Arch. Rational Mech. Anal. 240 (2021) 1487–1519
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-021-01640-9

Uniqueness of Plane Stationary Navier–Stokes Flow Past an Obstacle

Mikhail Korobkov & Xiao Ren

Communicated by P. Constantin

Abstract

We study the exterior problem for stationary Navier–Stokes equations in two dimensions describing a viscous incompressible fluid flowing past an obstacle. It is shown that, at small Reynolds numbers, the classical solutions constructed by Finn and Smith are unique in the class of D-solutions (that is, solutions with finite Dirichlet integral). No additional symmetry or decay assumptions are required. This result answers a long-standing open problem. In the proofs, we developed the ideas of the classical Ch. Amick paper (Acta Math. 1988).

1. Introduction

Let $\mathcal{E}$ be an exterior domain in $\mathbb{R}^2$, that is, $\mathcal{E} = \mathbb{R}^2 \setminus \overline{\mathcal{B}}$, where $\mathcal{B}$ is a bounded open set with sufficiently smooth boundary.[1] To be definite, we assume that the origin is $0 \in \mathcal{B}$. Without loss of generality, we also let $\mathbb{R}^2 \setminus B_1 \subset \mathcal{E}$, where B_1 is the open unit disk centered at 0. This paper studies the stationary Navier–Stokes equations in $\mathcal{E}$ with a Dirichlet boundary condition on $\partial\mathcal{E}$ and nonzero prescribed velocity at spatial infinity, that is,

$$\begin{cases} \Delta \mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla p = 0, \\ \nabla\cdot\mathbf{w} = 0, \\ \mathbf{w}|_{\partial\mathcal{E}} = 0, \\ \mathbf{w}(z) \to \mathbf{w}_\infty = \lambda\mathbf{e}_1 \ \text{ as } \ |z|\to\infty. \end{cases} \tag{1.1$_\lambda$}$$

The parameter $\lambda > 0$ will be referred to as the Reynolds number. Here $\mathbf{e}_1 = (1, 0)$ is the unit vector along x-axis. Physically, the system (1.1$_\lambda$) describes the stationary motion of a viscous incompressible fluid flowing past a rigid cylindrical body.

---

[1] $\partial\mathcal{E}$ being of class $C^{2+\alpha}$, $\alpha > 0$ would be sufficient. Such regularity is required in [7] for the construction of Finn-Smith solutions.

This problem has its origins in the 19th century, starting with the classical paper of Stokes [31], where the famous paradox of his name was discovered, that is, that the corresponding linearized system

$$\begin{cases} \Delta \mathbf{w} - \nabla p = 0, \\ \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}|_{\partial \mathcal{E}} = 0, \\ \mathbf{w}(z) \to \mathbf{e}_1 \ \text{ as } \ |z| \to \infty \end{cases} \tag{1.1}$$

has no solution.[2] The mathematical nature of the Stokes paradox was the subject of many investigations, see, for example, [4,26].

The celebrated J.Leray's paper [23] can be considered as a landmark point in the study of the *nonlinear* problem (1.1_λ). There, among many other results, Leray suggested an elegant approach which was called "*the invading domains method*". Denoting by $\mathbf{w}_k$ the solution to the problem

$$\begin{cases} -\Delta \mathbf{w}_k + (\mathbf{w}_k \cdot \nabla)\mathbf{w}_k + \nabla p_k = \mathbf{0} & \text{in } \mathcal{E} \cap B_{R_k}, \\ \operatorname{div} \mathbf{w}_k = 0 & \text{in } \mathcal{E} \cap B_{R_k}, \\ \mathbf{w}_k = \mathbf{0} & \text{on } \partial \mathcal{E}, \\ \mathbf{w}_k = \mathbf{w}_\infty & \text{for } |z| = R_k \end{cases} \tag{1.2}$$

on the intersection of $\mathcal{E}$ with the disk B_{R_k} of radius $R_k \geqq k (\gg R_0)$, whose existence he proved before, Leray showed that the sequence $\mathbf{w}_k$ satisfies the estimate

$$\int_{\mathcal{E} \cap B_{R_k}} |\nabla \mathbf{w}_k|^2 \leqq c \tag{1.3}$$

for some positive constant c independent of k. Hence, he observed that it is possible to extract a subsequence $\mathbf{w}_{k_n}$ which weakly converges to a solution $\mathbf{w}_L$ of problem $(1.1_\lambda)_{1,2,3}$ with $\int_{\mathcal{E}} |\nabla \mathbf{w}_L|^2 < +\infty$.[3] This solution was later called *Leray's solution* (see, *e.g.*, [1]).

This achievement of Leray immediately raises two crucial questions:

(1) Is the constructed solution $\mathbf{w}_L$ nontrivial, that is, can we exclude the identity $\mathbf{w}_L \equiv \mathbf{0}$?

This question is rather natural, since if we apply the Leray "invading domains" method to the corresponding Stokes system (1.1) (or even to the simplest Laplace equation), then the limiting solution will be identically zero.

---

[2] Stokes himself gave the following explanation: *the pressure of the cylinder on the fluid continuously tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of the fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on* ([31], p. 65).

[3] This convergence is uniform on every bounded set.

(2) If $\mathbf{w}_L$ is nontrivial, what can we say about its behavior at infinity? Namely, can we guarantee the desired convergence

$$\mathbf{w}_L(z) \to \mathbf{w}_\infty \text{ as } |z| \to \infty \quad ? \tag{1.4}$$

Many useful properties of Leray solutions were discovered in the classical papers by D. Gilbarg and H.F. Weinberger [11,12]. Further more, in the very deep paper [1] Ch. Amick proved, under an additional axial symmetry assumption, that the Leray solutions are nontrivial and they have some uniform limits at infinity, that is, there exists a constant vector $\mathbf{w}_0 \in \mathbb{R}^2$ such that

$$\mathbf{w}_L(z) \to \mathbf{w}_0 \text{ as } |z| \to \infty. \tag{1.5}$$

Very recently, in the joint papers by Korobkov–Pileckas–Russo [18–20], this additional symmetry assumption was removed, that is, they proved that Leray solutions are always nontrivial and have some uniform limit at infinity (1.5).

Nevertheless, despite the classical papers and the recent progress, the fundamental question, whether or not the Leray solutions satisfy the limiting condition (1.4), that is, whether the equality $\mathbf{w}_0 = \mathbf{w}_\infty$ holds, is still open. In other words, it is not clear whether one can construct the solution to the initial problem (1.1_λ) by Leray's method.

The brilliant success (for the small Reynolds numbers) was reached in 1967 by another approach. Namely, using the integral representations with the fundamental solution to Oseen linear system (see below (1.12)) and a contraction mapping argument in some suitable Banach spaces, R. Finn and D.R. Smith proved [7] the following remarkable result:

Theorem 1. (See [7] Corollary 4.2 and Theorem 7.1) *There exist constants λ_0, M_0, $\varepsilon_0 > 0$ depending only on the geometry of $\partial\mathcal{E}$ such that, for any $0 < \lambda < \lambda_0$, there exists a smooth solution $\boldsymbol{w}_{FS}(z;\lambda)$ to (1.1_λ) in $\mathcal{E}$ satisfying the pointwise estimate*

$$|(\boldsymbol{w}_{FS} - \lambda \boldsymbol{e}_1)_i(z)| \leqq M_0 |\log\lambda|^{-1} \lambda h_i(\lambda z), \ i = 1, 2 \tag{1.6}$$

for all $z \in \mathcal{E}$. Moreover, if $\boldsymbol{w}$ is a smooth solution to (1.1_λ) in $\mathcal{E}$ satisfying

$$|(\boldsymbol{w} - \lambda \boldsymbol{e}_1)_i(z)| \leqq \varepsilon_0 \lambda h_i(\lambda z), \ i = 1, 2 \tag{1.7}$$

for all $z \in \mathcal{E}$, then $\boldsymbol{w} \equiv \boldsymbol{w}_{FS}$.

(In fact, their construction allows nonzero small boundary data $\mathbf{w} = \mathbf{a}$ on $\partial\mathcal{E}$ with $|\mathbf{a} - \mathbf{w}_\infty|$ small enough, even without zero total flux condition.)

Here the majorant functions $h_i(\xi)$ are taken as

$$0 < |\xi| \leqq 1: \quad h_i(\xi) = \log\frac{2}{|\xi|}, \ i = 1, 2, \tag{1.8}$$

$$|\xi| > 1: \quad \begin{cases} h_1(\xi) = |\xi|^{-\frac{1}{2}} \\ h_2(\xi) = |\xi|^{-\frac{1}{2}-\mu}, \end{cases} \tag{1.9}$$

with $0 < \mu < \frac{1}{2}$ chosen arbitrarily and then fixed. To be definite, we simply take $\mu = \frac{1}{4}$.

We briefly describe the approach taken by Finn and Smith [7]. (The reader may also consult Galdi [10] Section XII.5 for another approach based on Sobolev norms instead of pointwise bounds.) Let $\mathbf{v}(z) = \lambda^{-1}\mathbf{w}(z) - \mathbf{e}_1$. To find a desired solution $\mathbf{w}(z)$, it is equivalent to finding $\mathbf{v}$ as a solution to

$$\begin{cases} \Delta\mathbf{v} - \lambda\partial_1\mathbf{v} - \nabla q = \lambda\mathbf{v}\cdot\nabla\mathbf{v}, \\ \nabla\cdot\mathbf{v} = 0, \\ \mathbf{v}|_{\partial\mathcal{E}} = -\mathbf{e}_1, \\ \mathbf{v} \to 0 \ \text{ as } \ |z| \to \infty. \end{cases} \tag{1.10}$$

In turn, this is reduced to solving the integral equation

$$\mathbf{v}(z) = \mathbf{v}_\ell(z;\lambda) - \lambda\int_{\mathcal{E}}\left(\mathbf{v}(z')\cdot\nabla_{z'}\right)\mathbf{G}(z,z';\lambda)\cdot\mathbf{v}(z')\mathrm{d}x'\mathrm{d}y' =: T_\lambda\mathbf{v}, \tag{1.11}$$

where $\mathbf{G}$ is the Green tensor with Dirichlet boundary conditions for the linearized problem

$$\begin{cases} \Delta\mathbf{v} - \lambda\partial_1\mathbf{v} - \nabla q = 0, \\ \nabla\cdot\mathbf{v} = 0 \end{cases} \tag{1.12}$$

in $\mathcal{E}$ and $\mathbf{v}_\ell(z;\lambda)$ is the linear solution to (1.12) with the same boundary conditions as $\mathbf{v}$ (both $\mathbf{G}$ and $\mathbf{v}_\ell$ were constructed and studied in [6]). It is proved in [7] that, when λ is sufficiently small (so that a crucial smallness estimate on $\mathbf{v}_\ell$ is valid, see [7, Lemma 2.1]), T_λ is a contraction mapping for $\mathbf{v}$ belonging to a small ball centered at 0 in the Banach space

$$X_\lambda = \left\{\mathbf{v}\in C(\mathcal{E},\mathbb{R}^2)\ :\ \|\mathbf{v}\|_{X_\lambda} := \max_{z\in\mathcal{E};i=1,2}\frac{|\mathbf{v}_i(z)|}{h_i(\lambda z)} < \infty\right\}. \tag{1.13}$$

Then, using standard perturbative arguments, the existence and (local) uniqueness for fixed points of T_λ under the conditions of Theorem 1 can be obtained.

Despite many other efforts (see, e.g., [8,10,21]), the existence problem in (1.1_λ) at high Reynolds numbers (for arbitrary $\lambda > 0$) remains open. In the famous lecture by professor V.I. Yudovich, which he gave at the University of Cambridge and published in [32], this task was included in the list of the most important open problems in mathematical fluid mechanics.

Since the work [7], another interesting problem has appeared: whether uniqueness in Theorem 1 is true globally, that is, without assumption (1.7)? Note, that for the same flow problem in three dimensions, such a smallness condition is not needed to prove uniqueness, see, e.g., [5,21]. In two dimensions, the problem appears to be much more difficult, mainly because the Dirichlet energy alone is not sufficient to control the behavior of functions at infinity. As shown in [10] Section XII.2, the usual energy estimate argument would run into immediate difficulties when applied to prove uniqueness for (1.1_λ) in the unbounded domain $\mathcal{E}$. We also point out that, when the Reynolds number λ is large, uniqueness is, in general, not expected to hold for the Navier–Stokes equations.[4] When $\lambda = 0$, uniqueness of the trivial solution $\mathbf{w} = 0$ is conjectured by Amick in [1, p.99] and still remains an open problem.

It is well known (see, e.g., [7]), that every Finn–Smith solution has a finite Dirichlet integral, that is, it is a D-solution. The purpose of our article is to give a positive answer to the uniqueness problem for small Reynolds numbers in the class of D-solutions. The main result is as follows:

Theorem 2. *There exists a positive constant λ_1 depending only on the geometry of $\partial\mathcal{E}$ such that, for $0 < \lambda < \lambda_1$ and for arbitrary D-solution $\boldsymbol{w}$ to (1.1_λ), the identity $\boldsymbol{w}(z) \equiv \boldsymbol{w}_{FS}(z;\lambda)$ holds. Here $\boldsymbol{w}_{FS}(z;\lambda)$ is the Finn-Smith solution given by Theorem 1.*

We prove Theorem 2 by deriving the estimate (1.7) for arbitrary D-solution to (1.1_λ) when λ is sufficiently small, thus invoking the uniqueness statement in Theorem 1. More precisely, Theorem 2 is an immediate corollary of the following lemma:

Lemma 3. *There exist positive constants λ_1 and M_1 depending only on the geometry of $\partial\mathcal{E}$ such that, for $0 < \lambda < \lambda_1$ and for arbitrary D-solution $\boldsymbol{w}$ to (1.1_λ), we have*

$$|(\boldsymbol{w} - \lambda\boldsymbol{e}_1)_i(z)| \leqq M_1|\log\lambda|^{-\frac{1}{2}}\lambda h_i(\lambda z) \leqq \varepsilon_0\lambda h_i(\lambda z), \ \ i = 1, 2. \tag{1.14}$$

Here λ_0 and ε_0 are those given by Theorem 1.

