Arch. Rational Mech. Anal. 239 (2021) 1809–1835
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-020-01604-5

Global Well-Posedness and Exponential Stability of 3D Navier–Stokes Equations with Density-Dependent Viscosity and Vacuum in Unbounded Domains

Cheng He, Jing Li & Boqiang Lü

Communicated by P.-L. Lions

Abstract

We consider the global existence and large-time asymptotic behavior of strong solutions to the Cauchy problem of the three-dimensional (3D) nonhomogeneous incompressible Navier–Stokes equations with density-dependent viscosity and vacuum. After establishing some key a priori exponential decay-in-time rates of the strong solutions, we obtain both the global existence and exponential stability of strong solutions in the whole three-dimensional space, provided that the initial velocity is suitably small in some homogeneous Sobolev space which may be optimal compared with the case of homogeneous Navier-Stokes equations. Note that this result is proved without any smallness conditions on the initial density which contains vacuum and even has compact support.

1. Introduction

The nonhomogeneous incompressible Navier–Stokes equations ([26]) read as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) d) + \nabla P = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1.1}$$

Here, $t \geq 0$ is time, $x \in \Omega \subset \mathbb{R}^3$ is the spatial coordinates, and the unknown functions $\rho = \rho(x,t)$, $u = (u^1, u^2, u^3)(x,t)$, and $P = P(x,t)$ denote the density, velocity, and pressure of the fluid, respectively. The deformation tensor is defined by

$$d = \frac{1}{2}\left[\nabla u + (\nabla u)^{\mathrm{T}}\right], \tag{1.2}$$

and the viscosity $\mu(\rho)$ satisfies the following hypothesis:

$$\mu \in C^1[0,\infty), \quad \mu(\rho) > 0. \tag{1.3}$$

We consider the Cauchy problem of (1.1) with (ρ, u) vanishing at infinity and the initial conditions

$$\rho(x,0) = \rho_0(x),\ \ \rho u(x,0) = m_0(x),\ \ x \in \mathbb{R}^3 \tag{1.4}$$

for given initial data ρ_0 and m_0.

There is a lot of literature on the mathematical study of nonhomogeneous incompressible flow. In particular, the system (1.1) with constant viscosity has been considered extensively. On the one hand, in the absence of a vacuum, the global existence of weak solutions and the local existence of strong ones were established in Kazhikov [4,23]. Ladyzhenskaya–Solonnikov [24] first proved the global well-posedness of strong solutions to the initial boundary value problems in both two-dimensional (2D) bounded domains (for large data) and 3D ones (with initial velocity small in suitable norms). Recently, the global well-posedness results with small initial data in critical spaces were considered by many people (see [1,10,11,18] and the references therein). On the other hand, when the initial density is allowed to vanish, the global existence of weak solutions is proved by Simon [31]. The local existence of strong solutions was obtained by Choe–Kim [8] (for 3D bounded and unbounded domains) and Lü–Wang–Zhong [27] (for 2D Cauchy problem) under some compatibility conditions. Recently, for the Cauchy problem in the whole 2D space, Lü–Shi–Zhong [28] obtained the global strong solutions for large initial data. For the 3D case, under some smallness conditions on the initial velocity, Craig–Huang–Wang [9] proved the following interesting result:

Proposition 1.1. ([9]) *Let* $\Omega = \mathbb{R}^3$. *For positive constants* $\bar{\rho}$ *and* μ, *assume that* $\mu(\rho) \equiv \mu$ *in* (1.1) *and the initial data* (ρ_0, m_0) *satisfy*

$$\begin{cases} 0 \le \rho_0 \le \bar{\rho},\ \ \rho_0 \in L^{3/2}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \\ u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \cap D^1_{0,\sigma}(\mathbb{R}^3) \cap D^{2,2}(\mathbb{R}^3),\ \ m_0 = \rho_0 u_0 \end{cases} \tag{1.5}$$

and the compatibility condition

$$-\mu \Delta u_0 + \nabla P_0 = \rho_0^{1/2} g, \quad \text{in } \mathbb{R}^3, \tag{1.6}$$

for some $(P_0, g) \in D^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. *Then, there exists some positive constant* ε *depending only on* $\bar{\rho}$ *such that there exists a unique global strong solution to the Cauchy problem* (1.1) (1.4) *provided* $\|u_0\|_{\dot{H}^{1/2}} \le \mu\varepsilon$. *Moreover, the following large time decay rate holds for* $t \ge 1$*:*

$$\|\nabla u(\cdot, t)\|_{L^2(\mathbb{R}^3)} \le \bar{C} t^{-1/2}, \tag{1.7}$$

where $\bar{C}$ *depends on* $\bar{\rho},\ \mu$, *and* $\|\rho_0^{1/2} u_0\|_{L^2(\mathbb{R}^3)}$.

When it comes to the case that the viscosity $\mu(\rho)$ depends on the density ρ, it is more difficult to investigate the global well-posedness of system (1.1) due to the strong coupling between viscosity coefficient and density. In fact, allowing the density to vanish initially, Lions [26] first obtained the global weak solutions whose uniqueness and regularity are still open even in two spatial dimensions. Later, Desjardins [12] established the global weak solution with higher regularity for 2D case provided that the viscosity $\mu(\rho)$ is a small perturbation of a positive constant in L^∞-norm. Recently, some progress has been made on the well-posedness of strong solutions to (1.1) (see [2,3,7,20,21,29,32] and the reference therein). In particular, on the one hand, when the initial density is strictly away from vacuum, Abidi–Zhang [2] obtained the global strong solutions in whole 2D space under smallness conditions on $\|\mu(\rho_0)-1\|_{L^\infty}$, and later for 3D case, they [3] obtained the global strong ones under the smallness conditions on both $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ and $\|\mu(\rho_0)-1\|_{L^\infty}$. On the other hand, for the case that the initial density contains vacuum, Huang–Wang [20] obtained the global strong solutions in 2D bounded domains when $\|\nabla\mu(\rho_0)\|_{L^p}(p\geq 2)$ is small enough; Huang–Wang [21] and Zhang [32] established the global strong solutions with small $\|\nabla u_0\|_{L^2}$ in 3D bounded domains. However, as pointed by Huang–Wang [21], the methods used in [21,32] depend heavily on the boundedness of the domains and little is known for the global well-posedness of strong solutions to the Cauchy problem (1.1)–(1.4) with density-dependent viscosity and vacuum.

Before stating the main results, we first explain the notations and conventions used throughout this paper. Set

$$\int f\,\mathrm{d}x \triangleq \int_{\mathbb{R}^3} f\,\mathrm{d}x.$$

Moreover, for $1\leq r\leq\infty, k\geq 1$, and $\beta>0$, the standard homogeneous and inhomogeneous Sobolev spaces are defined as follows:

$$\begin{cases} L^r=L^r(\mathbb{R}^3),\quad W^{k,r}=W^{k,r}(\mathbb{R}^3),\quad H^k=W^{k,2},\\ \|\cdot\|_{B_1\cap B_2}=\|\cdot\|_{B_1}+\|\cdot\|_{B_2},\ \text{for two Banach spaces } B_1 \text{ and } B_2,\\ D^{k,r}=D^{k,r}(\mathbb{R}^3)=\{v\in L^1_{\mathrm{loc}}(\mathbb{R}^3)|\nabla^k v\in L^r(\mathbb{R}^3)\},\\ D^1=\{v\in L^6(\mathbb{R}^3)|\nabla v\in L^2(\mathbb{R}^3)\},\\ C^\infty_{0,\sigma}=\{f\in C^\infty_0\mid \operatorname{div} f=0\},\quad D^1_{0,\sigma}=\overline{C^\infty_{0,\sigma}}\ \text{closure in the norm of } D^1,\\ \dot H^\beta=\left\{f:\mathbb{R}^3\to\mathbb{R}\,\middle|\,\|f\|^2_{\dot H^\beta}=\int|\xi|^{2\beta}|\hat f(\xi)|^2\mathrm{d}\xi<\infty\right\}, \end{cases}$$

where $\hat f$ is the Fourier transform of f.

Our main result can be stated as follows:

Theorem 1.2. *For constants $\bar\rho>0$, $q\in(3,\infty)$, and $\beta\in(\frac12,1]$, assume that the initial data (ρ_0,m_0) satisfy*

$$0\leq\rho_0\leq\bar\rho,\ \rho_0\in L^{3/2}\cap H^1,\ \nabla\mu(\rho_0)\in L^q,\ u_0\in\dot H^\beta\cap D^1_{0,\sigma},\ m_0=\rho_0u_0. \tag{1.8}$$

Then for

$$\underline{\mu}\triangleq\min_{\rho\in[0,\bar\rho]}\mu(\rho),\quad \bar\mu\triangleq\max_{\rho\in[0,\bar\rho]}\mu(\rho),\quad M\triangleq\|\nabla\mu(\rho_0)\|_{L^q},$$

there exists some small positive constant ε_0 depending only on $q,\beta,\bar\rho,\underline{\mu},\bar\mu$, $\|\rho_0\|_{L^{3/2}}$, and M such that if

$$\|u_0\|_{\dot H^\beta}\leq\varepsilon_0, \tag{1.9}$$

the Cauchy problem (1.1)–(1.4) *admits a unique global strong solution* (ρ, u, P) *satisfying that for any* $0 < \tau < T < \infty$ *and* $p \in [2, p_0)$ *with* $p_0 \triangleq \min\{6, q\}$,

$$\begin{cases} 0 \le \rho \in C([0,T]; L^{3/2} \cap H^1), \quad \nabla\mu(\rho) \in C([0,T]; L^q), \\ \nabla u \in L^\infty(0,T; L^2) \cap L^\infty(\tau, T; W^{1,p_0}) \cap C([\tau, T]; H^1 \cap W^{1,p}), \\ P \in L^\infty(\tau, T; W^{1,p_0}) \cap C([\tau, T]; H^1 \cap W^{1,p}), \\ \sqrt{\rho} u_t \in L^2(0,T; L^2) \cap L^\infty(\tau, T; L^2), \quad P_t \in L^2(\tau, T; L^2 \cap L^{p_0}), \\ \nabla u_t \in L^\infty(\tau, T; L^2) \cap L^2(\tau, T; L^{p_0}), \quad (\rho u_t)_t \in L^2(\tau, T; L^2). \end{cases} \tag{1.10}$$

Moreover, it holds that

$$\sup_{0 \le t < \infty} \|\nabla\rho\|_{L^2} \le 2\|\nabla\rho_0\|_{L^2}, \qquad \sup_{0 \le t < \infty} \|\nabla\mu(\rho)\|_{L^q} \le 2\|\nabla\mu(\rho_0)\|_{L^q}, \tag{1.11}$$

and that there exists some positive constant σ *depending only on* $\|\rho_0\|_{L^{3/2}}$ *and* $\underline{\mu}$ *such that, for all* $t \ge 1$,

$$\|\nabla u_t(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{H^1 \cap W^{1,p_0}}^2 + \|P(\cdot, t)\|_{H^1 \cap W^{1,p_0}}^2 \le Ce^{-\sigma t}, \tag{1.12}$$

where C *depends only on* $q, \beta, \bar{\rho}, \|\rho_0\|_{L^{3/2}}, \underline{\mu}, \bar{\mu}, M, \|\nabla u_0\|_{L^2}$, *and* $\|\nabla\rho_0\|_{L^2}$.

