$\|\Delta^{-1}(\varphi f)\|_{C^k(\Omega)} \lesssim_{k,d} \|f\|_{L^1(\text{supp}\,\varphi)}.$$

We also have

$$\|\mathbb{P}_\nabla(\varphi f)\|_{C^k(\Omega)}, \|\mathbb{P}_{\text{curl}}(\varphi f)\|_{C^k(\Omega)} \lesssim_{k,d} \|f\|_{L^1(\text{supp}\,\varphi)}.$$

3. Proof of the Main Results

In this section, we show that the Local Theorem 2 leads to the main results. First, we show the pivot quantity is indeed enough to bound $\nabla^n \omega$.

Lemma 5. *There exists $\eta_2 > 0$ such that the following holds. Let $\frac{11}{6} < p < 2$, $\frac{2-p}{p-1} < \nu \leq \frac{7p-12}{6-p}$. If u is a suitable solution to the Navier–Stokes equations in $(-9, 0) \times \mathbb{R}^3$ satisfying the condition*

$$\int_{B_1} u(t, x)\phi(x)\mathrm{d}x = 0, \qquad \text{a.e.}\ t \in (-9, 0), \tag{16}$$

$$\delta^{-\nu}\left(\int_{Q_3} |\nabla u|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}} \leq \eta_2, \tag{17}$$

$$\delta \int_{Q_3} |\nabla u|^2 \mathrm{d}x\mathrm{d}t \leq \eta_2, \tag{18}$$

for some $\delta \leq \eta_2$,
   then we have for any $n \geq 0$,

$$\|\nabla^n \omega\|_{L_{t,x}^\infty(Q_{8-n-2})} \leq C_n.$$

Here C_n is the same constant in Theorem 2.

Proof. First, we claim that

$$\delta\|\omega\|_{L^\infty(-4,0;L^1(B_2))} \leq C\eta_2. \tag{19}$$

Formally, we can take the dot product of both sides of the vorticity equation (1) with $\omega^0 := \frac{\omega}{|\omega|}$, and recalling the convexity inequality $\omega^0 \cdot \Delta\omega \leq \Delta|\omega|$, we have

$$(\partial_t + u \cdot \nabla - \Delta)|\omega| - \omega \cdot \nabla u \cdot \omega^0 \leq 0. \tag{20}$$

Let $\psi \in C_c^\infty((-9, 0] \times \mathbb{R}^3)$ be a cut-off function such that $\mathbf{1}_{Q_2} \leq \psi \leq \mathbf{1}_{Q_3}$. Multiply (20) by ψ and then integrate in space to get

$$\frac{d}{dt} \int \psi |\omega| dx \leq \int [(\partial_t + u \cdot \nabla + \Delta)\psi] |\omega| dx + \int \psi \omega \cdot \nabla u \cdot \omega^0 dx$$

$$\leq C \int_{B_3} 1 + |u|^2 + |\nabla u|^2 dx \leq C \left(1 + \int_{B_3} |\nabla u|^2 dx\right)$$

for some large universal constant $C > 1$. The last step uses Poincaré's inequaliting and (16). Integrate in time, we obtain

$$\|\omega\|_{L^\infty(-4,0;L^1(B_2))} \leq C \left(1 + \frac{\eta_2}{\delta}\right) \leq 2C \frac{\eta_2}{\delta}.$$

This proves the claim. A more rigorous proof can be obtained by difference quotient same as in Constantin [6] or Lions [13] Theorem 3.6, so we omit the details.

Now we interpolate between (17) and (19). Let $\theta = \frac{1}{1+\nu}$,

$$\|\omega\|_{L_t^{p_2} L_x^{q_2}(Q_2)} \leq \|\omega\|_{L^p(Q_2)}^\theta \|\omega\|_{L_t^\infty L_x^1(Q_2)}^{1-\theta} \leq (2C)^{1-\theta} \eta_2 \delta^{\theta\nu+\theta-1} \leq \frac{1}{2}\eta_1,$$

where we choose $\eta_2 = \frac{\eta_1}{4C+1} \leq \frac{1}{2}\eta_1$ from Theorem 2, and p_2, q_2 are determined by

$$\frac{1}{p_2} = \frac{\theta}{p}, \qquad \frac{1}{q_2} = \frac{\theta}{p} + 1 - \theta.$$

Combining the above with (17) we have

$$\|\nabla u\|_{L_t^p L_x^p(Q_2)} + \|\omega\|_{L_t^{p_2} L_x^{q_2}(Q_2)} \leq \frac{1}{2}\eta_1 + \frac{1}{2}\eta_1 = \eta_1. \tag{21}$$

By the choice of θ and the range of ν,

$$\frac{1}{p} + \frac{1}{p_2} = \frac{1}{p} + \frac{1}{p(1+\nu)} = \frac{2+\nu}{p(1+\nu)} < 1,$$

$$\frac{1}{p} + \frac{1}{q_2} = \frac{1}{p} + \frac{1+\nu p}{p(1+\nu)} = \frac{2+\nu+\nu p}{p(1+\nu)} \leq \frac{7}{6}.$$

One can also easily check that $p < 2$ implies $q_2 < 2$, and thus by (16) and (21) the requirements of the Local Theorem 2 are satisfied with $p_1 = q_1 = p$, and it completes the proof of the lemma. □

Now we transform this lemma into the global coordinate. Recall that $Q_\varepsilon(t, x)$ is defined by (7).

Corollary 3. *There exists $\eta_3 > 0$ such that, if for some $\delta \leq \eta_2$,*

$$\delta^{-2\nu} \left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^p dx dt\right)^{\frac{2}{p}} + \delta \fint_{Q_\varepsilon(t,x)} |\nabla u|^2 dx dt \leq \eta_3 \varepsilon^{-4}, \tag{22}$$

then

$$|\nabla^n \omega(t, x)| \leq C_n \varepsilon^{-n-2}.$$

Proof. Define $\tilde{u}$ by (3). Then (22) implies

$$\delta^{-2\nu}\left(\fint_{Q_3}|\nabla\tilde{u}|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p}}\leq\eta_3,\qquad\delta\fint_{Q_3}|\nabla\tilde{u}|^2\mathrm{d}x\mathrm{d}t\leq\eta_3$$

$$\Rightarrow\delta^{-\nu}\left(\int_{Q_3}|\nabla\tilde{u}|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\leq\eta_3^{\frac{1}{2}}|Q_3|^{\frac{1}{p}},\qquad\delta\int_{Q_3}|\nabla\tilde{u}|^2\mathrm{d}x\mathrm{d}t\leq\eta_3|Q_3|.$$

Moreover, (16) is satisfied by $\tilde{u}$. Therefore, if we choose η_3 such that

$$\max\left\{\eta_3^{\frac{1}{2}}|Q_3|^{\frac{1}{p}},\eta_3|Q_3|\right\}=\eta_2,$$

then by Lemma 5, $\tilde{\omega}:=\mathrm{curl}\,\tilde{u}$ has bounded derivatives at $(0,0)$, and thus finish the proof of the corollary by scaling. $\qquad\square$

Then we use the maximal function to go from the local bound to a global bound.

Proof of Theorem 1. First, we fix $\frac{11}{6}<p<2$, $\frac{2-p}{p-1}<\nu\leq\frac{7p-12}{6-p}$. Let $\eta\ll 1$ be a small constant to be specified later. Finally we fix a $0<\delta<\infty$. For $(t,x)\in(0,T)\times\mathbb{R}^3$, define

$$I(\varepsilon)=\varepsilon^4\left[\delta^{-2\nu}\left(\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^p\right)^{\frac{2}{p}}+\delta\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^2\right].$$

If (t,x) is both a $\mathcal{Q}$-Lebesgue point of $|\mathcal{M}(\nabla u)|^p$ and of $|\mathcal{M}(\nabla u)|^2$, then we claim that there exists a positive $\varepsilon=\varepsilon_{(t,x)}$ such that one of two cases is true:

Case 1. $3\varepsilon_{(t,x)}<t^{\frac{1}{2}}$, and $I(\varepsilon_{(t,x)})=\eta$.
Case 2. $3\varepsilon_{(t,x)}=t^{\frac{1}{2}}$, and $I(\varepsilon_{(t,x)})\leq\eta$.

This is because, by Theorem 4,

$$\lim_{\varepsilon\to0}I(\varepsilon)=0^4\left[\delta^{-2\nu}(|\mathcal{M}(|\nabla u|)(t,x)|^p)^{\frac{2}{p}}+\delta|\mathcal{M}(|\nabla u|)(t,x)|^2\right]=0,$$

and $I(\varepsilon)$ is a continuous function of ε.

On the one hand, in both cases we have $I(\varepsilon)\leq\eta$, which implies that

$$\delta^{-\nu}\varepsilon^2\left(\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^p\right)^{\frac{1}{p}}\leq\sqrt{\eta},\qquad\delta^{\frac{1}{2}}\varepsilon^2\left(\fint_{Q_\varepsilon(t,x)}|\mathcal{M}(\nabla u)|^2\right)^{\frac{1}{2}}\leq\sqrt{\eta}.$$

If we set $\eta<\eta_0^2$, then depending on $\delta\geq 1$ or $\delta\leq 1$, one of the two would imply admissibility condition (8) by Jensen's inequality. Therefore $Q_\varepsilon(t,x)$ is admissible and

$$I(\varepsilon)\leq\varepsilon^4\left[\delta^{-2\nu}\mathcal{M}_\mathcal{Q}(\mathcal{M}(\nabla u)^p)^{\frac{2}{p}}+\delta\mathcal{M}_\mathcal{Q}(\mathcal{M}(\nabla u)^2)\right],$$

so we can combine two cases and conclude

$$\varepsilon_{(t,x)}^{-4} \leq \max \left\{ \frac{1}{\eta} \left[\delta^{-2\nu} \mathcal{M}_Q (\mathcal{M}(\nabla u)^p)^{\frac{2}{p}} + \delta \mathcal{M}_Q (\mathcal{M}(\nabla u)^2) \right], 81t^{-2} \right\}. \quad (23)$$

On the other hand, if we set $\eta < \eta_3$, then in both cases $I(\varepsilon) \leq \eta_3$. If $\delta \leq \eta_2$ one would have

$$|\nabla^n \omega(t,x)| \leq C_n \varepsilon^{-n-2} \quad (24)$$

by Corollary 3. If $\delta > \eta_2$, notice that by Jensen's inequality,

$$\left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} \leq \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2,$$

so

$$I(\varepsilon) \geq \varepsilon^4 \left[(\delta^{-2\nu} + \delta - \eta_2) \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right]$$

$$\geq \varepsilon^4 \left[(1 - \eta_2) \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right]$$

$$\geq (1 - \eta_2) \eta_2^{2\nu} \varepsilon^4 \left[\eta_2^{-2\nu} \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right].$$

If we require $\eta < (1 - \eta_2)\eta_2^{2\nu}\eta_3$, then

$$\varepsilon^4 \left[\eta_2^{-2\nu} \left(\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^p \right)^{\frac{2}{p}} + \eta_2 \fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 \right] \leq \eta_3.$$

Again by Corollary 3, we would still have (24). In conclusion, we choose

$$\eta = \min \left\{ \eta_0^2, (1 - \eta_2)\eta_2^{2\nu}\eta_3 \right\}.$$

Then for any $0 < \delta < \infty$ one would have

$$|\nabla^n \omega(t,x)|^{\frac{4}{n+2}} \leq C_n^{\frac{4}{n+2}} \max \left\{ \frac{1}{\eta} \left[\delta^{-2\nu} \mathcal{M}_Q (\mathcal{M}(\nabla u)^p)^{\frac{2}{p}} + \delta \mathcal{M}_Q (\mathcal{M}(\nabla u)^2) \right], 81t^{-2} \right\},$$

by putting (24) and (23) together. Denote $f = |\nabla^n \omega|^{\frac{4}{n+2}}$, and we denote $f_1 = \mathcal{M}_Q(\mathcal{M}(|\nabla u|)^p)^{\frac{2}{p}}$, $f_2 = \mathcal{M}_Q(\mathcal{M}(|\nabla u|)^2)$. Then we have almost everywhere

$$f \mathbf{1}_{\{f > C_n t^{-2}\}} \lesssim_n \delta^{-2\nu} f_1 + \delta f_2.$$

By Theorem 3,

$$\|f_1\|_{L^1} \leq C_p \|\mathcal{M}(\nabla u)^p\|_{L^{\frac{2}{p}}}^{\frac{2}{p}} \lesssim C_p \|\mathcal{M}(\nabla u)^2\|_{L^1} \leq C_p \|\nabla u\|_{L^2}^2,$$

$$\|f_2\|_{L^{1,\infty}} \leq C_1 \|\mathcal{M}(\nabla u)^2\|_{L^1} \leq C_1 \|\nabla u\|_{L^2}^2.$$

Finally, by the interpolation between Lorentz spaces Lemma 1,

$$\|f\mathbf{1}_{\{f>C_n t^{-2}\}}\|_{L^{1,1+2\nu}} \lesssim_{p,n} \|\nabla u\|^2_{L^2((0,T)\times\mathbb{R}^3)} \le \|u_0\|^2_{L^2(\mathbb{R}^3)}.$$

This proves the theorem for $q \ge 1 + 2\nu$. Recall that p can be arbitrarily chosen between $\frac{11}{6}$ and 2, and ν can be chosen between $\frac{2-p}{p-1}$ and $\frac{7p-12}{6-p}$, so ν can be arbitrarily small, therefore we prove the theorem for any $q > 1$. $\square$

Estimates on $\nabla^2 u$ can be obtained by a Riesz transform of $\Delta u = -\operatorname{curl}\omega$.

Proof of Corollary 1. We can put $K \subset (t_0, T) \times B_R$ for some $t_0, T, R > 0$. Denote $Q = (t_0, T) \times B_{2R}$. Let $\rho \in C_c^\infty(\mathbb{R}^3)$ be a smooth spatial cut-off function between $\mathbf{1}_{B_R} \le \rho \le \mathbf{1}_{B_{2R}}$. Then

$$\|\Delta(\rho u)\|_{L^{\frac{4}{3},q}((t_0,T)\times\mathbb{R}^3)} \lesssim_\rho \|\Delta u\|_{L^{\frac{4}{3},q}(Q)} + \|\nabla u\|_{L^{\frac{4}{3},q}(Q)} + \|u\|_{L^{\frac{4}{3},q}(Q)}.$$

Since $\Delta u = -\operatorname{curl}\omega$, the case $n = 1$ of Theorem 1 gives

$$\|\Delta u \mathbf{1}_{\{|\Delta u|>C_1 t^{-\frac{3}{2}}\}}\|_{L^{\frac{4}{3},q}((0,T)\times\mathbb{R}^3)} \le C_q \|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)},$$

so

$$\|\Delta u\|_{L^{\frac{4}{3},q}(Q)} \le C_q \|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + C_1\|t^{-\frac{3}{2}}\|_{L^{\frac{4}{3}}(Q)} \lesssim C_q \|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + C_1\left(\frac{R^3}{t_0}\right)^{\frac{3}{4}}.$$

As for lower order terms,

$$\|\nabla u\|_{L^{\frac{4}{3}}(Q)} \lesssim \|\nabla u\|_{L^2(Q)},$$

$$\|u\|_{L^{\frac{4}{3}}(Q)} \le \|u\|_{L_t^\infty L_x^2(Q)}.$$

For Leray-Hopf solution, $\|\nabla u\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1((0,T)\times\mathbb{R}^3)} \le \|u_0\|_{L^2}$, so

$$\|\Delta(\rho u)\|_{L^{\frac{4}{3},q}((t_0,T)\times\mathbb{R}^3)} \lesssim_{q,K} \|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + 1 + \|u_0\|_{L^2(\mathbb{R}^3)} \lesssim \|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + 1.$$

Because Riesz transform is bounded from $L^{\frac{4}{3},q}((t_0, T) \times \mathbb{R}^3)$ to itself by Lemma 2,

$$\|\nabla^2 u\|_{L^{\frac{4}{3},q}(K)} \le \|\nabla^2(\rho u)\|_{L^{\frac{4}{3},q}(Q)} \lesssim_{q,K} \|u_0\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + 1.$$

$\square$

Remark 1. For smooth solutions to the Navier–Stokes equation, we have $L^{1,q}$ estimate for the third derivatives for any $q > 1$,

$$\left\|\nabla^2\omega \mathbf{1}_{\{|\nabla^2\omega|>Ct^{-2}\}}\right\|_{L^{1,q}((0,T)\times\mathbb{R}^3)} \le C_q \|u_0\|^2_{L^2}.$$

4. Local Study: Part One, Initial Energy

The next three sections are dedicated to the proof of the Local Theorem 2. In [22], the proof of the local theorem consists of the following three parts:

Step 1. Show the velocity u is locally small in the energy space $\mathcal{E} = L_t^\infty L_x^2 \cap L_t^2 H_x^1$.