Remark 4. In fact, we can prove Theorem 2 and Lemma 3 under more general boundary conditions for $\mathbf{w}$ on $\partial\mathcal{E}$. Namely, if $\mathbf{w}|_{\partial\mathcal{E}} = \lambda\mathbf{a}$ and $\mathbf{a}$ is a constant vector, then uniqueness as in Theorem 2 holds and our proof goes through with only minor modifications. See Remark 10 for further discussions.

Remark 5. The study of general D-solutions, that is, solutions to the Navier–Stokes system $(1.1_\lambda)_{1,2}$ with a finite Dirichlet integral in $\mathcal{E}$ without any a priori conditions at infinity, was initialized by the Leray paper [23].[5] Many useful and elegant properties of D-solutions were discovered in the classical papers [1,12]. Recently, based on these ideas, it was proved [18,19] that every D-solution is uniformly bounded and, moreover, it has a constant uniform limit at infinity.[6] Of course, any D-solution $\mathbf{w}$ to (1.1_λ) satisfies $\mathbf{w} \to \lambda\mathbf{e}_1 \neq 0$ at infinity by assumption. In the remarkable paper of L.I. Sazonov [27] it was proved that any D-solution to (1.1_λ) satisfies rather

[4] For example, in the recent paper [15] it was written: “The question of the uniqueness of weak solutions for small data is even more open in two-dimensional exterior domains... For two-dimensional exterior domains with nonempty boundary, we would a priori also expect the existence of infinitely many weak solutions parameterized by some parameter”.

[5] Physically, finiteness of the Dirichlet integral means that the total energy dissipation rate in the fluid is finite.

[6] This result is not trivial since in general the finiteness of the Dirichlet integral does not guarantee even the boundedness of the mapping, for example, the function $f(z) = \big(\ln(|z|)\big)^{\frac{1}{3}}$ has a finite Dirichlet integral in $\mathcal{E}$.

strong decay estimates at infinity, namely, it is *physically reasonable* (or *in the class of PR*) in the sense of [29, page 350]. In [29], Smith showed that the behavior of any PR-solution is essentially controlled at infinity by that of the fundamental Oseen solution (see also [9] and [10, Ch. XII]). In particular, using the results of [27,29], for any D-solution $\mathbf{w}$ to (1.1_λ) we have

$$|(\mathbf{w}-\lambda\mathbf{e}_1)_i(z)| \leqq M(\lambda,\mathbf{w})\lambda h_i(\lambda z), \;\; i=1,2 \tag{1.15}$$

for some constant $M(\lambda,\mathbf{w})$. Thus, the essence of Lemma 3 is that the dependence of M on $\mathbf{w}$ is removed and a smallness factor $|\log\lambda|^{-\frac{1}{2}}$ is achieved when λ is small.

Note, that some uniqueness results concerning the 2d exterior problem were obtained recently, see, e.g., [15,24,33,34], but in the quite different context and of quite different nature. Namely, in [24,33,34] it was considered the cases of zero limit at infinity under additional symmetry and smallness assumptions (including external force and energy inequality); and in [15] it was considered the case $\mathcal{E}=\mathbb{R}^2$ under additional assumptions of type (1.7) and in the presence of external force.

More detailed survey of results concerning boundary value problems for the stationary NS-system in plane exterior domains can be found, for example, in [10,13]. The subject is still a source of interest, as evidenced, *e.g.*, by the papers [14–16].

Now let us describe the main ideas and approaches of our paper. The proof of Lemma 3 starts with an interesting and very useful result stating that any D-solutions to (1.1_λ) have *extra* small Dirichlet energy when λ is small:

$$D_\lambda = \int_{\mathcal{E}} |\nabla\mathbf{w}|^2 \leqq C\varepsilon^2\lambda^2 \quad \text{with } \varepsilon = \frac{1}{\sqrt{|\log\lambda|}}. \tag{1.16}$$

The presence of ε is closely related to the Stokes paradox introduced earlier (see below Remark 10 for more detailed explanation of this fact). Note that similar logarithmic smallness was proved in [1, Theorem 22] for the Dirichlet energy of the Stokes solutions in bounded domains.

We now make a crucial observation: outside the critical circle $\{r \geqq \frac{1}{\lambda}\}$, pressure is uniformly small, that is,

$$|p(z)| \leqq C\varepsilon\lambda^2$$

(see Lemma 12). Also, we have to use a very elegant result of Amick [1] stating that Bernoulli pressure $p+\frac{\mathbf{w}^2}{2}-\frac{\lambda^2}{2}$ is increasing and decreasing along two regular vorticity level sets (curves) $\{\omega=0\}$ which are going from $\partial\mathcal{E}$ to infinity (see Lemma 13). The smallness of Dirichlet energy and pressure, along with the help of Amick's technique of working with level sets, gives us many initial estimates concerning the behavior of $\mathbf{w}$ outside (and near) the critical circle.

The estimates (1.14) are then considered separately, — inside and outside the critical circle $\{r=\frac{1}{\lambda}\}$. The inequalities inside the critical circle can be obtained by applying standard technique based on estimates for solutions to the Stokes system inside disks, and it is a kind of routine (see the Appendix II).

Outside the critical circle, our circumstance and the techniques developed in papers [1,12,18,20] allow to prove the uniform pointwise estimate

$$|\mathbf{w}(z) - (\lambda, 0)| \leqq C\varepsilon\lambda \tag{1.17}$$

(see Section 5, Step 2). The last estimates open the possibility to apply the technique of the Sazonov paper [27] on integral operators associated with the fundamental solution to the Oseen system in order to prove the estimate

$$|\mathbf{w}(z) - (\lambda, 0)| \leqq C\varepsilon\lambda|\lambda z|^{-\frac{1}{4}-\delta} \qquad \forall |z| \geqq \frac{1}{\lambda},$$

which coincides with the standard inequality using in the definition of the PR-solutions. Finally the well-known results of Smith's classical paper [29] allow to derive the required estimates (1.14) and to finish the proofs.

The rest of the paper is organized as follows. In Section 2, we introduce some frequently used notations and lemmas. A brief introduction to two-dimensional Oseen system is also included. In Section 3, a crucial smallness of Dirichlet energy is proved which has various interesting consequences. In Sections 4, 5 we prove pointwise bounds for $\mathbf{w}$ inside and outside the critical circle respectively. The final proof of the main result is summarized at the end of Section 5. For a reader's convenience, we moved the proof of the uniform estimates (1.17) (where the subtle real analysis arguments from [1] are used) into Appendix I.

2. Notations and Preliminaries

2.1. Notational Conventions

We work in the two dimensional setting and z will be a general point with Cartesian coordinates (x, y) in the plane. By *a domain* we mean an open connected set.

Let $\Omega_{r_1,r_2} := \{z : r_1 < |z| < r_2\}$ and $\mathcal{E}_r := \{z : |z| > r\}$. S_r will stand for the circle $\{z : |z| = r\}$ and B_r for the open disk $\{z : |z| < r\}$. We will often use $\bar{f}(r)$ to denote the average of a function f on the circle S_r, that is,

$$\bar{f}(r) := \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\mathrm{d}\theta.$$

We use standard notations for Sobolev spaces $W^{k,q}(\Omega)$, where $k \in \mathbb{N}$, $q \in [1, +\infty]$. In our proof we do not distinguish the function spaces for scalar valued or vector valued functions, since it will be clear from the context which one we mean.

2.2. Properties of D-Solutions

We present some standard facts about the behavior of general D-functions (=functions with bounded Dirichlet integral). For the proof of the next Lemma, see, e.g., section 2 in [18].

Lemma 6. *Let $f \in W^{1,2}_{\mathrm{loc}(\Omega)}$ and assume that*

$$D_f := \int_{\Omega_{r_1,r_2}} |\nabla f|^2 dxdy < \infty$$

for some ring $\Omega_{r_1,r_2} = \{z \in \mathbb{R}^2 : 0 < r_1 < |z| < r_2\} \subset \Omega$. Then we have

$$|\bar{f}(r_2) - \bar{f}(r_1)| \leqq \frac{1}{\sqrt{2\pi}} \left(D_f \ln \frac{r_2}{r_1} \right)^{\frac{1}{2}}. \tag{2.1}$$

Further more, if $r_2 < \beta r_1$, then there exists a number $r \in [r_1, r_2]$ such that

$$\sup_{|z|=r} |f(z) - \bar{f}(r)| \leqq C_\beta D_f^{\frac{1}{2}}, \tag{2.2}$$

with constant C_β depending on β only.

The circles S_r in Lemma 6 will often be called *good circles*.

Next, we present an elegant lemma from [12, Theorem 4, page 399] that allows us to control the direction of the averaged velocity on circles for D-solutions to the Navier–Stokes equations.

Lemma 7. ([12]) *Let $\boldsymbol{w}$ be a D-solution to the Navier–Stokes equations in some ring $\Omega_{r_1,r_2} = \{z \in \mathbb{R}^2 : 0 < r_1 < |z| < r_2\}$. Denote by $\bar{\boldsymbol{w}}$ the average of $\boldsymbol{w}$ over the circle S_r and let $\varphi(r)$ be the argument of the complex number associated with the vector $\bar{\boldsymbol{w}}(r) = (\bar{w}_1(r), \bar{w}_2(r))$, that is, $\bar{\boldsymbol{w}}(r) = |\bar{\boldsymbol{w}}(r)|\,(\cos\varphi(r), \sin\varphi(r))$. Assume also that $|\bar{\boldsymbol{w}}(r)| \geqq \sigma > 0$ for some constant σ and for all $r \in (r_1, r_2)$. Then the estimate*

$$\sup_{r_1 < \rho_1 \leqq \rho_2 < r_2} |\varphi(\rho_2) - \varphi(\rho_1)| \leqq \frac{1}{4\pi\sigma^2} \int_{\Omega_{r_1,r_2}} \left(\frac{1}{r} |\nabla \omega| + |\nabla \boldsymbol{w}|^2 \right) \tag{2.3}$$

holds. Here, $\omega = \partial_2 w_1 - \partial_1 w_2$ is the vorticity.

2.3. The Stokes Estimates

We recall the following classical local regularity estimate for the linear Stokes system (for the proof, see, for instance, [10, Theorem IV.4.1 and Remark IV.4.1]):

Lemma 8. *Let $\boldsymbol{w}_S$ be a local solution in B_1 to the Stokes system*

$$\begin{cases} \Delta \boldsymbol{w}_S - \nabla p_S = \boldsymbol{f}_S, \\ \nabla \cdot \boldsymbol{w}_S = 0. \end{cases} \tag{2.4}$$

Then there holds the following regularity estimates for $k = 0, 1, 2, \cdots$ and $1 < s < \infty$:

$$\|\nabla^{k+2}\boldsymbol{w}_S\|_{L^s(B_{\frac{1}{2}})} + \|\nabla^{k+1}p_S\|_{L^s(B_{\frac{1}{2}})} \leqq C(k,s)\left(\|\boldsymbol{w}_S\|_{W^{1,s}(B_1)} + \|\boldsymbol{f}_S\|_{W^{k,s}(B_1)}\right). \tag{2.5}$$

Moreover, the domains $B_{\frac{1}{2}}$ can be replaced by $B^+_{\frac{1}{2}} = B_{\frac{1}{2}} \cap \mathbb{R}^2_+$ if we assume that $\boldsymbol{w}_S = 0$ on $\partial\mathbb{R}^2_+ \cap \partial B^+_{\frac{1}{2}}$. Here $\mathbb{R}^2_+$ is the upper half plane $\{(x,y) : y > 0\}$.

It follows from this lemma and standard bootstrapping arguments that D-solutions to (1.1_λ) are locally smooth.

2.4. The Oseen System

For convenience of our presentation in Section 5, here we summarize some known results on the Oseen system (1.12) in two dimensions. The fundamental solution of the Ossen system $(\mathbf{E}, \mathbf{e})$, introduced in [25], consists of a symmetric tensor of rank two E_{ij} and a vector e_j, such that

$$\Delta E_{ij} - \partial_1 E_{ij} - \partial_i e_j = \delta_{ij}\delta_0, \qquad \sum_{i=1,2} \partial_i E_{ij} = 0, \tag{2.6}$$

where $i, j = 1, 2$ and δ_0 is the delta function supported at the origin. Explicitly, $(\mathbf{E}, \mathbf{e})$ are given by

$$\mathbf{E} = \begin{bmatrix} \partial_1(H+L) - L & \partial_2(H+L) \\ \partial_2(H+L) & -\partial_1(H+L) \end{bmatrix}, \quad \mathbf{e} = -\nabla H \tag{2.7}$$

where $\Delta H = \delta_0$ and $-\Delta L + \partial_1 L = \delta_0$. More explicitly, H and L are given by

$$H = \frac{1}{2\pi}\ln r, \quad L = \frac{1}{2\pi}e^{r\cos\theta/2}K_0(r/2) \tag{2.8}$$

where K_0 denotes the modified Bessel function of the second kind. Asymptotically, it holds that

$$K_0(\rho) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{\rho^{1/2}} + O\left(\frac{1}{\rho^{3/2}}\right)\right)e^{-\rho} \tag{2.9}$$

as $\rho \to \infty$. As a consequence, E_{11} exhibits a parabolic wake region $\{(x,y) : x \geqq 0, |y| \leqq \sqrt{x}\}$ in which the decay at infinity is slower than outside. It is also known that near the singularity of E at $z = 0$, it holds that

$$E_{ij}(z) = -\frac{1}{4\pi}\left(\delta_{ij}\ln\frac{1}{r} + \frac{z_i z_j}{r^2}\right) + o(1) \tag{2.10}$$

as $r = |z| \to 0$. Here $z_1 = x$, $z_2 = y$. The Fourier transform of $\mathbf{E}$ is given by

$$\hat{E}_{ij}(\xi) = \frac{\xi_i \xi_j - |\xi|^2 \delta_{ij}}{|\xi|^2(|\xi|^2 + i\xi_1)}. \tag{2.11}$$