As a direct consequence, our method can be applied to the case that $\mu(\rho) \equiv \mu$ is a positive constant and obtain the following global existence and large-time behavior of the strong solutions which improves slightly those due to Craig–Huang–Wang [9] (see Proposition 1.1).

Theorem 1.3. *For constants* $\bar{\rho} > 0$ *and* $\mu > 0$, *assume that* $\mu(\rho) \equiv \mu$ *in* (1.1) *and the initial data* (ρ_0, u_0) *satisfy* (1.5) *except* $u_0 \in D^{2,2}$. *Then, there exists some positive constant* ε *depending only on* $\bar{\rho}$ *such that there exists a unique global strong solution to the Cauchy problem* (1.1) (1.4) *satisfying* (1.10) *with* $p_0 = 6$ *provided* $\|u_0\|_{\dot{H}^{1/2}} \le \mu\varepsilon$. *Moreover, it holds that*

$$\sup_{0 \le t < \infty} \|\nabla\rho\|_{L^2} \le 2\|\nabla\rho_0\|_{L^2}, \tag{1.13}$$

and that there exists some positive constant σ *depending only on* $\|\rho_0\|_{L^{3/2}}$ *and* μ *such that, for* $t \ge 1$,

$$\|\nabla u_t(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{H^1 \cap W^{1,6}}^2 + \|P(\cdot, t)\|_{H^1 \cap W^{1,6}}^2 \le Ce^{-\sigma t}, \tag{1.14}$$

where C *depends only on* $\bar{\rho}, \mu, \|\rho_0\|_{L^{3/2}}, \|\nabla u_0\|_{L^2}$, *and* $\|\nabla\rho_0\|_{L^2}$.

A few remarks are in order.

Remark 1.1. To the best of our knowledge, the exponential decay-in-time properties (1.12) in Theorem 1.2 are new and somewhat surprising, since the known corresponding decay-in-time rates for the strong solutions to system (1.1) are algebraic even for the constant viscosity case [1,9] and the homogeneous case [6,15,16,22,30]. Moreover, as a direct consequence of (1.11), $\|\nabla\rho(\cdot, t)\|_{L^2}$ remains uniformly bounded with respect to time which is new even for the constant viscosity case (see [9] or Proposition 1.1).

Remark 1.2. It should be noted here that our Theorem 1.2 holds for any function $\mu(\rho)$ satisfying (1.3) and for arbitrarily large initial density with vacuum (even has compact support) with a smallness assumption only on the $\dot{H}^\beta$-norm of the initial velocity u_0 with $\beta \in (1/2, 1]$, which is in sharp contrast to Abidi-Zhang [3] where they need the initial density strictly away from vacuum and the smallness assumptions on both $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ and $\|\mu(\rho_0) - 1\|_{L^\infty}$.

Remark 1.3. For our case that the viscosity $\mu(\rho)$ depends on ρ, in order to bound the L^p-norm of the gradient of the density, we need the smallness conditions on the $\dot{H}^\beta$-norm ($\beta \in (1/2, 1]$) of the initial velocity. However, it seems that our conditions on the initial velocity may be optimal compared with the constant viscosity case considered by Craig–Huang–Wang [9] where they proved that the system (1.1) is globally wellposed for small initial data in the homogeneous Sobolev space $\dot{H}^{1/2}$ which is similar to the case of homogeneous Navier–Stokes equations (see [13]). Note that for the case of initial-boundary-value problem in 3D bounded domains, Huang–Wang [21] and Zhang [32] impose smallness conditions on $\|\nabla u_0\|_{L^2}$. Furthermore, in our Theorems 1.2 and 1.3, there is no need to imposed additional initial compatibility conditions, which is assumed in [9,21,32] for the global existence of strong solutions.

Remark 1.4. It is easy to prove that the strong-weak uniqueness theorem [26, Theorem 2.7] still holds for the initial data (ρ_0, u_0) satisfying (1.8) after modifying its proof slightly. Therefore, our Theorem 1.2 can be regarded as the uniqueness and regularity theory of Lions's weak solutions [26] with the initial velocity suitably small in the $\dot{H}^\beta$-norm.

Remark 1.5. In [7], Cho–Kim considered the initial boundary value problem in 3D bounded smooth domains. In addition to (1.8), assuming that the initial data satisfy the compatibility conditions

$$-\mathrm{div}\left(\mu(\rho_0)\left(\nabla u_0 + (\nabla u_0)^{\mathrm{T}}\right)\right) + \nabla P_0 = \rho_0^{1/2} g$$

for some $(P_0, g) \in H^1 \times L^2$, it is shown ([7]) that the local-in-time strong solution (ρ, u) satisfies

$$\rho u_t \in C\left([0, T]; L^2\right). \tag{1.15}$$

However, to obtain (1.15), it seems difficult to follow the proof of (1.15) as in [7]. Indeed, in our Proposition 3.7 (see [29] also), we give a complete new proof to show that $\rho u_t \in H^1\left(\tau, T; L^2\right)$ (for any $0 < \tau < T < \infty$) which directly implies [29]

$$\rho u_t \in C\left([\tau, T]; L^2\right). \tag{1.16}$$

In fact, (1.16) is crucial for deriving the time-continuity of ∇u and P, that is (see (1.10)),

$$\nabla u, P \in C\left([\tau, T]; H^1 \cap W^{1,p}\right). \tag{1.17}$$

We now make some comments on the analysis in this paper. To extend the local strong solutions whose existence is obtained by Lemma 2.1 globally in time, one needs to establish global a priori estimates on smooth solutions to (1.1)–(1.4) in suitable higher norms. It turns out that as in the 3D bounded case [21,32], the key ingredient here is to get the time-independent bounds on the $L^1(0, T; L^\infty)$-norm of ∇u and then the $L^\infty(0, T; L^q)$-norm of $\nabla\mu(\rho)$ and the $L^\infty(0, T; L^2)$-one of $\nabla\rho$. However, as mentioned by Huang–Wang [21], the methods used in [21,32] depend crucially on the boundedness of the domains. Hence, some new ideas are needed here. First, using the initial layer analysis (see [17,19]) and an interpolation argument (see [5]), we succeed in bounding the $L^1(0, \min\{1, T\}; L^\infty)$-norm of ∇u by $\|u_0\|_{\dot H^\beta}$ (see (3.34)). Then, in order to estimate the $L^1(\min\{1, T\}, T; L^\infty)$-norm of ∇u, we find that $\|\rho^{1/2}u(\cdot, t)\|_{L^2}^2$ in fact decays at the rate of $e^{-\sigma t}(\sigma > 0)$ for large time (see (3.21)), which can be achieved by combining the standard energy equality (see (3.25)) with the fact that

$$\left\|\rho^{1/2}u\right\|_{L^2}^2 \le \|\rho\|_{L^{3/2}}\|u\|_{L^6}^2 \le C\|\nabla u\|_{L^2}^2,$$

due to $(1.1)_1$ and the Sobolev inequality. With this key exponential decay-in-time rate at hand, we can obtain that both $\|\nabla u(\cdot, t)\|_{L^2}^2$ and $\|\rho^{1/2}u_t(\cdot, t)\|_{L^2}^2$ decay at the same rate as $e^{-\sigma t}(\sigma > 0)$ for large time (see (3.22) and (3.23)). In fact, all these exponential decay-in-time rates are the key to obtaining the desired uniform bound (with respect to time) on the $L^1(\min\{1, T\}, T; L^\infty)$-norm of ∇u (see (3.35)). Finally, using these a priori estimates and the fact that the velocity is divergent free, we establish the time-independent estimates on the gradients of the density and the velocity which guarantee the extension of local strong solutions (see Proposition 3.7).

The rest of this paper is organized as follows: in Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the a priori estimates. Finally, we will prove Theorems 1.2 and 1.3 in Section 4.

2. Preliminaries

In this section we shall enumerate some auxiliary lemmas.

We start with the local existence of strong solutions which has been proved in [29].

Lemma 2.1. *Assume that* (ρ_0, u_0) *satisfies* (1.8) *except* $u_0 \in \dot H^\beta$. *Then there exist a small time* $T_0 > 0$ *and a unique strong solution* (ρ, u, P) *to the problem* (1.1)–(1.4) *in* $\mathbb{R}^3 \times (0, T_0)$ *satisfying* (1.10).

Next, the following well-known Gagliardo–Nirenberg inequality will be used frequently later (see [25, Theorem 2.2]).

Lemma 2.2. ([25]) *For* $r \in (6/5, \infty]$ *and*

$$p \in \begin{cases} [2, \frac{3r}{3-r}], & \text{if } r \in (6/5, 3), \\ [2, \infty), & \text{if } r = 3, \\ [2, \infty], & \text{if } r \in (3, \infty], \end{cases} \tag{2.1}$$

there exists some generic constant $C > 0$ *that may depend on* p *and* r *such that for all* $f \in \{f | f \in L^2, \nabla f \in L^r\}$

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{\alpha}\|\nabla f\|_{L^r}^{1-\alpha}, \qquad \frac{1}{p} = \frac{\alpha}{2} + (1-\alpha)\left(\frac{1}{r} - \frac{1}{3}\right). \tag{2.2}$$

A direct consequence of Lemma 2.2 is the following inequality which will be useful for the next regularity results on the Stokes equations (Lemma 2.4):

Lemma 2.3. *For* $q > 3$ *and* $r \in [2q/(q+2), q]$, *there exists some generic constant* $C > 0$ *that may depend on* q *and* r *such that for all* $f \in L^q$ *and* $g \in \{g | g \in L^2, \nabla g \in L^r\}$

$$\|fg\|_{L^r} \leq C\|f\|_{L^q}\|g\|_{L^2}^{\alpha}\|\nabla g\|_{L^r}^{1-\alpha}, \tag{2.3}$$

with $\alpha = \frac{2r(q-3)}{q(5r-6)}$.

Proof. On the one hand, Holder's inequality shows that for $1 \leq r \leq q$

$$\|fg\|_{L^r} \leq C\|f\|_{L^q}\|g\|_{L^p}, \tag{2.4}$$

with

$$p \triangleq \frac{rq}{q-r},$$

where we agree with $p = \infty$ provided $r = q$.