Step 2. Use De Giorgi iteration and the truncation method developed in [23] to show u is locally bounded in L^∞.

Step 3. Bootstrap to higher regularity by differentiating the original equation.

In our case, directly working with u is difficult due to the lack of control on the pressure, which is nonlocal. Therefore, we would like to work on vorticity, whose evolution is governed by (6) and only involves local quantities. Since ω is one derivative of u, we have less integrability to do any parabolic regularization, and we don't have the local energy inequality to perform De Giorgi iteration. This motivates us to work on minus one derivative of ω, but instead of ω we use a localization of ω. Similar as [4], we introduce a new local quantity

$$\boxed{v := -\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi \operatorname{curl} u = -\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi\omega.}$$

where φ and $\varphi^\sharp$ are a pair of fixed smooth spatial cut-off functions, which are defined between $\mathbf{1}_{B_{\frac{6}{5}}} \le \varphi \le \mathbf{1}_{B_{\frac{5}{4}}}$, $\mathbf{1}_{B_{\frac{4}{3}}} \le \varphi^\sharp \le \mathbf{1}_{B_{\frac{3}{2}}}$. This v is divergence free and compactly supported. It will help us get rid of the pressure P, while staying in the same space as u: it scales the same as u, has the same regularity, inherit a local energy inequality from u, and its evolution only depends on local information. We will follow the same three steps above, but we will work on v instead of u.

For convenience, from now on we will use η to denote a small universal constant depending only on the smallness of η_1, such that $\lim_{\eta_1 \to 0} \eta = 0$. Similar as the constant C, the value of η may change from line to line. The purpose of this section is to obtain the smallness of v in the energy space $\mathcal{E}$, which is the following proposition:

Proposition 1. *Under the same assumptions of the Local Theorem 2, we have*

$$\|v\|_{\mathcal{E}(Q_1)}^2 = \sup_{t\in(-1,0)} \int_{B_1} |v(t)|^2 \mathrm{d}x + \int_{Q_1} |\nabla v|^2 \mathrm{d}x \le \eta. \tag{25}$$

For convenience, define q_3, q_4, q_5 by

$$\frac{1}{q_3} = \frac{1}{q_1} - \frac{1}{3}, \qquad \frac{1}{q_4} = \frac{1}{q_2} - \frac{1}{3}, \qquad \frac{1}{q_5} = \left(\frac{1}{q_3} - \frac{1}{3}\right)_+.$$

4.1. Equations of v

We use (10) in (1) to rewrite the equation of u, then take the curl to rewrite the equation of ω, finally apply $-\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi$ on the vorticity equation to obtain the equation of v:

$$\partial_t u + \mathbb{P}_{\mathrm{curl}}(\omega \times u) = \Delta u,$$

$$\partial_t \omega + \operatorname{curl}(\omega \times u) = \Delta \omega,$$

$$\partial_t v - \operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi \operatorname{curl}(\omega \times u) = -\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi \Delta \omega. \tag{26}$$

The second term of (26) is

$$\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi \operatorname{curl}(\omega \times u) = \mathbf{B} - \mathbb{P}_{\mathrm{curl}}(\varphi\omega \times u),$$

where $\mathbf{B}$ denotes the quadratic commutator

$$\mathbf{B} : = -\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi \operatorname{curl}(\omega \times u) + \operatorname{curl}\Delta^{-1}[\varphi, \operatorname{curl}](\omega \times u)$$

$$= -\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi \operatorname{curl}(\omega \times u) + \operatorname{curl}\Delta^{-1}(-\nabla\varphi \times (\omega \times u)).$$

Here we used (12). The right hand side of (26) is

$$-\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi \Delta \omega = \Delta v + \mathbf{L},$$

where $\mathbf{L}$ denotes the linear commutator

$$\mathbf{L} : = [-\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi, \Delta]\omega$$

$$= -\operatorname{curl}[\varphi^\sharp \Delta^{-1}\varphi, \Delta]\omega$$

$$= -\operatorname{curl}[\varphi^\sharp, \Delta]\Delta^{-1}\varphi\omega - \operatorname{curl}\varphi^\sharp \Delta^{-1}[\varphi, \Delta]\omega$$

$$= -\operatorname{curl}[\varphi^\sharp, \Delta]\Delta^{-1}\varphi\omega + \operatorname{curl}\varphi^\sharp \Delta^{-1}\left(2\operatorname{div}(\nabla\varphi \otimes \omega) - (\Delta\varphi)\omega\right).$$

Here we used (13). Therefore we have the equation for v as the following:

$$\partial_t v + \mathbb{P}_{\mathrm{curl}}(\varphi\omega \times u) = \mathbf{B} + \mathbf{L} + \Delta v. \tag{27}$$

We observe the following localization decomposition:

Lemma 6. *We can decompose*

$$\varphi u = v + w, \qquad \varphi\omega = \operatorname{curl} v + \varpi,$$

where w and ϖ are harmonic inside B_1.

Proof. We can compute v by

$$v = -\operatorname{curl}\varphi^\sharp \Delta^{-1}\varphi \operatorname{curl} u$$

$$= \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi \operatorname{curl} u - \operatorname{curl}\Delta^{-1}\varphi \operatorname{curl} u$$

$$= \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega - \operatorname{curl}\Delta^{-1}[\varphi, \operatorname{curl}]u + \mathbb{P}_{\mathrm{curl}}(\varphi u)$$

$$= \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega + \operatorname{curl}\Delta^{-1}(\nabla\varphi \times u) - \mathbb{P}_\nabla(\varphi u) + \varphi u$$

$$= \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega + \operatorname{curl}\Delta^{-1}(\nabla\varphi \times u) - \nabla\Delta^{-1}(\nabla\varphi \cdot u) + \varphi u,$$

using div $u = 0$. We denote

$$w := -\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega - \operatorname{curl}\Delta^{-1}(\nabla\varphi \times u) + \nabla\Delta^{-1}(\nabla\varphi \cdot u),$$

which implies the first decomposition $\varphi u = v + w$. By taking the curl,

$$\operatorname{curl}(\varphi u) = \operatorname{curl} v + \operatorname{curl} w,$$

$$\nabla\varphi \times u + \varphi\omega = \operatorname{curl} v - \operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega - \operatorname{curl}\operatorname{curl}\Delta^{-1}(\nabla\varphi \times u)$$

$$= \operatorname{curl} v - \operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega + \mathbb{P}_{\operatorname{curl}}(\nabla\varphi \times u).$$

We denote

$$\varpi := -\operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega - \mathbb{P}_{\nabla}(\nabla\varphi \times u)$$

$$= -\operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega - \nabla\Delta^{-1}\operatorname{div}(\nabla\varphi \times u)$$

$$= -\operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega + \nabla\Delta^{-1}(\nabla\varphi \cdot \omega),$$

which implies the second decomposition $\varphi\omega = \operatorname{curl} v + \varpi$. We can easily see that Δw and $\Delta\varpi$ are both the sum of a smooth function supported outside $B_{\frac{3}{2}}$ and the Newtonian potential of something supported inside $\operatorname{supp}(\nabla\varphi) \subset B_{\frac{5}{4}} \setminus B_{\frac{6}{5}}$, so they are harmonic inside B_1. $\square$

Using this decomposition, we can continue to expand

$$\mathbb{P}_{\operatorname{curl}}(\varphi\omega \times u) = \varphi\omega \times u - \mathbb{P}_{\nabla}(\varphi\omega \times u)$$

$$= \omega \times v + \omega \times w - \frac{1}{2}\mathbb{P}_{\nabla}\left((\operatorname{curl} v + \varpi) \times u + \omega \times (v + w)\right)$$

$$= \omega \times v - \frac{1}{2}\mathbb{P}_{\nabla}(\operatorname{curl} v \times u + \omega \times v) - \mathbf{W},$$

where $\mathbf{W}$ denotes the remainders involving w and ϖ,

$$\mathbf{W} := -\omega \times w + \frac{1}{2}\mathbb{P}_{\nabla}(\varpi \times u + \omega \times w).$$

By subtracting (11) from (10), for divergence free u, v we have

$$\operatorname{curl} v \times u + \operatorname{curl} u \times v = -\nabla(u \cdot v) + 2u \cdot \nabla v + \operatorname{curl}(u \times v),$$

so

$$\mathbb{P}_{\operatorname{curl}}(\varphi\omega \times u) = \omega \times v + \frac{1}{2}\nabla(u \cdot v) - \mathbb{P}_{\nabla}\operatorname{div}(u \otimes v) - \mathbf{W}$$

$$= \omega \times v + \nabla\left(\frac{1}{2}u \cdot v - \Delta^{-1}\operatorname{div}\operatorname{div}(u \otimes v)\right) - \mathbf{W}.$$

For convenience, denote the Riesz operator

$$\mathbf{R} = \frac{1}{2}\operatorname{tr} - \Delta^{-1}\operatorname{div}\operatorname{div}$$

Finally, we have the equation of v as

$$\partial_t v + \omega \times v + \nabla\mathbf{R}(u \otimes v) = \mathbf{B} + \mathbf{L} + \mathbf{W} + \Delta v, \qquad \operatorname{div} v = 0. \qquad (28)$$

We now check the spatial integrability of these new terms.

Lemma 7. *For any* $1 < p < \infty$,

$$\|v\|_{L^p}, \|\nabla w\|_{L^p}, \|\varpi\|_{L^p} \lesssim \|\omega\|_{L^1(B_2)} + \|u\|_{L^p(B_2)},$$
$$\|\nabla v\|_{L^p}, \|\nabla \varpi\|_{L^p} \lesssim \|\omega\|_{L^p(B_2)},$$
$$\|\nabla^2 w\|_{L^p} \lesssim \|u\|_{W^{1,p}}(B_2).$$

If we denote $q = (\frac{1}{p} - \frac{1}{3})_+^{-1}$, *then*

$$\|\mathbf{B}\|_{L^q(B_2)} \lesssim \|\omega \times u\|_{L^p(B_2)},$$
$$\|\mathbf{L}\|_{L^p(B_2)} \lesssim \|\omega\|_{L^p(B_2)},$$
$$\|\mathbf{W}\|_{L^p(B_2)} \lesssim \|\omega \times w\|_{L^p(B_2)} + \|\varpi \times u\|_{L^p(B_2)}.$$

Proof. v, w, ϖ are all supported inside B_2, so

$$\|v\|_{L^p} \leq \|\varphi u\|_{L^p} + \|w\|_{L^p} \lesssim \|u\|_{L^p(B_2)} + \|\nabla w\|_{L^p},$$
$$\|\nabla w\|_{L^p} \leq \|\nabla \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla \operatorname{curl}\Delta^{-1}(\nabla\varphi \times u)\|_{L^p}$$
$$+ \|\nabla^2 \Delta^{-1}(\nabla\varphi \cdot u)\|_{L^p}$$
$$\lesssim \|(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{C^2} + \|\nabla\varphi \times u\|_{L^p} + \|\nabla\varphi \cdot u\|_{L^p}$$
$$\lesssim \|\omega\|_{L^1(B_2)} + \|u\|_{L^p(B_2)},$$
$$\|\varpi\|_{L^p} \leq \|\operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla\Delta^{-1}(\nabla\varphi \cdot \omega)\|_{L^p}$$
$$\leq \|(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{C^2} + \|\nabla\Delta^{-1}\operatorname{div}(\nabla\varphi \times u)\|_{L^p}$$
$$\leq \|\omega\|_{L^1(B_2)} + \|\nabla\varphi \times u\|_{L^p}$$
$$\leq \|\omega\|_{L^1(B_2)} + \|u\|_{L^p(B_2)}.$$

Here we used Lemma 4 since φ and $1 - \varphi^\sharp$ are supported away from each other, and we also used the boundedness of Riesz transform by Lemma 2. Their derivatives are bounded by

$$\|\nabla v\|_{L^p} = \|\nabla \operatorname{curl}\varphi^\sharp\Delta^{-1}\varphi\omega\|_{L^p}$$
$$\leq \|\nabla \operatorname{curl}\Delta^{-1}\varphi\omega\|_{L^p} + \|\nabla \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)}$$
$$\lesssim \|\omega\|_{L^p(B_2)} + \|\omega\|_{L^1(B_2)} \lesssim \|\omega\|_{L^p(B_2)},$$
$$\|\nabla^2 w\|_{L^p} \leq \|\nabla^2 \operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla^2 \operatorname{curl}\Delta^{-1}(\nabla\varphi \times u)\|_{L^p}$$
$$+ \|\nabla^3 \Delta^{-1}(\nabla\varphi \cdot u)\|_{L^p}$$
$$\lesssim \|\omega\|_{L^1(B_2)} + \|u\|_{W^{1,p}(B_2)} \lesssim \|u\|_{W^{1,p}(B_2)},$$
$$\|\nabla\varpi\|_{L^p} \leq \|\nabla \operatorname{curl}\operatorname{curl}(1 - \varphi^\sharp)\Delta^{-1}\varphi\omega\|_{L^p(B_2)} + \|\nabla^2\Delta^{-1}(\nabla\varphi \cdot \omega)\|_{L^p}$$
$$\lesssim \|\omega\|_{L^1(B_2)} + \|\omega\|_{L^p(B_2)} \lesssim \|\omega\|_{L^p(B_2)}.$$

The proof for $\mathbf{B}$, $\mathbf{L}$, $\mathbf{W}$ are similar, so we omit it here. $\square$

Since $u \in \mathcal{E}$ and $\omega \in L_t^\infty L_x^1$, it can be seen from the above lemma that $v, \nabla w, \varpi \in \mathcal{E}$, thus

$$\|\mathbf{B}\|_{L^3(B_2)} \lesssim \|\omega \times u\|_{L^{\frac{3}{2}}(B_2)} \in L_t^1,$$

$$\|\mathbf{L}\|_{L^2(B_2)} \lesssim \|\omega\|_{L^2(B_2)} \in L_t^2,$$

$$\|\mathbf{W}\|_{L^{\frac{3}{2}}(B_2)} \lesssim \|\omega \times w\|_{L^{\frac{3}{2}}(B_2)} + \|\varpi \times u\|_{L^{\frac{3}{2}}(B_2)} \in L_t^2,$$

therefore $\mathbf{B}, \mathbf{L}, \mathbf{W} \in L_t^1 L_{\mathrm{loc},x}^3 + L_x^2 L_{\mathrm{loc},x}^{\frac{3}{2}}$. In the appendix we prove the suitability for v: it satisfies the following local energy inequality:

$$\partial_t \frac{|v|^2}{2} + |\nabla v|^2 + \mathrm{div}\,[v\mathbf{R}(u \otimes v)] \le \Delta \frac{|v|^2}{2} + v \cdot (\mathbf{B} + \mathbf{L} + \mathbf{W}). \quad (29)$$

