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Small Data Global Well-Posedness for a Boltzmann Equation via Bilinear Spacetime Estimates

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Abstract

We provide a new analysis of the Boltzmann equation with a constant collision kernel in two space dimensions. The scaling-critical Lebesgue space is $L^2_{x,v}$; we prove the global well-posedness and a version of scattering, assuming that the data f_0 is sufficiently smooth and localized, and the $L^2_{x,v}$ norm of f_0 is sufficiently small. The proof relies upon a new scaling-critical bilinear spacetime estimate for the collision "gain" term in Boltzmann's equation, combined with a novel application of the Kaniel–Shinbrot iteration.

Contents

1. Introduction and Main Results

1.1. Background

Boltzmann's equation describes the time-evolution of the phase-space density $f(t, x, v)$ of a dilute gas, accounting for both dispersion under the free flow and dissipation as the result of collisions. We are interested in the Boltzmann equation with *constant collision kernel* in the plane, $\mathbb{R}^2_x \times \mathbb{R}^2_v$, which is written as follows:

$$\begin{aligned}&(\partial_t + v \cdot \nabla_x)\, f(t, x, v)\\&\quad= \int_{\mathbb{S}^1} \mathrm{d}\omega \int_{\mathbb{R}^2} \mathrm{d}u \left\{ f(t, x, v^*) f(t, x, u^*) - f(t, x, v) f(t, x, u) \right\}, \qquad (1.1)\end{aligned}$$

with prescribed initial data $f(0, x, v) = f_0(x, v)$, and $(t, x, v) \in [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$. Here the symbols u^*, v^* are defined by the *collisional change of variables*

$$\begin{aligned}u^* &= u + (\omega \cdot (v - u))\omega\\ v^* &= v - (\omega \cdot (v - u))\omega\end{aligned}$$

and $\omega \in \mathbb{S}^1 \subset \mathbb{R}^2$ is a unit vector. We may also write

$$(\partial_t + v \cdot \nabla_x)\, f = Q(f, f) = Q^+(f, f) - Q^-(f, f), \qquad (1.2)$$

where

$$Q^+(f, g)(x, v) = \int_{\mathbb{S}^1} \mathrm{d}\omega \int_{\mathbb{R}^2} \mathrm{d}u f(x, v^*) g(x, u^*) \qquad (1.3)$$

$$Q^-(f, g)(x, v) = 2\pi f(x, v) \rho_g(x) \qquad (1.4)$$

and

$$\rho_f(x) = \int_{\mathbb{R}^2} \mathrm{d}v f(x, v).$$

The PDE (1.1) is scaling-critical, independently in x and v, for the $L^2\left(\mathbb{R}_x^2 \times \mathbb{R}_v^2\right)$ norm of f_0.

The Cauchy problem for (1.1), specifically with the constant collision kernel, is by now a mature subject and many different techniques are available. One of the oldest known techniques is the Kaniel–Shinbrot iteration [14], which will be explained in detail in Section 2; this is a monotonicity-based technique for producing a non-negative solution of Boltzmann's equation. Strichartz estimates have been used in [3] to solve equations related to (1.1) but containing a cut-off in the interaction at large velocities. Scattering was subsequently addressed in [13], again using Strichartz estimates. Global well-posedness has been proven near equilibrium by a variety of techniques [1,11,12,22], all of which rely somehow on a notion of Dirichlet form (and sometimes requiring the long-range version of (1.1), e.g., *true Maxwell molecules*). For more background on Boltzmann's equation we refer the reader to [6]. Weaker notions of solution are available globally in time due to DiPerna and Lions [9], but uniqueness remains an open problem for such solutions.

The difficulty with solving (1.1) at critical regularity is actually more challenging than appears to be customarily acknowledged, because though the two terms on the right hand side (known as "gain" Q^+ and "loss" Q^-, respectively) both scale the same way, they do *not* share the same estimates. In fact, the gain term exhibits a convolutive effect (similar to $f *_v g$) which is not observed with the loss term. This problem was acknowledged in [3] and dealt with by introducing a cutoff in the collision kernel at large velocities, thereby breaking the scale-invariance of the problem.

In the present work, we take the point of view that the data f_0 should be *sufficiently localized and regular enough* (in the sense of weighted L^2-based Sobolev spaces) to makes sense of both "gain" and "loss" terms, *but* that the theorem should only depend on the smallness of the critical norm, in this case L^2. The advantage of this approach is that the local iteration relies purely upon energy estimates in L^2-based spaces. In particular, we will prove a bilinear estimate of the form

$$L^2_{x,v} \times L^2_{x,v} \to L^1_{t\in\mathbb{R}} L^2_{x,v}$$

for the Q^+ operator (acting on the free flow), which is new to the best of our knowledge. Once this bilinear estimate is in hand, any space of *mixed* integrability in x, v, e.g. $L^p_x L^r_v$ with $p \neq r$, arises only as the result of Sobolev embedding applied to an L^2-based Sobolev norm.

In our analysis, we will invoke the approach that we introduced in [7,8], based on the Wigner transform of the Boltzmann equation, which makes the problem naturally accessible to a combination of techniques from both kinetic theory, and dispersive nonlinear PDEs.

1.2. Summary of the Present Work

The subject of this paper is a new treatment of the Boltzmann equation with constant collision kernel in $d = 2$, which is scaling-critical for the space $L^2_{x,v}$. We prove global well-posedness and scattering for solutions with small norm in the critical space $L^2_{x,v}$, whenever $\left\| \langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f_0 \right\|_{L^2_{x,v}}$ is *finite* but not necessarily small.

Our proof relies on the Kaniel–Shinbrot iteration, as recommended in the introduction to [3]. As far as we are aware, this is the first time that the Kaniel–Shinbrot iteration has been implemented outside Maxwellian-weighted L^∞ spaces. Moreover, a uniqueness result will be proven which does not require either non-negativity or Sobolev regularity of solutions. Therefore, the existence of a non-negative solution from Kaniel–Shinbrot will imply that any other local solution in the correct *integrability* class is automatically non-negative and coincides with the Kaniel–Shinbrot solution. From there, the extra regularity is propagated *a posteriori*, globally in time (with possibly large growth rate), by constructing sufficiently regular local solutions and employing standard commutation rules.

Our proof relies on the Wigner transform and endpoint Strichartz estimates due to Keel–Tao [15] for hyperbolic Schrödinger equations in the *doubled* dimension $2d = 4$. We point out that endpoint kinetic Strichartz estimates are *false* [4] in all dimensions. For this reason, there is no obvious analogue of our proof which employs the kinetic picture exclusively.

1.3. Main Results

Our main results are summarized as follows:

Theorem 1.1. *There exists a number $\eta_0 > 0$ such that all of the following are simultaneously true:*

Supposing that $f_0(x, v) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a non-negative*, measurable, locally integrable function such that*

$$\left\| \langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f_0(x, v) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} < \infty \tag{1.5}$$

and

$$\left\| f_0(x, v) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} < \eta_0, \tag{1.6}$$

then there exists a globally defined (for $t \geqq 0$) non-negative mild solution $f \in C\left([0, \infty), L^2_{x,v}\right)$ of Boltzmann's equation

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = Q(f, f), \tag{1.7}$$

where

$$Q(f, f) = Q^+(f, f) - Q^-(f, f),$$

with Q^+ and Q^- given in (1.3) and (1.4), respectively, such that $f(0) = f_0$ and the following bounds (1.8), (1.9), (1.10) hold for any $T \in (0, \infty]$ (noting that $T = +\infty$ is included):

$$\langle v \rangle^{\frac{1}{2}+} Q^+(f, f) \in L^1_{t \in [0,T]} L^2_{x,v} \tag{1.8}$$

$$\rho_f \in L^2_{t\in[0,T]} L^\infty_x \bigcap L^2_{t\in[0,T]} L^4_x \tag{1.9}$$

$$\langle v \rangle^{\frac{1}{2}+} f \in L^\infty_{t\in[0,T]} L^2_{x,v} \bigcap L^\infty_{t\in[0,T]} L^4_x L^2_v. \tag{1.10}$$

The solution $f(t)$ is unique in the class of all mild solutions, with the same initial data, satisfying the bounds (1.8)*,* (1.9)*,* (1.10) *for each $T \in (0,\infty)$. In particular, any mild solution with data f_0 satisfying* (1.8)*,* (1.9)*,* (1.10) *is automatically non-negative (since it is equal to f).*

The solution $f(t)$ also satisfies

$$\|f\|^2_{L^\infty_{t\geqq 0} L^2_{x,v}} + \left\| Q^+(f,f) \right\|_{L^1_{t\geqq 0} L^2_{x,v}} \leqq C \, \|f_0\|^2_{L^2_{x,v}} \tag{1.11}$$

Moreover, $f(t)$ scatters in $L^2_{x,v}$ as $t \to +\infty$; equivalently, $f_{+\infty} = \lim_{t\to+\infty} T(-t) f(t)$ exists in the norm topology in $L^2_{x,v}$. Here $T(t) = e^{-tv\cdot\nabla_x}$.

Finally, $f(t)$ carries (a posteriori) the same regularity as the initial data

$$\forall T > 0, \qquad \left\| \langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f(t) \right\|_{L^\infty_{t\in[0,T]} L^2_{x,v}} < \infty. \tag{1.12}$$

Remark 1.1. We note that no claim is made regarding the injectivity or non-injectivity for the map $f_0 \mapsto f_{+\infty}$. Moreover, no claim is made as to whether or not the bound in (1.12) is uniform as $T \to \infty$.

Remark 1.2. The constant C appearing in (1.11) is absolute, requiring only the imposed condition that $\|f_0\|_{L^2} < \eta_0$ for another absolute constant η_0. The existence of such an absolute C indicates that the behavior of Boltzmann's equation is effectively linear on long timescales if the $L^2_{x,v}$ norm of f_0 is sufficiently small. Note that the bound (1.11) appears to be new.

Remark 1.3. It is an easy consequence of the $Q^+(f,f)$ estimate (1.11), of Duhamel's formula, and Minkowski's inequality, along with the *homogeneous* Strichartz estimates, that the solution of (1.7) satisfies $f \in L^q_t L^r_x L^p_v \big([0,\infty) \times \mathbb{R}^2 \times \mathbb{R}^2\big)$, whenever $p, r \geqq 1$, $q > 2$, $\frac{1}{r} + \frac{1}{p} = 1$, and $\frac{1}{q} = \frac{1}{p} - \frac{1}{r}$. This is the full range of homogeneous Strichartz estimates expected for L^2 solutions of the free transport equation in $d = 2$. We do not mention estimates of this form in Theorem 1.1 because they are not relevant to the method of the proof.

1.4. The Local Well-Posedness Theorem

We will also prove a local well-posedness theorem, following a similar line of reasoning. We point out that while the data is required to have $\frac{1}{2}+$ regularity, the *time of existence* depends only on regularity at the s level for an arbitrary $s \in \left(0, \frac{1}{2}\right)$. We are not aware of any analogous theorem in the literature which works at arbitrarily small fractional (but non-zero) regularities for any Boltzmann equation; the proof relies on a novel interpolation strategy which would be difficult to implement in the usual framework of inhomogeneous Strichartz estimates. We also remark that the theorem is optimal because $s = 0$ is scaling critical, so we cannot expect a local theorem depending only on the size of the L^2 norm of the data.

Theorem 1.2. *Fix a number* $s \in \left(0, \frac{1}{2}\right)$. *Then there exists a function* $\lambda_s(\cdot) : \mathbb{R}^{\geqq 0} \to \mathbb{R}^{\geqq 0}$ *such that all of the following is true:*

Suppose $f_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is a* non-negative*, locally integrable function such that*

$$\left\| \langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f_0(x, v) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} < \infty. \tag{1.13}$$

Then, for some T_0 *satisfying*

$$T_0 > \lambda_s \left(\left\| \langle v \rangle^s \langle \nabla_x \rangle^s f_0 \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \right), \tag{1.14}$$

there exists a non-negative mild solution $f \in C\left([0, T_0), L^2_{x,v}\right)$ *of Boltzmann's equation*

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = Q(f, f), \tag{1.15}$$

where

$$Q(f, f) = Q^+(f, f) - Q^-(f, f),$$

with Q^+ *and* Q^- *given respectively in* (1.3) *and* (1.4)*, such that* $f(0) = f_0$ *and the following bounds* (1.16), (1.17), (1.18) *hold for any* $T \in (0, T_0)$*:*

$$\langle v \rangle^{\frac{1}{2}+} Q^+(f, f) \in L^1_{t \in [0,T]} L^2_{x,v} \tag{1.16}$$

$$\rho_f \in L^2_{t \in [0,T]} L^\infty_x \bigcap L^2_{t \in [0,T]} L^4_x \tag{1.17}$$

$$\langle v \rangle^{\frac{1}{2}+} f \in L^\infty_{t \in [0,T]} L^2_{x,v} \bigcap L^\infty_{t \in [0,T]} L^4_x L^2_v. \tag{1.18}$$

The solution $f(t)$ *is unique in the class of all mild solutions, with the same initial data, satisfying all the bounds* (1.16, 1.17, 1.18) *for each* $T \in (0, T_0)$. *In particular, any mild solution with data* f_0 *satisfying* (1.16, 1.17, 1.18) *is automatically non-negative (since it is equal to* f*).*

We are not able to show that the $\frac{1}{2}+$ regularity assumed at $t = 0$ is propagated, but we expect this to be true and state it as a conjecture.

Conjecture 1.1. *In the notation of Theorem* 1.2*, the local solution* $f(t)$ *carries the regularity of the data up to time* T_0. *More precisely, for any* $T \in (0, T_0)$*, it holds that*

$$\left\| \langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f(t) \right\|_{L^\infty_{t \in [0,T]} L^2_{x,v}} < \infty \tag{1.19}$$

Remark 1.4. It is possible to show that the regularity is propagated for a time that depends on the size of the $\frac{1}{2}+$ norm at time $t = 0$. The point of the conjecture is that the $\frac{1}{2}+$ regularity persists for a time depending on a *lower* regularity norm, namely the s norm.

Remark 1.5. In view of Theorem 1.2, where the time of existence depends on a norm which is very close to L^2, it is natural to ask whether it is possible to prove local well-posedness in a space like L^2 or $L^2 \cap L^1$ (note that the L^1 norm is conserved for Boltzmann's equation). Since L^2 is a critical norm for the Boltzmann equation with constant collision kernel, the best we can hope for is a local well-posedness time which depends on the profile of the initial data. Unfortunately, so far we have not been able to extract such a result using our method, though there is no obvious obstruction. Several *a priori* estimates are available in complete generality for L^2 solutions on a short time interval (assuming that a certain spacetime integral is finite in which case it is bounded quantatively), and they are presented in Appendix C.

2. Technical Preliminary: The Kaniel–Shinbrot Iteration

In this section, we present a brief review of the Kaniel–Shinbrot iteration method (see [14]) for proving existence of solutions for Boltzmann equations, and describe its typical use. Then we give a short preview of the new approach based on of the Kaniel–Shinbrot iteration method that we introduce in this paper.

2.1. The Method of Kaniel and Shinbrot in a Nutshell

The method of Kaniel and Shinbrot is based on three main steps:

(1) Construct a pair of functions satisfying the so-called *beginning condition*.
(2) Develop sequences of functions which act as *barriers* (above and below) which converge monotonically to upper and lower envelopes of a (hypothetical) true solution.
(3) Prove that the upper and lower envelopes coincide, hence defining a solution to the Boltzmann equation itself.

We note that there is no claim of uniqueness in the Kaniel–Shinbrot iteration, though the third step (convergence) is typically as hard to prove as uniqueness. Usually, one views Kaniel–Shinbrot as a proof of *existence by construction*, followed by a separate proof of uniqueness in a class of solutions containing the Kaniel–Shinbrot solution.

We start with two functions g_1, h_1, which are supposed to be upper and lower bounds (respectively) for a true solution of Boltzmann's equation. The first iterates g_2, h_2 are defined by the formulas

$$\begin{aligned} \left(\partial_t + v\cdot\nabla_x + 2\pi\rho_{h_1}\right) g_2 &= Q^+(g_1,g_1) \\ \left(\partial_t + v\cdot\nabla_x + 2\pi\rho_{g_1}\right) h_2 &= Q^+(h_1,h_1) \\ g_2(t=0) = h_2(t=0) &= f_0. \end{aligned} \tag{2.1}$$

Kaniel and Shinbrot [14] assume that g_1, h_1 are chosen to guarantee the following inequalities (for all times on the interval of interest):

$$0 \leqq h_1 \leqq h_2 \leqq g_2 \leqq g_1; \tag{2.2}$$

this is the so-called *beginning condition* of the Kaniel–Shinbrot iteration.

The beginning condition (2.2) secured, Kaniel and Shinbrot define the rest of the iteration (here $n \geqq 2$) as follows:

$$\begin{aligned} \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_{h_n}\right) g_{n+1} &= Q^+(g_n, g_n) \\ \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_{g_n}\right) h_{n+1} &= Q^+(h_n, h_n) \\ g_{n+1}(t=0) = h_{n+1}(t=0) &= f_0 \end{aligned} \tag{2.3}$$

They prove by induction that, as long as the beginning condition (2.2) is satisfied, the following inequalities hold for each n:

$$0 \leqq h_1 \leqq h_n \leqq h_{n+1} \leqq g_{n+1} \leqq g_n \leqq g_1. \tag{2.4}$$

In other words, there is a sequence h_n increasing from below and a decreasing sequence g_n, all bounded above by the fixed function g_1. This allows us to apply monotone convergence pointwise and conclude the existence (under mild regularity assumption) of limits g, h with $0 \leqq h \leqq g \leqq g_1$ satisfying the following equations:

$$\begin{aligned} (\partial_t + v \cdot \nabla_x + 2\pi \rho_h)\, g &= Q^+(g, g) \\ \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_g\right) h &= Q^+(h, h) \\ g(t=0) = h(t=0) &= f_0. \end{aligned} \tag{2.5}$$

This system is satisfied, of course, if $g = h = f$ is the (supposedly unique) solution of Boltzmann's equation; hence, if the system has a unique solution $(g, h = g)$, then that solution is exactly the unique solution of Boltzmann's equation. Thus the question of convergence of the Kaniel–Shinbrot scheme is closely related to a uniqueness question.

Remark 2.1. The method of Kaniel–Shinbrot [14] is applicable to the Boltzmann equation under an angular cutoff condition (Grad cut-off). We note that the Boltzmann equation with constant collision kernel satisfies Grad's cut-off (it is enough to note that Q^+ and $Q^- = f\rho_f$ each make sense taken separately, if f is nice enough).

Usually we do not prove that the system (2.5) has a unique solution, since this requires more effort than is actually necessary. In fact, if we can only prove that $g \equiv h$ (for instance by a Gronwall argument), then the function g (or equivalently h) is itself a solution of Boltzmann's equation, but there is no guarantee of uniqueness. In that case, uniqueness is usually proven by an independent argument. This is indeed the strategy employed in the present work.

The Kaniel–Shinbrot iteration has been applied to "large" initial conditions which are "squeezed" between two nearby Maxwellian distributions. This was first achieved by Toscani [21], using a clever choice of (locally Maxwellian) functions g_1, h_1 satisfying the beginning condition of Kaniel and Shinbrot. The approach was later adapted to soft potentials (with Grad cut-off) by Alonso and Gamba. [2]. Unfortunately, it is not clear to us how to adapt Toscani's proof to the scaling-critical ($L^2_{x,v}$) setting; the lower envelope h_1 should presumably be a Maxwellian, but the upper envelope g_1 must be some L^2 function which tracks the singularities of the data. There does not appear to be an obvious choice for upper envelope g_1 (satisfying the beginning condition) when the data f_0 is not small.

2.2. The Method of Kaniel and Shinbrot Revisited

The beginning conditions for Kaniel–Shinbrot is traditionally satisfied by taking g_1 to be a Maxwellian distribution which bounds f_0 from above, with $h_1 \equiv 0$; or, by "squeezing" f between two Maxwellians g_1, h_1 which need not be small (but must be close to each other). However these ideas do not work in our setting since f_0 does not need to be bounded above pointwise; indeed, the only *quantitative* estimate we are allowed is that $f_0 \in L^2_{x,v}$.

Instead, our strategy is to solve the **gain-term-only** Boltzmann equation using a bilinear estimate, and subsequently apply the Kaniel–Shinbrot iteration to the solution of the gain-only equation in order to develop a solution of the full Boltzmann equation. Thus, for us, h_1 is identically zero and g_1 satisfies

$$(\partial_t + v \cdot \nabla_x)\, g_1 = Q^+(g_1, g_1),$$

with initial data $g_1(t = 0) = f_0$. It would seem that the Kaniel–Shinbrot iteration gains us nothing, since we are initiating the iteration with the solution to a nonlinear equation. However, it turns out that at critical regularity, the gain-only equation is easier to solve than the full Boltzmann equation, as was observed by D. Arsenio, [3] In particular, the gain term Q^+ satisfies bilinear estimates which are not available for the loss term.

Remark 2.2. The suggestion to apply Kaniel–Shinbrot at low regularities is due to Arsenio in [3], who discussed the possibility in the introduction. However, Arsenio did *not* implement the Kaniel–Shinbrot iteration, instead relying on a compactness argument, apparently due to the lack of uniqueness in his formulation. We have overcome this limitation by propagating some auxiliary regularity and moment bounds for the gain-only equation, to the point that a uniqueness theorem for the full Boltzmann equation is indeed available, thereby allowing us to prove convergence of the Kaniel–Shinbrot iteration.

3. An Abstract Well-Posedness Theorem

In this section we present an abstract well-posedness theorem, which is inspired by "space-time" methods that are often used in the context of dispersive PDEs.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and let $k \geqq 2$ be an integer. Suppose that we have a map

$$\mathcal{A} : \mathcal{H}^{\times k} \to L^1(\mathbb{R}, \mathcal{H}) \tag{3.1}$$

such that $\mathcal{A}$ is linear with respect to each factor of $\mathcal{H}$ (keeping the others fixed), and an estimate of the following form holds:

$$\|\mathcal{A}(x_1, \ldots, x_k)(t)\|_{L^1_t \mathcal{H}} \leqq C_0 \prod_{j=1}^{k} \|x_j\|_{\mathcal{H}} \qquad x_1, \ldots, x_k \in \mathcal{H}. \tag{3.2}$$

We will say that $\mathcal{A}$ is a bounded k-linear map $\mathcal{H}^{\times k} \to L_t^1\mathcal{H}$, and we will generally write it equivalently as $\mathcal{A}(t, x_1, \ldots, x_k)$. We are interested in properly defining, and then solving, the equation

$$\frac{dx}{dt} = \mathcal{A}(t, x(t), \ldots, x(t)), \tag{3.3}$$

when $x(0) = x_0 \in \mathcal{H}$ is a given element of $\mathcal{H}$ with small norm. As we will see, the bound (3.2) along with the k-linearity is *sufficient* to solve (3.3) globally in time for small data; scattering will also follow automatically, in the sense that $\lim_{t\to+\infty} x(t)$ exists in the norm topology of $\mathcal{H}$. We will find that $x(t) \in W^{1,1}((0,T),\mathcal{H})$ for any $T > 0$, so equation (3.3) holds in a strong sense. The theorem, along with its proof, is inspired by certain methods due to Klainerman and Machedon for solving dispersive PDE [18,19].