On the other hand, since $r \in [2q/(q+2), q] \subseteq (6/5, q]$ due to $q > 3$, noticing that

$$p = \frac{rq}{q-r}\begin{cases} = \infty, & \text{if } r = q > 3, \\ < \infty, & \text{if } 3 \leq r < q, \\ < \frac{3r}{3-r}, & \text{if } 6/5 < 2q/(q+2) \leq r < 3, \end{cases}$$

which implies that p satisfies (2.1), after using the Gagliardo–Nirenberg inequality (2.2), we have

$$\|g\|_{L^p} \leq C\|g\|_{L^2}^{\alpha}\|\nabla g\|_{L^r}^{1-\alpha}, \qquad \frac{1}{p} = \frac{\alpha}{2} + (1-\alpha)\left(\frac{1}{r} - \frac{1}{3}\right). \tag{2.5}$$

Putting (2.5) into (2.4) leads to

$$\|fg\|_{L^r} \leq C\|f\|_{L^q}\|g\|_{L^2}^{\alpha}\|\nabla g\|_{L^r}^{1-\alpha}, \tag{2.6}$$

where

$$\frac{1}{r} - \frac{1}{q} = \frac{\alpha}{2} + (1-\alpha)\left(\frac{1}{r} - \frac{1}{3}\right). \tag{2.7}$$

It thus follows from (2.7) that

$$\alpha = \frac{2r(q-3)}{q(5r-6)} \in (0, 1],$$

which together with (2.6) proves (2.3). We thus finish the proof of Lemma 2.3. □

Next, the following regularity results on the Stokes equations will be useful for our derivation of higher order a priori estimates:

Lemma 2.4. *For positive constants* $\underline{\mu}, \bar{\mu}$, *and* $q \in (3, \infty)$, *in addition to* (1.3), *assume that* $\mu(\rho)$ *satisfies*

$$\nabla\mu(\rho) \in L^q, \quad 0 < \underline{\mu} \le \mu(\rho) \le \bar{\mu} < \infty. \tag{2.8}$$

Then, if $F \in L^{6/5} \cap L^r$ *with* $r \in [2q/(q+2), q]$, *there exists some positive constant* C *depending only on* $\underline{\mu}, \bar{\mu}, r$, *and* q *such that the unique weak solution* $(u, P) \in D^1_{0,\sigma} \times L^2$ *to the Cauchy problem*

$$\begin{cases} -\mathrm{div}(2\mu(\rho)d) + \nabla P = F, & x \in \mathbb{R}^3, \\ \mathrm{div} u = 0, & x \in \mathbb{R}^3, \\ u(x) \to 0, & |x| \to \infty \end{cases} \tag{2.9}$$

satisfies

$$\|\nabla u\|_{L^2} + \|P\|_{L^2} \le C\|F\|_{L^{6/5}}, \tag{2.10}$$

$$\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \le C\|F\|_{L^r} + C\|\nabla\mu(\rho)\|_{L^q}^{\frac{q(5r-6)}{2r(q-3)}}\|F\|_{L^{6/5}}. \tag{2.11}$$

Moreover, if $F = \mathrm{div} g$ *with* $g \in L^2 \cap L^{\tilde{r}}$ *for some* $\tilde{r} \in (6q/(q+6), q]$, *there exists a positive constant* C *depending only on* $\underline{\mu}, \bar{\mu}, q$, *and* $\tilde{r}$ *such that the unique weak solution* $(u, P) \in D^1_{0,\sigma} \times L^2$ *to* (2.9) *satisfies*

$$\|\nabla u\|_{L^2\cap L^{\tilde{r}}} + \|P\|_{L^2\cap L^{\tilde{r}}} \le C\|g\|_{L^2\cap L^{\tilde{r}}} + C\|\nabla\mu(\rho)\|_{L^q}^{\frac{3q(\tilde{r}-2)}{2\tilde{r}(q-3)}}\|g\|_{L^2}. \tag{2.12}$$

Proof. First, multiplying $(2.9)_1$ by u and integrating by parts, we obtain after using $(2.9)_2$ that

$$2\int \mu(\rho)|d|^2 \mathrm{d}x = \int F \cdot u \mathrm{d}x \le \|F\|_{L^{6/5}}\|u\|_{L^6} \le C\|F\|_{L^{6/5}}\|\nabla u\|_{L^2},$$

which, together with (2.8), yields

$$\|\nabla u\|_{L^2} \le C\|F\|_{L^{6/5}}, \tag{2.13}$$

due to

$$2\int |d|^2 \mathrm{d}x = \int |\nabla u|^2 \mathrm{d}x. \tag{2.14}$$

Furthermore, it follows from $(2.9)_1$ that

$$P = -(-\Delta)^{-1}\mathrm{div} F - (-\Delta)^{-1}\mathrm{div}\mathrm{div}(2\mu(\rho)d),$$

which, together with the Sobolev inequality and (2.14), gives

$$\|P\|_{L^2} \le \|(-\Delta)^{-1}\mathrm{div} F\|_{L^2} + \|2\mu(\rho)d\|_{L^2} \le C\|F\|_{L^{6/5}} + C\|\nabla u\|_{L^2}.$$

Combining this with (2.13) leads to (2.10).

Next, we rewrite $(2.9)_1$ as

$$-\Delta u + \nabla\left(\frac{P}{\mu(\rho)}\right) = \frac{F}{\mu(\rho)} + \frac{2d\cdot\nabla\mu(\rho)}{\mu(\rho)} - \frac{P\nabla\mu(\rho)}{\mu(\rho)^2}. \tag{2.15}$$

Applying the standard L^p-estimates to the Stokes system (2.15) $(2.9)_2$ $(2.9)_3$ yields that, for $r\in[2q/(q+2),q]$,

$$\begin{aligned}
\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} &\le \|\nabla^2 u\|_{L^r} + C\left\|\nabla\left(\frac{P}{\mu(\rho)}\right)\right\|_{L^r} + C\left\|\frac{P\nabla\mu(\rho)}{\mu(\rho)^2}\right\|_{L^r}\\
&\le C\|F\|_{L^r} + C\|2d\cdot\nabla\mu(\rho)\|_{L^r} + C\|P\nabla\mu(\rho)\|_{L^r}\\
&\le C\|F\|_{L^r} + C\|\nabla\mu(\rho)\|_{L^q}\|\nabla u\|_{L^2}^{\frac{2r(q-3)}{q(5r-6)}}\|\nabla^2 u\|_{L^r}^{1-\frac{2r(q-3)}{q(5r-6)}}\\
&\quad + C\|\nabla\mu(\rho)\|_{L^q}\|P\|_{L^2}^{\frac{2r(q-3)}{q(5r-6)}}\|\nabla P\|_{L^r}^{1-\frac{2r(q-3)}{q(5r-6)}}\\
&\le C\|F\|_{L^r} + C\|\nabla\mu(\rho)\|_{L^q}^{\frac{q(5r-6)}{2r(q-3)}}(\|\nabla u\|_{L^2} + \|P\|_{L^2})\\
&\quad + \frac{1}{2}\left(\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r}\right),
\end{aligned}$$

where in the third inequality we have used Lemma 2.3. Combining this with (2.10) yields (2.11).

Finally, we will prove (2.12). Multiplying $(2.9)_1$ by u and integrating by parts leads to

$$4\int\mu(\rho)|d|^2\mathrm{d}x = -2\int g\cdot\nabla u\mathrm{d}x \le \underline{\mu}\|\nabla u\|_{L^2}^2 + C\|g\|_{L^2}^2,$$

which, together with (2.14), gives

$$\|\nabla u\|_{L^2}\le C\|g\|_{L^2}. \tag{2.16}$$

It follows from $(2.9)_1$ that

$$P = -(-\Delta)^{-1}\mathrm{divdiv}g - (-\Delta)^{-1}\mathrm{divdiv}(2\mu(\rho)d),$$

which implies that, for any $p\in[2,\tilde r]$,

$$\|P\|_{L^p}\le C(p)\|\nabla u\|_{L^p} + C(p)\|g\|_{L^p}. \tag{2.17}$$

In particular, this, combined with (2.16), shows that

$$\|P\|_{L^2} + \|\nabla u\|_{L^2}\le C\|g\|_{L^2}. \tag{2.18}$$

Next, we rewrite $(2.9)_1$ as

$$-\Delta u + \nabla\left(\frac{P}{\mu(\rho)}\right) = \mathrm{div}\left(\frac{g}{\mu(\rho)}\right) + \tilde G, \tag{2.19}$$

where

$$\tilde G \triangleq \frac{g\cdot\nabla\mu(\rho)}{\mu(\rho)^2} + \frac{2d\cdot\nabla\mu(\rho)}{\mu(\rho)} - \frac{P\nabla\mu(\rho)}{\mu(\rho)^2}.$$

Holder's inequality thus gives

$$\begin{aligned}\left\|\frac{g\cdot\nabla\mu(\rho)}{\mu(\rho)^2}\right\|_{L^{\frac{3\tilde r}{3+\tilde r}}}&\le C\|\nabla\mu(\rho)\|_{L^q}\|g\|_{L^2}^{\frac{2\tilde r(q-3)}{3q(\tilde r-2)}}\|g\|_{L^{\tilde r}}^{1-\frac{2\tilde r(q-3)}{3q(\tilde r-2)}}\\&\le\varepsilon\|g\|_{L^{\tilde r}}+C(\varepsilon)\|\nabla\mu(\rho)\|_{L^q}^{\frac{3q(\tilde r-2)}{2\tilde r(q-3)}}\|g\|_{L^2}.\end{aligned}$$

Applying similar arguments to the other terms of $\tilde G$, we arrive at

$$\begin{aligned}\|\tilde G\|_{L^{\frac{3\tilde r}{3+\tilde r}}}&\le\varepsilon(\|g\|_{L^{\tilde r}}+\|\nabla u\|_{L^{\tilde r}}+\|P\|_{L^{\tilde r}})\\&\quad+C(\varepsilon)\|\nabla\mu(\rho)\|_{L^q}^{\frac{3q(\tilde r-2)}{2\tilde r(q-3)}}(\|g\|_{L^2}+\|\nabla u\|_{L^2}+\|P\|_{L^2}).\end{aligned}\tag{2.20}$$

Using (2.19) and $(2.9)_3$, we have

$$\begin{aligned}\|\nabla u\|_{L^{\tilde r}}&\le C\|\nabla\times u\|_{L^{\tilde r}}\\&=C\left\|(-\Delta)^{-1}\nabla\times\operatorname{div}\left(g(\mu(\rho))^{-1}\right)+(-\Delta)^{-1}\nabla\times\tilde G\right\|_{L^{\tilde r}}\\&\le C\|g\|_{L^{\tilde r}}+C\left\|\tilde G\right\|_{L^{\frac{3\tilde r}{3+\tilde r}}},\end{aligned}$$

which, together with (2.17), yields

$$\|\nabla u\|_{L^{\tilde r}}+\|P\|_{L^{\tilde r}}\le C\|g\|_{L^{\tilde r}}+C\left\|\tilde G\right\|_{L^{\frac{3\tilde r}{3+\tilde r}}}.$$

Combining this, (2.20), and (2.18) gives (2.12). The proof of Lemma 2.4 is finished. □

3. A Priori Estimates

In this section, we will establish some necessary a priori bounds of local strong solutions (ρ,u,P) to the Cauchy problem (1.1)–(1.4) whose existence is guaranteed by Lemma 2.1. Thus, let $T>0$ be a fixed time and (ρ,u,P) be the smooth solution to (1.1)–(1.4) on $\mathbb{R}^3\times(0,T]$ with smooth initial data (ρ_0,u_0) satisfying (1.8).

We have the following key a priori estimates on (ρ,u,P):

Proposition 3.1. *There exists some positive constant ε_0 depending only on q, β, $\bar\rho$, $\underline{\mu}$, $\bar\mu$, $\|\rho_0\|_{L^{3/2}}$, and M such that if (ρ,u,P) is a smooth solution of (1.1)–(1.4) on $\mathbb{R}^3\times(0,T]$ satisfying*

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|_{L^q}\le 4M,\quad\int_0^T\|\nabla u\|_{L^2}^4\mathrm{d}t\le 2\|u_0\|_{\dot H^\beta}^2,\tag{3.1}$$

the following estimates hold:

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|_{L^q}\le 2M,\quad\int_0^T\|\nabla u\|_{L^2}^4\mathrm{d}t\le\|u_0\|_{\dot H^\beta}^2,\tag{3.2}$$

provided that $\|u_0\|_{\dot H^\beta}\le\varepsilon_0$.