4.2. Energy Estimate

Multipling (29) by φ^4 then integrating over $\mathbb{R}^3$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x + \int \varphi^4 |\nabla v|^2 \mathrm{d}x$$

$$\le \int \frac{|v|^2}{2} \Delta \varphi^4 \mathrm{d}x + \int (v \cdot \nabla \varphi^4) \mathbf{R}(u \otimes v) \mathrm{d}x$$

$$+ \int \varphi^4 v \cdot \mathbf{B} \mathrm{d}x + \int \varphi^4 v \cdot \mathbf{L} \mathrm{d}x + \int \varphi^4 v \cdot \mathbf{W} \mathrm{d}x.$$

Let us discuss these terms. For the first four terms on the right hand side,

$$I_\Delta := \int \frac{|v|^2}{2} \Delta \varphi^4 \mathrm{d}x \le C \|\varphi^2 v\|_{L^2} \|v\|_{L^2}, \quad (30)$$

$$I_{\mathbf{R}} := \int (v \cdot \nabla \varphi^4) \mathbf{R}(u \otimes v) \mathrm{d}x \le C \|\varphi^2 v\|_{L^2} \|\mathbf{R}(u \otimes v)\|_{L^2}$$

$$\le C \|\varphi^2 v\|_{L^2} \|u \otimes v\|_{L^2}, \quad (31)$$

$$I_{\mathbf{B}} := \int \varphi^4 v \cdot \mathbf{B} \mathrm{d}x \le \|\varphi^2 v\|_{L^2} \|\varphi^2 \mathbf{B}\|_{L^2}$$

$$\le C \|\varphi^2 v\|_{L^2} \|\omega \times u\|_{L^{\frac{6}{5}}(B_2)}, \quad (32)$$

$$I_{\mathbf{L}} := \int \varphi^4 v \cdot \mathbf{L} \mathrm{d}x \le \|\varphi^{\frac{2}{3}} |v|^{\frac{1}{3}}\|_{L^6} \||v|^{\frac{2}{3}}\|_{L^{q_3}} \|\varphi^2 \mathbf{L}\|_{L^{q_2}}$$

$$\le \|\varphi^2 v\|_{L^2}^{\frac{1}{3}} \|v\|_{L^{q_3}}^{\frac{2}{3}} \|\omega\|_{L^{q_2}(B_2)}. \quad (33)$$

Here we use Hölder's inequality, and φ is compactly supported in B_2 and $\frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{6} \leq 1$. For the **W** term,

$$
\begin{aligned}
I_{\mathbf{W}} : &= \int \varphi^4 v \cdot \mathbf{W} \mathrm{d}x \\
&= -\int \varphi^4 v \cdot \omega \times w \mathrm{d}x + \frac{1}{2} \int \varphi^4 v \cdot \mathbb{P}_\nabla (\varpi \times u + \omega \times w) \, \mathrm{d}x \\
&= -I_{\mathbf{W}1} + \frac{1}{2} I_{\mathbf{W}2}.
\end{aligned}
$$

For the first one, we break it as

$$
I_{\mathbf{W}1} = \int \varphi^4 v \cdot \omega \times w \mathrm{d}x = \int \varphi^3 v \times \mathrm{curl}\, v \cdot w \mathrm{d}x + \int \varphi^3 v \cdot \varpi \times w \mathrm{d}x.
$$

Using (10),

$$
v \times \mathrm{curl}\, v = \frac{1}{2} \nabla |v|^2 - (v \cdot \nabla) v,
$$

we have

$$
\begin{aligned}
\int \varphi^3 v \times \mathrm{curl}\, v \cdot w \mathrm{d}x &= -\frac{1}{2} \int |v|^2 \, \mathrm{div}(\varphi^3 w) \mathrm{d}x + \int v \cdot \nabla(\varphi^3 w) \cdot v \mathrm{d}x \\
&\leq C \|\varphi^2 v\|_{L^2} \left(\|\nabla w \otimes v\|_{L^2} + \|w \otimes v\|_{L^2} \right).
\end{aligned}
$$

The remaining is of lower order:

$$
\int \varphi^3 v \cdot \varpi \times w \mathrm{d}x \leq C \|\varphi^2 v\|_{L^2} \|\varpi \times w\|_{L^2}.
$$

For the second one,

$$
\begin{aligned}
I_{\mathbf{W}2} &= \int \mathbb{P}_\nabla(\varphi^4 v) \cdot (\varpi \times u + \omega \times w) \, \mathrm{d}x \\
&\leq \|\mathbb{P}_\nabla(\varphi^4 v)\|_{L^6} \|\varpi \times u + \omega \times w\|_{L^{\frac{6}{5}}},
\end{aligned}
$$

where

$$
\|\mathbb{P}_\nabla(\varphi^4 v)\|_{L^6} = \|\nabla \Delta^{-1} \mathrm{div}(\varphi^4 v)\|_{L^6} = \|\nabla \Delta^{-1}(v \cdot \nabla \varphi^4)\|_{L^6} \leq C \|\varphi^2 v\|_{L^2}.
$$

Thus $I_{\mathbf{W}}$ can be bounded by

$$
I_{\mathbf{W}} \leq C \|\varphi^2 v\|_{L^2} \left(\|\nabla w \otimes v\|_{L^2} + \|\varpi \times w\|_{L^2} + \|\varpi \times u + \omega \times w\|_{L^{\frac{6}{5}}} \right). \tag{34}
$$

In summary, we conclude that, for $-4 \leq t \leq 0$,

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x + \int \varphi^4 |\nabla v|^2 \mathrm{d}x \leq I_\Delta + I_{\mathbf{R}} + I_{\mathbf{B}} + I_{\mathbf{L}} + I_{\mathbf{W}}, \tag{35}
$$

with good estimates on each of the term on the right.

4.3. Proof of Proposition 1

First we check the integrability of each terms.

Lemma 8. (Integrability) *Given conditions (4) and (5), we have*

$$\|u\|_{L_t^{p_1} L_x^{q_3}(Q_2)} \lesssim \eta,$$

$$\|\varphi\omega\|_{L_t^{p_1} L_x^{q_1}((-4,0)\times\mathbb{R}^3)} \lesssim \eta, \qquad \|\varphi\omega\|_{L_t^{p_2} L_x^{q_2}((-4,0)\times\mathbb{R}^3)} \lesssim \eta,$$

$$\|\nabla v\|_{L_t^{p_1} L_x^{q_1}((-4,0)\times\mathbb{R}^3)} \lesssim \eta, \qquad \|\nabla v\|_{L_t^{p_2} L_x^{q_2}((-4,0)\times\mathbb{R}^3)} \lesssim \eta,$$

$$\|v\|_{L_t^{p_1} L_x^{q_3}((-4,0)\times\mathbb{R}^3)} \lesssim \eta, \qquad \|v\|_{L_t^{p_2} L_x^{q_4}((-4,0)\times\mathbb{R}^3)} \lesssim \eta,$$

$$\|\nabla w\|_{L_t^{p_1} L_x^{q_3}((-4,0)\times\mathbb{R}^3)} \lesssim \eta,$$

$$\|w\|_{L_t^{p_1} L_x^{q_5}((-4,0)\times\mathbb{R}^3)} \lesssim \eta, \tag{36}$$

$$\|\varpi\|_{L_t^{p_1} L_x^{q_3}((-4,0)\times\mathbb{R}^3)} \lesssim \eta, \qquad \|\varpi\|_{L_t^{p_2} L_x^{q_4}((-4,0)\times\mathbb{R}^3)} \lesssim \eta. \tag{37}$$

Proof. Integrability of u is obtained by Sobolev embedding and that φu has average 0. Integrability of $\varphi\omega$ is given. The remaining are consequences of Lemma 7 and Sobolev embedding. □

Proof of Proposition 1. We prove Proposition 1 using a Grönwall argument. Multiply (35) by an increasing smooth function $\psi_1(t)$ with $\psi_1(t) = 0$ for $t \leq -2$, $\psi_1(t) = 1$ for $t \geq -1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\psi_1(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x \right) + \psi_1(t) \int \varphi^4 |\nabla v|^2 \mathrm{d}x$$

$$= \psi_1'(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x + \psi_1(t) \left(I_\Delta + I_\mathbf{R} + I_\mathbf{B} + I_\mathbf{L} + I_\mathbf{W} \right).$$

Formally, we can integrate from -4 to $t < 0$ and to get

$$\psi_1(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x + \int_{-2}^t \psi_1(s) \int \varphi^4 |\nabla v|^2 \mathrm{d}x$$

$$= \int_{-2}^t \psi_1'(s) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x \mathrm{d}t + \int_{-2}^t \psi_1(s) \left(I_{\Delta,\mathbf{R},\mathbf{B},\mathbf{L},\mathbf{W}} \right) \mathrm{d}t.$$

This integration is justified since v satisfies the local energy inequality (29) in distribution, and $\psi_1(t)\varphi^4(x) \in C_c^\infty((-4, 0] \times B_2)$. Because of (30), (31), (32), (33), (34), and

$$\|\varphi^2 v\|_{L^2(B_2)}, \|\varphi^2 v\|_{L^2(B_2)}^{\frac{1}{3}} \leq C \left(1 + \int \varphi^4 |v|^2 \mathrm{d}x \right),$$

we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\psi_1(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x \right) + \psi_1(t) \int \varphi^4 |\nabla v|^2 \mathrm{d}x$$

$$\leq C\Phi(t) \left(1 + \psi_1(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x \right),$$

where

$$\Phi(t) = \psi_1'(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x$$
$$+ \|v\|_{L^2} + \|u \otimes v\|_{L^2} + \|\omega \times u\|_{L^{\frac{6}{5}}(B_2)}$$
$$+ \|v\|_{L_x^{q_3}}^{\frac{2}{3}} \|\omega\|_{L_x^{q_2}(B_2)} + \|\nabla w \otimes v\|_{L^2}$$
$$+ \|\varpi \times w\|_{L^2} + \|\varpi \times u + \omega \times w\|_{L^{\frac{6}{5}}}$$
$$\leq \|v\|_{L^2}^2 + \|v\|_{L^2} + \|v\|_{L_x^{q_4}} \|u\|_{L_x^{q_3}(B_2)} + \|\omega\|_{L_x^{q_2}(B_2)} \|u\|_{L_x^{q_3}(B_2)}$$
$$+ \|v\|_{L_x^{q_3}}^{\frac{2}{3}} \|\omega\|_{L_x^{q_2}(B_2)} + \|\nabla w\|_{L_x^{q_3}} \|v\|_{L_x^{q_4}}$$
$$+ \|\varpi\|_{L_x^{q_4}} \|w\|_{L_x^{q_3}}$$
$$+ \|\varpi\|_{L_x^{q_4}} \|u\|_{L_x^{q_3}} + \|\omega\|_{L_x^{q_2}} \|w\|_{L_x^{q_3}}$$
$$\leq \left(\|v\|_{L_x^{q_3}} + \|v\|_{L_x^{q_3}}^{\frac{1}{2}} + \|u\|_{L_x^{q_3}(B_2)} + \|\nabla w\|_{L_x^{q_3}} + \|w\|_{L_x^{q_3}} \right)$$
$$\times \left(\|v\|_{L_x^{q_4}} + \|v\|_{L_x^{q_4}}^{\frac{1}{2}} + \|\omega\|_{L_x^{q_2}} + \|\varpi\|_{L_x^{q_4}} \right)$$

Here we used interpolation for $\|v\|_{L^2}^2 \leq \|v\|_{L_x^{q_3}} \|v\|_{L_x^{q_4}}$. Therefore

$$\|\Phi\|_{L_t^1} \lesssim \left\| \left(\|v\|_{L_x^{q_3}} + \|v\|_{L_x^{q_3}}^{\frac{1}{2}} + \|u\|_{L_x^{q_3}(B_2)} + \|\nabla w\|_{L_x^{q_3}} + \|w\|_{L_x^{q_3}} \right) \right\|_{L_t^{p_1}}$$
$$\times \left\| \left(\|v\|_{L_x^{q_4}} + \|v\|_{L_x^{q_4}}^{\frac{1}{2}} + \|\omega\|_{L_x^{q_2}} + \|\varpi\|_{L_x^{q_4}} \right) \right\|_{L_t^{p_2}} \leq \eta.$$

By Grönwall's lemma, we conclude that for every $-4 \leq t \leq 0$,

$$1 + \psi_1(t) \int \varphi^4 \frac{|v|^2}{2} \mathrm{d}x + \int_{-4}^{t} \psi_1(t) \int \varphi^4 |\nabla v|^2 \mathrm{d}x \leq e^{\int_{-4}^{t} C\Phi(s)\mathrm{d}s} \leq e^{C\eta}.$$

Therefore by taking the sup over $-1 \leq t \leq 0$ and $t = 0$ respectively, we conclude that

$$\sup_{-1 \leq t \leq 0} \int |v(t)|^2 \mathrm{d}x \leq \eta, \qquad \int_{Q_1} |\nabla v|^2 \mathrm{d}x \mathrm{d}t \leq \eta.$$

$\square$

5. Local Study: Part Two, De Giorgi Iteration

In this section, we derive the boundedness of v in $Q_{\frac{1}{2}}$ which is the following.