More detailed asymptotic behavior and summability of (E, e) are summarized in [10, Section VII.3]. Next, we write $\mathbf{T}(\mathbf{u}, p)$ for the stess tensor

$$\mathbf{T}(\mathbf{u}, p) = \nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathsf{T}} - p\mathbf{I}. \tag{2.12}$$

It is straightforward to check the following Green identity for the Oseen operator using integration by parts:

$$\begin{aligned} &\int_\Omega (\nabla \cdot \mathbf{T}(\mathbf{u}, p) + \partial_1 \mathbf{u}) \cdot \mathbf{v} - \int_\Omega (\nabla \cdot \mathbf{T}(\mathbf{v}, q) - \partial_1 \mathbf{v}) \cdot \mathbf{u} \\ &\quad = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n} - \mathbf{u} \cdot \mathbf{T}(\mathbf{v}, q) \cdot \mathbf{n} + (\mathbf{u} \cdot \mathbf{v})(\mathbf{n} \cdot \mathbf{e}_1), \end{aligned} \tag{2.13}$$

for any pairs $(\mathbf{u}, p)$, $(\mathbf{v}, q)$ such that $\mathbf{u}, \mathbf{v}$ are smooth solenoidal vector fields and p, q are smooth scalar functions in $\bar{\Omega}$. Here $\mathbf{n} = (n_1, n_2)$ is the normal vector of $\partial\Omega$ pointing outward with respect to Ω. Suppose that $(\mathbf{v}, q)$ is a solution to the forced Oseen system

$$\begin{aligned} \Delta \mathbf{v} - \partial_1 \mathbf{v} - \nabla q &= \mathbf{f}, \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned}$$

in a bounded domain Ω with sufficiently smooth boundary. Using (2.13) for $(\mathbf{E}(z - \cdot), -\mathbf{e}(z - \cdot))$ and $(\mathbf{v}, q)$ we have

$$\begin{aligned} \mathbf{v}(z) &= \int_\Omega \mathbf{f}(z')\mathbf{E}(z - z') \\ &\quad - \int_{\partial\Omega} \mathbf{v}(z') \cdot \mathbf{T}_z(\mathbf{E}, \mathbf{e})(z - z') \cdot \mathbf{n} \\ &\quad - \int_{\partial\Omega} \mathbf{E}(z - z') \cdot \mathbf{T}(\mathbf{v}, q)(z') \cdot \mathbf{n} + \int_{\partial\Omega} (\mathbf{v}(z') \cdot \mathbf{E}(z - z'))(\mathbf{n} \cdot \mathbf{e}_1) \end{aligned} \tag{2.14}$$

for any $z \in \bar{\Omega}$. Here the integrals are taken over the variable z' and $\mathbf{T}_z$ means that we take derivative in z when defining $\mathbf{T}$. For any D-solution $\mathbf{w}$ to the Navier–Stokes equations in the exterior domain $\mathcal{E}$ tending to the limiting velocity $\mathbf{e}_1$ at infinity, the following representation formula holds in $\mathcal{E}$ for $\mathbf{v} = \mathbf{w} - \mathbf{e}_1$:

$$\begin{aligned} \mathbf{v}(z) &= -\int_{\mathcal{E}} (\mathbf{v} \cdot \nabla_{z'})\mathbf{E}(z - z') \cdot \mathbf{v} \\ &\quad - \int_{\partial\mathcal{E}} \mathbf{v}(z') \cdot \mathbf{T}_z(\mathbf{E}, \mathbf{e})(z - z') \cdot \mathbf{n} \\ &\quad - \int_{\partial\mathcal{E}} \mathbf{E}(z - z') \cdot \mathbf{T}(\mathbf{v}, q)(z') \cdot \mathbf{n} + \int_{\partial\mathcal{E}} (\mathbf{v}(z') \cdot \mathbf{E}(z - z'))(\mathbf{n} \cdot \mathbf{w}(z')). \end{aligned} \tag{2.15}$$

In [29], pointwise asymptotic behavior of $\mathbf{v}$ is obtained through this representation formula for solutions of class PR. For more details, we refer to [29, Theorem 5] .

3. Smallness of D_λ and the Corresponding Estimates for Pressure and Bernoulli Pressure

From here and for the subsequent three sections, we shall always let $\mathbf{w}$ *be an arbitrary D-solution to* (1.1_λ) *with* $\lambda > 0$.

Unless otherwise specified, we use C, M to denote absolute positive constants, or positive constants that depend only on $\partial\mathcal{E}$. *It is important that they do not depend on* λ. The specific value of such constants may change from line to line.

Denote the total Dirichlet energy of $\mathbf{w}$ by D_λ. Using the celebrated *reductio ad absurdum* argument of Leray [23] or the Leray-Hopf extension method [17], it is possible to prove that, for any $0 < \lambda < \Lambda$, the apriori bound $D_\lambda < M_3(\Lambda, \partial\mathcal{E})$ holds for some constant M_3. It turns out, however, that when λ is small, we can use a special solenoidal extension of the boundary value in (1.1_λ) to prove an extra smallness of the Dirichlet energy. Note, that such an extension was also used earlier in [1, Theorem 22] to study the Dirichlet energy of the Stokes solutions in bounded domains.

The main result of the section is the following lemma:

Lemma 9. *There exist constants* $0 < \lambda_2 < \frac{1}{2}$, $M_2 > 0$ *depending only on the geometry of* $\partial\mathcal{E}$ *such that, for* $0 < \lambda < \lambda_2$, *we have*

$$D_\lambda = \int_{\mathcal{E}} |\nabla \boldsymbol{w}|^2 \leqq \frac{M_2\lambda^2}{|\log \lambda|}. \tag{3.1}$$

Proof. Let $\tau \in C^\infty(\mathbb{R})$ with $\tau(r) = 0$ for $r \leqq \frac{1}{2}$ and $\tau(r) = 1$ for $r \geqq 1$. Define $\mu(r) = \tau(\log r / \log R)$, so that $\mu(r) = 0$ when $r \leqq \sqrt{R}$. The parameter $R \geqq 2$ has to be chosen later. Define a solenoidal vector field $\mathbf{A} = (A_1, A_2)$ by

$$A_1 = \partial_y(\lambda y \mu(|z|)), \quad A_2 = -\partial_x(\lambda y \mu(|z|)).$$

Such $\mathbf{A}$ clearly satisfies the same boundary condition as in (1.1_λ). Set $\tilde{\mathbf{w}} = \mathbf{w} - \mathbf{A}$, then $\tilde{\mathbf{w}} = 0$ on $\partial\mathcal{E}$ and $\tilde{\mathbf{w}} \to 0$ at ∞. By (1.1_λ), $\tilde{\mathbf{w}}$ satisfies the equation

$$-\Delta\tilde{\mathbf{w}} - \Delta\mathbf{A} + (\tilde{\mathbf{w}} + \mathbf{A}) \cdot \nabla\tilde{\mathbf{w}} + \tilde{\mathbf{w}} \cdot \nabla\mathbf{A} + \mathbf{A} \cdot \nabla\mathbf{A} + \nabla p = 0.$$

Multiplying this equation by $\tilde{\mathbf{w}}$ and integrating in $\mathcal{E}$ gives

$$\int_{\mathcal{E}} |\nabla\tilde{\mathbf{w}}|^2 + \int_{\mathcal{E}} \nabla\mathbf{A} \cdot \nabla\tilde{\mathbf{w}} + \int_{\mathcal{E}} (\tilde{\mathbf{w}} \cdot \nabla)\mathbf{A} \cdot \tilde{\mathbf{w}} + \int_{\mathcal{E}} (\mathbf{A} \cdot \nabla)\mathbf{A} \cdot \tilde{\mathbf{w}} = 0. \tag{3.2}$$

(Note that we actually first carry out this energy estimate in $\mathcal{E} \cap B_\rho$ and then let $\rho \to \infty$. Due to the asymptotic behaviour of $\mathbf{w}$ at infinity proved in [29, Theorem 5] (see estimates (1.15) from our Remark 5), the boundary integrals on S_ρ converge to 0, thus leading to (3.2).)

The constructed extension $\mathbf{A}$ satisfies the pointwise bounds

$$|\mathbf{A}| \leqq C\lambda, \quad |\nabla\mathbf{A}|^2 \leqq \frac{C\lambda^2}{|z|^2(\log R)^2}.$$

As a consequence, the Dirichlet energy of $\mathbf{A}$ is bounded by

$$\int_{\mathcal{E}} |\nabla \mathbf{A}|^2 \leqq \frac{C\lambda^2}{\log R}. \tag{3.3}$$

The second term in (3.2) can be treated with Hölder's inequality and (3.3), while the fourth term in (3.2) is estimated by

$$\begin{aligned} -\int (\mathbf{A}\cdot\nabla)\mathbf{A}\cdot\tilde{\mathbf{w}} &= \int (\mathbf{A}\cdot\nabla)\tilde{\mathbf{w}}\cdot(\mathbf{A}-\mathbf{w}_\infty) \\ &\leqq \frac{1}{4}\int |\nabla\tilde{\mathbf{w}}|^2 + \int_{r\leqq R} |A|^2|A-\mathbf{w}_\infty|^2 \\ &\leqq \frac{1}{4}\int |\nabla\tilde{\mathbf{w}}|^2 + C\lambda^4 R^2. \end{aligned} \tag{3.4}$$

The use of (3.3) gives a bound for the third term in (3.2),

$$\begin{aligned} -\int (\tilde{\mathbf{w}}\cdot\nabla\mathbf{A})\cdot\tilde{\mathbf{w}} &\leqq \left(\int_{r\leqq R} |\tilde{\mathbf{w}}|^4\right)^{\frac{1}{2}} \left(\int |\nabla\mathbf{A}|^2\right)^{\frac{1}{2}} \\ &\leqq \frac{C\lambda}{\sqrt{\log R}} \left(\int_{r\leqq R} |\tilde{\mathbf{w}}|^4\right)^{\frac{1}{2}} \\ &\leqq \frac{C\lambda}{\sqrt{\log R}}\cdot R^2 \left(\int |\nabla\tilde{\mathbf{w}}|^2\right). \end{aligned} \tag{3.5}$$

In the last step we used the following Sobolev type inequality in $\Omega_R = B_R \cap \mathcal{E}$, $R \geqq 2$ for functions f defined in Ω_R with the property that f vanishes on the boundary $\partial\mathcal{E}$:

$$\|f\|_{L^4(\Omega_R)} \leqq CR\|\nabla f\|_{L^2(\Omega_R)}.$$

This can be easily checked using Sobolev imbeddings for unit disk, scaling, and the inequality (2.1).

Now we let $R = \lambda^{-1/4}$. Choose $\lambda_2 < \frac{1}{2}$ sufficiently small such that for any $0 < \lambda < \lambda_2$ we have

$$\frac{CR^2\lambda}{\sqrt{\log R}} = \frac{2C\lambda^{1/2}}{\sqrt{|\log\lambda|}} < \frac{1}{4}, \qquad \lambda^4 R^2 = \lambda^{3\frac{1}{2}} < \frac{\lambda^2}{\sqrt{\log\lambda}}.$$

Then, combining the estimates (3.2), (3.3), (3.4), and (3.5), we obtain for $0 < \lambda < \lambda_2$ that

$$\int |\nabla\tilde{\mathbf{w}}|^2 \leqq \frac{C\lambda^2}{|\log\lambda|}. \tag{3.6}$$

Together with (3.3), inequality (3.6) gives the conclusion. □

Remark 10. The small factor $|\log\lambda|^{-1}$ will be essential for our remaining arguments. Its presence is, in some sense, a reflection of the Stokes paradox. For example, it guarantees that for any sequence of solutions $\mathbf{w}^{(k)}$ to (1.1_λ) with $\lambda_k \to 0$, the modified sequence $\lambda_k^{-1}\mathbf{w}^{(k)}$, which is solution to the system

$$\begin{cases} \Delta\mathbf{w} - \lambda_k(\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla p = 0, \\ \nabla\cdot\mathbf{w} = 0, \\ \mathbf{w}|_{\partial\mathcal{E}} = 0, \\ \mathbf{w}(z) \to \mathbf{e}_1 \ \text{ as } \ |z| \to \infty, \end{cases} \tag{3.7}$$

converges (on every bounded set) to the zero function which is the only D-solution to the Stokes system $(1.1)_{1,2,3}$. If, instead of $\mathbf{w}|_{\partial\mathcal{E}} = 0$ in (1.1_λ), one prescribes a more general boundary condition $\mathbf{w}|_{\partial\mathcal{E}} = \lambda\mathbf{a}$ for a general fixed **non**-constant function $\mathbf{a}$, then due to [7, Theorem 4.2], one can still construct solutions to the Navier–Stokes equations when λ is sufficiently small. However, for such solutions only a weaker bound $\int_{\mathcal{E}}|\nabla\mathbf{w}|^2 < M_4\lambda^2$ can be obtained, see [7, Lemma 5.2]. Thus the uniqueness (or non-uniqueness) of solutions under such inhomogeneous boundary conditions among all D-solutions is a more subtle problem, and lies still beyond the reach of our methods. Nevertheless, when $\mathbf{a}$ is a fixed **constant** vector on $\partial\mathcal{E}$, the proof of Lemma 9 works by a slight modification in the definition of $\mathbf{A}$. The rest of our proof (including the uniqueness result) also works in this situation.