Before proving Proposition 3.1, we establish some necessary a priori estimates, see Lemmas 3.2–3.5.

We start with the following time-weighted estimates on the $L^\infty(0, \min\{1, T\}; L^2)$-norm of the gradient of velocity:

Lemma 3.2. *Let (ρ, u, P) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a generic positive constant C depending only on q, β, $\bar{\rho}$, $\underline{\mu}$, $\bar{\mu}$, $\|\rho_0\|_{L^{3/2}}$, and M such that*

$$\sup_{t\in[0,\zeta(T)]}\left(t^{1-\beta}\|\nabla u\|_{L^2}^2\right)+\int_0^{\zeta(T)} t^{1-\beta}\|\rho^{1/2}u_t\|_{L^2}^2\mathrm{d}t \le C\|u_0\|_{\dot{H}^\beta}^2, \tag{3.3}$$

with $\zeta(t)=\min\{1,t\}$.

Proof. First, standard arguments ([26]) imply that

$$0\le\rho\le\bar{\rho}, \qquad \|\rho\|_{L^{3/2}}=\|\rho_0\|_{L^{3/2}}. \tag{3.4}$$

Next, for fixed (ρ, u) with $\rho \ge 0$ and $\mathrm{div}u = 0$, we consider the following linear Cauchy problem for $(w, \tilde{P})$:

$$\begin{cases} \rho w_t+\rho u\cdot\nabla w-\mathrm{div}\left(\mu(\rho)\left[\nabla w+(\nabla w)^{\mathrm{T}}\right]\right)+\nabla\tilde{P}=0, & x\in\mathbb{R}^3,\\ \mathrm{div}w=0, & x\in\mathbb{R}^3,\\ w(x,0)=w_0, & x\in\mathbb{R}^3. \end{cases} \tag{3.5}$$

It follows from Lemma 2.4, $(3.5)_1$, (3.1), (3.4), and the Garliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla w\|_{H^1}+\|\tilde{P}\|_{H^1} &\le C\left(\|\rho w_t+\rho u\cdot\nabla w\|_{L^2}+\|\rho w_t+\rho u\cdot\nabla w\|_{L^{6/5}}\right)\\ &\le C(\bar{\rho}^{1/2}+\|\rho\|_{L^{3/2}}^{1/2})\left(\|\rho^{1/2}w_t\|_{L^2}+\bar{\rho}^{1/2}\|u\cdot\nabla w\|_{L^2}\right)\\ &\le C\|\rho^{1/2}w_t\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{1/2}\|\nabla^2 w\|_{L^2}^{1/2}\\ &\le C\|\rho^{1/2}w_t\|_{L^2}+C\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}+\frac{1}{2}\|\nabla^2 w\|_{L^2},\end{aligned}$$

which directly yields that

$$\begin{aligned}&\|\nabla w\|_{H^1}+\|\tilde{P}\|_{H^1}+\|\rho w_t+\rho u\cdot\nabla w\|_{L^{6/5}\cap L^2}\\ &\le C\|\rho^{1/2}w_t\|_{L^2}+C\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}.\end{aligned} \tag{3.6}$$

Multiplying $(3.5)_1$ by w_t and integrating the resulting equality by parts leads to

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int\mu(\rho)\left|\nabla w+(\nabla w)^{\mathrm{T}}\right|^2\mathrm{d}x+\int\rho|w_t|^2\mathrm{d}x\\ &=-\int\rho u\cdot\nabla w\cdot w_t\mathrm{d}x+\frac{1}{4}\int\mu(\rho)u\cdot\nabla\left|\nabla w+(\nabla w)^{\mathrm{T}}\right|^2\mathrm{d}x\\ &\le\bar{\rho}^{1/2}\|\rho^{1/2}w_t\|_{L^2}\|u\|_{L^6}\|\nabla w\|_{L^3}+C\bar{\mu}\|u\|_{L^6}\|\nabla w\|_{L^3}\|\nabla^2 w\|_{L^2}\\ &\le C\|\rho^{1/2}w_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{1/2}\|\nabla^2 w\|_{L^2}^{1/2}+C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{1/2}\|\nabla^2 w\|_{L^2}^{3/2}\\ &\le\frac{3}{4}\|\rho^{1/2}w_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla w\|_{L^2}^2,\end{aligned} \tag{3.7}$$

where in the last inequality one has used (3.6). This combined with Grönwall's inequality and (3.1) yields

$$\sup_{t\in[0,\zeta(T)]}\int|\nabla w|^2\mathrm{d}x+\int_0^{\zeta(T)}\int\rho|w_t|^2\mathrm{d}x\mathrm{d}t\le C\|\nabla w_0\|_{L^2}^2. \tag{3.8}$$

Furthermore, multiplying (3.7) by t leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(t\int\mu(\rho)\left|\nabla w+(\nabla w)^{\mathrm{T}}\right|^2\mathrm{d}x\right)+t\int\rho|w_t|^2\mathrm{d}x\\&\le Ct\|\nabla w\|_{L^2}^2\|\nabla u\|_{L^2}^4+C\|\nabla w\|_{L^2}^2.\end{aligned}$$

Combining this with Grönwall's inequality and (3.1) shows that

$$\sup_{t\in[0,\zeta(T)]}t\int|\nabla w|^2\mathrm{d}x+\int_0^{\zeta(T)}t\int\rho|w_t|^2\mathrm{d}x\mathrm{d}t\le C\|w_0\|_{L^2}^2, \tag{3.9}$$

where one has used the simple fact that

$$\sup_{t\in[0,\zeta(T)]}\|\rho^{1/2}w\|_{L^2}^2+\int_0^{\zeta(T)}\|\nabla w\|_{L^2}^2\mathrm{d}t\le C\|w_0\|_{L^2}^2,$$

which can be obtained by multiplying $(3.5)_1$ by w and integrating by parts.

Hence, the standard Stein-Weiss interpolation arguments (see [5, Theorem 5.4.1]) together with (3.8) and (3.9) imply that, for any $\theta\in[\beta,1]$,

$$\sup_{t\in[0,\zeta(T)]}t^{1-\theta}\int|\nabla w|^2\mathrm{d}x+\int_0^{\zeta(T)}t^{1-\theta}\int\rho|w_t|^2\mathrm{d}x\mathrm{d}t\le C(\theta)\|w_0\|_{\dot H^\theta}^2. \tag{3.10}$$

Finally, taking $w_0=u_0$, the uniqueness of strong solutions to the linear problem (3.5) implies that $w\equiv u$. The estimate (3.3) thus follows from (3.10). The proof of Lemma 3.2 is finished. □

As an application of Lemma 3.2, we have the following time-weighted estimates on $\|\rho^{1/2}u_t\|_{L^2}^2$ for small time:

Lemma 3.3. *Let (ρ,u,P) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a generic positive constant C depending only on q, β, $\bar\rho$, $\underline{\mu}$, $\bar\mu$, $\|\rho_0\|_{L^{3/2}}$, and M such that*

$$\sup_{t\in[0,\zeta(T)]}\left(t^{2-\beta}\|\rho^{1/2}u_t\|_{L^2}^2\right)+\int_0^{\zeta(T)}t^{2-\beta}\|\nabla u_t\|_{L^2}^2\mathrm{d}t\le C\|u_0\|_{\dot H^\beta}^2. \tag{3.11}$$

Proof. First, operating ∂_t to $(1.1)_2$ yields that

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t-\operatorname{div}(2\mu(\rho)d_t)+\nabla P_t\\&=-\rho_t u_t-(\rho u)_t\cdot\nabla u+\operatorname{div}(2(\mu(\rho))_t d).\end{aligned} \tag{3.12}$$

Multiplying the above equality by u_t, we obtain after using integration by parts and (1.1)$_1$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \int 2\mu(\rho)|d_t|^2\mathrm{d}x \\
&= -2\int \rho u\cdot\nabla u_t\cdot u_t\mathrm{d}x - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\mathrm{d}x \\
&\quad - \int \rho u_t\cdot\nabla u\cdot u_t\mathrm{d}x + 2\int (u\cdot\nabla\mu(\rho))\, d\cdot\nabla u_t\mathrm{d}x \triangleq \sum_{i=1}^{4} J_i.
\end{aligned} \tag{3.13}$$

Now, we will use the Gagliardo–Nirenberg inequality, (3.1), and (3.4) to estimate each term on the right hand of (3.13) as follows:

$$\begin{aligned}
|J_1|+|J_3| &\le C\|\rho^{1/2}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\|u\|_{L^6} + C\|\rho^{1/2}u_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6} \\
&\le C\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{3/2}\|\nabla u\|_{L^2} \\
&\le \frac{1}{4}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
|J_2| &= \left|\int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\mathrm{d}x\right| \\
&\le C\int \rho|u||u_t|\left(|\nabla u|^2 + |u||\nabla^2 u|\right)\mathrm{d}x + \int \rho|u|^2|\nabla u||\nabla u_t|\mathrm{d}x \\
&\le C\|u\|_{L^6}\|u_t\|_{L^6}\left(\|\nabla u\|_{L^3}^2 + \|u\|_{L^6}\|\nabla^2 u\|_{L^2}\right) + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\le C\|\nabla u_t\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}^2 \\
&\le \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^4,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
|J_4| &\le C\|\nabla\mu(\rho)\|_{L^q}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^{\frac{2q}{q-2}}} \\
&\le C(q,M)\|u\|_{L^6}^{1/2}\|\nabla u\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{q-3}{q}}\|\nabla^2 u\|_{L^2}^{\frac{3}{q}} \\
&\le \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^4.
\end{aligned} \tag{3.16}$$

Substituting (3.14)–(3.16) into (3.13) gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \underline{\mu}\int |\nabla u_t|^2\mathrm{d}x \\
&\le C\left(\|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right)\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^4 \\
&\le C\|\rho^{1/2}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 + C\|\rho^{1/2}u_t\|_{L^2}^3\|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}^{10} + C\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.17}$$

where in the last inequality one has used

$$\begin{aligned}&\|\nabla u\|_{H^1}+\|P\|_{H^1}+\|\rho(u_t+u\cdot\nabla u)\|_{L^{6/5}\cap L^2}\\&\le C\left(\|\rho^{1/2}u_t\|_{L^2}+\|\nabla u\|_{L^2}^3\right),\end{aligned}\tag{3.18}$$

which can be obtained by taking $w\equiv u$ in (3.6). It thus follows from (3.17) and (3.3) that, for $t\in(0,\zeta(T)]$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u_t|^2\mathrm{d}x+\underline{\mu}\int|\nabla u_t|^2\mathrm{d}x\\&\le C\|\rho^{1/2}u_t\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4+\|\rho^{1/2}u_t\|_{L^2}\|\nabla u\|_{L^2}\right)\\&\quad+Ct^{3(\beta-1)}\|\nabla u\|_{L^2}^4+C\|\nabla u\|_{L^2}^2.\end{aligned}\tag{3.19}$$