Proposition 2. *Let v solves (28). If (25) holds for sufficiently small η, and we have integrability bounds in Lemma 8, then we have*

$$\|v\|_{L^\infty(Q_{\frac{1}{2}})} = \sup_{t \in (-1,0)} \|v(t)\|_{L^\infty(B_{\frac{1}{2}})} \leq 1.$$

The proof uses De Giorgi technique and the truncation method. First, we set a dyadically shrinking radius

$$r_k^\flat = \frac{1}{2}(1 + 8^{-k}), \qquad r_k^\natural = \frac{1}{2}(1 + 2 \times 8^{-k}), \qquad r_k^\sharp = \frac{1}{2}(1 + 4 \times 8^{-k}).$$

Then we define dyadically shrinking cylinder Q_k's

$$
\begin{array}{lll}
T_k^\flat = r_k^{\flat 2}, & B_k^\flat = B_{r_k^\flat}(0), & Q_k^\flat = (-T_k^\flat, 0) \times B_k^\flat, \\[4pt]
T_k^\natural = r_k^{\natural 2}, & B_k^\natural = B_{r_k^\natural}(0), & Q_k^\natural = (-T_k^\natural, 0) \times B_k^\natural, \\[4pt]
T_k^\sharp = r_k^{\sharp 2}, & B_k^\sharp = B_{r_k^\sharp}(0), & Q_k^\sharp = (-T_k^\sharp, 0) \times B_k^\sharp.
\end{array}
$$

We also introduce positive smooth space-time cut-off functions ρ_k and $\rho_k^\sharp$ with

$$\mathbf{1}_{Q_k^\flat} \leq \rho_k \leq \mathbf{1}_{Q_k^\natural}, \qquad \mathbf{1}_{Q_k^\sharp} \leq \rho_k^\sharp \leq \mathbf{1}_{Q_{k-1}^\flat}.$$

Then, let c_k denote a sequence of rising energy level

$$
\begin{array}{lll}
c_k = 1 - 2^{-k}, & v_k = (|v| - c_k)_+, & \beta_k = \dfrac{v_k}{|v|}, \\[8pt]
\Omega_k = \{v_k > 0\}, & \mathbf{1}_k = \mathbf{1}_{\Omega_k}, & \alpha_k = 1 - \beta_k.
\end{array}
$$

We define analogous of vector derivative d_k and energy quantity U_k to get

$$d_k^2 = \mathbf{1}_k \left(\alpha_k |\nabla|v||^2 + \beta_k |\nabla v|^2 \right),$$
$$U_k = \|v_k\|_{L^\infty(-T_k^\flat, 0; L^2(B_k^\flat))}^2 + \|d_k\|_{L^2(Q_k^\flat)}^2.$$

We have the following truncation estimates:

Lemma 9.

$$\alpha_k v \leq c_k \leq 1,$$
$$\|\beta_k v\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{k-1}^\flat)}^2 \leq 9 U_{k-1},$$
$$\|\mathbf{1}_k\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^6(Q_{k-1}^\flat)}^2 \leq C^k U_{k-1}.$$

Proof. The first estimate follows from the definition. By Lemma 4 in [23], we have $|\nabla v_k| \leq d_k$ and $|\nabla(\beta_k v)| \leq 3d_k$. Moreover, since $|\nabla|v|| \leq |\nabla v|^2$, we see $d_k \leq d_{k-1}$, as v_k and β_k are monotonously decreasing, so

$$\|\nabla(\beta_k v)\|_{L^2(Q^\flat_{k-1})} \leq 3\|d_k\|_{L^2(Q^\flat_{k-1})} \leq 3\|d_{k-1}\|_{L^2(Q^\flat_{k-1})}.$$

Moreover, the truncation gives $|\beta_k v| + 2^{-k}\mathbf{1}_k = v_k + 2^{-k}\mathbf{1}_k = \mathbf{1}_k v_{k-1}$, so

$$\|\beta_k v\|_{L^\infty_t L^2_x(Q^\flat_{k-1})} \leq \|v_{k-1}\|_{L^\infty_t L^2_x(Q^\flat_{k-1})},$$

$$2^{-k}\|\mathbf{1}_k\|_{L^\infty_t L^2_x(Q^\flat_{k-1})} \leq \|v_{k-1}\|_{L^\infty_t L^2_x(Q^\flat_{k-1})},$$

$$2^{-k}\|\mathbf{1}_k\|_{L^2_t L^6_x(Q^\flat_{k-1})} \leq \|v_{k-1}\|_{L^2_t L^6_x(Q^\flat_{k-1})}$$

$$\leq \|v_{k-1}\|_{L^\infty_t L^2_x(Q^\flat_{k-1})} + \|\nabla v_{k-1}\|_{L^2(Q^\flat_{k-1})}$$

$$\leq \|v_{k-1}\|_{L^\infty_t L^2_x(Q^\flat_{k-1})} + \|d_{k-1}\|_{L^2(Q^\flat_{k-1})}.$$

$\square$

Corollary 4. (Nonlinearization) *If* $f \in L^p_t L^q_x(Q_{k-1})$, *with*

$$\frac{1}{p} + \gamma\left(\frac{\theta}{2} + \frac{1-\theta}{\infty}\right) = 1, \quad \frac{1}{q} + \gamma\left(\frac{\theta}{6} + \frac{1-\theta}{2}\right) = 1$$

for some $0 \leq \theta \leq 1$, $0 < \sigma \leq \gamma$, *then, uniformly in* σ,

$$\int_{Q^\flat_{k-1}} |\beta_k v|^\sigma |f| \mathrm{d}x\mathrm{d}t \leq C^k \|f\|_{L^p_t L^q_x(Q_{k-1})} U_{k-1}^{\frac{\gamma}{2}}.$$

Proof. By interpolation,

$$\|\beta_k v\|, \|\mathbf{1}_k\|_{L^{p_\theta}_t L^{q_\theta}_x(Q_{k-1})} \leq U_{k-1}^{\frac{1}{2}},$$

where

$$\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{\infty}, \quad \frac{1}{q_\theta} = \frac{\theta}{6} + \frac{1-\theta}{2}.$$

Therefore, using Hölder's inequality,

$$\int_{Q_{k-1}} |\beta_k v|^\sigma |f| \mathrm{d}x\mathrm{d}t \leq \|f\|_{L^p_t L^q_x} \|\beta_k v\|^\sigma_{L^{p_\theta}_t L^{q_\theta}_x} \|\mathbf{1}_k\|^{\gamma-\sigma}_{L^{p_\theta}_t L^{q_\theta}_x} \leq \|f\|_{L^p_t L^q_x} U_{k-1}^{\frac{\gamma}{2}}.$$

$\square$

First, we recall the following identities from [23]:

$$\alpha_k v \cdot \partial_\bullet v = \partial_\bullet\left(\frac{|v|^2 - v_k^2}{2}\right), \tag{38}$$

$$\alpha_k v \cdot \Delta v = \Delta\left(\frac{|v|^2 - v_k^2}{2}\right) + d_k^2 - |\nabla v|^2. \tag{39}$$

Since $\alpha_k v$ is bounded, we can multiply equation (28) by $\alpha_k v$ and obtain

$$
\partial_t \left(\frac{|v|^2 - v_k^2}{2} \right) + \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v)
$$

$$
= \Delta \left(\frac{|v|^2 - v_k^2}{2} \right) + d_k^2 - |\nabla v|^2 + \alpha_k v \cdot (\mathbf{B} + \mathbf{L} + \mathbf{W}), \qquad (40)
$$

using (38) and (39). Denote $\mathbf{C}_v = \mathbf{B} + \mathbf{L} + \mathbf{W}$. Subtracting (40) from (29), we have

$$
\partial_t \frac{v_k^2}{2} + d_k^2 + \operatorname{div}(v\mathbf{R}(u \otimes v)) - \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \le \Delta \frac{v_k^2}{2} + \beta_k v \cdot \mathbf{C}_v.
$$

Multipling by ρ_k, then integrating in space and from σ to τ in time,

$$
\left[\int \rho_k \frac{v_k^2}{2} \mathrm{d}x \right]_\sigma^\tau + \int_\sigma^\tau \int \rho_k d_k^2 \mathrm{d}x\mathrm{d}t
$$

$$
\le \int_\sigma^\tau \int (\partial_t \rho_k + \Delta \rho_k) \frac{v_k^2}{2} \mathrm{d}x\mathrm{d}t - \int_\sigma^\tau \int \rho_k \operatorname{div}(v\mathbf{R}(u \otimes v)) \mathrm{d}x\mathrm{d}t
$$

$$
+ \int_\sigma^\tau \int \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \mathrm{d}x\mathrm{d}t + \int_\sigma^\tau \int \rho_k \beta_k v \cdot \mathbf{C}_v \mathrm{d}x\mathrm{d}t.
$$

Taking the sup over $\tau > -T_k^\flat$, and set $\sigma < -T_{k-1}^\flat$, we obtain

$$
U_k \le \sup_{\tau \in (-T_k^\flat, 0)} \int \rho_k \frac{v_k^2}{2} \mathrm{d}x + \int_{-T_{k-1}^\flat}^0 \int \rho_k d_k^2 \mathrm{d}x\mathrm{d}t
$$

$$
\le C^k \int_{Q_k^\natural} v_k^2 \mathrm{d}x\mathrm{d}t + \sup_{\tau \in (-T_k^\flat, 0)} \left\{ \int_{-T_k^\natural}^\tau \int \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \mathrm{d}x\mathrm{d}t \right.
$$

$$
- \int_{-T_k^\natural}^\tau \int \rho_k \operatorname{div}(v\mathbf{R}(u \otimes v)) \mathrm{d}x\mathrm{d}t
$$

$$
\left. + \int_{-T_k^\natural}^\tau \int \rho_k \beta_k v \cdot \mathbf{C}_v \mathrm{d}x\mathrm{d}t \right\}. \qquad (41)
$$

Using Corollary 4, the first one is bounded by

$$
\int_{Q_k^\natural} v_k^2 \mathrm{d}x\mathrm{d}s \le \int_{Q_{k-1}^\flat} |\beta_k v|^2 \mathrm{d}x\mathrm{d}s \le U_{k-1}^{\frac{5}{3}}. \qquad (42)
$$

Now let us deal with the last few terms. For simplicity, we use $\iint \mathrm{d}x\mathrm{d}t$ to denote $\int_{-T_k^\natural}^\tau \int_{\mathbb{R}^3} \mathrm{d}x\mathrm{d}t$ in the rest of this section.

5.1. Highest Order Nonlinear Term

Define three trilinear forms:

$$\mathbf{T}_\circ[v_1, v_2, v_3] = \iint \rho_k \operatorname{div}(v_1 \mathbf{R}(v_2 \otimes v_3)) \mathrm{d}x\mathrm{d}t,$$

$$\mathbf{T}_\nabla[v_1, v_2, v_3] = \iint \rho_k v_1 \cdot \nabla \mathbf{R}(v_2 \otimes v_3) \mathrm{d}x\mathrm{d}t,$$

$$\mathbf{T}_{\mathrm{div}}[v_1, v_2, v_3] = \iint \rho_k \operatorname{div} v_1 \mathbf{R}(v_2 \otimes v_3) \mathrm{d}x\mathrm{d}t.$$

There are symmetric on v_2, v_3 positions. When we have enough integrability, that is, when

$$|\nabla v_1||v_2||v_3|, \ |v_1||\nabla v_2||v_3|, \ |v_1||v_2||\nabla v_3| \in L^1_{t,x},$$

we have Leibniz is rule

$$\mathbf{T}_\circ = \mathbf{T}_\nabla + \mathbf{T}_{\mathrm{div}}.$$

The goal is to estimate the first two double integrals in (41),

$$\iint \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \mathrm{d}x\mathrm{d}t - \iint \rho_k \operatorname{div}(v \mathbf{R}(u \otimes v)) \mathrm{d}x\mathrm{d}t$$
$$= \mathbf{T}_\nabla[\alpha_k v, u, v] - \mathbf{T}_\circ[v, u, v].$$

We first separate $w \otimes v$ from $u \otimes v$, and we have

$$\mathbf{T}_\nabla[\alpha_k v, w, v] - \mathbf{T}_\circ[v, w, v] = \mathbf{T}_\nabla[\alpha_k v, w, v] - \mathbf{T}_\nabla[v, w, v] - \mathbf{T}_{\mathrm{div}}[v, w, v]$$
$$= -\mathbf{T}_\nabla[\beta_k v, w, v]$$
$$= - \iint \rho_k \beta_k v \cdot \nabla \mathbf{R}(w \otimes v) \mathrm{d}x\mathrm{d}t.$$

Denote $-\nabla \mathbf{R}(w \otimes v) =: \mathbf{W}_2$ and we will deal with it later. The remaining $(u - w) \otimes v$ can be separated into interior part and exterior part,

$$(u - w) \otimes v = \rho_k^\sharp v \otimes v + (1 - \rho_k^\sharp)(u - w) \otimes v.$$

The exterior part is bounded and smooth in space over the support of ρ_k:

$$\|\rho_k \mathbf{R}((1 - \rho_k^\sharp)(u - w) \otimes v)\|_{L^{p_3}_t C^\infty_x} \leq C \|(u - w) \otimes v\|_{L^{p_3}_t L^2_x}$$
$$\leq C \|u - w\|_{L^{p_1}_t L^{q_3}_x(Q_2)} \|v\|_{L^{p_2}_t L^{q_4}_x} \leq \eta.$$

Here, we denote

$$\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} < 1.$$

Therefore we can use Leibniz is rule similarly as to as w and

$$
\begin{aligned}
\mathbf{T}_\nabla[\alpha_k v, &(1 - \rho_k^\sharp)(u - w), v] - \mathbf{T}_\circ[v, (1 - \rho_k^\sharp)(u - w), v] \\
&= \mathbf{T}_\nabla[\alpha_k v, (1 - \rho_k^\sharp)(u - w), v] - \mathbf{T}_\nabla[v, (1 - \rho_k^\sharp)(u - w), v] \\
&= -\mathbf{T}_\nabla[\beta_k v, (1 - \rho_k^\sharp)(u - w), v] \\
&= -\iint \rho_k \beta_k v \cdot \nabla \mathbf{R}(v \otimes (1 - \rho_k^\sharp)(u - w)) \mathrm{d}x \mathrm{d}t \\
&\le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3 p_3}}
\end{aligned}
$$

by nonlinearization Corollary 4. The interior part is

$$
\begin{aligned}
\mathbf{T}_\nabla[\alpha_k v, &\rho_k^\sharp v, v] - \mathbf{T}_\circ[v, \rho_k^\sharp v, v] \\
&= \mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v] + 2\mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \alpha_k v, \beta_k v] \\
&\quad + \mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \alpha_k v, \alpha_k v] - \mathbf{T}_\circ[v, \rho_k^\sharp v, v] \\
&= \mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v] \\
&\quad + 2\mathbf{T}_\circ[\alpha_k v, \rho_k^\sharp \alpha_k v, \beta_k v] - 2\mathbf{T}_{\text{div}}[\alpha_k v, \rho_k^\sharp \alpha_k v, \beta_k v] \\
&\quad + \mathbf{T}_\circ[\alpha_k v, \rho_k^\sharp \alpha_k v, \alpha_k v] - \mathbf{T}_{\text{div}}[\alpha_k v, \rho_k^\sharp \alpha_k v, \alpha_k v] \\
&\quad - \mathbf{T}_\circ[v, \rho_k^\sharp v, v] \\
&= \mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v] \\
&\quad + 2\mathbf{T}_{\text{div}}[\beta_k v, \rho_k^\sharp \alpha_k v, \beta_k v] + \mathbf{T}_{\text{div}}[\beta_k v, \rho_k^\sharp \alpha_k v, \alpha_k v] \\
&\quad + 2\mathbf{T}_\circ[\alpha_k v, \rho_k^\sharp \alpha_k v, \beta_k v] + \mathbf{T}_\circ[\alpha_k v, \rho_k^\sharp \alpha_k v, \alpha_k v] \\
&\quad - \mathbf{T}_\circ[v, \rho_k^\sharp v, v] \\
&= \mathbf{T}_\nabla[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v] + \mathbf{T}_{\text{div}}[\beta_k v, \rho_k^\sharp \alpha_k v, (\beta_k + 1)v] \\
&\quad - \mathbf{T}_\circ[\alpha_k v, \rho_k^\sharp \beta_k v, \beta_k v] - \mathbf{T}_\circ[\beta_k v, \rho_k^\sharp v, v].
\end{aligned}
$$

Notice that the boundedness of $\alpha_k v$ guarantees enough integrability to switch between trilinear forms. Then

$$
\begin{aligned}
|\mathbf{T}_\nabla[\alpha_k v, &\rho_k^\sharp \beta_k v, \beta_k v]|, \ |\mathbf{T}_{\text{div}}[\beta_k v, \rho_k^\sharp \alpha_k v, (\beta_k + 1)v]| \\
&\lesssim \|\nabla(\beta_k v)\|_{L^2(Q_{k-1})} U_{k-1}^{\frac{5}{6}} \le U_{k-1}^{\frac{4}{3}}, \\
|\mathbf{T}_\circ[\alpha_k v, &\rho_k^\sharp \beta_k v, \beta_k v]|, \ |\mathbf{T}_\circ[\beta_k v, \rho_k^\sharp v, v]| \lesssim U_{k-1}^{\frac{5}{3}}.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
\bigg| \iint \rho_k \alpha_k v \cdot \nabla \mathbf{R}(u \otimes v) \mathrm{d}x \mathrm{d}t &- \iint \rho_k \operatorname{div}(v \mathbf{R}(u \otimes v)) \mathrm{d}x \mathrm{d}t \\
&- \iint \rho_k \beta_k v \cdot \mathbf{W}_2 \mathrm{d}x \mathrm{d}t \bigg| \lesssim C^k U_{k-1}^{\min\{\frac{4}{3}, \frac{5}{3} - \frac{2}{3} p_3\}}.
\end{aligned}
\tag{43}
$$