Next, using (1.1_λ), (3.1), Lemma 8, and standard bootstrapping arguments, we obtain explicit bounds for $\mathbf{w}$ near $\partial\mathcal{E}$, namely, in $\Omega_3 = \mathcal{E}\cap B_3$. For simplicity, we work with infinitely smooth $\partial\mathcal{E}$ here. If $\partial\mathcal{E}$ has only finite regularity, then the following estimates near the boundary are valid up to a finite k (nevertheless still sufficient for the rest of the paper):

Lemma 11. *Let $\mathbf{w}$ be an arbitrary D-solution to (1.1_λ) for some $0 < \lambda < \lambda_2$. Then $\|\mathbf{w}\|_{C^k(\Omega_3)} \leqq C(k,\partial\mathcal{E})D_\lambda^{\frac{1}{2}}$ for any integer $k \geqq 0$. Moreover, up to the subtraction of a suitable constant, the associated pressure $p \to 0$ at infinity and $\|p\|_{C^k(\Omega_3)} \leqq C(k,\partial\mathcal{E})D_\lambda^{\frac{1}{2}}$ for any integer $k \geqq 0$.*[7]

Proof. The regularity estimate of $\mathbf{w}$ is standard. We have used that D_λ is small, so that $D_\lambda^{\frac{m}{2}}$, $m \geqq 2$ coming from the nonlinear term are dominated by $D_\lambda^{\frac{1}{2}}$. Let us explain the last statement. It is proved in [12] that the pressure p has a uniform limit at infinity which, after the subtraction of a suitable constant, may be taken as 0. Denote $\bar{p}(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r,\theta)\mathrm{d}\theta$. It is shown in [12, Lemma 4.1] that

$$2\pi|\bar{p}(r_2) - \bar{p}(r_1)| \leqq \int_{r>r_1} |\nabla\mathbf{w}|^2\mathrm{d}x\mathrm{d}y \leqq D_\lambda \tag{3.8}$$

[7] Throughout the rest of the paper, we always assume that such subtraction of a constant has been carried out.

for any $1 \leqq r_1 \leqq r_2 < \infty$. Sending $r_2 \to \infty$, (3.8) gives, for any $r_1 \geqq 1$,

$$|\bar{p}_{r_1}| \leqq \frac{D_\lambda}{2\pi}. \tag{3.9}$$

Note that $\|\nabla p\|_{C^k(\Omega_3)} \leqq C(k, \partial\mathcal{E}) D_\lambda^{\frac{1}{2}}$ follows from (2.5) and bootstrapping arguments. Using (3.9), we get

$$\begin{aligned}\|p\|_{C^{k+1}(\Omega_3)} &\leqq C(k, \partial\mathcal{E})(D_\lambda^{\frac{1}{2}} + D_\lambda) \\ &\leqq C(k, \partial\mathcal{E}) D_\lambda^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

□

The next lemma plays the crucial role in many estimates near and outside the critical circle $|z| = \frac{1}{\lambda}$.

Lemma 12. *Let* $\mathbf{w}$ *be an arbitrary D-solution to (1.1$_\lambda$) for some* $0 < \lambda < \lambda_2$. *Then the pressure* p *can be decomposed as* $p = p_1 + p_2$ *such that*

$$\lim_{z\to\infty} p_1 = \lim_{z\to\infty} p_2 = 0, \tag{3.11}$$

$$\|p_1\|_{C^0} \leqq C D_\lambda, \tag{3.12}$$

$$|p_2(z)| \leqq \frac{C D_\lambda^{\frac{1}{2}}}{|z|}, \quad \forall |z| \geqq 2. \tag{3.13}$$

Proof. By equations $(1.1_\lambda)_1$ and $(1.1_\lambda)_2$, the pressure solves the Poisson equation in $\mathcal{E}$:

$$\Delta p = -\nabla \mathbf{w} \cdot (\nabla \mathbf{w})^\intercal. \tag{3.14}$$

Let p_1 be the potential solution to (3.14), that is,

$$p_1(z) = -\frac{1}{2\pi} \int_{\mathcal{E}} \log|z - \zeta| (\nabla \mathbf{w} \cdot (\nabla \mathbf{w})^\intercal)(\zeta)\, \mathrm{d}\zeta_1\, \mathrm{d}\zeta_2.$$

By the classical div-curl lemma (see, e.g., [3]), $\nabla \mathbf{w} \cdot (\nabla \mathbf{w})^\intercal$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$. Hence, by the Calderón-Zygmund theorem for Hardy spaces [30], $\nabla^2 p_1 \in L^1(\mathbb{R}^2)$, and $\nabla p \in L^2(\mathbb{R}^2)$. Moreover, $p_1 \in C^0(\mathbb{R}^2)$ and converges to 0 at infinity. In particular,

$$\sup_{\mathbb{R}^2} |p_1| \leqq C \|\nabla \mathbf{w} \cdot (\nabla \mathbf{w})^\intercal\|_{\mathcal{H}^1} \leqq C D_\lambda. \tag{3.15}$$

Let $p_2 = p - p_1$, a function defined in $\mathcal{E}$. Clearly, p_2 is a harmonic function and satisfies

$$\lim_{r\to\infty} p_2 = 0. \tag{3.16}$$

By Lemma 11 and (3.15), we have

$$\begin{aligned}\sup_{S_1}|p_2| &\leqq \sup_{S_1}(|p|+|p_1|)\\ &\leqq C(D_\lambda^{\frac{1}{2}}+D_\lambda)\\ &\leqq CD_\lambda^{\frac{1}{2}}.\end{aligned} \tag{3.17}$$

Since $z \mapsto \frac{1}{z}$ is a conformal mapping in the extended complex plane, $p_2(\frac{1}{z})$ is a harmonic function in $z \in B_1$ with $p_2(\frac{1}{0}) = 0$. Then by the classical Schwarz lemma and by estimate (3.17), we have $|p_2(\frac{1}{z})| \leqq CD_\lambda^{\frac{1}{2}}|z|$ for $|z| \leqq \frac{1}{2}$. This proves (3.13). □

Let $\Phi = \frac{|\mathbf{w}|^2}{2} + p$ be the Bernoulli function. To proceed, we need an elegent observation made by Amick [1] for a general D-solution $\mathbf{w}$ to $(1.1_\lambda)_{1,2,3}$ in $\mathcal{E}$. This result concerns topological behaviour of certain level sets of the vorticity $\omega = \partial_2 w_1 - \partial_1 w_2$. Namely, Amick proved that there exist finitely many distinct unbounded connected components of the set $\{z \in \mathcal{E} : \omega(z) \neq 0\}$. These components are denoted by V_+ and V_- in his paper [1]. Each V_+ and V_- is a simply-connected domain not separated from $\partial\mathcal{E}$, essentially due to the maximum principle for ω. Then one may take the two unbounded components of ∂V_+ for some V_+ as $\mathcal{C}_i$, $i = 1, 2$, so that the statements in the following lemma are satisfied:

Lemma 13. (See [1] Theorem 11) *For any D-solution $\boldsymbol{w}$ to (1.1_λ), there exist two unbounded continuous curves $\mathcal{C}_i$, parametrized by arc length as $\mathcal{C}_i = \{(x_i(s), y_i(s) : s \in (0,\infty))\}$, $i = 1, 2$. The functions $x_i(\cdot)$ and $y_i(\cdot)$ are real-analytic on $(0,\infty)$ except possibly at isolated points, and they satisfy $(x_i(0), y_i(0)) \in \{|z| = 1\}$ and $|(x_i(s), y_i(s))| \to \infty$ as $s \to \infty$. The vorticity ω vanishes on these two curves. Moreover, the maps $s \to \Phi(x_i(s), y_i(s))$ are monotone decreasing and increasing in $s \in (0,\infty)$ respectively for $i = 1, 2$.*

Two immediate consequences of Lemma 13 and the known fact that $\Phi \to \frac{\lambda^2}{2}$ at infinity are as follows: for any $r \geqq 1$ we have

$$\max_{|z|=r}\Phi(z) \geqq \frac{\lambda^2}{2}, \tag{3.18}$$

and

$$\min_{|z|=r}\Phi(z) \leqq \frac{\lambda^2}{2}. \tag{3.19}$$

Hence, using Lemma 12, for any $r = |z| \geqq \lambda^{-1}$, it holds that

$$\max_{|z|=r}|\mathbf{w}|^2 \geqq \lambda^2 - C\lambda D_\lambda^{\frac{1}{2}}, \tag{3.20}$$

and

$$\min_{|z|=r} |\mathbf{w}|^2 \leqq \lambda^2 + C\lambda D_\lambda^{\frac{1}{2}} \tag{3.21}$$

for arbitrary D-solution $\mathbf{w}$ to (1.1_λ) with $0 < \lambda < \lambda_2$.

Due to Lemma 6, there exists a sequence of "good" radii $R_n \in [2^n\lambda^{-1}, 2^{n+1}\lambda^{-1})$ for $n = -4, -3, \cdots, 1, 2, \cdots$, such that

$$|\mathbf{w}(R_n, \theta) - \bar{\mathbf{w}}(R_n)| \leqq C D_\lambda^{\frac{1}{2}} \tag{3.22}$$

for all $0 \leqq \theta < 2\pi$. From (3.20), (3.21) and by the triangle inequality we obtain

$$\big||\bar{\mathbf{w}}(R_n)| - \lambda\big| \leqq C D_\lambda^{\frac{1}{2}}. \tag{3.23}$$

Hence, by (2.1) of Lemma 6, we have

$$\Big||\bar{\mathbf{w}}(r)| - \lambda\Big| \leqq C D_\lambda^{\frac{1}{2}} \tag{3.24}$$

for any $r \geqq \frac{1}{16\lambda}$. When λ is sufficiently small, this implies, in particular, that

$$|\bar{\mathbf{w}}(r)| \geqq \frac{\lambda}{2}. \tag{3.25}$$

In order to use Lemma 7 to control the direction of the vector $\bar{\mathbf{w}}$, below we need to establish some suitable estimates for $\mathbf{w}$ in the region $r \gtrsim \lambda^{-1}$.

Lemma 14. *Let $\mathbf{w}$ be an arbitrary D-solution to (1.1_λ) with $0 < \lambda < \min\{\lambda_2, \frac{1}{16}\}$. Then the estimate $|\mathbf{w}| \leqq C\lambda$ holds for all $r \geqq (8\lambda)^{-1}$.*

Proof. By Lemma 12, we have $|p| \leqq C\lambda^2 |\log\lambda|^{-\frac{1}{2}}$ for $r \geqq (16\lambda)^{-1}$. According to the estimates (3.1) and (3.22), (3.23), we obtain

$$\Phi \leqq C\lambda^2 \tag{3.26}$$

on the good circle $S_{R_{-4}}$. Recall, that Φ satisfies the classical identity

$$\Delta\Phi = \omega^2 + \mathbf{w} \cdot \nabla\Phi.$$

Therefore, from the maximum principle for Φ and from the convergence $\Phi \to \frac{\lambda^2}{2}$ at infinity, we obtain $\Phi \leqq C\lambda^2$ in the region $r \geqq R_{-4}$. Combined with the mentioned estimate of p, this clearly implies $|\mathbf{w}| < C\lambda$ in the region $r \geqq (8\lambda)^{-1}$. □

Lemma 15. *Let $\mathbf{w}$ be as in Lemma 14, then $\int_{\mathcal{E}\setminus B_{(4\lambda)^{-1}}} r|\nabla\omega|^2\,dxdy \leqq C\lambda D_\lambda$.*

Proof. Let $\rho_* \in ((8\lambda)^{-1}, (4\lambda)^{-1})$ to be chosen later. The classical vorticity equation $\Delta\omega = \mathbf{w} \cdot \nabla\omega$ implies the identity

$$\operatorname{div}(r\omega\nabla\omega) = r|\nabla\omega|^2 + \omega\partial_r\omega - (\mathbf{w}\cdot\mathbf{e}_r)\frac{\omega^2}{2} + \operatorname{div}\Big(r\mathbf{w}\frac{\omega^2}{2}\Big). \tag{3.27}$$

Then we have the energy estimate

$$\begin{aligned}&\int_{r \geqq \rho_*} r|\nabla\omega|^2 \mathrm{d}x\mathrm{d}y + \int_{r \geqq \rho_*} \omega\partial_r\omega\, \mathrm{d}x\mathrm{d}y - \int_{r \geqq \rho_*} (\mathbf{w}\cdot\mathbf{e}_r)\frac{\omega^2}{2}\, \mathrm{d}x\mathrm{d}y \\ &+ \rho_* \int_{S_{\rho_*}} \partial_r \frac{\omega^2}{2} ds - \rho_* \int_{S_{\rho_*}} (\mathbf{w}\cdot\mathbf{e}_r)\frac{\omega^2}{2} ds = 0.\end{aligned} \tag{3.28}$$

(Strictly speaking, to obtain the last formula, we have to integrate (3.27) on bounded domains $B_\rho \setminus B_{\rho_*}$ first and then let $\rho \to +\infty$, that is, the outer boundary goes to infinity. Using boundedness of Dirichlet energy of $\mathbf{w}$, it can be easily checked that the boundary terms on large circles $|z| = \rho$ disappear, at least for a sequence of radii going to infinity.)

