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Analysis of the Viscosity of Dilute Suspensions Beyond Einstein's Formula

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Abstract

We provide a mathematical analysis of the effective viscosity of suspensions of spherical particles in a Stokes flow, at low solid volume fraction ϕ. Our objective is to go beyond Einstein's approximation $\mu_{eff} = (1+\frac{5}{2}\phi)\mu$. Assuming a lower bound on the minimal distance between the N particles, we are able to identify the $O(\phi^2)$ correction to the effective viscosity, which involves pairwise particle interactions. Applying the methodology developped over the last years on Coulomb gases, we are able to tackle the limit $N \to +\infty$ of the $O(\phi^2)$-correction, and provide an explicit formula for this limit when the particles centers can be described by either periodic or stationary ergodic point processes.

1. Setting of the Problem

Our general concern is the computation of the effective viscosity generated by a suspension of N particles in a fluid flow. We consider spherical particles of small radius a, centered at $x_{i,N}$, with $N \geqq 1$ and $1 \leqq i \leqq N$. To lighten notations, we write x_i instead of $x_{i,N}$, and $B_i = B(x_i, a)$. We assume that the Reynolds number of the fluid flow is small, so that the fluid velocity is governed by the Stokes equation. Moreover, the particles are assumed to be force- and torque-free. If $\mathcal{F} = \mathbb{R}^3 \setminus (\cup_i B_i)$ is the fluid domain, governing equations are

$$\begin{cases} -\mu \Delta u + \nabla p = 0, & x \in \mathcal{F}, \\ \operatorname{div} u = 0, & x \in \mathcal{F}, \\ u|_{B_i} = u_i + \omega_i \times (x - x_i), \end{cases} \tag{1.1}$$

where μ is the kinematic viscosity, while the constant vectors u_i and ω_i are Lagrange multipliers associated to the constraints

$$\int_{\partial B_i} \sigma_\mu(u, p) n \, \mathrm{d}s = 0, \qquad \int_{\partial B_i} \sigma_\mu(u, p) n \times (x - x_i) \, \mathrm{d}s = 0. \tag{1.2}$$

Here, $\sigma_\mu(u, p) := 2\mu D(u) - pI$ is the usual Cauchy stress tensor. The boundary condition at infinity will be specified later on.

We are interested in a situation where the number of particles is large, $N \gg 1$. We want to understand the additional viscosity created by the particles. Ideally, our goal is to replace the viscosity coefficient μ in (1.1) by an effective viscosity tensor μ' that would encode the average effect induced by the particles. Note that such replacement can only make sense in the flow region $\mathcal{O}$ in which the particles are distributed in a dense way. For instance, a finite number of isolated particles will not contribute to the effective viscosity, and should not be taken into account in $\mathcal{O}$. The selection of the flow region is formalized through the following hypothesis on the empirical measure:

$$\begin{gathered}\delta_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \xrightarrow[N\to+\infty]{} f(x)\mathrm{d}x \quad \text{weakly},\\ support(f) = \overline{\mathcal{O}}, \quad \mathcal{O} \text{ smooth, bounded and open, } f|_{\mathcal{O}} \in C^1(\overline{\mathcal{O}}).\end{gathered} \tag{H1}$$

Note that we do not ask for regularity of the limit density f over $\mathbb{R}^3$, but only in restriction to $\mathcal{O}$. Hence, our assumption covers the important case $f = \frac{1}{|\mathcal{O}|}1_{\mathcal{O}}$.

We investigate the classical regime of dilute suspensions, in which the solid volume fraction

$$\phi = \frac{4}{3}N\pi a^3/|\mathcal{O}| \tag{1.3}$$

is small, but independent of N. In addition to (H1), we make the separation hypothesis

$$\min_{i\neq j}|x_i - x_j| \geqq cN^{-1/3} \text{ for some constant } c > 0 \text{ independent of } N. \tag{H2}$$

Let us stress that (H2) is compatible with (H1) only if the L^∞ norm of f is small enough (roughly less than $1/c^3$), which in turn forces $\mathcal{O}$ to be large enough.

Our hope is to replace a model of type (1.1) by a model of the form

$$\begin{cases} -\mu\Delta u + \nabla p = 0, \quad \mathrm{div}\, u = 0, \quad x \in \mathbb{R}^3\setminus\mathcal{O},\\ -2\mathrm{div}(\mu' D(u')) + \nabla p' = 0, \quad \mathrm{div}\, u' = 0, \quad x\in\mathcal{O}, \end{cases} \tag{1.4}$$

with the usual continuity conditions on the velocity and the stress

$$u = u' \quad \text{at } \partial\mathcal{O}, \quad \sigma_\mu(u,p)n = \sigma_{\mu'}(u',p')n \quad \text{at } \partial\mathcal{O}. \tag{1.5}$$

A priori, μ' could be inhomogeneous (and should be if the density f seen above is itself non-constant over $\mathcal{O}$). It could also be anisotropic, if the cloud of particles favours some direction. With this in mind, it is natural to look for $\mu' = \mu'(x)$ as a general 4-tensor, with $\sigma' = 2\mu' D(u)$ given in coordinates by $\sigma_{ij} = \mu'_{ijkl}D(u)_{kl}$. By standard classical considerations of mechanics, μ' should satisfy the relations

$$\mu'_{ijkl} = \mu'_{jikl} = \mu'_{jilk} = \mu'_{lkji};$$

namely, μ' should define a symmetric isomorphism over the space of 3×3 symmetric matrices.

As we consider a situation in which ϕ is small, we may expect μ' to be a small perturbation of μ, and hopefully admit an expansion in powers of ϕ:

$$\mu' = \mu \mathrm{Id} + \phi\mu_1 + \phi^2\mu_2 + \cdots + \phi^k\mu_k + o(\phi^k). \tag{1.6}$$

The main mathematical questions are:

- Can solutions u_N of (1.1)–(1.2) be approximated by solutions $u_{eff} = 1_{\mathbb{R}^3\setminus\mathcal{O}}u + 1_{\mathcal{O}}u'$ of (1.4)–(1.5), for an appropriate choice of μ' and an appropriate topology ?
- If so, does μ' admit an expansion of type (1.6), for some k ?
- If so, what are the values of the viscosity coefficients μ_i, $1 \leqq i \leqq k$?

Let us stress that, in most articles about the effective viscosity of suspensions, it is implicitly assumed that the first two questions have a positive answer, at least for $k = 1$ or 2. In other words, the existence of an effective model is taken for granted, and the point is then to answer the third question, or at least to determine the mean values

$$\nu_i := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \mu_i(x)\mathrm{d}x \tag{1.7}$$

of the viscosity coefficients. As we will see in Section 2, these mean values can be determined from the asymptotic behaviour of some integral quantities $\mathcal{I}_N$ as $N \to +\infty$. These integrals involve the solutions u_N of (1.1)–(1.2) with condition at infinity

$$\lim_{|x|\to+\infty} u(x) - Sx = 0, \tag{1.8}$$

where S is an arbitrary symmetric trace-free matrix.

The effective viscosity problem for dilute suspensions of spherical particles has a long history, mostly focused on the first order correction created by the suspension, that is $k = 1$ in (1.6). The pioneering work on this problem was due to Einstein [15], not mentioning earlier contributions on the similar conductivity problem by Maxwell [29], Clausius [11], Mossotti [32]. The celebrated Einstein's formula,

$$\mu' = \mu + \frac{5}{2}\phi\mu + o(\phi), \tag{1.9}$$

was derived under the assumption that the particles are homogeneously and isotropically distributed, and neglecting the interactions between particles. In other words, the correction $\mu_1 = \frac{5}{2}\mu$ is obtained by summing N times the contribution of one spherical particle to the effective stress. The calculation of Einstein will be seen in Section 2. It was later extended to the case of an inhomogeneous suspension by Almog and Brenner [1, p. 16], who found that

$$\mu_1 = \frac{5}{2}|\mathcal{O}|f(x)\mu. \tag{1.10}$$

The mathematical justification of formula (1.9) came much later. As far as we know, the first step in this direction was due to Sanchez-Palencia [38] and Levy and Sanchez-Palencia [28], who recovered Einstein's formula from homogenization techniques, when the suspension is periodically distributed in a bounded domain.

Another justification, based on variational principles, is due to Haines and Mazzucato [19]. They also consider a periodic array of spherical particles in a bounded domain Ω, and define the viscosity coefficient of the suspension in terms of the energy dissipation rate:

$$\mu_N = \frac{\mu}{|S|^2}\int_{\mathcal{F}} |D(u_N)|^2,$$

where u_N is the solution of (1.1)–(1.2)–(1.8), replacing $\mathbb{R}^3$ by Ω. Their main result is that

$$\mu_N = \mu + \frac{5}{2}\phi\mu + O(\phi^{3/2}).$$

For preliminary results in the same spirit, see Keller-Rubenfeld [27]. Eventually, a recent work [21] by the second author and Di Wu shows the validity of Einstein's formula under general assumptions of type (H1)–(H2). See also [33] for a similar recent result.

Our goal in the present paper is to go beyond this famous formula, and to study the second order correction to the effective viscosity, that is $k = 2$ in (1.6). Results on this problem have split so far into two settings: periodic distributions, and random distributions of spheres. Many different formulas have emerged in the literature, after analytical, numerical and experimental studies. In the periodic case, one can refer to the works [2,34,37,42], or to the more recent work [2], dedicated to the case of spherical inclusions of another Stokes fluid with viscosity $\tilde{\mu} \neq \mu$. Still, in the simple case of a primitive cubic lattice, the expressions for the second order correction differ. In the random case, the most reknowned analysis is due to Batchelor and Green [5], who consider a homogeneous and stationary distribution of spheres, and express the correction μ_2 as an ensemble average that involves the N-point correlation function of the process. As pointed out by Batchelor and Green, the natural idea when investigating the effective viscosity up to $O(\phi^2)$ is to replace the N-point correlation function by the two-point correlation function, but this leads to a divergent integral. To overcome this difficulty, Batchelor and Green develop what they call a renormalization technique, that was developed earlier by Batchelor to determine the sedimentation speed of a dilute suspension. After further analysis of the expression of the two-point correlation function of spheres in a Stokes flow [6], completed by numerical computations, they claim that under a pure strain, the particles induce a viscosity of the form

$$\mu' = \mu + \frac{5}{2}\phi\mu + 7.6\phi^2\mu + o(\phi^2). \tag{1.11}$$

Although the result of Batchelor and Green is generally accepted by the fluid mechanics community, the lack of clarity about their renormalization technique has led to debate; see [1,22,35].

One main objective in the present paper is to give a rigorous and global mathematical framework for the computation of

$$\nu_2 = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \mu_2(x)\mathrm{d}x, \tag{1.12}$$

leading to explicit formula in periodic and stationary random settings. We will adopt the point of view of the studies mentioned before: we will assume the validity of an effective model of type (1.4)–(1.5)–(1.6) with $k = 2$, and will identify the averaged coefficient ν_2.

More precisely, our analysis is divided into two parts. The first part, conducted in Section 2, has as its main consequence the following:

Theorem 1.1. *Let $(x_i)_{1\leq i\leq N}$ a family of points supported in a fixed compact set $\mathbb{R}^3$, and satisfying (H1)–(H2). For any trace-free symmetric matrix S and any $\phi > 0$, let u_N, resp. u_{eff}, the solution of (1.1)–(1.2)–(1.8) with the radius a of the balls defined through (1.3), resp. the solution of (1.4)–(1.5)–(1.8) where μ' obeys (1.6) with $k = 2$, μ_1 being given in (1.10).*

If $u_N - u_{eff} = o(\phi^2)$ in $H^{-\infty}_{loc}(\mathbb{R}^3)$, meaning that for all of bounded open set U, there exists $s \in \mathbb{R}$ such that

$$\limsup_{N\to+\infty} \|u_N - u_{eff}\|_{H^s(U)} = o(\phi^2), \quad \text{as } \phi \to 0,$$

then, necessarily, the coefficient ν_2 defined in (1.12) satisfies $\nu_2 S : S = \mu \lim_{N\to+\infty} \mathcal{V}_N$ where ν_2 was defined in (1.12), and

$$\mathcal{V}_N := \frac{75|\mathcal{O}|}{16\pi}\left(\frac{1}{N^2}\sum_{i\neq j} g_S(x_i - x_j) - \int_{\mathbb{R}^3\times\mathbb{R}^3} g_S(x-y) f(x) f(y) \mathrm{d}x\mathrm{d}y\right) \tag{1.13}$$

with the Calderón–Zygmund kernel

$$g_S := -D\left(\frac{S : (x\otimes x)x}{|x|^5}\right) : S. \tag{1.14}$$

Roughly, this theorem states that *if there is an effective model at order* ϕ^2, the mean quadratic correction ν_2 is given by the limit of $\mathcal{V}_N$, defined in (1.13). Note that the integral at the right-hand side of (1.13) is well-defined: $f \in L^2(\mathbb{R}^3)$ and $f \to g_S \star f$ is a Calderón–Zygmund operator, therefore continuous on $L^2(\mathbb{R}^3)$. We insist that our result is an *if theorem*: the limit of (1.13) does not necessarily exist for any configuration of particles $x_i = x_{i,N}$ satisfying (H1)–(H2). In particular, it is not clear that an effective model at order ϕ^2 is available for all such configurations.

Still, the second part of our analysis shows that for points associated to stationary random processes (including periodic patterns or Poisson hard core processes), the limit of the functional does exist, and is given by an explicit formula. We shall leave for later investigation the problem of approximating u_N by u_{eff} when the limit of $\mathcal{V}_N$ exists.

Our study of functional (1.13) is detailed in Sections 3 to 5. It borrows a lot from the mathematical analysis of Coulomb gases, as developped over the last years by Sylvia Serfaty and her coauthors [9,36,40]. Although our paper is self-contained, we find useful to give a brief account of this analysis here. As explained in the lecture notes [41], one of its main goals is to understand what configurations of points minimize Coulomb energies of the form

$$H_N = \frac{1}{N^2} \sum_{i \neq j} g(x_i - x_j) + \frac{1}{N} \sum_{i=1}^{N} V(x_i),$$

where $g(x) = \frac{1}{|x|}$ is a repulsive potential of Coulomb type, and V is typically a confining potential. It is well-known, see [41, chapter 2], that under suitable assumptions on V, the sequence of functionals H_N (seen as a functionals over probability measures by extension by $+\infty$ outside the set of empirical measures) Γ-converges to the functional

$$H(\lambda) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(x-y) \mathrm{d}\lambda(x) \mathrm{d}\mu(y) + \int_{\mathbb{R}^3} V(x) \mathrm{d}\lambda(x).$$

Hence, the empirical measure $\delta_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$ associated to the minimizer $(x_1, \ldots, x_N)$ of H_N converges weakly to the minimizer λ of H.

In the series of works [36,40], see also [39] on the Ginzburg-Landau model, Serfaty and her coauthors investigate the next order term in the asymptotic expansion of $\min_{x_1,\ldots,x_N} H_N$. A keypoint in these works is understanding the behaviour of (the minimum of)

$$\mathcal{H}_N = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \setminus \mathrm{Diag}} g(x-y) \mathrm{d}(\delta_N - \lambda)(x) \mathrm{d}(\delta_N - \lambda)(y) \tag{1.15}$$

as $N \to +\infty$. This is done through the notion of renormalized energy. Roughly, the starting point behind this notion is the (abusive) formal identity

$$\text{"} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(x-y) \mathrm{d}(\delta_N - \lambda)(x) \mathrm{d}(\delta_N - \lambda)(y) = \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla h_N|^2 \text{"}, \tag{1.16}$$

where h_N is the solution of $\Delta h_N = 4\pi(\delta_N - \lambda)$ in $\mathbb{R}^3$. Of course, this identity does not make sense, as both sides are infinite. On one hand, the left-hand side is not well-defined: the potential g is singular at the diagonal, so that the integral with respect to the product of the empirical measures diverges. On the other hand, the right-hand side is not better defined: as the empirical measure does not belong to $H^{-1}(\mathbb{R}^3)$, h_N is not in $\dot{H}^1(\mathbb{R}^3)$.

Still, as explained in [41, chapter 3], one can modify this identity, and show a formula of the form

$$\mathcal{H}_N = \lim_{\eta \to 0} \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla h_N^\eta|^2 - N g(\eta) \right), \tag{1.17}$$

where h_N^η is an approximation of h_N obtained by regularization of the Dirac masses at the right-hand side of the Laplace equation: $\Delta h_N^\eta = 4\pi(\delta_N^\eta - \lambda)$ in $\mathbb{R}^3$. Note the removal of the term $N g(\eta)$ at the right-hand side of (1.17). This term, which goes to infinity as the parameter $\eta \to 0$, corresponds to the self-interaction of the Dirac masses: it must be removed, consistently with the fact that the integral defining

$\mathcal{H}_N$ excludes the diagonal. This explains the term renormalized energy. See [41, chapter 3] for more details.

From there (omitting to discuss the delicate commutation of the limits in N and η !), the asymptotics of $\min_{x_1,\dots,x_N} \mathcal{H}_N$ can be deduced from the one of $\min_{x_1,\dots,x_N} \int_{\mathbb{R}^3} |\nabla h_N^\eta|^2$, for fixed η. The next step is to show that such minimum can be expressed as spatial averages of (minimal) microscopic energies, expressed in terms of solutions of the so-called *jellium problems*: see [41, chapter 4]. These problems, obtained through rescaling and blow-up of the equation on h_N^η, are an analogue of cell problems in homogenization. More will be said in Section 4, and we refer to the lecture notes [41] for all necessary complements.

Thus, the main idea in the second part of our paper is to take advantage of the analogy between the functionals $\mathcal{V}_N$ and $\mathcal{H}_N$ to apply the strategy just described. Doing so, we face specific difficulties: our distribution of points is not minimizing an energy, the potential g_S is much more singular than g, the reformulation of the functional in terms of an energy is less obvious, *etc*. Still, we are able to reproduce the same kind of scheme. We introduce in Section 3 an analogue of the renormalized energy. The analogue of the jellium problem is discussed in Section 4. Finally, in Section 5, we are able to tackle the convergence of $\mathcal{V}_N$, and give explicit formula for the limit in two cases: the case of a (properly rescaled) $L\mathbb{Z}^3$-periodic pattern of M-spherical particles with centers $a_1, \dots, a_M$, and the case of a (properly rescaled) hardcore stationary random process with locally integrable two points correlation function $\rho_2(y, z) = \rho(y - z)$. In the first case, we show that

$$\lim_{N\to+\infty} \mathcal{V}_N = \frac{25L^3}{2M^2}\Big(\sum_{i\neq j} S\nabla \cdot G_{S,L}(a_i - a_j) + K S\nabla \cdot (G_{S,L} - G_S)(0)\Big), \quad (1.18)$$

where G_S and $G_{S,L}$ are the whole space and $L\mathbb{Z}^3$-periodic kernels defined respectively in (3.12) and (5.18); see Proposition 5.4. In the special case of a primitive cubic lattice, for which $M = L = 1$, we can push the calculation further, finding that

$$\nu_2 S : S = \mu\big(\alpha \sum_{i=1}^{3} |S_{ii}|^2 + \beta \sum_{i\neq j} |S_{ij}|^2\big),$$

with $\alpha \approx 9.48$ and $\beta \approx -2, 15$, *cf.* Proposition 5.5 for precise expressions. Our result is in agreement with [42]. In the random stationary case, if the process has mean intensity one, we show that

$$\begin{aligned}\lim_N \mathcal{V}_N &= \frac{25}{2} \lim_{L\to+\infty} \frac{1}{L^3} \sum_{z\neq z' \in \Lambda\cap K_L} S\nabla \cdot G_{S,L}(z - z') \\ &= \frac{25}{2} \lim_{L\to+\infty} \frac{1}{L^3} \int_{K_L\times K_L} S\nabla \cdot G_{S,L}(z - z')\rho(z - z')dzdz'. \quad (1.19)\end{aligned}$$

These formula open the road to numerical computations of the viscosity coefficients of specific processes, and should in particular allow us to check the formula found in the literature [5,35].

Let us conclude this introduction by pointing out that our analysis falls into the general scope of deriving macroscopic properties of dilute suspensions. From this perspective, it can be related to mathematical studies on the drag or sedimentation speed of suspensions; see [13,23–25,30] among many. See also the recent work [14] on the conductivity problem.

2. Expansion of the Effective Viscosity

The aim of this section is to understand the origin of the functional $\mathcal{V}_N$ introduced in (1.13), and to prove Theorem 1.1. The outline is the following. We first consider the effective model (1.4)–(1.5)–(1.6). Given S a symmetric trace-free matrix, and a solution u_{eff} with condition at infinity (1.8), we exhibit an integral quantity $\mathcal{I}_{eff} = \mathcal{I}_{eff}(S)$ that involves u_{eff} and allows us to recover (partially) the mean viscosity coefficient ν_2. In the next paragraph, we introduce the analogue $\mathcal{I}_N$ of $\mathcal{I}_{eff}$, that involves this time the solution u_N of (1.1)–(1.2) and (1.8). In brief, we show that if u_N is $o(\phi^2)$ close to u_{eff}, then $\mathcal{I}_N$ is $o(\phi^2)$ close to $\mathcal{I}_{eff}$. Finally, we provide an expansion of $\mathcal{I}_N$, allowing us to express ν_2 in terms of $\mathcal{V}_N$. Theorem 1.1 follows.