Since (3.3) implies

$$\begin{aligned}&\int_0^{\zeta(T)}\|\rho^{1/2}u_t\|_{L^2}\|\nabla u\|_{L^2}\mathrm{d}tt\\&\le C\sup_{0\le t\le\zeta(T)}\left(t^{\frac{1-\beta}{2}}\|\nabla u\|_{L^2}\right)\left(\int_0^{\zeta(T)}t^{1-\beta}\|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}t\right)^{1/2}\left(\int_0^{\zeta(T)}t^{2\beta-2}\mathrm{d}t\right)^{1/2}\\&\le C\|u_0\|_{\dot H^\beta}^2,\end{aligned}$$

we multiply (3.19) by $t^{2-\beta}$ and use Grönwall's inequality, (3.1), and (3.3) to obtain (3.11). The proof of Lemma 3.3 is finished. □

Next, we will prove the following exponential decay-in-time estimates on the solutions for large time, which plays a crucial role in our analysis:

Lemma 3.4. *Let (ρ, u, P) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then for*

$$\sigma\triangleq 3\underline{\mu}/(4\|\rho_0\|_{L^{3/2}}),\tag{3.20}$$

there exists a generic positive constant C depending only on q, β, $\bar\rho$, $\underline{\mu}$, $\bar\mu$, $\|\rho_0\|_{L^{3/2}}$, and M such that

$$\sup_{t\in[0,T]}e^{\sigma t}\|\rho^{1/2}u\|_{L^2}^2+\int_0^T e^{\sigma t}\int|\nabla u|^2\mathrm{d}x\mathrm{d}t\le C\|u_0\|_{\dot H^\beta}^2,\tag{3.21}$$

$$\sup_{t\in[\zeta(T),T]}e^{\sigma t}\int|\nabla u|^2\mathrm{d}x+\int_{\zeta(T)}^T e^{\sigma t}\int\rho|u_t|^2\mathrm{d}x\mathrm{d}t\le C\|u_0\|_{\dot H^\beta}^2,\tag{3.22}$$

$$\sup_{t\in[\zeta(T),T]}e^{\sigma t}\int\rho|u_t|^2\mathrm{d}x+\int_{\zeta(T)}^T e^{\sigma t}\int|\nabla u_t|^2\mathrm{d}x\mathrm{d}t\le C\|u_0\|_{\dot H^\beta}^2,\tag{3.23}$$

and

$$\sup_{t\in[\zeta(T),T]}e^{\sigma t}\left(\|\nabla u\|_{H^1}^2+\|P\|_{H^1}^2\right)\le C\|u_0\|_{\dot H^\beta}^2.\tag{3.24}$$

Proof. First, multiplying $(1.1)_2$ by u and integrating by parts leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{1/2}u\|_{L^2}^2+\int 2\mu(\rho)|d|^2\mathrm{d}x=0. \tag{3.25}$$

It follows from the Sobolev inequality [14, (II.3.11)], (3.4), and (2.14) that

$$\|\rho^{1/2}u\|_{L^2}^2\le\|\rho\|_{L^{3/2}}\|u\|_{L^6}^2\le\frac{4}{3}\|\rho_0\|_{L^{3/2}}\|\nabla u\|_{L^2}^2\le\sigma^{-1}\int 2\mu(\rho)|d|^2\mathrm{d}x, \tag{3.26}$$

with σ as in (3.20). Putting (3.26) into (3.25) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{1/2}u\|_{L^2}^2+\sigma\|\rho^{1/2}u\|_{L^2}^2+\int 2\mu(\rho)|d|^2\mathrm{d}x\le 0,$$

which together with Grönwall's inequality gives

$$\begin{aligned}&\sup_{t\in[0,T]}e^{\sigma t}\|\rho^{1/2}u\|_{L^2}^2+\int_0^T e^{\sigma t}\int|\nabla u|^2\mathrm{d}x\mathrm{d}t\\&\le C\|\rho_0^{1/2}u_0\|_{L^2}^2\le C\|\rho_0\|_{L^{\frac{3}{2\beta}}}\|u_0\|_{L^{\frac{6}{3-2\beta}}}^2\le C\|u_0\|_{\dot H^\beta}^2,\end{aligned} \tag{3.27}$$

due to $\beta\in(1/2,1]$.

Next, similar to (3.7), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int 2\mu(\rho)|d|^2\mathrm{d}x+\int\rho|u_t|^2\mathrm{d}x\le C\|\nabla u\|_{L^2}^4\|\nabla u\|_{L^2}^2, \tag{3.28}$$

which combined with Grönwall's inequality, (3.27), (3.3), and (3.1) gives (3.22).

Furthermore, multiplying (3.17) by $e^{\sigma t}$, we obtain (3.23) after using Grönwall's inequality, (3.11), (3.1), (3.21), and (3.22).

Finally, it follows from (3.18), (3.22), and (3.23) that (3.24) holds. The proof of Lemma 3.4 is completed. □

We will use Lemmas 3.2–3.4 to prove the following time-independent bound on the $L^1(0,T;L^\infty)$-norm of ∇u which is important for obtaining the uniform one (with respect to time) on the $L^\infty(0,T;L^q)$-norm of the gradient of $\mu(\rho)$:

Lemma 3.5. *Let (ρ,u,P) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a generic positive constant C depending only on q, β, $\bar\rho$, $\underline{\mu}$, $\bar\mu$, $\|\rho_0\|_{L^{3/2}}$, and M such that*

$$\int_0^T\|\nabla u\|_{L^\infty}\mathrm{d}t\le C\|u_0\|_{\dot H^\beta}. \tag{3.29}$$

Proof. First, it follows from the Gagliardo–Nirenberg inequality that for any $p\in[2,\min\{6,q\}]\cap[2,6)$,

$$\begin{aligned}&\|\rho u_t+\rho u\cdot\nabla u\|_{L^p}\\&\le C\|\rho^{1/2}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\rho^{1/2}u_t\|_{L^6}^{\frac{3p-6}{2p}}+C\|u\|_{L^6}\|\nabla u\|_{L^{\frac{6p}{6-p}}}\\&\le C\|\rho^{1/2}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{p}{5p-6}}\|\nabla^2 u\|_{L^p}^{\frac{4p-6}{5p-6}}.\end{aligned} \tag{3.30}$$

Moreover, the Gagliardo–Nirenberg inequality also gives

$$\begin{aligned}&\|\rho u_t+\rho u\cdot\nabla u\|_{L^6}\\&\le C\|u_t\|_{L^6}+C\|u\|_{L^6}\|\nabla u\|_{L^\infty}\\&\le C\|\nabla u_t\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{1/4}\|\nabla^2 u\|_{L^6}^{3/4},\end{aligned}$$

which implies (3.30) holds for all $p\in[2,\min\{6,q\}]$. Combining (3.30), (2.11), and (3.18) yields that for any $p\in[2,\min\{6,q\}]$,

$$\begin{aligned}\|\nabla^2 u\|_{L^p}+\|\nabla P\|_{L^p}&\le C\|\rho u_t+\rho u\cdot\nabla u\|_{L^{6/5}\cap L^p}\\&\le C\|\rho^{1/2}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}}+C\|\nabla u\|_{L^2}^{\frac{6p-6}{p}}\\&\quad+\frac12\|\nabla^2 u\|_{L^p}+C\|\rho^{1/2}u_t\|_{L^2}+C\|\nabla u\|_{L^2}^3.\end{aligned}\tag{3.31}$$

Then, setting

$$r\triangleq\frac12\min\left\{q+3,\frac{3(5-2\beta)}{3-2\beta}\right\}\in\left(3,\min\left\{q,\frac{6}{3-2\beta}\right\}\right),\tag{3.32}$$

one derives from the Sobolev inequality and (3.31) that

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\le C\|\nabla u\|_{L^2}+C\|\nabla^2 u\|_{L^r}\\&\le C\|\nabla u\|_{L^2}+C\|\rho^{1/2}u_t\|_{L^2}+C\|\rho^{1/2}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}\\&\quad+C\|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}.\end{aligned}\tag{3.33}$$

Finally, on the one hand, it follows from (3.3) and (3.11) that for $t\in(0,\zeta(T)]$,

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\le C\|u_0\|_{\dot H^\beta}t^{\frac{\beta-2}{2}}+C\|u_0\|_{\dot H^\beta}^{\frac{6-r}{2r}}t^{\frac{\beta-2}{2}}\left(t^{2-\beta}\|\nabla u_t\|_{L^2}^2\right)^{\frac{3r-6}{4r}}\\&\quad+C\|u_0\|_{\dot H^\beta}^2t^{2r(\beta-1)/3}+C\|\nabla u\|_{L^2}^4,\end{aligned}$$

which, together with (3.1), (3.11), and (3.32), gives

$$\begin{aligned}&\int_0^{\zeta(T)}\|\nabla u\|_{L^\infty}\mathrm{d}t\\&\le C\|u_0\|_{\dot H^\beta}+C\|u_0\|_{\dot H^\beta}^{\frac{6-r}{2r}}\left(\int_0^1 t^{\frac{2(\beta-2)r}{r+6}}\mathrm{d}t\right)^{\frac{r+6}{4r}}\left(\int_0^1 t^{2-\beta}\|\nabla u_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{3r-6}{4r}}\\&\le C\|u_0\|_{\dot H^\beta}.\end{aligned}\tag{3.34}$$

On the other hand, using (3.33), (3.22), and (3.23), we obtain that for $t\in[\zeta(T),T]$,

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\le C\|\rho^{1/2}u_t\|_{L^2}+C\|\nabla u_t\|_{L^2}+C\|\nabla u\|_{L^2}+C\|\nabla u\|_{L^2}^6\\&\le C\|u_0\|_{\dot H^\beta}e^{-\sigma t/2}+C\|\nabla u_t\|_{L^2},\end{aligned}$$

and thus

$$\int_{\zeta(T)}^{T} \|\nabla u\|_{L^\infty} \mathrm{d}t \le C\|u_0\|_{\dot{H}^\beta} + C\left(\int_{\zeta(T)}^{T} e^{-\sigma t}\mathrm{d}t\right)^{1/2}\left(\int_{\zeta(T)}^{T} e^{\sigma t}\|\nabla u_t\|_{L^2}^2\mathrm{d}t\right)^{1/2} \\ \le C\|u_0\|_{\dot{H}^\beta}. \tag{3.35}$$

Combining this with (3.34) gives (3.29) and finishes the proof of Lemma 3.5. □

With Lemmas 3.2–3.5 at hand, we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Since $\mu(\rho)$ satisfies

$$(\mu(\rho))_t + u \cdot \nabla\mu(\rho) = 0,$$

standard calculations show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu(\rho)\|_{L^q} \le q\|\nabla u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}, \tag{3.36}$$

which together with Grönwall's inequality and (3.29) yields

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|_{L^q} \le \|\nabla\mu(\rho_0)\|_{L^q}\exp\left\{q\int_0^T\|\nabla u\|_{L^\infty}\mathrm{d}t\right\} \\ \le \|\nabla\mu(\rho_0)\|_{L^q}\exp\left\{C\|u_0\|_{\dot{H}^\beta}\right\} \\ \le 2\|\nabla\mu(\rho_0)\|_{L^q}, \tag{3.37}$$

provided that

$$\|u_0\|_{\dot{H}^\beta} \le \varepsilon_1 \triangleq C^{-1}\ln 2. \tag{3.38}$$

Moreover, it follows from (3.3) and (3.22) that

$$\int_0^T \|\nabla u\|_{L^2}^4\mathrm{d}t \le \sup_{t\in[0,\zeta(T)]}\left(t^{1-\beta}\|\nabla u\|_{L^2}^2\right)^2\int_0^{\zeta(T)} t^{2\beta-2}\mathrm{d}t \\ + \sup_{t\in[\zeta(T),T]}\left(e^{\sigma t}\|\nabla u\|_{L^2}^2\right)^2\int_{\zeta(T)}^T e^{-2\sigma t}\mathrm{d}t \\ \le C\|u_0\|_{\dot{H}^\beta}^4 \le \|u_0\|_{\dot{H}^\beta}^2, \tag{3.39}$$

provided that

$$\|u_0\|_{\dot{H}^\beta} \le \varepsilon_2 \triangleq C^{-1/2}. \tag{3.40}$$

Choosing $\varepsilon_0 \triangleq \min\{1, \varepsilon_1, \varepsilon_2\}$, we directly obtain (3.2) from (3.37)–(3.40). The proof of Proposition 3.1 is finished. □