Remark 3.1. In the complex case, it is acceptable for $\mathcal{A}$ to be conjugate linear with respect to some or all entries; the changes to the proof are trivial so we only discuss the linear case.

Note that *a priori* we can only evaluate $\mathcal{A}(t, x_1, \ldots, x_k)$ for a.e. t given *fixed* elements $x_1, \ldots, x_k$ of $\mathcal{H}$; in particular, the exceptional set in t may depend on $x_1, \ldots, x_k$. However, if $x(t)$ is a C^1 curve, then near any given time t_0, x is almost a constant. This observation motivates the following result:

Lemma 3.1. *Let $\mathcal{H}$ be a separable Hilbert space and suppose $\mathcal{A} : \mathcal{H}^{\times k} \to L_t^1\mathcal{H}$ is a mapping which is linear or conjugate linear in each entry; furthermore, suppose that the estimate (3.2) holds. Then, for any $T \in (0,\infty)$, there exists a unique k-linear map*

$$\tilde{\mathcal{A}} : \left(W^{1,1}((0,T),\mathcal{H})\right)^{\times k} \to L^1((0,T),\mathcal{H}) \tag{3.4}$$

which satisfies

$$\tilde{\mathcal{A}}(t, f_1x_1, \ldots, f_kx_k) = \left(\prod_{j=1}^{k} f_j(t)\right)\mathcal{A}(t, x_1, \ldots, x_k) \tag{3.5}$$

for any $x_1, \ldots, x_k \in \mathcal{H}$ and any smooth bounded real-valued functions $f_1, \ldots, f_k$ on $[0,T]$; here, f_jx_j denotes the function $(f_jx_j)(t) = f_j(t)x_j$. It holds as well that

$$\begin{aligned}\left\|\tilde{\mathcal{A}}(t, x_1(\cdot), \ldots, x_k(\cdot))\right\|_{L^1_{t\in(0,T)}\mathcal{H}} \\ \leqq (1+k)C_0 \prod_{j=1}^{k}\left(\left\|x_j(t)\right\|_{L^\infty_{t\in(0,T)}\mathcal{H}} + \left\|\frac{dx_j}{dt}\right\|_{L^1_{t\in(0,T)}\mathcal{H}}\right)\end{aligned} \tag{3.6}$$

for any $x_1(\cdot), \ldots, x_j(\cdot) \in W^{1,1}((0,T),\mathcal{H})$.

Proof. (Sketch.) It is possible to prove this result by expanding each x_j via Duhamel's formula and using k-linearity. However, it is much easier to simply differentiate $\mathcal{A}$ directly as follows, denoting $\zeta_j = \frac{dx_j}{dt}$:

$$\frac{\partial}{\partial\sigma}\mathcal{A}\left(t, x_1(\sigma), \ldots, x_k(\sigma)\right) = \mathcal{A}\left(t, \zeta_1(\sigma), x_2(\sigma), \ldots, x_k(\sigma)\right) + \cdots + \mathcal{A}\left(t, x_1(\sigma), \ldots, x_{k-1}(\sigma), \zeta_k(\sigma)\right). \tag{3.7}$$

We can integrate both sides in σ from 0 to t, in order to relate the diagonal $\sigma = t$ in terms of quantities off the diagonal:

$$\begin{aligned}\mathcal{A}\left(t, x_1(t), \ldots, x_k(t)\right) = {}& \mathcal{A}\left(t, x_1(0), \ldots, x_k(0)\right)\\ &+ \int_0^t \mathcal{A}\left(t, \zeta_1(\sigma), x_2(\sigma), \ldots, x_k(\sigma)\right) \mathrm{d}\sigma\\ &+ \cdots + \int_0^t \mathcal{A}\left(t, x_1(\sigma), \ldots, x_{k-1}(\sigma), \zeta_k(\sigma)\right) \mathrm{d}\sigma.\end{aligned} \tag{3.8}$$

The first term is obviously bounded in $L^1_t\mathcal{H}$ due to (3.2). We demonstrate how to estimate the first integral term (the others are treated similarly):

$$\begin{aligned}&\left\| \int_0^t \mathcal{A}\left(t, \zeta_1(\sigma), x_2(\sigma), \ldots, x_k(\sigma)\right) \mathrm{d}\sigma \right\|_{L^1_{t\in(0,T)}\mathcal{H}}\\ &\leqq \left\| \int_0^t \left\|\mathcal{A}\left(t, \zeta_1(\sigma), x_2(\sigma), \ldots, x_k(\sigma)\right)\right\|_{\mathcal{H}} \mathrm{d}\sigma \right\|_{L^1_{t\in(0,T)}}\\ &\leqq \left\| \int_0^T \left\|\mathcal{A}\left(t, \zeta_1(\sigma), x_2(\sigma), \ldots, x_k(\sigma)\right)\right\|_{\mathcal{H}} \mathrm{d}\sigma \right\|_{L^1_{t\in(0,T)}}\\ &\leqq \int_0^T \left\|\mathcal{A}\left(t, \zeta_1(\sigma), x_2(\sigma), \ldots, x_k(\sigma)\right)\right\|_{L^1_{t\in(0,T)}\mathcal{H}} \mathrm{d}\sigma\\ &\leqq C_0 \int_0^T \left\|\zeta_1(\sigma)\right\|_{\mathcal{H}} \left\|x_2(\sigma)\right\|_{\mathcal{H}} \ldots \left\|x_k(\sigma)\right\|_{\mathcal{H}} \mathrm{d}\sigma\\ &\leqq C_0 \left\|\zeta_1\right\|_{L^1_{t\in(0,T)}\mathcal{H}} \left\|x_2\right\|_{L^\infty_{t\in(0,T)}\mathcal{H}} \cdots \left\|x_k\right\|_{L^\infty_{t\in(0,T)}\mathcal{H}}.\end{aligned}$$

Gathering terms together, we are able to conclude. □

Remark 3.2. The map $\tilde{\mathcal{A}}$ clearly extends $\mathcal{A}$, in the sense that we can view any $x_0 \in \mathcal{H}$ as a function of time by calling it a constant function. Since there is no ambiguity, we will refer to both operators using the common notation $\mathcal{A}$.

Theorem 3.2. *Let $\mathcal{H}$ be a separable Hilbert space, fix an integer $k \geqq 2$, and let $\mathcal{A} : \mathcal{H}^{\times k} \to L^1(\mathbb{R}, \mathcal{H})$ be a mapping which is linear or conjugate linear in each entry, and satisfies the estimate*

$$\left\|\mathcal{A}(t, x_1, \ldots, x_k)\right\|_{L^1_t\mathcal{H}} \leqq C_0 \prod_{j=1}^{k} \left\|x_j\right\|_{\mathcal{H}} \qquad x_1, \ldots, x_k \in \mathcal{H}. \tag{3.9}$$

Then, defining

$$M = \left(\frac{1}{4^k k(1+k)C_0} \right)^{1/(k-1)},$$

we find that for any $x_0 \in \mathcal{H}$ *with* $\|x_0\|_{\mathcal{H}} \leqq M$ *there exists a global solution* $x(t) \in \bigcap_{T>0} W^{1,1}((0,T),\mathcal{H})$ *of the integral equation*

$$x(t) = x_0 + \int_0^t \mathcal{A}(\sigma, x(\sigma), \ldots, x(\sigma))\, \mathrm{d}\sigma, \tag{3.10}$$

and this solution is unique in the regularity class $\bigcap_{T>0} W^{1,1}_{t\in[0,T]}\mathcal{H}$*. Moreover, for the solutions arising in this way, the following estimate holds:*

$$\|x(t)\|^k_{L^\infty_{t\geqq 0}\mathcal{H}} + \|\mathcal{A}(t, x(t), \ldots, x(t))\|_{L^1_{t\geqq 0}\mathcal{H}} \leqq C_1 \|x_0\|^k_{\mathcal{H}} \tag{3.11}$$

for some constant C_1 *depending only on k and* C_0*. In particular,* (3.11) *implies that* $\lim_{t\to+\infty} x(t)$ *exists strongly in* $\mathcal{H}$ *(i.e., the solution scatters).*

Proof. (Sketch) We will use the following norm on $W^{1,1}((0,T),\mathcal{H})$:

$$\|x(\cdot)\|_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}} = \|x(t)\|_{L^\infty_{t\in(0,T)}\mathcal{H}} + \left\| \frac{dx}{dt}(t) \right\|_{L^1_{t\in(0,T)}\mathcal{H}}. \tag{3.12}$$

This norm is equivalent to the usual norm on $W^{1,1}$ for fixed finite T by Sobolev embedding, but exhibits better scaling properties in this context for large T.

Define the map $\mathfrak{F} : W^{1,1}((0,T),\mathcal{H}) \to W^{1,1}((0,T),\mathcal{H})$ by the formula

$$[\mathfrak{F}(x(\cdot))](t) = x_0 + \int_0^t \mathcal{A}(\sigma, x(\sigma), \ldots, x(\sigma))\, d\sigma. \tag{3.13}$$

This is well-defined by Lemma 3.1.

Using Lemma 3.1, we easily derive the following boundedness and locally Lipschitz estimates:

$$\|[\mathfrak{F}(x(\cdot))](t)\|_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}} \leqq \|x_0\|_{\mathcal{H}} + 2(1+k)C_0 \|x(\cdot)\|^k_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}}$$

and

$$\begin{aligned} &\|[\mathfrak{F}(x_2(\cdot)) - \mathfrak{F}(x_1(\cdot))](t)\|_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}} \\ &\leqq 2k(1+k)C_0 \left(\sum_{i=1,2} \|x_i(t)\|_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}} \right)^{k-1} \|x_2(t) - x_1(t)\|_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}}. \end{aligned}$$

Therefore, defining the closed ball

$$\mathfrak{B} = \left\{ x(\cdot) \in W^{1,1}((0,T),\mathcal{H}) \;\middle|\; \|x(t)\|_{\mathcal{W}^{1,1}_{t\in(0,T)}\mathcal{H}} \leqq 2M \right\},$$

with M as in the statement of the theorem, we find that $\mathfrak{F}\mathfrak{B} \subset \mathfrak{B}$ and $\mathfrak{F}$ is a strict contraction of $\mathfrak{B}$. Hence, we may apply the Banach fixed point theorem and thereby extract a unique fixed point of $\mathfrak{F}$ within $\mathfrak{B}$. □

Theorem 3.3. *Let $\mathcal{H}$, $\mathcal{A}$, M, k, C_0 be as in the statement of Theorem 3.2. Consider the integral equation*

$$x(t) = x_0 + \int_0^t \mathcal{A}(\sigma, x(\sigma), \ldots, x(\sigma))\, \mathrm{d}\sigma \tag{3.14}$$

with unique solutions $x \in \bigcap_{T>0} W^{1,1}((-T,T),\mathcal{H})$, $x(0) = x_0$, as given by Theorem 3.2 for any $x_0 \in \mathcal{H}$ such that $\|x_0\|_{\mathcal{H}} \leqq M$. Define the map

$$\mathcal{S} : \overline{B_M^{\mathcal{H}}(0)} \to \bigcap_{T>0} W^{1,1}((-T,T),\mathcal{H}) \tag{3.15}$$

such that

$$[\mathcal{S}(x_0)](t) = x_0 + \int_0^t \mathcal{A}(\sigma, [\mathcal{S}(x_0)](\sigma), \ldots, [\mathcal{S}(x_0)](\sigma))\, \mathrm{d}\sigma. \tag{3.16}$$

The map $\mathcal{S}$ is well-defined by the statement and proof of Theorem 3.2. For any $r \in (0, M)$ define the maps $\mathfrak{S}_r^+$, $\mathfrak{S}_r^-$,

$$\mathfrak{S}_r^{\pm} : B_r^{\mathcal{H}}(0) \to \mathcal{H} \tag{3.17}$$

$$\mathfrak{S}_r^{\pm}(x_0) = \lim_{t\to\pm\infty} [\mathcal{S}(x_0)](t), \tag{3.18}$$

where the limit is taken in the norm topology of $\mathcal{H}$; this is possible by Theorem 3.2. Let $\mathfrak{U}_r^{\pm}$ denote the image of $\mathfrak{S}_r^{\pm}$, and note that $0 \in \mathfrak{U}_r^+ \bigcap \mathfrak{U}_r^-$.

Then, there exists $r_0 = r_0(k, C_0) > 0$ such that if $0 < r < r_0$ then $\mathfrak{U}_r^+$, $\mathfrak{U}_r^-$ are each open in the norm topology of $\mathcal{H}$, and $\mathfrak{S}_r^+$, $\mathfrak{S}_r^-$ are each bijective and bi-Lipschitz. As a consequence, the composite maps

$$\mathfrak{S}_r^+ \circ \left(\mathfrak{S}_r^-\right)^{-1} : \mathfrak{U}_r^- \to \mathfrak{U}_r^+ \tag{3.19}$$

$$\mathfrak{S}_r^- \circ \left(\mathfrak{S}_r^+\right)^{-1} : \mathfrak{U}_r^+ \to \mathfrak{U}_r^- \tag{3.20}$$

are bijective and bi-Lipschitz.

Proof. (Sketch.) The key estimate states that x_0 expresses a Lipschitz depencence on $x_{+\infty} = \lim_{t\to+\infty}[\mathcal{S}(x_0)](t)$, at least within sufficiently small neighborhoods of $0 \in \mathcal{H}$.

Let $T > 0$ and consider the solution $x(t) = [\mathcal{S}(x_0)](t)$ for $t \in (0, T)$. As long as $\|x_0\|_{\mathcal{H}}$ is sufficiently small (depending only on k, C_0), we can guarantee that $\|x(T)\|_{\mathcal{H}} < M$, so that Theorem 3.2 can be applied *backwards* in time with data $x(T)$. Considering two solutions $x(t) = [\mathcal{S}(x_0)](t)$, $y(t) = [\mathcal{S}(y_0)](t)$, we can apply this procedure to each of them and derive the following identity:

$$\begin{aligned} x(t) - y(t) = x(T) - y(T) &- \int_t^T \mathcal{A}(\sigma, x(\sigma), \ldots, x(\sigma))\, \mathrm{d}\sigma \\ &+ \int_t^T \mathcal{A}(\sigma, y(\sigma), \ldots, y(\sigma))\, d\sigma. \end{aligned}$$

Hence, using the norms defined in the proof of Theorem 3.2, along with Lemma 3.1, we have

$$\begin{aligned} \|x(t) - y(t)\|_{\mathcal{W}^{1,1}((0,T),\mathcal{H})} &\leqq \|x(T) - y(T)\|_{\mathcal{H}} \\ &+ 2k(1+k)C_0 \left(\sum_{z\in\{x,y\}} \|z(t)\|_{\mathcal{W}^{1,1}((0,T),\mathcal{H})} \right)^{k-1} \|x(t) - y(t)\|_{\mathcal{W}^{1,1}((0,T),\mathcal{H})} . \end{aligned} \tag{3.21}$$

In view of the statement and proof of Theorem 3.2, under the above assumptions we can deduce the quantitative estimate

$$\|x(t) - y(t)\|_{\mathcal{W}^{1,1}((0,T),\mathcal{H})} \leqq 2\, \|x(T) - y(T)\|_{\mathcal{H}} , \tag{3.22}$$

as long as $\|x_0\|_{\mathcal{H}}$, $\|y_0\|_{\mathcal{H}}$ are sufficiently small (depending on only k, C_0). This immediately implies that

$$\|x_0 - y_0\|_{\mathcal{H}} \leqq 2\, \|x(T) - y(T)\|_{\mathcal{H}} . \tag{3.23}$$

Taking strong limits in $\mathcal{H}$ as $T \to +\infty$, we obtain

$$\|x_0 - y_0\|_{\mathcal{H}} \leqq 2\, \|x_{+\infty} - y_{+\infty}\|_{\mathcal{H}} , \tag{3.24}$$

which is the desired Lipschitz estimate.

The last claim is the following: for all $q \in (0, M)$, $\mathfrak{S}_q^{\pm}\left[B_q^{\mathcal{H}}(0)\right]$ contains a neighborhood of $0 \in \mathcal{H}$. This is routine to check by adapting the proof of Theorem 3.2. □

Remark 3.3. If, instead of the "critical" estimate (3.2), $\mathcal{A}$ satisfies a "subcritical" estimate of the form

$$\|\mathcal{A}(t, x_1, \ldots, x_k)\|_{L_t^p \mathcal{H}} \leqq \tilde{C} \prod_{j=1}^{k} \|x_j\|_{\mathcal{H}} \qquad x_1, \ldots, x_k \in \mathcal{H} \tag{3.25}$$

for some $p > 1$, then we can always convert $\mathcal{A}$ into a form suitable for the application of Theorem 3.2 by multiplying $\mathcal{A}$ by a bump function in time which is equal to one on an interval $[0, T]$. In that case, the constant C_0 in the theorem would be $C_0 \approx \tilde{C} T^{\frac{1}{p'}}$, so that the allowable size of the data tends to infinity as T tends to zero. Hence, Theorem 3.2 can be used to prove a wide range of local well-posedness results in the large for the strictly scaling-subcritical case.

Remark 3.4. There is a version of Theorem 3.2 when $k = 1$, i.e. linear equations, but only if $C_0 < \frac{1}{4}$.

Remark 3.5. Local well-posedness for arbitrary $x_0 \in \mathcal{H}$ is **not** recovered under the sole assumption (3.2); this is because the equation for $\tilde{x}(t) = x(t) - x_0$ contains linear terms, and we can only solve the linear case when $C_0 < \frac{1}{4}$ per the previous remark. (The forcing term $\mathcal{A}(t, x_0, \ldots, x_0)$ can always be made negligible, for fixed x_0, by localizing to a small time interval depending on x_0.) If, for any $T > 0$ and any $x_0 \in \mathcal{H}$, estimates of the following form are satisfied for open intervals $I \subset (-T, T)$:

$$\limsup_{\delta \to 0^+} \sup_{I \subset (-T,T) \,:\, |I| \leqq \delta} \sup_{y_0 \in \mathcal{H} \backslash \{0\}} \frac{1}{\|y_0\|_{\mathcal{H}}} \|\mathcal{A}(t, y_0, x_0, \ldots, x_0)\|_{L^1_{t \in I} \mathcal{H}} = 0$$

(and similarly for the other entries of $\mathcal{A}$), then large data LWP can be recovered in the limited sense that the time of existence depends on $x_0 \in \mathcal{H}$ instead of $\|x_0\|_{\mathcal{H}}$.

4. Example: Cubic NLS in $d = 2$

In this section we illustrate how Theorem 3.2 can be used to recover small data global well-posedness and scattering for the L^2 critical nonlinear Schrödinger equation in spatial dimension $d = 2$. Furthermore, we illustrate an approach to study propagation of regularity for the same equation. Although these results themselves are well known, we illustrate how they can be recovered using the tools of Section 3. This will form a footprint for our study of the Boltzmann equation in subsequent sections.

Consider the nonlinear Schrödinger equation (NLS)

$$(i\partial_t + \Delta)\varphi = |\varphi|^2\varphi \qquad \varphi(t, x) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}, \tag{4.1}$$

where $\Delta \equiv \Delta_x$ and $\varphi(0, x) = \varphi_0(x) \in L^2(\mathbb{R}^2)$. The nonlinearity can be written $\varphi\overline{\varphi}\varphi$, so it is either linear or conjugate linear in each entry.

4.1. Small Data Global Existence and Scattering

We wish to solve this equation for *small data* $\varphi_0(x)$ in the scaling-critical space $L^2(\mathbb{R}^2)$. We point out that the method as formulated in the statement of Theorem 3.2 yields *no conclusion* for (4.1) given initial data outside a small ball of the origin in L^2; this is expected due to the fact that (4.1) is L^2-critical with respect to scaling.

We impose the unitary change of variables

$$\psi(t) = e^{-it\Delta}\varphi(t), \tag{4.2}$$

which implies $\psi(0) = \varphi(0) = \varphi_0$ and

$$\partial_t\psi(t) = -ie^{-it\Delta}g\left(e^{it\Delta}\psi\right), \tag{4.3}$$

where $g(u) = u\overline{u}u$. Let us define the more general nonlinearity

$$g(u, v, w) = u\overline{v}w,$$

and estimate, for given $u_0, v_0, w_0 \in L^2\left(\mathbb{R}^2\right)$,

$$\begin{aligned}\left\|e^{-it\Delta} g\left(e^{it\Delta}u_0, e^{it\Delta}v_0, e^{it\Delta}w_0\right)\right\|_{L^1_t L^2_x} &= \left\|g\left(e^{it\Delta}u_0, e^{it\Delta}v_0, e^{it\Delta}w_0\right)\right\|_{L^1_t L^2_x} \\ &= \left\|\left(e^{it\Delta}u_0\right)\left(\overline{e^{it\Delta}v_0}\right)\left(e^{it\Delta}w_0\right)\right\|_{L^1_t L^2_x} \\ &\leqq \left\|e^{it\Delta}u_0\right\|_{L^3_t L^6_x}\left\|e^{it\Delta}v_0\right\|_{L^3_t L^6_x}\left\|e^{it\Delta}w_0\right\|_{L^3_t L^6_x} \\ &\leqq C\left\|u_0\right\|_{L^2_x}\left\|v_0\right\|_{L^2_x}\left\|w_0\right\|_{L^2_x}. \end{aligned} \tag{4.4}$$

We have used the unitarity, the Hölder, and the Strichartz estimates, in that order. In other words, we have shown that

$$\left\|e^{-it\Delta} g\left(e^{it\Delta}u_0, e^{it\Delta}v_0, e^{it\Delta}w_0\right)\right\|_{L^1_t L^2_x} \leqq C\left\|u_0\right\|_{L^2_x}\left\|v_0\right\|_{L^2_x}\left\|w_0\right\|_{L^2_x}. \tag{4.5}$$

Applying Theorem 3.2 with

$$\mathcal{A}(t, u_0, v_0, w_0) = -ie^{-it\Delta} g\left(e^{it\Delta}u_0, e^{it\Delta}v_0, e^{it\Delta}w_0\right), \tag{4.6}$$

we find that solutions of (4.1) are globally well-posed and scatter, as long as the data $\varphi_0 \in L^2\left(\mathbb{R}^2\right)$ has sufficiently small norm. Theorem 3.2 guarantees that, at the very least, uniqueness of *small* solutions holds within the class of *all* mild solutions satisfying the bound $g(\varphi) \in L^1_{t\in[0,T]} L^2_x$; this uniqueness criterion can be equivalently written $\varphi \in L^3_{t\in[0,T]} L^6_x$ by definition of g.

Theorem 4.1. *There exists a number $\eta > 0$ such that, for any $\varphi_0 \in L^2\left(\mathbb{R}^2\right)$ satisfying*

$$\|\varphi_0\|_{L^2(\mathbb{R}^2)} < \eta,$$

it follows that equation (4.1) has a global solution which scatters in $L^2\left(\mathbb{R}^2\right)$. The solution is unique in the class of all L^2 mild solutions for which $\varphi \in L^3_{t,loc} L^6_x$.

Remark 4.1. It is crucial to remember that the space $W^{1,1}$ (in time) appearing in Theorem 3.2 is *not* the usual Sobolev norm of the solution. This is because we only have $W^{1,1}$ *after* intertwining with the free evolution. For this reason, to avoid confusion, in practice it is often better to use unitarity in order to state the uniqueness criterion in terms of an equivalent estimate on the nonlinearity, cf. (3.11).