2.1. Recovering the Viscosity Coefficients in the Effective Model

Let $k \geqq 2$, μ' satisfying (1.6), with viscosity coefficients μ_i that may depend on x. Let S symmetric and trace-free. We denote $u_0(x) = Sx$. Let $u_{eff} = 1_{\mathbb{R}^3\setminus\mathcal{O}} u + 1_{\mathcal{O}} u'$ the weak solution in $u_0 + \dot{H}^1(\mathbb{R}^3)$ of (1.4)–(1.5)–(1.8). By a standard energy estimate, one can show the expansion

$$u_{eff} - u_0 = \phi\, u_{eff,1} + \cdots + \phi^k u_{eff,k} + o(\phi^k) \quad \text{in } \dot{H}^1(\mathbb{R}^3),$$

where the system satisfied by $u_{eff,i} = 1_{\mathbb{R}^3\setminus\mathcal{O}} u_i + 1_{\mathcal{O}} u'_i$ is derived by plugging the expansion in (1.4)–(1.5) and keeping terms with power ϕ^i only. Notably, we find that

$$\begin{cases} -\mu\Delta u_1 + \nabla p_1 = 0, \quad \operatorname{div} u_1 = 0, \quad x \in \mathbb{R}^3 \setminus \mathcal{O}, \\ -\mu\Delta u'_1 + \nabla p'_1 = 2\operatorname{div}(\mu_1 D(u_0)) \quad \operatorname{div} u'_1 = 0, \quad x \in \mathcal{O}, \end{cases} \tag{2.1}$$

together with the conditions $u_1 = 0$ at infinity,

$$u_1 = u'_1 \ \text{ at } \partial\mathcal{O}, \quad \sigma_\mu(u_1, p_1)n \ = \ \sigma_\mu(u'_1, p'_1)n \ + \ 2\mu_1 D(u_0)n \text{ at } \partial\mathcal{O}.$$

Similarly,

$$\begin{cases} -\mu\Delta u_2 + \nabla p_2 = 0, \quad \operatorname{div} u_2 = 0, \quad x \in \mathbb{R}^3 \setminus \mathcal{O}, \\ -\mu\Delta u'_2 + \nabla p'_2 = 2\operatorname{div}(\mu_2 D(u_0)) + 2\operatorname{div}(\mu_1 D(u'_1)), \quad \operatorname{div} u'_2 = 0, \quad x \in \mathcal{O}, \end{cases} \tag{2.2}$$

together with $u_2 = 0$ at infinity,

$$\begin{aligned} u_2 = u'_2 \ \text{ at } \partial\mathcal{O}, \quad & \sigma_\mu(u_2, p_2)n \\ & = \sigma_\mu(u'_2, p'_2)n \ + \ 2\mu_2 D(u_0)n + 2\mu_1 D(u'_1)n \ \text{ at } \partial\mathcal{O}. \end{aligned}$$

Now, inspired by formula (4.11.16) in [4], we define

$$\mathcal{I}_{eff} := \int_{\partial\mathcal{O}} \sigma_\mu(u - u_0, p_{eff})n \cdot Sx\mathrm{d}s - 2\mu \int_{\partial\mathcal{O}} (u - u_0) \cdot Sn\mathrm{d}s, \tag{2.3}$$

where n refers to the outward normal. We will show that

$$\mathcal{I}_{eff} = 2|\mathcal{O}| \left(\phi\nu_1 S : S + \phi^2\nu_2 S : S\right) + 2\phi^2 \int_{\mathcal{O}} \mu_1 D(u_1') : S + o(\phi^2). \tag{2.4}$$

We first use (1.5) to write

$$\begin{aligned}
\mathcal{I}_{eff} &= \int_{\partial\mathcal{O}} \sigma_{\mu'}(u' - u_0, p')n \cdot Sx\mathrm{d}s + \int_{\partial\mathcal{O}} \sigma_{\mu'-\mu}(u_0, 0)n \cdot Sx\mathrm{d}s \\
&\quad - 2\mu \int_{\partial\mathcal{O}} (u' - u_0) \cdot Sn\mathrm{d}s \\
&= \int_{\partial\mathcal{O}} \sigma_{\mu'}(\phi u_1' + \phi^2 u_2', \phi p_1 + \phi^2 p_2)n \cdot Sx\mathrm{d}s \\
&\quad + 2\int_{\partial\mathcal{O}} (\phi\mu_1 + \phi^2\mu_2)Sn \cdot Sx\mathrm{d}s - 2\mu \int_{\partial\mathcal{O}} (\phi u_1' + \phi^2 u_2') \cdot Sn\mathrm{d}s + o(\phi^2) \\
&= \int_{\partial\mathcal{O}} \sigma_{\mu}(\phi u_1' + \phi^2 u_2', \phi p_1 + \phi^2 p_2)n \cdot Sx\mathrm{d}s + \phi \int_{\partial\mathcal{O}} \sigma_{\mu_1}(\phi u_1', 0)n \cdot Sx\mathrm{d}s \\
&\quad + 2\int_{\partial\mathcal{O}} (\phi\mu_1 + \phi^2\mu_2)Sn \cdot Sx\mathrm{d}s - 2\mu \int_{\partial\mathcal{O}} (\phi u_1' + \phi^2 u_2') \cdot Sn\mathrm{d}s + o(\phi^2).
\end{aligned}$$

Using the equations satisfied by u_1' and u_2', after integration by parts, we get

$$\begin{aligned}
&\int_{\partial\mathcal{O}} \sigma_{\mu}(\phi u_1' + \phi^2 u_2', \phi p_1 + \phi^2 p_2)n \cdot Sx\mathrm{d}s \\
&\quad = -\int_{\mathcal{O}} 2\mathrm{div}(\phi\mu_1 S + \phi^2\mu_2 S) \cdot Sx\mathrm{d}x - \int_{\mathcal{O}} 2\mathrm{div}(\phi^2\mu_1 D(u_1')) \cdot Sx\mathrm{d}x \\
&\qquad + 2\mu \int_{\mathcal{O}} D(\phi u_1' + \phi^2 u_2') : S\mathrm{d}x \\
&\quad = 2|\mathcal{O}|(\phi\nu_1 S : S + \phi^2\nu_2 S : S) - 2\int_{\partial\mathcal{O}} (\phi\mu_1 + \phi^2\mu_2)Sn \cdot Sx\mathrm{d}s \\
&\qquad + 2\int_{\mathcal{O}} \phi^2\mu_1 D(u_1') : S - 2\int_{\partial\mathcal{O}} \phi^2\mu_1 D(u_1')n \cdot Sx\mathrm{d}s \\
&\qquad + 2\mu \int_{\mathcal{O}} (\phi u_1' + \phi^2 u_2') \cdot Sn\mathrm{d}x.
\end{aligned}$$

Plugging this last line in the expression for $\mathcal{I}_{eff}$ yields (2.4).

We see through formula (2.4) that the expansion of $\mathcal{I}_{eff}$ in powers of ϕ gives access to ν_1, and, if μ_1 is known, it further gives access to ν_2. On the basis of the works [1,33] and of the recent paper [21], which considers the same setting as ours, it is natural to assume that μ_1 is given by (1.10). This implies $\nu_1 = \frac{5}{2}\mu$.

With such expression of μ_1, and the form of f specified in (H1), we can check that $u_S = (5|\mathcal{O}|)^{-1} u_{eff,1}$ satisfies

$$-\Delta u_S + \nabla p = \operatorname{div}(Sf) = S\nabla f, \quad \operatorname{div} u_S = 0 \quad \text{in } \mathbb{R}^3, \quad \lim_{|x|\to\infty} u_S(x) = 0. \quad (2.5)$$

It follows that

$$\mathcal{I}_{eff} = 5\phi\mu|\mathcal{O}||S|^2 + 2\phi^2|\mathcal{O}|\nu_2 S : S - 50\mu\phi^2|\mathcal{O}|^2 \int_{\mathbb{R}^3} |D(u_S)|^2 + o(\phi^2). \quad (2.6)$$

2.2. Recovering the Viscosity Coefficients in the Model with Particles

To determine the possible value of the mean viscosity coefficient ν_2, we must now relate the functional $\mathcal{I}_{eff}$, based on the effective model, to a functional $\mathcal{I}_N$ based on the real model with spherical rigid particles. From now on, we place ourselves under the assumptions of Theorem 1.1. Note that, thanks to hypothesis (H2), the spherical particles do not overlap for ϕ small enough, so that a weak solution $u_N \in u_0 + \dot{H}^1(\mathbb{R}^3)$ of (1.1)–(1.2)–(1.8) exists and is unique.

By integration by parts, for any R such that $\mathcal{O} \Subset B_R$, we have

$$\mathcal{I}_{eff} = \int_{\partial B_R} \sigma_\mu(u_{eff} - u_0, p_{eff})n \cdot Sx\,\mathrm{d}s - 2\mu \int_{\partial B_R} (u_{eff} - u_0) \cdot Sn\,\mathrm{d}s. \quad (2.7)$$

By analogy with (2.3), and as all particles remain in a fixed compact $K \supset \mathcal{O}$ independent of N, we set for any R such that $K \subset B_R$:

$$\mathcal{I}_N := \int_{\partial B_R} \sigma_\mu(u_N - u_0, p_N)n \cdot Sx\,\mathrm{d}s - 2\mu \int_{\partial B_R} (u_N - u_0) \cdot Sn\,\mathrm{d}s, \quad (2.8)$$

which again does not depend on our choice of R by integration by parts. Now, if u_{eff} and u_N are $o(\phi^2)$-close in the sense of Theorem 1.1, then

$$\limsup_{N\to+\infty} |\mathcal{I}_N - \mathcal{I}_{eff}| = o(\phi^2). \quad (2.9)$$

Indeed, $u_N - u_{eff}$ is a solution of a homogenenous Stokes equation outside K. By elliptic regularity, we find that $\limsup_{N\to+\infty} \|u_{eff} - u_N\|_{H^s(K')} = 0$, for any compact $K' \subset \mathbb{R}^3 \setminus K$ and any positive s. Relation (2.9) follows.

We now turn to the most difficult part of this section, that is expanding $\mathcal{I}_N$ in powers of ϕ. We aim to prove

Proposition 2.1. *Let* $(x_i)_{1\leqq i\leqq N}$, *satisfying* (H1)–(H2). *For S trace-free and symmetric, for* $\phi > 0$, *let u_N the solution of* (1.1)–(1.2)–(1.8) *with the ball radius a defined through* (1.3). *Let $\mathcal{I}_N$ as in* (2.8), *$\mathcal{V}_N$ as in* (1.13), *and u_S the solution of* (2.5). *One has*

$$\mathcal{I}_N = 5\phi\mu|\mathcal{O}||S|^2 + 2\phi^2\mu|\mathcal{O}|\mathcal{V}_N - 50\mu\phi^2|\mathcal{O}|^2 \int_{\mathbb{R}^3} |D(u_S)|^2 + o(\phi^2). \quad (2.10)$$

As before, notation $A_N = B_N + o(\phi^2)$ means $\limsup_N |A_N - B_N| = o(\phi^2)$. Obviously, Theorem 1.1 follows directly from (2.6), (2.9) and from the proposition.

To start the proof, we set $v_N := u_N - u_0$. Note that $v_N \in \dot{H}^1(\mathcal{F})$ still satisfies the Stokes equation outside the ball, with $v_N = 0$ at infinity, and $v_N = -Sx + u_i + \omega_i \times (x - x_i)$ inside B_i. Moreover, taking into account the identities

$$\int_{\partial B_i} \sigma_\mu(u_0, 0)n \, \mathrm{d}s = 2\mu \int_{\partial B_i} Sn = 2\mu \int_{B_i} \operatorname{div} S = 0$$

and

$$\int_{\partial B_i} \sigma_\mu(u_0, 0)n \times (x - x_i) \, \mathrm{d}s = 2\mu \int_{\partial B_i} Sn \times (x - x_i) \, \mathrm{d}s = 2\mu \int_{\partial B_i} S(x - x_i) \times n \, \mathrm{d}s$$
$$= 2\mu \int_{B_i} \operatorname{curl}(S(x - x_i)) \, \mathrm{d}s = 0, \tag{2.11}$$

one has for all i that

$$\int_{\partial B_i} \sigma_\mu(v_N, p_N)n \, \mathrm{d}s = 0, \quad \int_{\partial B_i} \sigma_\mu(v_N, p_N)n \times (x - x_i) \, \mathrm{d}s = 0.$$

From the definition (2.8), we can re-express $\mathcal{I}_N$ as

$$\mathcal{I}_N = \sum_{i=1}^N \int_{\partial B_i} \sigma_\mu(v_N, p_N)n \cdot Sx \, \mathrm{d}s - 2\mu \sum_{i=1}^N \int_{\partial B_i} v_N \cdot Sn \, \mathrm{d}s. \tag{2.12}$$

To obtain an expansion of $\mathcal{I}_N$ in powers of ϕ, we will now approximate (v_N, p_N) by some explicit field (v_{app}, p_{app}), inspired by the method of reflections. This approximation involves the elementary problem

$$\begin{cases} -\mu \Delta v + \nabla p = 0 & \text{outside } B(0, a), \\ \operatorname{div} v = 0 & \text{outside } B(0, a), \\ v(x) = -Sx, & x \in B(0, a). \end{cases} \tag{2.13}$$

The solution of (2.13) is explicit [18], and given by

$$v^s[S](x) := -\frac{5}{2} S : (x \otimes x) \frac{a^3 x}{|x|^5} - Sx \frac{a^5}{|x|^5} + \frac{5}{2}(S : x \otimes x) \frac{a^5 x}{|x|^7}$$
$$= v[S] + O(a^5 |x|^{-4}), \tag{2.14}$$

with

$$v[S](x) := -\frac{5}{2} S : (x \otimes x) \frac{a^3 x}{|x|^5}. \tag{2.15}$$

The pressure is

$$p^s[S](x) := -5\mu a^3 \frac{S : (x \otimes x)}{|x|^5}. \tag{2.16}$$

We now introduce

$$(v_{app}, p_{app})(x) := \sum_{i=1}^{N}(v^s[S], p^s[S])(x - x_i) + \sum_{i=1}^{N}(v^s[S_i], p^s[S_i])(x - x_i), \tag{2.17}$$

where

$$S_i \; := \sum_{j \neq i} D(v[S])(x_i - x_j). \tag{2.18}$$

In short, the first sum at the right-hand side of (2.17) corresponds to a superposition of N elementary solutions, meaning that the interaction between the balls is neglected. This sum satisfies the Stokes equation outside the ball, but creates an error at each ball B_i, whose leading term is $S_i x$. This explains the correction by the second sum at the right-hand side of (2.17). One could of course reiterate the process: as the distance between particles is large compared to their radius, we expect the interactions to be smaller and smaller. This is the principle of the method of reflections that is investigated in [24]. From there, Proposition 2.1 will follow from two facts. Defining

$$\mathcal{I}_{app} := \sum_{i=1}^{N} \int_{\partial B_i} \sigma_\mu(v_{app}, p_{app})n \cdot Sx \, \mathrm{d}s - 2\mu \sum_{i=1}^{N} \int_{\partial B_i} v_{app} \cdot Sn \, \mathrm{d}s,$$

we will show first that

$$\mathcal{I}_{app} = 5\phi\mu|S|^2 + 2\phi^2\mu|\mathcal{O}|\mathcal{V}_N - \; 50\mu\phi^2|\mathcal{O}|^2 \int_{\mathbb{R}^3} |D(u_S)|^2, \tag{2.19}$$

and then

$$\limsup_{N\to+\infty} |\mathcal{I}_N - \mathcal{I}_{app}| = o(\phi^2). \tag{2.20}$$

Identity (2.19) follows from a calculation that we now detail. We define

$$\mathcal{I}_i(v, p) := \int_{\partial B_i} \big((\sigma(v, p)n \otimes x) - 2\mu(v \otimes n)\big) \, \mathrm{d}s.$$

We have

$$\begin{aligned}
\mathcal{I}_{app} = & \sum_i \mathcal{I}_i(v^s[S](\cdot - x_i), p^s[S](\cdot - x_i)) : S \\
& + \sum_i \sum_{j \neq i} \mathcal{I}_i(v^s[S](\cdot - x_j), p^s[S](\cdot - x_j)) : S \\
& + \sum_i \mathcal{I}_i(v^s[S_i](\cdot - x_i), p^s[S_i](\cdot - x_i)) : S \\
& + \sum_i \sum_{j \neq i} \mathcal{I}_i(v^s[S_j](\cdot - x_j), p^s[S_j](\cdot - x_j)) : S \\
=: & \; I_a + I_b + I_c + I_d.
\end{aligned}$$

To treat I_b and I_d, we rely on the following property, which is checked easily through integration by parts: *for any* (v, p) *solution of Stokes in* B_i, *and any trace-free symmetric matrix* S, $\mathcal{I}_i(v, p) : S = 0$. As for all i and all $j \neq i$, $v^s[S](\cdot - x_j)$ or $v^s[S_j](\cdot - x_j)$ is a solution of Stokes inside B_i, we deduce

$$I_b = I_d = 0. \tag{2.21}$$

As regards I_a, we use the following formula, which follows from a tedious calculation [18]: for any traceless matrix S,

$$\mathcal{I}_i(v^s[S](\cdot - x_i)) = \frac{20\pi}{3}\mu a^3 S. \tag{2.22}$$

It follows that

$$I_a = N\frac{20\pi}{3}\mu a^3|S|^2 = 5\phi|\mathcal{O}|\mu|S|^2. \tag{2.23}$$

This term corresponds to the famous Einstein formula for the mean effective viscosity. It is coherent with the expression (1.10) for μ_1, which implies $\nu_1 = \frac{5}{2}\mu$.

Eventually, as regards I_c, we can use (2.22) again, replacing S by S_i:

$$\begin{aligned}
I_c &= \frac{20\pi}{3}\mu a^3 \sum_i S_i : S = \frac{20\pi}{3}\mu a^3 \sum_i \sum_{j\neq i} D(v[S])(x_i - x_j) : S \\
&= \frac{75|\mathcal{O}|^2}{8\pi}\mu\phi^2 \frac{1}{N^2}\sum_i \sum_{j\neq i} g_S(x_i - x_j) \\
&= 2\phi^2\mu|\mathcal{O}|\mathcal{V}_N + \phi^2\frac{75|\mathcal{O}|^2}{8\pi}\mu \int_{\mathbb{R}^3\times\mathbb{R}^3} g_S(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{2.24}$$

with g_S defined in (1.14). In view of (2.21)–(2.23)–(2.24), to conclude that (2.19) holds, it is enough to prove

Lemma 2.2. *For any* $f \in L^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} g_S(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y = -\frac{16\pi}{3}\int_{\mathbb{R}^3}|D(u_S)|^2, \tag{2.25}$$

with g_S *defined in* (1.14), *and* $u_S \in \dot{H}^1(\mathbb{R}^3)$ *the solution of* (2.5).

Proof. Note that both sides of the identity are continuous over L^2: the left-hand side is continuous as the Calderón–Zygmund operator $f \to g_S \star f$ is continuous over L^2, while the right-hand side is continuous by classical elliptic estimates for the Stokes operator. By density, this is therefore enough to assume that $f \in C_c^\infty(\mathbb{R}^3)$. We denote by $U = (U_{ij})$, $Q = (Q_j)$ the fondamental solution of the Stokes operator. This means that for all j, the vector field $U_j = (U_{ij})_{1\leqq i\leqq 3}$ and the scalar field Q_j satisfy the Stokes equation

$$-\Delta U_j + \nabla Q_j = \delta e_j, \quad \mathrm{div}\, U_j = 0 \quad \text{in } \mathbb{R}^3. \tag{2.26}$$

It is well-known, (see [16, p. 239]), that

$$U(x) = \frac{1}{8\pi}\left(\frac{1}{|x|}Id + \frac{x\otimes x}{|x|^3}\right), \quad Q(x) = \frac{1}{4\pi}\frac{x}{|x|^3}.$$

From there, one can deduce the following formula, *cf* [16, p. 290, equation (IV.8.14)]:

$$\sigma(U_j, Q_j) = -\frac{3}{4\pi}\frac{(x\otimes x)x_j}{|x|^5}.$$

Using the Einstein convention for summation, this implies in turn that

$$\begin{aligned} g_S(x) &= -S_{kl}\partial_{x_k}\left(\frac{S:(x\otimes x)x_l}{|x|^5}\right) = \frac{4\pi}{3}S : S_{kl}\partial_{x_k}\sigma(U_l, Q_l)(x) \\ &= \frac{8\pi}{3}S : DS_{kl}\partial_{x_k}U_l = (S\nabla)\cdot(S_{kl}\partial_{x_k}U_l), \end{aligned} \tag{2.27}$$

where we have used that S is trace-free to obtain the third equality. Hence,

$$\begin{aligned} \int\int g_S(x-y)f(x)\mathrm{d}x f(y)\mathrm{d}y &= \frac{8\pi}{3}\int_{\mathbb{R}^3}\big((S:DS_{kl}\partial_{x_k}U_l)\star f\big)(y)f(y)\mathrm{d}y \\ &= \frac{8\pi}{3}\int S : DS_{kl}\partial_{x_k}(U_l\star f)(y)\, f(y)\mathrm{d}y. \end{aligned} \tag{2.28}$$

Note that the permutations between the derivatives and the convolution product do not raise any difficulty, as $f\in C_c^\infty(\mathbb{R}^3)$. Now, using $S_{kl}=S_{lk}$, and denoting by St^{-1} the convolution with the fundamental solution (inverse of the Stokes operator), we get

$$S_{kl}\partial_{x_k}\int U_l(y-x)f(x)\mathrm{d}x = \mathrm{St}^{-1}(S\nabla f)(y). \tag{2.29}$$

Eventually,

$$\begin{aligned} \int\int g_S(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y &= \frac{8\pi}{3}\int S:\nabla\mathrm{St}^{-1}(S\nabla f)(y)\, f(y)\mathrm{d}y \\ &= -\frac{8\pi}{3}\int \mathrm{St}^{-1}(S\nabla f)(y)\cdot(S\nabla f)(y)\,\mathrm{d}y \\ &= -\frac{16\pi}{3}\int_{\mathbb{R}^3}|D(u_S)|^2. \end{aligned}$$

This concludes the proof of the lemma. □

Remark 2.3. By polarization of the previous identity, at least for $f, \tilde{f}$ smooth and decaying enough, one has

$$\begin{aligned} \int\int g_S(x-y)f(y)\tilde{f}(x)\mathrm{d}x &= -\frac{8\pi}{3}\int \mathrm{St}^{-1}(S\nabla f)(x)\cdot(S\nabla\tilde{f})(x)\,\mathrm{d}x \\ &= \frac{8\pi}{3}\int (S\nabla)\cdot\big(\mathrm{St}^{-1}(S\nabla f)\big)(x)\,\tilde{f}(x)\,\mathrm{d}x. \end{aligned} \tag{2.30}$$

The last step in proving Proposition 2.1, hence Theorem 1.1, is to show the bound (2.20). If $w := v_N - v_{app}$, $q := p_N - p_{app}$,

$$\mathcal{I}_N - \mathcal{I}_{app} = \sum_{i=1}^{N} \int_{\partial B_i} \sigma_\mu(w, q)n \cdot Sx \, \mathrm{d}s - 2\mu \sum_{i=1}^{N} \int_{\partial B_i} w \cdot Sn \, \mathrm{d}s$$

Direct verifications show that v_{app}, hence w, satisfies the same force- and torque-free conditions as v. This means that for any family of constant vectors u_i and ω_i, $1 \leqq i \leqq N$,

$$\mathcal{I}_N - \mathcal{I}_{app} = \sum_{i=1}^{N} \int_{\partial B_i} \sigma_\mu(w, q)n \cdot (Sx - u_i - \omega_i \times (x - x_i)) \, \mathrm{d}s \\ -2\mu \sum_{i=1}^{N} \int_{\partial B_i} w \cdot Sn \, \mathrm{d}s.$$

By a proper choice of u_i and ω_i, we find

$$\begin{aligned} \mathcal{I}_N - \mathcal{I}_{app} &= -\sum_{i=1}^{N} \int_{\partial B_i} \sigma_\mu(w, q)n \cdot v_N \, \mathrm{d}s - 2\mu \sum_{i=1}^{N} \int_{\partial B_i} w \cdot Sn \, \mathrm{d}s \\ &= -\int_{\mathcal{F}} 2\mu D(w) : D(v_N) \, \mathrm{d}x - 2\mu \sum_{i=1}^{N} \int_{B_i} D(w) : S \, \mathrm{d}x \\ &= -\sum_{i=1}^{N} \int_{\partial B_i} \sigma_\mu(v_N, p_N)n \cdot w \, \mathrm{d}s - 2\mu \sum_{i=1}^{N} \int_{B_i} D(w) : S \, \mathrm{d}x \\ &= -\sum_{i=1}^{N} \int_{\partial B_i} \sigma_\mu(v_N, p_N)n \cdot (w + \tilde{u}_i + \tilde{\omega}_i \times (x - x_i)) \, \mathrm{d}s \\ &\quad - 2\mu \sum_{i=1}^{N} \int_{B_i} D(w) : S \, \mathrm{d}x \end{aligned} \tag{2.31}$$

for any family $(\tilde{u}_i, \tilde{\omega}_i)$, using this time that v_N is force- and torque-free. Let $q \geqq 2$. By a proper choice of $(\tilde{u}_i, \tilde{\omega}_i)$, by Poincaré and Korn inequalities, one can ensure that for all i,

$$\|w + \tilde{u}_i + \tilde{\omega}_i \times (x - x_i)\|_{W^{1-\frac{1}{q},q}(\partial B_i)} \leqq C \|D(w)\|_{L^q(B_i)},$$

where

$$\|g\|_{W^{1-\frac{1}{q},q}(\partial B_i)} = \inf \left\{ \frac{1}{a} \|G\|_{L^q(B_i)} + \|\nabla G\|_{L^q(B_i)}, \quad G|_{\partial B_i} = g \right\}.$$

Note that the factor $\frac{1}{a}$ at the right-hand side is consistent with scaling considerations. Moreover, by standard use of the Bogovskii operator, see [16], there exists a constant

C (depending only on the constant c in (H2)) and a field $W \in W^{1,q}(\mathcal{F})$, zero outside $\cup_{i=1}^N B(x_i, 2a)$ satisfying

$$\operatorname{div} W = 0 \quad \text{in } \mathcal{F}, \quad W|_{B_i} = (w + \tilde{u}_i + \tilde{\omega}_i \times (x - x_i))|_{B_i},$$
$$\|D(W)\|^q_{L^q(\mathcal{F})} \leqq \sum_i \|w + \tilde{u}_i + \tilde{\omega}_i \times (x - x_i)\|^q_{W^{1-\frac{1}{q},q}(B_i)}.$$

We deduce, with $p \leqq 2$ the conjugate exponent of q, that

$$\Big|\sum_{i=1}^N \int_{\partial B_i} \sigma_\mu(v_N, p_N)n \cdot (w + \tilde{u}_i + \tilde{\omega}_i \times (x - x_i))\, \mathrm{d}s\Big| = 2\mu\Big|\int_{\mathcal{F}} D(v_N) : D(W)\Big|$$
$$\leqq 2\mu \|D(v_N)\|_{L^p(\cup B(x_i,2a))} \|D(W)\|_{L^q(\mathcal{F})}$$
$$\leqq C\phi^{1/p-1/2} \|D(v_N)\|_{L^2(\mathbb{R}^3)} \Big(\sum_i \|D(w)\|^q_{L^q(B_i)}\Big)^{1/q}.$$