The following Lemma 3.6 is necessary for further estimates on the higher-order derivatives of the strong solution (ρ, u, P):

Lemma 3.6. *Let (ρ, u, P) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant C depending only on $q, \beta, \bar\rho, \underline{\mu}, \bar\mu, M, \|\rho_0\|_{L^{3/2}}$, and $\|\nabla u_0\|_{L^2}$ such that for $p_0 \triangleq \min\{6, q\}$,*

$$\begin{aligned}&\sup_{t\in[0,T]} e^{\sigma t}\left(\|\nabla u\|_{L^2}^2 + \zeta\|\nabla u\|_{H^1}^2 + \zeta\|P\|_{H^1}^2\right) + \int_0^T \zeta e^{\sigma t}\|\nabla u_t\|_{L^2}^2 \mathrm{d}t \\ &+ \int_0^T e^{\sigma t}\left(\|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 + \zeta\|\nabla u\|_{W^{1,p_0}}^2 + \zeta\|P\|_{W^{1,p_0}}^2\right)\mathrm{d}t \le C.\end{aligned} \tag{3.41}$$

Proof. First, multiplying (3.28) by $e^{\sigma t}$, we get after using Grönwall's inequality, (3.21), and (3.2) that

$$\sup_{t\in[0,T]} e^{\sigma t}\|\nabla u\|_{L^2}^2 + \int_0^T e^{\sigma t}\|\rho^{1/2}u_t\|_{L^2}^2 \mathrm{d}t \le C. \tag{3.42}$$

Combining this with (3.17) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \underline{\mu}\int |\nabla u_t|^2\mathrm{d}x \le C\|\rho^{1/2}u_t\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2,$$

which along with Grönwall's inequality, (3.42), and (3.21) implies that

$$\sup_{t\in[0,T]} \zeta e^{\sigma t}\|\rho^{1/2}u_t\|_{L^2}^2 + \int_0^T \zeta e^{\sigma t}\|\nabla u_t\|_{L^2}^2 \mathrm{d}t \le C. \tag{3.43}$$

Combining this, (3.18), and (3.42) gives

$$\sup_{t\in[0,T]} \zeta e^{\sigma t}\left(\|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2\right) + \int_0^T e^{\sigma t}\left(\|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2\right)\mathrm{d}t \le C. \tag{3.44}$$

Finally, it follows from (3.18), (3.31), (3.42), and (3.4) that, for $p_0 \triangleq \min\{6, q\}$,

$$\|\nabla u\|_{H^1\cap W^{1,p_0}} + \|P\|_{H^1\cap W^{1,p_0}} \le C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}, \tag{3.45}$$

which, together with (3.43) and (3.21), implies

$$\int_0^T \zeta e^{\sigma t}\left(\|\nabla u\|_{W^{1,p_0}}^2 + \|P\|_{W^{1,p_0}}^2\right)\mathrm{d}t \le C.$$

This combined with (3.42)–(3.44) gives (3.41) and completes the proof of Lemma 3.6. □

The following Proposition 3.7 is concerned with the estimates on the higher-order derivatives of the strong solution (ρ, u, P) which in particular imply the continuity in time of both $\nabla^2 u$ and ∇P in the $L^2 \cap L^p$-norm:

Proposition 3.7. *Let* (ρ, u, P) *be a smooth solution to* (1.1)–(1.4) *satisfying* (3.1). *Then there exists a positive constant* C *depending only on* $q, \beta, \bar{\rho}, \underline{\mu}, \bar{\mu}, \|\rho_0\|_{L^{3/2}}, M$, $\|\nabla u_0\|_{L^2}$, *and* $\|\nabla \rho_0\|_{L^2}$ *such that for* $p_0 \triangleq \min\{6, q\}$ *and* $q_0 \triangleq 4q/(q-3)$,

$$\begin{aligned} &\sup_{t\in[0,T]} \zeta^{q_0} e^{\sigma t} \left(\|\nabla u\|_{W^{1,p_0}}^2 + \|P\|_{W^{1,p_0}}^2 + \|\nabla u_t\|_{L^2}^2 \right) \\ &+ \int_0^T \zeta^{q_0+1} e^{\sigma t} \left(\|(\rho u_t)_t\|_{L^2}^2 + \|\nabla u_t\|_{L^{p_0}}^2 + \|P_t\|_{L^2\cap L^{p_0}}^2 \right) dt \le C. \end{aligned} \tag{3.46}$$

Proof. First, in a similar way to (3.36) and (3.37), we have

$$\sup_{0\le t\le T} \|\nabla \rho\|_{L^2} \le 2\|\nabla \rho_0\|_{L^2}, \tag{3.47}$$

which together with the Sobolev inequality and (3.42) gives

$$\begin{aligned} \|\rho_t\|_{L^2\cap L^{3/2}} &= \|u \cdot \nabla \rho\|_{L^2\cap L^{3/2}} \\ &\le C\|\nabla \rho\|_{L^2} \|\nabla u\|_{L^2}^{1/2} \|\nabla u\|_{H^1}^{1/2} \le C\|\nabla u\|_{H^1}^{1/2}. \end{aligned} \tag{3.48}$$

Next, it follows from (3.12) that u_t satisfies

$$\begin{cases} -\mathrm{div}(2\mu(\rho)d_t) + \nabla P_t = \tilde{F} + \mathrm{div} g, \\ \mathrm{div} u_t = 0, \end{cases}$$

with

$$\tilde{F} \triangleq -\rho u_{tt} - \rho u \cdot \nabla u_t - \rho_t u_t - (\rho u)_t \cdot \nabla u, \quad g \triangleq -2u \cdot \nabla \mu(\rho) d.$$

Hence, one can deduce from Lemma 2.4 and the Sobolev inequality that

$$\|\nabla u_t\|_{L^2\cap L^{p_0}} + \|P_t\|_{L^2\cap L^{p_0}} \le C\|\tilde{F}\|_{L^{6/5}\cap L^{\frac{3p_0}{p_0+3}}} + C\|g\|_{L^2\cap L^{p_0}}. \tag{3.49}$$

Using (3.1), (3.4), (3.48), (3.41), and (3.45), we get by direct calculations that

$$\begin{aligned} &\|\tilde{F}\|_{L^{6/5}\cap L^{\frac{3p_0}{p_0+3}}} \\ &\le C\|\rho\|_{L^{3/2}\cap L^{\frac{3p_0}{6-p_0}}}^{1/2} \|\rho^{1/2} u_{tt}\|_{L^2} + C\|\rho\|_{L^3\cap L^{\frac{6p_0}{6-p_0}}} \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \\ &\quad + C\|\rho_t\|_{L^2\cap L^{3/2}} \left(\|u_t\|_{L^6\cap L^{\frac{6p_0}{6-p_0}}} + \|\nabla u\|_{H^1}^2 + \|\nabla u\|_{H^1}\|\nabla u\|_{W^{1,p_0}} \right) \\ &\quad + C\|\rho\|_{L^2\cap L^{p_0}} \|u_t\|_{L^6} \|\nabla u\|_{L^6} \\ &\le C\|\sqrt{\rho} u_{tt}\|_{L^2} + \varepsilon\|\nabla u_t\|_{L^{p_0}} + C(\varepsilon)\|\nabla u_t\|_{L^2}(1 + \|\nabla u\|_{H^1}^{3/2}) + C\|\nabla u\|_{H^1}^{5/2}, \end{aligned} \tag{3.50}$$

and that

$$\begin{aligned} \|g\|_{L^2\cap L^{p_0}} &\le C\|\nabla \mu(\rho)\|_{L^q} \|u\|_{L^6\cap L^\infty} \|\nabla u\|_{L^2\cap L^\infty} \\ &\le C\|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} + C\|\nabla u\|_{H^1}^2, \end{aligned} \tag{3.51}$$

where in the second inequality one has used the following simple fact that

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{H^1\cap W^{1,p_0}} \le C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}, \tag{3.52}$$

due to the Sobolev inequality and (3.45). Then, putting (3.50) and (3.51) into (3.49), we obtain after choosing ε suitably small that

$$\begin{aligned}&\|\nabla u_t\|_{L^2\cap L^{p_0}}+\|P_t\|_{L^2\cap L^{p_0}}\\&\le C\|\sqrt{\rho}u_{tt}\|_{L^2}+C\|\nabla u_t\|_{L^2}(1+\|\nabla u\|_{H^1}^2)+C\|\nabla u\|_{H^1}+C\|\nabla u\|_{H^1}^3.\end{aligned}\tag{3.53}$$

Now, multiplying (3.12) by u_{tt} and integrating the resulting equality by parts lead to

$$\begin{aligned}&\int\rho|u_{tt}|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int\mu(\rho)|d_t|^2\mathrm{d}x\\&=\int\operatorname{div}(\mu(\rho)u)|d_t|^2\mathrm{d}x-\int\rho(u\cdot\nabla u_t+u_t\cdot\nabla u)\cdot u_{tt}\mathrm{d}x-\int\rho_t u_t^j u_{tt}^j\mathrm{d}x\\&\quad-\int\rho_t u\cdot\nabla u^j u_{tt}^j\mathrm{d}x-2\int\partial_i(u^k\partial_k\mu(\rho)d_i^j)u_{tt}^j\mathrm{d}x\triangleq\sum_{i=1}^5 I_i.\end{aligned}\tag{3.54}$$

We will use (3.41), (3.53), and the Sobolev inequality to estimate each term on the righthand side of (3.54) as follows:

First, it follows from (3.1), (3.41), and (3.53) that

$$\begin{aligned}|I_1|&\le C\|u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}\|\nabla u_t\|_{L^2}^{\frac{2(p_0q-p_0-2q)}{q(p_0-2)}}\|\nabla u_t\|_{L^{p_0}}^{\frac{2p_0}{q(p_0-2)}}\\&\le\varepsilon\|\nabla u_t\|_{L^{p_0}}^2+C(\varepsilon)\|\nabla u\|_{H^1}^{\frac{q(p_0-2)}{p_0q-p_0-2q}}\|\nabla u_t\|_{L^2}^2\\&\le C\varepsilon\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C(\varepsilon)\left(1+\|\nabla u\|_{H^1}^{q_0}\right)\|\nabla u_t\|_{L^2}^2\\&\quad+C(\varepsilon)\|\nabla u\|_{H^1}^2+C(\varepsilon)\|\nabla u\|_{H^1}^6,\end{aligned}\tag{3.55}$$

where in the last inequality we have used

$$\frac{q(p_0-2)}{p_0q-p_0-2q}\in[1,q_0].$$

Next, Hölder's inequality gives

$$|I_2|\le\varepsilon\int\rho|u_{tt}|^2\mathrm{d}x+C(\varepsilon)\|\nabla u\|_{H^1}^2\|\nabla u_t\|_{L^2}^2.\tag{3.56}$$

Then, direct calculations show

$$\begin{aligned}I_3&=-\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_t|u_t|^2\mathrm{d}x+\int(\rho u^i)_t\partial_i u_t^j u_t^j\mathrm{d}x\\&\le-\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_t|u_t|^2\mathrm{d}x+C\|\rho\|_{L^6}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6}^2\\&\quad+C\|\rho_t\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^3}\|u_t\|_{L^6}\\&\le-\frac{\mathrm{d}}{\mathrm{d}t}\int\rho u\cdot\nabla u_t^j u_t^j\mathrm{d}x+C(\varepsilon)(1+\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}^4)\|\nabla u_t\|_{L^2}^2\\&\quad+\varepsilon\int\rho|u_{tt}|^2\mathrm{d}x+C(\varepsilon)\|\nabla u\|_{H^1}^2+C(\varepsilon)\|\nabla u\|_{H^1}^6,\end{aligned}\tag{3.57}$$

where in the last inequality one has used (3.48) and (3.53).