4.2. Regularity

Regularity is a subtle question because it hides two *separate* questions.

- The first, which is easy to answer, is whether any $\varphi_0 \in H^1\left(\mathbb{R}^2\right)$, say, yields a global solution when the H^1 norm is small enough. The answer is *yes* because, by Leibniz' rule and standard commutation formulae, and $\mathcal{A}$ as in (4.6), we have

$$\left\|\mathcal{A}\left(t, u_0, v_0, w_0\right)\right\|_{L^1_t H^1_x} \leqq \tilde{C}\left\|u_0\right\|_{H^1_x}\left\|v_0\right\|_{H^1_x}\left\|w_0\right\|_{H^1_x}. \tag{4.7}$$

Now as long as $\|\varphi_0\|_{H^1}$ is smaller than some number which depends explicitly on $\tilde{C}$, the cubic NLS will have a global solution which scatters in $H^1\left(\mathbb{R}^2\right)$, as a direct consequence of Theorem 3.2 and Theorem 3.3.

- The second, more difficult, question is whether H^1 regularity is propagated for smooth solutions which are *only small in* L^2. This can be seen as a persistence of regularity question, since we know that any small L^2 data will lead to a global L^2 solution. The answer, perhaps surprisingly, is *yes*, as we now show.

The key is to introduce a new norm, H^1_ε, parameterized by $\varepsilon \in (0, 1]$, which is equivalent to H^1 up to an ε-dependent factor, but tends to the L^2 norm as $\varepsilon \to 0^+$. The goal is to prove a bound of the form

$$\|\mathcal{A}\left(t, u_0, v_0, w_0\right)\|_{L^1_t H^1_\varepsilon} \leqq \tilde{C} \, \|u_0\|_{H^1_\varepsilon} \, \|v_0\|_{H^1_\varepsilon} \, \|w_0\|_{H^1_\varepsilon} \, , \tag{4.8}$$

where the constant $\tilde{C}$ is independent of ε. Now as long as $\varphi_0 \in H^1$ has L^2 norm smaller than some constant depending explicitly on $\tilde{C}$ (*not* the original C from (4.5)), we can pick a value of ε depending on φ_0 so that the H^1_ε norm is small enough. The key here is that the constants appearing in Theorems 3.2 and 3.3 are quantitative.

The simplest norm which makes the above argument work seems to be the following one:

$$\|\varphi_0\|^2_{H^1_\varepsilon} = \|\varphi_0\|^2_{L^2} + \varepsilon^2 \, \|\varphi_0\|^2_{\dot{H}^1} \, . \tag{4.9}$$

Now if $\|\varphi_0\|_{L^2} < \eta$ and $\varphi_0 \in H^1$, then there exists a value of ε (depending explicitly on $\|\varphi_0\|_{L^2}$ and $\|\varphi_0\|_{\dot{H}^1}$) such that $\|\varphi_0\|_{H^1_\varepsilon} < \eta$. We have only to choose η according to the constant $\tilde{C}$ instead of the constant C; unfortunately, the "gap" between C and $\tilde{C}$ seems to be unrecoverable by this approach.

In order to establish (4.8) for the norm (4.9), we estimate the L^2 and $\dot{H}^1$ norms separately, tracking the location of ε throughout. The important observation is a power of ε is always accompanied by a single derivative on one of the factors (u_0, v_0 or w_0), while the remaining factors remain in L^2. Thus we may estimate as follows, where $\lesssim$ allows an arbitrary constant which is *independent of* ε:

$$\begin{aligned}
&\|\mathcal{A}\left(t, u_0, v_0, w_0\right)\|_{L^1_t H^1_\varepsilon} \\
&\quad\lesssim \|\mathcal{A}\left(t, u_0, v_0, w_0\right)\|_{L^1_t L^2} + \varepsilon \, \|\mathcal{A}\left(t, u_0, v_0, w_0\right)\|_{L^1_t \dot{H}^1} \\
&\quad\lesssim \|u_0\|_{L^2} \, \|v_0\|_{L^2} \, \|w_0\|_{L^2} + \varepsilon \, \|u_0\|_{\dot{H}^1} \, \|v_0\|_{L^2} \, \|w_0\|_{L^2} \\
&\qquad + \varepsilon \, \|u_0\|_{L^2} \, \|v_0\|_{\dot{H}^1} \, \|w_0\|_{L^2} + \varepsilon \, \|u_0\|_{L^2} \, \|v_0\|_{L^2} \, \|w_0\|_{\dot{H}^1} \\
&\quad\lesssim \|u_0\|_{H^1_\varepsilon} \, \|v_0\|_{H^1_\varepsilon} \, \|w_0\|_{H^1_\varepsilon} \, .
\end{aligned} \tag{4.10}$$

As a result of this calculation, we can conclude the following:

Theorem 4.2. *There exists a number* $\tilde{\eta} > 0$ *such that all of the following is true:*
Let $\varphi_0 \in H^1\left(\mathbb{R}^2\right)$ *be such that*

$$\|\varphi_0\|_{L^2(\mathbb{R}^2)} < \tilde{\eta} \, .$$

Then equation (4.1) *has a global solution which scatters in* $H^1\left(\mathbb{R}^2\right)$. *The solution is unique in the class of all* L^2 *mild solutions for which* $\varphi \in L^3_{t,loc} L^6_x$.

5. The Gain-Only Boltzmann Equation

In this section, we focus on the gain-only Boltzmann equation.[1] We employ the inverse Wigner transform which converts this kinetic equation into a hyperbolic Schrödinger equation, a technique we explored in [7,8]. Subsequently, we can prove a certain bilinear Strichartz estimate (stated in Proposition 5.2), based on which we can use Theorem 3.2 to establish small data global well-posedness for this hyperbolic Schrödinger equation. The bilinear Strichartz estimate is obtained from a certain bilinear estimate based on Lorentz spaces, and the validity of the endpoint Strichartz estimate for the hyperbolic Schrödinger equation (which is crucial for our argument, since the endpoint Strchartz estimate fails on the kinetic side). However, once we obtain the bilinear Strichartz estimate on the dispersive side, we can convert it to a *bilinear* Strichartz estimate on the kinetic side, see Proposition 5.4. Consequently, this proposition combined with Theorem 3.2 provide us with small data global well-posedness for the gain-only Boltzmann equation, which is the main result of this section.

Everything below only applies to the gain-only Boltzmann equation with constant collision kernel in dimension $d = 2$.

5.1. Hyperbolic Schrödinger Equation Associated with the Gain-Only Boltzmann Equation

We will require the Wigner transform, which we shall now define. Given a function $f \in L^2_{x,v}$, the Wigner (or Wigner-Weyl) transformation is defined by the following formula:

$$\gamma\left(x, x'\right) = \int_{\mathbb{R}^d} f\left(\frac{x + x'}{2}, v\right) e^{iv\cdot(x-x')}\, \mathrm{d}v. \tag{5.1}$$

Up to a linear change of variables, this is equivalent to a partial Fourier transform accounting for only the velocity variable. The inverse transformation is defined by

$$f\left(x, v\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \gamma\left(x + \frac{y}{2}, x - \frac{y}{2}\right) e^{-iv\cdot y}\, \mathrm{d}y. \tag{5.2}$$

One of the main interests driving the use of the Wigner transform is that it converts the free transport generator $-v \cdot \nabla_x$ into the hyperbolic Schrödinger generator $i\Delta_x - i\Delta_{x'}$. Aside from being the starting point for semiclassical limits (up to scaling), the Wigner transform allows for the transfer of ideas from the literature of nonlinear Schrödinger equations (NLS) into the kinetic realm. For the present study, the big ideas which we wish to adapt are largely related to $X^{s,b}$ spaces (also known as Bourgain spaces), which are well-studied for NLS and hyperbolic-NLS, but have not been fully utilized in the kinetic theory literature. We note that the spaces used in this paper are not actually Bourgain spaces (which are typically L^2

[1] The gain-only Boltzmann equation refers to the Boltzmann equation having the Q^+ term only.

in time), but rather, they are scale-invariant spaces (which are L^1 in time) inspired by Bourgain spaces; see Section 3.

In our situation, namely the Boltzmann equation with constant collision kernel in $d = 2$, $L^2_{x,x'}$ is a scaling critical space for γ, and corresponds to $L^2_{x,v}$ for f.

Remark 5.1. The use of the Wigner transform is *necessary* for the type of proof used here. Indeed, if one were to execute the corresponding steps on the kinetic side (and thereby produce the needed bilinear bound for Q^+ acting on the freely transported solution), the proof would *fail* because the endpoint kinetic Strichartz estimates are false in all dimensions. [4] By contrast, we will be using the usual endpoint Strichartz estimates for the free hyperbolic Schrödinger equation in $d = 4$ (note the dimension doubling!), which are indeed *true* by Keel–Tao, [15].

We use the notation $\eta_{\|} = P_\omega \eta$ and $\eta_\perp = \eta - P_\omega \eta$ where for any vector η and any unit vector ω in the plane,

$$P_\omega \eta = \omega\omega \cdot \eta.$$

Then

$$Q^+(f,g)(v) = \int_{\mathbb{S}^1} \mathrm{d}\omega \int_{\mathbb{R}^2} \mathrm{d}u f(v^*) g(u^*) \tag{5.3}$$

$$Q^-(f,g)(v) = \int_{\mathbb{S}^1} \mathrm{d}\omega \int_{\mathbb{R}^2} \mathrm{d}u f(v) g(u) \tag{5.4}$$

$$\left(Q^+(f,g)\right)^\wedge(\eta) = \int_{\mathbb{S}^1} \mathrm{d}\omega \hat{f}(\eta_\perp)\, \hat{g}\left(\eta_{\|}\right). \tag{5.5}$$

The Wigner transform of the Boltzmann gain operator Q^+ is

$$\begin{aligned} B^+(\gamma_1,\gamma_2)(x,x') = i \int_{\mathbb{S}^1} \mathrm{d}\omega & \\ \times \gamma_1 \left(x - \frac{1}{2} P_\omega \left(x - x'\right), x' + \frac{1}{2} P_\omega \left(x - x'\right)\right) & \\ \times \gamma_2 \left(\frac{x + x'}{2} + \frac{1}{2} P_\omega \left(x - x'\right), \frac{x + x'}{2} - \frac{1}{2} P_\omega \left(x - x'\right)\right). & \end{aligned} \tag{5.6}$$

Theorem 5.1. *For any $\gamma_0 \in L^2_{x,x'}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$ with sufficiently small $L^2_{x,x'}$ norm, there exists a unique global mild solution to the equation*

$$\left(i\partial_t + \Delta_x - \Delta_{x'}\right) \gamma(t) = B^+\left(\gamma(t), \gamma(t)\right) \tag{5.7}$$

with $\gamma(0) = \gamma_0$ such that $\gamma \in C_t L^2_{x,x'}$ and $B^+(\gamma,\gamma) \in L^1_{t,loc} L^2_{x,x'}$. For this solution, it holds that $\gamma \in L^\infty_{t\in\mathbb{R}} L^2_{x,x'}$ and $B^+(\gamma,\gamma) \in L^1_{t\in\mathbb{R}} L^2_{x,x'}$, and the solution scatters in $L^2_{x,x'}$ as $t \to \pm\infty$.

Theorem 5.1 follows from Theorem 3.2 along with the following estimate for the gain term B^+:

Proposition 5.2. *There is a constant* $C > 0$ *such that, for any* $\gamma_{0,1}, \gamma_{0,2} \in L^2_{x,x'}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$,

$$\left\| B^+ \left(e^{it\Delta_\pm}\gamma_{0,1}, e^{it\Delta_\pm}\gamma_{0,2} \right) \right\|_{L^1_t L^2_{x,x'}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)} \leqq C \prod_{i=1,2} \left\| \gamma_{0,i} \right\|_{L^2_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)}, \tag{5.8}$$

where $\Delta_\pm = \Delta_x - \Delta_{x'}$.

We will need the Lorentz spaces $L^{p,q}$ defined by the following quasi-norm, for any function $h(\xi) : \mathbb{R}^n \to \mathbb{C}$:

$$\|h(\xi)\|_{L^{p,q}_*(\mathbb{R}^n)} = p^{\frac{1}{q}} \left\| \lambda \left| \left\{ \xi \in \mathbb{R}^n \ : \ |h(\xi)| \geqq \lambda \right\} \right|^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}^+, \frac{d\lambda}{\lambda}\right)}. \tag{5.9}$$

Note that $L^{p,p} = L^p$ for $1 < p < \infty$. In all cases of interest here, the Lorentz quasi-norm above can be shown to be equivalent to a Banach space norm.

Lemma 5.3. *For any Schwartz functions* $f, g : \mathbb{R}^2 \to \mathbb{C}$, *it holds that*

$$\left\| \left(Q^+(f,g) \right)^\wedge (\eta) \right\|_{L^2_\eta(\mathbb{R}^2)} \leqq C \left\| \hat{f}(\eta) \right\|_{L^{4,2}_\eta(\mathbb{R}^2)} \left\| \hat{g}(\eta) \right\|_{L^{4,2}_\eta . (\mathbb{R}^2)}. \tag{5.10}$$

Also, if $\gamma_{0,1}, \gamma_{0,2} \in L^{4,2}_{x,x'}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, *it holds that*

$$\left\| B^+ \left(\gamma_{0,1}, \gamma_{0,2} \right) \right\|_{L^2_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)} \leqq C \prod_{i=1,2} \left\| \gamma_{0,i} \right\|_{L^{4,2}_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)}. \tag{5.11}$$

Proof. (Lemma 5.3)

We apply Minkowski, Hölder, and Fubini (twice), as follows:

$$\begin{aligned}
\left\| \left(Q^+(f,g) \right)^\wedge (\eta) \right\|_{L^2_\eta} &= \left\| \int_{\mathbb{S}^1} \mathrm{d}\omega \hat{f}(\eta_\perp) \hat{g}\left(\eta_\parallel\right) \right\|_{L^2_\eta} \\
&\leqq \int_{\mathbb{S}^1} \mathrm{d}\omega \left\| \hat{f}(\eta_\perp) \hat{g}\left(\eta_\parallel\right) \right\|_{L^2_\eta} \\
&= \int_{\mathbb{S}^1} \mathrm{d}\omega \left\| \hat{f}(\eta_\perp) \right\|_{L^2_{\eta_\perp}} \left\| \hat{g}\left(\eta_\parallel\right) \right\|_{L^2_{\eta_\parallel}} \\
&\leqq \left\| \hat{f}(\eta_\perp) \right\|_{L^2_\omega L^2_{\eta_\perp}} \left\| \hat{g}\left(\eta_\parallel\right) \right\|_{L^2_\omega L^2_{\eta_\parallel}} \\
&= C \left\| \frac{1}{|\eta|^{\frac{1}{2}}} \hat{f}(\eta) \right\|_{L^2_\eta} \left\| \frac{1}{|\eta|^{\frac{1}{2}}} \hat{g}(\eta) \right\|_{L^2_\eta}.
\end{aligned}$$

Then again, because $|\eta|^{-1} \in L^{2,\infty}\left(\mathbb{R}^2\right)$, we may apply the duality $\left(L^{2,1}\right)' = L^{2,\infty}$ ([10] Theorem 1.4.17 (v)), combined with the "power property," to deduce that

$$\left\| \frac{1}{|\eta|^{\frac{1}{2}}} \hat{f}(\eta) \right\|_{L^2_\eta} = \left\| \frac{1}{|\eta|} \left| \hat{f}(\eta) \right|^2 \right\|^{\frac{1}{2}}_{L^1_\eta} \lesssim \left\| \left| \hat{f}(\eta) \right|^2 \right\|^{\frac{1}{2}}_{L^{2,1}_\eta} \lesssim \left\| \hat{f}(\eta) \right\|_{L^{4,2}_\eta(\mathbb{R}^2)},$$

hence we obtain

$$\left\| \left(Q^+(f,g) \right)^\wedge (\eta) \right\|_{L^2_\eta(\mathbb{R}^2)} \lesssim \left\| \hat{f}(\eta) \right\|_{L^{4,2}_\eta(\mathbb{R}^2)} \left\| \hat{g}(\eta) \right\|_{L^{4,2}_\eta(\mathbb{R}^2)}, \tag{5.12}$$

which is (5.10). *Remark:* The full duality of Lorentz spaces is not actually necessary at this stage; in fact, a simple application of the Hardy-Littlewood rearrangement inequality is sufficient.

Using the change of variables

$$w = \frac{x + x'}{2} \qquad z = \frac{x - x'}{2},$$

we find that (5.11) follows immediately from (5.10) and Hölder's inequality, as long as we can show that

$$L^{4,2}_{w,z}\left(\mathbb{R}^2 \times \mathbb{R}^2\right) \subset L^4_w\left(\mathbb{R}^2, L^{4,2}_z\left(\mathbb{R}^2\right)\right). \tag{5.13}$$

The $L^4_w L^{4,2}_z$ norm of a function $F(w,z)$ can be controlled directly from the definition of $L^{p,q}$ as follows:

$$\left\{ \int_{\mathbb{R}^2} \mathrm{d}w \int_0^\infty \frac{d\lambda}{\lambda} \lambda^2 \left| \left\{ z \in \mathbb{R}^2 \ : \ |F(w,z)| \geqq \lambda \right\} \right|^{\frac{1}{2}} \right.$$
$$\left. \times \int_0^\infty \frac{d\lambda'}{\lambda'} \left(\lambda'\right)^2 \left| \left\{ z \in \mathbb{R}^2 \ : \ |F(w,z)| \geqq \lambda' \right\} \right|^{\frac{1}{2}} \right\}^{\frac{1}{4}}.$$

Now the idea is to move the dw integral to the *inside* and apply Cauchy-Schwarz in w, followed by Fubini; this leads us to the quantity

$$\left\{ \int_0^\infty \frac{d\lambda}{\lambda} \lambda^2 \left| \left\{ (w,z) \in \mathbb{R}^4 \ : \ |F(w,z)| \geqq \lambda \right\} \right|^{\frac{1}{2}} \right.$$
$$\left. \times \int_0^\infty \frac{d\lambda'}{\lambda'} \left(\lambda'\right)^2 \left| \left\{ (w,z) \in \mathbb{R}^4 \ : \ |F(w,z)| \geqq \lambda' \right\} \right|^{\frac{1}{2}} \right\}^{\frac{1}{4}},$$

but this is comparable to the $L^{4,2}_{w,z}$ norm of F, so we are done. □

Finally we are ready to prove our main result for this section.

Proof. (Proposition 5.2)

We estimate by Lemma 5.3, combined with Hölder's inequality in time, to get that

$$\left\| B^+ \left(e^{it\Delta_\pm} \gamma_{0,1}, e^{it\Delta_\pm} \gamma_{0,2} \right) \right\|_{L^1_t L^2_{x,x'}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)}$$
$$\leqq C \prod_{i=1,2} \left\| e^{it\Delta_\pm} \gamma_{0,i} \right\|_{L^2_t L^{4,2}_{x,x'}(\mathbb{R}^2 \times \mathbb{R}^2)}. \tag{5.14}$$

We apply Theorem 10.1 of Keel–Tao [15], with $H = L^2_{x,x'}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, $B_0 = L^2_{x,x'}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, $B_1 = L^1_{x,x'}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, and $(q, \sigma, \theta) = \left(2, 2, \frac{1}{2}\right)$ to deduce the Strichartz estimate (see Appendix A)

$$\left\| e^{it\Delta_\pm}\gamma_0 \right\|_{L^2_t L^{4,2}_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)} \lesssim \|\gamma_0\|_{L^2_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)} . \tag{5.15}$$

Here we have used the real interpolation space

$$\left(\left(L^2_{x,x'}, L^1_{x,x'}\right)_{\frac{1}{2},2}\right)' = \left(L^{\frac{4}{3},2}_{x,x'}\right)' = L^{4,2}_{x,x'}; \tag{5.16}$$

e.g. see Chapter 5 of the book [5].

Combining (5.14) and (5.15), we are able to conclude. □

5.2. Back to the Gain-Only Boltzmann Equation

Combining Proposition 5.2 and Plancherel's theorem, and defining $T(t) = e^{-tv\cdot\nabla_x}$, we easily deduce the following bound stated in the spatial domain:

Proposition 5.4. *There is a constant $C > 0$ such that, for any $f_0, g_0 \in L^2_{x,v}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$,*

$$\left\| Q^+\left(T(t)f_0, T(t)g_0\right)\right\|_{L^1_t L^2_{x,v}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)} \leqq C\, \|f_0\|_{L^2_{x,v}(\mathbb{R}^2\times\mathbb{R}^2)}\, \|g_0\|_{L^2_{x,v}(\mathbb{R}^2\times\mathbb{R}^2)} \tag{5.17}$$

The following theorem is an immediate consequence of Proposition 5.4 and Theorem 3.2:

Theorem 5.5. *For any $f_0 \in L^2_{x,v}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$ with sufficiently small $L^2_{x,v}$ norm, there exists a unique global ($t \in \mathbb{R}$) mild solution to the equation*

$$\left(\partial_t + v\cdot\nabla_x\right) f(t) = Q^+\left(f(t), f(t)\right), \tag{5.18}$$

with $f(0) = f_0$ such that $f \in C_t L^2_{x,v}$ and $Q^+(f, f) \in L^1_{t,loc} L^2_{x,v}$. For this solution, it holds that $f \in L^\infty_{t\in\mathbb{R}} L^2_{x,v}$ and $Q^+(f, f) \in L^1_{t\in\mathbb{R}} L^2_{x,v}$, and the solution scatters in $L^2_{x,v}$ as $t \to \pm\infty$.

Remark 5.2. It is not necessary in Theorem 5.5 for f_0 to be non-negative. However, assuming f_0 is non-negative, we can show that the solution $f(t)$ of the Q^+ equation (5.18) is non-negative for a.e. $(t, x, v) \in (0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$. Indeed, there is a globally convergent expansion of $f(t)$ in terms of f_0, which comes from iterating Duhamel's formula:

$$\begin{aligned} f(t) =& T(t)f_0 + \int_0^t T(t-t_1)Q^+\left(T(t_1)f_0, T(t_1)f_0\right) \mathrm{d}t_1 \\ &+ \int_0^t \int_0^{t_1} T(t-t_1)Q^+\left(T(t_1-t_2)Q^+\left(T(t_2)f_0, T(t_2)f_0\right), T(t_1)f_0\right) \mathrm{d}t_2\, \mathrm{d}t_1 + \ldots. \end{aligned} \tag{5.19}$$

If $f_0 \geqq 0$ then all the terms in the series are non-negative for $t \geqq 0$; hence, the solution $f(t)$ is non-negative at positive times.

5.3. Short-Time Estimates

The bilinear estimates above will not be suitable for every result we wish to prove, e.g. uniqueness, where we must rely upon integrability properties instead of regularity. For this reason we will require the following "short-time" estimates which follow essentially from the dominated convergence theorem.