5.2. Lower Order Terms

For the bilinear and linear term, recall that inside B_1,

$$\mathbf{B} = -\operatorname{curl} \Delta^{-1}(\nabla\varphi \times (\omega \times u)),$$

$$\mathbf{L} = \operatorname{curl} \Delta^{-1}\left(2\operatorname{div}(\nabla\varphi \otimes \omega) - (\Delta\varphi)\omega\right).$$

Therefore,

$$\|\rho_k\mathbf{B}\|_{L_t^{p_3}L_x^\infty} \leq \|\omega \times u\|_{L_t^{p_3}L_x^{\frac{6}{5}}(Q_2)} \leq \|u\|_{L_t^{p_1}L_x^{q_3}}\|\omega\|_{L_t^{p_2}L_x^{q_2}} \leq \eta,$$

$$\|\rho_k\mathbf{L}\|_{L_t^{p_2}L_x^\infty} \leq \|\omega\|_{L_t^{p_2}L_x^{q_2}(Q_2)} \leq \eta,$$

Thus

$$\iint \mathbf{B} \cdot \rho_k\beta_k v \, dx \, dt \leq C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_3}}, \tag{44}$$

$$\iint \mathbf{L} \cdot \rho_k\beta_k v \, dx \, dt \leq C^k U_{k-1}^{\frac{5}{3}-\frac{2}{3p_2}}. \tag{45}$$

5.3. W Terms

Finally, let us deal with

$$\mathbf{W} + \mathbf{W}_2 = -\omega \times w + \frac{1}{2}\mathbb{P}_\nabla\left(\varpi \times u + \omega \times w\right) - \nabla\mathbf{R}(w \otimes v).$$

Here $\nabla\mathbf{R} = \frac{1}{2}\nabla\operatorname{tr} - \mathbb{P}_\nabla\operatorname{div}$, so

$$\nabla\mathbf{R}(w \otimes v) = \frac{1}{2}\nabla(w \cdot v) - \mathbb{P}_\nabla\operatorname{div}(v \otimes w)$$

$$= \frac{1}{2}\left(w \cdot \nabla v + v \cdot \nabla w + w \times \operatorname{curl} v + v \times \operatorname{curl} w\right) - \mathbb{P}_\nabla(v \cdot \nabla w)$$

$$= \frac{1}{2}\left(w \cdot \nabla v - v \cdot \nabla w\right) + \mathbb{P}_{\operatorname{curl}}(v \cdot \nabla w)$$

$$\quad + \frac{1}{2}\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right),$$

$$\nabla\mathbf{R}(w \otimes v) = \mathbb{P}_\nabla(\nabla\mathbf{R}(w \otimes v))$$

$$= \frac{1}{2}\mathbb{P}_\nabla\left(w \cdot \nabla v - v \cdot \nabla w\right) + \frac{1}{2}\mathbb{P}_\nabla\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right)$$

$$= \frac{1}{2}\mathbb{P}_\nabla\left(\operatorname{curl}(v \times w) - v\operatorname{div} w + w\operatorname{div} v\right)$$

$$\quad + \frac{1}{2}\mathbb{P}_\nabla\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right)$$

$$= -\frac{1}{2}\mathbb{P}_\nabla\left(v(u \cdot \nabla\varphi)\right) + \frac{1}{2}\mathbb{P}_\nabla\left(w \times \operatorname{curl} v + v \times \operatorname{curl} w\right).$$

Hence

$$\mathbf{W} + \mathbf{W}_2 = -\omega \times w + \frac{1}{2}\mathbb{P}_\nabla\left(v(u \cdot \nabla\varphi)\right)$$
$$+ \frac{1}{2}\mathbb{P}_\nabla\left(\varpi \times u + \omega \times w + \operatorname{curl} v \times w + \operatorname{curl} w \times v\right).$$

Again, we separate $\mathbf{W} + \mathbf{W}_2$ into exterior and interior part, with

$$\mathbf{W} + \mathbf{W}_2 = \mathbf{W}_{\text{ext}} + \mathbf{W}_{\text{int}},$$

where

$$\mathbf{W}_{\text{ext}} = -(1 - \rho_k^\sharp)\omega \times w + \frac{1}{2}\mathbb{P}_\nabla\left(v(u \cdot \nabla\varphi)\right)$$
$$+ \frac{1}{2}\mathbb{P}_\nabla\left((1 - \rho_k^\sharp)\left(\varpi \times u + \omega \times w + \operatorname{curl} v \times w + \operatorname{curl} w \times v\right)\right),$$
$$\mathbf{W}_{\text{int}} = -\rho_k^\sharp \omega \times w$$
$$+ \frac{1}{2}\mathbb{P}_\nabla\left(\rho_k^\sharp\left(\varpi \times u + \omega \times w + \operatorname{curl} v \times w + \operatorname{curl} w \times v\right)\right)$$
$$= -\rho_k^\sharp \operatorname{curl} v \times w - \rho_k^\sharp \varpi \times w$$
$$+ \frac{1}{2}\mathbb{P}_\nabla\left(\rho_k^\sharp\left(\varpi \times u + \operatorname{curl} w \times v + \varpi \times w\right)\right)$$
$$+ \frac{1}{2}\mathbb{P}_\nabla\left(\rho_k^\sharp\left(\omega \times w + \operatorname{curl} v \times w - \varpi \times w\right)\right)$$
$$= -\rho_k^\sharp \operatorname{curl} v \times w - \rho_k^\sharp \varpi \times w$$
$$+ \mathbb{P}_\nabla\left(\rho_k^\sharp \varpi \times u\right) + \mathbb{P}_\nabla\left(\rho_k^\sharp \operatorname{curl} v \times w\right)$$
$$= -\mathbb{P}_{\text{curl}}(\rho_k^\sharp \operatorname{curl} v \times w) - \mathbb{P}_{\text{curl}}(\rho_k^\sharp \varpi \times w) + \mathbb{P}_\nabla\left(\rho_k^\sharp \varpi \times v\right).$$

Similarly as to the bilinear terms, $\rho_k \mathbf{W}_{\text{ext}}$ is small in $L_t^{p_3} L_x^\infty$. Among the three terms in $\mathbf{W}_{\text{int}}$, $\rho_k^\sharp \varpi \times w$ is bounded in $L_t^{p_3} L_x^\infty$, and $\rho_k^\sharp \varpi$ is in $L_t^{p_2} L_x^\infty$. Finally, for the first term,

$$\mathbb{P}_{\text{curl}}(\operatorname{curl} v \times \rho_k^\sharp w) = -\mathbb{P}_{\text{curl}}(\operatorname{curl} \rho_k^\sharp w \times v) + \mathbb{P}_{\text{curl}}(v \cdot \nabla \rho_k^\sharp w + \rho_k^\sharp w \cdot \nabla v),$$
$$\mathbb{P}_{\text{curl}}(\rho_k^\sharp w \cdot \nabla v) = \mathbb{P}_{\text{curl}}(\operatorname{curl}(v \times \rho_k^\sharp w) + v \cdot \nabla \rho_k^\sharp w - v \operatorname{div} \rho_k^\sharp w)$$
$$= \operatorname{curl}(v \times \rho_k^\sharp w) + \mathbb{P}_{\text{curl}}(v \cdot \nabla \rho_k^\sharp w - v \operatorname{div} \rho_k^\sharp w),$$
$$\operatorname{curl}(v \times \rho_k^\sharp w) = v \operatorname{div} \rho_k^\sharp w + \rho_k^\sharp w \cdot \nabla v - v \cdot \nabla \rho_k^\sharp w.$$

Every term is a product of v and $\nabla \rho_k^\sharp w$ (possibly with a Riesz transform) except $\rho_k^\sharp w \cdot \nabla v$. Because in Ω_k, $\nabla |v| = \nabla v_k$ are the same, we have

$$\int \rho_k \beta_k v \cdot (\rho_k^\sharp w \cdot \nabla) v \, dx = \int \rho_k \beta_k (w \cdot \nabla) \frac{|v|^2}{2} dx$$

$$= \int \rho_k \beta_k |v| (w \cdot \nabla) |v| dx$$

$$= \int \rho_k v_k (w \cdot \nabla) v_k \, dx$$

$$= \int \rho_k (w \cdot \nabla) \frac{v_k^2}{2} dx$$

$$= -\int \frac{v_k^2}{2} \operatorname{div}(\rho_k w) dx.$$

Therefore, every term of $\mathbb{P}_{\mathrm{curl}}(\operatorname{curl} v \times \rho_k^\sharp w)$ is a product of v and $\nabla \rho_k w$ or $\nabla \rho_k^\sharp w$. Inside B_1, $w \in L_t^{p_1} C_x^\infty$. In conclusion,

$$\iint \rho_k \beta_k v \cdot \mathbf{W}_{\mathrm{ext}} dx dt \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_3}},$$

$$\iint \rho_k \beta_k v \cdot \mathbb{P}_{\mathrm{curl}}(\rho_k^\sharp \operatorname{curl} v \times w) dx dt \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_1}},$$

$$\iint \rho_k \beta_k v \cdot \mathbb{P}_{\mathrm{curl}}(\rho_k^\sharp \varpi \times w) dx dt \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_3}},$$

$$\iint \rho_k \beta_k v \cdot \mathbb{P}_\nabla(\rho_k^\sharp \varpi \times v) dx dt \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_2}},$$

so the sum is bounded in

$$\iint \rho_k \beta_k v \cdot (\mathbf{W} + \mathbf{W}_2) dx dt = \iint \rho_k \beta_k v \cdot (\mathbf{W}_{\mathrm{int}} + \mathbf{W}_{\mathrm{ext}}) dx dt \le C^k U_{k-1}^{\frac{5}{3} - \frac{2}{3p_3}}, \tag{46}$$

provided that $U_{k-1} < 1$.

5.4. Proof of Proposition 2

Proof of Proposition 2. Coming back to (41), by estimates (42) on the first term, (43) on the trilinear terms, (44), (45) on the $\mathbf{B}$, $\mathbf{L}$ terms and (46) on the $\mathbf{W}$ terms, we conclude that

$$U_k \le C^k U_{k-1}^{\min\{\frac{5}{3} - \frac{2}{3p_3}, \frac{4}{3}\}},$$

provided that $U_{k-1} < 1$. Here $p_3 > 1$ ensures the index is strictly greater than 1. Since

$$U_0 = \sup_{t \in (-1,0)} \int |v_0|^2 dx + \int_{-1}^0 \int_{B_1} d_0^2 dx dt$$

$$= \sup_{t \in (-1,0)} \int |v|^2 dx + \int_{-1}^0 \int_{B_1} |\nabla v|^2 dx dt \leq \eta$$

by Proposition 1, we know that if η is small enough, $U_k \to 0$ as $k \to \infty$. Thus in $Q_{\frac{1}{2}}$, $|v| \leq 1$ a.e.. This finishes the proof of Proposition 2. $\quad\square$

6. Local Study: Part Three, More Regularity

In this section, we will show that the vorticity ω is smooth in space. We will only work with the vorticity equation from now on. After the previous two steps, in $B_{\frac{1}{2}}$ we should always decompose $u = v + w$, because v is bounded and w is harmonic.

For convenience, given a vector ω, we denote

$$\omega^0 := \frac{\omega}{|\omega|}, \qquad \omega^\alpha := |\omega|^\alpha \omega^0, \alpha \in \mathbb{R}.$$

Let $\partial_\bullet$ be the partial derivative in any space direction or time. Then we have

$$\partial_\bullet(|\omega|^\alpha) = \alpha \omega^{\alpha-1} \cdot \partial_\bullet \omega,$$

$$\partial_\bullet(\omega^\alpha) = |\omega|^{\alpha-1} \partial_\bullet \omega + (\alpha-1)(\omega^{\alpha-2} \cdot \partial_\bullet \omega)\omega,$$

$$\frac{1}{\alpha} \partial_\bullet \partial_\bullet(|\omega|^\alpha) = |\omega|^{\alpha-2}|\partial_\bullet \omega|^2 + (\alpha-2)(\omega^{\frac{\alpha}{2}-1} \cdot \partial_\bullet \omega)^2 + \omega^{\alpha-1} \cdot \partial_\bullet \partial_\bullet \omega$$

$$\geq (\alpha-1)(\omega^{\frac{\alpha}{2}-1} \cdot \partial_\bullet \omega)^2 + \omega^{\alpha-1} \cdot \partial_\bullet \partial_\bullet \omega$$

$$= \frac{4(\alpha-1)}{\alpha^2} \left| \partial_\bullet \omega^{\frac{\alpha}{2}} \right|^2 + \omega^{\alpha-1} \cdot \partial_\bullet \partial_\bullet \omega.$$

6.1. Bound Vorticity in the Energy Space

We will first show ω is bounded in the energy space.

Proposition 3. *If $u = v + w$ in $Q_{\frac{1}{2}}$, where v, w are bounded in*

$$\|v\|_{L^\infty(Q_{\frac{1}{2}})} + \|\nabla v\|_{L^2(Q_{\frac{1}{2}})} \leq 2, \tag{47}$$

$$\| \operatorname{curl} w \|_{L_t^2 L_x^{\frac{3}{2}}(Q_{\frac{1}{2}})} + \|w\|_{L_t^{\frac{4}{3}} \operatorname{Lip}_x(Q_{\frac{1}{2}})} \leq 2, \tag{48}$$

$\omega = \operatorname{curl} u$ solves the vorticity equation (6), then

(a) $\|\omega^{\frac{3}{4}}\|_{\mathcal{E}(Q_{\frac{1}{4}})} \leq C$,

(b) $\|\omega\|_{\mathcal{E}(Q_{\frac{1}{8}})} \leq C$,

Proof of Proposition 3 (a). We fix a pair of smooth space-time cut-off functions ϱ and ς which satisfy

$$1_{Q_{\frac{1}{8}}} \leq \varsigma \leq 1_{Q_{\frac{1}{4}}} \leq \varrho \leq 1_{Q_{\frac{1}{2}}}.$$

Take the dot product of the vorticity equation (6) with $\frac{3}{2}\omega^{\frac{1}{2}}$:

$$\frac{3}{2}\omega^{\frac{1}{2}} \cdot \partial_t \omega = \partial_t(|\omega|^{\frac{3}{2}}),$$

$$\frac{3}{2}\omega^{\frac{1}{2}} \cdot (u \cdot \nabla)\omega = (u \cdot \nabla)(|\omega|^{\frac{3}{2}}),$$

$$\frac{3}{2}\omega^{\frac{1}{2}} \cdot \Delta\omega \leq \Delta(|\omega|^{\frac{3}{2}}) - \frac{4}{3}|\nabla\omega^{\frac{3}{4}}|^2.$$

Therefore,

$$(\partial_t + u \cdot \nabla - \Delta)(|\omega|^{\frac{3}{2}}) + \frac{3}{2}\omega \cdot \nabla u \cdot \omega^{\frac{1}{2}} + \frac{4}{3}|\nabla\omega^{\frac{3}{4}}|^2 \leq 0.$$