By well-known variational properties of the Stokes solution, $\|D(v_N)\|_{L^2}$ minimizes $\|D(v)\|_{L^2}$ over the set of all v in $\dot{H}^1(\mathbb{R}^3)$ satisfying a boundary condition of the form $v|_{B_i} = -Sx + u_i + \omega_i \times (x - x_i)$ for all i. By the same considerations as before, based on the Bogovski operator, we infer that

$$\|D(v_N)\|^2_{L^2(\mathbb{R}^3)} \leqq C \sum_{i=1}^N \|D(-Sx)\|^2_{L^2(B_i)} \leqq C'\phi,$$

so that

$$\Big|\sum_{i=1}^N \int_{\partial B_i} \sigma_\mu(v_N, p_N)n \cdot (w + \tilde{u}_i + \tilde{\omega}_i \times (x - x_i))\, \mathrm{d}s\Big|$$
$$\leqq C\phi^{1/p} \Big(\sum_i \|D(w)\|^q_{L^q(B_i)}\Big)^{1/q}.$$

Using this inequality with the first term in (2.31) and applying the Hölder inequality to the second term, we end up with

$$|\mathcal{I}_N - \mathcal{I}_{app}| \leqq C\phi^{1/p} \Big(\sum_i \|D(w)\|^q_{L^q(B_i)}\Big)^{1/q}. \tag{2.32}$$

To deduce (2.20), it is now enough to prove that for all $q > 1$, there exists a constant C independent of N or ϕ such that

$$\sum_i \|D(w)\|^q_{L^q(B_i)} \leqq C(\phi^{1+\frac{2q}{p}} + \phi^{1+\frac{4q}{3}}). \tag{2.33}$$

Indeed, taking $q > 2$, meaning $p < 2$, and combining this inequality with (2.32) yields (2.20), more precisely

$$|\mathcal{I}_N - \mathcal{I}_{app}| \leqq C(\phi^{1+\frac{2}{p}} + \phi^{\frac{7}{3}}).$$

In order to show the bound (2.33), we must write down the expression for $w|_{B_i} = v_N|_{B_i} - v_{app}|_{B_i}$, where v_{app} was introduced in (2.17). A little calculation, using Taylor's formula with an integral remainder, shows that

$$w|_{B_i}(x) = w_i^r(x) - D_i(x - x_i) - E_i(x - x_i) - \mathbf{F}_i|_x(x - x_i, x - x_i), \quad (2.34)$$

with w_i^r being a rigid vector field (that disappears when taking the symmetric gradient), with

$$D_i := \sum_{j \neq i} D(v[S_j])(x_i - x_j), \quad E_i := \sum_{j \neq i} D(v^s[S + S_j] - v[S + S_j])(x_i - x_j)$$

and with the bilinear application

$$\mathbf{F}_i|_x := \sum_{j \neq i} \int_0^1 (1 - t)\nabla^2 v^s[S + S_j](t(x - x_i) + x_i - x_j)\, \mathrm{d}t.$$

We remind that $v^s[S]$ and $v[S]$ were introduced in (2.14) and (2.15), while the matrices S_j are defined in (2.18). Note that the matrices D_i and S_i have the same kind of structure. More precisely, we can define for a collection $(A_1, \ldots, A_N)$ of N symmetric matrices, an application

$$\mathcal{A} : (A_1, \ldots, A_N) \to (A'_1, \ldots, A'_N), \quad A'_i = \sum_{j \neq i} D(v[A_j])(x_i - x_j).$$

Then, $(S_1, \ldots, S_N) = \mathcal{A}(S, \ldots, S)$ and $(D_1, \ldots, D_N) = \mathcal{A}(S_1, \ldots, S_N) = \mathcal{A}^2(S, \ldots, S)$. Note that for any matrix A, the kernel $D(v[A])$, homogeneous of degree -3, is of Calderón–Zygmund type. Using this property, we are able to prove in the appendix the following lemma, which is an adaptation of a result by the second author and Di Wu [21]:

Lemma 2.4. *For all $1 < q < +\infty$, there exists a constant C, depending on q and on the constant c in (H2), such that, if $(A'_1, \ldots, A'_N) = \mathcal{A}(A_1, \ldots, A_N)$, then*

$$\sum_{i=1}^N |A'_i|^q \leqq C\phi^{\frac{q}{p}} \sum_{i=1}^N |A_i|^q.$$

We can now proceed to the proof of (2.33). Denoting $w_i^1 := D_i(x - x_i)$, we find by the lemma that

$$\sum_i \|D(w_i^1)\|^q_{L^q(B_i)} \leqq Ca^3 \sum_i |D_i|^q \leqq C' a^3 \phi^{\frac{q}{p}} \sum_{i=1}^N |S_i|^q$$
$$\leqq C'' a^3 \phi^{\frac{2q}{p}} \sum_{i=1}^N |S|^q \leqq \mathcal{C}\phi^{1+\frac{2q}{p}}.$$

Then, we notice that for any matrix A, $|D(v^s[A] - v[A])(x)| = O(a^5|x|^{-5})$. This implies that $w_i^2 := E_i(x - x_i)$ satisfies

$$\sum_i \|D(w_i^2)\|^q_{L^q(B_i)} \leqq Ca^3 \sum_i |E_i|^q \leqq C'a^3 a^{5q} \sum_i \Big(\sum_{j\neq i} \frac{|S_j| + |S|}{|x_i - x_j|^5}\Big)^q.$$

By assumption (H2), the points $y_i := N^{1/3}x_i$ satisfy, for all $i \neq j$, that

$$|y_i - y_j| \geqq \frac{1}{2}(c + |y_i - y_j|) \geqq c.$$

In particular,

$$\sum_i \|D(w_i^2)\|^q_{L^q(B_i)} \leqq Ca^3\phi^{5q/3} \sum_i (\sum_j \frac{|S| + |S_j|}{(c + |y_i - y_j|)^5})^q.$$

We then make use of the following easy generalization of Young's convolution inequality: $\forall q \geqq 1$,

$$\sum_i (\sum_j |a_{ij} b_j|)^q \leqq \max\big(\sup_i \sum_j |a_{ij}|, \sup_j \sum_i |a_{ij}|\big)^q \sum_i |b_i|^q. \tag{2.35}$$

Applied with $a_{ij} = \frac{1}{(c+|y_i-y_j|)^5}$ and $b_j = |S| + |S_j|$, together with Lemma 2.4, this yields

$$\sum_i \|D(w_i^2)\|^q_{L^q(B_i)} \leqq Ca^3\phi^{5q/3}\big(\sum_j |S|^q + |S_j|^q\big)$$
$$\leqq C'a^3\phi^{5q/3}(1 + \phi^{\frac{q}{p}})N \leqq \mathcal{C}\phi^{1+\frac{5q}{3}}.$$

It remains to bound the symmetric gradient of $w_i^3 := \mathbf{F}_i|_x(x - x_i, x - x_i)$. By the expression of v^s, we get that, in B_i

$$|D(w_i^3)| \leqq C \sum_{j\neq i} \left(\frac{a^5}{|x_i - x_j|^5} + \frac{a^4}{|x_i - x_j|^4}\right)(|S| + |S_j|).$$

Proceeding as above, we find

$$\sum_i \|D(w_i^3)\|^q_{L^q(B_i)} \leqq Ca^3(\phi^{5q/3} + \phi^{4q/3})(1 + \phi^{\frac{q}{p}})N \leqq C'\phi^{1+\frac{4q}{3}}.$$

As $D(w) = D(w_i^1) + D(w_i^2) + D(w_i^3)$, *cf.* (2.34), the previous estimates yield (2.33). This concludes the proof of Proposition 2.1, and therefore the proof of Theorem 1.1.

3. The ϕ^2 Correction $\mathcal{V}_N$ as a Renormalized Energy

We start in this section the asymptotic analysis of the viscosity coefficient

$$\mathcal{V}_N = \frac{75|\mathcal{O}|}{16\pi}\Big(\frac{1}{N^2}\sum_{i\neq j} g_S(x_i - x_j) - \int_{\mathbb{R}^3\times\mathbb{R}^3} g_S(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y\Big).$$

As a preliminary step, we will show that there is no loss of generality in assuming

$$\forall i \in \{1,\ldots,N\}, \quad \mathrm{dist}(x_i, \mathcal{O}^c) \geqq \frac{1}{\ln N}. \tag{3.1}$$

We introduce the set

$$I_{N,ext} = \Big\{1 \leqq i \leqq N,\ \mathrm{dist}(x_i, \mathcal{O}^c) \leqq \frac{1}{\ln N}\Big\}, \quad \text{and } N_{ext} = N_{ext}(N) := |I_{N,ext}|.$$

By (H1)–(H2), it is easily seen that $N_{ext} = o(N)$ as $N \to +\infty$. We now show

Lemma 3.1. *$\mathcal{V}_N$ is uniformly bounded in N, and*

$$\mathcal{V}_{N,ext} := \mathcal{V}_N - \frac{75|\mathcal{O}|}{16\pi}\Big(\frac{1}{(N-N_{ext})^2}\sum_{\substack{i\neq j\\ i,j\notin I_{N,ext}}} g_S(x_i - x_j) - \int_{\mathbb{R}^3\times\mathbb{R}^3} g_S(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y\Big)$$

goes to zero as $N \to +\infty$.

Proof: For any open set U, we denote $⨍_U = \frac{1}{|U|}\int_U$.
Let $d := \frac{c}{4}N^{-1/3} \leqq \min_{i\neq j}\frac{|x_i - x_j|}{4}$ by (H2). We write

$$\begin{aligned}\frac{1}{N^2}\sum_{i\neq j} g_S(x_i-x_j) &= \frac{1}{N^2}\sum_{i\neq j}\Big(g_S(x_i-x_j) - ⨍_{B(x_j,d)} g_S(x_i-y)\mathrm{d}y\Big)\\ &\quad + \frac{1}{N^2}\sum_{i\neq j}\Big(⨍_{B(x_j,d)} g_S(x_i-y)\mathrm{d}y - ⨍_{B(x_i,d)}⨍_{B(x_j,d)} g_S(x-y)\mathrm{d}x\mathrm{d}y\Big)\\ &\quad + \frac{1}{N^2}\sum_{i\neq j}⨍_{B(x_i,d)}⨍_{B(x_j,d)} g_S(x-y)\mathrm{d}x\mathrm{d}y \ := I + II + III.\end{aligned}$$

For the first term, with $y_i := N^{1/3}x_i$ and with (H2) in mind, that is $|y_i - y_j| \geqq c$ for $i \neq j$, we get that

$$\begin{aligned}\Big|g_S(x_i-x_j) - ⨍_{B(x_j,d)} g_S(x_i-y)\mathrm{d}y\Big| &\leqq ⨍_{B(x_j,d)} \sup_{z\in[x_j,y]} |\nabla g_S|(x_i - z)||x_j-y|\mathrm{d}y\\ &\leqq CN^{4/3}\frac{d}{(c+|y_i-y_j|)^4};\end{aligned}$$

see (1.14). This yields, by a discrete convolution inequality,

$$|I| \leqq \frac{CN^{7/3}}{N^2} d \sup_i \sum_j \frac{1}{(c+|y_i - y_j|)^4} \leqq C' N^{1/3} d \leqq \mathcal{C},$$

where we have used that $\sum_{j=1}^N \frac{1}{(c+|y_i-y_j|)^4}$ is uniformly bounded in N and in the index i thanks to the separation assumption. By similar arguments, $|II| \leqq \mathcal{C}$. As regards the last term, we notice that

$$|III| \leqq \frac{1}{N^2 d^6} \Big| \int_{\mathbb{R}^3 \times \mathbb{R}^3} g_S(x-y) F_N(x) F_N(y) \mathrm{d}y \\ - \sum_{i=1}^N \int_{\mathbb{R}^3 \times \mathbb{R}^3} g_S(x-y) 1_{B(x_i,d)}(x) 1_{B(x_i,d)}(y) \mathrm{d}x \mathrm{d}y \Big|,$$

where $F_N = \sum_{i=1}^N 1_{B(x_i,d)}$. The operator $\mathcal{T}F(x) = \int g_S(x-y)F(y)\mathrm{d}y$ is a Calderón–Zygmund operator, and therefore continuous over L^2. As $F_N^2 = F_N$ (the balls are disjoint), we find that the L^2 norm of F_N is $(Nd^3)^{1/2}$ and

$$\Big| \int_{\mathbb{R}^3 \times \mathbb{R}^3} g_S(x-y) F_N(x) F_N(y) \mathrm{d}y \Big| \leqq \|\mathcal{T}\| \|F_N\|_{L^2}^2 \leqq \|\mathcal{T}\| N d^3.$$

Similarly,

$$\sum_{i=1}^N \Big| \int_{\mathbb{R}^3 \times \mathbb{R}^3} g_S(x-y) 1_{B(x_i,\eta)}(x) 1_{B(x_i,\eta)}(y) \mathrm{d}x \mathrm{d}y \Big| \leqq N \|\mathcal{T}\| d^3.$$

It follows that $|III| \leqq \frac{C}{Nd^3}$. With our choice of d, the first part of the lemma is proved.

From there, to prove that $\mathcal{V}_{N,ext}$ goes to zero, as $N_{ext} = o(N)$, it is enough to show that

$$\frac{1}{N^2} \Big(\sum_{i \neq j} g_S(x_i - x_j) - \sum_{\substack{i \neq j, \\ i,j \notin I_{N,ext}}} g_S(x_i - x_j) \Big) \rightarrow 0.$$

By symmetry, it is enough that

$$\frac{1}{N^2} \sum_{\substack{i \neq j, \\ i \in I_{N,ext}}} g_S(x_i - x_j) \rightarrow 0.$$

This can be shown by a similar decomposition as the previous one. Namely,

$$\frac{1}{N^2}\sum_{i\neq j} g_S(x_i-x_j) = \frac{1}{N^2}\sum_{\substack{i\neq j\\ i\in I_{N,ext}}}\left(g_S(x_i-x_j) - \fint_{B(x_j,d)} g_S(x_i-y)\mathrm{d}y\right)$$
$$+\frac{1}{N^2}\sum_{\substack{i\neq j\\ i\in I_{N,ext}}}\left(\fint_{B(x_j,d)} g_S(x_i-y)\mathrm{d}y - \fint_{B(x_i,d)}\fint_{B(x_j,d)} g_S(x-y)\mathrm{d}x\mathrm{d}y\right)$$
$$+\frac{1}{N^2}\sum_{\substack{i\neq j\\ i\in I_{N,ext}}}\fint_{B(x_i,d)}\fint_{B(x_j,d)} g_S(x-y)\mathrm{d}x\mathrm{d}y := I_{ext}+II_{ext}+III_{ext}.$$

Proceeding as above, we find this time that

$$|I_{ext}|+|II_{ext}|+|III_{ext}| \leqq \mathcal{C}\frac{N_{ext}}{N} \to 0 \quad \text{as } N\to+\infty,$$

which concludes the proof. □

Remark 3.2. By Lemma 3.1, there is no restriction assuming (3.1) when studying the asymptotic behaviour of $\mathcal{V}_N$. *Therefore, we make from now on the assumption* (3.1).

As explained in the introduction, the analysis of $\mathcal{V}_N$ will rely on the mathematical methods introduced over the last years for Coulomb gases, the core problem being the analysis of a functional of the form (1.15). We shall first reexpress $\mathcal{V}_N$ in a similar form. More precisely, we will show

Proposition 3.3. *Denoting*

$$\mathcal{W}_N := \frac{75|\mathcal{O}|}{16\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\big(\mathrm{d}\delta_N(x)-f(x)\mathrm{d}x\big)\big(\mathrm{d}\delta_N(y)-f(y)\mathrm{d}y\big),$$

we have $\mathcal{V}_N = \mathcal{W}_N + \varepsilon(N)$ *where* $\varepsilon(N)\to 0$ *as* $N\to\infty$.

Remark 3.4. In the definition of $\mathcal{W}_N$, the integrals of the form

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\mathrm{d}\delta_N(x)f(y)\mathrm{d}y,$$
$$\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)f(x)\mathrm{d}x\mathrm{d}\delta_N(y),$$

which appear when expanding the product, are understood as

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\mathrm{d}\delta_N(x)f(y)\mathrm{d}y = \frac{8\pi}{3}\frac{1}{N}\sum_{i=1}^N S\nabla\cdot \mathrm{St}^{-1}S\nabla f(x_i),$$
$$\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)f(x)\mathrm{d}x\mathrm{d}\delta_N(y) = \frac{8\pi}{3}\frac{1}{N}\sum_{i=1}^N S\nabla\cdot \mathrm{St}^{-1}S\nabla f(x_i),$$

where St is the Stokes operator; see (2.30) and the proof below for an explanation.

Proof. Clearly,

$$\mathcal{V}_N = \frac{75|\mathcal{O}|}{16\pi} \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\Big(\mathrm{d}\delta_N(x)\mathrm{d}\delta_N(y) - f(x)f(y)\mathrm{d}x\mathrm{d}y\Big),$$

so that, formally,

$$\mathcal{V}_N = \mathcal{W}_N + \frac{75|\mathcal{O}|}{16\pi} \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)(\mathrm{d}\delta_N(x) - f(x)\mathrm{d}x) f(y)\mathrm{d}y$$
$$+ \frac{75|\mathcal{O}|}{16\pi} \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y) f(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y).$$

Note that it is not obvious that this formal decomposition makes sense, because all three quantities at the right-hand side involve integrals of $g_S(x-y)$ against product measures of the form $\mathrm{d}\delta_N(x)f(y)\mathrm{d}y$ (or the symmetric one), which may fail to converge because of the singularity of g_S. To solve this issue, a rigorous path consists in approximating, at fixed N, each Dirac mass δ_{x_i} by a (compactly suppported) approximation of unity $\rho_\eta(x-x_i)$, where $\eta > 0$ is the approximation parameter and goes to zero. One can then set, for each η, $\delta_N^\eta(x) := \frac{1}{N}\sum_{i=1}^N \rho_\eta(x-x_i)$, leading to the rigorous decomposition

$$\mathcal{V}_N^\eta = \mathcal{W}_N^\eta + \frac{75|\mathcal{O}|}{16\pi} \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)(\delta_N^\eta(x)d(x) - f(x)\mathrm{d}x) f(y)\mathrm{d}y$$
$$+ \frac{75|\mathcal{O}|}{16\pi} \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y) f(x)\mathrm{d}x(\delta_N^\eta(y)\mathrm{d}y - f(y)\mathrm{d}y),$$

where $\mathcal{V}_N^\eta$, $\mathcal{W}_N^\eta$ are deduced from $\mathcal{V}_N$, $\mathcal{W}_N$ replacing the empirical measure by its regularization. It is easy to show that $\lim_{\eta\to 0}\mathcal{V}_N^\eta = \mathcal{V}_N$. To conclude the proof, we shall establish the following: first,

$$\lim_{\eta\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\delta_N^\eta(x)\mathrm{d}x f(y)\mathrm{d}y = \frac{8\pi}{3}\frac{1}{N}\sum_{i=1}^N S\nabla \mathrm{St}^{-1} S\nabla f(x_i); \tag{3.2}$$

the same limit holding for the symmetric term. In particular, (3.2) will show that $\mathcal{W}_N = \lim_{\eta\to 0}\mathcal{W}_N^\eta$ exists, in the sense given in Remark 3.4. Then, we will prove

$$\lim_{N\to+\infty}\frac{8\pi}{3}\frac{1}{N}\sum_{i=1}^N S\nabla\mathrm{St}^{-1}S\nabla f(x_i) = \int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y, \tag{3.3}$$

which, together with (3.2), will complete the proof of the proposition.

The limit (3.2) follows from identity (2.30). Indeed, for $\eta > 0$, this formula yields

$$\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\delta_N^\eta(x)\mathrm{d}x f(y)\mathrm{d}y = -\frac{8\pi}{3}\int_{\mathbb{R}^3}\mathrm{St}^{-1}(S\nabla f)(x)\cdot S\nabla\delta_N^\eta(x)\mathrm{d}x.$$

Now, we remark that due to our assumptions on f, by elliptic regularity, $h = \mathrm{St}^{-1}(S\nabla f)(x)$ is C^1 inside $\mathcal{O}$. Moreover, in virtue of Remark (3.2), we can assume (3.1). Hence, as $\eta \to 0$,

$$-\frac{8\pi}{3}\int_{\mathbb{R}^3} h(x)\cdot S\nabla\delta_N^\eta(x)\mathrm{d}x \;\to\; -\frac{8\pi}{3}\langle S\nabla\delta_N, h\rangle = \frac{8\pi}{3}\frac{1}{N}\sum_{i=1}^N S\nabla\cdot h(x_i).$$

It remains to prove (3.3). In the special case where $f \in C^r(\mathbb{R}^3)$ for some $r \in (0,1)$ (implying that it vanishes at $\partial\mathcal{O}$), classical results on Calderón–Zygmund operators yield that the function $\int_{\mathbb{R}^3} g_S(x-y)f(x)\mathrm{d}x = \frac{8\pi}{3}S\nabla\cdot h(y)$ is a continuous (even Hölder) bounded function, so (H1) implies straightforwardly that

$$\begin{aligned}&\int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathrm{Diag}} g_S(x-y)f(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\ &\quad= \int_{\mathbb{R}^3}\frac{8\pi}{3}S\nabla\cdot h(y)(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y) \to 0.\end{aligned}$$

In the general case where f is discontinuous across $\partial\mathcal{O}$, the proof is a bit more involved. The difficulty lies in the fact that some points x_i get closer to the boundary as $N \to +\infty$.

Let $\varepsilon > 0$. Under (H2), there exists $c' > 0$ (depending on c only) such that for $N^{-1/3} \leqq \varepsilon$,

$$\big|\{i,\ x_i \text{ belongs to the } c'\varepsilon \text{ neighborhood of } \partial\mathcal{O}\}\big| \leqq \varepsilon N. \tag{3.4}$$

Let $\chi_\varepsilon : \mathbb{R}^3 \to [0,1]$ be a smooth function such that $\chi_\varepsilon = 1$ in a $c'\varepsilon/4$ neighborhood of $\partial\mathcal{O}$, $\chi_\varepsilon = 0$ outside a $c'\varepsilon/2$ neighborhood of $\partial\mathcal{O}$. We write

$$\begin{aligned}&\int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathrm{Diag}} g_S(x-y)f(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\ &\quad= \int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathrm{Diag}} g_S(x-y)(\chi_\varepsilon f)(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\ &\qquad+ \int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathrm{Diag}} g_S(x-y)((1-\chi_\varepsilon)f)(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y).\end{aligned}$$

By formula (2.30), the second term reads as

$$\begin{aligned}&\int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus\mathrm{Diag}} g_S(x-y)(1-\chi_\varepsilon f)(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\ &\quad= \frac{8\pi}{3}\int_{\mathbb{R}^3} S\nabla\cdot u_\varepsilon(y)\,(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y),\end{aligned}$$

with $u_\varepsilon = \mathrm{St}^{-1}S\nabla((1-\chi_\varepsilon)f)$. The source term $(1-\chi_\varepsilon)f$ being C^1 and compactly supported, $S\nabla\cdot u_\varepsilon$ is Hölder and bounded, so that, as $N \to +\infty$, the integral goes to zero by the weak convergence assumption (H1), for any fixed $\varepsilon > 0$. As regards the first term, we split it again into

$$\begin{aligned}
&\int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus \mathrm{Diag}} g_S(x-y)(\chi_\varepsilon f)(x)\mathrm{d}x(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\
&\quad = \int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus \mathrm{Diag}} g_S(x-y)(\chi_\varepsilon f)(x)\mathrm{d}x\,\chi_\varepsilon(y)(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\
&\qquad + \int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus \mathrm{Diag}} g_S(x-y)(\chi_\varepsilon f)(x)\mathrm{d}x(1-\chi_\varepsilon)(y)(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\
&\quad = \frac{8\pi}{3}\int_{\mathbb{R}^3} S\nabla\cdot v_\varepsilon(y)\chi_\varepsilon(y)(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\\
&\qquad + \frac{8\pi}{3}\int_{\mathbb{R}^3} S\nabla\cdot v_\varepsilon(y)(1-\chi_\varepsilon)(y)(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y),
\end{aligned}$$

where v_ε is this time the solution of the Stokes equation with source $S\nabla(\chi_\varepsilon f)$. It is Hölder away from $\partial\mathcal{O}$, so that the last term at the right-hand side goes again to zero as $N \to +\infty$, by assumption (H1).