The second term in (3.28) can be treated using Hölder's inequality as follows:

$$\begin{aligned}\left|\int_{r \geqq \rho_*} \omega\partial_r\omega\, \mathrm{d}x\mathrm{d}y\right| &\leqq \frac{1}{2\rho_*}\int_{r \geqq \rho_*} \omega^2\, \mathrm{d}x\mathrm{d}y + \frac{\rho_*}{2}\int_{r \geqq \rho_*} |\partial_r\omega|^2\, \mathrm{d}x\mathrm{d}y \\ &\leqq \frac{D_\lambda}{\rho_*} + \frac{1}{2}\int_{r \geqq \rho_*} r|\nabla\omega|^2.\end{aligned}$$

The third term on the left of (3.28) is controlled by $C\lambda D_\lambda$ since by Lemma 14 $|\mathbf{w}| \leqq C\lambda$ holds true in $\mathcal{E}\backslash B_{(8\lambda)^{-1}}$. Hence, we just need to treat the boundary terms. Using

$$\int_{r \geqq \rho} \omega^2 \mathrm{d}x\, \mathrm{d}y = \int_\rho^\infty dr\Big(\int_{S_r} \omega^2 ds\Big) \leqq 2D_\lambda$$

(with $ds := rd\theta$), it is easy to show that there exists a $\rho_* \in ((8\lambda)^{-1}, (4\lambda)^{-1})$ such that

$$\int_{S_{\rho_*}} \omega^2 ds \leqq C\lambda D_\lambda, \qquad -\Big[\partial_r \int_{S_r} \omega^2 ds\Big]_{r=\rho_*} \leqq C\lambda^2 D_\lambda \tag{3.29}$$

for some constant C. Then (3.29) and Lemma 14 together imply that

$$-\rho_* \int_{S_{\rho_*}} \partial_r \frac{\omega^2}{2} ds \leqq C\lambda D_\lambda, \qquad \rho_* \int_{S_{\rho_*}} (\mathbf{w}\cdot\mathbf{e}_r)\frac{\omega^2}{2} ds \leqq C\lambda D_\lambda. \tag{3.30}$$

Hence, from (3.28) we deduced $\int_{r \geqq \rho_*} r|\nabla\omega|^2\, \mathrm{d}x\mathrm{d}y \leqq C\lambda D_\lambda$ for such ρ_*. This proves the lemma. □

Using (3.25) and Lemma 7 with $\sigma = \frac{\lambda}{2}$, as well as the assumed convergence $\mathbf{w} \to \lambda \mathbf{e}_1$ at infinity, we have

$$|\varphi(r)| \leqq \frac{C}{\lambda^2} \int_{\mathcal{E} \setminus B_{(4\lambda)^{-1}}} \left(\frac{1}{r} |\nabla \omega| + |\nabla \mathbf{w}|^2 \right) \mathrm{d}x \mathrm{d}y$$

for any $r \geqq \frac{1}{4\lambda}$. Here $\varphi(r)$ is the angle of $\bar{\mathbf{w}}(r)$. From Lemma 15, and using Hölder's inequality, we obtain

$$\begin{aligned} |\varphi(r)| &\leqq \frac{C}{\lambda^2} \int_{\mathcal{E} \setminus B_{(4\lambda)^{-1}}} \left(\frac{\mu}{r^3} + \mu^{-1} r |\nabla \omega|^2 + |\nabla \mathbf{w}|^2 \right) \\ &\leqq C \left(\frac{\mu}{\lambda} + \frac{D_\lambda}{\mu \lambda} + \frac{D_\lambda}{\lambda^2} \right) \\ &\leqq C \frac{D_\lambda^{\frac{1}{2}}}{\lambda} \end{aligned}$$

for any $r \geqq \frac{1}{4\lambda}$ (taking $\mu = D_\lambda^{\frac{1}{2}}$ in the penultimate inequality). Together with (3.24), this implies that

$$|\bar{\mathbf{w}}(r) - \lambda \mathbf{e}_1| \leqq C D_\lambda^{\frac{1}{2}} \tag{3.31}$$

for all $r \geqq \frac{1}{4\lambda}$. By (3.22), we immediately have that

$$|\mathbf{w} - \lambda \mathbf{e}_1| \leqq C D_\lambda^{\frac{1}{2}} \tag{3.32}$$

on S_{R_n} for $n = -2, -1, \cdots$. These estimates on the good circles S_{R_n} will play important roles in the next two sections.

4. Pointwise estimates inside the critical circle (Stokes regime)

Let $\mathbf{w}$ be an arbitrary solution to (1.1_λ) with $0 < \lambda < \min\{\lambda_2, \frac{1}{16}\}$ so that all the bounds in Section 3 are valid. In the present section the pointwise upper estimates for $|\mathbf{w} - \lambda \mathbf{e}_1|$ will be derived within the bounded region $\{r \leqq \lambda^{-1}\} \cap \mathcal{E}$. They imply, in particular, that the required crucial estimates (1.14) are valid in this region.

We prove the following:

Lemma 16. *The inequality* $|\boldsymbol{w}(z) - \lambda \boldsymbol{e}_1| \leqq C D_\lambda^{\frac{1}{2}} \left(\log \frac{2}{\lambda r} \right)^{\frac{1}{2}}$ *holds for all* $z \in \{r \leqq \lambda^{-1}\} \cap \mathcal{E}$.

The method here is quite standard. It is based on the inequality (3.31) and accurate direct applications of linear Stokes estimates introduced in Section 2.3. The proof on Lemma 16 contains no surprising methods or ideas, and the reader can omit it on the first reading. So we moved this proof to the Appendix II.

The combination of Lemmas 16, 14 immediately implies

Corollary 17. $|\boldsymbol{w}| \leqq C\lambda$ *throughout* $\mathcal{E}$.

5. Pointwise Estimates Outside the Critical Circle (Oseen Regime)

We assume $0 < \lambda < \min\{\lambda_2, \frac{1}{16}\}$ so that the estimates in Sections 3, 4 are valid. In this section, we prove the required decay estimates (1.14) for $|\mathbf{w} - \lambda \mathbf{e}_1|$ in the unbounded region $r \gtrsim \lambda^{-1}$.

To make some notations simpler, we define a rescaled Navier–Stokes solution $\mathbf{u}(z) = \lambda^{-1}\mathbf{w}(\lambda^{-1}z)$, $q(z) = \lambda^{-2}p(\lambda^{-1}z)$ in the rescaled domain $\lambda\mathcal{E}$. Let

$$\mathbf{v} = \mathbf{u} - \mathbf{e}_1 \qquad \text{and} \qquad \varepsilon = |\log \lambda|^{-\frac{1}{2}}.$$

As usual, the components of $\mathbf{u}$, $\mathbf{v}$ will be denoted by u_i, v_i, $i = 1, 2$.

Lemma 18. *There exists a constant* $0 < \lambda_3 < \min\{\lambda_2, \frac{1}{16}\}$, *such that when* $0 < \lambda < \lambda_3$, *there holds* $|v_i(z)| \leqq C\varepsilon h_i(z)$ *in* $|z| \geqq 1$, $i = 1, 2$.

Here the majorant functions $h_i(\xi)$ are the same as in Theorem 1, that is,

$$|\xi| > 1: \quad \begin{cases} h_1(\xi) = |\xi|^{-\frac{1}{2}} \\ h_2(\xi) = |\xi|^{-\frac{3}{4}}. \end{cases}$$

Proof. With slight abuse of notation, in this proof, we still denote the Bernoulli function for $\mathbf{u}$ by $\Phi = \frac{|\mathbf{u}^2|}{2} + q$, and denote the vorticity by $\omega = \partial_2 u_1 - \partial_1 u_2$. Hence Φ and ω here are different from those in previous sections. Define the stream function ψ for $\mathbf{u}$ in $\lambda\mathcal{E}$ by the relation $\nabla\psi = \mathbf{u}^{\perp} = (-u_2, u_1)$. In particular, we have

$$\omega = \Delta\psi.$$

For definiteness, put $\psi(1, 0) = 0$.

Let

$$\gamma = \Phi - \omega\psi.$$

It is known that γ satisfies the two-sided maximum principle [1]. Now we divide the proof into a few steps.

Step 1: preparations. For convenience, in this step we collect and list some crucial information that we know on $\mathbf{u}$ in the region $\{r \geqq \frac{1}{4}\}$. By (3.1) we have

$$\int_{\lambda\mathcal{E}} |\nabla \mathbf{u}|^2 \mathrm{d}x\mathrm{d}y \leqq C\varepsilon^2. \tag{5.1}$$

Further more, by Lemma 15 we obtain

$$\int_{r \geqq \frac{1}{4}} r|\nabla\omega|^2 \mathrm{d}x\mathrm{d}y \leqq C\varepsilon^2. \tag{5.2}$$

As a consequence $\int_{S_{r_1}} |\partial_\theta \omega|^2 \leqq C\varepsilon^2$ for some $\frac{1}{4} < r_1 < \frac{1}{2}$. Then by virtue of Newton–Leibniz formula, since ω changes sign on any circle surrounding the origin (see [1]), we get $|\omega| \leqq C\varepsilon$ on S_{r_1}. Using convergence $\omega \to 0$ at infinity and the two-sided maximum principle again, we have $|\omega| \leqq C\varepsilon$ in $r \geqq r_1$. By Lemma 12, the pressure satisfies the inequality

$$|q| \leqq C\varepsilon \tag{5.3}$$

in $r \geqq \frac{1}{8}$. By the discussion on good circles S_{R_n} in Section 3, in particular (3.32), we have a sequence of radii $r_n = \lambda R_n \in [2^n, 2^{n+1})$, $n = -2, -1, \cdots, 1, 2, \cdots$ such that

$$|\mathbf{u} - \mathbf{e}_1| \leqq C\varepsilon \quad \text{on every } S_{r_n}. \tag{5.4}$$

Combining (5.3) and (5.4), we get $|\Phi - \frac{1}{2}| \leqq C\varepsilon$ on $S_{r_{-2}}$. By Corollary 17, we have

$$|\mathbf{u}| \leqq C \tag{5.5}$$

in $\lambda\mathcal{E}$. Due to the definition of ψ, we have also $|\psi| \leqq C$ in $\Omega_{\frac{1}{4},4}$. Since $\gamma = \Phi - \omega\psi$, on $S_{r_{-2}}$ we obtain $|\gamma - \frac{1}{2}| \leqq |\Phi - \frac{1}{2}| + |\omega\psi| \leqq C\varepsilon$. It is proved in [1, Theorem 14(b)] that $\gamma \to \frac{1}{2}$. By the two-sided maximum principle of γ, we find that

$$|\gamma - \frac{1}{2}| \leqq C\varepsilon \tag{5.6}$$

for any $r \geqq r_{-2}$. We also point out here that by (5.1), (5.5) and the local regularity theory of Lemma 8, we have the pointwise control on derivatives

$$|\nabla \mathbf{u}|, |\nabla^2 \mathbf{u}| \leqq C\varepsilon \tag{5.7}$$

in $r \geqq \frac{1}{4}$.

Step 2: pointwise smallness of v. In $\mathcal{E}_1$, we claim that

$$|\mathbf{v}| \leqq C\varepsilon. \tag{5.8}$$

The proof of this claim is essentially based on the classical works [1,12], and on the recent work [18]. In [1], Amick proved that the absolute value $|\mathbf{u}|$ is close to the limiting value 1 using the smallness of three quantities: the Dirichlet energy, $|\gamma - \frac{1}{2}|$, and the pressure. We refer to the proof of Theorem 21(a) in [1] for details. In our situation, such smallness is evidently provided by (5.1)–(5.6). Further, in [18, Lemma 3.3(ii)] Korobkov et al. proved the smallness of $|\mathbf{v}| = |\mathbf{u} - \mathbf{e}_1|$ using Amick's result and the additional observation that the angle of velocity is also under control.

In our situation, we have to modify some of the arguments in [1] so that the smallness factor ε can be fully preserved. The complete proof of (5.8) is presented in our Appendix I. In fact, many technical moments of Amick's proof can be simplified using [18, Remark 4.1].

Step 3: summability of v. For this step, we mainly use the potential estimates for Oseen system which are developed in [27, Lemmas 1–2] by L.I.Sazonov (see also [9], [28]). As will be shown, the pointwise smallness of **v** and the smallness of Dirichlet energy together are sufficient to control certain Lebesgue norms of **v** in the exterior domain by the regularity estimates near the boundary.

We write $\mathcal{E}_\rho = \{r \geqq \rho\}$ and write $\mathbf{E}_i = (E_{1i}, E_{2i}),\ i = 1, 2$, where E_{ij} is the Oseen tensor introduced in Section 2.4. Using (2.15) in the domain $\mathcal{E}_1$ and applying suitable integration by parts (and using $\partial_1 v_1 + \partial_2 v_2 = 0$), we obtain

$$\begin{aligned} v_2(z) = &\int_{\mathcal{E}_1} (v_1\partial_2 E_{12} - v_1\partial_1 E_{22} - v_2\partial_2 E_{22} + 2\partial_2 v_1\, E_{12})v_2 \\ &- \int_{S_1} (v_1^2 n_1 + 2v_1 v_2 n_2)E_{12} - \int_{S_1} \mathbf{v}(z')\cdot \mathbf{T}_z(\mathbf{E}_2, e_2)(z-z')\cdot\mathbf{n} \\ &- \int_{S_1} \mathbf{E}_2(z-z')\cdot\mathbf{T}(\mathbf{v}, q)(z')\cdot\mathbf{n} + \int_{S_1} (\mathbf{v}(z')\cdot\mathbf{E}_2(z-z'))(\mathbf{n}\cdot\mathbf{u}(z')) \\ =:&\ A_2(z) + B_2(z). \end{aligned} \tag{5.9}$$

Here A_2 and B_2 stand for the area integrals and the boundary integrals respectively. The area integrals are taken over the variable z'. Note that E_{ij} appearing in the integrals should be understood as $E_{ij}(z - z')$ and the derivatives on E_{ij} act on the variable z'. For representation of v_1, we simply use (2.15) without integration by parts to get

$$\begin{aligned} v_1(z) = &- \int_{\mathcal{E}_1} (v_1\partial_1 E_{11} + v_2\partial_1 E_{21} + v_2\partial_2 E_{11})v_1 + v_2^2\partial_2 E_{21} \\ &- \int_{S_1} \mathbf{v}(z')\cdot\mathbf{T}_z(\mathbf{E}_1, e_1)(z-z')\cdot\mathbf{n} \\ &- \int_{S_1} \mathbf{E}_1(z-z')\cdot\mathbf{T}(\mathbf{v}, q)(z')\cdot\mathbf{n} + \int_{S_1} (\mathbf{v}(z')\cdot\mathbf{E}_1(z-z'))(\mathbf{n}\cdot\mathbf{u}(z')) \\ =:&\ A_1(z) + B_1(z). \end{aligned} \tag{5.10}$$

As before, A_1 stands for the area integrals and B_1 stands for the boundary integrals.