Proposition 5.6. *Let* $f_0 \in L^2_{x,v}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$. *Then it holds that*

$$\limsup_{T \to 0^+} \sup_{g_0 \in L^2_{x,v},\ \|g_0\|_{L^2_{x,v}}=1} \left\| Q^+\left(T(t) f_0, T(t) g_0\right)\right\|_{L^1_{t\in[-T,T]} L^2_{x,v}} = 0 \quad (5.20)$$

$$\limsup_{T \to 0^+} \sup_{g_0 \in L^2_{x,v},\ \|g_0\|_{L^2_{x,v}}=1} \left\| Q^+\left(T(t) g_0, T(t) f_0\right)\right\|_{L^1_{t\in[-T,T]} L^2_{x,v}} = 0, \quad (5.21)$$

where we note that $T(t)$ *is an operator whereas* $T > 0$ *is real valued.*

Proof. We only prove the first bound; the second proceeds similarly. By the proof of Proposition 5.2, for any two density matrices $\gamma_{0,1}, \gamma_{0,2} \in L^2_{x,x'}$, B^+ (the Wigner transform of Q^+) satisfies the bilinear estimates

$$\begin{aligned} &\left\| B^+\left(e^{it\Delta_\pm}\gamma_{0,1}, e^{it\Delta_\pm}\gamma_{0,2}\right)\right\|_{L^1_{t\in[-T,T]} L^2_{x,x'}} \\ &\leqq C \prod_{i\in\{1,2\}} \left\| e^{it\Delta_\pm}\gamma_{0,i}\right\|_{L^2_{t\in[-T,T]} L^{4,2}_{x,x'}} \end{aligned} \quad (5.22)$$

Apply Strichartz in the *second entry only* to yield

$$\begin{aligned} &\left\| B^+\left(e^{it\Delta_\pm}\gamma_{0,1}, e^{it\Delta_\pm}\gamma_{0,2}\right)\right\|_{L^1_{t\in[-T,T]} L^2_{x,x'}} \\ &\leqq C \left\| e^{it\Delta_\pm}\gamma_{0,1}\right\|_{L^2_{t\in[-T,T]} L^{4,2}_{x,x'}} \left\|\gamma_{0,2}\right\|_{L^2_{x,x'}}. \end{aligned} \quad (5.23)$$

Now observe that since $\gamma_{0,1} \in L^2_{x,x'}$ by assumption, it follows that $e^{it\Delta_\pm}\gamma_{0,1} \in L^2_t L^{4,2}_{x,x'}$ by Strichartz; therefore, by the dominated convergence theorem,

$$\limsup_{T\to 0^+} \left\| e^{it\Delta_\pm}\gamma_{0,1}\right\|_{L^2_{t\in[-T,T]} L^{4,2}_{x,x'}} = 0 \quad (5.24)$$

We take the sup in $\gamma_{0,2}$, followed by the limsup in T, and then conclude by Plancherel. □

6. Tools for the Analysis of the Full Boltzmann Equation

In this section, we present key tools that will allow us to treat the full Boltzmann equation in subsequent sections.

We start this section by presenting Strichartz estimates for the spatial density

$$\rho_f(x) = \int_{\mathbb{R}^2} f(x, v)\, \mathrm{d}v \tag{6.1}$$

in Section 6.1.

The main challenge for solving Boltzmann's equation (with a constant collision kernel) in $L^2_{x,v}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$ is that the spatial density ρ_f is not necessarily well-defined when $f \in L^2_{x,v}$; therefore, since the loss term has the form $Q^-(f, f) = f\rho_f$, we find that Q^- might not make sense. The ideal way to deal with this situation would be to realize that Q^- *subtracts* from f, and therefore view the loss term as an *unbounded operator* at least when $t \to 0^+$. However, it is not clear to us how to implement this strategy, nor whether it would produce enough integrability to prove uniqueness (and we are not aware of any full treatment of this problem in the literature). The simplest way to avoid the issue of unbounded operators is to introduce an auxiliary norm; one natural possibility would be the $L^1_{x,v}$ norm of f (since it is conserved if f_0 has enough smoothness and decay), but we have instead elected to impose moment and regularity bounds on f_0 so that we can employ Strichartz estimates *in the auxiliary space*, which we introduce in Section 6.2.

6.1. Strichartz Estimates for the Spatial Density

The following lemma follows from a velocity averaging argument. We present the details following the dispersive context [18] for the reader's convenience.

Lemma 6.1. *Fix a sufficiently small number $\delta > 0$. Let $I \subseteq \mathbb{R}$ be an open interval and let $f(t, x, v) : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a measurable and locally integrable function. Then the following estimates hold whenever the respective norms are finite:*

$$\begin{aligned} \left\|\rho_f\right\|_{L^2_{t\in I} L^\infty_x} \leqq & \tilde{C}_\delta \left(\left\| \langle v\rangle^{\frac{1}{2}+\delta} \langle \nabla_x\rangle^{\frac{1}{2}+\delta} f \right\|_{L^\infty_{t\in I} L^2_{x,v}} \right. \\ & \left. + \left\| \langle v\rangle^{\frac{1}{2}+\delta} \langle \nabla_x\rangle^{\frac{1}{2}+\delta} (\partial_t + v\cdot\nabla_x) f \right\|_{L^1_{t\in I} L^2_{x,v}} \right) \end{aligned} \tag{6.2}$$

$$\left\|\rho_f\right\|_{L^2_{t\in I} L^4_x} \leqq C_\delta \left(\left\| \langle v\rangle^{\frac{1}{2}+\delta} f \right\|_{L^\infty_{t\in I} L^2_{x,v}} + \left\| \langle v\rangle^{\frac{1}{2}+\delta} (\partial_t + v\cdot\nabla_x) f \right\|_{L^1_{t\in I} L^2_{x,v}} \right). \tag{6.3}$$

The constants $C_\delta, \tilde{C}_\delta$ do not depend on the interval I.

Proof. Observe that (6.2) follows immediately from (6.3) due to Morrey inequalities [20] and the fact that $\langle\nabla_x\rangle$ commutes with the operators $(\partial_t + v\cdot\nabla_x)$ and $f \mapsto \rho_f$. Therefore, we will prove only the estimate (6.3); moreover, up to possibly increasing the constant C_δ by a fixed factor, we are free to assume that $I = \mathbb{R}$ by standard approximation arguments. If the right hand side of (6.3) is finite, then it immediately follows that $\langle v\rangle^{\frac{1}{2}+\delta} f \in C\left(I, L^2_{x,v}\right)$, so we can assume f is as regular as necessary by standard approximation arguments. Finally, by Duhamel's formula, we have

$$f(t) = e^{-tv\cdot\nabla_x} f_0 + \int_0^t e^{-(t-\sigma)v\cdot\nabla_x} \{(\partial_t + v\cdot\nabla_x) f\}(\sigma)\, \mathrm{d}\sigma.$$

Using Duhamel, along with the linearity of the map $f \mapsto \rho_f$ and Minkowski's inequality (first in x, then in t), we obtain

$$\begin{aligned}\|\rho_f\|_{L_t^2 L_x^4} \leqq & \left\|\rho\left[e^{-tv\cdot\nabla_x} f_0\right]\right\|_{L_t^2 L_x^4} \\ & + \int_{\mathbb{R}} \left\|\rho\left[e^{-(t-\sigma)v\cdot\nabla_x}\{(\partial_t + v\cdot\nabla_x) f\}(\sigma)\right]\right\|_{L_t^2 L_x^4} \mathrm{d}\sigma\end{aligned}$$

and therefore we immediately deduce (6.3), once the same inequality holds with

$$(\partial_t + v\cdot\nabla_x) f = 0.$$

In words, we can assume f is a solution of the free transport equation.

Altogether, we only need to show that if $f_0(x, v)$ is smooth and compactly supported in $\mathbb{R}^2 \times \mathbb{R}^2$, then

$$\|\rho_{T(t)f_0}\|_{L_t^2 L_x^4(\mathbb{R}\times\mathbb{R}^2)} \leqq C_\delta \left\|\langle v\rangle^{\frac{1}{2}+\delta} f_0\right\|_{L_{x,v}^2(\mathbb{R}^2\times\mathbb{R}^2)}, \tag{6.4}$$

where $T(t) f_0 = e^{-tv\cdot\nabla_x} f_0$. By the fractional Gagliardo-Nirenberg-Sobolev inequality [20], it suffices to show that

$$\left\|(-\Delta_x)^{\frac{1}{4}} \rho_{T(t)f_0}\right\|_{L_t^2 L_x^2(\mathbb{R}\times\mathbb{R}^2)} \leqq C_\delta \left\|\langle v\rangle^{\frac{1}{2}+\delta} f_0\right\|_{L_{x,v}^2(\mathbb{R}^2\times\mathbb{R}^2)}, \tag{6.5}$$

whenever f_0 is smooth and compactly supported in $\mathbb{R}^2 \times \mathbb{R}^2$.[2] We will establish (6.5) using the spacetime Fourier transform to conclude the lemma.

To prove (6.5), we apply Plancherel in (t, x) on the left-hand side, and in x on the right-hand side; hence, an equivalent bound is

$$\left\|\mathcal{F}_{t,x}\left\{(-\Delta_x)^{\frac{1}{4}} \rho_{T(t)f_0}\right\}(\tau, \xi)\right\|_{L_\tau^2 L_\xi^2} \leqq C_\delta \left\|\langle v\rangle^{\frac{1}{2}+\delta} \mathcal{F}_x\{f_0\}(\xi, v)\right\|_{L_{\xi,v}^2}. \tag{6.6}$$

Let us define

$$H(\xi, v) = \mathcal{F}_x\{f_0\}(\xi, v). \tag{6.7}$$

Then (6.6) may be re-cast as the following inequality:

$$\left\||\xi|^{\frac{1}{2}} \int_{\mathbb{R}^2} \mathrm{d}v\,\delta(\tau + v\cdot\xi)\, H(\xi, v)\right\|_{L_\tau^2 L_\xi^2}^2 \leqq C_\delta^2 \left\|\langle v\rangle^{\frac{1}{2}+\delta} H(\xi, v)\right\|_{L_{\xi,v}^2}^2. \tag{6.8}$$

[2] Note that if f_0 is smooth and compactly supported, then for any fixed $t \in \mathbb{R}$, $T(t) f_0$ is also smooth and compactly supported.

The quantity on the left can be equivalently written as

$$\int_{\mathbb{R}} \mathrm{d}\tau \int_{\mathbb{R}^2} \mathrm{d}\xi \int_{\mathbb{R}^2} dv \int_{\mathbb{R}^2} \mathrm{d}u \delta\left(\tau + v\cdot\xi\right)\delta\left(\tau + u\cdot\xi\right)|\xi|\, H\left(\xi, v\right)\overline{H\left(\xi, u\right)},$$

which is the same as

$$\int_{\mathbb{R}} \mathrm{d}\tau \int_{\mathbb{R}^2} \mathrm{d}\xi \int_{\mathbb{R}^2} \mathrm{d}v \int_{\mathbb{R}^2} \mathrm{d}u \delta\left(\tau + v\cdot\xi\right)\delta\left(\tau + u\cdot\xi\right)|\xi| \\ \times \left(\frac{1}{\langle u\rangle^{\frac{1}{2}+\delta}}\langle v\rangle^{\frac{1}{2}+\delta} H\left(\xi, v\right)\right)\left(\frac{1}{\langle v\rangle^{\frac{1}{2}+\delta}}\langle u\rangle^{\frac{1}{2}+\delta}\overline{H\left(\xi, u\right)}\right)$$

The idea of [18] is to apply the Cauchy-Schwarz inequality, $AB \leqq \frac{A^2}{2} + \frac{B^2}{2}$, but *pointwise* in (τ, ξ, v, u) (not in the integral sense!) to the two terms in the large parentheses. Thus we will end up with the sum of two terms, one involving only $H(\xi, v)$ and the other only involving $H(\xi, u)$; under the obvious symmetry $u \leftrightarrow v$, we can discard one of them up to a factor of 2.

Thus we now only need to prove that

$$\int_{\mathbb{R}} \mathrm{d}\tau \int_{\mathbb{R}^2} \mathrm{d}\xi \int_{\mathbb{R}^2} \mathrm{d}v \int_{\mathbb{R}^2} \mathrm{d}u \delta\left(\tau + v\cdot\xi\right)\delta\left(\tau + u\cdot\xi\right)|\xi| \\ \times \left(\frac{1}{\langle u\rangle^{1+2\delta}}\langle v\rangle^{1+2\delta}\left|H\left(\xi, v\right)\right|^2\right) \\ \leqq C_\delta^2 \left\| \langle v\rangle^{\frac{1}{2}+\delta} H\left(\xi, v\right)\right\|^2_{L^2_{\xi,v}}$$

(we can assume H vanishes for ξ close to the origin, so that the integral on the left certainly makes sense). Hence if we can show that

$$\sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^2_{\neq 0}} \int_{\mathbb{R}^2} \mathrm{d}u \delta\left(\tau + u\cdot\xi\right)\frac{|\xi|}{\langle u\rangle^{1+2\delta}} < \infty, \tag{6.9}$$

then we will be done (note that the other δ-function, $\delta\left(\tau + v\cdot\xi\right)$, is absorbed by the integral in τ, but only *after* using the supremum bound).

Let us define

$$I\left(\tau, \xi\right) = \int_{\mathbb{R}^2} \mathrm{d}u \delta\left(\tau + u\cdot\xi\right)\frac{|\xi|}{\langle u\rangle^{1+2\delta}}.$$

If we denote the line

$$P\left(\tau, \xi\right) = \left\{u \in \mathbb{R}^2 \mid \tau + u\cdot\xi = 0\right\},$$

then it follows that

$$I\left(\tau, \xi\right) = \int_{u\in P(\tau,\xi)} d\ell(u)\frac{1}{\langle u\rangle^{1+2\delta}},$$

where $d\ell(u)$ is the induced linear measure. We can only increase the value of the integral of $\langle u\rangle^{-1-2\delta}$ by translating the line $P\left(\tau, \xi\right)$ toward the origin of $\mathbb{R}^2$. Therefore,

$$\sup_{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^2_{\neq 0}} I\left(\tau, \xi\right) \leqq \int_{q\in\mathbb{R}} \frac{dq}{\left(1+q^2\right)^{\frac{1}{2}+\delta}} < \infty,$$

so we are able to conclude. □

6.2. Weights and Regularity

Let $\varepsilon \in (0, 1]$ and define the norm

$$\| f_0 \|^2_{H^{1,1}_\varepsilon} = \| f_0 \|^2_{L^2_{x,v}} + \varepsilon^2 \| v f_0 \|^2_{L^2_{x,v}} + \varepsilon^2 \| \nabla_x f_0 \|^2_{L^2_{x,v}} + \varepsilon^4 \| v \otimes \nabla_x f_0 \|^2_{L^2_{x,v}} . \quad (6.10)$$

Note that the space $H^{1,1}_\varepsilon$ is independent of $\varepsilon > 0$, but the norm of a fixed element $f_0 \in H^{1,1}_\varepsilon$ does depend on ε in general. The norm on $H^{1,1}_\varepsilon$ is equivalently written as

$$\| f_0 \|_{H^{1,1}_\varepsilon} = \left\| \left(1 + \varepsilon^2 |v|^2\right)^{\frac{1}{2}} \left(1 + \varepsilon^2 |\xi|^2\right)^{\frac{1}{2}} \mathcal{F}_x f_0(\xi, v) \right\|_{L^2_{\xi,v}} , \quad (6.11)$$

where $\mathcal{F}_x f_0$ is the Fourier transform of f_0 in the spatial variable only. This may also be written as

$$\| f_0 \|_{H^{1,1}_\varepsilon} = \| \langle \varepsilon v \rangle \langle \varepsilon \nabla_x \rangle f_0 \|_{L^2_{x,v}} , \quad (6.12)$$

where $\langle v \rangle = \left(1 + |v|^2\right)^{\frac{1}{2}}$. We will use the notation $H^{1,1} \equiv H^{1,1}_1$ when the dependence on ε is unimportant.

More generally, we also define the norms

$$\| f_0 \|_{H^{\alpha,\beta}_\varepsilon} = \left\| \langle \varepsilon v \rangle^\beta \langle \varepsilon \nabla_x \rangle^\alpha f_0 \right\|_{L^2_{x,v}} , \quad (6.13)$$

where the exponents $\alpha, \beta \geqq 0$ are chosen independently.

The following commutation relations are standard:

$$\begin{aligned} \nabla_x Q^+ (f, g) &= Q^+ (\nabla_x f, g) + Q^+ (f, \nabla_x g) \\ \nabla_x T(t) f_0 &= T(t) \nabla_x f_0 \\ v T(t) f_0 &= T(t) (v f_0) . \end{aligned}$$

Additionally, from conservation of energy, we have

$$\left| v Q^+ (f, g) \right| \lesssim Q^+ (|vf| , |g|) + Q^+ (|f| , |vg|) .$$

Using the commutation relations and Proposition 5.4, we have

$$\left\| \nabla_x Q^+ (T(t) f_0, T(t) g_0) \right\|_{L^1_t L^2_{x,v}} \lesssim \| \nabla_x f_0 \|_{L^2_{x,v}} \| g_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2_{x,v}} \| \nabla_x g_0 \|_{L^2_{x,v}} \quad (6.14)$$

$$\left\| v Q^+ (T(t) f_0, T(t) g_0) \right\|_{L^1_t L^2_{x,v}} \lesssim \| v f_0 \|_{L^2_{x,v}} \| g_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2_{x,v}} \| v g_0 \|_{L^2_{x,v}} \quad (6.15)$$

and

$$\begin{aligned} \left\| v \otimes \nabla_x Q^+ (T(t) f_0, T(t) g_0) \right\|_{L^1_t L^2_{x,v}} &\lesssim \| v \otimes \nabla_x f_0 \|_{L^2_{x,v}} \| g_0 \|_{L^2_{x,v}} \\ &+ \| \nabla_x f_0 \|_{L^2_{x,v}} \| v g_0 \|_{L^2_{x,v}} + \| v f_0 \|_{L^2_{x,v}} \| \nabla_x g_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2_{x,v}} \| v \otimes \nabla_x g_0 \|_{L^2_{x,v}} . \end{aligned} \quad (6.16)$$

Using (6.14), (6.15) and (6.16), and the definition of $H^{1,1}_\varepsilon$, we obtain the following estimate:

$$\left\| Q^+ (T(t) f_0, T(t) g_0) \right\|_{L^1_t H^{1,1}_\varepsilon} \leqq C \| f_0 \|_{H^{1,1}_\varepsilon} \| g_0 \|_{H^{1,1}_\varepsilon} . \quad (6.17)$$

Here, the constant C does not depend on $\varepsilon \in (0, 1]$.

Proposition 6.2. *For any $f_0, g_0 \in H^{1,1}$, it holds that*

$$\left\| Q^+ \left(T(t) f_0, T(t) g_0\right)\right\|_{L^1_t H^{1,1}_\varepsilon} \leqq C \left\| f_0 \right\|_{H^{1,1}_\varepsilon} \left\| g_0 \right\|_{H^{1,1}_\varepsilon} \tag{6.18}$$

where $T(t) = e^{-tv\cdot\nabla_x}$. The constant C is independent of $\varepsilon \in (0, 1]$.

Similarly, we also have

Proposition 6.3. *For any $f_0, g_0 \in H^{0,1}$, it holds that*

$$\left\| \langle \varepsilon v \rangle\, Q^+ \left(T(t) f_0, T(t) g_0\right)\right\|_{L^1_t L^2_{x,v}} \leqq C \left\| \langle \varepsilon v \rangle\, f_0 \right\|_{L^2_{x,v}} \left\| \langle \varepsilon v \rangle\, g_0 \right\|_{L^2_{x,v}} \tag{6.19}$$

where $T(t) = e^{-tv\cdot\nabla_x}$. The constant C is independent of $\varepsilon \in (0, 1]$.

The bounds in the preceding two propositions can be interpolated against Proposition 5.4, using Theorem 5.1.2 of the book [5], to obtain

Proposition 6.4. *Let $\alpha \in (0, 1)$. For any $f_0, g_0 \in H^{\alpha,\alpha}$, it holds that*

$$\left\| Q^+ \left(T(t) f_0, T(t) g_0\right)\right\|_{L^1_t H^{\alpha,\alpha}_\varepsilon} \leqq C \left\| f_0 \right\|_{H^{\alpha,\alpha}_\varepsilon} \left\| g_0 \right\|_{H^{\alpha,\alpha}_\varepsilon} \tag{6.20}$$

where $T(t) = e^{-tv\cdot\nabla_x}$. The constant C is independent of ε, α.

Proposition 6.5. *Let $\alpha \in (0, 1)$. For any $f_0, g_0 \in H^{0,\alpha}$, it holds that*

$$\left\| \langle \varepsilon v \rangle^\alpha\, Q^+ \left(T(t) f_0, T(t) g_0\right)\right\|_{L^1_t L^2_{x,v}} \leqq C \left\| \langle \varepsilon v \rangle^\alpha\, f_0 \right\|_{L^2_{x,v}} \left\| \langle \varepsilon v \rangle^\alpha\, g_0 \right\|_{L^2_{x,v}} \tag{6.21}$$

where $T(t) = e^{-tv\cdot\nabla_x}$. The constant C is independent of ε, α.

6.3. A Useful Lemma

The next lemma is a consequence of Section 3; we record it here to help clarify the main ideas underlying the present work. Note that the theory of Section 3 cannot be applied "out of box" to the Boltzmann equation accounting for the loss term. For this reason, it is crucial to observe that the theory of Section 3 rests upon a single bound which can be applied to the Q^+ term in any estimate.

Lemma 6.6. *Let $I = (a, b) \subset \mathbb{R}$ be a nonempty open interval with $-\infty \leqq a < b \leqq +\infty$. Furthermore, for $i = 1, 2$, suppose $f_i(t, x, v) : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $f_i \in L^\infty_{t\in I} L^2_{x,v}$ and $(\partial_t + v \cdot \nabla_x) f_i \in L^1_{t\in I} L^2_{x,v}$. Then the following estimate holds:*

$$\begin{aligned} &\left\| Q^+ \left(f_1(t), f_2(t)\right)\right\|_{L^1_{t\in I} L^2_{x,v}} \\ &\leqq C \prod_{i=1,2} \left(\left\| f_i(t) \right\|_{L^\infty_{t\in I} L^2_{x,v}} + \left\| (\partial_t + v \cdot \nabla_x) f_i(t) \right\|_{L^1_{t\in I} L^2_{x,v}} \right), \end{aligned} \tag{6.22}$$

for some constant C which does not depend on f_1, f_2 or the interval I.

Proof. We may assume without loss that $I = (0, T)$ for some $T > 0$. The lemma then follows from Proposition 6.4 and Lemma 3.1, under the following assignments: $\mathcal{H} = L^2_{x,v}$, $x_j(t) = e^{tv\cdot\nabla_x} f_j(t)$, and

$$\mathcal{A}(t, x_1, x_2) = e^{tv\cdot\nabla_x} Q^+ \left(e^{-tv\cdot\nabla_x} x_1, e^{-tv\cdot\nabla_x} x_2\right).$$

Here we have used that $e^{-tv\cdot\nabla_x}$ is an isometry on $L^2_{x,v}$ for any $t \in \mathbb{R}$. □

Similarly, we deduce the following result as a consequence of Proposition 6.4 and Lemma 3.1:

Lemma 6.7. *Let $\varepsilon \in (0, 1]$ and let $\alpha \in (0, 1)$. Let $I = (a, b) \subset \mathbb{R}$ be a nonempty open interval with $-\infty \leqq a < b \leqq +\infty$. Furthermore, for $i = 1, 2$, suppose $f_i(t, x, v) : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $f_i \in L^\infty_{t\in I} H^{\alpha,\alpha}$ and $(\partial_t + v \cdot \nabla_x) f_i \in L^1_{t\in I} H^{\alpha,\alpha}$. Then the following estimate holds:*

$$\begin{aligned}&\left\| Q^+ (f_1(t), f_2(t))\right\|_{L^1_{t\in I} H^{\alpha,\alpha}_\varepsilon} \\ &\leqq C \prod_{i=1,2} \left(\| f_i(t)\|_{L^\infty_{t\in I} H^{\alpha,\alpha}_\varepsilon} + \|(\partial_t + v \cdot \nabla_x) f_i(t)\|_{L^1_{t\in I} H^{\alpha,\alpha}_\varepsilon}\right),\end{aligned} \tag{6.23}$$

for some constant C which does not depend on $f_1, f_2, \alpha, \varepsilon$ or the interval I.