Multiply by ϱ^6 then integrate over space to get

$$\int \varrho^6(\partial_t + u \cdot \nabla - \Delta)(|\omega|^{\frac{3}{2}})dx + \frac{4}{3}\int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2 dx \leq -\frac{3}{2}\int \varrho^6\omega \cdot \nabla u \cdot \omega^{\frac{1}{2}} dx. \tag{49}$$

For the left hand side, we can integrate by parts to get

$$\int \varrho^6(\partial_t + u \cdot \nabla - \Delta)(|\omega|^{\frac{3}{2}})dx$$

$$= \frac{d}{dt}\int \varrho^6|\omega|^{\frac{3}{2}}dx - \int \left((\partial_t + u \cdot \nabla + \Delta)\varrho^6\right)|\omega|^{\frac{3}{2}}dx, \tag{50}$$

where the latter can be controlled by

$$\int \left((\partial_t + u \cdot \nabla + \Delta)\varrho^6\right)|\omega|^{\frac{3}{2}}dx \leq C\left(1 + \|u\|_{L^\infty(B_{\frac{1}{2}})}\right)\int \varrho^4|\omega|^{\frac{3}{2}}dx. \tag{51}$$

For the right hand side, using $u = v + w$ over the support of ϱ, we can separate

$$\int \varrho^6\omega \cdot \nabla u \cdot \omega^{\frac{1}{2}}dx = \int \varrho^6\omega \cdot \nabla v \cdot \omega^{\frac{1}{2}}dx + \int \varrho^6\omega \cdot \nabla w \cdot \omega^{\frac{1}{2}}dx, \tag{52}$$

The ∇v term can be controlled by

$$\int \varrho^6\omega \cdot \nabla v \cdot \omega^{\frac{1}{2}}dx = -\int \omega \cdot \nabla(\varrho^6\omega^{\frac{1}{2}}) \cdot v dx$$

$$= -\int \varrho^6\omega \cdot \nabla(\omega^{\frac{1}{2}}) \cdot v dx - \int \omega \cdot (\omega^{\frac{1}{2}} \otimes \nabla\varrho^6) \cdot v dx, \tag{53}$$

where

$$\omega \cdot \nabla(\omega^{\frac{1}{2}}) = |\omega|^{-\frac{1}{2}}\omega \cdot \nabla\omega - \frac{1}{2}(\omega \cdot \nabla\omega \cdot \omega^{-\frac{3}{2}})\omega = \omega^{\frac{1}{2}} \cdot \nabla\omega - \frac{1}{2}(\omega^{\frac{1}{2}} \cdot \nabla\omega \cdot \omega^0)\omega^0$$

$$\Rightarrow |\omega \cdot \nabla(\omega^{\frac{1}{2}})| \leq \left| \frac{3}{2}\omega^{\frac{1}{2}} \cdot \nabla\omega \right| = 2|\omega|^{\frac{3}{4}} \left| \frac{3}{4}\omega^{-\frac{1}{4}} \cdot \nabla\omega \right| = 2|\omega|^{\frac{3}{4}} \left| \nabla|\omega|^{\frac{3}{4}} \right|$$

$$\leq 2|\omega|^{\frac{3}{4}}|\nabla\omega^{\frac{3}{4}}| \leq |\omega|^{\frac{3}{2}} + |\nabla\omega^{\frac{3}{4}}|^2.$$

Here the second to the last inequality is due to $\partial_i|\omega|^{\frac{3}{4}} = \partial_i\omega^{\frac{3}{4}} \cdot \omega^0$. Since $|v| \leq 1$ over the support of ϱ,

$$\int \varrho^6\omega \cdot \nabla(\omega^{\frac{1}{2}}) \cdot v\mathrm{d}x \leq \int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x + \int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2\mathrm{d}x. \tag{54}$$

By using (50)–(54) in (49), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x + \frac{4}{3} \int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2\mathrm{d}x$$

$$\leq \int \left[(\partial_t + u \cdot \nabla + \Delta)\varrho^6\right]|\omega|^{\frac{3}{2}}\mathrm{d}x$$

$$+ \int \varrho^6\omega \cdot \nabla w \cdot \omega^{\frac{1}{2}}\mathrm{d}x$$

$$+ \int \omega \cdot (\omega^{\frac{1}{2}} \otimes \nabla\varrho^6) \cdot v\mathrm{d}x$$

$$+ \int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x + \int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2\mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x + \frac{1}{3} \int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2\mathrm{d}x$$

$$\leq C \left(1 + \|u(t)\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{2}})} \right) \int \varrho^4|\omega|^{\frac{3}{2}}\mathrm{d}x.$$

By Hölder's inequality,

$$\int \varrho^4|\omega|^{\frac{3}{2}}\mathrm{d}x \leq \|\omega(t)\|_{L^{\frac{3}{2}}(B_{\frac{1}{2}})}^{\frac{1}{2}} \left(\int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x \right)^{\frac{2}{3}}.$$

Therefore we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x + \frac{1}{3} \int \varrho^6|\nabla\omega^{\frac{3}{4}}|^2\mathrm{d}x \leq C\Phi(t) \left(1 + \int \varrho^6|\omega|^{\frac{3}{2}}\mathrm{d}x \right),$$

where

$$\Phi(t) = \left(1 + \|u(t)\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{2}})} \right) \|\omega(t)\|_{L^{\frac{3}{2}}(B_{\frac{1}{2}})}^{\frac{1}{2}}$$

$$\leq \left(2 + \|w(t)\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{2}})} \right)$$

$$\times \left(\|\operatorname{curl} v(t)\|_{L^{\frac{3}{2}}(B_{\frac{1}{2}})}^{\frac{1}{2}} + \|\operatorname{curl} w(t)\|_{L^{\frac{3}{2}}(B_{\frac{1}{2}})}^{\frac{1}{2}} \right).$$

since $u = w + v$, and $|v| \leq 1$ inside $B_{\frac{1}{2}}$. By (47),

$$\int_{-\frac{1}{4}}^{0} \Phi(t) dt \lesssim \left(1 + \|w\|_{L_t^{\frac{4}{3}} \mathrm{Lip}_x(Q_{\frac{1}{2}})} \right) \left(\|\nabla v\|_{L^2(Q_{\frac{1}{2}})}^{\frac{1}{2}} + \| \operatorname{curl} w(t) \|_{L_t^2 L_x^{\frac{3}{2}}(Q_{\frac{1}{2}})}^{\frac{1}{2}} \right) \leq C.$$

Thus, by Grönwall's inequality,

$$\|\omega^{\frac{3}{4}}\|_{L_t^{\infty} L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{\frac{1}{4}})}^2 \leq e^C - 1.$$

$\square$

Proof of Proposition 3(b). From Proposition 3(a) and Sobolev embedding,

$$\|\omega\|_{L_t^{\infty} L_x^{\frac{3}{2}} \cap L_t^{\frac{3}{2}} L_x^{\frac{9}{2}}(Q_{\frac{1}{4}})} \leq C,$$

this interpolates the space

$$\|\omega\|_{L_t^4 L_x^2(Q_{\frac{1}{4}})} \leq C.$$

Multiply the vorticity equation (6) by $\varsigma^2 \omega$ then integrate over $\mathbb{R}^3$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varsigma^2 \frac{|\omega|^2}{2} \mathrm{d}x + \int \varsigma^2 |\nabla \omega|^2 \mathrm{d}x = \int (\partial_t \varsigma^2 + \Delta \varsigma^2) \frac{|\omega|^2}{2} \mathrm{d}x$$
$$- \int (u \cdot \nabla \omega) \cdot \varsigma^2 \omega \mathrm{d}x$$
$$+ \int (\omega \cdot \nabla u) \cdot \varsigma^2 \omega \mathrm{d}x.$$

The first integral is L^1 in time because $\omega \in L_t^4 L_x^2$. For the second,

$$\int (u \cdot \nabla \omega) \cdot \varsigma^2 \omega \mathrm{d}x = \int \varsigma^2 u \cdot \nabla \frac{|\omega|^2}{2} \mathrm{d}x$$
$$= - \int \frac{|\omega|^2}{2} u \cdot \nabla \varsigma^2 \mathrm{d}x$$
$$= - \int \varsigma |\omega|^2 u \cdot \nabla \varsigma$$
$$\leq \|\varsigma \omega\|_{L^2} \|u \cdot \nabla \varsigma |\omega|\|_{L^2};$$

the latter is bounded L^1 in time, by $u \in L_t^{\frac{4}{3}} L_x^{\infty}$ and $\omega \in L_t^4 L_x^2$. For the third integral,

$$\int (\omega \cdot \nabla u) \cdot \varsigma^2 \omega \mathrm{d}x = \int (\omega \cdot \nabla v) \cdot \varsigma^2 \omega \mathrm{d}x + \int (\omega \cdot \nabla w) \cdot \varsigma^2 \omega \mathrm{d}x.$$

w is bounded in $L_t^{\frac{4}{3}}\mathrm{Lip}_x$, and for v,

$$\int (\omega \cdot \nabla v) \cdot \varsigma^2 \omega \mathrm{d}x = \int v \cdot (\omega \cdot \nabla)(\varsigma^2 \omega) \mathrm{d}x$$

$$= \int v \cdot \omega (\omega \cdot \nabla \varsigma^2) \mathrm{d}x + \int v \cdot (\varsigma^2 \omega \cdot \nabla \omega) \mathrm{d}x.$$

The former is L^1 in time, while the latter can be bounded by Cauchy-Schwartz,

$$\int v \cdot (\varsigma^2 \omega \cdot \nabla \omega) \mathrm{d}x \le \frac{1}{2} \int |v \otimes \varsigma \omega|^2 \mathrm{d}x + \frac{1}{2} \int \varsigma^2 |\nabla \omega|^2 \mathrm{d}x.$$

In conclusion,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varsigma^2 \frac{|\omega|^2}{2} \mathrm{d}x + \frac{1}{2} \int \varsigma^2 |\nabla \omega|^2 \mathrm{d}x$$

$$\le C \|\omega(t)\|_{L^2(B_{\frac{1}{4}})}^2 + C \|u(t)\|_{L^\infty(B_{\frac{1}{4}})} \|\omega(t)\|_{L^2(B_{\frac{1}{4}})} \|\varsigma \omega(t)\|_{L^2}$$

$$+ C \|\nabla w\|_{L^\infty(B_{\frac{1}{4}})} \|\varsigma \omega(t)\|_{L^2}^2$$

$$\le C\Phi(t) \left(1 + \int \varsigma^2 \frac{|\omega|^2}{2} \mathrm{d}x\right),$$

where

$$\Phi(t) = \|\omega(t)\|_{L^2(B_{\frac{1}{4}})}^2 + \|u(t)\|_{L^\infty(B_{\frac{1}{4}})} \|\omega(t)\|_{L^2(B_{\frac{1}{4}})} + \|\nabla w(t)\|_{L^\infty(B_{\frac{1}{4}})},$$

whose integral is bounded using (47),

$$\int_{-\frac{1}{16}}^0 \Phi(t) \mathrm{d}t \le \|\omega\|_{L^2(Q_{\frac{1}{4}})}^2 + \|u\|_{L_t^{\frac{4}{3}} L_x^\infty(Q_{\frac{1}{4}})} \|\omega\|_{L_t^4 L_x^2(Q_{\frac{1}{4}})} + \|\nabla w\|_{L_t^{\frac{4}{3}} L_x^\infty(Q_{\frac{1}{4}})} \le C.$$

By a Grönwall is argument, we have

$$\|\omega\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{\frac{1}{8}})}^2 \le e^C - 1.$$

$\square$

6.2. Bound Higher Derivatives in the Energy Space

Now we bootstrap to higher regularity of ω using similar ideas as in the proof of Proposition 3.

Proposition 4. *For any $n \ge 1$, if $u = v + w$ in $Q_{8^{-n}}$, where v, w are bounded in*

$$\|v\|_{L^\infty(Q_{8^{-n}/2})} + \|v\|_{L_t^2 H_x^{n+1}(Q_{8^{-n}/2})} \le c_n, \tag{55}$$

$$\|w\|_{L_t^{\frac{4}{3}} C_x^{n+1}(Q_{8^{-n}/2})} \le c_n, \tag{56}$$

for some constant c_n, $\omega = \mathrm{curl}\, u$ solves the vorticity equation (6), and is bounded in

$$\|\omega\|_{L_t^\infty H_x^{n-1} \cap L_t^2 H_x^n (Q_{8-n/2})} \le c_n, \tag{57}$$

then for any multiindex α with $|\alpha| = n$,

(a) $\|\nabla^\alpha \omega^{\frac{3}{4}}\|_{\mathcal{E}(Q_{8-n/4})} \le C_n$
(b) $\|\nabla^\alpha \omega\|_{\mathcal{E}(Q_{8-n-1})} \le C_n$

for some C_n depending on c_n and n.

Proof of Proposition 4(a). Similarly, we fix smooth cut-off functions ϱ_n and ς_n which satisfy

$$\mathbf{1}_{Q_{8-n-1}} \le \varsigma_n \le \mathbf{1}_{Q_{8-n/4}} \le \varrho_n \le \mathbf{1}_{Q_{8-n/2}}.$$

Differentiate (6) by ∇^α to get

$$\partial_t \nabla^\alpha \omega + u \cdot \nabla \nabla^\alpha \omega - \nabla^\alpha \omega \cdot \nabla u + \mathbf{P}_\alpha = \Delta \nabla^\alpha \omega, \tag{58}$$

where

$$\mathbf{P}_\alpha = \sum_{\beta < \alpha} \binom{\alpha}{\beta} \mathrm{curl}\left(\nabla^\beta \omega \times \nabla^{\alpha-\beta} u\right).$$

Multiply (58) by $\frac{3}{2}\varrho_n^6 (\nabla^\alpha \omega)^{\frac{1}{2}}$ then integrate in space to get

$$\frac{d}{dt} \int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} dx + \frac{4}{3} \int \varrho_n^6 |\nabla \nabla^\alpha \omega^{\frac{3}{4}}|^2 dx$$

$$\le \int \left[(\partial_t + u \cdot \nabla + \Delta)\varrho_n^6\right] |\nabla^\alpha \omega|^{\frac{3}{2}} dx$$

$$+ \int \varrho_n^6 \nabla^\alpha \omega \cdot \nabla w \cdot (\nabla^\alpha \omega)^{\frac{1}{2}} dx$$

$$+ \int \nabla^\alpha \omega \cdot ((\nabla^\alpha \omega)^{\frac{1}{2}} \otimes \nabla \varrho_n^6) \cdot v dx$$

$$+ \|v\|_{L^\infty(Q_{8-n})}^2 \int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} dx + \int \varrho_n^6 |\nabla \nabla^\alpha \omega^{\frac{3}{4}}|^2 dx$$

$$+ \frac{3}{2} \int \varrho_n^6 (\nabla^\alpha \omega)^{\frac{1}{2}} \cdot \mathbf{P}_\alpha dx,$$

the same as in the proof of Proposition 3(a). Therefore,

$$\frac{d}{dt} \int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} dx + \frac{1}{3} \int \varrho_n^6 |\nabla \nabla^\alpha \omega^{\frac{3}{4}}|^2 dx$$

$$\le C \left(1 + \|u(t)\|_{L^\infty(B_{8-n})} + \|\nabla w(t)\|_{L^\infty(B_{8-n})}\right) \int \varrho_n^4 |\nabla^\alpha \omega|^{\frac{3}{2}} dx$$

$$+ \frac{3}{2} \int \varrho_n^6 (\nabla^\alpha \omega)^{\frac{1}{2}} \cdot \mathbf{P}_\alpha dx.$$