Of course, the key issue here is to estimate the area integrals, because decay of boundary integrals outside the unit disk can be estimated relatively easily (using the uniform smallness of $\mathbf{v}$).

Let us recall some estimates of $\mathbf{E}$. By the exact form of $\mathbf{E}$ and the asymptotic of K_0 at infinity (see Section 2.4), we have

$$\begin{gathered} E_{11} \in L^{3,\infty} \cap L^{3+\delta}, \\ E_{12} = E_{21}, E_{22} \in L^{2,\infty} \cap L^{2+\delta} \end{gathered}$$

in $\mathbb{R}^2$ for any finite $\delta > 0$, and

$$\partial_2 E_{11} \in L^{\frac{3}{2},\infty}$$

in $\mathbb{R}^2$. Here $L^{s,\infty}$ is the weak L^s-space. Moreover, by the Fourier transform (2.11) of $\mathbf{E}$ and by the Mikhlin multiplier theorem (see, e.g., [30, Chapter VI,§4–5]) we have

$$\|\partial_k E_{ij} * f\|_{L^s(\mathbb{R}^2)} \leqq C_s\, \|f\|_{L^s(\mathbb{R}^2)}$$

for any $f \in L^s(\mathbb{R}^2)$, $1 < s < \infty$, $(i, j, k) \neq (1, 1, 2)$.[8] It is also known that $v_1 \in L^{3+\delta}(\mathcal{E}_1)$ and $v_2 \in L^{2+\delta}(\mathcal{E}_1)$ for any $\delta > 0$ by the estimates (1.15). With the above bounds, using weak Young inequality for convolutions (see, e.g., [22, Section 4.3]), we deduce from (5.9) that

$$\begin{aligned}
\|v_2\|_{L^s(\mathcal{E}_1)} &\leqq \|A_2\|_{L^s(\mathcal{E}_1)} + \|B_2\|_{L^s(\Omega_{1,2})} + \|B_2\|_{L^s(\mathcal{E}_2)} \\
&\leqq 2\|A_2\|_{L^s(\mathcal{E}_1)} + \|v_2\|_{L^s(\Omega_{1,2})} + \|B_2\|_{L^s(\mathcal{E}_2)} \\
&\leqq C_s \left(\|\mathbf{v}\|_{L^\infty(\mathcal{E}_1)} + \|\partial_2 v_1\|_{L^2(\mathcal{E}_1)}\right) \|v_2\|_{L^s(\mathcal{E}_1)} \\
&\quad + \|v_2\|_{L^s(\Omega_{1,2})} + \|B_2\|_{L^s(\mathcal{E}_2)}
\end{aligned} \tag{5.11}$$

for any $2 < s < \infty$. Using information from Step 1 and Step 2, we have

$$\|\mathbf{v}\|_{L^\infty(\mathcal{E}_1)} + \|\partial_2 v_1\|_{L^2(\mathcal{E}_1)} \leqq C\varepsilon$$

and

$$\|v_2\|_{L^s(\Omega_{1,2})} + \|B_2\|_{L^s(\mathcal{E}_2)} \leqq C\|\mathbf{v}\|_{C^1(\Omega_{\frac{1}{2},2})} + \|q\|_{L^\infty(\Omega_{\frac{1}{2},2})} \leqq C_s\varepsilon$$

for any $2 < s < \infty$. Hence, when ε is sufficiently small (depending on the choice of s), or equivalently, when λ is sufficiently small, we obtain from (5.11) that

$$\|v_2\|_{L^s(\mathcal{E}_1)} \leqq C_s\varepsilon \tag{5.12}$$

for any $2 < s < \infty$. Similarly, using (5.10), we have

$$\begin{aligned}
\|v_1\|_{L^m(\mathcal{E}_1)} &\leqq \|A_1\|_{L^m(\mathcal{E}_1)} + \|B_1\|_{L^m(\Omega_{1,2})} + \|B_1\|_{L^m(\mathcal{E}_2)} \\
&\leqq 2\|A_1\|_{L^m(\mathcal{E}_1)} + \|v_1\|_{L^m(\Omega_{1,2})} + \|B_1\|_{L^m(\mathcal{E}_2)} \\
&\leqq C_m \left(\|\mathbf{v}\|_{L^\infty(\mathcal{E}_1)} + \|v_2\|_{L^3(\mathcal{E}_1)}\right) \|v_1\|_{L^m(\mathcal{E}_1)} + C_m\|v_2\|^2_{L^{2m}(\mathcal{E}_1)} \\
&\quad + \|v_1\|_{L^m(\Omega_{1,2})} + \|B_2\|_{L^m(\mathcal{E}_2)}
\end{aligned} \tag{5.13}$$

for any $3 < m < \infty$. Using (5.12) and similar arguments as those for v_2, we deduce

$$\|v_1\|_{L^m(\mathcal{E}_1)} \leqq C_m\varepsilon \tag{5.14}$$

for any $3 < m < \infty$, when λ is sufficiently small (depending on the choice of m). **Step 4: pointwise decay of v**. First, we prove a pointwise decay estimate for vorticity using an idea of Gilbarg and Weinberger [12]. By Hölder's inequality,

$$\begin{aligned}
\int_1^\infty \frac{dr}{r} \int_0^{2\pi} |\partial_\theta(r^{\frac{3}{2}}\omega^2)| \mathrm{d}\theta &= 2\int_{\mathcal{E}_1} r^{-\frac{1}{2}} |\omega\partial_\theta\omega| \mathrm{d}x\,\mathrm{d}y \\
&\leqq \int_{\mathcal{E}_1} \omega^2 \mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{E}_1} r|\nabla\omega|^2 \mathrm{d}x\,\mathrm{d}y \\
&\leqq C\varepsilon^2.
\end{aligned}$$

[8] Similar facts concerning the integrability of E_{ij} are collected in [27, § 2–3].

Hence, for each $n = 0, 1, 2, \cdots$, there exists $r \in [2^n, 2^{n+1})$ such that

$$\int_0^{2\pi} |\partial_\theta (r^{\frac{3}{2}} \omega^2)| \mathrm{d}\theta \leqq C\varepsilon^2.$$

Recall that there are two curves $\lambda\mathcal{C}_i, i = 1, 2$ with $\mathcal{C}_i$ given by Lemma 13 such that ω vanishes on them. Evidently, S_r intersects $\lambda\mathcal{C}_i$ for any $r \geqq 1$. Hence, for $r \in [2^n, 2^{n+1})$ given above, we have

$$r^{\frac{3}{4}} \max_{S_r} |\omega| \leqq C\varepsilon. \tag{5.15}$$

By the two-sided maximum principle for ω, the above estimate holds for any $r \geqq 2$.

For any disk $B_\rho(z) \subset \mathcal{E}$, the following standard identity holds:

$$\mathbf{v}(z) = \frac{1}{2\pi} \int_{\partial B_\rho(z)} \frac{\mathbf{v}(\zeta)}{\rho} ds - \frac{1}{2\pi} \int_{B_\rho(z)} \frac{\omega(\zeta)(z-\zeta)^\perp}{|z-\zeta|^2} \mathrm{d}\zeta.$$

Here $(z-\zeta)^\perp = (-(z_2 - \zeta_2), z_1 - \zeta_1) \in \mathbb{R}^2$. By virtue of this identity, following [27, Section 5], using $\|\mathbf{v}\|_{L^m(r \geqq 1)} \leqq C_m\varepsilon,\ m > 3$ from Step 3, and (5.15), we immediately reach the pointwise bound

$$|\mathbf{v}(z)| \leqq C_\delta \varepsilon\, r^{-\frac{3}{10}+\delta} \tag{5.16}$$

in $\mathcal{E}_1$ for any $\delta > 0$ (when λ is sufficiently small). Choose and fix a small δ such that $\frac{3}{10} - \delta > \frac{1}{4}$. Now the meaning of "$\lambda$ *being sufficiently small*" in Step 3 is also fixed.

Next, we use the representation formula (2.15) in $\mathcal{E}_1$ again to get

$$\begin{aligned}
\mathbf{v}(z) = &- \int_{\mathcal{E}_1} (\mathbf{v} \cdot \nabla_{z'})\mathbf{E}(z - z') \cdot \mathbf{v} \\
&- \int_{S_1} \mathbf{v}(z') \cdot \mathbf{T}_z(\mathbf{E}, \mathbf{e})(z - z') \cdot \mathbf{n} \\
&- \int_{S_1} \mathbf{E}(z - z') \cdot \mathbf{T}(\mathbf{v}, p)(z') \cdot \mathbf{n} + \int_{S_1} (\mathbf{v}(z') \cdot \mathbf{E}(z - z'))(\mathbf{n} \cdot \mathbf{u}(z')) \\
&=: \mathbf{N} + \mathbf{L}.
\end{aligned} \tag{5.17}$$

Here $\mathbf{L}$ are the sum of all boundary integrals and $\mathbf{N}$ is the area integral. By the asymptotic form of $\mathbf{E}$, we have

$$|L_i| \leqq C\varepsilon h_i(z) \tag{5.18}$$

in $r \geqq 2$. It is easy to check that $|\mathbf{N}| \leqq C\varepsilon^2$ in $\Omega_{\frac{1}{2},2}$. Hence $|\mathbf{L}| \leqq |\mathbf{v}| + |\mathbf{N}| \leqq C\varepsilon$ in $\Omega_{1,2}$, and as a consequence, (5.18) holds in $r \geqq 1$.

Now it remains to estimate the second term $\mathbf{N}$ in (5.17) (area integrals). This can be done using Lemmas 1 and 2 from the classical Smith paper [29, p.361]. These lemmas give some self-improving estimates for term $\mathbf{N}$, that is, if we assume a priori that $\mathbf{v}$ has the uniform decay of type (5.16), then a posteriori $\mathbf{N}$ has better decay, etc. So we can use these two lemmas (with parameter $\sigma = 0$ there) finitely many times to improve the bound (5.16) until $\mathbf{N}$ is shown to be decaying faster than the right of (5.18). This concludes the proof of Lemma 18. □

Now we are ready to give the

Proof of Lemma 3. Lemma 18 implies that, when λ is sufficiently small,

$$|(\mathbf{w}(z) - \lambda \mathbf{e}_1)_i(z)| \leqq C|\log\lambda|^{-\frac{1}{2}}\lambda h_i(\lambda z), i = 1, 2$$

in the exterior region $\mathcal{E}_{\lambda^{-1}} = \{z : r \geqq \lambda^{-1}\}$. By Lemmas 16 and 9, the above also holds true in the bounded region $\{z : r \leqq \lambda^{-1}\} \cap \mathcal{E}$ (with a different positive constant C). Hence, (1.14) holds throughout $\mathcal{E}$ for some positive constant M_1. It remains to take λ sufficiently small so that $M_1|\log\lambda|^{-\frac{1}{2}} \leqq \varepsilon_0$ where ε_0 is given by Theorem 1.

□

Now the assertion of the main Theorem 2 follows immediately from Lemma 3 and from the conditional uniqueness result in Finn–Smith Theorem 1. The proof is finished.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Appendix I

Proof of Step 2. In this section we discuss the proof of the uniform pointwise estimate (5.8), that is,

$$|\mathbf{u} - (1, 0)| = |\mathbf{v}| \leqq C\varepsilon$$

for $r \geqq 1$.

Step 2a. First of all, consider the "good" cone

$$\mathcal{K}^{\pm y} = \Big\{r \geqq 1, |\theta| \in \big(\frac{1}{5}\pi, \frac{4}{5}\pi\big)\Big\},$$

which is separated from the x-axis. Here one can simply follow the proof of [1, Theorem 19]. The key observation is that $|\psi| \geqq cr > 0$ in such a cone, and this fact, by virtue of smallness, gives

$$|\Phi - \frac{1}{2}| \leqq C\varepsilon, \qquad |\gamma - \frac{1}{2}| = |\Phi - \frac{1}{2} - \omega\psi| \leqq C\varepsilon$$

on good circles and by maximum principle for ω, implies $|\omega| \leqq C\varepsilon r^{-1}$ in $\mathcal{K}^{\pm y}$, which easily gives the required smallness $|\mathbf{u} - (1,0)| \leqq C\varepsilon$ here.
Now consider the more complicated region

$$\left\{r \geqq 1, |\theta| \leqq \frac{1}{5}\pi\right\}.$$

Here the previous estimate $|\omega| \leqq C\varepsilon r^{-1}$ does not hold in general, so the arguments should be more delicate and subtle. The main ideas here are due to Amick [1] and Korobkov et al. [18, Remark 4.1].