The following result is similarly straightforward to prove by omitting spatial derivatives throughout the argument:

Lemma 6.8. *Let $\varepsilon \in (0, 1]$ and let $\alpha \in (0, 1)$. Let $I = (a, b) \subset \mathbb{R}$ be a nonempty open interval with $-\infty \leqq a < b \leqq +\infty$. Furthermore, for $i = 1, 2$, suppose $f_i(t, x, v) : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\langle v\rangle^\alpha f_i \in L^\infty_{t\in I} L^2_{x,v}$ and $\langle v\rangle^\alpha (\partial_t + v \cdot \nabla_x) f_i \in L^1_{t\in I} L^2_{x,v}$. Then the following estimate holds:*

$$\begin{aligned}&\left\|\langle \varepsilon v\rangle^\alpha Q^+ (f_1(t), f_2(t))\right\|_{L^1_{t\in I} L^2_{x,v}} \\ &\leqq C \prod_{i=1,2} \left(\left\|\langle \varepsilon v\rangle^\alpha f_i(t)\right\|_{L^\infty_{t\in I} L^2_{x,v}} + \left\|\langle \varepsilon v\rangle^\alpha (\partial_t + v \cdot \nabla_x) f_i(t)\right\|_{L^1_{t\in I} L^2_{x,v}}\right),\end{aligned} \tag{6.24}$$

for some constant C which does not depend on $f_1, f_2, \alpha, \varepsilon$ or the interval I.

7. Uniqueness

In this section, we present our main uniqueness result.

Theorem 7.1. *There is at most one mild solution of the full Boltzmann equation on an interval $[0, T]$, with given initial data f_0, such that the estimates*

$$\langle v\rangle^{\frac{1}{2}+} Q^+(f, f) \in L^1_{t\in[0,T]} L^2_{x,v} \tag{7.1}$$

$$\rho_f \in L^2_{t\in[0,T]} L^\infty_x \bigcap L^2_{t\in[0,T]} L^4_x \tag{7.2}$$

$$\langle v\rangle^{\frac{1}{2}+} f \in L^\infty_{t\in[0,T]} L^2_{x,v} \bigcap L^\infty_{t\in[0,T]} L^4_x L^2_v \tag{7.3}$$

are all verified.

Remark 7.1. Theorem 7.1 makes no assumptions about the non-negativity of either $f(t)$ or $f(0) = f_0$; in particular, neither f nor ρ_f needs to be non-negative anywhere on their respective domains of definition.

Remark 7.2. A similar statement is discussed in [13]. We note that the analysis in [13] does not cover the case studied in our work (constant collision kernel in d = 2) because the parameter range (in their notation) $r_v \leqq p_v$ needed for Lemma 2.4 in [13] does not match the required parameter range $r_v > p_v$ for the Strichartz estimate controlling the loss term.

7.1. Proof of Theorem 7.1

Let f, g be two mild solutions of Boltzmann's equation on the given interval $[0, T]$ (each satisfying the bounds stated in the theorem), and consider the difference

$$w = f - g. \tag{7.4}$$

The function w satisfies the difference equation

$$(\partial_t + v \cdot \nabla_x)\, w = Q^+(f, w) + Q^+(w, g) - w\rho_f - g\rho_w, \tag{7.5}$$

with $w(0) = 0$. Now we apply the lemma to follow (it is not hard to check that all necessary bounds follow from the hypotheses of the uniqueness theorem and the fact that f, g solve Boltzmann's equation with w being their difference).

Lemma 7.2. *Assume that* f_i, $i = 1, 2, 3, 4$, *satisfy the bounds*

$$\langle v\rangle^{\frac{1}{2}+} f_i \in L^\infty_{t\in[0,T]} L^2_{x,v} \bigcap L^\infty_{t\in[0,T]} L^4_x L^2_v \tag{7.6}$$

$$\langle v\rangle^{\frac{1}{2}+} (\partial_t + v \cdot \nabla_x)\, f_i \in L^1_{t\in[0,T]} L^2_{x,v} \tag{7.7}$$

$$\rho_{f_i} \in L^2_{t\in[0,T]} L^\infty_x \bigcap L^2_{t\in[0,T]} L^4_x. \tag{7.8}$$

Also assume that w *is a mild solution of the equation*

$$(\partial_t + v \cdot \nabla_x)\, w = Q^+(f_1, w) + Q^+(w, f_2) + w\rho_{f_3} + f_4\rho_w \tag{7.9}$$

for $t \in [0, T]$, *and satisfies the bounds*

$$\langle v\rangle^{\frac{1}{2}+} w \in L^\infty_{t\in[0,T]} L^2_{x,v} \tag{7.10}$$

$$\langle v\rangle^{\frac{1}{2}+} (\partial_t + v \cdot \nabla_x)\, w \in L^1_{t\in[0,T]} L^2_{x,v}. \tag{7.11}$$

Then if $w(t = 0) = 0$ *then* $w \equiv 0$ *for* $0 \leqq t \leqq T$.

Proof. The bounds imposed on w immediately imply that $\langle v\rangle^{\frac{1}{2}+} w \in C\big([0, T], L^2_{x,v}\big)$. Let us suppose that the conclusion fails and define

$$t_0 = \inf \left\{ t \in [0, T] \,\middle|\, \|w(t)\|_{L^2_{x,v}} > 0 \right\}. \tag{7.12}$$

Then $0\leqq t_0 < T$, and $w \equiv 0$ for all $0\leqq t\leqq t_0$ by continuity.

Let us define the error, for $0\leqq s\leqq T - t_0$,

$$e_{t_0}(s) = \left\| \langle v\rangle^{\frac{1}{2}+} w \right\|_{L^\infty_{t\in[t_0,t_0+s]} L^2_{x,v}} + \left\| \langle v\rangle^{\frac{1}{2}+} (\partial_t + v\cdot\nabla_x)\, w \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}}, \quad (7.13)$$

and note that $e_{t_0}(s) < +\infty$, by hypothesis. We re-write the equation for w as follows:

$$\begin{aligned}
&(\partial_t + v\cdot\nabla_x)\, w \\
&\quad = Q^+ (f_1 - T(t-t_0) f_1(t_0), w) + Q^+ (T(t-t_0) f_1(t_0), w) \\
&\qquad + Q^+ (w, f_2 - T(t-t_0) f_2(t_0)) + Q^+ (w, T(t-t_0) f_2(t_0)) \\
&\qquad + w\rho_{f_3} + f_4 \rho_w.
\end{aligned} \quad (7.14)$$

The most dangerous terms are

$$Q^+ (T(t-t_0) f_1(t_0), w) \quad (7.15)$$

and

$$Q^+ (w, T(t-t_0) f_2(t_0)), \quad (7.16)$$

because a quantitative estimate will always be proportional to

$$\| f_i(t_0)\|_{L^2_{x,v}} \times e_{t_0}(s),$$

which is not necessarily a small multiple of $e_{t_0}(s)$ (unless $\| f_i(t_0)\|_{L^2_{x,v}}$ is small). We will address this problem using the short-time estimates from Proposition 5.6.

We will show how to estimate (7.15); the alternative term (7.16) is dealt with similarly. To begin, let us define

$$\zeta = (\partial_t + v\cdot\nabla_x)\, w,$$

and then use Duhamel's formula to write

$$w(t) = \int_{t_0}^{t} T\,(t-\sigma)\,\zeta(\sigma) d\sigma, \quad (7.17)$$

since $w(t_0) = 0$. Due to the bilinearity of Q^+, we can now write

$$\begin{aligned}
Q^+ (T(t-t_0) f_1(t_0), w) &= \int_{t_0}^{t} Q^+ (T(t-t_0) f_1(t_0), T(t-\sigma)\zeta(\sigma))\, \mathrm{d}\sigma \\
&= \int_{t_0}^{t} Q^+ (T(t-t_0) f_1(t_0), T(t-t_0) T(t_0-\sigma)\zeta(\sigma))\, \mathrm{d}\sigma.
\end{aligned} \quad (7.18)$$

Now, by Minkowski's inequality, we have

$$\begin{aligned}
&\left\| Q^+ (T(t-t_0) f_1(t_0), w)\right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \\
&\leqq \int_{t_0}^{t_0+s} \left\| Q^+ (T(t-t_0) f_1(t_0), T(t-t_0) T(t_0-\sigma)\zeta(\sigma))\right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \mathrm{d}\sigma.
\end{aligned} \quad (7.19)$$

Apply Proposition 5.6 to obtain

$$\begin{aligned}
&\left\| Q^{+}\left(T(t-t_0) f_1(t_0), w\right)\right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \\
&\leqq \int_{t_0}^{t_0+s} \delta_{f_1(t_0)}(s)\, \| T(t_0-\sigma)\zeta(\sigma)\|_{L^2_{x,v}}\, \mathrm{d}\sigma \\
&= \int_{t_0}^{t_0+s} \delta_{f_1(t_0)}(s)\, \|\zeta(\sigma)\|_{L^2_{x,v}}\, \mathrm{d}\sigma = \delta_{f_1(t_0)}(s)\, \|\zeta(t)\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \\
&\leqq \delta_{f_1(t_0)}(s) e_{t_0}(s),
\end{aligned} \tag{7.20}$$

where, for each $f_1(t_0) \in L^2_{x,v}$,

$$\limsup_{s\to 0^+} \delta_{f_1(t_0)}(s) = 0$$

The same argument can be applied with a weight $\langle v\rangle^{\frac{1}{2}+}$ to yield

$$\left\| \langle v\rangle^{\frac{1}{2}+} Q^{+}\left(T(t-t_0) f_1(t_0), w\right)\right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \leqq \tilde{\delta}_{f_1(t_0)}(s) e_{t_0}(s), \tag{7.21}$$

where for each $f_1(t_0)$ with $\langle v\rangle^{\frac{1}{2}+} f_1(t_0) \in L^2_{x,v}$ it holds that

$$\limsup_{s\to 0^+} \tilde{\delta}_{f_1(t_0)}(s) = 0.$$

Next we consider the term

$$Q^{+}\left(f_1 - T(t-t_0) f_1(t_0), w\right) \tag{7.22}$$

(the corresponding term involving Q^+ and f_2 is dealt with similarly). Here we use Lemma 6.8 to write

$$\begin{aligned}
&\left\| \langle v\rangle^{\frac{1}{2}+} Q^{+}\left(f_1 - T(t-t_0) f_1(t_0), w\right)\right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \\
&\leqq C \left(\left\| \langle v\rangle^{\frac{1}{2}+} \left(f_1 - T(t-t_0) f_1(t_0)\right)\right\|_{L^\infty_{t\in[t_0,t_0+s]} L^2_{x,v}} \right. \\
&\quad \left. + \left\| \langle v\rangle^{\frac{1}{2}+} \left(\partial_t + v\cdot\nabla_x\right) f_1 \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \right) \\
&\quad \times \left(\left\| \langle v\rangle^{\frac{1}{2}+} w \right\|_{L^\infty_{t\in[t_0,t_0+s]} L^2_{x,v}} + \left\| \langle v\rangle^{\frac{1}{2}+} \left(\partial_t + v\cdot\nabla_x\right) w \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \right) \\
&\leqq C \left\| \langle v\rangle^{\frac{1}{2}+} \left(\partial_t + v\cdot\nabla_x\right) f_1 \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \times e_{t_0}(s).
\end{aligned}$$

Now let us consider the term

$$w\rho_{f_3}.$$

We have, by Hölder's inequality,

$$\left\| \langle v \rangle^{\frac{1}{2}+} w \rho_{f_3} \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \leqq \left\| \langle v \rangle^{\frac{1}{2}+} w \right\|_{L^\infty_{t\in[t_0,t_0+s]} L^2_{x,v}} \left\| \rho_{f_3} \right\|_{L^1_{t\in[t_0,t_0+s]} L^\infty_x}$$
$$\leqq s^{\frac{1}{2}} \left\| \rho_{f_3} \right\|_{L^2_{t\in[0,T]} L^\infty_x} e_{t_0}(s).$$

Finally, consider the term

$$f_4 \rho_w.$$

We have, by Hölder's inequality, and Lemma 6.1,

$$\begin{aligned} \left\| \langle v \rangle^{\frac{1}{2}+} f_4 \rho_w \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} &\leqq \left\| \langle v \rangle^{\frac{1}{2}+} f_4 \right\|_{L^\infty_{t\in[t_0,t_0+s]} L^4_x L^2_v} \left\| \rho_w \right\|_{L^1_{t\in[t_0,t_0+s]} L^4_x} \\ &\leqq s^{\frac{1}{2}} \left\| \langle v \rangle^{\frac{1}{2}+} f_4 \right\|_{L^\infty_{t\in[0,T]} L^4_x L^2_v} \left\| \rho_w \right\|_{L^2_{t\in[t_0,t_0+s]} L^4_x} \\ &\leqq C s^{\frac{1}{2}} \left\| \langle v \rangle^{\frac{1}{2}+} f_4 \right\|_{L^\infty_{t\in[0,T]} L^4_x L^2_v} \times e_{t_0}(s). \end{aligned} \tag{7.23}$$

Altogether we can conclude the bound

$$e_{t_0}(s) \leqq c(s) e_{t_0}(s),$$

where

$$\begin{aligned} c(s) =& C \sum_{i=1,2} \left(\tilde{\delta}_{f_i(t_0)}(s) + \left\| \langle v \rangle^{\frac{1}{2}+} (\partial_t + v \cdot \nabla_x) f_i \right\|_{L^1_{t\in[t_0,t_0+s]} L^2_{x,v}} \right) \\ &+ C s^{\frac{1}{2}} \left\| \rho_{f_3} \right\|_{L^2_{t\in[0,T]} L^\infty_x} + C s^{\frac{1}{2}} \left\| \langle v \rangle^{\frac{1}{2}+} f_4 \right\|_{L^\infty_{t\in[0,T]} L^4_x L^2_v}. \end{aligned} \tag{7.24}$$

Clearly, $c(s) \to 0$ as $s \to 0^+$; hence, taking s small enough, we shall have $c(s) < 1$. This implies that $e_{t_0}(s) < e_{t_0}(s)$; since $e_{t_0}(s)$ is finite, we can conclude that $e_{t_0}(s) = 0$ for some $s > 0$ sufficiently small. This contradicts the definition of t_0, so we are done. □

8. The Kaniel–Shinbrot Iteration

The problem we encounter in trying to solve Boltzmann's equation is that we are unable to prove Proposition 5.4 with Q^- in place of Q^+. Indeed, it is not even clear whether $Q^-(f, f)$ is meaningful, in general, when f is a mild solution of the gain-only equation obtained from Theorem 5.5. On the other hand, it is definitely possible to solve uniquely the full Boltzmann equation (with constant collision kernel in $d = 2$) *locally in time* if we assume that

$$\langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f_0 \in L^2_{x,v} \left(\mathbb{R}^2 \times \mathbb{R}^2 \right).$$

The challenge, therefore, is to propagate sufficient regularity for the gain-only equation, *assuming a smallness condition only for the $L^2_{x,v}$ norm*. To this end, we will need to employ a small parameter $\varepsilon \in (0,1]$ to encode the fact that higher derivatives may be much larger than the $L^2_{x,v}$ norm of f_0.

We proceed by first establishing regularity of the gain-only equation in Section 8.1. Then, in Section 8.2, we present a novel application of the iterative method of Kaniel–Shinbrot to establish existence of global solution to the Boltzmann equation.

8.1. Regularity for the Gain-Only Equation

Theorem 8.1. *There exists a number $\eta \in (0,1)$ such that of all the following is true:*

(i) For any $f_0 \in H^{\alpha,\alpha}$, $\alpha \in (0,1)$, with $\| f_0 \|_{L^2_{x,v}} < \eta$, there exists a unique global ($t \in \mathbb{R}$) mild solution to the gain-only *Boltzmann equation*

$$(\partial_t + v \cdot \nabla_x) f(t) = Q^+ (f(t), f(t)) \tag{8.1}$$

with $f(0) = f_0$ such that $f \in C_{t,loc} H^{\alpha,\alpha}$ and $Q^+(f,f) \in L^1_{t,loc} H^{\alpha,\alpha}$. For this solution, it holds that $f \in L^\infty_t H^{\alpha,\alpha}$ and $Q^+(f,f) \in L^1_t H^{\alpha,\alpha}$, and the solution scatters in $H^{\alpha,\alpha}$ as $t \to \pm\infty$.

(ii) For any $f_0 \in H^{\alpha,\alpha}$ with $\| f_0 \|_{L^2_{x,v}} < \eta$, we have the estimate

$$\| f \|^2_{L^\infty_{t\in\mathbb{R}} L^2_{x,v}} + \left\| Q^+ (f,f) \right\|_{L^1_{t\in\mathbb{R}} L^2_{x,v}} \leqq C \, \| f_0 \|^2_{L^2_{x,v}} \tag{8.2}$$

for the solution f of the gain-only Boltzmann equation (note, this bound only depends on the $L^2_{x,v}$ norm of f_0). Also, if $f_0(x,v) \geqq 0$ a.e. $-(x,v)$ then $f(t,x,v) \geqq 0$ for a.e. $-(t,x,v)$ such that $t \geqq 0$.

(iii) If $\alpha > \frac{1}{2}$, then we have $\langle v \rangle^{\frac{1}{2}+} f \in L^\infty_t L^4_x L^2_v$ and $\rho_f \in L^2_t L^\infty_x \bigcap L^2_t L^4_x$. Combining these estimates, the loss term $Q^-(f,f) = \rho_f f$ (although not *appearing in the equation for f) satisfies*

$$\langle v \rangle^{\frac{1}{2}+} Q^-(f,f) \in L^2_{t\in\mathbb{R}} L^2_{x,v}.$$

Proof. Parts *(i)* and *(ii)* are direct consequences of Proposition 6.4, combined with Theorem 3.2 taking $\mathcal{H} = H^{\alpha,\alpha}_\varepsilon$ where $\varepsilon = \varepsilon(f_0)$ is sufficiently small; here we have used the fact that the constant C in Proposition 6.4 does not depend on ε. Note that $L^2_{x,v} \subset H^{\alpha,\alpha}$, so the uniqueness in $L^2_{x,v}$ implies that $L^2_{x,v}$ and $H^{\alpha,\alpha}$ solutions coincide globally in time (as long as the $L^2_{x,v}$ norm of f_0 is small enough).

For part *(iii)*, to see that $\langle v \rangle^{\frac{1}{2}+} f \in L^\infty_t L^4_x L^2_v$, we may observe that $\langle v \rangle^{\frac{1}{2}+} \langle \nabla_x \rangle^{\frac{1}{2}+} f \in L^\infty_t L^2_{x,v}$ and apply the Sobolev embedding theorem in the x variable. On the other hand, the estimate $\rho_f \in L^2_t L^\infty_x \bigcap L^2_t L^4_x$ follows directly from Lemma 6.1 and the estimates from part *(i)*. The estimate on $Q^-(f,f)$ then follows from Hölder's inequality. Note that, contrary to part *(ii)*, all the bounds from part *(iii)* depend explicitly on the $H^{\alpha,\alpha}$ norm of f_0. □

8.2. The Full Equation via Kaniel–Shinbrot Iteration

The iteration of Kaniel and Shinbrot constructs a decreasing sequence $g_n(t, x, v)$ and an increasing sequence $h_n(t, x, v)$ with $0 \leqq h_n \leqq g_n$. The goal is to show that $\lim_n g_n = \lim_n h_n = f$, with f being a solution of the full Boltzmann equation. One can view the functions g_n, h_n as being "barriers" which progressively limit the possible oscillation of f, until eventually there is no room left in which to wiggle.

Recall the convenient notation

$$\rho_f(x) = \int_{\mathbb{R}^2} f(x, v)\,\mathrm{d}v. \tag{8.3}$$

The iteration is as follows:

$$\begin{aligned} \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_{g_n}\right) h_{n+1} &= Q^+(h_n, h_n) \\ \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_{h_n}\right) g_{n+1} &= Q^+(g_n, g_n) \\ g_{n+1}(0) = h_{n+1}(0) &= f_0. \end{aligned}$$

For each n, observe that we are simply solving *linear* differential equations (with the initial data always fixed at f_0), so the existence of the iteration is typically not a big problem. It is possible to show (see e.g. [6]), using monotonicity, that if

$$0 \leqq h_{n-1} \leqq h_n \leqq g_n \leqq g_{n-1} \tag{8.4}$$

holds globally, then

$$0 \leqq h_n \leqq h_{n+1} \leqq g_{n+1} \leqq g_n. \tag{8.5}$$

Hence, in order to exploit monotonicity, we must at least have

$$0 \leqq h_1 \leqq h_2 \leqq g_2 \leqq g_1, \tag{8.6}$$

where

$$\begin{aligned} \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_{g_1}\right) h_2 &= Q^+(h_1, h_1) \\ \left(\partial_t + v \cdot \nabla_x + 2\pi \rho_{h_1}\right) g_2 &= Q^+(g_1, g_1) \\ g_2(0) = h_2(0) &= f_0, \end{aligned} \tag{8.7}$$

and this is the so-called *beginning condition* (note that no initial conditions are imposed for (h_1, g_1)). Note that the beginning condition has to be verified *for all time* (or at least on the full time interval for which the iteration is to be employed). For this reason, establishing the beginning condition is considered the most difficult part of the Kaniel–Shinbrot iteration.

We choose h_1 as follows:

$$h_1 \equiv 0,$$

and we choose g_1 to solve the gain only equation

$$(\partial_t + v \cdot \nabla_x)\, g_1 = Q^+(g_1, g_1), \qquad g_1(0) = f_0. \tag{8.8}$$

Then we compute h_2 and g_2 according to (8.7) to obtain

$$h_2(t) = [T(t) f_0]\, e^{-\int_0^t T(t-\tau)\rho_{g_1}(\tau)\,\mathrm{d}\tau} \tag{8.9}$$

and

$$g_2(t) = T(t) f_0 + \int_0^t T(t-\tau) Q^+(g_1, g_1)(\tau)\,\mathrm{d}\tau. \tag{8.10}$$

Therefore, the condition

$$0 \leqq h_1(t) \leqq h_2(t) \leqq g_2(t) \tag{8.11}$$

is satisfied for all $t \geqq 0$. On the other hand, since $h_1 \equiv 0$ we see from (8.7) and (8.8) that g_2 and g_1 solve the same initial value problem. Therefore

$$g_2(t) = g_1(t) \tag{8.12}$$

for all $t \geqq 0$, for which we can make sense of the gain only equation. We conclude that for our choice of h_1 and g_1, the beginning condition follows (8.11) and (8.12).