Terms other than $\mathbf{P}_\alpha$ are dealt with in the same way as in Proposition 3:

$$\int \varrho_n^4 |\nabla^\alpha \omega|^{\frac{3}{2}} dx \leq \|\nabla^\alpha \omega(t)\|_{L^{\frac{3}{2}}(B_{8-n})}^{\frac{1}{2}} \left(\int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} dx \right)^{\frac{2}{3}}.$$

The induction condition (57) ensures that $\|\nabla^\alpha \omega\|_{L^2(Q_{8-n})} \leq c_n$. Therefore

$$\int_{-8^{-2n}}^{0} \left(1 + \|u(t)\|_{L^\infty(B_{8-n})} + \|\nabla w(t)\|_{L^\infty(B_{8-n})}\right) \|\nabla^\alpha \omega(t)\|_{L^{\frac{3}{2}}(B_{8-n})}^{\frac{1}{2}} dt$$

$$\lesssim \left(1 + \|v\|_{L^\infty(B_{8-n})} + \|w\|_{L_t^{\frac{4}{3}} C_x^1(B_{8-n})}\right) \|\nabla^\alpha \omega\|_{L^2(Q_{8-n})}^{\frac{1}{2}} \leq C_n.$$

Now let's focus on $\mathbf{P}_\alpha$:

$$|\mathbf{P}_\alpha| \lesssim \sum_{k=0}^{n} |\nabla^k \omega||\nabla^{n-k+1} u| \leq \sum_{k=0}^{n} |\nabla^k \omega||\nabla^{n-k+1} v| + \sum_{k=0}^{n} |\nabla^k \omega||\nabla^{n-k+1} w|.$$

We denote

$$\mathbf{P}_{v,k} = |\nabla^k \omega||\nabla^{n-k+1} v|, \qquad \mathbf{P}_{w,k} = |\nabla^k \omega||\nabla^{n-k+1} w|.$$

First we estimate $\mathbf{P}_{v,k}$. By (55) and (57), when $k = 0$,

$$\|\mathbf{P}_{v,0}\|_{L_t^1 L_x^{\frac{3}{2}}(Q_{8-n})} \leq \|\omega\|_{L_t^2 L_x^6(Q_{8-n})} \|\nabla^{n+1} v\|_{L_t^2 L_x^2(Q_{8-n})} \leq C_n,$$

and when $0 < k \leq n$,

$$\|\mathbf{P}_{v,k}\|_{L_t^1 L_x^{\frac{3}{2}}(Q_{8-n})} \leq \|\nabla^k \omega\|_{L_t^2 L_x^2(Q_{8-n})} \|\nabla^{n+1-k} v\|_{L_t^2 L_x^6(Q_{8-n})} \leq C_n.$$

Next we estimate $\mathbf{P}_{w,k}$. When $0 \leq k < n$,

$$\|\mathbf{P}_{w,k}\|_{L_t^1 L_x^{\frac{3}{2}}(Q_{8-n})} \leq \|\nabla^k \omega\|_{L_t^\infty L_x^2(Q_{8-n})} \|\nabla^{n+1-k} w\|_{L_t^{\frac{4}{3}} L_x^\infty(Q_{8-n})} \leq C_n.$$

Finally, when $k = n$,

$$|\mathbf{P}_{w,n}|_{L_x^{\frac{3}{2}}(B_{8-n})} \leq |\nabla^n \omega||\nabla w|.$$

Therefore,

$$\int \varrho_n^6 (\nabla^\alpha \omega)^{\frac{1}{2}} \cdot \mathbf{P}_\alpha dx \leq \left(1 + \int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} dx\right)$$

$$\times \left(\sum_{k=0}^{n} \|\mathbf{P}_{v,k}\|_{L_x^{\frac{3}{2}}(B_{8-n})} + \sum_{k=0}^{n-1} \|\mathbf{P}_{w,k}\|_{L_x^{\frac{3}{2}}(B_{8-n})} + \|\nabla w\|_{L_x^\infty(B_{8-n})} \right).$$

In conclusion, we have shown that

$$\frac{d}{dt} \int \varrho_n^6 |\nabla^\alpha \omega|^{\frac{3}{2}} dx + \frac{1}{3} \int \varrho_n^6 |\nabla \nabla^\alpha \omega^{\frac{3}{4}}|^2 dx \leq C \Phi(t) \left(1 + \int \varrho_n^6 |\nabla^n \omega|^{\frac{3}{2}} dx\right),$$

where

$$\Phi(t) = \big(1 + \|u(t)\|_{L^\infty(B_{8-n})} + \|\nabla w(t)\|_{L^\infty(B_{8-n})}\big) \|\nabla^\alpha \omega(t)\|_{L^{\frac{3}{2}}(B_{8-n})}^{\frac{1}{2}}$$

$$+ \sum_{k=0}^{n} \|\mathbf{P}_{v,k}\|_{L_x^{\frac{3}{2}}(B_{8-n})} + \sum_{k=0}^{n-1} \|\mathbf{P}_{w,k}\|_{L_x^{\frac{3}{2}}(B_{8-n})} + \|\nabla w\|_{L_x^\infty(B_{8-n})}$$

with integral

$$\int_{-8^{-2n}/4}^{0} \Phi(t)\mathrm{d}t \le C_n.$$

Taking the sum over all multi-index α with size $|\alpha| = n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho_n^6 |\nabla^n \omega|^{\frac{3}{2}} \mathrm{d}x + \frac{1}{3} \int \varrho_n^6 |\nabla^{n+1} \omega^{\frac{3}{4}}|^2 \mathrm{d}x \le C\Phi(t)\left(1 + \int \varrho_n^6 |\nabla^{n+1}\omega|^{\frac{3}{2}} \mathrm{d}x\right).$$

Finally, Grönwall inequality gives

$$\||\nabla^{n+1}\omega|^{\frac{3}{4}}\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{8-n}/4)} \le C_n.$$

$\square$

Proof of Proposition 4(b). Now we multiply (58) by $\varsigma_n^2 \nabla^\alpha \omega$ then integrate over $\mathbb{R}^3$ to get that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x + \int \varsigma_n^2 |\nabla \nabla^\alpha \omega|^2 \mathrm{d}x = \int (\partial_t \varsigma_n^2 + \Delta \varsigma_n^2)\frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x$$

$$- \int (u \cdot \nabla \nabla^\alpha \omega) \cdot \varsigma_n^2 \nabla^\alpha \omega \mathrm{d}x$$

$$+ \int (\nabla^\alpha \omega \cdot \nabla u) \cdot \varsigma_n^2 \nabla^\alpha \omega \mathrm{d}x$$

$$+ \int \varsigma_n^2 \nabla^\alpha \omega \cdot \mathbf{P}_\alpha \mathrm{d}x.$$

For the same reason, the only term that we need to take care of is $\mathbf{P}_\alpha$ term, and the others are dealt the same as with Proposition 3(b):

$$\int (\partial_t \varsigma_n^2 + \Delta \varsigma_n^2)\frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x - \int (u \cdot \nabla \nabla^\alpha \omega) \cdot \varsigma_n^2 \nabla^\alpha \omega \mathrm{d}x + \int (\nabla^\alpha \omega \cdot \nabla u) \cdot \varsigma_n^2 \nabla^\alpha \omega \mathrm{d}x$$

$$\lesssim_n \|\nabla^\alpha \omega\|_{L^2(Q_{8-n}/4)}^2 + \|u\|_{L^\infty(Q_{8-n}/4)} \|\nabla^\alpha \omega\|_{L^2(Q_{8-n}/4)} \left(\int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x\right)^{\frac{1}{2}}$$

$$+ \|\nabla w\|_{L^\infty(Q_{8-n}/4)} \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x + \|v\|_{L^\infty(Q_{8-n}/4)} \|\nabla^\alpha \omega\|_{L^2(Q_{8-n}/4)}^2$$

$$+ \frac{1}{\varepsilon} \|v\|_{L^\infty(Q_{8-n}/4)}^2 \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x + \varepsilon \int \varsigma_n^2 |\nabla \nabla^\alpha \omega|^2 \mathrm{d}x.$$

The last term can be absorbed into the left, and we will use Grönwall on the remaining terms.

Now we shall focus on the $\mathbf{P}_\alpha$ term. From Proposition 4(a), we have

$$\|\nabla^n \omega\|_{L_t^\infty L_x^{\frac{3}{2}} \cap L_x^{\frac{3}{2}} L_t^{\frac{9}{2}}(Q_{8-n/4})} \leq C_n. \tag{59}$$

Again, by interpolation,

$$\|\nabla^n \omega\|_{L_t^4 L_x^2(Q_{8-n/4})} \leq C_n, \qquad \|\nabla^n \omega\|_{L_t^2 L_x^3(Q_{8-n/4})} \leq C_n.$$

First we estimate $\mathbf{P}_{w,k}$. In this case, for any $0 \leq k \leq n$,

$$\|\mathbf{P}_{w,k}\|_{L_t^1 L_x^2(Q_{8-n/4})} \leq \|\nabla^k \omega\|_{L_t^4 L_x^2(Q_{8-n/4})} \|\nabla^{n+1-k} w\|_{L_t^{\frac{4}{3}} L_x^\infty(Q_{8-n})} \leq C_n.$$

Then we estimate $\mathbf{P}_{v,k}$. When $0 < k \leq n$,

$$\|\mathbf{P}_{v,k}\|_{L_t^1 L_x^2(Q_{8-n})} \leq \|\nabla^k \omega\|_{L_t^2 L_x^3(Q_{8-n})} \|\nabla^{n+1-k} v\|_{L_t^2 L_x^6(Q_{8-n})} \leq C_n.$$

For the case $k = 0$ of the v term, we put the curl on $\nabla^\alpha \omega$ to obtain

$$\int \varsigma_n^2 \nabla^\alpha \omega \cdot \mathrm{curl}\,(\omega \times \nabla^\alpha v)\,\mathrm{d}x$$

$$= \int (\omega \times \nabla^\alpha v) \cdot \mathrm{curl}(\varsigma_n^2 \nabla^\alpha \omega)\mathrm{d}x$$

$$\leq \int \varsigma_n^2 |\omega||\nabla^\alpha v||\nabla\nabla^\alpha \omega| + \varsigma_n |\nabla \varsigma_n||\omega||\nabla^\alpha v||\nabla^\alpha \omega|\mathrm{d}x$$

$$\leq \int \varsigma_n^2 |\omega|^2 |\nabla^\alpha v|^2 \mathrm{d}x + \varepsilon \int \varsigma_n^2 |\nabla\nabla^\alpha \omega|^2 \mathrm{d}x + \frac{1}{\varepsilon} \int |\nabla \varsigma_n|^2 |\nabla^\alpha \omega|^2 \mathrm{d}x,$$

where $|\nabla\nabla^\alpha \omega|$ term can be absorbed to the left. By (59) and Sobolev embedding,

$$\|\omega\|_{L_t^\infty L_x^3(Q_{8-n/4})} \leq C_n.$$

Therefore

$$\iint \varsigma_n^2 |\omega|^2 |\nabla^\alpha v|^2 \mathrm{d}x\mathrm{d}t \leq \|\omega\|^2_{L_t^\infty L_x^3(Q_{8-n/4})} \|\nabla^\alpha v\|^2_{L_t^2 L_x^6(Q_{8-n/4})} \leq C_n.$$

In conclusion,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x + \int \varsigma_n^2 |\nabla\nabla^\alpha \omega|^2 \mathrm{d}x \leq C\Phi(t) \left(1 + \int \varsigma_n^2 \frac{|\nabla^\alpha \omega|^2}{2} \mathrm{d}x\right),$$

where

$$\Phi(t) = \|\nabla^\alpha \omega(t)\|^2_{L^2(B_{8-n/4})} + \|u\|_{L^\infty(B_{8-n/4})} \|\nabla^\alpha \omega\|_{L^2(B_{8-n/4})}$$

$$+ \|\nabla w\|_{L^\infty(B_{8-n/4})} + \|v\|_{L^\infty(B_{8-n/4})} \|\nabla^\alpha \omega\|^2_{L^2(B_{8-n/4})}$$

$$+ \frac{1}{\varepsilon} \|v\|^2_{L^\infty(B_{8-n/4})} + \sum_{k=0}^n \|\mathbf{P}_{w,k}\|_{L^2(B_{8-n/4})} + \sum_{k=0}^{n-1} \|\mathbf{P}_{v,k}\|_{L^2(B_{8-n/4})}$$

$$+ \|\omega\|^2_{L^3(B_{8-n/4})} \|\nabla^\alpha v\|^2_{L^6(B_{8-n/4})} + \frac{1}{\varepsilon} \|\nabla^\alpha \omega(t)\|^2_{L^2(B_{8-n/4})}$$

has the integral $\int_{-8^{-2n}/16}^{0} \Phi(t)\mathrm{d}t \leq C_n$. Finally, Grönwall is inequality gives

$$\|\nabla^\alpha \omega\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{8^{-n-1}})} \leq C_{n+1}.$$

<div align="right">□</div>

6.3. Proof of the Local Theorem

Proof of the Local Theorem 2. First, Proposition 1 gives

$$\|v\|_{\mathcal{E}(Q_1)} \leq \eta,$$

where η can be chosen arbitrarily small if we pick η_1 small. Next, by Proposition 2, we know that

$$\|v\|_{L^\infty(Q_{\frac{1}{2}})} \leq 1.$$

These two steps implies (47). As for (48), curl $w = \varpi$ in B_1, so we use interpolation in (37) to get that

$$\|\operatorname{curl} w\|_{L_t^2 L_x^{\frac{3}{2}}(Q_{\frac{1}{2}})} \leq \|\varpi\|_{L_t^2 L_x^{\frac{12}{7}}} \leq \|\varpi\|_{L_t^{p_1} L_x^{q_3}}^{\frac{1}{2}} \|\varpi\|_{L_t^{p_2} L_x^{q_4}}^{\frac{1}{2}} \leq \eta,$$

w is harmonic inside B_1, therefore

$$\|w\|_{L_t^{\frac{4}{3}} C_x^n(Q_{\frac{1}{2}})} \lesssim_n \|w\|_{L_t^{\frac{4}{3}} L_x^1(Q_1)} \leq \eta,$$

due to (36) and $p_1 \geq \frac{4}{3}$. Therefore, we can use Proposition 3 to obtain

$$\|\omega\|_{\mathcal{E}(Q_{\frac{1}{8}})} \leq C.$$

The next step is to use Proposition 4 iteratively. Suppose that for $n \geq 1$ we know that

$$\|\nabla^{n-1} \omega\|_{\mathcal{E}(Q_{8^{-n}})} \leq c_n,$$

which is equivalent to (57). Let φ_n and $\varphi_n^\sharp$ be a pair of smooth spatial cut-off functions, with

$$\mathbf{1}_{B_{\frac{1}{8^n+4}}} \leq \varphi_n \leq \mathbf{1}_{B_{\frac{1}{8^n+3}}}, \qquad \mathbf{1}_{B_{\frac{1}{8^n+2}}} \leq \varphi_n^\sharp \leq \mathbf{1}_{B_{\frac{1}{8^n+1}}},$$

and set

$$v_n := -\operatorname{curl} \varphi_n^\sharp \Delta^{-1} \varphi_n \omega, \qquad w_n = \varphi_n u - v_n.$$

On the one hand, ∇v_n is a Riesz transform of $\varphi_n \omega$ up to lower order terms, so by the boundedness of Riesz transform we know that

$$\|\nabla^{n+1} v_n\|_{L^2(Q_{8^{-n}/2})} \leq \|\nabla^n \omega\|_{L^2(Q_{8^{-n}})} \leq c_{n-1}.$$