It remains to handle the first term at the right-hand side. We shall show below that for a proper choice of χ_ε one has

$$\|\nabla v_\varepsilon\|_{L^\infty} \leqq C, \quad C \text{ independent of } \varepsilon. \tag{3.5}$$

Taking advantage of this fact, we write

$$\begin{aligned}
&\left|\frac{8\pi}{3}\int_{\mathbb{R}^3} S\nabla\cdot v_\varepsilon(y)\chi_\varepsilon(y)(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y)\right|\\
&\quad \leqq \frac{8\pi}{3}\|S\cdot\nabla v_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\left(\frac{1}{N}\,|\{i,\ \chi_\varepsilon(x_i) > 0\}| + \|\chi_\varepsilon f\|_{L^1}\right) \leqq C\varepsilon,
\end{aligned}$$

where we used property (3.4) to obtain the last inequality. With this bound and the convergence to zero of the other terms for fixed ε and $N \to +\infty$, the limit (3.3) follows.

We still have to show that ∇v^ε is uniformly bounded in L^∞ for a good choice of χ_ε. We borrow here to the analysis of vortex patches in the Euler equation, initiated by Chemin in 2-d [10], extended by Gamblin and Saint-Raymond in 3-d [17]. First, as $\mathcal{O}$ is smooth, one can find a family of five smooth divergence-free vector fields $w_1, \dots, w_5$, tangent at $\partial\mathcal{O}$ and non-degenerate in the sense that

$$\inf_{x\in\mathbb{R}^3}\sum_{i\neq j}|w_i \times w_j| > 0;$$

see [17, Proposition 3.2]. We take χ_ε in the form $\chi(t/\varepsilon)$, for a coordinate t transverse to the boundary, meaning that ∂_t is normal at $\partial\mathcal{O}$. With this choice and the assumptions on f, one checks easily that $\chi_\varepsilon f$ is bounded uniformly in ε in $L^\infty(\mathbb{R}^3)$ and that for all i, $w_i\cdot\nabla(\chi_\varepsilon f)$ is bounded uniformly in ε in $C^0(\mathbb{R}^3) \subset C^{r-1}(\mathbb{R}^3)$ for all $r \in (0,1)$. Hence, the norm $\|\chi_\varepsilon f\|_{r,W}$ introduced in [17, p. 395], where $W = (w_1, \dots, w_5)$, is bounded uniformly in ε.

We then split the Stokes system

$$-\Delta v_\varepsilon + \nabla p_\varepsilon = S\nabla(\chi_\varepsilon f), \quad \operatorname{div} v_\varepsilon = 0$$

into the equations

$$\operatorname{curl} v_\varepsilon = \Omega_\varepsilon, \quad \operatorname{div} v_\varepsilon = 0$$

and

$$-\Delta\Omega_\varepsilon = \operatorname{curl} S\nabla(\chi_\varepsilon f).$$

Let us show that $\partial_i\partial_j\Delta^{-1}(\chi_\varepsilon f)$ is bounded uniformly in ε in L^∞. Let $\chi \in C_c^\infty(\mathbb{R}^3)$, $\chi \geqq 0$, $\chi = 1$ near zero. Let for all $m \in \mathbb{R}$, $\Lambda^m(\xi) := (\chi(\xi) + |\xi|^2)^{m/2}$. It is easily seen through the Fourier transform that, for all $s \in \mathbb{N}$,

$$\|\partial_i\partial_j\chi(D)\Lambda^{-2}(D)\Delta^{-1}(\chi_\varepsilon f)\|_{H^s} \leqq C_s\|\chi_\varepsilon f\|_{L^2} \leqq C'_s. \tag{3.6}$$

Moreover, by the calculations in [17, p. 401], replacing ω with $\chi_\varepsilon f$, we get

$$\|\partial_i\partial_j\Lambda^{-2}(D)(\chi_\varepsilon f)\|_{L^\infty} \leqq C\|\chi_\varepsilon f\|_{L^\infty}\ln\Big(2 + \frac{\|\chi_\varepsilon f\|_{r,W}}{\|\chi_\varepsilon f\|_{L^\infty}}\Big) \leqq C'_r, \quad \forall 0 < r < 1. \tag{3.7}$$

Combining (3.6) and (3.7), we find that

$$\partial_i\partial_j\Delta^{-1}(\chi_\varepsilon f) = \partial_i\partial_j\left(\chi(D)\Lambda^{-2}(D)\Delta^{-1} + \Lambda^{-2}(D)\right)(\chi_\varepsilon f)$$

is bounded uniformly in ε in L^∞, and consequently that

$$\|\Omega_\varepsilon\|_{L^\infty} \leqq C.$$

Also, by continuity of Riesz transforms over L^p, we have

$$\forall 1 < p < \infty, \quad \|\Omega_\varepsilon\|_{L^2} \leqq C_p\|\chi_\varepsilon f\|_{L^p} \leqq C'_p.$$

Now, applying $w_k \cdot \nabla$ to the equation satisfied by Ω_ε, we obtain for all $1 \leqq k \leqq 5$,

$$\begin{aligned} -\Delta(w_k\cdot\nabla\Omega_\varepsilon) &= \operatorname{curl} S\nabla(w_k\cdot\nabla(\chi_\varepsilon f)) + [w_k\cdot\nabla, \operatorname{curl} S\nabla](\chi_\varepsilon f) + [w_k\cdot\nabla, \Delta]\Omega_\varepsilon \\ &= \sum_{i,j}\partial_i\partial_j F_{i,j,\varepsilon} + \sum_i \partial_i G_{i,\varepsilon} + H_\varepsilon, \end{aligned} \tag{3.8}$$

where $F_{i,j,\varepsilon}$, $G_{i,\varepsilon}$ and H_ε are combinations of Ω_ε, $\chi_\varepsilon f$ and $w_k \cdot \nabla(\chi_\varepsilon f)$. In particular, they are bounded uniformly in ε in $L^\infty \cap L^p$, for any $1 < p < \infty$.

For the first term at the r.h.s., we write, with the same cut-off function χ as before,

$$(-\Delta)^{-1}\sum_{i,j}\partial_i\partial_j F_{i,j,\varepsilon} = \chi(D)(-\Delta)^{-1}\sum_{i,j}\partial_i\partial_j F_{i,j,\varepsilon} + (1-\chi(D))\sum_{i,j}\partial_i\partial_j F_{i,j,\varepsilon}.$$

By continuity of $(-\Delta)^{-1}\partial_i\partial_j$ over L^2, the first term, with low frequencies, belongs to H^s for any s, with uniform bound in ε. By the continuity of $(1-\chi(D))(-\Delta)^{-1}\partial_i\partial_j$

over Hölder spaces (Coifman-Meyer theorem), the second term, with high frequencies, is uniformly bounded in ε in $C^{r-1}(\mathbb{R}^3)$, for any $0 < r < 1$.

For the second and third terms in (3.8), we claim that

$$\|(-\Delta)^{-1}\sum_i \partial_i G_{i,\varepsilon}\|_{L^\infty} \leqq C, \quad \|(-\Delta)^{-1} H_\varepsilon\|_{L^\infty} \leqq C.$$

This can be easily seen by expressing these fields as $\sum_i \partial_i \Phi \star G_{i,\varepsilon}$ and $\Phi \star H_\varepsilon$ with Φ the fundamental solution, and by using the uniform L^p bounds on $G_{i,\varepsilon}$ and H_ε. Eventually, we find that

$$\|w_k \cdot \nabla\Omega_\varepsilon\|_{C^{r-1}} \leqq C_r, \quad \forall 1 \leqq k \leqq 5, \quad \forall 0 < r < 1.$$

We conclude by [17, Proposition 3.3] that ∇v_ε is bounded in $L^\infty(\mathbb{R}^3)$ uniformly in ε. □

3.1. Smoothing

By Proposition 3.3, we are left with understanding the asymptotic behaviour of

$$\mathcal{W}_N := \frac{75|\mathcal{O}|}{16\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus Diag} g_S(x-y)\big(\mathrm{d}\delta_N(x) - f(x)\mathrm{d}x\big)\big(\mathrm{d}\delta_N(y) - f(y)\mathrm{d}y\big). \tag{3.9}$$

The following field will play a crucial role: for U, Q defined in (2.26), we set

$$G_S(x) := S_{kl}\partial_k U_l(x), \quad p_S(x) = S_{kl}\partial_k Q_l(x). \tag{3.10}$$

From (2.27), we have $g_S = \frac{8\pi}{3}(S\nabla)\cdot G_S$, and that G_S solves, in the sense of distributions,

$$-\Delta G_S + \nabla p_S = S\nabla\delta, \quad \operatorname{div} G_S = 0 \text{ in } \mathbb{R}^3. \tag{3.11}$$

Moreover, from the explicit expression

$$U_l(x) = \frac{1}{8\pi}\left(\frac{1}{|x|}e_l + \frac{x_l}{|x|^3}x\right), \quad Q_l(x) = \frac{1}{4\pi}\frac{x_l}{|x|^3},$$

and taking into account the fact that S is symmetric and trace-free, we get

$$G_S(x) = -\frac{3}{8\pi}S_{kl}x_l x_k\frac{x}{|x|^5} = -\frac{3}{8\pi}(Sx\cdot x)\frac{x}{|x|^5}, \quad p_S(x) = -\frac{3}{4\pi}\frac{(Sx\cdot x)}{|x|^5}. \tag{3.12}$$

Let us note that G_S is called a point stresslet in the literature, see [18]. It can be interpreted as the velocity field created in a fluid of viscosity 1 by a point particle whose resistance to a strain is given by $-S$.

We now come back to the analysis of (3.9). Formal replacement of the function f in Lemma 2.2 by $\delta_N - f$ yields the formula

$$"\int_{\mathbb{R}^3\times\mathbb{R}^3} g_S(x-y)\big(\mathrm{d}\delta_N(x)-f(x)\mathrm{d}x\big)\big(\mathrm{d}\delta_N(y)-f(y)\mathrm{d}y\big) = -\frac{16\pi}{3N^2}\int_{\mathbb{R}^3}|D(h_N)|^2", \tag{3.13}$$

where

$$h_N(x) := \sum_{i=1}^{N} G_S(x - x_i) - N\mathrm{St}^{-1}(S\nabla f)$$
$$= \sum_{i=1}^{N} G_S(x - x_i) - N \int_{\mathbb{R}^3} G_S(x - y) f(y) \mathrm{d}y \tag{3.14}$$

satisfies

$$-\Delta h_N + \nabla q_N = S\nabla \sum_i \delta_{x_i} \; - \; NS\nabla f, \quad \operatorname{div} h_N = 0 \quad \text{in } \mathbb{R}^3. \tag{3.15}$$

The formula (3.13) is similar to the formula (1.16), and is as much abusive, as both sides are infinite. Still, by an appropriate regularization of the source term $S\nabla \sum_i \delta_{x_i}$, we shall be able in the end to obtain a rigorous formula, convenient for the study of $\mathcal{W}_N$. This regularization process is the purpose of the present paragraph.

For any $\eta > 0$, we denote $B_\eta = B(0, \eta)$, and define G_S^η by

$$G_S^\eta = G_S, \; p_S^\eta = p_S \; \text{ outside } B_\eta, \tag{3.16}$$
$$-\Delta G_S^\eta + \nabla p_S^\eta = 0, \quad \operatorname{div} G_S^\eta = 0, \quad G_S^\eta|_{\partial B_\eta} = G_S|_{\partial B_\eta} \text{ in } B_\eta. \tag{3.17}$$

Note that, by homogeneity,

$$G_S^\eta(x) = \frac{1}{\eta^2} G_S^1(x/\eta). \tag{3.18}$$

The field G_S^η belongs to $\dot{H}^1(\mathbb{R}^3)$, and solves

$$-\Delta G_S^\eta + \nabla p_S^\eta = S^\eta, \tag{3.19}$$

where S^η is the measure on the sphere defined by

$$S^\eta := -\left[2D(G_S^\eta)n - p_S^\eta n\right] s^\eta = -\left[\partial_n G_S^\eta - p_S^\eta n\right] s^\eta, \tag{3.20}$$

with $n = \frac{x}{|x|}$ the unit normal vector pointing *outward* B_η, $[F] := F|_{\partial B_\eta^+} - F|_{\partial B_\eta^-}$ the jump at ∂B_η (with ∂B_η^+, resp. ∂B_η^-, the outer, resp. inner boundary of the ball), and s^η the standard surface measure on ∂B_η. We claim the following:

Lemma 3.5. *For all $\eta > 0$, $S^\eta = \operatorname{div} \Psi^\eta$ in $\mathbb{R}^3$, where*

$$\Psi^\eta := \frac{3}{\pi \eta^5} \left(Sx \otimes x + x \otimes Sx - 5\frac{|x|^2}{2} S + \frac{5}{4}\eta^2 S \right)$$
$$- 2D(G_S^\eta)(x) + p_S^\eta(x)\mathrm{Id}, \quad x \in B_\eta,$$
$$\Psi^\eta := 0 \text{ outside}. \tag{3.21}$$

Moreover, $\Psi^\eta \to S\delta$ in the sense of distributions as $\eta \to 0$, so that $S^\eta \to S\nabla\delta$.

Proof of the lemma.. From the explicit formula (3.12) for G_S and p_S, we find

$$2D(G_S) = -\frac{3}{4\pi}\frac{Sx\otimes x + x\otimes Sx}{|x|^5} + \frac{15}{4\pi}\frac{(Sx\cdot x)x\otimes x}{|x|^7} - \frac{3}{4\pi}\frac{Sx\cdot x}{|x|^5}\mathrm{Id.},$$

so that

$$(2D(G_S^\eta)n - p_S^\eta n)|_{\partial B_\eta^+} = (2D(G_S)n - p_S n)|_{\partial B_\eta^+} = \frac{3}{4\pi|\eta|^3}\left(4(Sn\cdot n)n - Sn\right). \tag{3.22}$$

Using that S is trace-free, one can check from definition (3.21) that $\operatorname{div}\Psi^\eta = 0$ in the complement of ∂B_η, while

$$\begin{aligned}[\Psi^\eta n] &= -\Psi^\eta n|_{\partial B_\eta^-} \\ &= \frac{3}{\pi\eta^3}\Big((Sn\otimes n)n + (n\otimes Sn)n - \frac{5}{4}Sn\Big) - (2D(G_S^\eta)n + p_S^\eta n)|_{\partial B_\eta^-} \\ &= (2D(G_S^\eta)n - p_S^\eta n)|_{\partial B_\eta^+} - (2D(G_S^\eta)n + p_S^\eta n)|_{\partial B_\eta^-},\end{aligned}$$

where the last equality comes from (3.22). Together with (3.20), this implies the first claim of the lemma.

To compute the limit of Ψ^η as $\eta\to 0$, we write $\Psi^\eta = \Psi_1^\eta + \Psi_2^\eta$, with

$$\begin{aligned}\Psi_1^\eta &= \frac{3}{\pi\eta^5}\left(Sx\otimes x + x\otimes Sx - 5\frac{|x|^2}{2}S + \frac{5}{4}\eta^2 S\right), \\ \Psi_2^\eta &= -2D(G_S^\eta)(x) + p_S^\eta(x)\mathrm{Id}..\end{aligned}$$

Let $\varphi\in C_c^\infty(\mathbb{R}^3)$ be a test function. We can write $\langle\Psi_1^\eta,\varphi\rangle = \langle\Psi_1^\eta,\varphi(0)\rangle + \langle\Psi_1^\eta,\varphi - \varphi(0)\rangle$. The second term is $O(\eta)$, while the first term can be computed using the elementary formula $\int_{B_1} x_i x_j \mathrm{d}x = \frac{4\pi}{15}\delta_{ij}$. We find

$$\lim_{\eta\to 0}\langle\Psi_1^\eta,\varphi\rangle = \frac{3}{5}S\varphi(0) = \langle\frac{3}{5}S\delta,\varphi\rangle. \tag{3.23}$$

For the second term, using the homogeneity (3.18), we find again that $\lim_\eta\langle\Psi_2^\eta,\varphi\rangle = \langle\Psi_2^1,\varphi(0)\rangle$. Note that the pressure p_S^1 is defined up to a constant, so that we can always select the one with zero average. With this choice, we find

$$\begin{aligned}\langle\Psi_2^1,\varphi(0)\rangle &= \int_{B_\eta}\big(-2D(G_S^\eta) + p_S^\eta\mathrm{Id}\big)\varphi(0) = -2\int_{B_1} D(G_S^1)\,\varphi(0) \\ &= -\int_{\partial B_1}(n\otimes G_S^1 + G_S^1\otimes n)\,\varphi(0) = -\int_{\partial B_1}(n\otimes G_S + G_S\otimes n)\,\varphi(0) \\ &= \frac{3}{4\pi}\int_{\partial B_1}(Sn\cdot n)n\otimes n\,\varphi(0) = \frac{2}{5}S\varphi(0) = \langle\frac{2}{5}S\delta,\varphi\rangle,\end{aligned} \tag{3.24}$$

where the sixth equality comes from the elementary formula $\int_{\partial B_1} n_i n_j n_k n_l \mathrm{d}s^1 = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The result follows. □

For later purposes, we also prove here

Lemma 3.6.

$$\int_{\partial B_\eta} G_S^\eta dS^\eta = \int_{\partial B_\eta} G_S dS^\eta = \frac{1}{\eta^3}\left(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right).$$

Proof.

$$\begin{aligned}\int_{\partial B_\eta} G_S^\eta dS^\eta &= \int_{\partial B_\eta} G_S^\eta \left(\partial_n G_S^\eta - p_S n\right)|_{\partial B_\eta^-} \mathrm{d}s^\eta \quad - \int_{\partial B_\eta} G_S^\eta \left(\partial_n G_S^\eta - p_S n\right)|_{\partial B_\eta^+} \mathrm{d}s^\eta \\ &= \int_{B_\eta} |\nabla G_S^\eta|^2 \mathrm{d}x \quad - \int_{\partial B_\eta} G_S \left(\partial_r G_S - p_S e_r\right)|_{\partial B_\eta} \mathrm{d}s^\eta.\end{aligned}$$

By (3.18), $\int_{B_\eta} |\nabla G_S^\eta|^2 \mathrm{d}x = \frac{1}{\eta^3}\int_{B_1} |\nabla G_S^1|^2 \mathrm{d}x$. The second term can be computed with (3.12):

$$\begin{aligned}&\int_{\partial B_\eta} G_S \left(\partial_r G_S - p_S e_r\right)|_{\partial B_\eta} \mathrm{d}s^\eta = \int_{\partial B_\eta} \left(-\frac{3}{8\pi\eta^2}(Sn\cdot n)n\right)\left(\frac{3}{2\pi\eta^3}(Sn\cdot n)n\right)\mathrm{d}s^\eta \\ &= -\frac{9}{16\pi^2\eta^3}\int_{\partial B_1} (Sn\cdot n)^2 \mathrm{d}s^1 = -\frac{3}{10\pi}|S|^2.\end{aligned}$$

□

3.2. The Renormalized Energy

Thanks to the regularization of $S\nabla\delta$ introduced in the previous paragraph, *cf.* Lemma 3.5, we shall be able to set a rigorous alternative to the abusive formula (3.13). Specifically, we shall state an identity involving $\mathcal{W}_N$, defined in (3.9), and the energy of the function

$$\begin{aligned}h_N^\eta(x) &:= \sum_{i=1}^N G_S^\eta(x-x_i) + N\mathrm{St}^{-1}(S\nabla f) \\ &= \sum_{i=1}^N G_S^\eta(x-x_i) - N\int_{\mathbb{R}^3} G_S(x-y)f(y)\mathrm{d}y.\end{aligned} \tag{3.25}$$

This function solves

$$-\Delta h_N^\eta + \nabla p_N^\eta = \sum_{i=1}^N S^\eta(x-x_i) - NS\nabla f, \quad \mathrm{div} h_N^\eta = 0, \tag{3.26}$$

and is a regularization of h_N, *cf.* (3.14)–(3.15).

The main result of this section is

Proposition 3.7.

$$\mathcal{W}_N = -\frac{25|\mathcal{O}|}{2N^2}\lim_{\eta\to 0}\left(\int_{\mathbb{R}^3} |\nabla h_N^\eta|^2 - \frac{N}{\eta^3}(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2)\right). \tag{3.27}$$

Proof. We assume that η is small enough so that $2\eta < \min_{i\neq j}|x_i - x_j|$. From the explicit expressions (3.14), (3.25), we find that $h_N, h_N^\eta = O(|x|^{-2})$, $\nabla(h_N, h_N^\eta) = O(|x|^{-3})$ and $p_N, p_N^\eta = O(|x|^{-3})$ at infinity. As these quantities decay fast enough, we can perform an integration by parts to find

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla h_N^\eta|^2 &= \langle -\Delta h_N^\eta, h_N^\eta\rangle = \langle -\Delta h_N^\eta + \nabla p_N^\eta, h_N^\eta\rangle \\ &= \langle \sum_i S^\eta(x-x_i) - NS\nabla f, h_N\rangle \\ &\quad + \langle \sum_i S^\eta(x-x_i) - NS\nabla f, h_N^\eta - h_N\rangle \\ &= \sum_i \langle S^\eta(x-x_i), h_N^i\rangle + \sum_i \langle S^\eta(x-x_i), G_S(x-x_i)\rangle - \langle NS\nabla f, h_N\rangle \\ &\quad + \langle \sum_i S^\eta(x-x_i) - NS\nabla f, h_N^\eta - h_N\rangle =: a+b+c+d,\end{aligned}$$

where we defined $h_N^i := h_N - G_S(x-x_i)$.

As h_N^i is smooth over the support of $S^\eta(\cdot - x_i)$, we can apply Lemma 3.5 to obtain

$$\lim_{\eta\to 0} a = -\sum_i S\nabla \cdot h_N^i(x_i).$$

We can then apply Lemma 3.6 to obtain

$$b = \frac{N}{\eta^3}\Big(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\Big).$$

As regards the fourth term, we notice that by our definition (3.16)–(3.17) of G_S^η, and the fact that the balls $B(x_i,\eta)$ are disjoint, the function $h_N - h_N^\eta = \sum_i (G_S(x-x_i) - G_S^\eta(x-x_i))$ is zero over $\cup_i \partial B(x_i,\eta)$, which is the support of $\sum_i S^\eta(x-x_i)$. It follows that

$$\begin{aligned} d = -N\langle S\nabla f, h_N^\eta - h_N\rangle &= N\sum_i \int_{B(x_i,\eta)} S\nabla\cdot G_S^\eta(x-x_i)\,(f(x)-f(x_i))\,\mathrm{d}x \\ &\quad - N\sum_i \int_{B(x_i,\eta)} S\nabla\cdot G_S(x-x_i)\,(f(x)-f(x_i))\,\mathrm{d}x,\end{aligned}$$

where we integrated by parts, using that $G_S - G_S^\eta$ is zero outside the balls. Let us notice that the second integral at the right-hand side converges despite the singularity of $S\nabla\cdot G_S$, using the smoothness of f near x_i (by assumption (3.1) and Remark 3.2). Moreover, it goes to zero as $\eta\to 0$. Using the homogeneity and smoothness properties of G_S^η inside B^η, we also find that the first sum goes to zero with η, resulting in

$$\lim_{\eta\to 0} d = 0.$$

We end up with

$$\lim_{\eta\to 0}\left(\int_{\mathbb{R}^3}|\nabla h_N^\eta|^2-\frac{N}{\eta^3}\Big(\int_{B^1}|\nabla G_S^1|^2+\frac{3}{10\pi}|S|^2\Big)\right)$$
$$=-\sum_i S\nabla\cdot h_N^i(x_i)-\langle NS\nabla f,h_N\rangle$$

It remains to rewrite properly the right-hand side. We first get

$$-\sum_i S\nabla\cdot h_N^i(x_i)=-\sum_{i\neq j}S\nabla\cdot G_S(x_i-x_j)+N\sum_i\int_{\mathbb{R}^3}S\nabla\cdot G_S(x_i-y)f(y)\mathrm{d}y$$
$$=-\frac{3N^2}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\setminus\mathrm{Diag}}g_S(x-y)d\delta_N(x)(\mathrm{d}\delta_N(y)-f(y)\mathrm{d}y),$$

and, integrating by parts,

$$-\langle NS\nabla f,h_N\rangle=N\int_{\mathbb{R}^3}S\nabla\cdot h_N(x)f(x)\mathrm{d}x$$
$$=N\int_{\mathbb{R}^3}\left(\sum_i S\nabla\cdot G_S(x-x_i)-N\int_{\mathbb{R}^3}S\nabla\cdot G_S(x-y)f(y)\mathrm{d}y\right)f(x)\mathrm{d}x$$
$$=\frac{3N^2}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}g_S(x-y)f(x)\mathrm{d}x(\mathrm{d}\delta_N(y)-f(y)\mathrm{d}y)\mathrm{d}x.$$

The last equality was deduced from the identity $g_S=\frac{8\pi}{3}(S\nabla)\cdot G_S$, see the line after (3.10). The proposition follows: □

We can refine the previous proposition as follows:

Proposition 3.8. *Let $c>0$ the constant in* (H2). *There exists $C>0$ such that: for all $\alpha<\eta<\frac{c}{2}N^{-1/3}$,*

$$\left|\int_{\mathbb{R}^3}|\nabla h_N^\eta|^2-\int_{\mathbb{R}^3}|\nabla h_N^\alpha|^2-N\left(\frac{1}{\eta^3}-\frac{1}{\alpha^3}\right)\Big(\int_{B^1}|\nabla G_S^1|^2+\frac{3}{10\pi}|S|^2\Big)\right|\leqq CN^2\eta.$$

Proof. One has from (3.25) that

$$h_N^\eta=h_N^\alpha+\sum_{i=1}^N(G_S^\eta-G_S^\alpha)(x-x_i).$$

It follows that

$$\int_{\mathbb{R}^3}|\nabla h_N^\eta|^2-\int_{\mathbb{R}^3}|\nabla h_N^\alpha|^2=\sum_{i,j}\int_{\mathbb{R}^3}\nabla(G_S^\eta-G_S^\alpha)(x-x_i):\nabla(G_S^\eta-G_S^\alpha)(x-x_j)$$
$$+2\sum_i\int_{\mathbb{R}^3}\nabla h_N^\alpha:\nabla(G_S^\eta-G_S^\alpha)(x-x_i).$$

After integration by parts,

$$\int_{\mathbb{R}^3} \nabla(G_S^\eta - G_S^\alpha)(\cdot - x_i) : \nabla(G_S^\eta - G_S^\alpha)(\cdot - x_j)$$
$$= \langle (S^\eta - S^\alpha)(\cdot - x_i), (G_S^\eta - G_S^\alpha)(\cdot - x_j)\rangle,$$

while

$$\int_{\mathbb{R}^3} \nabla h_N^\alpha : \nabla(G_S^\eta - G_S^\alpha)(x - x_i) = \langle \sum_j S^\alpha(\cdot - x_i) - NS\nabla f, (G_S^\eta - G_S^\alpha)(\cdot - x_i)\rangle.$$

We get

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla h_N^\eta|^2 - \int_{\mathbb{R}^3} |\nabla h_N^\alpha|^2 &= \sum_{i\neq j} \langle (S^\alpha + S^\eta)(\cdot - x_i), (G_S^\eta - G_S^\alpha)(\cdot - x_j)\rangle \\ &- 2\sum_i N\langle S\nabla f, (G_S^\eta - G_S^\alpha)(\cdot - x_i)\rangle \\ &+ N\langle (S^\alpha + S^\eta), (G_S^\eta - G_S^\alpha)\rangle =: a + b + c. \end{aligned} \tag{3.28}$$

We note that $G_S^\eta - G_S^\alpha$ is zero outside B_η, while $S^\alpha + S^\eta$ is supported in B_η. Moreover, thanks to (H2), for $\alpha < \eta < \frac{c}{2}$, the balls $B(x_i, \eta)$ are disjoint. We deduce: $a = 0$.