Since all the g_n, h_n are bounded by g_1, under the conditions of Theorem 8.1 with $f_0 \in H^{\frac{1}{2}+,\frac{1}{2}+}$ we automatically have

$$\begin{aligned}
&\sup_n \|h_n\|_{L_t^\infty L_{x,v}^2} \leqq \sup_n \|g_n\|_{L_t^\infty L_{x,v}^2} < \infty \\
&\sup_n \left\| Q^+(h_n, h_n)\right\|_{L^1_{t\geqq 0} L^2_{x,v}} \leqq \sup_n \left\| Q^+(g_n, g_n)\right\|_{L^1_{t\geqq 0} L^2_{x,v}} < \infty \\
&\sup_n \left\| Q^-(h_n, h_n)\right\|_{L^1_{t\in[0,T]} L^2_{x,v}} \leqq \sup_n \left\| Q^-(g_n, g_n)\right\|_{L^1_{t\in[0,T]} L^2_{x,v}} < \infty,
\end{aligned}$$

assuming the iteration makes sense. Moreover, since the functions h_n are increasing and the g_n are decreasing, we can define their pointwise limits

$$g = \lim_n g_n \qquad h = \lim_n h_n.$$

Since $0 \leqq h_n \leqq g_n \leqq g_1$, and $Q^\pm(g_1, g_1) \in \left(L^1_{t,x,v}\right)_{\mathrm{loc}}$, an easy application of the dominated convergence theorem shows that

$$Q^\pm(h_n, h_n) \to Q^\pm(h, h) \qquad Q^\pm(g_n, g_n) \to Q^\pm(g, g),$$

in the sense of distributions. Mixed terms such as $Q^-(h_n, g_n)$ are handled similarly. Altogether we conclude that the limits g, h satisfy

$$\begin{aligned}
&\left(\partial_t + v\cdot\nabla_x + \rho_g\right) h = Q^+(h, h) \\
&\left(\partial_t + v\cdot\nabla_x + \rho_h\right) g = Q^+(g, g) \\
&g(0) = h(0) = f_0,
\end{aligned}$$

in the sense of distributions.

We have yet to show that $h = g$ in order to conclude the convergence of the Kaniel–Shinbrot iteration. Let us define

$$w(t, x, v) = g(t, x, v) - h(t, x, v) \geqq 0, \tag{8.13}$$

and note that $w \leqq g_1$. The function w satisfies the following equation in the sense of distributions:

$$\begin{aligned}&(\partial_t + v \cdot \nabla_x)\, w = Q^+ (g, w) + Q^+ (w, h) + \rho_w h - \rho_h w \\ &w(0) = 0.\end{aligned}$$

The goal is to show that $w = 0$ globally in $t \geqq 0$. This follows from Lemma 7.2 as long as we can show

$$\langle v \rangle^{\frac{1}{2}+} Q^+(g_1, g_1) \in L^1_{t\in[0,T]} L^2_{x,v} \tag{8.14}$$

$$\rho_{g_1} \in L^2_{t\in[0,T]} L^\infty_x \bigcap L^2_{t\in[0,T]} L^4_x \tag{8.15}$$

$$\langle v \rangle^{\frac{1}{2}+} g_1 \in L^\infty_{t\in[0,T]} L^2_{x,v} \bigcap L^\infty_{t\in[0,T]} L^4_x L^2_v, \tag{8.16}$$

but these bounds follow from Theorem 8.1 since we assume $f_0 \in H^{\frac{1}{2}+,\frac{1}{2}+}$. We can conclude that the Kaniel–Shinbrot iteration converges to a solution of Boltzmann's equation.

As a final crucial remark, let us note that since $0 \leqq f \leqq g_1$ (by construction), and $f_0 \in H^{\frac{1}{2}+,\frac{1}{2}+}$, by Theorem 8.1 we have the following estimates for the *full Boltzmann equation* with small $L^2_{x,v}$ norm:

$$\begin{aligned}&\langle v \rangle^{\frac{1}{2}+} Q^+(f, f) \in L^1_t L^2_{x,v} \\ &\langle v \rangle^{\frac{1}{2}+} Q^-(f, f) \in L^2_t L^2_{x,v} \\ &\langle v \rangle^{\frac{1}{2}+} f \in L^\infty_t L^2_{x,v} \bigcap L^\infty_t L^4_x L^2_v \\ &\rho_f \in L^2_t L^\infty_x \bigcap L^2_t L^4_x \\ &\| f \|^2_{L^\infty_t L^2_{x,v}} + \left\| Q^+(f, f) \right\|_{L^1_t L^2_{x,v}} \leqq C \, \| f_0 \|^2_{L^2_{x,v}} .\end{aligned}$$

Let us emphasize that we have *not* established that $f \in L^\infty_{t\in[0,T]} H^{\frac{1}{2}+,\frac{1}{2}+}$ so it is not valid to apply Lemma 6.1 directly to the solution f of Boltzmann's equation in order to deduce that $\rho_f \in L^2_{t\in[0,T]} L^\infty_x$. Rather, we are using the fact that $0 \leqq \rho_f \leqq \rho_{g_1}$ combined with the propagation of regularity for the gain-only equation, $g_1 \in L^\infty_t H^{\frac{1}{2}+,\frac{1}{2}+}$, and applying Lemma 6.1 to g_1. Indeed, to obtain the best possible bounds, we are required to convert all regularity information on g_1 into *integrability* information via the Sobolev embedding, at which point it becomes useful information for the solution f of the full Boltzmann equation. This is a strange situation because we are using the regularity condition $f_0 \in H^{\frac{1}{2}+,\frac{1}{2}+}$ to construct global solutions $f(t)$ for which *a priori* the $H^{\frac{1}{2}+,\frac{1}{2}+}$ norm could blow up to $+\infty$ in finite time (we will show later by an independent argument that this blow-up scenario cannot happen).

9. Scattering in $L^2_{x,v}$

The idea is to use the non-negativity of f in a rather strong way. We can write the solution of Boltzmann's equation as follows:

$$\begin{aligned} &T(-t)f(t) + \int_0^t T(-\sigma)Q^-\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma \\ &= f_0 + \int_0^t T(-\sigma)Q^+\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma \end{aligned} \tag{9.1}$$

Everything on either side is non-negative (we are assuming $t \geqq 0$), so we can write

$$\int_0^t T(-\sigma)Q^-\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma \leqq f_0 + \int_0^t T(-\sigma)Q^+\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma, \tag{9.2}$$

which implies

$$\int_0^t T(-\sigma)Q^-\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma \leqq f_0 + \int_0^\infty T(-\sigma)Q^+\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma. \tag{9.3}$$

Then, by monotone convergence in t, for almost every (x, v) we have

$$\int_0^\infty T(-\sigma)Q^-\left(f(\sigma), f(\sigma)\right) d\sigma \leqq f_0 + \int_0^\infty T(-\sigma)Q^+\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma. \tag{9.4}$$

Taking the $L^2_{x,v}$ norm of both sides and applying Minkowski on the *right hand side* only, and using the fact that $T(t)$ preserves $L^2_{x,v}$, we obtain

$$\left\| T(-t)Q^-\left(f(t), f(t)\right)\right\|_{L^2_{x,v}L^1_{t\geqq 0}} \leqq \|f_0\|_{L^2_{x,v}} + \left\| Q^+\left(f(t), f(t)\right)\right\|_{L^1_{t\geqq 0}L^2_{x,v}} \tag{9.5}$$

We have

$$Q^+\left(f(t), f(t)\right) \in L^1_{t\geqq 0}L^2_{x,v}, \tag{9.6}$$

because (9.6) holds for the solution of the gain-only Boltzmann equation (with small data $f_0 \in L^2_{x,v}$), and the solution of the full Boltzmann equation is bounded above by the solution of the gain-only Boltzmann equation as a result of the Kaniel–Shinbrot construction.

We can combine (9.5) and (9.6) to conclude

$$T(-t)Q^\pm\left(f(t), f(t)\right) \in L^2_{x,v}L^1_{t\geqq 0}, \tag{9.7}$$

and this implies that the limit in norm

$$\lim_{t\to+\infty} \int_0^t T(-\sigma)Q\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma \tag{9.8}$$

exists in $L^2_{x,v}$, by the dominated convergence theorem. Indeed, the $L^2_{x,v}$ remainder is bounded by

$$\int \mathrm{d}x\, \mathrm{d}v \int_t^\infty \mathrm{d}\sigma \int_t^\infty \mathrm{d}\sigma' \left\{T(-\sigma)\left|Q\left(f(\sigma), f(\sigma)\right)\right|\right\}\left\{T(-\sigma')\left|Q\left(f(\sigma'), f(\sigma')\right)\right|\right\}, \tag{9.9}$$

and this clearly tends to zero as $t \to +\infty$.

As a result of the convergence argument detailed above, if we define

$$f_{+\infty} = f_0 + \lim_{t \to +\infty} \int_0^t T(-\sigma) Q\left(f(\sigma), f(\sigma)\right) \mathrm{d}\sigma, \tag{9.10}$$

then it follows that $f_{+\infty} \in L^2_{x,v}$ and

$$\lim_{t \to +\infty} \| T(-t) f(t) - f_{+\infty} \|_{L^2_{x,v}} = 0. \tag{9.11}$$

The same argument implies the following slightly more general result (which does not require uniqueness, nor that f_0 necessarily have small $L^2_{x,v}$ norm):

Theorem 9.1. *Suppose $f \in \bigcap_{T>0} L^\infty_{t \in [0,T]} L^2_{x,v}$ is a* non-negative *mild solution of the full Boltzmann equation,*

$$\left(\partial_t + v \cdot \nabla_x\right) f = Q^+\left(f, f\right) - f \rho_f, \tag{9.12}$$

such that, along the solution $f(t)$, the gain operator *Q^+ satisfies*

$$T(-t) Q^+\left(f(t), f(t)\right) \in L^2_{x,v} L^1_{t \geqq 0}. \tag{9.13}$$

Then $f(t)$ scatters in $L^2_{x,v}$ as $t \to +\infty$; that is, there exists a function $f_{+\infty} \in L^2_{x,v}$ such that the limit

$$\lim_{t \to +\infty} \| f(t) - T(t) f_{+\infty} \|_{L^2_{x,v}} = 0 \tag{9.14}$$

holds.

Remark 9.1. The gain-only Boltzmann equation scatters in $H^{\frac{1}{2}+,\frac{1}{2}+}$, assuming only that $f_0 \in H^{\frac{1}{2}+,\frac{1}{2}+} \cap B^{L^2}_\eta$; of course, this implies that solutions of the gain-only equation remain uniformly bounded in $H^{\frac{1}{2}+,\frac{1}{2}+}$ as $t \to +\infty$. However, we do not know whether the full Boltzmann equation scatters in $H^{\frac{1}{2}+,\frac{1}{2}+}$; indeed, whereas we show in Section 10 that the solution of the full Boltzmann equation propagates $H^{\frac{1}{2}+,\frac{1}{2}+}$ for small $L^2_{x,v}$ solutions, we do not even know whether the $H^{\frac{1}{2}+,\frac{1}{2}+}$ norm (for the full Boltzmann equation) remains *bounded* in time as $t \to +\infty$.

Remark 9.2. Due to the lack of $L^1_t L^2_{x,v}$ *bilinear* spacetime estimates for $Q^-(f, f)$, we cannot use Theorem 3.3 (or its proof) to describe qualitatively the correspondence between f_0 and $f_{+\infty}$ for the full Boltzmann equation (though Theorem 3.3 clearly applies to the *gain-only* equation).

10. Propagation of Regularity for the Full Equation

Recall that some extra regularity for the *gain-only* equation was required to produce enough *integrability* to close the Kaniel–Shinbrot iteration and prove uniqueness. However, so far we have said nothing about the regularity of the full Boltzmann equation. The point of this section is to prove that, for all the regularity which we required to construct a solution, such regularity is indeed propagated by the solution itself.

Remark 10.1. It is important to observe that it is *not necessary* to propagate regularity for the full Boltzmann equation in order to close the Kaniel–Shinbrot iteration. Thus, the regularity for the full equation is propagated *a posteriori*.

10.1. Loss Operator Bounds

Recall the loss operator

$$Q^{-}(f,g)=f\rho_g. \tag{10.1}$$

Lemma 10.1. *Let $\alpha\in\left(\frac{1}{2},1\right]$. For any two measurable and locally integrable functions $f_0(x,v)$, $g_0(x,v)$ such that $\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} f_0$, $\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} g_0\in L^2_{x,v}$, the function $Q^{-}(T(t)f_0,T(t)g_0)$ is in $L^2_{t,x,v}\left(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2\right)$ and the following estimate holds:*

$$\begin{aligned}&\left\|\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} Q^{-}(T(t)f_0,T(t)g_0)\right\|_{L^2_{t,x,v}}\\ &\leqq C\left\|\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} f_0\right\|_{L^2_{x,v}}\left\|\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} g_0\right\|_{L^2_{x,v}}.\end{aligned} \tag{10.2}$$

Proof. We will assume $\alpha\in\left(\frac{1}{2},1\right)$; the case $\alpha=1$ follows in a similar manner by using the Leibniz differentiation rule (note that $H^1_x=L^2_x\cap\dot{H}^1_x$, and that $|\nabla_x|$ can be replaced by ∇_x in defining the $\dot{H}^1_x$ semi-norm).

We begin with the L^2_x estimate. We have

$$\begin{aligned}\left\|\langle v\rangle^{\alpha} Q^{-}(T(t)f_0,T(t)g_0)\right\|_{L^2_{t,x,v}}&=\left\|\langle v\rangle^{\alpha}\{T(t)f_0\}\rho_{T(t)g_0}\right\|_{L^2_{t,x,v}}\\ &\leqq\left\|\langle v\rangle^{\alpha} T(t)f_0\right\|_{L^\infty_t L^2_{x,v}}\left\|\rho_{T(t)g_0}\right\|_{L^2_t L^\infty_x}\\ &\leqq\left\|\langle v\rangle^{\alpha} f_0\right\|_{L^2_{x,v}}\left\|\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} g_0\right\|_{L^2_{x,v}}\\ &\leqq\left\|\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} f_0\right\|_{L^2_{x,v}}\left\|\langle v\rangle^{\alpha}\langle\nabla_x\rangle^{\alpha} g_0\right\|_{L^2_{x,v}},\end{aligned}$$

where we have used that $\alpha>\frac{1}{2}$ in order to apply Lemma 6.1.

Let us now turn to the $\dot{H}^{\alpha}_x$ estimate; by Theorem B.1 (due to Kenig-Ponce-Vega [17]) we have

$$\left\|\langle v\rangle^{\alpha}|\nabla_x|^{\alpha} Q^{-}(T(t)f_0,T(t)g_0)\right\|_{L^2_x}=\left\||\nabla_x|^{\alpha}\left(\left\{\langle v\rangle^{\alpha} T(t)f_0\right\}\rho_{T(t)g_0}\right)\right\|_{L^2_x}$$

$$\leqq\left\|\left(\langle v\rangle^{\alpha} T(t)f_0\right)|\nabla_x|^{\alpha}\rho_{T(t)g_0}\right\|_{L^2_x}+$$

$$+ C \left\| \rho_{T(t)g_0} \right\|_{L^\infty_x} \left\| \langle v \rangle^\alpha |\nabla_x|^\alpha T(t) f_0 \right\|_{L^2_x} .$$

Now we take the $L^2_{t,v}$ norm of both sides, and then apply Hölder's inequality and Lemma 6.1 (which is justified because $\alpha > \frac{1}{2}$) to get

$$\begin{aligned}
&\left\| \langle v \rangle^\alpha |\nabla_x|^\alpha Q^- \left(T(t) f_0, T(t) g_0\right) \right\|_{L^2_{t,x,v}} \\
&\leqq \left\| \left(\langle v \rangle^\alpha T(t) f_0 \right) |\nabla_x|^\alpha \rho_{T(t)g_0} \right\|_{L^2_{t,x,v}} \\
&\quad + C \left\| \left\| \rho_{T(t)g_0} \right\|_{L^\infty_x} \left\| \langle v \rangle^\alpha |\nabla_x|^\alpha T(t) f_0 \right\|_{L^2_x} \right\|_{L^2_{t,v}} \\
&\leqq \left\| \langle v \rangle^\alpha T(t) f_0 \right\|_{L^\infty_t L^4_x L^2_v} \left\| |\nabla_x|^\alpha \rho_{T(t)g_0} \right\|_{L^2_t L^4_x} \\
&\quad + C \left\| \rho_{T(t)g_0} \right\|_{L^2_t L^\infty_x} \left\| \langle v \rangle^\alpha |\nabla_x|^\alpha T(t) f_0 \right\|_{L^\infty_t L^2_{x,v}} \\
&\leqq \left\| \langle v \rangle^\alpha T(t) f_0 \right\|_{L^\infty_t L^4_x L^2_v} \left\| \langle v \rangle^\alpha |\nabla_x|^\alpha g_0 \right\|_{L^2_{x,v}} \\
&\quad + C \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha g_0 \right\|_{L^2_{x,v}} \left\| \langle v \rangle^\alpha |\nabla_x|^\alpha T(t) f_0 \right\|_{L^\infty_t L^2_{x,v}} \\
&\leqq \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha T(t) f_0 \right\|_{L^\infty_t L^2_{x,v}} \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha g_0 \right\|_{L^2_{x,v}} \\
&\quad + C \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha g_0 \right\|_{L^2_{x,v}} \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha T(t) f_0 \right\|_{L^\infty_t L^2_{x,v}} .
\end{aligned}$$

Note that $|\nabla_x|$ commutes with $\rho_{(\cdot)}$, and we have used the Sobolev embedding $H^{\frac{1}{2}}_x \left(\mathbb{R}^2\right) \subset L^4_x \left(\mathbb{R}^2\right)$ in the last step. We finally use the fact that $T(t)$ preserves $H^{\alpha,\beta}$ to obtain

$$\begin{aligned}
&\left\| \langle v \rangle^\alpha |\nabla_x|^\alpha Q^- \left(T(t) f_0, T(t) g_0\right) \right\|_{L^2_{t,x,v}} \\
&\leqq C \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha f_0 \right\|_{L^2_{x,v}} \left\| \langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha g_0 \right\|_{L^2_{x,v}} .
\end{aligned}$$

Combining the L^2_x and $\dot{H}^\alpha_x$ estimates allows us to conclude. □

The next lemma is a refinement of Lemma 10.1 which only places a spatial gradient on one argument at a time.

Lemma 10.2. *Let $\alpha \in \left(\frac{1}{2}, 1\right]$, and let $I \subseteq \mathbb{R}$ be an open interval (either bounded or unbounded). Let $f(t,x,v) : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ be a measurable and locally integrable function such that*

$$\langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha f \in L^\infty \left(I, L^2_{x,v}\right) \tag{10.3}$$

and

$$\langle v \rangle^\alpha \langle \nabla_x \rangle^\alpha \left(\partial_t + v \cdot \nabla_x\right) f \in L^1 \left(I, L^2_{x,v}\right). \tag{10.4}$$

Then the following estimate holds:

$$\begin{aligned}
&\left\| \langle v \rangle^\alpha |\nabla_x|^\alpha Q^-(f,f) \right\|_{L^2(I, L^2_{x,v})} \\
&\leqq C \times \left\{ \left\| \rho_f \right\|_{L^2(I, L^\infty_x)} \left\| \langle v \rangle^\alpha |\nabla_x|^\alpha f \right\|_{L^\infty(I, L^2_{x,v})} \right.
\end{aligned}$$

$$+ \left\| \langle v \rangle^{\alpha} f \right\|_{L^{\infty}(I, L^4_x L^2_v)} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^{\infty}(I, L^2_{x,v})}$$
$$+ \left\| \langle v \rangle^{\alpha} f \right\|_{L^{\infty}(I, L^4_x L^2_v)} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} (\partial_t + v \cdot \nabla_x) f \right\|_{L^1(I, L^2_{x,v})} \Big\} . \quad (10.5)$$

The constant C does not depend on the interval I, but it may depend on α.

Proof. As in the proof of Lemma 10.1, we will assume $\alpha \in \left(\frac{1}{2}, 1\right)$. The case $\alpha = 1$ may be checked directly in a similar fashion.

We begin by applying Theorem B.1, which is due to Kenig-Ponce-Vega [17]:

$$\left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} Q^{-}(f, f) \right\|_{L^2_x} = \left\| |\nabla_x|^{\alpha} \left(\langle v \rangle^{\alpha} f \rho_f \right) \right\|_{L^2_x}$$
$$\leqq \left\| \langle v \rangle^{\alpha} f |\nabla_x|^{\alpha} \rho_f \right\|_{L^2_x} + C \left\| \rho_f \right\|_{L^{\infty}_x} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^2_x} .$$

We take the $L^2_{t\in I} L^2_v$ norm of both sides, followed by Hölder's inequality, to get

$$\left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} Q^{-}(f, f) \right\|_{L^2_{t\in I} L^2_{x,v}}$$
$$\leqq \left\| \langle v \rangle^{\alpha} f |\nabla_x|^{\alpha} \rho_f \right\|_{L^2_{t\in I} L^2_{x,v}} + C \left\| \left\| \rho_f \right\|_{L^{\infty}_x} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^2_x} \right\|_{L^2_{t\in I} L^2_v}$$
$$= C \left\| \left\| \rho_f \right\|_{L^{\infty}_x} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^2_x} \right\|_{L^2_{t\in I} L^2_v} + \left\| \langle v \rangle^{\alpha} f |\nabla_x|^{\alpha} \rho_f \right\|_{L^2_{t\in I} L^2_{x,v}} .$$

Finally we apply Hölder's inequality, followed by Lemma 6.1 since $\alpha > \frac{1}{2}$; we are using the fact that $|\nabla_x|^{\alpha}$ commutes with $\rho_{(\cdot)}$. This yields

$$\left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} Q^{-}(f, f) \right\|_{L^2_{t\in I} L^2_{x,v}}$$
$$\leqq C \left\| \rho_f \right\|_{L^2_{t\in I} L^{\infty}_x} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^{\infty}_{t\in I} L^2_{x,v}}$$
$$+ \left\| \langle v \rangle^{\alpha} f \right\|_{L^{\infty}_{t\in I} L^4_x L^2_v} \left\| |\nabla_x|^{\alpha} \rho_f \right\|_{L^2_{t\in I} L^4_x}$$
$$\leqq C \left\| \rho_f \right\|_{L^2_{t\in I} L^{\infty}_x} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^{\infty}_{t\in I} L^2_{x,v}}$$
$$+ C \left\| \langle v \rangle^{\alpha} f \right\|_{L^{\infty}_{t\in I} L^4_x L^2_v} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} f \right\|_{L^{\infty}_{t\in I} L^2_{x,v}}$$
$$+ C \left\| \langle v \rangle^{\alpha} f \right\|_{L^{\infty}_{t\in I} L^4_x L^2_v} \left\| \langle v \rangle^{\alpha} |\nabla_x|^{\alpha} (\partial_t + v \cdot \nabla_x) f \right\|_{L^1_{t\in I} L^2_{x,v}} ,$$

hence the conclusion. □

10.2. Gain Operator Bounds

The proof of Lemma 10.2, which allows us to apply spatial gradients to one entry at a time in $Q^{-}(f, f)$, does not work for the gain operator $Q^{+}(f, f)$ in our formulation. The difficulty is that we do not have an exact commutation rule for $|\nabla_x|^{\alpha}$ and $Q^{+}(f, f)$, and the multilinear Riesz–Thorin theorem does not apply.