On the other hand, we have similar boundedness estimates to those of Proposition 2 as before, so

$$\|v_n\|_{L^\infty(Q_{8^{-n}/2})} \le 1.$$

w_n is harmonic in $B_{\frac{1}{8^n+4}}$, so we also have

$$\|w_n\|_{L_t^{\frac{4}{3}} C_x^{n+1}(Q_{8^{-n}/2})} \lesssim_n \|w_n\|_{L_t^{\frac{4}{3}} L_x^1(Q_{\frac{1}{8^n+4}})} \le \eta.$$

Therefore, by Proposition 4,

$$\|\nabla^n \omega\|_{\mathcal{E}(Q_{8^{-n-1}})} \le C_n.$$

By induction, we have

$$\|\nabla^n \omega\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(Q_{8^{-n-1}})} \le C_n$$

for any n. By Sobolev embedding, this implies, for any n, that

$$\|\nabla^n \omega\|_{L^\infty(Q_{8^{-n-3}})} \lesssim \|\nabla^n \omega\|_{L_t^\infty L_x^2(Q_{8^{-n-3}})} + \|\nabla^{n+2} \omega\|_{L_t^\infty L_x^2(Q_{8^{-n-3}})} \le C_n.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

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A. Suitability of Solutions

Theorem 5. *Let u be a suitable weak solution to the Navier–Stokes equation in* $\mathbb{R}^3$. *That is,* $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ *solves the following equation in the sense of distribution:*

$$\partial_t u + u \cdot \nabla u + \nabla P = \Delta u, \qquad \mathrm{div}\, u = 0, \qquad\qquad (60)$$

where P is the pressure, and u satisfies the following local energy inequality in the sense of distribution:

$$\partial_t \frac{|u|^2}{2} + \mathrm{div}\left(u \left(\frac{|u|^2}{2} + P \right) \right) + |\nabla u|^2 \le \Delta \frac{|u|^2}{2}. \qquad\qquad (61)$$

Suppose $v \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ *is compactly supported in space and solves the following equation,*

$$\partial_t v + \omega \times v + \nabla \mathbf{R}(u \otimes v) = \Delta v + \mathbf{C}_v, \qquad \mathrm{div}\, v = 0 \qquad\qquad (62)$$

where $\omega = \operatorname{curl} u$ *is the vorticity,* $\mathbf{C}_v \in L_t^1 L_{\mathrm{loc},x}^2 + L_t^2 L_{\mathrm{loc},x}^{\frac{6}{5}}$ *is a force term, and*

$$\mathbf{R} = \frac{1}{2}\operatorname{tr} - \Delta^{-1}\operatorname{div}\operatorname{div}$$

is a symmetric Riesz operator. Moreover, suppose v *differs from* φu *by*

$$\varphi u - v = w \in L_t^\infty H_x^1 \cap L_t^2 H_x^2$$

for some fixed $\varphi \in C_c^\infty(\mathbb{R}^3)$. *Then* v *satisfies the following local energy inequality:*

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}(v\mathbf{R}(u \otimes v)) + |\nabla v|^2 \leq \Delta \frac{|v|^2}{2} + v \cdot \mathbf{C}_v. \tag{63}$$

Proof. It is well-known that the pressure P can be recovered from u by

$$P = -\Delta^{-1}\operatorname{div}\operatorname{div}(u \otimes u).$$

Since

$$u \cdot \nabla u + \nabla P = \nabla \frac{|u|^2}{2} + \omega \times u - \nabla \Delta^{-1}\operatorname{div}\operatorname{div}(u \otimes u)$$
$$= \omega \times u + \nabla \mathbf{R}(u \otimes u),$$

the Navier–Stokes equation (60) can be rewritten as

$$\partial_t u + \omega \times u + \nabla \mathbf{R}(u \otimes u) = \Delta u, \tag{64}$$

and local energy inequality (61) can be rewritten as

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(u\mathbf{R}(u \otimes u)) + |\nabla u|^2 \leq \Delta \frac{|u|^2}{2}. \tag{65}$$

First, multiply (64) by φ, to get

$$\partial_t \varphi u + \omega \times \varphi u + \nabla \mathbf{R}(u \otimes \varphi u) = \Delta(\varphi u) + [\nabla \mathbf{R}, \varphi](u \otimes u) + [\varphi, \Delta]u.$$

Denote

$$\mathbf{C}_u = [\nabla \mathbf{R}, \varphi](u \otimes u) + [\varphi, \Delta]u$$

for these commutator terms. Subtracting the equation of v from this equation of φu, we will have the equation for w. In summary,

$$\partial_t \varphi u + \omega \times \varphi u + \nabla \mathbf{R}(u \otimes \varphi u) = \Delta(\varphi u) + \mathbf{C}_u, \tag{66}$$
$$\partial_t v + \omega \times v + \nabla \mathbf{R}(u \otimes v) = \Delta v + \mathbf{C}_v, \tag{67}$$
$$\partial_t w + \omega \times w + \nabla \mathbf{R}(u \otimes w) = \Delta w + \mathbf{C}_u - \mathbf{C}_v. \tag{68}$$

Recall from [22] that $\Delta u \in L_{\text{loc}(t,x)}^{\frac{4}{3}-\varepsilon}$. Since $\Delta w \in L_{t,x}^2$, we have $\Delta v \in L_{\text{loc}(t,x)}^{\frac{4}{3}-\varepsilon}$.
Moreover, $\mathbf{C}_u, \mathbf{C}_v \in L_t^1 L_{\text{loc},x}^2 + L_t^2 L_{\text{loc},x}^{\frac{6}{5}}$, and $\varphi u, v \in \mathcal{E}$ are compactly supported.
Therefore, we can multiply (66) and (67) by w, and (68) by φu and v, to get

$$w \cdot \partial_t(\varphi u) + w \cdot \omega \times \varphi u + w \cdot \nabla \mathbf{R}(u \otimes \varphi u) = w \cdot \Delta(\varphi u) + w \cdot \mathbf{C}_u, \qquad (69)$$

$$w \cdot \partial_t v + w \cdot \omega \times v + w \cdot \nabla \mathbf{R}(u \otimes v) = w \cdot \Delta v + w \cdot \mathbf{C}_v \qquad (70)$$

$$\varphi u \cdot \partial_t w + \varphi u \cdot \omega \times w + \varphi u \cdot \nabla \mathbf{R}(u \otimes w) = \varphi u \cdot \Delta w + \varphi u \cdot (\mathbf{C}_u - \mathbf{C}_v). \qquad (71)$$

$$v \cdot \partial_t w + v \cdot \omega \times w + v \cdot \nabla \mathbf{R}(u \otimes w) = v \cdot \Delta w + v \cdot (\mathbf{C}_u - \mathbf{C}_v). \quad (72)$$

Now take the sum of (69)–(72). The ∂_t terms are

$$\varphi u \cdot \partial_t w + w \cdot \partial_t(\varphi u) + v \cdot \partial_t w + w \cdot \partial_t v$$
$$= \partial_t(\varphi u \cdot w) + \partial_t(w \cdot v)$$
$$= \partial_t(|\varphi u|^2 - |v|^2).$$

The $\omega \times$ terms are

$$w \cdot \omega \times \varphi u + \varphi u \cdot \omega \times w + w \cdot \omega \times v + v \cdot \omega \times w = 0.$$

The $\nabla \mathbf{R}$ terms are

$$w \cdot \nabla \mathbf{R}(u \otimes \varphi u) + v \cdot \nabla \mathbf{R}(u \otimes w) + \varphi u \cdot \nabla \mathbf{R}(u \otimes w) + w \cdot \nabla \mathbf{R}(u \otimes v)$$
$$= \operatorname{div}(w \mathbf{R}(u \otimes \varphi u)) + \operatorname{div}(v \mathbf{R}(u \otimes w))$$
$$\quad + \operatorname{div}(\varphi u \mathbf{R}(u \otimes w)) + \operatorname{div}(w \mathbf{R}(u \otimes v))$$
$$\quad - \operatorname{div}(w) \nabla \mathbf{R}(u \otimes \varphi u) - \operatorname{div}(v) \nabla \mathbf{R}(u \otimes w)$$
$$\quad - \operatorname{div}(\varphi u) \nabla \mathbf{R}(u \otimes w) - \operatorname{div}(\varphi) \nabla \mathbf{R}(u \otimes v)$$
$$= 2 \operatorname{div}(\varphi u \mathbf{R}(u \otimes \varphi u) - v \mathbf{R}(u \otimes v))$$
$$\quad - (u \cdot \nabla \varphi)(\nabla \mathbf{R}(u \otimes \varphi u) + \nabla \mathbf{R}(u \otimes w) + \nabla \mathbf{R}(u \otimes v))$$
$$= 2 \operatorname{div}(\varphi u \mathbf{R}(u \otimes \varphi u) - v \mathbf{R}(u \otimes v)) - 2(u \cdot \nabla \varphi) \mathbf{R}(u \otimes \varphi u).$$

Here we use $\operatorname{div} v = 0$, $\operatorname{div}(\varphi u) = \operatorname{div} w = u \cdot \nabla \varphi$. The Δ terms are

$$\varphi u \cdot \Delta w + w \cdot \Delta(\varphi u) + v \cdot \Delta w + w \cdot \Delta v$$
$$= \Delta(u \cdot w) - 2\nabla(\varphi u) : \nabla w + \Delta(v \cdot w) - 2\nabla v : \nabla w$$
$$= \Delta(|\varphi u|^2 - |v|^2) - 2(|\nabla(\varphi u)|^2 - |\nabla v|^2).$$

The commutator terms are

$$w \cdot \mathbf{C}_u + \varphi u \cdot (\mathbf{C}_u - \mathbf{C}_v) + w \cdot \mathbf{C}_v + v \cdot (\mathbf{C}_u - \mathbf{C}_v) = 2\varphi u \cdot \mathbf{C}_u - 2v \cdot \mathbf{C}_v.$$

In summary, half the sum of these four identities (69)–(72) gives

$$\partial_t \frac{|\varphi u|^2 - |v|^2}{2} + \operatorname{div}(\varphi u \mathbf{R}(u \otimes \varphi u) - v \mathbf{R}(u \otimes v)) + |\nabla(\varphi u)|^2 - |\nabla v|^2$$
$$= \Delta \frac{|\varphi u|^2 - |v|^2}{2} + \varphi u \cdot \mathbf{C}_u - v \cdot \mathbf{C}_v + (u \cdot \nabla \varphi) \mathbf{R}(\varphi u \otimes u). \qquad (73)$$

Next, multiply local energy inequality of u (65) by φ^2. Then

$$\partial_t \frac{|\varphi u|^2}{2} + |\varphi \nabla u|^2 + \operatorname{div}\left(\varphi^2 u \mathbf{R}(u \otimes u)\right)$$

$$\leq \Delta \frac{|\varphi u|^2}{2} + [\varphi^2, \Delta]\frac{|u|^2}{2} + [\operatorname{div}, \varphi^2]\,(u\mathbf{R}(u \otimes u)),$$

$$\partial_t \frac{|\varphi u|^2}{2} + |\nabla(\varphi u)|^2 + \operatorname{div}\left(\varphi u \mathbf{R}(u \otimes \varphi u)\right)$$

$$\leq \Delta \frac{|\varphi u|^2}{2} + [\varphi^2, \Delta]\frac{|u|^2}{2} + |u \otimes \nabla\varphi|^2 + 2(u \otimes \nabla\varphi) : (\varphi\nabla u)$$

$$+ [\operatorname{div}, \varphi^2]\,(u\mathbf{R}(u \otimes u)) + \operatorname{div}(\varphi u[\mathbf{R}, \varphi](u \otimes u)). \tag{74}$$

The quadratic commutator terms in (74) are

$$[\varphi^2, \Delta]\frac{|u|^2}{2} + |u \otimes \nabla\varphi|^2 + 2(u \otimes \nabla\varphi) : (\varphi\nabla u)$$

$$= [\varphi^2, \Delta]\frac{|u|^2}{2} + |u|^2|\nabla\varphi|^2 + 2\nabla\varphi \cdot \varphi\nabla u \cdot u$$

$$= -2\nabla(\varphi^2) \cdot \nabla\frac{|u|^2}{2} - \Delta(\varphi^2)\frac{|u|^2}{2} + |u|^2|\nabla\varphi|^2 + 2\nabla\varphi \cdot \nabla u \cdot \varphi u$$

$$= -4\varphi\nabla\varphi \cdot \nabla\frac{|u|^2}{2} - \frac{1}{2}\Delta(\varphi^2)|u|^2 + |u|^2|\nabla\varphi|^2 + 2\nabla\varphi \cdot \nabla u \cdot \varphi u$$

$$= -2\varphi\nabla\varphi \cdot \nabla u \cdot u - \varphi\Delta\varphi|u|^2$$

$$= \varphi u \cdot (-2\nabla\varphi \cdot \nabla u - (\Delta\varphi)u)$$

$$= \varphi u \cdot [\varphi, \Delta]u.$$

The cubic commutator terms in (74) are

$$[\operatorname{div}, \varphi^2]\,(u\mathbf{R}(u \otimes u)) + \operatorname{div}(\varphi u[\mathbf{R}, \varphi](u \otimes u))$$

$$= 2\varphi\nabla\varphi \cdot u\mathbf{R}(u \otimes u) + \varphi u \cdot \nabla[\mathbf{R}, \varphi](u \otimes u) + \operatorname{div}(\varphi u)[\mathbf{R}, \varphi](u \otimes u)$$

$$= 2\varphi(u \cdot \nabla\varphi)\mathbf{R}(u \otimes u) + \varphi u \cdot \nabla[\mathbf{R}, \varphi](u \otimes u) + (u \cdot \nabla\varphi)[\mathbf{R}, \varphi](u \otimes u)$$

$$= 2\varphi(u \cdot \nabla\varphi)\mathbf{R}(u \otimes u) + \varphi u \cdot \nabla[\mathbf{R}, \varphi](u \otimes u)$$

$$\quad + (u \cdot \nabla\varphi)\mathbf{R}(\varphi u \otimes u) - (u \cdot \nabla\varphi)\varphi\mathbf{R}(u \otimes u)$$

$$= \varphi u \cdot \nabla\varphi\mathbf{R}(u \otimes u) + \varphi u \cdot \nabla[\mathbf{R}, \varphi](u \otimes u) + (u \cdot \nabla\varphi)\mathbf{R}(\varphi u \otimes u)$$

$$= \varphi u \cdot [\nabla, \varphi]\mathbf{R}(u \otimes u) + \varphi u \cdot \nabla[\mathbf{R}, \varphi](u \otimes u) + (u \cdot \nabla\varphi)\mathbf{R}(\varphi u \otimes u)$$

$$= \varphi u \cdot ([\nabla, \varphi]\mathbf{R} - \nabla[\varphi, \mathbf{R}])\,(u \otimes u) + (u \cdot \nabla\varphi)\mathbf{R}(\varphi u \otimes u)$$

$$= \varphi u \cdot [\nabla\mathbf{R}, \varphi](u \otimes u) + (u \cdot \nabla\varphi)\mathbf{R}(\varphi u \otimes u).$$

Therefore, local energy inequality for φu can be simplified as

$$\partial_t \frac{|\varphi u|^2}{2} + |\nabla(\varphi u)|^2 + \operatorname{div}\left(\varphi u \mathbf{R}(u \otimes \varphi u)\right)$$

$$\leq \Delta \frac{|\varphi u|^2}{2} + \varphi u \cdot \mathbf{C}_u + (u \cdot \nabla\varphi)\mathbf{R}(\varphi u \otimes u).$$

Subtracting (73) from this, we obtain (63). □

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