After integration by parts, taking into account that $G_S^\eta - G_S^\alpha$ vanishes outside B_η, we can write $b = b_\eta - b_\alpha$ with

$$b_\alpha := 2\sum_i N \int_{B(x_i,\eta)} S\nabla \cdot G_S^\alpha(\cdot - x_i)\,(f - f(x_i))$$
$$b_\eta := 2\sum_i N \int_{B(x_i,\eta)} S\nabla \cdot G_S^\eta(\cdot - x_i)\,(f - f(x_i)).$$

By assumption (3.1), for N large enough, for all $1 \leqq i \leqq N$ and all $\eta \leqq \frac{c}{2}N^{-1/3}$, $B(x_i, \eta)$ is included in $\mathcal{O}$. Hence, f is C^1 in $B(x_i, \eta)$, and

$$\Big|\int_{B(x_i,\eta)} S\nabla \cdot G_S^\eta(\cdot - x_i)\,(f - f(x_i)\Big| \leqq \frac{C}{\eta^3}\|\nabla f|_{\mathcal{O}}\|_\infty \int_{B(x_i,\eta)} |x - x_i|\mathrm{d}x \leqq C\eta.$$

This results in $b_\eta \leqq CN^2\eta$.

Similarly, decomposing $B(x_i, \eta) = B(x_i, \alpha) \cup \Big(B(x_i, \eta) \setminus B(x_i, \alpha)\Big)$, we find that

$$\Big|\int_{B(x_i,\eta)} S\nabla \cdot G_S^\alpha(\cdot - x_i)\,(f - f(x_i))\Big| \leqq C\left(\alpha + \int_{B(x_i,\eta)} \frac{1}{|x - x_i|^2}\mathrm{d}x\right) \leqq C'\eta,$$

using again that f is Lipschitz over $B(x_i, \eta)$. We end up with $b_\alpha \leqq CN^2\eta$, and finally $b \leqq CN^2\eta$.

For the last term c in (3.28), we first notice that as $G_S^\eta - G_S^\alpha$ is zero outside B_η:

$$\begin{aligned}
&\langle (S^\alpha + S^\eta), (G_S^\eta - G_S^\alpha)\rangle = \langle S^\alpha, (G_S^\eta - G_S^\alpha)\rangle \\
&= \langle S^\alpha, G_S^\eta\rangle \; - \; \langle S^\alpha, G_S\rangle \\
&= \langle S^\alpha, G_S^\eta\rangle \; - \; \frac{1}{\alpha^3}\left(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right),
\end{aligned} \tag{3.29}$$

where we used Lemma 3.6 in the last line. By the definition of S^α, the remaining term splits into

$$\langle S^\alpha, G_S^\eta\rangle = -\int_{\partial B_\alpha^+} (\partial_r G_S - p_S e_r)\cdot G_S^\eta \mathrm{d}s^\alpha \; + \int_{\partial B_\alpha^-} \left(\partial_r G_S^\alpha - p_S^\alpha e_r\right)\cdot G_S^\eta \mathrm{d}s^\alpha.$$

By integration by parts, applied in $B_\eta \setminus B_\alpha$ for the first term and in B_α for the second term, we get

$$\begin{aligned}
\langle S^\alpha, G_S^\eta\rangle &= -\int_{\partial B_\eta^-} (\partial_r G_S - p_S e_r)\cdot G_S^\eta \mathrm{d}s^\eta + \int_{B_\eta\setminus B_\alpha} \nabla G_S : \nabla G_S^\eta \\
&\quad + \int_{B_\alpha} \nabla G_S^\alpha : \nabla G_S^\eta \\
&= -\int_{\partial B_\eta} (\partial_r G_S - p_S e_r)\cdot G_S^\eta \mathrm{d}s^\eta + \int_{B_\eta} \nabla G_S^\alpha \cdot \nabla G_S^\eta \\
&= -\int_{\partial B_\eta} (\partial_r G_S - p_S e_r)\cdot G_S^\eta \mathrm{d}s^\eta + \int_{\partial B_\eta^-} G_S^\alpha \cdot \left(\partial_r G_S^\eta - p_S^\eta e_r\right) \\
&= \langle S^\eta, G_S\rangle = \frac{1}{\eta^3}\left(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right).
\end{aligned}$$

From there, the conclusion follows easily. □

If we let $\alpha \to 0$ in Proposition 3.8, combining with Propositions 3.27 and 3.3, we find

Corollary 3.9. *For all* $\eta < \frac{c}{2}N^{-1/3}$,

$$\left|\mathcal{V}_N + \frac{25|\mathcal{O}|}{2N^2}\left(\int_{\mathbb{R}^3} |\nabla h_N^\eta|^2 - \frac{N}{\eta^3}\Big(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\Big)\right)\right| \leqq \varepsilon(N),$$

where $\varepsilon(N) \to 0$ *as* $N \to +\infty$.

This corollary shows that to understand the limit of $\mathcal{V}_N$, it is enough to study the limit of

$$\frac{25|\mathcal{O}|}{2N^2}\left(\int_{\mathbb{R}^3} |\nabla h_N^{\eta_N}|^2 - \frac{N}{\eta_N^3}\Big(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\Big)\right)$$

for $\eta_N := \eta N^{-1/3}$, $\eta < \frac{c}{2}$ fixed. For periodic and more general stationary point processes, this will be possible through a homogenization approach. This homogenization approach involves an analogue of a cell equation, called jellium in the literature on Coulomb gases. We will motivate and introduce this system in the next section.

4. Blown-up System

Formula (3.27) suggests to understand at first the behaviour of $\int_{\mathbb{R}^3} |\nabla h_N^\eta|^2$ at fixed η, when $N \to +\infty$. To analyze the system (3.26), a useful intuition can be taken from classical homogenization problems of the form

$$\begin{aligned} - \Delta h_\varepsilon + \nabla p_\varepsilon = S\nabla \left(\frac{1}{\varepsilon^3} F(x, x/\varepsilon) - \frac{1}{\varepsilon^3} \overline{F}(x) \right), \ \operatorname{div} h_\varepsilon \\ = 0 \ \text{ in a domain } \Omega, \quad h_\varepsilon|_{\partial\Omega} = 0, \end{aligned} \tag{4.1}$$

with $F(x, y)$ periodic in variable y, and $\overline{F}(x) := \int_{\mathbb{T}} F(x, y)\mathrm{d}y$. Roughly, Ω would be like $\mathcal{O}$, the small scale ε like $N^{-1/3}$, the term $\frac{1}{\varepsilon^3} F(x, x/\varepsilon)$ would correspond to the sum of (regularized) Dirac masses, while the term $\frac{1}{\varepsilon^3}\overline{F}$ would be an analogue of Nf. The factor $\frac{1}{\varepsilon^3}$ in front of F is put consistently with the fact that $\sum_i \delta_{x_i}$ has mass N. The dependence on x of the source term in (4.1) mimics the possible macroscopic inhomogeneity of the point distribution $\{x_i\}$.

In the much simpler model (4.1), standard arguments show that h_ε behaves like

$$h_\varepsilon(x) \approx \frac{1}{\varepsilon^2} H(x, x/\varepsilon), \tag{4.2}$$

where $H(x, y)$ satisfies the cell problem

$$-\Delta_y H(x, \cdot) + \nabla_y P(x, \cdot) = S\nabla_y F(x, \cdot), \quad \operatorname{div}_y H(x, \cdot) = 0, \quad y \in \mathbb{T}^3.$$

Let us stress that substracting the term $\frac{1}{\varepsilon^3}\overline{F}(x)$ in the source term of (4.1) is crucial for the asymptotics (4.2) to hold. It follows that

$$\varepsilon^6 \int_\Omega |\nabla h_\varepsilon|^2 \approx \int_\Omega |\nabla_y H(x, x/\varepsilon)|^2 \mathrm{d}x \xrightarrow[\varepsilon \to 0]{} \int_\Omega \int_{\mathbb{T}^3} |\nabla_y H(x, y)|^2 \mathrm{d}y\mathrm{d}x.$$

Note that the factor ε^6 in front of the left-hand side is coherent with the factor $\frac{1}{N^2}$ at the right-hand side of (3.27). Note also that

$$\int_{\mathbb{T}^3} |\nabla_y H(x, y)|^2 \mathrm{d}y = \lim_{R \to +\infty} \frac{1}{R^3} \int_{(-R,R)^3} |\nabla_y H(x, y)|^2 \mathrm{d}y.$$

Such average over larger and larger boxes may be still meaningful in more general settings, typically in stochastic homogenization.

Inspired by those remarks, and back to system (3.26), the hope is that some homogenization process may take place, at least locally near each $x \in \mathcal{O}$. More precisely, we hope to recover $\lim_N \mathcal{W}_N$ by summing over $x \in \mathcal{O}$ some microscopic energy, locally averaged around x. This microscopic energy will be deduced from an analogue of the cell problem, called a *jellium* in the literature on the Ginzburg-Landau model and Coulomb gases.

4.1. Setting of the Problem

We will call *point distribution* a locally finite subset of $\mathbb{R}^3$. Given a point distribution Λ, we consider the following problem in $\mathbb{R}^3$:

$$-\Delta H + \nabla P = \sum_{z\in\Lambda} S\nabla\delta_{-z}$$
$$\operatorname{div} H = 0. \tag{4.3}$$

Given a solution $H = H(y)$, $P = P(y)$, we introduce, for any $\eta > 0$,

$$H^\eta := H + \sum_{z\in\Lambda}(G_S^\eta - G_S)(\cdot + z), \tag{4.4}$$

which satisfies, by (3.11), (3.19), that

$$-\Delta H^\eta + \nabla P^\eta = \sum_{z\in\Lambda} S^\eta(\cdot + z)$$
$$\operatorname{div} H^\eta = 0. \tag{4.5}$$

We remark that, the set Λ being locally finite, the sum at the right-hand side of (4.3) or (4.5) is well-defined as a distribution. Also, the sum at the right-hand side of (4.4) is well-defined pointwise, because $G_S^\eta - G_S$ is supported in B_η.

As discussed at the beginning of Section 4, we expect the limit of $\int_{\mathbb{R}^3} |\nabla h_N^\eta|^2$ to be described in terms of quantities of the form

$$\lim_{R\to+\infty} \frac{1}{R^3}\int_{K_R} |\nabla H^\eta(y)|^2 \, dy,$$

where $K_R := (-\frac{R}{2}, \frac{R}{2})^3$, for various Λ and solutions H^η of (4.5). Broadly, the energy concentrated locally around a point x should be understood from a blow-up of the original system (3.26), zooming at scale $N^{-1/3}$ around x. Let $x \in \mathcal{O}$ (the center of the blow-up), and $\eta_N := \eta N^{-1/3}$, for a fixed $\eta > 0$. If we introduce

$$H_N^\eta(y) := N^{-2/3} h_N^{\eta_N}(x + N^{-1/3}y), \quad P_N^\eta(y) := N^{-1} p_N^{\eta_N}(x + N^{-1/3}y),$$
$$z_{i,N} := N^{1/3}(x - x_{i,N}) \tag{4.6}$$

we find that

$$-\Delta H_N^\eta + \nabla P_N^\eta = \sum_{i=1}^N S^\eta(\cdot + z_{i,N}) - N^{-1/3} S\nabla_x f(x + N^{-1/3}y), \quad \operatorname{div} H_N^\eta = 0. \tag{4.7}$$

System (4.5) corresponds to a formal asymptotics where one replaces $\sum_{i=1}^N \delta_{z_{i,N}}$ by $\sum_{i=1}^\infty \delta_{z_i}$, with $\Lambda = \{z_i\}$ a point distribution. Note that, under (H2), we expect this point distribution to be *well-separated*, meaning that there is $c > 0$ such that: for all $z' \neq z \in \Lambda$, $|z' - z| \geqq c$. Still, we insist that this asymptotics is purely formal and requires much more to be made rigorous. Such rigorous asymptotics will be carried in Section 5 for various classes of point configurations.

We now collect several general remarks on the blown-up system (4.3). We start by defining a renormalized energy. For any $L > 0$, we denote $K_L := (-\frac{L}{2}, \frac{L}{2})^3$.

Definition 4.1. Given a point distribution Λ, we say that a solution H of (4.3) is admissible if for all $\eta > 0$, the field H^η defined by (4.4) satisfies $\nabla H^\eta \in L^2_{loc}(\mathbb{R}^3)$.

Given an admissible solution H and $\eta > 0$, we say that H^η is of finite renormalized energy if

$$\mathcal{W}^\eta(\nabla H) := -\lim_{R\to+\infty} \frac{1}{R^3}\left(\int_{K_R} |\nabla H^\eta|^2 - \frac{1}{\eta^3}|\Lambda \cap K_R|\left(\int_{B^1} |\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\right)\right)$$

exists in $\mathbb{R}$. We say that H is of finite renormalized energy if H^η is for all η, and

$$\mathcal{W}(\nabla H) := \lim_{\eta\to 0} \mathcal{W}^\eta(\nabla H)$$

exists in $\mathbb{R}$.

Remark 4.2. From formula (4.4), it is easily seen that H is admissible if and only if there exists one $\eta > 0$ with $\nabla H^\eta \in L^2_{loc}(\mathbb{R}^3)$.

Proposition 4.3. *If H_1 and H_2 are admissible solutions of* (4.3) *satisfying, for some $\eta > 0$, that*

$$\limsup_{R\to+\infty} \frac{1}{R^3}\int_{K_R} |\nabla H_1^\eta|^2 < +\infty, \quad \limsup_{R\to+\infty} \frac{1}{R^3}\int_{K_R} |\nabla H_2^\eta|^2 < +\infty,$$

then ∇H_1 and ∇H_2 differ from a constant matrix.

Proof. We set $H := H_1 - H_2 = H_1^\eta - H_2^\eta$. It is a solution of the homogeneous Stokes equation with

$$\limsup_{R\to+\infty} \frac{1}{R^3}\int_{K_R} |\nabla H|^2 < +\infty.$$

By standard elliptic regularity, any solution v of the Stokes equation in the unit ball

$$-\Delta v + \nabla p = 0, \quad \operatorname{div} v = 0 \text{ in } B(0,1)$$

satisfies, for some absolute constant C,

$$|\nabla^2 v(0)| \leqq C\|\nabla v\|_{L^2(B(0,1))}.$$

We apply this inequality to $v(x) = H(x_0 + Rx)$, x_0 arbitrary. After rescaling, we find that

$$|\nabla^2 H(x_0)| \leqq \frac{C}{R}\left(\frac{1}{R^{3/2}}\|\nabla H(x_0+\cdot)\|_{L^2(B(0,R))}\right).$$

As $R \to +\infty$, the right hand-side goes to zero, which concludes the proof. □

Proposition 4.4. *Let Λ be a well-separated point distribution, meaning there exists $c > 0$ such that for all $z' \neq z \in \Lambda$, $|z' - z| \geqq c$. Let $0 < \alpha < \eta < \frac{\min(c,1)}{4}$. Let H be an admissible solution of* (4.3) *such that H^η is of finite renormalized energy. Then, H^α is also of finite renormalized energy, and*

$$\mathcal{W}^\alpha(\nabla H) = \mathcal{W}^\eta(\nabla H).$$

In particular, H is of finite renormalized energy as soon as H^η is for some $\eta \in (0, \frac{c}{4})$, and $\mathcal{W}(\nabla H) = \mathcal{W}^\eta(\nabla H)$ for all $\eta < \frac{\min(c,1)}{4}$.

Proof. Let $R > 0$. As Λ is well-separated,

$$|\Lambda \cap (K_{R+2} \setminus K_{R-2})| \leqq C R^2. \tag{4.8}$$

From this and the fact that the limit $\mathcal{W}^\eta(\nabla H)$ exists (in $\mathbb{R}$), it follows that

$$\lim_{R \to +\infty} \frac{1}{R^3} \int_{K_{R+2} \setminus K_{R-2}} |\nabla H^\eta|^2 = 0. \tag{4.9}$$

Let Ω_R be an open set such that $K_{R-1} \subset \Omega_R \subset K_R$ and such that

$$\mathrm{dist}\big(\partial\Omega_R\,,\, \cup_{z\in\Lambda} B(-z,\eta)\big) \geqq c' > 0, \tag{4.10}$$

where c' depends on c only. This implies that $G^\eta(\cdot + z)$, $G^\alpha(\cdot + z)$ are smooth at $\partial\Omega_R$ for all $z \in \Lambda$, and that H^η, H^α are smooth at $\partial\Omega_R$.

We now proceed as in the proof of Proposition 3.8. We write

$$H^\eta = H^\alpha + \sum_{z\in\Lambda} (G^\eta_S - G^\alpha_S)(\cdot + z),$$

$$\int_{\Omega_R} |\nabla H^\eta|^2 = \int_{\Omega_R} |\nabla H^\alpha|^2 + 2 \sum_{z\in\Lambda} \int_{\Omega_R} \nabla H^\alpha : \nabla(G^\eta_S - G^\alpha_S)(\cdot + z)$$
$$+ \sum_{z,z'\in\Lambda} \int_{\Omega_R} \nabla(G^\eta_S - G^\alpha_S)(\cdot + z) : \nabla(G^\eta_S - G^\alpha_S)(\cdot + z').$$

After integration by parts, and manipulations similar to those used to show Proposition 3.8, we end up with

$$\int_{\Omega_R} |\nabla H^\eta|^2 - \int_{\Omega_R} |\nabla H^\alpha|^2 = \sum_{z\in\Lambda} \int_{\Omega_R} (G^\eta_S - G^\alpha_S)(\cdot + z) dS^\alpha(\cdot + z). \tag{4.11}$$

Let us emphasize that the contribution of the boundary terms at $\partial\Omega_R$ is zero: indeed, thanks to (4.10), $(G^\eta_S - G^\alpha_S)(\cdot + z)$ is zero at $\partial\Omega_R$ for any $z \in \Lambda$. Similarly,

$$\sum_{z\in\Lambda} \int_{\Omega_R} (G^\eta_S - G^\alpha_S)(\cdot + z) dS^\alpha(\cdot + z) = \sum_{z\in\Lambda\cap\Omega_R} \int_{\Omega_R} (G^\eta_S - G^\alpha_S)(\cdot + z) dS^\alpha(\cdot + z)$$
$$= \sum_{z\in\Lambda\cap\Omega_R} \int_{\mathbb{R}^3} (G^\eta_S - G^\alpha_S)(\cdot + z) dS^\alpha(\cdot + z).$$

The integral in the right-hand side was computed above, (see (3.29) and the lines after):

$$\sum_{z\in\Lambda\cap\Omega_R} \int_{\mathbb{R}^3} (G^\eta_S - G^\alpha_S)(\cdot + z) dS^\alpha(\cdot + z)$$
$$= |\Lambda \cap \Omega_R| \left(\frac{1}{\eta^3} - \frac{1}{\alpha^3}\right) \left(\int_{B_1} |\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\right).$$

Back to (4.11), we find that

$$\int_{\Omega_R} |\nabla H^\eta|^2 - \int_{\Omega_R} |\nabla H^\alpha|^2 = |\Lambda \cap \Omega_R| \left(\frac{1}{\eta^3} - \frac{1}{\alpha^3}\right) \left(\int_{B_1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right).$$

We deduce from this identity, (4.8) and (4.9) that

$$\lim_{R\to+\infty} \frac{1}{R^3} \left(\int_{\Omega_R} |\nabla H^\alpha|^2 - \frac{|\Lambda \cap K_R|}{\alpha^3} \left(\int_{B_1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\right) = \mathcal{W}^\eta(\nabla H),$$

and replacing R by $R + 1$,

$$\lim_{R\to+\infty} \frac{1}{R^3} \left(\int_{\Omega_{R+1}} |\nabla H^\alpha|^2 - \frac{|\Lambda \cap K_R|}{\alpha^3} \left(\int_{B_1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\right) = \mathcal{W}^\eta(\nabla H).$$

As $\Omega_R \subset K_R \subset \Omega_{R+1}$, the result follows. □

4.2. Resolution of the Blown-up System for Stationary Point Processes

As pointed out several times, we follow the strategy described in [41] for the treatment of minimizers and minima of Coulomb energies, but in our effective viscosity problem, the points $x_{i,N}$ do not minimize the analogue $\mathcal{V}_N$ of the Coulomb energy $\mathcal{H}_N$. Actually, although we consider the steady Stokes equation, our point distribution may be time dependent. More precisely, in many settings, the dynamics of the suspension evolves on a timescale associated with viscous transport (scaling like a^2, with a the radius of the particle), which is much smaller than the convective time scale (scaling like a). This allows us to neglect the time derivative in the Stokes equation: system (1.1)–(1.2) corresponds then to a snapshot of the flow at a given time t. Even when one is interested in the long time behaviour, the existence of an equilibrium measure for the system of particles is a very difficult problem. To bypass this issue, a usual point of view in the physics literature is to assume that the distribution of points is given by a stationary random process (whose refined description is an issue *per se*).

We will follow this point of view here, and introduce a class of random point processes for which we can solve (4.3). Let $X = \mathbb{R}$ or $X = \mathbb{T}_L := \mathbb{R}/(L\mathbb{Z})$ for some $L > 0$. We denote by $Point_X$ the set of point distributions in X^3: an element of $Point_X$ is a locally finite subset of X^3, in particular a finite subset when $X = \mathbb{T}_L$. We endow $Point_X$ with the smallest σ-algebra $\mathcal{P}_X$ which makes measurable all the mappings

$$Point_X \to \mathbb{N}, \quad \omega \to |A \cap \omega|, \quad A \text{ borelian bounded subset of } X.$$

Given a probability space $(\Omega, \mathcal{A}, P)$, a random point process Λ with values in X^3 is a measurable map from Ω to $Point_X$, see [12]. By pushing forward the probability P with Λ, we can always assume that the process is in canonical form, that is $\Omega = Point_X$, $\mathcal{A} = \mathcal{P}_X$, and $\Lambda(\omega) = \omega$.