Nevertheless, it *is* possible to recover a useful inequality in "Peter-Paul" form (before optimizing) which estimates fractional spatial derivatives of the gain operator, which will be essential for the global propagation of regularity to be proven in Section 10.4. The strategy is to apply the multilinear Riesz–Thorin theorem to well-chosen *inhomogeneous* norms with a suitable ε-dependent weight; then, we divide out powers of ε from both sides, and optimize over ε. In this way, we are able to avoid any problem-specific commutator estimates, which would not be in keeping with the spirit of our approach.

Lemma 10.3. *Let $\alpha \in [0,1]$, and let $I \subseteq \mathbb{R}$ be an open interval (either bounded or unbounded). Let $f(t,x,v) : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ be a measurable and locally integrable function such that*

$$\langle v\rangle^\alpha \langle \nabla_x\rangle^\alpha f \in L^\infty \left(I, L^2_{x,v}\right) \tag{10.6}$$

and

$$\langle v\rangle^\alpha \langle \nabla_x\rangle^\alpha (\partial_t + v\cdot\nabla_x) f \in L^1 \left(I, L^2_{x,v}.\right) \tag{10.7}$$

Then, for any $q \in (0,\infty)$, the following estimate holds:

$$\begin{aligned}
&\left\| \langle qv\rangle^\alpha |\nabla_x|^\alpha Q^+(f,f)\right\|_{L^1(I,L^2_{x,v})} \\
&\leqq C \left\|\langle qv\rangle^\alpha f\right\|_{L^\infty(I,L^2_{x,v})} \left\|\langle qv\rangle^\alpha |\nabla_x|^\alpha f\right\|_{L^\infty(I,L^2_{x,v})} \\
&\quad + C \left\|\langle qv\rangle^\alpha (\partial_t + v\cdot\nabla_x) f\right\|_{L^1(I,L^2_{x,v})} \left\|\langle qv\rangle^\alpha |\nabla_x|^\alpha f\right\|_{L^\infty(I,L^2_{x,v})} \\
&\quad + C \left\|\langle qv\rangle^\alpha f\right\|_{L^\infty(I,L^2_{x,v})} \left\|\langle qv\rangle^\alpha |\nabla_x|^\alpha (\partial_t + v\cdot\nabla_x) f\right\|_{L^1(I,L^2_{x,v})} \\
&\quad + C \left\|\langle qv\rangle^\alpha (\partial_t + v\cdot\nabla_x) f\right\|_{L^1(I,L^2_{x,v})} \left\|\langle qv\rangle^\alpha |\nabla_x|^\alpha (\partial_t + v\cdot\nabla_x) f\right\|_{L^1(I,L^2_{x,v})}.
\end{aligned} \tag{10.8}$$

The constant C is independent of I, q, α.

Proof. Adapting the proof of Proposition 6.4 as necessary, by using the multilinear Riesz–Thorin theorem we are able to show that, for any $f_0, g_0 \in H^{\alpha,\alpha}$, $\alpha \in [0,1]$, and $q, \varepsilon \in (0,\infty)$,

$$\begin{aligned}
&\left\|\langle qv\rangle^\alpha \langle \varepsilon\nabla_x\rangle^\alpha Q^+\left(T(t)f_0, T(t)g_0\right)\right\|_{L^1_t L^2_{x,v}} \\
&\leqq C \left\|\langle qv\rangle^\alpha \langle \varepsilon\nabla_x\rangle^\alpha f_0\right\|_{L^2_{x,v}} \left\|\langle qv\rangle^\alpha \langle \varepsilon\nabla_x\rangle^\alpha g_0\right\|_{L^2_{x,v}},
\end{aligned} \tag{10.9}$$

where the constant C does not depend on α, q, ε. It suffices to check the endpoints $\alpha = 0$ and $\alpha = 1$, viewing $\varepsilon, q \in (0,\infty)$ as arbitrary constants.

Having verified (10.9), let f, g be time-dependent functions as in the statement of the lemma. Combining (10.9) and Lemma 3.1, and using the fact that $T(t)$ is an isometry on $L^2_{x,v}$ for each $t \in \mathbb{R}$, we deduce the following estimate, up to increasing the constant by an absolute factor:

$$\begin{aligned}
&\left\|\langle qv\rangle^\alpha \langle \varepsilon\nabla_x\rangle^\alpha Q^+(f,g)\right\|_{L^1(I,L^2_{x,v})} \\
&\leqq C \prod_{h\in\{f,\,g\}} \Bigg(\left\|\langle qv\rangle^\alpha \langle \varepsilon\nabla_x\rangle^\alpha h\right\|_{L^\infty(I,L^2_{x,v})}
\end{aligned}$$

$$+ \left\| \langle qv \rangle^\alpha \langle \varepsilon \nabla_x \rangle^\alpha (\partial_t + v \cdot \nabla_x) h \right\|_{L^1(I, L^2_{x,v})} \Bigg). \tag{10.10}$$

Now may we specialize to the case $g = f$:

$$\begin{aligned} &\left\| \langle qv \rangle^\alpha \langle \varepsilon \nabla_x \rangle^\alpha Q^+(f, f) \right\|_{L^1(I, L^2_{x,v})} \\ &\leqq C \Bigg(\left\| \langle qv \rangle^\alpha \langle \varepsilon \nabla_x \rangle^\alpha f \right\|^2_{L^\infty(I, L^2_{x,v})} \\ &\qquad + \left\| \langle qv \rangle^\alpha \langle \varepsilon \nabla_x \rangle^\alpha (\partial_t + v \cdot \nabla_x) f \right\|^2_{L^1(I, L^2_{x,v})} \Bigg). \end{aligned} \tag{10.11}$$

At this point we need to estimate $\varepsilon^\alpha |\nabla_x|^\alpha \lesssim \langle \varepsilon \nabla_x \rangle^\alpha$ on the *left*, and $\langle \varepsilon \nabla_x \rangle^\alpha \lesssim 1 + \varepsilon^\alpha |\nabla_x|^\alpha$ on the *right* (and note the squares!), and finally, divide throughout by ε^α. Hence we obtain the following "Peter-Paul" inequality:

$$\begin{aligned} &\left\| \langle qv \rangle^\alpha |\nabla_x|^\alpha Q^+(f, f) \right\|_{L^1(I, L^2_{x,v})} \\ &\leqq \frac{C}{\varepsilon^\alpha} \Bigg(\left\| \langle qv \rangle^\alpha f \right\|^2_{L^\infty(I, L^2_{x,v})} + \left\| \langle qv \rangle^\alpha (\partial_t + v \cdot \nabla_x) f \right\|^2_{L^1(I, L^2_{x,v})} \Bigg) \\ &\quad + C\varepsilon^\alpha \Bigg(\left\| \langle qv \rangle^\alpha |\nabla_x|^\alpha f \right\|^2_{L^\infty(I, L^2_{x,v})} + \left\| \langle qv \rangle^\alpha |\nabla_x|^\alpha (\partial_t + v \cdot \nabla_x) f \right\|^2_{L^1(I, L^2_{x,v})} \Bigg). \end{aligned} \tag{10.12}$$

The conclusion then follows by optimal choice of ε and trivial manipulations. □

10.3. Local Propagation of Regularity

The idea for proving local propagation of regularity is to construct a local solution in the more regular space $H^{\alpha,\alpha}$ with $\alpha > \frac{1}{2}$, and then appeal to uniqueness via Theorem 7.1 to conclude that the $H^{\alpha,\alpha}$ solution coincides with the small $L^2_{x,v}$ solution obtained from Kaniel–Shinbrot. The various estimates required to apply Theorem 7.1 to $H^{\alpha,\alpha}$ solutions follow immediately from the local well-posedness theory in $H^{\alpha,\alpha}$ for $\alpha > \frac{1}{2}$, combined with the Sobolev embedding theorem and Lemma 6.1.[3] The local theory presented here relies on the $L^2_{x,v}$ norm remaining small, which is parallel to the assumption for the uniqueness theorem, Theorem 7.1; however, the $H^{\alpha,\alpha}$ norm may be very large and the local theory will still be valid. The time of existence for local solutions given $f_0 \in H^{\alpha,\alpha} \cap B^{L^2}_\eta$ is determined solely by the magnitude of the $H^{\alpha,\alpha}$ norm. A separate argument (discussed in Section 10.4) is required to obtain the propagation of regularity on arbitrarily large time intervals.

Recall the $H^{\alpha,\alpha}$ norm with ε dependence

$$\|f\|_{H^{\alpha,\alpha}_\varepsilon} = \left\| \langle \varepsilon v \rangle^\alpha \langle \varepsilon \nabla_x \rangle^\alpha f \right\|_{L^2_{x,v}}. \tag{10.13}$$

[3] Interestingly, it was the local $H^{\alpha,\alpha}$ theory with $\alpha > \frac{1}{2}$ which served as the inspiration for Theorem 7.1 in the first place (and, by extension, the proof of convergence of the Kaniel–Shinbrot scheme).

We know that the gain term Q^+ obeys, by Proposition 6.4, the following estimate:

$$\left\| Q^+ \left(T(t) f_0, T(t) g_0\right) \right\|_{L^1_t H^{\alpha,\alpha}_\varepsilon} \leqq C \left\| f_0 \right\|_{H^{\alpha,\alpha}_\varepsilon} \left\| g_0 \right\|_{H^{\alpha,\alpha}_\varepsilon}, \tag{10.14}$$

and the constant does not depend on $\alpha, \varepsilon \in (0, 1]$. With respect to the loss term, we cannot expect bounds independent of ε, but we can use Lemma 10.1 to prove the following:

$$\left\| Q^- \left(T(t) f_0, T(t) g_0\right) \right\|_{L^2_t H^{\alpha,\alpha}_\varepsilon} \leqq C_\varepsilon \left\| f_0 \right\|_{H^{\alpha,\alpha}_\varepsilon} \left\| g_0 \right\|_{H^{\alpha,\alpha}_\varepsilon}. \tag{10.15}$$

Hence, by Hölder's inequality,

$$\left\| Q^- \left(T(t) f_0, T(t) g_0\right) \right\|_{L^1_{t\in[0,T]} H^{\alpha,\alpha}_\varepsilon} \leqq C_\varepsilon T^{\frac{1}{2}} \left\| f_0 \right\|_{H^{\alpha,\alpha}_\varepsilon} \left\| g_0 \right\|_{H^{\alpha,\alpha}_\varepsilon}. \tag{10.16}$$

Note that the size of the constant C_ε in (10.16) is irrelevant for our analysis, but it can be estimated as $C_\varepsilon \lesssim \varepsilon^{-4\alpha}$ when $\varepsilon \to 0^+$. The point is that the large factor of C_ε can always be balanced in (10.16) by letting T be small. Since the parameter ε reflects (in this instance) the size of the $H^{\alpha,\alpha}$ norm at a given time t_0, we can apply Theorem 3.2 using (10.14) and (10.16) to deduce local well-posedness for the full Boltzmann equation in $H^{\alpha,\alpha} \cap B_\eta^{L^2_{x,v}}$ (for some constant $\eta > 0$), with existence time depending only on the $H^{\alpha,\alpha}$ norm.

Remark 10.2. We can say nothing for f_0 outside the η-ball of L^2 by the above logic, due to the fact that the constant C in (10.14) remains fixed regardless of any localization in time.

As a result of the preceding discussion, we may conclude as follows:

Theorem 10.4. *There exists a number $\eta > 0$ such that all of the following is true: Let $\alpha \in \left(\frac{1}{2}, 1\right]$ and $f_0 \in H^{\alpha,\alpha}$, and, further, suppose that*

$$\| f_0(x, v) \|_{L^2_{x,v}} < \eta. \tag{10.17}$$

Then there exists a time $T > 0$ such that, for $t \in [0, T]$, the full Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f = Q^+(f, f) - f \rho_f \tag{10.18}$$

has a unique mild solution $f(t)$ such that $f \in L^\infty_{t\in[0,T]} H^{\alpha,\alpha}$, $Q^\pm(f, f) \in L^1_{t\in[0,T]} H^{\alpha,\alpha}$ and $f(0) = f_0$ all hold. The solution is continuous, in the sense that $f \in C\left([0, T], H^{\alpha,\alpha}\right)$. Additionally, the time T may be chosen to depend only on the $H^{\alpha,\alpha}$ norm of f_0; that is, the lower bound

$$T \geqq T_0 \left(\| f_0 \|_{H^{\alpha,\alpha}}\right) > 0 \qquad \textit{assuming} \qquad \| f_0(x, v) \|_{L^2_{x,v}} < \eta$$

may be assumed.

10.4. Global Propagation of Regularity

The key observation to round out our discussion of regularity is that we do not have to propagate the *entire* $H^{\alpha,\alpha}$ norm, because part of it is given to us *for free* by the Kaniel–Shinbrot iteration. Indeed we already know that $\langle v\rangle^\alpha f \in L^\infty_{t\geqq 0} L^2_{x,v}$, and similarly $\langle v\rangle^\alpha Q^+(f,f) \in L^1_{t\geqq 0} L^2_{x,v}$ and $\langle v\rangle^\alpha Q^-(f,f) \in L^2_{t\geqq 0} L^2_{x,v}$. (See Theorem 8.1 and Section 8.2.) Hence, we have only to show that

$$\forall T \in (0,\infty), \qquad \left\| \langle v\rangle^\alpha |\nabla_x|^\alpha f \right\|_{L^\infty_{t\in[0,T]} L^2_{x,v}} < \infty \tag{10.19}$$

and

$$\forall T \in (0,\infty), \qquad \left\| \langle v\rangle^\alpha |\nabla_x|^\alpha Q^\pm(f,f) \right\|_{L^1_{t\in[0,T]} L^2_{x,v}} < \infty. \tag{10.20}$$

Note that Theorem 7.1, combined with Sobolev embedding, implies that the local $H^{\alpha,\alpha}$ solution from Theorem 10.4 coincides with the solution obtained via Kaniel–Shinbrot. (This is due to the fact that Theorem 7.1 refers only to *integrability* properties, not *regularity* properties, in the (x,v) domain.) Therefore, we can assume that the $H^{\alpha,\alpha}$ norms are finite on small time intervals. We can then use continuity arguments, combined with Lemma 10.2 and Lemma 10.3, to extend the $H^{\alpha,\alpha}$ time up to a *larger* small time interval which now only depends on controlled quantities which do not contain $|\nabla_x|^\alpha$. Finally, a standard iteration in time provides the desired result.

Theorem 10.5. *There exists an absolute constant $\eta > 0$ such that the following is true:*

Let $T \in (0,\infty)$ and $\alpha \in \left(\frac{1}{2}, 1\right]$, and suppose $f(t) \in C\left([0,T], L^2_{x,v}\right)$ is a mild solution of the full Boltzmann equation satisfying all of the following estimates:

$$\|f\|_{L^\infty_{t\in[0,T]} L^2_{x,v}} + \left\|Q^+(f,f)\right\|_{L^1_{t\in[0,T]} L^2_{x,v}} < \eta \tag{10.21}$$

$$\langle v\rangle^\alpha Q^+(f,f) \in L^1_{t\in[0,T]} L^2_{x,v} \tag{10.22}$$

$$\rho_f \in L^2_{t\in[0,T]} L^\infty_x \cap L^2_{t\in[0,T]} L^4_x \tag{10.23}$$

$$\langle v\rangle^\alpha f \in L^\infty_{t\in[0,T]} L^2_{x,v} \cap L^\infty_{t\in[0,T]} L^4_x L^2_v \tag{10.24}$$

and $f(0) = f_0$. If in addition $f_0 \in H^{\alpha,\alpha}$, then $f \in L^\infty_{t\in[0,T]} H^{\alpha,\alpha}$ and $Q^\pm(f,f) \in L^1_{t\in[0,T]} H^{\alpha,\alpha}$.

Remark 10.3. We emphasize the ordering of quantifiers: A single $\eta > 0$ works simultaneously for all $T > 0$. Also, the supplied estimates automatically imply $\langle v\rangle^\alpha Q^-(f,f) \in L^2_{t\in[0,T]} L^2_{x,v}$, by Hölder's inequality.

Proof. In view of Theorem 10.4, Theorem 7.1, and the Sobolev embedding theorem, we only need to formally estimate $\langle v\rangle^\alpha |\nabla_x|^\alpha f \in L^\infty_{t\in[0,T]} L^2_{x,v}$ and $\langle v\rangle^\alpha |\nabla_x|^\alpha Q^\pm(f,f) \in L^2_{x,v}$. Additionally, due to Lemma 10.1, Proposition 6.4, Lemma 3.1, and Duhamel's formula with $f_0 \in H^{\alpha,\alpha}$, it will be enough to show that

$$\langle v\rangle^\alpha |\nabla_x|^\alpha (\partial_t + v\cdot\nabla_x) f \in L^1_{t\in[0,T]} L^2_{x,v}. \tag{10.25}$$

Suppose that $0 \leqq t_0 < T$ and $0 < s \leqq T - t_0$, and let $e_{t_0}(s)$ denote the quantity

$$e_{t_0}(s) = \left\| \langle qv \rangle^{\alpha} |\nabla_x|^{\alpha} (\partial_t + v \cdot \nabla_x) f \right\|_{L^1_{t \in [t_0, t_0+s]} L^2_{x,v}} \tag{10.26}$$

whenever it is well-defined, or $+\infty$ otherwise. Note that $e_0(s) < +\infty$ for some $s > 0$ by Theorem 10.4 and Theorem 7.1. Additionally, if $e_{t_0}(s) < +\infty$, then $\lim_{s \to 0^+} e_{t_0}(s) = 0$ by the dominated convergence theorem. We want to show that $e_0(T) < +\infty$.

We define, for convenience,

$$M = \left\| \langle v \rangle^{\alpha} f \right\|_{L^\infty_{t \in [0,T]} L^2_{x,v}} + \left\| \langle v \rangle^{\alpha} f \right\|_{L^\infty_{t \in [0,T]} L^4_x L^2_v} + \left\| \rho_f \right\|_{L^2_{t \in [0,T]} L^\infty_x} + \left\| \rho_f \right\|_{L^2_{t \in [0,T]} L^4_x}. \tag{10.27}$$

Pick a number $q \in (0, 1)$ such that

$$\left\| \langle qv \rangle^{\alpha} f \right\|_{L^\infty_{t \in [0,T]} L^2_{x,v}} + \left\| \langle qv \rangle^{\alpha} Q^+ (f, f) \right\|_{L^1_{t \in [0,T]} L^2_{x,v}} < \eta, \tag{10.28}$$

where η is as in the statement of the theorem (the size of η may be determined by tracking constants through the proof).

Suppose t_0, s are such that $e_0(s + t_0) = e_0(t_0) + e_{t_0}(s) < +\infty$ (here the allowable values of t_0, s are determined by the solution f itself, not necessarily by the statement of Theorem 10.4). Since f solves Boltzmann's equation, we clearly have

$$\begin{aligned} &e_{t_0}(s) \\ &\leqq \left\| \langle qv \rangle^{\alpha} |\nabla_x|^{\alpha} Q^+(f, f) \right\|_{L^1_{t \in [t_0, t_0+s]} L^2_{x,v}} + \left\| \langle qv \rangle^{\alpha} |\nabla_x|^{\alpha} Q^-(f, f) \right\|_{L^1_{t \in [t_0, t_0+s]} L^2_{x,v}}. \end{aligned} \tag{10.29}$$

We have, as an immediate consequence of Lemma 10.2, the estimate

$$\begin{aligned} &\left\| \langle qv \rangle^{\alpha} |\nabla_x|^{\alpha} Q^-(f, f) \right\|_{L^1_{t \in [t_0, t_0+s]} L^2_{x,v}} \\ &\leqq C M s^{\frac{1}{2}} \left(\| f_0 \|_{H^{\alpha,\alpha}} + \frac{1}{q^{\alpha}} e_0(t_0) + \frac{1}{q^{\alpha}} e_{t_0}(s) \right). \end{aligned} \tag{10.30}$$

Note that s can be chosen, depending only on the parameters M, q fixed as above, to make the prefactor on $e_{t_0}(s)$ as small as we like.

By Lemma 10.3, we have

$$\begin{aligned} &\left\| \langle qv \rangle^{\alpha} |\nabla_x|^{\alpha} Q^+(f, f) \right\|_{L^1_{t \in [t_0, t_0+s]} L^2_{x,v}} \\ &\leqq \text{const.} \times \left(\eta + M s^{\frac{1}{2}} \right) \left(\| f_0 \|_{H^{\alpha,\alpha}} + e_0(t_0) + e_{t_0}(s) \right). \end{aligned} \tag{10.31}$$

Combining estimates (and picking η small enough once and for all), we find that there exists a number $\tilde{s} > 0$, depending on the solution f only through M, q, with the following property: if $e_0(t_0) < \infty$, then $e_{t_0}(s) < \infty$. This is sufficient to conclude the theorem. □

11. The Local Well-Posedness Theorem

In view of the Kaniel–Shinbrot iteration, in order to prove Theorem 1.2 it suffices to prove a suitable local well-posedness theorem for the *gain-only* Boltzmann equation. This theorem will require $\alpha = \frac{1}{2}+$ regularity on f_0 but the time of existence will depend only on the $H^{s,s}$ norm for given $s \in (0, \frac{1}{2})$.

Let us define the norms, for $\alpha \in (\frac{1}{2}, 1)$, $0 < \theta < 1$ and $\varepsilon > 0$,

$$\begin{aligned} &\|f\|_{H^{\alpha,\alpha}_{\varepsilon,\theta}} \\ &= \left\| \left(1+\varepsilon^2|v|^2\right)^{\frac{\alpha}{2}(1-\theta)} \left(1+\varepsilon^2|\xi|^2\right)^{\frac{\alpha}{2}(1-\theta)} \left(1+|v|^2\right)^{\frac{\alpha}{2}\theta} \left(1+|\xi|^2\right)^{\frac{\alpha}{2}\theta} \mathcal{F}_x f(\xi, v) \right\|_{L^2_{\xi,v}} . \end{aligned} \tag{11.1}$$

The $H^{\alpha,\alpha}_{\varepsilon,\theta}$ norm is equivalent (up to powers of ε) to the $H^{\alpha,\alpha}$ norm, but for small ε the $H^{\alpha,\alpha}_{\varepsilon,\theta}$ norm is nearly equal to the $H^{s,s}$ norm where $s = \alpha\theta$. Also note that

$$\left(H^{\alpha,\alpha}_{\varepsilon}, H^{\alpha,\alpha}\right)_{\theta} = H^{\alpha,\alpha}_{\varepsilon,\theta},$$

with equality of norms.

The following bilinear estimate is proven in [7]:

Proposition 11.1. *Let $\alpha > \frac{1}{2}$. Then there is a constant $C = C(\alpha)$ such that, for the constant collision kernel in dimension $d = 2$,*

$$\left\| Q^+(T(t) f_0, T(t) g_0) \right\|_{L^2_{t\in\mathbb{R}} H^{\alpha,\alpha}} \leqq C \|f_0\|_{H^{\alpha,\alpha}} \|g_0\|_{H^{\alpha,\alpha}} . \tag{11.2}$$

On the other hand, from Proposition 6.4 we know that

$$\left\| Q^+(T(t) f_0, T(t) g_0) \right\|_{L^1_{t\in\mathbb{R}} H^{\alpha,\alpha}_{\varepsilon}} \leqq C \|f_0\|_{H^{\alpha,\alpha}_{\varepsilon}} \|g_0\|_{H^{\alpha,\alpha}_{\varepsilon}} , \tag{11.3}$$

where the constant C is independent of ε.