Step 2b. We are going to use the uniform smallness of γ-function

$$\left|\gamma - \frac{1}{2}\right| = \left|\Phi - \frac{1}{2} - \omega\psi\right| \leqq C\varepsilon$$

and the level sets of the stream function ψ. On the first step here we show that the set $\mathcal{C} = \{r \geqq \frac{1}{2}, \psi = 0\}$ consists of exactly two smooth curves $\mathcal{C}_{\pm}$. Indeed, from Step 1, we know that

$$|\psi - y| \leqq C\varepsilon \tag{A.1}$$

in $\Omega_{\frac{1}{2},2}$. Using (5.1), there exists an angle $\theta_1 \in [\frac{\pi}{10}, \frac{\pi}{5})$ such that

$$\int_{1/4}^{+\infty} |\partial_r \mathbf{u}(r,\theta)|^2 r dr \leqq C\varepsilon^2$$

for $\theta = \theta_1, -\theta_1, \pi - \theta_1, -\pi + \theta_1$. Hence, for such θ,

$$\int_{r_n}^{r_{n+1}} |\partial_r \mathbf{u}(r,\theta)| dr \leqq C\varepsilon \left(\int_{r_n}^{r_{n+1}} \frac{dr}{r}\right)^{\frac{1}{2}},$$
$$\leqq C\varepsilon$$

with $r_n, n = -2, -1, \cdots$ given in Step 1. In view of (5.4), we obtain

$$|\mathbf{v}(r,\theta)| \leqq C\varepsilon$$

for any $r \geqq \frac{1}{2}$ and $\theta = \theta_1, -\theta_1, \pi - \theta_1, -\pi + \theta_1$. By the definition of ψ and (A.1), we have

$$|\psi - y| \leqq C\varepsilon r \tag{A.2}$$

pointwise on the set

$$\mathcal{N} := \left(\cup_{n \geqq -2} S_{r_n}\right) \cup \left\{r \geqq \frac{1}{2}, \theta = \theta_1, -\theta_1, \pi - \theta_1, -\pi + \theta_1\right\}. \tag{A.3}$$

When λ is small, this implies that $\mathcal{C}$ must intersect $\mathcal{N}$ within the cone $\mathcal{K} := \{r \geqq \frac{1}{2}, |\theta| < \frac{\pi}{10} \text{ or } |\pi - \theta| < \frac{\pi}{10}\}$. Moreover, by (5.4), $\mathcal{C}$ intersects each S_{r_n}, $n \geqq -2$

at exactly two points, one with $x > 0$ and another with $x < 0$. On the level set $\mathcal{C}$ we have $\left|\frac{|\mathbf{u}|^2}{2} - \frac{1}{2}\right| = |\gamma - \frac{1}{2} - q| \leqq C\varepsilon$ (see (5.3), (5.6)), hence

$$\Big||\mathbf{u}| - 1\Big| \leqq C\varepsilon \quad \text{on } \mathcal{C}. \tag{A.4}$$

As a consequence, $|\nabla\psi| = |\mathbf{u}| \neq 0$ on the set $\mathcal{C}$, that is, $\mathcal{C}$ is a regular curve in the case when λ is small.

Further, when λ is small, $\mathcal{C}$ cannot contain any closed curve $\mathcal{L}$ by an elegent idea of Amick's. To explain this, suppose $\mathcal{C}$ contains a closed curve $\mathcal{L}$. Let $\mathcal{U}$ denote the domain bounded by $\mathcal{L}$, then

$$\left(\int_{\mathcal{U}} \omega^2 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} |\mathcal{U}|^{\frac{1}{2}} \geqq \left|\int_{\mathcal{U}} \omega \,\mathrm{d}x\mathrm{d}y\right| = \left|\int_{\mathcal{U}} \Delta\psi \,\mathrm{d}x\mathrm{d}y\right| = \left|\int_{\mathcal{L}} \partial_n\psi \, ds\right| =$$
$$= \int_{\mathcal{L}} |\nabla\psi| \, ds = \int_{\mathcal{L}} |\mathbf{u}| \, ds \overset{(A.4)}{\geqq} \frac{1}{2}|\mathcal{L}|.$$

(Note, that on $\mathcal{L}$ one of the identities $\partial_n\psi \equiv |\nabla\psi|$ or $\partial_n\psi \equiv -|\nabla\psi|$ holds, because $\mathcal{L}$ is regular closed level set of ψ.) This implies

$$|\mathcal{L}| \leqq C\varepsilon|\mathcal{U}|^{\frac{1}{2}},$$

which contradicts with the isoperimetric inequality when ε is small. Hence $\mathcal{C}$ must consist of two smooth curves starting from $r = \frac{1}{2}$ and extending to infinity, each contained in $\mathcal{K} \cap \{x > 0\}$ and $\mathcal{K} \cap \{x < 0\}$ respectively. We denote these two curves by $\mathcal{C}_{\pm}$.

Step 2c. Here we show that

$$|\mathbf{v}| \leqq C\varepsilon \quad \text{along } \mathcal{C}_{\pm}. \tag{A.5}$$

Take any point $z \in \mathcal{C}_+$ ($\mathcal{C}_-$ would be similar) with $r = |z| \geqq 1$. By Lemma 6, there exist good circles $S_{\tilde{r}_n}$ centered at z with $\tilde{r}_n \in [2^{-n-1}r, 2^{-n}r)$, $n = 1, 2, \cdots$, such that

$$|\mathbf{u} - \bar{\mathbf{u}}^{(n)}| \leqq C\varepsilon \quad \text{on } S_{\tilde{r}_n}.$$

Here $\bar{\mathbf{u}}^{(n)}$ is the average of $\mathbf{u}$ on $S_{\tilde{r}_n}$. Since $S_{\tilde{r}_n}$ must intersect $\mathcal{C}_+$ and on $\mathcal{C}_+$ the inequality (A.4) holds, we obtain

$$\Big||\bar{\mathbf{u}}^{(n)}| - 1\Big| \leqq C\varepsilon$$

on each $S_{\tilde{r}_n}$. From this, (2.1) of Lemma 6 implies

$$\Big||\bar{\mathbf{u}}^{(\rho)}| - 1\Big| \leqq C\varepsilon,$$

where $\bar{\mathbf{u}}^{(\rho)}$ is the mean value of $\mathbf{u}$ on circles $S_\rho(z)$ centered at z with radius $\rho \leqq \frac{r}{2}$. Let $\varphi^{(n)}$ be the angle of $\bar{\mathbf{u}}^{(n)}$. By Lemma 7, we have, for any $n, m \geqq 1$,

$$|\varphi^{(n)} - \varphi^{(m)}| \leqq C \int_{|\zeta - z| \leqq \frac{r}{2}} \frac{|\nabla \omega|}{|\zeta - z|} + |\nabla \mathbf{u}|^2 \, \mathrm{d}\zeta_1 \mathrm{d}\zeta_2$$

$$\leqq C \left(\int_{1 \leqq |\zeta - z| \leqq \frac{r}{2}} \frac{|\nabla \omega|}{|\zeta - z|} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \right) \tag{A.6}$$

$$+ \, C \left(\int_{|\zeta - z| \leqq 1} \frac{|\nabla \omega|}{|\zeta - z|} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \right) + C\varepsilon^2$$

$$\leqq C\varepsilon. \tag{A.7}$$

In this last line we have used (5.2) and (5.7) for the first and second terms in the penultimate inequality respectively. By construction, the circle $S_{\tilde{r}_1}$ must intersect the set $\mathcal{N}$ (see (A.3)) on which $|\mathbf{v}| \leqq C\varepsilon$. Hence, $|\bar{\mathbf{u}}^{(1)} - \mathbf{e}_1| \leqq C\varepsilon$. Letting $n = 1, m \to \infty$ in (A.6), and using (A.4), we get $|\mathbf{v}(z)| \leqq C\varepsilon$. Note that this implies that the slope of $\mathcal{C}_\pm$ is small.

Step 2d. Now take arbitrary point $z_1 = (x_1, y_1) \in \{r \geqq 1, |\theta| < \frac{\pi}{5}, x > 0\}$ (the case $x < 0, |\pi - \theta| < \frac{\pi}{5}$ is similar) and show that

$$|\mathbf{v}(z_1)| \leqq C\varepsilon.$$

Let $z_2 = (x_1, y_2) \in \mathcal{C}_+$. To simplify notations, let's change the coordinate system, namely, let's move the coordinate origin to the point z_2, so now

$$z_2 = (x_1, y_2) = (0, 0) \in \mathcal{C}_+, \quad z_1 = (x_1, y_1) = (0, y_1).$$

Consider the case $y_1 > 0$, that is, when the point z_1 is above the $\mathcal{C}_+$ curve, so that $\psi(z_1) > 0$ (the opposite case $y_1 < 0$ case is quite similar). Let

$$R = y_1.$$

Using Lemma 6, we find two good circles $S_{R_1}(z_1)$ and $S_{R_2}(z_2)$ centered at z_1 and z_2 respectively, with radii

$$R_1, R_2 \in \left(\frac{2R}{3}, \frac{3R}{4}\right).$$

Clearly,

$$S_{R_1}(z_1) \cap S_{R_2}(z_2) \neq \emptyset \neq \mathcal{C}_+ \cap S_{R_2}(z_2).$$

As a result, on both $S_{R_1}(z_1)$ and $S_{R_2}(z_2)$ we have

$$|\mathbf{v}| \leqq C\varepsilon. \tag{A.8}$$

Note that $\psi = 0$ on $\mathcal{C}_+$. We can integrate $\nabla(\psi - y) = \mathbf{v}^\perp$ starting from $z_2 \in \mathcal{C}_+$ along arcs of $\mathcal{C}_+, S_{R_2}(z_2)$ and $S_{R_1}(z_1)$. This process gives $|\psi - y| \leqq C\varepsilon R$ on $S_{R_1}(z_1)$. As a consequence, by virtue of the evident inequalities

$$\frac{1}{4} R \leqq \min_{(x,y) \in S_{R_1}(z_1)} y < \max_{(x,y) \in S_{R_1}(z_1)} y < \frac{7}{4} R,$$

we have

$$\left|\frac{\psi}{y}-1\right| \leqq C\varepsilon \tag{A.9}$$

on $S_{R_1}(z_1)=\partial B_{R_1}(z_1)$.

Now we are going to prove that the last estimate is valid not only for boundary circle $S_{R_1}(z_1)$, but for **all** points of the disk $B_{R_1}(z_1)$. Recall that

$$|\gamma-q-\frac{1}{2}| = \left|\frac{|\nabla\psi|^2}{2}-\psi\Delta\psi-\frac{1}{2}\right| \leqq C\varepsilon \tag{A.10}$$

pointwisely holds in the region $\{r \geqq 1\}$ (see (5.3), (5.6)). Denote $\psi = y(1+S)$ in $B_{R_1}(z_1)$. Since $\mathcal{C}_+ \cap B_{R_1}(z_1)=\emptyset$ and ψ is positive on $B_{R_1}(z_1)$, by construction we have that the values $(1+S)$ are positive on $B_{R_1}(z_1)$ as well. Assume at the moment, that S has positive maximum at the interior point of the disk $B_{R_1}(z_1)$. Then at this maximum point $\nabla S=0$ and $\Delta S \leqq 0$, therefore,

$$|\nabla\psi|^2-2\psi\Delta\psi=(1+S)^2-2\psi y\Delta S \geqq (1+S)^2,$$

which, by virtue of (A.10), implies $(1+S)^2 \leqq 1+C\varepsilon$, and consequently,

$$S \leqq C\varepsilon$$

at any maximum point of S inside the disk $B_{R_1}(z_1)$. Similarly, a consideration for the negative minimal points of S gives $S \geqq -C\varepsilon$. Hence, taking into account (A.9), we have proved

$$\left|\frac{\psi}{y}-1\right| \leqq C\varepsilon \tag{A.11}$$

in $B_{R_1}(z_1)$. In particular,

$$|\psi-y| \leqq C\varepsilon R \qquad \text{in } \ B_{R_1}(z_1).$$

Next, one observes that

$$\Delta(\sqrt{\psi}-\sqrt{y}) = \frac{2\psi\Delta\psi-|\nabla\psi|^2}{4\psi^{\frac{3}{2}}}+\frac{1}{4y^{\frac{3}{2}}}.$$

Hence, using (A.10) and (A.11), we get

$$|\Delta(\sqrt{\psi}-\sqrt{y})| \leqq C\varepsilon y^{-\frac{3}{2}} \leqq C\varepsilon R^{-\frac{3}{2}} \tag{A.12}$$

in the disc $B_{R_1}(z_1)$. On the other hand, (A.11) implies

$$|\sqrt{\psi}-\sqrt{y}| \leqq C\varepsilon R^{\frac{1}{2}} \tag{A.13}$$

in $B_{R_1}(z_1)$. Now, applying the standard estimates for Laplac operator in the unit disk and scaling to the function $\sqrt{\psi}-\sqrt{y}$ with (A.12)–(A.13), we obtain

$$|\nabla(\sqrt{\psi}-\sqrt{y})| \leqq C\varepsilon R^{-\frac{1}{2}}$$

inside $\frac{1}{2}B_{R_1}(z_1)$, that implies the required estimate $|\nabla\psi(z_1)-\mathbf{e}_1^{\perp}| = |\mathbf{u}-\mathbf{e}_1| \leqq C\varepsilon$, see [1, Proof of Theorem 27, page 118].