Next, it follows from $(1.1)_1$ and (3.48) that

$$
\begin{aligned}
I_4 =& -\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_t u\cdot\nabla u^j u_t^j \mathrm{d}x + \int (\rho u^i)_t \partial_i(u\cdot\nabla u^j u_t^j)\mathrm{d}x + \int \rho_t (u\cdot\nabla u^j)_t u_t^j \mathrm{d}x \\
=& -\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_t u\cdot\nabla u^j u_t^j \mathrm{d}x + \int \rho u_t^i (u\cdot\nabla u^j \partial_i u_t^j + \partial_i(u\cdot\nabla u^j) u_t^j)\mathrm{d}x \\
& + \int \rho_t u^i (u\cdot\nabla u^j \partial_i u_t^j + \partial_i(u\cdot\nabla u^j)u_t^j)\mathrm{d}x + \int \rho_t(u\cdot\nabla u^j)_t u_t^j \mathrm{d}x \\
\le& -\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_t u\cdot\nabla u^j u_t^j \mathrm{d}x + C\|u_t\|_{L^6}\|\nabla u\|_{H^1}^2(\|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}) \\
& + C\|\rho_t\|_{L^2}\|\nabla u\|_{H^1}^{1/2}\left(\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}\|\nabla u\|_{H^1\cap W^{1,p_0}} + \|u_t\|_{L^6}\|\nabla u\|_{H^1}^2\right) \\
& + C\|\rho_t\|_{L^2}\|u_t\|_{L^6}\left(\|u_t\|_{L^6}\|\nabla u\|_{H^1} + \|\nabla u_t\|_{L^3}\|\nabla u\|_{H^1}\right) \\
\le& -\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_t u\cdot\nabla u^j u_t^j \mathrm{d}x + C(\varepsilon)\left(1 + \|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^4\right)\|\nabla u_t\|_{L^2}^2 \\
& + \varepsilon\int \rho|u_{tt}|^2\mathrm{d}x + C(\varepsilon)\|\nabla u\|_{H^1}^2 + C(\varepsilon)\|\nabla u\|_{H^1}^6.
\end{aligned}
\tag{3.58}
$$

Finally, direct calculations lead to

$$
\begin{aligned}
I_5 =& -2\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_i(u^k\partial_k\mu(\rho)d_i^j)u_t^j\mathrm{d}x - 2\int \partial_i(u^k\mu(\rho)\partial_k d_i^j)_t u_t^j \mathrm{d}x \\
& + 2\int \partial_i(u^k\partial_k(\mu(\rho)d_i^j))_t u_t^j\mathrm{d}x \\
=& 2\frac{\mathrm{d}}{\mathrm{d}t}\int u^k\partial_k\mu(\rho)d_i^j\partial_i u_t^j\mathrm{d}x + 2\int (\mu(\rho)u^k\partial_k d_i^j)_t\partial_i u_t^j\mathrm{d}x \\
& - 2\int(\partial_i u^k\mu(\rho)d_i^j)_t\partial_k u_t^j\mathrm{d}x - 2\int u_t^k\partial_i(\mu(\rho)d_i^j)\partial_k u_t^j\mathrm{d}x \\
& - 2\int u^k(\partial_i(\mu(\rho)d_i^j))_t\partial_k u_t^j\mathrm{d}x \\
\triangleq& 2\frac{\mathrm{d}}{\mathrm{d}t}\int u^k\partial_k\mu(\rho)d_i^j\partial_i u_t^j\mathrm{d}x + \sum_{i=1}^4 I_{5,i}.
\end{aligned}
\tag{3.59}
$$

We estimate each $I_{5,i}(i=1,\cdots,4)$ as follows:

First, integration by parts gives

$$
\begin{aligned}
I_{5,1} &= 2\int(\mu(\rho)u^k)_t\partial_k d_i^j\partial_i u_t^j\mathrm{d}x + 2\int \mu(\rho)u^k\partial_k(d_i^j)_t\partial_i u_t^j\mathrm{d}x \\
&= -2\int u\cdot\nabla\mu(\rho)u^k\partial_k d_i^j\partial_i u_t^j\mathrm{d}x + 2\int\mu(\rho)u_t^k\partial_k d_i^j\partial_i u_t^j\mathrm{d}x \\
&\quad - \int \operatorname{div}(\mu(\rho)u)|d_t|^2\mathrm{d}x \\
&\le C\|u\|_{L^{6q/(q-3)}}^2\|\nabla\mu(\rho)\|_{L^q}\|\nabla^2 u\|_{L^3}\|\nabla u_t\|_{L^3}
\end{aligned}
\tag{3.60}
$$

$$
\begin{aligned}
&+C\|\nabla^2 u\|_{L^3}\|\nabla u_t\|_{L^2}^2+|I_1|\\
&\le C\varepsilon\|\rho^{1/2}u_{tt}\|_{L^2}^2+C(\varepsilon)(1+\|\nabla u\|_{H^1}^{q_0}+\|\nabla u_t\|_{L^2})\|\nabla u_t\|_{L^2}^2\\
&+C(\varepsilon)\|\nabla u\|_{H^1}^6+C(\varepsilon)\|\nabla u\|_{H^1}^2
\end{aligned}
$$

where in the last inequality we have used (3.41), (3.45), (3.53), and (3.55).

Then, it follows from (3.1) and (3.52) that

$$
\begin{aligned}
|I_{5,2}| &\le C\|u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}\|\nabla u\|_{L^{3q/(q-3)}}\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\quad + C\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2}^2\\
&\le C\|\nabla u\|_{H^1\cap W^{1,p_0}}\left(\|\nabla u\|_{H^1}^2\|\nabla u_t\|_{L^2}+C\|\nabla u_t\|_{L^2}^2\right)\\
&\le C\|\nabla u\|_{H^1}^4+C(1+\|\nabla u\|_{H^1}^2+\|\nabla u_t\|_{L^2})\|\nabla u_t\|_{L^2}^2.
\end{aligned}
\tag{3.61}
$$

Similarly, combining Hölder's inequality and (3.45) leads to

$$
\begin{aligned}
|I_{5,3}| &\le C\|u_t\|_{L^6}\|\nabla u_t\|_{L^2}(\|\nabla\mu(\rho)\|_{L^q}\|\nabla u\|_{L^{3q/(q-3)}}+\|\nabla^2 u\|_{L^3})\\
&\le C\|\nabla u_t\|_{L^2}^2(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}).
\end{aligned}
\tag{3.62}
$$

Finally, using $(1.1)_2$ and $(1.1)_3$, we obtain after integrating by parts that

$$
\begin{aligned}
I_{5,4} &= -2\int u^k\partial_j P_t\partial_k u_t^j\,\mathrm{d}x-2\int u^k(\rho u_t^j+\rho u\cdot\nabla u^j)_t\partial_k u_t^j\,\mathrm{d}x\\
&=2\int\partial_j u^k P_t\partial_k u_t^j\,\mathrm{d}x-2\int u^k\rho u_{tt}^j\partial_k u_t^j\,\mathrm{d}x\\
&\quad-2\int u^k(\rho_t u_t^j+(\rho u\cdot\nabla u^j)_t)\partial_k u_t^j\,\mathrm{d}x\\
&\le C\|\nabla u\|_{L^6}\|P_t\|_{L^3}\|\nabla u_t\|_{L^2}+C\|\sqrt{\rho}u_{tt}\|_{L^2}\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}\\
&\quad+C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\rho_t\|_{L^2}(\|u_t\|_{L^\infty}+\|\nabla u\|_{H^1}\|\nabla u\|_{L^\infty})\\
&\quad+C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}(\|\nabla u_t\|_{L^2}+\|u_t\|_{L^6})\|\nabla u\|_{H^1}\\
&\le C\varepsilon\int\rho|u_{tt}|^2\mathrm{d}x+C(\varepsilon)(1+\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}^4)\|\nabla u_t\|_{L^2}^2\\
&\quad+C(\varepsilon)\|\nabla u\|_{H^1}^2+C(\varepsilon)\|\nabla u\|_{H^1}^6,
\end{aligned}
\tag{3.63}
$$

where in the last inequality one has used (3.53) and (3.48).

Substituting (3.55)–(3.63) into (3.54), we get after choosing ε suitably small that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\mu(\rho)|d_t|^2\mathrm{d}x+\Psi'(t)+\frac12\int\rho|u_{tt}|^2\mathrm{d}x\\
&\le C(1+\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}^{q_0})\|\nabla u_t\|_{L^2}^2+C\|\nabla u\|_{H^1}^2+C\|\nabla u\|_{H^1}^6,
\end{aligned}
\tag{3.64}
$$

where

$$
\Psi(t)\triangleq-\int\rho u\cdot\nabla u_t^j u_t^j\,\mathrm{d}x-\int\rho_t u\cdot\nabla u^j u_t^j\,\mathrm{d}x+2\int u^k\partial_k\mu(\rho)d_i^j\partial_i u_t^j\,\mathrm{d}x
$$

satisfies

$$
\begin{aligned}
|\Psi(t)| \le & C\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1} + C\|\rho_t\|_{L^2}\|u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^6} \\
& + C\|\nabla\mu(\rho)\|_{L^q}\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}^2 \\
\le & \frac{1}{4}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2 + C\|\nabla u\|_{H^1}^4,
\end{aligned}
\tag{3.65}
$$

due to (3.1) and (3.48).