Interpolating these two estimates yields

$$\left\| Q^+(T(t) f_0, T(t) g_0) \right\|_{L^{p_\theta}_{t\in\mathbb{R}} H^{\alpha,\alpha}_{\varepsilon,\theta}} \leqq C \|f_0\|_{H^{\alpha,\alpha}_{\varepsilon,\theta}} \|g_0\|_{H^{\alpha,\alpha}_{\varepsilon,\theta}} , \tag{11.4}$$

where C is independent of ε and $1/p_\theta = 1 - \frac{1}{2}\theta$; note that $p_\theta > 1$ for each $\theta \in (0, 1)$.

The chain of reasoning is as follows: let $\alpha = \frac{1}{2}+$ and fix a desired regularity $s \in (0, \frac{1}{2})$; then, θ is fixed so that $s = \alpha\theta$. Let $f_0 \in H^{\alpha,\alpha}$ be an arbitrary initial datum. By choosing ε very small, we can let the $H^{\alpha,\alpha}_{\varepsilon,\theta}$ norm approach the $H^{s,s}$ norm of f_0, while the constant C remains fixed. This implies that the local time of existence depends only on the $H^{s,s}$ norm of f_0, by an application of Theorem 3.2 (we have localized in time using that $p_\theta > 1$). Altogether, we will be able to conclude as follows:

Theorem 11.2. *Let* $f_0 \in H^{\frac{1}{2}+,\frac{1}{2}+}$ *and fix* $s \in (0, \frac{1}{2})$*. The gain-only Boltzmann equation*

$$(\partial_t + v \cdot \nabla_x) f = Q^+(f, f) \tag{11.5}$$

has a mild solution $f \in C\left([0, T], H^{\frac{1}{2}+,\frac{1}{2}+}\right)$ *such that* $Q^+(f, f) \in L^1_{t\in[0,T]} H^{\frac{1}{2}+,\frac{1}{2}+}$ *and* $f(t = 0) = f_0$*. The solution is unique in the class of all mild solutions with the same initial data satisfying* $Q^+(f, f) \in L^1_{t\in[0,T]} H^{\frac{1}{2}+,\frac{1}{2}+}$*. The existence time* T *depends only on* s *and the* $H^{s,s}$ *norm of* f_0*.*

Once we have Theorem 11.2, we repeat the Kaniel–Shinbrot iteration as in Section 8.2 to conclude Theorem 1.2.

Proof. (Theorem 11.2) Since $s \in (0, \frac{1}{2})$ and $\alpha = \frac{1}{2}+$, we can fix $\theta \in (0, 1)$ so that $s = \alpha\theta$; then, we have $p_\theta > 1$ where $1/p_\theta = 1 - \frac{1}{2}\theta$.

Under the change of variables

$$\tilde{f}(t) = T(-t) f(t),$$

the equation (11.5) is transformed into

$$\partial_t \tilde{f}(t) = T(-t) Q^+ \left(T(t)\tilde{f}(t), T(t)\tilde{f}(t)\right)$$

Fix a smooth, even function $\psi(t) : \mathbb{R} \to \mathbb{R}$, which is decreasing on $(0, \infty)$, equals 1 on $(0, T)$, and equals 0 on $(2T, \infty)$. Then consider the equation

$$\partial_t \tilde{f}(t) = \mathcal{A}(t, \tilde{f}(t), \tilde{f}(t)), \tag{11.6}$$

where

$$\mathcal{A}(t, f_0, g_0) = \psi(t) T(-t) Q^+ (T(t) f_0, T(t), g_0)$$

and $\tilde{f}(t = 0) = f_0$. By (11.4), the definition of ψ, and Hölder's inequality, it holds that

$$\| \mathcal{A}(t, f_0, g_0) \|_{L^1_t H^{\alpha,\alpha}_{\varepsilon,\theta}} \lessapprox C T^{\theta/2} \| f_0 \|_{H^{\alpha,\alpha}_{\varepsilon,\theta}} \| g_0 \|_{H^{\alpha,\alpha}_{\varepsilon,\theta}}, \tag{11.7}$$

where the constant C is independent of ε. By Theorem 3.2, equation (11.6) is well-posed as long as

$$\| f_0 \|_{H^{\alpha,\alpha}_{\varepsilon,\theta}} \lessapprox C T^{-\theta/2}, \tag{11.8}$$

where the constant C is again independent of ε. Letting ε tend to zero, this condition becomes

$$\| f_0 \|_{H^{s,s}} \lessapprox C T^{-\theta/2}, \tag{11.9}$$

which is what we wanted. □

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Appendix A. An Endpoint Strichartz Estimate

We recall Theorem 10.1 from [15].

Theorem A.1. [15] *Let $\sigma > 0$, H be a Hilbert space and B_0, B_1 be Banach spaces. Suppose that for each time t we have an operator $U(t) : H \to B_0^*$ such that*

$$\|U(t)\|_{H\to B_0^*} \lesssim 1 \tag{A.1}$$

$$\left\|U(t)\,(U(s))^*\right\|_{B_1\to B_1^*} \lesssim |t-s|^{-\sigma}. \tag{A.2}$$

Let B_θ denote the real interpolation space $(B_0, B_1)_{\theta,2}$. Then we have the estimates

$$\|U(t)f\|_{L_t^q B_\theta^*} \lesssim \|f\|_H \tag{A.3}$$

$$\left\|\int (U(s))^*\, F(s)\,\mathrm{d}s\right\|_H \lesssim \|F\|_{L_t^{q'} B_\theta} \tag{A.4}$$

$$\left\|\int_{s<t} U(t)\,(U(s))^*\, F(s)\,\mathrm{d}s\right\|_{L_t^q B_\theta^*} \lesssim \|F\|_{L_t^{\tilde{q}'} B_{\tilde{\theta}}^*}, \tag{A.5}$$

whenever $0\leqq\theta\leqq1$, $2\leqq q = \frac{2}{\sigma\theta}$, $(q,\theta,\sigma) \neq (2,1,1)$, and similarly for $(\tilde{q},\tilde{\theta})$. If the decay estimate is strengthened to

$$\left\|U(t)\,(U(s))^*\right\|_{B_1\to B_1^*} \lesssim (1+|t-s|)^{-\sigma}, \tag{A.6}$$

then the requirement $q = \frac{2}{\sigma\theta}$ can be relaxed to $q \geqq \frac{2}{\sigma\theta}$, and similarly for $(\tilde{q},\tilde{\theta})$.

For our application, we will need to think of $\gamma_0(x,x')$ as an arbitrary measurable complex-valued function of $x, x' \in \mathbb{R}^2$. Let us take $H = L^2_{x,x'}\left(\mathbb{R}^2\times\mathbb{R}^2\right)$, $B_0 = L^2_{x,x'}\left(\mathbb{R}^2\times\mathbb{R}^2\right)$, and $B_1 = L^1_{x,x'}\left(\mathbb{R}^2\times\mathbb{R}^2\right)$. We employ the notation $\Delta_\pm = \Delta_x - \Delta_{x'}$. The energy estimate

$$\left\|e^{it\Delta_\pm}\gamma_0\right\|_{L^2_{x,x'}} \lesssim \|\gamma_0\|_{L^2_{x,x'}} \tag{A.7}$$

is immediate. The dispersive estimate

$$\left\|e^{i(t-s)\Delta_\pm}\gamma_0\right\|_{L^\infty_{x,x'}} \lesssim |t-s|^{-2}\left\|\gamma_0(x,x')\right\|_{L^1_{x,x'}} \tag{A.8}$$

follows from writing the fundamental solution of $(i\partial_t + \Delta_\pm)\,\gamma = 0$, that is

$$\frac{1}{t^2}e^{i\left(|x|^2-|x'|^2\right)/t}$$

for initial data $\delta(x)\delta(x')$, and applying Young's inequality. The relevant parameters for Theorem A.1 are $q = 2$, $\theta = \frac{1}{2}$ and $\sigma = 2$. The real interpolation space $(B_0, B_1)_{\theta,2}$ is the Lorentz space $L^{\frac{4}{3},2}_{x,x'}$ ([5] Theorem 5.3.1), and its dual is $L^{4,2}_{x,x'}$ ([10] Theorem 1.4.17 (vi)), so we obtain

$$\left\|e^{it\Delta_\pm}\gamma_0\right\|_{L_t^2 L^{4,2}_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)} \lesssim \|\gamma_0\|_{L^2_{x,x'}(\mathbb{R}^2\times\mathbb{R}^2)}, \tag{A.9}$$

which is the desired inequality.

Appendix B. Fractional Leibniz Formulas

Theorem B.1. *Let $s \in (0,1)$ and $n \in \{2,3,4,5,\dots\}$. Then if $f(x), g(x) : \mathbb{R}^n \to \mathbb{R}$ are measurable functions such that $f \in H^s(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then $(-\Delta)^{\frac{s}{2}}(fg)$ and $f(-\Delta)^{\frac{s}{2}} g$ are canonically identified with well-defined tempered distributions, and their difference is in $L^2(\mathbb{R}^n)$ and the following estimate holds:*

$$\left\| (-\Delta)^{\frac{s}{2}}(fg) - f(-\Delta)^{\frac{s}{2}} g \right\|_{L^2(\mathbb{R}^n)} \leqq C(n,s) \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)}. \tag{B.1}$$

Proof. The estimate follows formally from [17], Appendix A, Theorem A.12, in the one-dimensional case, for Schwartz functions f, g. (Also see [16] problem 5.1 and pp. 105–110 for the multidimensional case.) The objective here is to ensure that the result remains true in suitable inhomogeneous Sobolev spaces; the argument is broken into three parts.

(i) For f, g in the Schwartz class, the estimate (B.1) is true due to [17].

(ii) Keeping f fixed in the Schwartz class, we can pass to the *distributional* limit $g_n \rightharpoonup g \in L^\infty(\mathbb{R}^n)$ in (B.1), where each g_n is Schwartz and uniformly bounded in L^∞. Every term makes sense because g is a tempered distribution and f is Schwartz.

(iii) We need to pass to the limit $f_n \to f \in H^s(\mathbb{R}^n)$ in (B.1), where the f_n are Schwartz and uniformly bounded in H^s, but $g \in L^\infty(\mathbb{R}^n)$ is now *fixed*. Now f_n, f are uniformly bounded in $H^s(\mathbb{R}^n)$, hence uniformly bounded in $L^r(\mathbb{R}^n)$ where $\frac{1}{2} - \frac{s}{n} = \frac{1}{r}$, by the Sobolev embedding theorem. Hence $f_n g$ and fg are uniformly bounded in $L^r(\mathbb{R}^n)$, so they are well-defined tempered distributions, as is $(-\Delta)^{\frac{s}{2}}(fg)$. For any Schwartz function ψ, the estimate

$$\begin{aligned} \int \psi (-\Delta)^{\frac{s}{2}}(fg) &\leqq C \left\| (-\Delta)^{\frac{s}{2}} \psi \right\|_{L^{r'}(\mathbb{R}^n)} \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} \\ &\leqq C \left\| (-\Delta)^{\frac{s}{2}} \psi \right\|_{L^{r'}(\mathbb{R}^n)} \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)}, \end{aligned} \tag{B.2}$$

where $\frac{1}{2} - \frac{s}{n} = \frac{1}{r}$, follows from duality, Hölder's inequality, and Sobolev's inequality.

Finally we deal with the term $f(-\Delta)^{\frac{s}{2}} g$. The idea is to re-write it as

$$f(-\Delta)^{\frac{s}{2}} g = -\left\{ (-\Delta)^{\frac{s}{2}}(fg) - f(-\Delta)^{\frac{s}{2}} g \right\} + (-\Delta)^{\frac{s}{2}}(fg), \tag{B.3}$$

so it is a difference of two things which apparently make sense. Using this difference formula and the commutator estimate of Kenig-Ponce-Vega from the theorem statement, we can prove the estimate

$$\begin{aligned} &\int \psi f (-\Delta)^{\frac{s}{2}} g \\ &\leqq C \left(\|\psi\|_{L^2(\mathbb{R}^n)} + \left\| (-\Delta)^{\frac{s}{2}} \psi \right\|_{L^{r'}(\mathbb{R}^n)} \right) \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)}, \end{aligned} \tag{B.4}$$

where $g \in L^\infty(\mathbb{R}^n)$ and f, ψ are in the Schwartz class. We conclude (by density of Schwartz functions in $H^s(\mathbb{R}^n)$) that $f(-\Delta)^{\frac{s}{2}} g$ is canonically identified with a well-defined tempered distribution whenever $f \in H^s(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$; moreover, we can take distributional limits in f_n where needed (keeping $g \in L^\infty(\mathbb{R}^n)$ fixed) to derive the desired estimate in this class. □

Appendix C. Some general estimates in $L^2 \bigcap L^1$

Assume throughout this appendix that $0 \leqq f_0 \in L^2_{x,v} \cap L^1_{x,v}$. As is typical for a kinetic equation, we will consider a suitable *mollification* (with the same, i.e. *unmollified*, initial data f_0), which takes the following form:

$$(\partial_t + v \cdot \nabla_x) f^n = \frac{Q(f^n, f^n)}{1 + \frac{1}{n}\rho_{f^n}}. \tag{C.1}$$

Here $n = 1, 2, 3, \ldots$ and $f^n(t = 0) = f_0$. Note that we are not allowed to mollify the data in general, because that would change the profile of the data, and we are looking for local well-posedness in the critical space L^2 (with an auxiliary L^1 estimate). It is well-known that the mollified equation (C.1) is globally well-posed for initial data $f_0 \in L^1$; the proof is by a Picard iteration and time-stepping procedure. [9]

Since $Q = Q^+ - Q^-$ (both non-negative) and $\rho_{f^n} \geqq 0$, we can conclude

$$(\partial_t + v \cdot \nabla_x) f^n \leqq Q^+(f^n, f^n), \tag{C.2}$$

which implies

$$f^n(t) \leqq T(t) f_0 + \int_0^t T(t-t') Q^+(f^n, f^n), (t')\,\mathrm{d}t' \tag{C.3}$$

where $T(t) = e^{-tv\cdot\nabla_x}$. In particular, for $0 \leqq t \leqq T$,

$$f^n(t) \leqq T(t) f_0 + \int_0^T T(t-t') Q^+(f^n, f^n).(t')\,\mathrm{d}t' \tag{C.4}$$

Apply $Q^+(\cdot,\cdot)$ to both sides of this inequality and apply monotonicity to obtain

$$\begin{aligned} Q^+(f^n, f^n) &\leqq Q^+\left(T(t) f_0, T(t) f_0\right) \\ &+ \int_0^T Q^+\left(T(t) f_0, T(t-t') Q^+(f^n, f^n)(t')\right) \mathrm{d}t' \\ &+ \int_0^T Q^+\left(T(t-t') Q^+(f^n, f^n)(t'), T(t) f_0\right) \mathrm{d}t' \\ &+ \int_0^T \int_0^T Q^+\left(T(t-t') Q^+(f^n, f^n)(t'), T(t-t'') Q^+(f^n, f^n)(t'')\right) \mathrm{d}t'\,\mathrm{d}t''. \end{aligned} \tag{C.5}$$

Now we take the $L^1_{t\in[0,T]}L^2_{x,v}$ norm of both sides (noting that this quantity might be infinite), and apply Minkowski's inequality to get

$$\begin{aligned}
\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} &\leqq \left\| Q^+\left(T(t)f_0, T(t)f_0\right)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} \\
&+ \int_0^T \left\| Q^+\left(T(t)f_0, T(t-t')Q^+(f^n, f^n)(t')\right)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} \mathrm{d}t' \\
&+ \int_0^T \left\| Q^+\left(T(t-t')Q^+(f^n, f^n)(t'), T(t)f_0\right)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} \mathrm{d}t' \\
&+ \int_0^T\int_0^T \mathrm{d}t'\,\mathrm{d}t'' \\
&\quad\times \left\| Q^+\left(T(t-t')Q^+(f^n, f^n)(t'), T(t-t'')Q^+(f^n, f^n)(t'')\right)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}}.
\end{aligned} \tag{C.6}$$

Apply Proposition 5.4 to the last term, and Proposition 5.6 to the first three terms, to obtain

$$\begin{aligned}
\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} \leqq& \,\delta_{f_0}(T)\left\|f_0\right\|_{L^2_{x,v}} \\
&+ 2\int_0^T \delta_{f_0}(T)\left\|T(-t')Q^+(f^n, f^n)(t')\right\|_{L^2_{x,v}} \mathrm{d}t' \\
&+ \int_0^T\int_0^T C\left\|T(-t')Q^+(f^n, f^n)(t')\right\|_{L^2_{x,v}}\left\|T(-t'')Q^+(f^n, f^n)(t'')\right\|_{L^2_{x,v}} \mathrm{d}t'\,\mathrm{d}t'',
\end{aligned} \tag{C.7}$$

where $\limsup_{T\to 0^+}\delta_{f_0}(T) = 0$ (note that $\delta_{f_0}(T)$ depends on the profile of the data for any fixed $T > 0$).

Overall, we conclude that

$$\begin{aligned}
\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} \leqq& \,\delta_{f_0}(T)\left\|f_0\right\|_{L^2_{x,v}} \\
&+ 2\delta_{f_0}(T)\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,T]}L^2_{x,v}} + C\left\| Q^+(f^n, f^n)\right\|^2_{L^1_{t\in[0,T]}L^2_{x,v}},
\end{aligned} \tag{C.8}$$

where $\limsup_{T\to 0^+}\delta_{f_0}(T) = 0$. In the case that $\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,T_0]}L^2_{x,v}}$ is finite for *some* $T_0 > 0$, a standard continuity argument allows us to bound this quantity *uniformly in n* up to some other small time $T > 0$ which depends on f_0. We can state this is an alternative: there are numbers $C(f_0), T(f_0)$ such that, for each n, exactly one of the following holds:

(1) *Case 1:* $\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,\sigma]}L^2_{x,v}} = \infty$ for every $\sigma > 0$

(2) *Case 2:* $\left\| Q^+(f^n, f^n)\right\|_{L^1_{t\in[0,T(f_0)]}L^2_{x,v}} \leqq C(f_0)$

In particular $C(f_0), T(f_0)$ are independent of n; hence, as long as Case 2 holds for infinitely many n, we can hope for a compactness argument. Note that once $Q^+(f^n, f^n)$ is placed uniformly in $L^1_{t\in[0,T(f_0)]}L^2_{x,v}$, the method of Section 9 implies that

$$T(-t)\frac{Q^-(f^n, f^n)(t)}{1+\frac{1}{n}\rho_{f^n(t)}}$$

is uniformly bounded in $L^2_{x,v}L^1_{t\in[0,T(f_0)]}$; in particular, $(\partial_t + v \cdot \nabla_x) f^n$ is locally integrable in (t, x, v), boundedly with respect to n. Moreover, on $[0, T(f_0)]$, f^n satisfies the full range of Strichartz estimates expected for L^2 solutions of the free transp1ort equation, uniformly in n.

Remark C.1. The classical L^1 velocity averaging lemma used in [9] requires that *both* f^n and $(\partial_t + v \cdot \nabla_x) f^n$ are relatively weakly compact in $L^1(K)$ for compact sets $K \subset [0, \infty) \times \mathbb{R}^2_x \times \mathbb{R}^2_v$. However, a refinement cited as Lemma 4.1 in [3] states that, under the condition that f^n is relatively weakly compact in $L^1(K)$ for compact sets K, it suffices for $(\partial_t + v \cdot \nabla_x) f^n$ to be uniformly *bounded* in $L^1(K)$ for compact K.

References

1. Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T.: Global existence and full regularity of the Boltzmann equation without angular cutoff. *Commun. Math. Phys.* **304**(2), 513–581, 2011
2. Alonso, R.J., Gamba, I.M.: Distributional and classical solutions to the Cauchy Boltzmann problem for soft potentials with integrable angular cross section. *J. Stat. Phys.* **137**(5), 1147, 2009
3. Arsenio, D.: On the global existence of mild solutions to the Boltzmann equation for small data in L^D. *Commun. Math. Phys.* **302**(2), 453–476, 2011
4. Bennett, J., Bez, N., Gutiérrez, S., Lee, S.: On the Strichartz estimates for the kinetic transport equation. *Commun. Partial Differ. Equ.* **39**(10), 1821–1826, 2014
5. Bergh, J., Löfström, J.: *Interpolation Spaces, Grundlehren der mathematischen Wissenschaften*, vol. 223. Springer, Berlin 1976
6. Cercignani, C., Illner, R., Pulvirenti, M.: *The Mathematical Theory of Dilute Gases*. Springer, Berlin 1994
7. Chen, T., Denlinger, R., Pavlovic, N.: Local well-posedness for Boltzmann's equation and the Boltzmann hierarchy via Wigner transform. *Commun. Math. Phys.* **368**, 427–465, 2019
8. Chen, T., Denlinger, R., Pavlovic, N.: Moments and regularity for a Boltzmann equation via Wigner transform. *Discrete Contin. Dyn. Syst. A* **39**(9), 4979–5015, 2019
9. DiPerna, R.J., Lions, P.-L.: On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. Math.* **130**(2), 321–366, 1989
10. Grafakos, L.: *Classical Fourier Analysis, Graduate Texts in Mathematics*, vol. 249, 2nd edn. Springer, New York 2008
11. Gressman, P.T., Strain, R.M.: Global classical solutions of the Boltzmann equation without angular cut-off. *J. Am. Math. Soc.* **24**(3), 771–847, 2011
12. Guo, Y.: Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.* **169**(4), 305–353, 2003
13. He, L., Jiang, J.-C.: Well-posedness and scattering for the Boltzmann equations: soft potential with cut-off. *J. Stat. Phys.* **168**(2), 470–481, 2017
14. Kaniel, S., Shinbrot, M.: The Boltzmann equation: I. Uniqueness and local existence. *Commun. Math. Phys.* **58**(1), 65–84, 1978
15. Keel, M., Tao, T.: Endpoint Strichartz estimates. *Am. J. Math.* **120**(5), 955–980, 1998
16. Kenig, C.: The Cauchy problem for the quasilinear Schrodinger equation, (September 2013). arXiv:1309.3291
17. Kenig, C.E., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Commun. Pure Appl. Math.* **46**(4), 527–620, 1993

18. Klainerman, S., Machedon, M.: On the uniqueness of solutions to the Gross-Pitaevskii hierarchy. *Commun. Math. Phys*. **279**(1), 169–185, 2008
19. Klainerman, S., Machedon, M.: Space-time estimates for null forms and the local existence theorem. *Commun. Pure Appl. Math*. **46**(9), 1221–1268, 1993
20. Stein, E.M.: *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton 1970
21. Toscani, G.: Global solution of the initial value problem for the Boltzmann equation near a local Maxwellian. *Arch. Ration. Mech. Anal*. **102**(3), 231–241, 1988
22. Ukai, S.: On the existence of global solutions of mixed problem for the non-linear Boltzmann equation. *Proc. Jpn. Acad*. **50**(3), 179–184, 1974

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