We shall consider processes that, once in canonical form, are

(P1) stationary: the probability P on Ω is invariant by the shifts

$$\tau_y : \Omega \to \Omega, \quad \omega \to y + \omega, \quad y \in X^3.$$

(P2) ergodic: if $A \in \mathcal{A}$ satisfies $\tau_y(A) = A$ for all y, then $P(A) = 0$ or $P(A) = 1$.
(P3) uniformly well-separated: we mean that there exists $c > 0$ such that almost surely, $|z - z'| \geqq c$ for all $z \neq z'$ in ω.

These properties are satisfied in two important contexts:

Example 4.5. (Periodic point distributions). Namely, for $L > 0$, $a_1, \dots, a_M$ in K_L, we introduce the set $\Lambda_0 := \{a_1, \dots, a_M\} + L\mathbb{Z}^d$. We can of course identify Λ_0 with a point distribution in X^3 with $X = \mathbb{T}_L$. We then take $\Omega = \mathbb{T}^3_L$, P the normalized Lebesgue measure on $\mathbb{T}^3_L$, and set $\Lambda(\omega) := \Lambda_0 + \omega$. It is easily checked that this random process satisfies all assumptions. Moreover, a realization of this process is a translate of the initial periodic point distribution Λ_0. By translation, the almost sure results that we will show below (well-posedness of the blown-up system, convergence of $\mathcal{W}_N$) will actually yield results for Λ_0 itself.

Example 4.6. (Poisson hard core processes). These processes are obtained from Poisson point processes, by removing balls in order to guarantee the hypothesis (P3). For instance, given $c > 0$, one can remove from the Poisson process all points z which are not alone in $B(z, c)$. This leads to the so-called *Matérn I* hard-core process. To increase the density of points while keeping (P3), one can refine the removal process in the following way: for each point z of the Poisson process, one associates an "age" u_z, with (u_z) a family of i.i.d. variables, uniform over (0, 1). Then, one retains only the points z that are (strictly) the "oldest" in $B(z, c)$. This leads to the so-called *Matérn II* hard-core process. Obviously, these two processes satisfy (P1) by stationarity of the Poisson process, and satisfy (P2) because they have only short range of correlations. For much more on hard core processes, we refer to [8].

The point is now to solve almost surely the blown-up system (4.3) for point processes with properties (P1)–(P2)–(P3). We first state

Proposition 4.7. *Let $\Lambda = \Lambda(\omega)$ a random point process with properties (P1)–(P2)–(P3). Let $\eta > 0$. For almost every ω, there exists a solution $\mathbf{H}^\eta(\omega, \cdot)$ of (4.5) in $H^1_{loc}(X^3)$ such that*

$$\nabla \mathbf{H}^\eta(\omega, y) = D^\eta_{\mathbf{H}}(\tau_y \omega),$$

where $D^\eta_{\mathbf{H}} \in L^2(\Omega)$ is the unique solution of the variational formulation (4.12) below.

Remark 4.8. In the case $X = \mathbb{T}_L$, point distributions and solutions H^η over X^3 can be identified with $L\mathbb{Z}^3$-periodic point distributions and $L\mathbb{Z}^3$-periodic solutions defined on $\mathbb{R}^3$. This identification is implicit here and in all that follows.

Proof. We treat the case $X = \mathbb{R}$, the case $X = \mathbb{T}_L$ follows the same approach. We remind that the process is in canonical form: $\Omega = Point_{\mathbb{R}}$, $\mathcal{A} = \mathcal{P}_{\mathbb{R}}$, $\Lambda(\omega) = \omega$. The idea is to associate to (4.5) a probabilistic variational formulation. This approach is inspired by works of Kozlov [7,26], see also [3]. Prior to the statement of this variational formulation, we introduce some vocabulary and functional spaces. First, for any $\mathbb{R}^d$-valued measurable $\phi = \phi(\omega)$, we call a realization of ϕ an application

$$R_\omega[\phi](y) := \phi(\tau_y \omega), \quad \omega \in \Omega.$$

For $p \in [1, +\infty)$, $\phi \in L^p(\Omega)$, as τ_y is measure preserving, we have for all $R > 0$ that $\mathbb{E} \int_{K_R} |R_\omega[\phi]|^p = R^3\, \mathbb{E}|\phi|^p$. Hence, almost surely, $R_\omega[\phi]$ is in $L^p_{loc}(\mathbb{R}^3)$. Also, for $\phi \in L^\infty(\Omega)$, one finds that almost surely $R_\omega[\phi] \in L^\infty_{loc}(\mathbb{R}^3)$. It is a consequence of Fatou's lemma: for all $R > 0$,

$$\begin{aligned}
\mathbb{E}\|R_\omega[\phi]\|_{L^\infty(K_R)} &= \mathbb{E} \liminf_{p\to+\infty} \|R_\omega[\phi]\|_{L^p(K_R)} \leqq \liminf_{p\to+\infty} \mathbb{E}\|R_\omega[\phi]\|_{L^p(K_R)} \\
&\leqq \liminf_{p\to+\infty} \left(\mathbb{E}\|R_\omega[\phi]\|^p_{L^p(K_R)}\right)^{1/p} \\
&= \liminf_{p\to+\infty} \left(\mathbb{E}|\phi|^p\right)^{1/p} = \|\phi\|_{L^\infty(\Omega)}.
\end{aligned}$$

We say that ϕ is smooth if, almost surely, $R_\omega[\phi]$ is. For a smooth function ϕ, we can define its stochastic gradient $\nabla_\omega \phi$ by the formula

$$\nabla_\omega \phi(\omega) := \nabla R_\omega[\phi]|_{y=0},$$

where here and below, $\nabla = \nabla_y$ refers to the usual gradient (in space). Note that $\nabla_\omega \phi(\tau_y \omega) = \nabla R_\omega[\phi](y)$. One can define similarly the stochastic divergence, curl, *etc*, and reiterate to define partial stochastic derivatives ∂^α_ω.

Starting from a function $V \in L^p(\Omega)$, $p \in [1, +\infty]$ one can build smooth functions through convolution. Namely, for $\rho \in C^\infty_c(\mathbb{R}^3)$, one can define

$$\rho \star V(\omega) := \int_{\mathbb{R}^3} \rho(y) V(\tau_y \omega) \mathrm{d}y,$$

which is easily seen to be in $L^p(\Omega)$, as

$$\begin{aligned}
\mathbb{E}|\rho \star V(\omega)|^p &\leqq \mathbb{E}\left(\int_{\mathbb{R}^3} |\rho(y)|\mathrm{d}y\right)^{p-1} \left(\int_{\mathbb{R}^3} |\rho(y)||V(\tau_y\omega)|^p \mathrm{d}y\right) \\
&= \left(\int_{\mathbb{R}^3} |\rho(y)|\mathrm{d}y\right)^p \mathbb{E}|V(\omega)|^p,
\end{aligned}$$

using that τ_y is measure-preserving. Moreover, it is smooth: we leave to the reader to check

$$R_\omega[\rho \star V] = \check{\rho} \star R_\omega[V], \quad \nabla_\omega(\rho \star V) = \nabla \check{\rho} \star V, \quad \check{\rho}(y) := \check{\rho}(-y).$$

We are now ready to introduce the functional spaces we need. We set

$$\mathcal{D}_\sigma := \{\phi : \Omega \to \mathbb{R}^3 \text{ smooth}, \partial_\omega^\alpha \phi \in L^2(\Omega)\ \forall \alpha,\ \nabla_\omega \cdot \phi = 0\},$$
$$\mathcal{V}_\sigma := \text{ the closure of } \{\nabla_\omega \phi,\ \phi \in \mathcal{D}_\sigma\} \text{ in } L^2(\Omega).$$

We remind that $S^\eta = \operatorname{div} \Psi^\eta$, with Ψ^η defined in (3.21). We introduce

$$\Pi^\eta(\omega) := \sum_{z \in \omega} \Psi^\eta(z)$$

Note that it is well-defined, as Ψ^η is supported in B_η and ω is a discrete subset. It is measurable: indeed, Ψ^η is the pointwise limit of a sequence of simple functions of the form $\sum_i \alpha_i 1_{A_i}$, where A_i are Borel subsets of $\mathbb{R}^3$. As

$$\omega \to \sum_{z \in \omega} \sum_i \alpha_i 1_{A_i}(z) = \sum_i \alpha_i |A_i \cap \omega|$$

is measurable by definition of the σ-algebra $\mathcal{A}$, we find that Π^η is. Moreover, as Λ is uniformly well-separated, one has $|\Pi^\eta(\omega)| \leqq C \|\Psi^\eta\|_{L^\infty}$ for a constant C that does not depend on ω, so that Π^η belongs to $L^\infty(\Omega)$.

We now introduce the variational formulation: *find* $D^\eta_{\mathbf{H}} \in \mathcal{V}_\sigma$ *such that for all* $D_\phi \in \mathcal{V}_\sigma$,

$$\mathbb{E}\, D^\eta_{\mathbf{H}} : D_\phi = -\mathbb{E}\, \Pi^\eta : D_\phi. \tag{4.12}$$

As $\mathcal{V}_\sigma$ is a closed subspace of $L^2(\Omega)$, existence and uniqueness of a solution comes from the Riesz theorem.

It remains to build a solution of (4.5) almost surely, based on $D^\eta_{\mathbf{H}}$. Let $\phi_k = \phi_k(\omega)$ a sequence in $\mathcal{D}_\sigma$ such that $\nabla_\omega \phi_k$ converges to $D^\eta_{\mathbf{H}}$ in $L^2(\Omega)$. Let $\rho \in C_c^\infty(\mathbb{R}^3)$. It is easily seen that $\rho \star \phi_k$ also belongs to $\mathcal{D}_\sigma$ and that $\partial_\omega^\alpha \nabla_\omega(\rho \star \phi_k) = \partial_\omega^\alpha(\rho \star \nabla_\omega \phi_k)$ converges to the smooth function $\partial_\omega^\alpha(\rho \star D^\eta_{\mathbf{H}})$ in $L^2(\Omega)$, for all α. In particular, as $\nabla_\omega \times \nabla_\omega(\rho \star \phi_k) = 0$, we find that $\nabla_\omega \times (\rho \star D^\eta_{\mathbf{H}}) = 0$. Applying the realization operator R_ω, we deduce that

$$\nabla \times (\check{\rho} \star R_\omega[D^\eta_{\mathbf{H}}]) = \check{\rho} \star \nabla \times R_\omega[D^\eta_{\mathbf{H}}] = 0.$$

We recall that $R_\omega[D^\eta_{\mathbf{H}}]$ belongs almost surely to $L^2_{loc}(\mathbb{R}^3)$, so that $\nabla \times R_\omega[D^\eta_{\mathbf{H}}]$ is well-defined in $H^{-1}_{loc}(\mathbb{R}^3)$. Taking $\rho = \rho_n$ an approximation of the identity, and sending n to infinity, we end up with $\nabla \times R_\omega[D^\eta_{\mathbf{H}}] = 0$ in $\mathbb{R}^3$. As curl-free vector fields on $\mathbb{R}^3$ are gradients, it follows that almost surely, there exists $\mathbf{H}^\eta = \mathbf{H}^\eta(\omega, y)$ with

$$\nabla \mathbf{H}^\eta(\omega, y) = R_\omega[D^\eta_{\mathbf{H}}](y) = D^\eta_{\mathbf{H}}(\tau_y(\omega)), \quad \forall y \in \mathbb{R}^3.$$

In the case $X = \mathbb{T}_L$, one can show that the mean of $R_\omega[D^\eta_{\mathbf{H}}]$ is almost surely zero, so that the same result holds. In addition, because the matrices $\nabla_\omega \phi$, $\phi \in \mathcal{D}_\sigma$, have zero trace, the same holds for $D^\eta_{\mathbf{H}}$. Hence,

$$\operatorname{div} \mathbf{H}^\eta(\omega, y) = \operatorname{trace}(\nabla \mathbf{H}^\eta(\omega, y)) = \operatorname{trace}(D^\eta_{\mathbf{H}})(\tau_y(\omega)) = 0.$$

One still has to prove that the first equation of (4.5) is satisfied. Therefore, we use (4.12) with test function $D_\phi = \nabla_\omega \phi$, where the smooth function ϕ is of the form

$$\phi = \rho \star (\nabla_\omega \times \varphi), \ \varphi : \Omega \to \mathbb{R}^3 \text{ a smooth function.}$$

Note that for smooth functions $\varphi, \tilde{\varphi}$, a stochastic integration by parts formula holds:

$$\begin{aligned}\mathbb{E}\, \partial^i_\omega \varphi\, \tilde{\varphi} = \mathbb{E} \int_{K_1} \partial_i R_\omega[\varphi]\, R_\omega[\tilde{\varphi}] &= -\mathbb{E} \int_{K_1} R_\omega[\varphi]\, \partial_i R_\omega[\tilde{\varphi}] \\ &+ \mathbb{E} \int_{\partial K_1} n_i R_\omega[\varphi]\, R_\omega[\tilde{\varphi}] \\ &= -\mathbb{E} \int_{K_1} R_\omega[\varphi]\, \partial_i R_\omega[\tilde{\varphi}] = -\mathbb{E}\, \varphi\, \partial_{\omega,i} \tilde{\varphi}.\end{aligned}$$

Thanks to this formula, we may write

$$\begin{aligned}\mathbb{E}\, D^\eta_{\mathbf{H}} : \nabla_\omega(\rho \star (\nabla_\omega \times \varphi)) &= \mathbb{E}\, \check{\rho} \star D^\eta_{\mathbf{H}} : \nabla_\omega(\nabla_\omega \times \varphi) \\ &= -\mathbb{E}\, \nabla_\omega \times (\nabla_\omega \cdot (\check{\rho} \star D^\eta_{\mathbf{H}})) \cdot \varphi.\end{aligned}$$

Similarly, we find

$$-\mathbb{E}\, \Pi^\eta : \nabla_\omega(\rho \star \nabla_\omega \times \varphi) = \mathbb{E} \nabla_\omega \times (\nabla_\omega \cdot (\check{\rho} \star \Pi^\eta)) \cdot \varphi.$$

As this identity is valid for all smooth test fields φ, we end up with

$$-\nabla_\omega \times (\nabla_\omega \cdot (\check{\rho} \star D^\eta_{\mathbf{H}})) = \nabla_\omega \times (\nabla_\omega \cdot (\check{\rho} \star \Pi^\eta)).$$

Proceeding as above, we find that, almost surely,

$$-\nabla \times \mathrm{div} R_\omega[D^\eta_{\mathbf{H}}] = \nabla \times \mathrm{div} R_\omega[\Pi^\eta],$$

which can be written as

$$\nabla \times (-\Delta \mathbf{H}^\eta) = \nabla \times \mathrm{div} \sum_{z \in \Omega} \Psi^\eta(\cdot + z).$$

It follows that there exists $\mathbf{P}^\eta = \mathbf{P}^\eta(\omega, y)$ such that

$$-\Delta \mathbf{H}^\eta + \nabla \mathbf{P}^\eta = \mathrm{div} \sum_{z \in \omega} \Psi^\eta(\cdot + z) = \sum_{z \in \omega} S^\eta(\cdot + z),$$

which concludes the proof of the proposition. □

Corollary 4.9. *For random point processes with properties (P1)–(P2)–(P3), there exists almost surely a solution H of* (4.3) *with finite renormalized energy and such that for all $\eta > 0$, the gradient field ∇H^η, where H^η is given by* (4.4)*, coincides with the gradient field $\nabla \mathbf{H}^\eta$ of Proposition* 4.7*. Moreover,*

$$\mathcal{W}(\nabla H) = -\lim_{\eta \to 0} \left(\mathbb{E} \int_{K_1} |\nabla H^\eta|^2 - \frac{m}{\eta^3} \Big(\int_{B^1} |\nabla G^1_S|^2 + \frac{3}{10\pi} |S|^2 \Big) \right),$$

where $m := \mathbb{E}|\Lambda \cap K_1|$ is the mean intensity of the point process, the expression at the right-hand side being actually constant for η small enough.

Proof. By the definition of the mean intensity and by property (P2), which allows us to apply the ergodic theorem (*cf.* [12, Corollary 12.2.V]), we have, almost surely, that

$$\lim_{R\to\infty} \frac{|\Lambda \cap K_R|}{R^3} = m. \tag{4.13}$$

Let $\eta_0 < \frac{\min(c,1)}{4}$ fixed, and $\mathbf{H}^{\eta_0}$ given by the previous proposition. We set

$$H(\omega, y) := \mathbf{H}^{\eta_0}(\omega, y) + \sum_{z\in\omega} (G_S - G_S^{\eta_0})(y + z). \tag{4.14}$$

It is clearly an admissible solution of (4.3). By Proposition 4.4, in order to show that H has almost surely finite renormalized energy, it is enough to show that for one $\eta < \frac{\min(c,1)}{4}$, almost surely, the function H^η given by (4.4), namely,

$$\begin{aligned} H^\eta(\omega, y) &:= H(\omega, y) + \sum_{z\in\omega} (G_S^\eta - G_S)(y + z) \\ &= \mathbf{H}^{\eta_0}(\omega, y) + \sum_{z\in\omega} (G_S^\eta - G_S^{\eta_0})(y + z), \end{aligned}$$

has finite renormalized energy. This holds for $\eta = \eta_0$, as $H^{\eta_0} = \mathbf{H}^{\eta_0}$ and the ergodic theorem applies. We then notice that

$$\nabla H^\eta(\omega, y) = D_H^\eta(\tau_y(\omega)), \quad D_H^\eta(\omega) := D_{\mathbf{H}}^{\eta_0}(\omega) + \sum_{z\in\omega} \nabla(G_S^\eta - G_S^{\eta_0})(z). \tag{4.15}$$

We remark that $G_S^\eta - G_S^{\eta_0} = 0$ outside $B_{\max(\eta,\eta_0)}$, so that the sum at the r.h.s. has only a finite number of non-zero terms. In the same way as we proved that the function Π^η belongs to $L^\infty(\Omega)$, we get that $\sum_{z\in\omega} \nabla(G_S^\eta - G_S^{\eta_0})(z)$ defines an element of $L^\infty(\Omega)$. Hence, by the ergodic theorem, we have, almost surely, that

$$\lim_{R\to+\infty} \frac{1}{R^3} \int_{K_R} |\nabla H^\eta|^2 \to \mathbb{E} \int_{K_1} |\nabla H^\eta|^2.$$

Combining this with (4.13) and Proposition 4.4, we obtain the formula for $\mathcal{W}(\nabla H)$.

The last step is to prove that for all $\eta > 0$, $\nabla H^\eta = \nabla \mathbf{H}^\eta$ almost surely. As a consequence of the ergodic theorem, one has, almost surely, that

$$\limsup_{R\to+\infty} \frac{1}{R^3} \int_{K_R} |\nabla H^\eta|^2 < +\infty, \quad \limsup_{R\to+\infty} \frac{1}{R^3} \int_{K_R} |\nabla \mathbf{H}^\eta|^2 < +\infty.$$

Reasoning as in the proof of Proposition 4.3, we find that their gradients differ by a constant:

$$\nabla H^\eta(\omega, y) = \nabla \mathbf{H}^\eta(\omega, y) + C(\omega).$$

Applying again the ergodic theorem, we get that almost surely $\mathbb{E} D_H^\eta = \mathbb{E} D_{\mathbf{H}}^\eta + C(\omega)$. As $D_{\mathbf{H}}^\eta$ belongs to $\mathcal{V}_\sigma$, its expectation is easily seen to be zero. To conclude, it remains to prove that $\mathbb{E}D_H^\eta = \mathbb{E}\sum_{z\in\omega}\nabla(G_S^\eta - G_S^{\eta_0})(z)$ is zero. Using stationarity, we write, for all $R > 0$,

$$\mathbb{E}\sum_{z\in\omega}\nabla(G_S^\eta - G_S^{\eta_0})(z) = \frac{1}{R^3}\mathbb{E}\sum_{z\in\omega}\int_{K_R}\nabla(G_S^\eta - G_S^{\eta_0})(z+y)\mathrm{d}y.$$

We remark that for all z outside a $\max(\eta,\eta_0)$-neighborhood of ∂K_R, $\int_{K_R}\nabla(G_S^\eta - G_S^{\eta_0})(z+\cdot) = \int_{\partial K_R} n\otimes(G_S^\eta - G_S^{\eta_0})(z+\cdot) = 0$. It follows from the separation assumption and the L^∞ bound on $\nabla(G_S^\eta - G_S^{\eta_0})$ that

$$\frac{1}{R^3}\mathbb{E}\sum_{z\in\omega}\int_{K_R}\nabla(G_S^\eta - G_S^{\eta_0})(z+y)\mathrm{d}y = O(1/R)\to 0 \quad \text{as} \quad R\to+\infty.$$

□

5. Convergence of $\mathcal{V}_N$

This section concludes our analysis of the quadratic correction to the effective viscosity. From Theorem 1.1, we know that this quadratic correction should be given by the limit of $\mathcal{V}_N$ as N goes to infinity, where $\mathcal{V}_N$ was introduced in (1.13). We show here that the functional $\mathcal{V}_N$ has indeed a limit, when the particles are given by the kind of stationary point processes seen in Section 4.

5.1. Proof of Convergence

Let $\varepsilon > 0$ a small parameter, and $\Lambda = \Lambda(\omega)$ a random point process with properties (P1)–(P2)–(P3): stationarity, ergodicity, and uniform separation. As seen in Examples 4.5 and 4.6, this setting covers the case of *periodic patterns of points* as well as *classical hard core processes*. We set $N = N(\varepsilon)$ the cardinal of the set

$$\{x\in\varepsilon\check{\Lambda},\ B(x,\varepsilon)\subset\mathcal{O}\} = \{x_{1,N},\ldots,x_{N,N}\},$$

where $\check{\Lambda} := -\Lambda$ and where we label the elements arbitrarily. Note that N depends on ω, although it does not appear explicitly. From the fact that Λ is uniformly well-separated and from the ergodic theorem (*cf.* [12, Corollary 12.2.V]), we can deduce that, almost surely,

$$\lim_{\varepsilon\to 0} N(\varepsilon)\varepsilon^3 = \lim_{\varepsilon\to 0}|\varepsilon\check{\Lambda}(\omega)\cap\mathcal{O}|\,\varepsilon^3 = \lim_{\varepsilon\to 0}\frac{|\check{\Lambda}(\omega)\cap\varepsilon^{-1}\mathcal{O}|}{\varepsilon^{-3}|\mathcal{O}|}|\mathcal{O}| = m|\mathcal{O}|, \quad (5.1)$$

so that we shall note indifferently $\lim_{\varepsilon\to 0}$ or $\lim_{N\to+\infty}$. Note that, strictly speaking, $N = N(\varepsilon)$ does not necessarily cover all integer values when $\varepsilon\to 0$, but this is no difficulty.

More generally, for all φ smooth and compactly supported in $\mathbb{R}^3$, ergodicity implies

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{i=1}^{N}\varphi(x_i) = \lim_{N\to+\infty}\frac{1}{N}\sum_{x_i\in\mathcal{O}}\varphi(x_i)$$
$$= \lim_{N\to+\infty}\frac{1}{\varepsilon^3 N}m\int_{\mathcal{O}}\varphi(x)\mathrm{d}x = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}}\varphi(x)\mathrm{d}x,$$

which shows that (H1) is satisfied with $f = \frac{1}{|\mathcal{O}|}1_{\mathcal{O}}$. The hypothesis (H2) is also trivially satisfied, as well as (3.1). Our main theorem is

Theorem 5.1. *Almost surely,*

$$\lim_{N\to+\infty}\mathcal{V}_N = \frac{25}{2m^2}\mathcal{W}(\nabla H),$$

with m the mean intensity of the process, and H the solution of (4.3) *given in Corollary* 4.9.

The rest of the paragraph is dedicated to the proof of this theorem.