□

B. Appendix II

Proof of Lemma 16. Let $w_r = \mathbf{w} \cdot \mathbf{e}_r$ be the radial component of $\mathbf{w}$. We need the following classical inequality:

$$\begin{aligned}\frac{d}{dr}\int_0^{2\pi} |\mathbf{w}(r,\theta) - \bar{\mathbf{w}}(r)|^2 \mathrm{d}\theta &= \int_0^{2\pi} 2w_r \cdot (\mathbf{w} - \bar{\mathbf{w}}(r))\mathrm{d}\theta \\ &\leqq \int_0^{2\pi} \left[r|w_r|^2 + \frac{|\mathbf{w} - \bar{\mathbf{w}}(r)|^2}{r} \right] \mathrm{d}\theta \\ &\leqq \int_0^{2\pi} |\nabla \mathbf{w}|^2 r \mathrm{d}\theta. \end{aligned} \tag{B.1}$$

By our assumption on the domain $\mathcal{E}$, we have $\{r \geqq 1\} \subset \mathcal{E}$. By integrating (B.1) on the interval $[r, \infty)$, and using the fact that $\mathbf{w} \to \lambda \mathbf{e}_1$ uniformly at infinity, we obtain

$$\int_0^{2\pi} |\mathbf{w}(r,\theta) - \bar{\mathbf{w}}(r)|^2 \mathrm{d}\theta \leqq D_\lambda \tag{B.2}$$

for any $r \geqq 1$. By (2.1) and (3.31), for $1 \leqq r \leqq \frac{3}{2}\lambda^{-1}$, we have

$$\begin{aligned}|\bar{\mathbf{w}}(r) - \lambda \mathbf{e}_1| &\leqq |\bar{\mathbf{w}}(r) - \bar{\mathbf{w}}(R_1)| + |\bar{\mathbf{w}}(R_1) - \lambda \mathbf{e}_1| \\ &\leqq C D_\lambda^{\frac{1}{2}} \left(\log \frac{2}{\lambda r} \right)^{\frac{1}{2}} + C D_\lambda^{\frac{1}{2}} \leqq 2C D_\lambda^{\frac{1}{2}} \left(\log \frac{2}{\lambda r} \right)^{\frac{1}{2}}. \end{aligned} \tag{B.3}$$

Here we have used that $\log \frac{R_1}{r} \leqq \log \frac{4}{\lambda r} \leqq C \log \frac{2}{\lambda r}$ and $\log \frac{2}{\lambda r} \geqq c$ for any $r \leqq \frac{3}{2}\lambda^{-1}$, for some absolute positive constants c, C. Combining (B.2) and (B.3), we obtain

$$\begin{aligned}\int_0^{2\pi} |\mathbf{w} - \lambda \mathbf{e}_1|^2 \mathrm{d}\theta &\leqq 2\int_0^{2\pi} |\mathbf{w} - \bar{\mathbf{w}}|^2 \mathrm{d}\theta + 2\int_0^{2\pi} |\bar{\mathbf{w}} - \lambda \mathbf{e}_1|^2 \mathrm{d}\theta \\ &\leqq 2D_\lambda + C D_\lambda \log \frac{2}{\lambda r} \\ &\leqq C D_\lambda \log \frac{2}{\lambda r}\end{aligned}$$

for $1 \leqq r \leqq \frac{3}{2}\lambda^{-1}$. Integrating the above in r with respect to the measure $r dr$ gives

$$\int_{\Omega_{\frac{1}{2}r, \frac{3}{2}r}} |\mathbf{w} - \lambda \mathbf{e}_1|^2 \, \mathrm{d}x\mathrm{d}y \leqq C r^2 D_\lambda \log \frac{2}{\lambda r}$$

for any $2 \leqq r \leqq \lambda^{-1}$. (Recall our notation $\Omega_{r_1,r_2} := \{z : r_1 < |z| < r_2\}$.) Now we use Ladyzhenskaya's inequality to obtain

$$\begin{aligned}\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\mathbf{w}-\lambda\mathbf{e}_1|^4 \,\mathrm{d}x\mathrm{d}y &\leqq C \int_{\Omega_{\frac{1}{2}r,\frac{3}{2}r}} |\mathbf{w}-\lambda\mathbf{e}_1|^2 \,\mathrm{d}x\mathrm{d}y \Big(\int_{\Omega_{\frac{1}{2}r,\frac{3}{2}r}} |\nabla\mathbf{w}|^2 \,\mathrm{d}x\mathrm{d}y \\ &\qquad + \frac{1}{r^2} \int_{\Omega_{\frac{1}{2}r,\frac{3}{2}r}} |\mathbf{w}-\lambda\mathbf{e}_1|^2 \,\mathrm{d}x\mathrm{d}y \Big) \\ &\leqq C r^2 D_\lambda^2 \left(\log\frac{2}{\lambda r}\right)^2 \end{aligned} \tag{B.4}$$

for any $2 \leqq r \leqq \lambda^{-1}$. By Hölder's inequality and (B.4), we have

$$\begin{aligned}\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\mathbf{w}\cdot\nabla\mathbf{w}|^{\frac{4}{3}} \,\mathrm{d}x\mathrm{d}y &\leqq \left(\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\mathbf{w}|^4 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{3}} \left(\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\nabla\mathbf{w}|^2 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{2}{3}} \\ &\lesssim \left(\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\mathbf{w}-\lambda\mathbf{e}_1|^4 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{3}} \left(\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\nabla\mathbf{w}|^2 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{2}{3}} \\ &\quad + \left(\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} \lambda^4 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{3}} \left(\int_{\Omega_{\frac{2}{3}r,\frac{4}{3}r}} |\nabla\mathbf{w}|^2 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{2}{3}} \\ &\leqq C r^{\frac{2}{3}} D_\lambda^{\frac{4}{3}} \left(\log\frac{2}{\lambda r}\right)^{\frac{2}{3}} + C r^{\frac{2}{3}} \lambda^{\frac{4}{3}} D_\lambda^{\frac{2}{3}} \\ &\leqq C r^{\frac{2}{3}} \lambda^{\frac{4}{3}} D_\lambda^{\frac{2}{3}} \max\left\{ |\log\lambda|^{-\frac{2}{3}} \left(\log\frac{2}{\lambda r}\right)^{\frac{2}{3}}, 1 \right\} \\ &\leqq C r^{\frac{2}{3}} \lambda^{\frac{4}{3}} D_\lambda^{\frac{2}{3}} \end{aligned} \tag{B.5}$$

for any $2 \leqq r \leqq \lambda^{-1}$. Local regularity theory for Stokes system as shown in Section 2.3 yields the estimate

$$\begin{aligned}\|\nabla^2\mathbf{w}\|_{L^{\frac{4}{3}}(\Omega_{\frac{3}{4}r,\frac{5}{4}r})} &\leqq C\Big(\frac{1}{r^2}\|\mathbf{w}-\lambda\mathbf{e}_1\|_{L^{\frac{4}{3}}(\Omega_{\frac{2}{3}r,\frac{4}{3}r})} + \frac{1}{r}\|\nabla\mathbf{w}\|_{L^{\frac{4}{3}}(\Omega_{\frac{2}{3}r,\frac{4}{3}r})} \\ &\qquad + \|\mathbf{w}\cdot\nabla\mathbf{w}\|_{L^{\frac{4}{3}}(\Omega_{\frac{2}{3}r,\frac{4}{3}r})}\Big)\end{aligned}$$

where C is independent of r. Applying (B.4), (3.1) and (B.5) to the above inequality, we obtain

$$\|\nabla^2\mathbf{w}\|_{L^{\frac{4}{3}}(\Omega_{\frac{3}{4}r,\frac{5}{4}r})} \leqq C r^{-\frac{1}{2}} D_\lambda^{\frac{1}{2}} \left(\log\frac{2}{\lambda r}\right)^{\frac{1}{2}}$$

for any $2 \leqq r \leqq \lambda^{-1}$, which clearly implies

$$\|\nabla^2 \mathbf{w}\|_{L^1(\Omega_{\frac{3}{4}r,\frac{5}{4}r})} \leqq C D_\lambda^{\frac{1}{2}} \left(\log \frac{2}{\lambda r}\right)^{\frac{1}{2}}.$$

Now, Sobolev space theory (see, e.g., [2, Lemma 4.3]) gives the following bound for the variation of $\mathbf{w}$:

$$\begin{aligned} \operatorname{diam} \mathbf{w}(\Omega_{\frac{3}{4}r,\frac{5}{4}r}) &\leqq C\Big(\|\nabla \mathbf{w}\|_{L^2(\Omega_{\frac{3}{4}r,\frac{5}{4}r})} + \|\nabla^2 \mathbf{w}\|_{L^1(\Omega_{\frac{3}{4}r,\frac{5}{4}r})}\Big) \\ &\leqq C D_\lambda^{\frac{1}{2}} \left(\log \frac{2}{\lambda r}\right)^{\frac{1}{2}}, \end{aligned} \tag{B.6}$$

for any $2 \leqq r \leqq \lambda^{-1}$. Together with (B.3), (B.6) gives the desired bound

$$|\mathbf{w}(z) - \lambda \mathbf{e}_1| \leqq C D_\lambda^{\frac{1}{2}} \left(\log \frac{2}{\lambda r}\right)^{\frac{1}{2}} \tag{B.7}$$

in the region $2 \leqq r \leqq \lambda^{-1}$. To finish the proof, we point out that for the region $\{r \leqq 2\} \cap \mathcal{E}$, due to Lemma 11, (B.7) also holds true. □

References

1. Amick, C.J.: On Leray's problem of steady Navier-Stokes flow past a body in the plane. *Acta Math*. **161**, 71–130, 1988
2. Bourgain, J., Korobkov, M., Kristensen, J.: On the Morse-Sard property and level sets of Sobolev and BV functions Rev. *Mat. Iberoam*. **29**, 1–23, 2013
3. Coifman, R.R., Lions, J.L., Meier, Y., Semmes, S.: Compensated compactness and Hardy spaces. *J. Math. Pures App. IX Sér*. **72**, 247–286, 1993
4. Ferone, A., Russo, R., Tartaglione, A.: The Stokes Paradox in Inhomogeneous Elastostatics. *J. Elasticity* **142**(1), 35–52, 2020
5. Finn, R.: On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Rational. Mech. Anal*. **19**, 363–406, 1965
6. Finn, R., Smith, D.R.: On the linearized hydrodynamical equations in two dimensions. *Arch. Ration. Mech. Anal*. **25**, 1–25, 1967
7. Finn, R., Smith, D.R.: On the stationary solutions of the Navier-Stokes equations in two dimensions. *Arch. Ration. Mech. Anal*. **25**, 26–39, 1967
8. Fujita, H.: On the existence and regularity of the steady-state solutions of the Navier-Stokes squation. *J. Fac. Sci. Univ. Tokyo* **9**(1A), 59–102, 1961
9. Galdi, G.P., Sohr, H.: On the asymptotic structure of plane steady flow of a viscous fluid in exterior domains. *Arch. Ration. Mech. Anal*. **131**, 101–119, 1995
10. Galdi, G.P.: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems*. Springer, Berlin 2011
11. Gilbarg, D., Weinberger, H.F.: Asymptotic properties of Leray's solution of the stationary two-dimensional Navier-Stokes equations. *Russian Math. Surv*. **29**, 109–123, 1974
12. Gilbarg, D., Weinberger, H.F.: Asymptotic properties of steady plane solutions of the Navier–Stokes equation with bounded Dirichlet integral. *Ann. Scuola Norm. Sup. Pisa Cl. Sci*. **5**(4), 381–404, 1978
13. Guillod, J.: Steady solutions of the Navier–Stokes equations in the plane. arXiv:1511.03938

14. Guillod, J., Wittwer, P.: Optimal asymptotic behavior of the vorticity of a viscous flow past a two-dimensional body. *J. Math. Pures Appl.* **118**, 481–499, 2017
15. Guillod, J., Wittwer, P.: Existence and uniqueness of steady weak solutions to the Navier-Stokes equations in $\mathbb{R}^2$. *Proc. Am. Math. Soc.* **146**, 4429–4445, 2018
16. Hillaret, M., Wittwer, P.: On the existence of solutions to the planar exterior Navier Stokes system. *J. Differ. Equ.* **255**, 2996–3019, 2013
17. Hopf, E.: Ein Allgemeiner Endlichkeitsatz der Hydrodynamik. *Math. Ann.* **117**, 764–775, 1941
18. Korobkov, M.V., Pileckas, K., Russo, R.: On convergence of arbitrary D-solution of steady Navier-Stokes system in $2D$ exterior domains. *Arch. Ration. Mech. Anal.* **233**(1), 385–407, 2019
19. Korobkov, M.V., Pileckas, K., Russo, R.: On the steady Navier-Stokes equations in $2D$ exterior domains. *J. Differ. Equ.* **269**(3), 1796–1828, 2020
20. Korobkov, M.V., Pileckas, K., Russo, R.: Leray's plane steady state solutions are nontrivial. *Adv. Math.* **269**(3), 1796–1828, 2020
21. Ladyzhenskaya, O.A.: *The Mathematical Theory of Viscous Incompressible Flow*. Gordon & Breach, New York 1969
22. Lieb, E.H., Loss, M.: *Analysis, Graduate Studies in mathematics*, vol. 14, 2nd edn. AMS, Providence 2001
23. Leray, J.: Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.* **12**, 1–82, 1933
24. Nakatsuka, T.: On uniqueness of symmetric Navier-Stokes flows around a body in the plane. *Adv. Differ. Equ.* **20**(3/4), 193–212, 2013
25. Oseen, C.W.: *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Akademische Verlagsgesellschaft m.b.H, Leipzig 1927
26. Russo, R.: On Stokes' problem. In: Rannacher, R., Sequeira, A. (eds.) *Advances in Mathematical Fluid Mechanics*, pp. 473–511. Springer, Berlin 2010
27. Sazonov, L.I.: On the asymptotic behavior of the solution to the two-dimensional stationary problem of the flow past a body far from it. *Math. Notes* **65**(2), 202–207, 1999
28. Sazonov, L.I.: The justification of an asymptotic expansion for the solution of the two-dimensional flow problem at small Reynolds numbers. *Izvestiya Math.* **67**(5), 975–1006, 1999
29. Smith, D.R.: Estimates at infinity for stationary solutions of the Navier-Stokes equations in two dimensions. *Arch. Rational Mech. Anal.* **20**, 341–372, 1965
30. Stein, E.: *Harmonic Analysis: Real-Variables Methods*. Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton 1993
31. Stokes, G.G.: On the effect of the internal friction of fluids on the motion of pendulums. *Trans. Cambridge Phil. Soc.* **9**, 8–106, 1851
32. Yudovich, V.I.: Eleven great problems of mathematical hydrodynamics. *Mosc. Math. J.* **3**, 711–737, 2003
33. Yamazaki, M.: Unique existence of stationary solutions to the two-dimensional Navier-Stokes equations on exterior domains. In: Adachi, T., et al. (eds.) *Mathematical Aspects on the Navier-Stokes Equations and Related Topics, Past and Future - In memory of Professor Tetsuro Miyakawa, Gakuto International Series, Mathematical Sciences and Applications*, vol. 35, pp. 220–241. Gakkotosho, Tokyo 2011
34. Yamazaki, M.: Existence and uniqueness of weak solutions to the two-dimensional stationary Navier-Stokes exterior problem. *J. Math. Fluid Mech.* **20**(4), 2019–2051, 2018

M. Korobkov, X. Ren
School of Mathematical Sciences,
Fudan University,
Shanghai
200433 People's Republic of China.
e-mail: xiaoren18@fudan.edu.cn

and

M. Korobkov
Sobolev Institute of Mathematics,
pr-t Ac. Koptyug, 4,
Novosibirsk
Russia. 630090
e-mail: korob@math.nsc.ru

(*Received December 10, 2020 / Accepted March 11, 2021*)
Published online March 23, 2021