Then, multiplying (3.64) by $\zeta^{q_0}e^{\sigma t}$ and noticing that (3.41) gives

$$
\zeta^{q_0}(1+\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}^{q_0})\|\nabla u_t\|_{L^2}^2 \le C\zeta^{q_0+1}\|\nabla u_t\|_{L^2}^4 + C\zeta\|\nabla u_t\|_{L^2}^2,
$$

we get after using Grönwall's inequality, (3.65), (3.41), and (3.43) that

$$
\sup_{0\le t\le T}\zeta^{q_0}e^{\sigma t}\|\nabla u_t\|_{L^2}^2 + \int_0^T \zeta^{q_0}e^{\sigma t}\int \rho|u_{tt}|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{3.66}
$$

Furthermore, it follows from (3.48) and (3.41) that

$$
\|(\rho u_t)_t\|_{L^2}^2 \le C\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2\cap L^{p_0}}^2 + C\|\rho^{1/2}u_{tt}\|_{L^2}^2,
$$

which together with (3.66), (3.45), (3.53), and (3.41) gives (3.46) and thus completes the proof of Proposition 3.7. □

4. Proofs of Theorems 1.2 and 1.3

With all the a priori estimates in Section 3 at hand, we are now in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. First, by Lemma 2.1, there exists a $T_* > 0$ such that the Cauchy problem (1.1)–(1.4) has a unique local strong solution (ρ, u, P) on $\mathbb{R}^3 \times (0, T_*]$. It follows from (1.8) that there exists a $T_1 \in (0, T_*]$ such that (3.1) holds for $T = T_1$.

Next, set

$$
T^* \triangleq \sup\{T|(\rho, u, P) \text{ is a strong solution on } \mathbb{R}^3 \times (0, T] \text{ and (3.1) holds}\}. \tag{4.1}
$$

Then $T^* \ge T_1 > 0$. Hence, for any $0 < \tau < T \le T^*$ with T finite, one deduces from (3.41) and (3.46) that

$$
\nabla u,\ P \in C\left([\tau, T]; L^2\right) \cap C\left(\overline{\mathbb{R}^3} \times [\tau, T]\right), \tag{4.2}
$$

where one has used the standard embedding

$$
L^\infty(\tau, T; H^1 \cap W^{1,p_0}) \cap H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^2) \cap C(\overline{\mathbb{R}^3} \times [\tau, T]).
$$

Moreover, it follows from (3.1), (3.4), (3.47), and [26, Lemma 2.3] that

$$
\rho \in C([0, T]; L^{3/2} \cap H^1), \quad \nabla\mu(\rho) \in C([0, T]; L^q). \tag{4.3}
$$

Thanks to (3.42) and (3.46), the standard arguments yield that

$$\rho u_t \in H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^2),$$

which, together with (4.2) and (4.3), gives

$$\rho u_t + \rho u \cdot \nabla u \in C([\tau, T]; L^2). \tag{4.4}$$

Since (ρ, u) satisfies (2.15) with $F \equiv \rho u_t + \rho u \cdot \nabla u$, we deduce from (1.1), (4.2), (4.3), (4.4), and (3.46) that

$$\nabla u,\ P \in C([\tau, T]; D^1 \cap D^{1,p}), \tag{4.5}$$

for any $p \in [2, p_0)$.

Now, we claim that

$$T^* = \infty. \tag{4.6}$$

Otherwise, $T^* < \infty$. Proposition 3.1 implies that (3.2) holds at $T = T^*$. It follows from (4.2), (4.3), and (4.5) that

$$(\rho^*, u^*)(x) \triangleq (\rho, u)(x, T^*) = \lim_{t\to T^*} (\rho, u)(x, t)$$

satisfies

$$\rho^* \in L^{3/2} \cap H^1, \quad u^* \in D^1_{0,\sigma} \cap D^{1,p}$$

for any $p \in [2, p_0)$. Therefore, one can take $(\rho^*, \rho^* u^*)$ as the initial data and apply Lemma 2.1 to extend the local strong solution beyond T^*. This contradicts the assumption of T^* in (4.1). Hence, (4.6) holds. We thus finish the proof of Theorem 1.2 since (1.11) and (1.12) follow directly from (3.47) and (3.46), respectively. □

Proof of Theorem 1.3. With the global existence result at hand (see Proposition 1.1), one can modify slightly the proofs of Lemma 3.4 and (3.47) to obtain (1.13) and (1.14). □

Acknowledgements. The authors would like to thank the referee for his/her careful reading and helpful suggestions on the manuscript. The research of J. LI is partially supported by the National Center for Mathematics and Interdisciplinary Sciences, CAS, National Natural Science Foundation of China Grant Nos. 11688101, 11525106, and 12071200, and Double-Thousand Plan of Jiangxi Province (No. jxsq2019101008). The research of B. LÜ is partially supported by Natural Science Foundation of Jiangxi Province (No. 20202ACBL211002), Science and Technology Project of Jiangxi Provincial Education Department (No. GJJ160719), and National Natural Science Foundation of China (Grant Nos. 11601218 and 11971217).

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Informed Consent The authors claim that none of the material in the paper has been published or is under consideration for publication elsewhere. Informed consent has been obtained from all authors in this paper.

References

1. Abidi, H., Gui, G.L., Zhang, P.: On the decay and stability to global solutions of the 3-D inhomogeneous Navier–Stokes equations. *Commun. Pure Appl. Math.* **64**, 832–881, 2011
2. Abidi, H., Zhang, P.: On the global well-posedness of 2-D density-dependent Navier–Stokes system with variable viscosity. *J. Differ. Equ.* **259**, 3755–3802, 2015
3. Abidi, H., Zhang, P.: Global well-posedness of 3-D density-dependent Navier–Stokes system with variable viscosity. *Sci. China Math.* **58**(6), 1129–1150, 2015
4. Antontesv, S.A., Kazhikov, A.V.: *Mathematical Study of Flows of Nonhomogeneous Fluids*. Lecture Notes. Novosibirsk State University, Novosibirsk, 1973
5. Bergh, J., Lofstrom, J.: *Interpolation Spaces: an Introduction*. Springer, Berlin, 1976
6. Chen, Z.M.: A sharp decay result on strong solutions of the Navier–Stokes equations in the whole space. *Commun. Partial Differ. Equ.* **16**, 801–820, 1991
7. Cho, Y., Kim, H.: Unique solvability for the density-dependent Navier–Stokes equations. *Nonlinear Anal.* **59**(4), 465–489, 2004
8. Choe, H.Y., Kim, H.: Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids. *Commun. Partial Differ. Equ.* **28**, 1183–1201, 2003
9. Craig, W., Huang, X.D., Wang, Y.: Global strong solutions for 3D nonhomogeneous incompressible Navier–Stokes equations. *J. Math. Fluid Mech.* **15**, 747–758, 2013
10. Danchin, R.: Local and global well-posedness results for flows of inhomogeneous viscous fluids. *Adv. Differ. Equ.* **9**, 353–386, 2004
11. Danchin, R., Mucha, P.B.: A Lagrangian approach for the incompressible Navier–Stokes equations with variable density. *Commun. Pure Appl. Math.* **65**, 1458–1480, 2012
12. Desjardins, B.: Regularity results for two-dimensional flows of multiphase viscous fluids. *Arch. Ration. Mech. Anal.* **137**, 135–158, 1997
13. Fujita, H., Kato, T.: On the Navier–Stokes initial value problem. I. *Arch. Ration. Mech. Anal.* **16**, 269–315, 1964
14. Galdi, G.P.: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, 2nd edn. Springer, New York, 2011
15. He, C., Hsiao, L.: The decay rates of strong solutions for Navier–Stokes equations. *J. Math. Anal. Appl.* **268**, 417–425, 2002
16. He, C., Xin, Z.: On the decay properties of solutions to the nonstationary Navier–Stokes equations in R^3. *Proc. R. Edinb. Soc. Sect. A* **131**, 597–619, 2001
17. Hoff, D.: Compressible flow in a half-space with Navier boundary conditions. *J. Math. Fluid Mech.* **7**(3), 315–338, 2005
18. Huang, J.C., Paicu, M., Zhang, P.: Global solutions to 2-D inhomogeneous Navier–Stokes system with general velocity. *J. Math. Pures Appl.* **100**, 806–831, 2013
19. Huang, X.D., Li, J., Xin, Z.P.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations. *Commun. Pure Appl. Math.* **65**, 549–585, 2012
20. Huang, X.D., Wang, Y.: Global strong solution with vacuum to the two-dimensional density-dependent Navier–Stokes system. *SIAM J. Math. Appl.* **46**, 1771–1788, 2014
21. Huang, X.D., Wang, Y.: Global strong solution of 3D inhomogeneous Navier–Stokes equations with density-dependent viscosity. *J. Differ. Equ.* **259**, 1606–1627, 2015
22. Kato, T.: Strong L^p-solutions of the Navier–Stokes equation in R^m, with applications to weak solutions. *Math. Z.* **187**(4), 471–480, 1984
23. Kazhikov, A.V.: Resolution of boundary value problems for nonhomogeneous viscous fluids. *Dokl. Akad. Nauk.* **216**, 1008–1010, 1974
24. Ladyzhenskaya, O.A., Solonnikov, V.A.: Unique solvability of an initial and boundary value problem for viscous incompressible nonhomogeneous fluids. *J. Soviet Math.* **9**, 697–749, 1978

25. Ladyzhenskaya, O.A., Solonnikov, V.A., Ural'ceva, N.N.: *Linear and Quasilinear Equations of Parabolic Type*, vol. 23. Translations of Mathematical Monographs, American Mathematical Society, Providence, 1968
26. Lions, P.L.: *Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models*. Oxford University Press, Oxford, 1996
27. Lü, B.Q., Wang, X., Zhong, X.: Strong solutions to the 2D Cauchy problem of nonhomogeneous magnetohydrodynamic equations with vacuum. *J. Math. Phys*. **61**, 101501, 2020
28. Lü, B.Q., Shi, X.D., Zhong, X.: Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier-Stokes equations with vacuum. *Nonlinearity* **31**, 2617–2632, 2018
29. Lü, B.Q., Song, S.S.: On local strong solutions to the three-dimensional nonhomogeneous incompressible Navier–Stokes equations with density-dependent viscosity and vacuum. *Nonlinear Anal. Real World Appl*. **46**, 58–81, 2019
30. Schonbek, M.E.: Large time behaviour of solutions to the Navier–Stokes equations in H^m spaces. *Commun. Partial Differ. Equ*. **20**, 103–117, 1995
31. Simon, J.: Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. *SIAM J. Math. Anal*. **21**, 1093–1117, 1990
32. Zhang, J.W.: Global well-posedness for the incompressible Navier–Stokes equations with density-dependent viscosity coefficient. *J. Differ. Equ*. **259**, 1722–1742, 2015

C. He
Division of Mathematics, Department of Mathematical and Physical Sciences,
National Natural Science Foundation of China,
Beijing
100085 People's Republic of China.
e-mail: hecheng@nsfc.gov.cn

and

J. Li
Department of Mathematics,
Nanchang University,
Nanchang
330031 People's Republic of China.
e-mail: ajingli@gmail.com

and

Institute of Applied Mathematics, AMSS, and Hua Loo-Keng Key Laboratory of Mathematics,
Chinese Academy of Sciences,
Beijing
100190 People's Republic of China.

and

School of Mathematical Sciences,
University of Chinese Academy of Sciences,
Beijing
100049 People's Republic of China.

and

B. Lü
College of Mathematics and Information Science,
Nanchang Hangkong University,
Nanchang
330063 People's Republic of China.
e-mail: lvbq86@163.com

(*Received April 25, 2018 / Accepted December 5, 2020*)
Published online January 5, 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH, DE part of Springer Nature (*2021*)