Let η satisfying $\eta < \frac{\min(c,1)}{4}$ and $\eta < \frac{c}{2}(m|\mathcal{O}|)^{-1/3}$. By (5.1), it follows that, almost surely, for ε small enough, $\varepsilon\eta < \frac{c}{2}N^{-1/3}$. By Corollary 3.9,

$$\lim_{N\to+\infty}\quad \mathcal{V}_N + \frac{25|\mathcal{O}|}{2N^2}\Big(\int_{\mathbb{R}^3}|\nabla h_N^{\eta\varepsilon}|^2 - \frac{N}{(\eta\varepsilon)^3}\Big(\int_{B^1}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\Big)\Big) = 0. \tag{5.2}$$

We denote $h_\varepsilon^\eta := h_N^{\eta\varepsilon}$, see (3.25)–(3.26). Let H be the solution of the blown-up system (4.3) provided by Corollary 4.9, H^η given in (4.4), and P^η as in (4.5). We define new fields $\bar{h}_\varepsilon^\eta$, $\bar{p}_\varepsilon^\eta$ by the following conditions: $\bar{h}_\varepsilon^\eta \in \dot{H}^1(\mathbb{R}^3)$,

$$\bar{h}_\varepsilon^\eta(\omega,x) = \frac{1}{\varepsilon^2}H^\eta\Big(\frac{x}{\varepsilon}\Big) - \fint_{\mathcal{O}}\frac{1}{\varepsilon^2}H^\eta\Big(\frac{\cdot}{\varepsilon}\Big),\ x\in\mathcal{O}$$
$$\bar{p}_\varepsilon^\eta(\omega,x) = \frac{1}{\varepsilon^3}P^\eta\Big(\frac{x}{\varepsilon}\Big) - \fint_{\mathcal{O}}\frac{1}{\varepsilon^3}P^\eta\Big(\frac{\cdot}{\varepsilon}\Big),\ x\in\mathcal{O}$$
$$-\Delta\bar{h}_\varepsilon^\eta + \nabla\bar{p}_\varepsilon^\eta = 0,\quad \operatorname{div}\bar{h}_\varepsilon^\eta = 0 \ \text{ in }\ \operatorname{ext}\mathcal{O}.$$

We omit indication of the dependence in ω to lighten notations. We claim

Proposition 5.2.

$$\lim_{\varepsilon\to0}-\frac{1}{N^2}\Big(\int_{\mathbb{R}^3}|\nabla\bar{h}_\varepsilon^\eta|^2 - \frac{N}{(\eta\varepsilon)^3}\Big(\int_{B^1}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\Big)\Big) = \frac{1}{m^2|\mathcal{O}|}\mathcal{W}^\eta(\nabla H).$$

Proposition 5.3.

$$\lim_{\varepsilon\to0}\varepsilon^6\int_{\mathbb{R}^3}|\nabla(h_\varepsilon^\eta - \bar{h}_\varepsilon^\eta)|^2 = 0.$$

Note that, by Proposition 4.4 and our choice of η, $\mathcal{W}^\eta(\nabla H) = \mathcal{W}(\nabla H)$. Theorem 5.1 follows directly from this fact, (5.2), and the propositions.

Proof of Proposition 5.2.. We know from Corollary 4.9 that

$$\mathcal{W}^\eta(\nabla H) = -\Big(\mathbb{E}\int_{K_1} |\nabla H^\eta|^2 - \frac{m}{\eta^3}\big(\int_{B^1} |\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\big)\Big).$$

From this and relation (5.1), we see that the proposition amounts to the statement

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^6}{|\mathcal{O}|}\int_{\mathbb{R}^3} |\nabla \bar{h}^\eta_\varepsilon|^2 = \mathbb{E}\int_{K_1} |\nabla H^\eta|^2.$$

A simple application of the ergodic theorem shows that, almost surely,

$$\frac{\varepsilon^6}{|\mathcal{O}|}\int_{\mathcal{O}} |\nabla \bar{h}^\eta_\varepsilon|^2 = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} |\nabla_y H^\eta(\frac{x}{\varepsilon})|^2 \mathrm{d}y \to \mathbb{E}\int_{K_1} |\nabla H^\eta|^2.$$

It remains to show that

$$\lim_{\varepsilon\to 0} \varepsilon^6 \int_{\mathrm{ext}\,\mathcal{O}} |\nabla \bar{h}^\eta_\varepsilon|^2 = 0. \tag{5.3}$$

It will be deduced from the well-known fact that the Stokes solution $\bar{h}^\eta_\varepsilon$ minimizes

$$\int_{\mathrm{ext}\,\mathcal{O}} |\nabla \bar{h}|^2$$

among divergence-free fields $\bar{h}$ in ext $\mathcal{O}$ satisfying the Dirichlet condition $\bar{h}|_{\partial\mathcal{O}} = \bar{h}^\eta_\varepsilon|_{\partial\mathcal{O}}$.

First, we prove that the $H^{1/2}(\partial\mathcal{O})$-norm of $\varepsilon^3\bar{h}^\eta_\varepsilon$ goes to zero. In this perspective, we introduce for all $\delta > 0$ a function χ_δ with $\chi_\delta = 1$ in a $\frac{\delta}{2}$-neighborhood of $\partial\mathcal{O}$, $\chi_\delta = 0$ outside a δ-neighborhood of $\partial\mathcal{O}$. We write

$$\begin{aligned}\|\varepsilon^3\bar{h}^\eta_\varepsilon\|_{H^{1/2}(\partial\mathcal{O})} &= \|\varepsilon^3\bar{h}^\eta_\varepsilon\chi_\delta\|_{H^{1/2}(\partial\mathcal{O})}\\ &\leqq C\left(\|\varepsilon^3\bar{h}^\eta_\varepsilon\chi_\delta\|_{L^2(\mathcal{O})} + \|\varepsilon^3\nabla\bar{h}^\eta_\varepsilon\chi_\delta\|_{L^2(\mathcal{O})} + \|\varepsilon^3\bar{h}^\eta_\varepsilon\nabla\chi_\delta\|_{L^2(\mathcal{O})}\right).\end{aligned}$$

By the ergodic theorem and Corollary 4.9, $\varepsilon^3\nabla\bar{h}^\eta_\varepsilon = \nabla_y H^\eta(\frac{\cdot}{\varepsilon})$ converges almost surely weakly in $L^2(\mathcal{O})$ to $\mathbb{E}D_{\mathbf{H}^\eta} = 0$. Let $\varphi \in L^2(\mathcal{O})$. By standard results on the divergence operator, *cf* [16], there exists $v \in H^1_0(\mathcal{O})$ with $\mathrm{div}\, v = \varphi - \fint_{\mathcal{O}}\varphi$, $\|v\|_{H^1(\mathcal{O})} \leqq C_{\mathcal{O}}\|\varphi\|_{L^2(\mathcal{O})}$. As by definition $\bar{h}^\eta_\varepsilon$ has zero mean over $\mathcal{O}$, it follows that

$$\int_{\mathcal{O}} \varepsilon^3\bar{h}^\eta_\varepsilon\varphi = \int_{\mathcal{O}} \varepsilon^3\bar{h}^\eta_\varepsilon\,(\varphi - \fint_{\mathcal{O}}\varphi) = -\int_{\mathcal{O}} \varepsilon^3\nabla\bar{h}^\eta_\varepsilon\, v \to 0 \ \text{ as } \varepsilon \to 0.$$

Hence, $\varepsilon^3\bar{h}^\eta_\varepsilon$ converges weakly to zero in $H^1(\mathcal{O})$ and therefore strongly in $L^2(\mathcal{O})$. It follows that, for any given δ,

$$\|\varepsilon^3\bar{h}^\eta_\varepsilon\chi_\delta\|_{L^2(\mathcal{O})} \to 0, \quad \|\varepsilon^3\bar{h}^\eta_\varepsilon\nabla\chi_\delta\|_{L^2(\mathcal{O})} \to 0 \quad \text{as } \varepsilon \to 0.$$

To conclude, it is enough to show that $\limsup_{\varepsilon\to 0}\|\varepsilon^3\nabla\bar{h}^\eta_\varepsilon\chi_\delta\|_{L^2(\mathcal{O})}$ goes to zero as $\delta \to 0$. This comes from

$$\|\varepsilon^3\nabla\bar{h}^\eta_\varepsilon\chi_\delta\|^2_{L^2(\mathcal{O})} = \int_{\mathcal{O}} |\nabla H^\eta(\cdot/\varepsilon)|^2\,\chi_\delta^2 \xrightarrow[\varepsilon\to 0]{} \mathbb{E}|D^\eta_{\mathbf{H}}|^2\int_{\mathcal{O}}\chi_\delta^2 \leqq C\delta. \tag{5.4}$$

Finally, $\|\varepsilon^3 \bar{h}_\varepsilon^\eta\|_{H^{1/2}(\partial\mathcal{O})} = o(1)$. To conclude that (5.3) holds, we notice that

$$\int_{\partial\mathcal{O}} \bar{h}_\varepsilon^\eta \cdot n = \int_{\mathcal{O}} \mathrm{div}\, \bar{h}_\varepsilon^\eta = 0.$$

By classical results on the right inverse of the divergence operator, see [16], one can find for R such that $\mathcal{O} \Subset B(0, R)$ a solution $\bar{h}$ of the equation

$$\mathrm{div}\bar{h} = 0 \quad \text{in } \operatorname{ext}\mathcal{O} \cap B(0, R), \quad \bar{h}|_{\partial\mathcal{O}} = \bar{h}_\varepsilon^\eta|_{\partial\mathcal{O}}, \quad \bar{h}|_{\partial B(0,R)} = 0,$$

and such that

$$\|\bar{h}\|_{H^1(\operatorname{ext}\mathcal{O} \cap B(0,R))} \leqq C\|\bar{h}_\varepsilon^\eta\|_{H^{1/2}(\partial\mathcal{O})} = o(\varepsilon^{-3}).$$

Extending $\bar{h}$ by zero outside $B(0, R)$, we find

$$\int_{\operatorname{ext}\mathcal{O}} |\nabla \bar{h}_\varepsilon^\eta|^2 \leqq \int_{\operatorname{ext}\mathcal{O}} |\nabla \bar{h}|^2 = o(\varepsilon^{-6}). \tag{5.5}$$

This concludes the proof of the proposition. □

Proof of Proposition 5.3.. Let $h := h_\varepsilon^\eta - \bar{h}_\varepsilon^\eta$. It satisfies an equation of the form

$$-\Delta h + \nabla p = R_1 + R_2 + R_3, \quad \mathrm{div}\, h = 0 \text{ in } \mathbb{R}^3,$$

where the various source terms will now be defined. First,

$$R_1 := \sigma(\bar{h}_\varepsilon^\eta, \bar{p}_\varepsilon^\eta) n|_{\partial(\operatorname{ext}\mathcal{O})}\, s_\partial.$$

Here, the value of the stress is taken from $\operatorname{ext}\mathcal{O}$, n refers to the normal vector pointing outward $\mathcal{O}$ and s_∂ refers to the surface measure on $\partial\mathcal{O}$. We remind that $\bar{h}_\varepsilon^\eta \in \dot{H}^1(\mathbb{R}^3)$ does not jump at the boundary, but its derivatives do, so that one must specify from which side the stress is considered. Then,

$$R_2 := -\sigma(\bar{h}_\varepsilon^\eta, \bar{p}_\varepsilon^\eta) n|_{\partial\mathcal{O}}\, s_\partial = -\frac{1}{\varepsilon^3}\sigma\Big(H^\eta, P^\eta - \fint_{\mathcal{O}} P^\eta(\omega, \cdot/\varepsilon)\Big)\big(\frac{\cdot}{\varepsilon}\big)|_{\partial\mathcal{O}} n\, s_\partial,$$

with the value of the stress taken from $\mathcal{O}$, and n as before. Noticing that $S\nabla f = -\frac{1}{|\mathcal{O}|} Sn\, s_\partial$, we finally set

$$R_3 := -\mathbf{1}_{\mathcal{O}} \sum_{i \in I_\varepsilon^\eta} S^{\eta\varepsilon}(x - x_i) + \frac{N}{|\mathcal{O}|} Sn\, s_\partial,$$

where

$$I_\varepsilon^\eta = \{i, B(x_i, \varepsilon) \not\subset \mathcal{O},\ B(x_i, \eta\varepsilon) \cap \mathcal{O} \neq \emptyset\}.$$

Note that the term R_3 is supported in pieces of spheres. From (3.20), we know that for all $\eta > 0$,

$$\int_{\mathbb{R}^3} S^\eta = \int_{\mathbb{R}^3} \big(-\Delta G_S^\eta + \nabla p_S^\eta\big) = 0. \tag{5.6}$$

This allows us to show that the integral of $R_2 + R_3$ is zero. Indeed,

$$\int_{\mathbb{R}_3} R_2 = \frac{1}{\varepsilon^4}\int_{\mathcal{O}}(-\Delta H^\eta + \nabla P^\eta)(\cdot/\varepsilon) = \int_{\mathcal{O}} \sum_{i, B(x_i,\eta\varepsilon)\cap\mathcal{O}\neq\emptyset} S^{\eta\varepsilon}(\cdot - x_i)$$
$$= \sum_{i\in I^\eta_\varepsilon}\int_{\mathcal{O}} S^{\eta\varepsilon}(\cdot - x_i),$$

so that

$$\int_{\mathbb{R}^3}(R_2 + R_3) = \frac{N}{|\mathcal{O}|}\int_{\partial\mathcal{O}} Sn\, \mathrm{d}s_\partial = 0. \tag{5.7}$$

The point is now to prove that $\varepsilon^3\|\nabla h\|_{L^2(\mathbb{R}^3)} \to 0$ as $\varepsilon \to 0$. From a simple energy estimate, and taking (5.7) into account, we find

$$\|\nabla h\|^2_{L^2(\mathbb{R}^3)} = \langle R_1, h\rangle + \langle R_2, h - \fint_{\mathcal{O}} h\rangle + \langle R_3, h - \fint_{\mathcal{O}} h\rangle. \tag{5.8}$$

As $(\bar{h}^\eta_\varepsilon, \bar{p}^\eta_\varepsilon)$ is a solution of a homogeneous Stokes equation in ext $\mathcal{O}$, we get, from an integration by parts, that

$$\langle R_1, h\rangle = \int_{\mathrm{ext}\,\mathcal{O}} \nabla\bar{h}^\eta_\varepsilon \cdot \nabla h \leqq \nu(\varepsilon)\varepsilon^{-3}\|\nabla h\|_{L^2(\mathbb{R}^3)}, \quad \nu(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0, \tag{5.9}$$

using the Cauchy–Schwarz inequality and the bound (5.5).

We now wish to show that

$$\langle (R_2 + R_3), h - \fint_{\mathcal{O}} h\rangle \leqq \nu(\varepsilon)\varepsilon^{-3}\|\nabla h\|_{L^2(\mathbb{R}^3)} \tag{5.10}$$

for some $\nu(\varepsilon)$ going to zero with ε. More precisely, we will prove that for any divergence-free $\varphi \in \dot{H}^1(\mathbb{R}^3)$,

$$\langle (R_2 + R_3), \varphi\rangle \leqq \nu(\varepsilon)\varepsilon^{-3}(\|\nabla\varphi\|_{L^2(\mathbb{R}^3)} + \|\varphi\|_{H^1(\mathcal{O})}), \quad \nu(\varepsilon) \to 0 \text{ as } \varepsilon \to 0, \tag{5.11}$$

which implies (5.10), by Poincaré inequality. We first notice that

$$\langle R_2, \varphi\rangle = \frac{1}{\varepsilon^3}\langle n \cdot F^\varepsilon_2, \varphi\rangle_{\langle H^{-1/2}(\partial\mathcal{O}), H^{1/2}(\partial\mathcal{O})\rangle}, \tag{5.12}$$

where

$$F^\varepsilon_2 := \varepsilon^3\left(2D(\bar{h}^\eta_\varepsilon) - \overline{p}^\eta_\varepsilon\mathrm{Id}\right) = 2D(H^\eta)(\omega, \cdot/\varepsilon) + \left(P^\eta(\omega, \cdot/\varepsilon) - \fint_{\mathcal{O}} P^\eta(\omega, \cdot/\varepsilon)\right)\mathrm{Id}. \tag{5.13}$$

Then, we use the relation $S^\eta = \mathrm{div}\,\Psi^\eta$, *cf.* Lemma 3.5 and integrate by parts to get

$$\langle R_3, \varphi\rangle = \frac{1}{\varepsilon^3}\sum_{i\in I^\varepsilon_\eta}\left(-\int_{\partial\mathcal{O}} n \cdot \Psi^\eta\left(\frac{x - x_i}{\varepsilon}\right) \cdot \varphi(x)\mathrm{d}s_\partial(x)\right.$$
$$\left. + \int_{\mathcal{O}} \Psi^\eta\left(\frac{x - x_i}{\varepsilon}\right) : \nabla\varphi(x)\mathrm{d}x\right) + \frac{N}{|\mathcal{O}|}\int_{\partial\mathcal{O}} Sn(x) \cdot \varphi(x)\,\mathrm{d}s_\partial(x).$$

For a fixed η, there is a constant C (depending on η) such that

$$\begin{aligned}
&\sum_{i\in I_\eta^\varepsilon}\int_{\mathcal{O}}\Psi^\eta\left(\frac{x-x_i}{\varepsilon}\right):\nabla\varphi(x)\mathrm{d}x\\
&\leqq C\sum_{i\in I_\eta^\varepsilon}\int_{B(x_i,\eta\varepsilon)\cap\mathcal{O}}|\nabla\varphi|(x)\mathrm{d}x \leqq C|\cup_{i\in I_\varepsilon^\eta}B(x_i,\eta\varepsilon)|^{1/2}\|\nabla\varphi\|_{L^2(\mathbb{R}^3)}\\
&\leqq C\varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For the last inequality, we have used that all x_i's with $i\in I_\varepsilon^\eta$ belong to an ε-neighborhood of $\partial\mathcal{O}$, so that $|I_\varepsilon^\eta| = O(\varepsilon^{-2})$. Hence,

$$\begin{aligned}
\langle R_3,\varphi\rangle \leqq \frac{1}{\varepsilon^3}\sum_{i\in I_\eta^\varepsilon} -\int_{\partial\mathcal{O}} n\cdot\Psi^\eta\left(\frac{x-x_i}{\varepsilon}\right)\cdot\varphi(x)\mathrm{d}s_\partial(x)\\
+\frac{N}{|\mathcal{O}|}\int_{\partial\mathcal{O}} Sn(x)\cdot\varphi(x)\,\mathrm{d}s_\partial(x)+\nu(\varepsilon)\varepsilon^{-5/2}\|\nabla\varphi\|_{L^2(\mathcal{O})}.
\end{aligned}\tag{5.14}$$

Let

$$F_3(\omega) := -\sum_{z\in\Lambda}\Psi^\eta(z)+mS,\quad F_3^\varepsilon(x) := F_3(\tau_{x/\varepsilon}(\omega)).\tag{5.15}$$

We claim that $\mathbb{E}\int_{K_1}F_3 = 0$. Indeed, by stationarity, for all $R>0$

$$\begin{aligned}
\mathbb{E}\sum_{z\in\Lambda}\Psi^\eta(z) &= \frac{1}{R^3}\mathbb{E}\sum_{z\in\Lambda}\int_{K_R}\Psi^\eta(y+z)\mathrm{d}y\\
&= \frac{1}{R^3}\mathbb{E}\sum_{\substack{z\in\Lambda,\\ K_R\supset B(-z,\eta)}}\int_{K_R}\Psi^\eta(y+z)\mathrm{d}y\\
&\quad+\frac{1}{R^3}\mathbb{E}\sum_{\substack{z\in\Lambda,\\ \partial K_R\cap B(-z,\eta)\neq\emptyset}}\int_{K_R}\Psi^\eta(y+z)\mathrm{d}y\\
&\quad+\frac{1}{R^3}\mathbb{E}\sum_{\substack{z\in\Lambda,\\ K_R\cap B(-z,\eta)=\emptyset}}\int_{K_R}\Psi^\eta(y+z)\mathrm{d}y\\
&= \frac{1}{R^3}\mathbb{E}\sum_{\substack{z\in\Lambda,\\ K_R\supset B(-z,\eta)}}\int_{K_R}\Psi^\eta(y+z)\mathrm{d}y\\
&\quad+\frac{1}{R^3}\mathbb{E}\sum_{\substack{z\in\Lambda,\\ \partial K_R\cap B(-z,\eta)\neq\emptyset}}\int_{K_R}\Psi^\eta(y+z)\mathrm{d}y\\
&= \frac{1}{R^3}\mathbb{E}\big|\{z,\,K_R\supset B(0,\eta)-z\}\big|\int_{B(0,\eta)}\Psi^\eta(y)\mathrm{d}y\;+\;O\big(\frac{1}{R}\big),\quad R\gg 1.
\end{aligned}$$

We have used crucially the fact that Ψ^η is supported in $B(0,\eta)$. The $O(\frac{1}{R})$-term is associated to the points $z\in\Lambda$ which lie in a δ-neighborhood of ∂K_R: see the end of the proof of Corollary 4.9 for similar reasoning. By sending R to infinity, we find that, almost surely,

$$\mathbb{E}F_3 = -m \int_{B(0,\eta)} \Psi^\eta \, + mS.$$

The last step is to compute $\int_{B(0,\eta)} \Psi^\eta$, which is independent of η by homogeneity. It is in particular equal to $\lim_{\eta\to 0}\langle \Psi^\eta, 1\rangle$, a limit that was already computed in the proof of Lemma 3.5, *cf.* (3.23)–(3.24). We get $\int_{B(0,\eta)} \Psi^\eta = S$, which shows that $\mathbb{E}F_3 = 0$.

By the definition of F_3^ε, we can write

$$\begin{aligned}
&\frac{1}{\varepsilon^3}\sum_{i\in I_\eta^\varepsilon} \int_{\partial\mathcal{O}\cap B(x_i,\eta\varepsilon)} -n\cdot\Psi^\eta\left(\frac{x-x_i}{\varepsilon}\right)\cdot\varphi(x)\mathrm{d}s_\partial(x) \\
&\quad + \frac{N}{|\mathcal{O}|}\int_{\partial\mathcal{O}} Sn(x)\cdot\varphi(x)\,\mathrm{d}s_\partial(x) \\
&= \frac{1}{\varepsilon^3}\int_{\partial\mathcal{O}} n(x)\cdot F_3^\varepsilon(x)\cdot\varphi(x)\mathrm{d}s_\partial(x) + \left(\frac{N}{|\mathcal{O}|} - \frac{m}{\varepsilon^3}\right)\int_{\partial\mathcal{O}} Sn\cdot\varphi \\
&\leqq \frac{1}{\varepsilon^3}\int_{\partial\mathcal{O}} n(x)\cdot F_3^\varepsilon(x)\cdot\varphi(x)\mathrm{d}s_\partial(x) + \nu(\varepsilon)\varepsilon^{-3}\|\varphi\|_{H^1(\mathcal{O})}, \quad \nu(\varepsilon)\xrightarrow[\varepsilon\to 0]{} 0,
\end{aligned}$$

where the last inequality follows from (5.1). Plugging this inequality in (5.14), and combining with (5.12), we see that to derive (5.11), it remains to show that almost surely, for all divergence-free fields $\varphi\in H^1(\mathcal{O})$,

$$|\langle n\cdot F^\varepsilon, \varphi\rangle_{\langle H^{-1/2}(\partial\mathcal{O}),H^{1/2}(\partial\mathcal{O})\rangle}| \leqq \nu(\varepsilon)\|\varphi\|_{H^1(\mathcal{O})}, \quad \nu(\varepsilon)\to 0 \text{ as } \varepsilon\to 0, \quad (5.16)$$

where $F^\varepsilon := F_2^\varepsilon + F_3^\varepsilon$. Notice that $\mathrm{div}\,(F_2^\varepsilon + F_3^\varepsilon) = 0$. We introduce again the functions χ_δ, $\delta > 0$, seen above. We get

$$\begin{aligned}
&\langle n\cdot F^\varepsilon, \varphi\rangle_{\langle H^{-1/2}(\partial\mathcal{O}),H^{1/2}(\partial\mathcal{O})\rangle} = \langle n\cdot\chi_\delta F^\varepsilon, \varphi\rangle_{\langle H^{-1/2}(\partial\mathcal{O}),H^{1/2}(\partial\mathcal{O})\rangle} \\
&\quad = \int_{\mathcal{O}} (\nabla\chi_\delta\cdot F^\varepsilon)\cdot\varphi - \int_{\mathcal{O}} \chi_\delta F^\varepsilon\cdot\nabla\varphi
\end{aligned}$$

For the last term, we take into account that φ is divergence-free, so that the pressure disappears. We find that

$$\left|\int_{\mathcal{O}} \chi_\delta F^\varepsilon\cdot\nabla\varphi\right| \leqq \left(\|2\chi_\delta D(H)(\cdot/\varepsilon)\|_{L^2(\mathcal{O})} + \|\chi_\delta F_3(\cdot/\varepsilon)\|_{L^2(\mathcal{O})}\right)\,\|\varphi\|_{H^1(\mathcal{O})}.$$

As seen in (5.4), we have

$$\lim_{\varepsilon\to 0}\|2\chi_\delta D(H)(\cdot/\varepsilon)\|^2_{L^2(\mathcal{O})} \leqq C\delta,$$

and similarly,

$$\lim_{\varepsilon\to 0}\|\chi_\delta F_3^\varepsilon\|^2_{L^2(\mathcal{O})} \leqq C\delta.$$

For the first term, we write

$$\int_{\mathcal{O}} (\nabla\chi_\delta \cdot F^\varepsilon)\cdot\varphi = 2\int_{\mathcal{O}} \nabla\chi_\delta \cdot \varepsilon^3 D(\bar{h}^\eta_\varepsilon)\cdot\varphi - \int_{\mathcal{O}} (\nabla\chi_\delta\, \varepsilon^3 \overline{p}^\eta_\varepsilon)\cdot\varphi + \int_{\mathcal{O}} (\nabla\chi_\delta \cdot F^\varepsilon_3)\cdot\varphi.$$

We know that $\varepsilon^3 D(\bar{h}^\eta_\varepsilon)$ goes weakly to zero in $L^2(\mathcal{O})$, so that it converges strongly to zero in $H^{-1}(\mathcal{O})$. As $\nabla\chi_\delta\otimes\varphi$ belongs to $H^1_0(\mathcal{O})$, we find that, for a fixed δ,

$$|2\int_{\mathcal{O}} \nabla\chi_\delta \cdot \varepsilon^3 D(\bar{h}^\eta_\varepsilon)\cdot\varphi| \leqq C\|\varepsilon^3 D(\bar{h}^\eta_\varepsilon)\|_{H^{-1}(\mathcal{O})}\|\nabla\chi_\delta\varphi\|_{H^1(\mathcal{O})} \leqq \nu(\varepsilon)\|\varphi\|_{H^1(\mathcal{O})}.$$

Similarly, as $\mathbb{E}F_3 = 0$, F^ε_3 converges weakly to zero in $L^2(\mathcal{O})$ and we get

$$|\int_{\mathcal{O}} (\nabla\chi_\delta \cdot F^\varepsilon_3)\cdot\varphi| \leqq \nu(\varepsilon)\|\varphi\|_{H^1(\mathcal{O})}.$$

The last step is to prove that $\varepsilon^3\overline{p}^\eta_\varepsilon$ converges weakly to zero in $L^2(\mathcal{O})$, which will yield

$$|\int_{\mathcal{O}} (\nabla\chi_\delta\, \varepsilon^3\overline{p}^\eta_\varepsilon)\cdot\varphi| \leqq \nu(\varepsilon)\|\varphi\|_{H^1(\mathcal{O})}.$$

As above, for $\phi\in L^2(\mathcal{O})$, we introduce $v\in H^1_0(\mathcal{O})$ such that $\operatorname{div} v = \phi - \fint_{\mathcal{O}}\phi$, $\|v\|_{H_1(\mathcal{O})} \leqq C_{\mathcal{O}}\|\phi\|_{L^2(\mathcal{O})}$. Then, using the equation satisfied by $\overline{p}^\eta_\varepsilon$ in $\mathcal{O}$,

$$-\Delta\varepsilon^3\bar{h}^\eta_\varepsilon + \nabla\varepsilon^3\overline{p}^\eta_\varepsilon = \operatorname{div} F^\varepsilon_3,$$

we find, after integration by parts, that

$$\int_{\mathcal{O}} \varepsilon^3\overline{p}^\eta_\varepsilon\phi = \int_{\mathcal{O}} \varepsilon^3\overline{p}^\eta_\varepsilon(\phi - \fint_{\mathcal{O}}\phi) = \int_{\mathcal{O}} \varepsilon^3\nabla\bar{h}^\eta_\varepsilon : \nabla v + \int_{\mathcal{O}} F^\varepsilon_3 : \nabla v \xrightarrow[\varepsilon\to 0]{} 0.$$

This concludes the proof of (5.16), of Proposition 5.3 and of the theorem. □

5.2. Formula for Periodic Point Distributions

Theorem 5.1 gives the limit of $\mathcal{V}_N$ for properly rescaled stationary and ergodic point processes, under uniform separation of the points. Such setting includes periodic point distributions, as well as Poisson hard core processes. We focus here on the periodic case, for which further explicit formula can be given. For $L > 0$, we consider distinct points $a_1, \dots, a_M$ in K_L, and set $\Lambda_0 := \{a_1, \dots, a_M\} + L\mathbb{Z}^d$, which can be seen as a subset of $\mathbb{T}^3_L$. In Example 4.5, we explained how to build a process on $\mathbb{T}^3_L$ out of Λ_0, with $\Lambda(\omega) = \Lambda_0 + \omega$, $\omega\in\mathbb{T}^3_L$. By a simple translation, the results above, that are valid for $\Lambda_0 + \omega$ for almost everywhere ω, are still valid for $\omega = 0$. Thus, for $\Lambda = \Lambda_0$, we deduce from Proposition 4.7 the existence of an $L\mathbb{Z}^3$-periodic solution $\mathbf{H}^\eta$ of (4.5) with $\nabla\mathbf{H}^\eta \in L^2_{loc}$. If we further assume that $\mathbf{H}^\eta$

is mean-free, it is clearly unique. Then, following Corollary 4.9 and Theorem 5.1, there exists an $L\mathbb{Z}^3$-periodic solution H of (4.3), such that

$$\lim_{N\to+\infty} \mathcal{V}_N = \frac{25L^6}{2M^2}\mathcal{W}(\nabla H), \quad \mathcal{W}(\nabla H) = \lim_{\eta\to 0} \mathcal{W}^\eta(\nabla H),$$
$$\mathcal{W}^\eta(\nabla H) = -\left(\fint_{K_L} |\nabla H^\eta|^2 - \frac{M}{L^3\eta^3}\left(\int_{B^1} |\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\right), \qquad (5.17)$$

where H^η is associated to H by (4.4). We have used that in the periodic case, the intensity of the process is $m = \frac{M}{L^3}$, while the expectation is simply the average over K_L.

To make things more explicit, we introduce the periodic Green function $G_{S,L} : \mathbb{R}^3 \to \mathbb{R}^3$, satisfying

$$-\Delta G_{S,L} + \nabla p_{S,L} = S\nabla\delta_0\,, \quad \operatorname{div} G_{S,L} = 0 \text{ in } K_L, \quad G_{S,L}\ L\mathbb{Z}^3\text{-periodic}, \quad \int_{K_L} G_{S,L} = 0. \qquad (5.18)$$

The Green function $G_{S,L}$ is easily expressed in Fourier series. If we write

$$G_{S,L}(y) = \sum_{k\in\mathbb{Z}^3_*} e^{\frac{2i\pi k}{L}\cdot y}\widehat{G}_{S,L}(k),$$

a straightforward calculation shows that, for all $k \in \mathbb{Z}^3_*$,

$$\widehat{G}_{S,L}(k) = \frac{i}{2\pi L^2|k|}\left(S\frac{k}{|k|} - \frac{Sk\cdot k}{|k|^2}\frac{k}{|k|}\right) = \frac{i}{2\pi L^2|k|^2}\pi_k^\perp Sk,$$

where $\pi_k^\perp$ denotes the projection orthogonally to the line $\mathbb{R}k$. Note that the Fourier series for $G_{S,L}$ converges, for instance, in the quadratic sense as follows:

Proposition 5.4.

$$\lim_{N\to+\infty} \mathcal{V}_N = \frac{25L^3}{2M^2}\left(\sum_{i\neq j\in\{1,\dots,M\}} S\nabla\cdot G_{S,L}(a_i - a_j) + M\lim_{y\to 0} S\nabla\cdot(G_{S,L}(y) - G_S(y))\right).$$

Proof. Clearly, the $L\mathbb{Z}^3$-periodic field defined on K_L by $\tilde{H}(y) := \sum_{i=1}^M G_{S,L}(y + a_i)$ is a solution of (4.3), and by Proposition 4.3 $\nabla\tilde{H}$ and ∇H differ from a constant matrix. As $\nabla(\tilde{H} - H) = \nabla(\tilde{H}^\eta - H^\eta)$ is the gradient of a periodic function, we have eventually $\nabla\tilde{H} = \nabla H$. Up to adding a constant field to H, we can assume that

$$H(y) = \sum_{i=1}^M G_{S,L}(y + a_i).$$

Then, if η is small enough so that $B(a_i, \eta) \subset K_L$ for all i, H^η is the L-periodic field given on K_L by

$$H^\eta(y) = \sum_{i=1}^M \left(G_{S,L}(y + a_i) + (G_S^\eta - G_S)(y + a_i)\right).$$

We integrate by parts to find

$$\begin{aligned}\frac{1}{L^3}\int_{K_L}|\nabla H^\eta|^2 &= \frac{1}{L^3}\int_{K_L}\sum_{i=1}^{M} H^\eta \mathrm{d}S^\eta(\cdot+a_i)\\ &= \frac{1}{L^3}\int_{K_L}\sum_{i,j} G_{S,L}(\cdot+a_j)\mathrm{d}S^\eta(\cdot+a_i)\\ &\quad+\frac{1}{L^3}\int_{K_L}\sum_{i,j}(G_S^\eta-G_S)(\cdot+a_j)\mathrm{d}S^\eta(\cdot+a_i)\\ &= \frac{1}{L^3}\sum_{i\neq j}\int_{K_L} G_{S,L}(\cdot+a_j)\mathrm{d}S^\eta(\cdot+a_i)\\ &\quad+\frac{1}{L^3}\sum_{i}\int_{K_L} G_{S,L}(\cdot+a_i)\mathrm{d}S^\eta(\cdot+a_i),\end{aligned}$$

where we have used that the last term of the second line vanishes identically. We then write $G_{S,L}=G_S+\phi_{S,L}$ with $\phi_{S,L}$ smooth near 0 to obtain

$$\begin{aligned}\frac{1}{L^3}\int_{K_L}|\nabla H^\eta|^2 &= \sum_{i\neq j}\frac{1}{L^3}\int_{K_L} G_{S,L}(\cdot+a_j)\mathrm{d}S^\eta(\cdot+a_i)\\ &\quad+\frac{1}{L^3}\sum_{i}\int_{K_L}\phi_{S,L}(\cdot+a_i)\mathrm{d}S^\eta(\cdot+a_i)+\frac{M}{L^3}\int_{\mathbb{R}^3} G_S\mathrm{d}S^\eta.\end{aligned}$$

Combining this with Lemma 3.6 and (5.17), we get

$$\begin{aligned}\lim_{N\to\infty}\mathcal{V}_N = -\frac{25L^6}{2M^2}\lim_{\eta\to 0}\Big(&\sum_{i\neq j}\frac{1}{L^3}\int_{K_L} G_{S,L}(\cdot+a_j)\mathrm{d}S^\eta(\cdot+a_i)\\ &+\frac{1}{L^3}\sum_{i}\int_{K_L}\phi_{S,L}(\cdot+a_i)\mathrm{d}S^\eta(\cdot+a_i)\Big).\end{aligned}$$

We conclude by the last point of Lemma 3.5 that

$$\lim_{N\to\infty}\mathcal{V}_N = \frac{25L^3}{2M^2}\Big(\sum_{i\neq j} S\nabla\cdot G_{S,L}(a_i-a_j)\ +\ MS\nabla\cdot\phi_{S,L}(0)\Big).$$

□

Proposition 5.5. (Simple cubic lattice). *In the special case where $L=M=1$, we find*

$$\lim_{N\to\infty}\mathcal{V}_N=\alpha\sum_i S_{ii}^2+\beta\sum_{i\neq j}S_{ij}^2,$$

with $\alpha=\frac{5}{2}(1-60a)$, $\beta=\frac{5}{2}(1+40a)$, and $a\approx -0,04655$ is defined in (5.19).

Proof. When $M = L = 1$, the formula from the last proposition simplifies into $\lim_N \mathcal{V}_N = \frac{25}{2} S\nabla \cdot \phi_{S,1}(0)$, with $\phi_{S,1} = G_{S,1} - G_S$. The periodic Green function $G_{S,1}$ was computed using the Fourier series in the last paragraph. We found

$$G_{S,1}(y) = \sum_{k\in\mathbb{Z}^3_*} \frac{i}{2\pi|k|}\left(S\frac{k}{|k|} - \frac{Sk\cdot k}{|k|^2}\frac{k}{|k|}\right)e^{2i\pi k\cdot y}$$
$$= S\nabla\left(\sum_{k\in\mathbb{Z}^3_*}\frac{1}{4\pi^2|k|^2}e^{2i\pi k\cdot y}\right) + S:(\nabla\otimes\nabla)\nabla\left(\sum_{k\in\mathbb{Z}^3_*}\frac{1}{16\pi^4|k|^4}e^{2i\pi k\cdot y}\right).$$

We use formulas from [20], (see also [42, Eqs. (64)–(65)]) to get

$$\sum \frac{1}{4\pi^2|k|^2}e^{2i\pi k\cdot y} = \frac{1}{4\pi}\left(\frac{1}{|y|} - c_1 + \frac{2\pi}{3}|y|^2 + O(|y|^4)\right)$$

and

$$\sum \frac{1}{16\pi^4|k|^4}e^{2i\pi k\cdot y} = -\frac{1}{4\pi}\left(\frac{|y|}{2} - c_2 - \frac{c_1}{6}|y|^2 + \frac{\pi}{30}|y|^4 + aP(y) + O(|y|^6)\right), \tag{5.19}$$

where c_1 and c_2 are constants, and

$$P(y) = \frac{4\pi}{3}\Big(\frac{5}{8}(y_1^4 + y_2^4 + y_3^4) - \frac{15}{4}(y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2) + \frac{3}{8}|y|^4\Big).$$

Note that the formula (5.19) defines implicitly a. A numerical computation was carried in [42], see also [34], giving $a \approx -0,04655$.

Inserting in the expression for $G_{S,1}$, we find, after a tedious calculation, that

$$S\nabla\cdot G_{S,1}(y) = S\nabla\cdot\big(-\frac{3}{8\pi}\frac{(Sy\cdot y)y}{|y|^5}\big)$$
$$+\frac{1}{5}|S|^2 - 12a\sum_i S_{ii}^2 + 8a\sum_{i\neq j}|S_{ij}|^2 + O(|y|).$$

Note that to carry out this calculation, we used the fact that S is trace-free, which leads to the identity

$$0 = \left(\sum_i S_{ii}\right)^2 = \sum_i S_{ii}^2 + \sum_{i\neq j} S_{ii}S_{jj}.$$

Moreover, we know from (3.12) that

$$G_S(y) = -\frac{3}{8\pi}\frac{(Sy\cdot y)y}{|y|^5}.$$

We end up with

$$S\nabla\cdot\phi_{S,1}(0) = \frac{1}{5}|S|^2 - 12a\sum_i S_{ii}^2 + 8a\sum_{i\neq j}|S_{ij}|^2,$$

and the right formula for $\lim_N \mathcal{V}_N$. □

5.3. Formula in the Stationary Case with the 2-Point Correlation Function

We consider here the case of random point processes in $\mathbb{R}^3$ ($X = \mathbb{R}$), such that (P1)–(P2)–(P3) hold. We further assume that the mean density is $m = 1$. We assume moreover that this point process admits a 2-point correlation function, that is a function $\rho_2 = \rho_2(x, y) \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$ such that for all bounded sets K and all smooth F in a neighborhood of K,

$$\mathbb{E} \sum_{z \neq z' \in K} F(z, z') = \int_{K \times K} F(x, y)\rho_2(x, y)\mathrm{d}x\mathrm{d}y.$$

As the process is stationary, one can write $\rho_2(x, y) = \rho(x - y)$. Our goal is to prove the following formula:

Proposition 5.6. *Almost surely,*

$$\begin{aligned}\lim_N \mathcal{V}_N &= \frac{25}{2} \lim_{L \to +\infty} \frac{1}{L^3} \sum_{z \neq z' \in \Lambda \cap K_{L-1}} S\nabla \cdot G_{S,L}(z - z') \\ &= \frac{25}{2} \lim_{L \to +\infty} \frac{1}{L^3} \int_{K_{L-1} \times K_{L-1}} S\nabla \cdot G_{S,L}(z - z')\rho(z - z')dzdz',\end{aligned}$$

where $G_{S,L}$ refers to the $L\mathbb{Z}^3$-periodic Green function introduced in (5.18).

Remark 5.7. We remind the reader that the periodic Green function $G_{S,L}$ has singularities at each point of $L\mathbb{Z}^d$. But as the sum is restricted to points z, z' in $\Lambda \cap K_{L-1}$, $z - z'$ is always away from this set of singularities. In the same way, the integral over $K_{L-1} \times K_{L-1}$ in the second equality is well-defined. Under further assumption on the two-point correlation function ρ, one could make sense of the integral over $K_L \times K_L$ and replace the former by the latter.

Proof. Let η small enough so that Proposition 4.4 holds. We have

$$\mathcal{W}(\nabla H) = \mathcal{W}^\eta(\nabla H) = -\left(\mathbb{E} \int_{K_1} |\nabla H^\eta|^2 - \frac{1}{\eta^3}\Big(\int_{B^1} |\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\Big)\right).$$

Let $H_L = \sum_{i=1}^M G_{S,L}(\cdot + a_i)$, where $\{a_1, \ldots, a_M\} = \Lambda \cap K_{L-1}$. Note that H_L is associated to the point process Λ_L obtained by $L\mathbb{Z}^d$-periodization of $\Lambda \cap K_{L-1}$. We shall prove below that

$$\mathbb{E} \int_{K_1} |\nabla H^\eta|^2 = \lim_{L \to +\infty} \frac{1}{L^3} \int_{K_L} |\nabla H^\eta_L|^2, \text{ almost surely.} \tag{5.20}$$

As $\frac{M}{L^3} = \frac{|\Lambda \cap K_L|}{L^3} \to 1$ as $L \to +\infty$, it follows from (5.20) that

$$\begin{aligned}\mathcal{W}(\nabla H) &= \lim_{L \to +\infty} -\left(\frac{1}{L^3} \int_{K_L} |\nabla H^\eta_L|^2 - \frac{M}{(\eta L)^3}\Big(\int_{B^1} |\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\Big)\right), \\ &= \lim_{L \to +\infty} \mathcal{W}^\eta(\nabla H_L) = \lim_{L \to +\infty} \mathcal{W}(\nabla H_L),\end{aligned} \tag{5.21}$$

where the last equality comes from Proposition 4.4. One can apply such proposition because the $L\mathbb{Z}^d$-periodized network Λ_L has a minimal distance between points that is independent of L. This is the reason why we used K_{L-1} instead of K_L in the definition of Λ_L. Eventually, by Proposition 5.4,

$$\begin{aligned}&\lim_{L\to+\infty} \mathcal{W}(\nabla H_L)\\ &= \lim_{L\to+\infty}\Big(\frac{1}{L^3}\sum_{i\neq j\in\{1,\dots,M\}} S\nabla\cdot G_{S,L}(a_i-a_j) + \frac{M}{L^3}\lim_{y\to 0} S\nabla\cdot(G_{S,L}(y)-G_S(y))\Big).\end{aligned}$$

Using that

$$G_{S,L}(y) = \frac{1}{L^2}G_{S,1}\left(\frac{\cdot}{L}\right), \quad G_S(y) = \frac{1}{L^2}G_S\left(\frac{\cdot}{L}\right),$$

we get that

$$\begin{aligned}\frac{M}{L^3}\Big|\lim_{y\to0} S\nabla\cdot(G_{S,L}-G_S)(y)\Big| &\leqq C\Big|\lim_{y\to0} S\nabla\cdot(G_{S,L}-G_S)(y)\Big|\\ &\leqq \frac{C'}{L^3}\Big|\lim_{y\to0} S\nabla\cdot(G_{S,1}-G_S)(y/L)\Big| = O(L^{-3}).\end{aligned}$$

We obtain

$$\mathcal{W}(\nabla H) = \lim_{L\to+\infty}\frac{1}{L^3}\sum_{i\neq j\in\{1,\dots,M\}} S\nabla\cdot G_{S,L}(a_i-a_j). \tag{5.22}$$

This is the first formula of the proposition. To prove the second one, one can go back to formula (5.21) and take the expectation of both sides. The left-hand side, which is deterministic, is of course unchanged. As regards the r.h.s., one can swap the limit in L and the expectation by invoking the dominated convergence theorem. Indeed, both terms $\frac{1}{L^3}\int_{K_L}|\nabla H_L^\eta|^2$ and $\frac{M}{(\eta L)^3}\Big(\int_{B^1}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\Big)$ are bounded uniformly in n and in the random parameter ω (but not uniformly on η): the first term is bounded through a simple energy estimate, while the second one is bounded thanks to the almost sure separation assumption.

The final step is to prove (5.20), almost surely. We set $\varepsilon := \frac{1}{L}$, and introduce, for all $x \in K_1$,

$$h_\varepsilon^\eta(x) = \frac{1}{\varepsilon^2}H_L^\eta\left(\frac{x}{\varepsilon}\right), \quad p_\varepsilon^\eta(x) = \frac{1}{\varepsilon^3}p_L^\eta\left(\frac{x}{\varepsilon}\right),$$

and similarly, for all $x \in K_1$,

$$\begin{aligned}\bar{h}_\varepsilon^\eta(x) &= \frac{1}{\varepsilon^2}H^\eta\left(\frac{x}{\varepsilon}\right) - \fint_{K_1}\frac{1}{\varepsilon^2}H^\eta\left(\frac{\cdot}{\varepsilon}\right),\\ \overline{p}_\varepsilon^\eta(x) &= \frac{1}{\varepsilon^3}P^\eta\left(\frac{x}{\varepsilon}\right) - \fint_{K_1}\frac{1}{\varepsilon^3}P^\eta\left(\frac{\cdot}{\varepsilon}\right),\end{aligned}$$

where (H^η, P^η) refers to the field built in Proposition 4.7. Clearly,

$$\varepsilon^6\int_{K_1}|\nabla h_\varepsilon^\eta|^2 = \frac{1}{L^3}\int_{K_L}|\nabla H_L^\eta|^2,$$

while, by the ergodic theorem, one has, almost surely, that

$$\varepsilon^6 \int_{K_1} |\nabla \bar{h}_\varepsilon^\eta|^2 = \frac{1}{L^3} \int_{K_L} |\nabla H^\eta|^2 \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E} \int_{K_1} |\nabla H^\eta|^2.$$

It remains to show that

$$\varepsilon^6 \int_{K_1} |\nabla(\bar{h}_\varepsilon^\eta - h_\varepsilon^\eta)|^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

We notice that the difference $h_\varepsilon = \bar{h}_\varepsilon^\eta - h_\varepsilon^\eta$ satisfies the Stokes equation

$$-\Delta h_\varepsilon + \nabla p_\varepsilon = \frac{1}{\varepsilon^3} \mathrm{div}\,(R_\varepsilon - R_{\varepsilon,L}), \quad \mathrm{div}\, h_\varepsilon = 0 \quad \text{in } K_1,$$

where

$$R_\varepsilon := \sum_{z\in\Lambda} \Psi^\eta(x/\varepsilon + z), \qquad R_{\varepsilon,L} := \sum_{z\in\Lambda_L} \Psi^\eta(x/\varepsilon + z),$$

and where we recall that Λ_L is obtained by $L\mathbb{Z}^3$-periodization of $\Lambda \cap K_{L-1}$. Testing against $\varepsilon^6 h_\varepsilon$, we find

$$\begin{aligned}
\varepsilon^6 \int_{K_1} |\nabla h_\varepsilon|^2 = &- \int_{K_1} (R_\varepsilon - R_{\varepsilon,L}) \varepsilon^3 \nabla h_\varepsilon \\
&+ \int_{\partial K_1} F_\varepsilon n \cdot \varepsilon^3 \Big(h_\varepsilon - \fint_{K_1} h_\varepsilon\Big) - \int_{\partial K_1} G_\varepsilon n \cdot \varepsilon^3 h_\varepsilon, \quad (5.23)
\end{aligned}$$

where

$$\begin{aligned}
F_\varepsilon(x) &:= \nabla H^\eta\Big(\frac{x}{\varepsilon}\Big) - P^\eta\Big(\frac{x}{\varepsilon}\Big) I_d + \int_{K_1} P^\eta\Big(\frac{\cdot}{\varepsilon}\Big) I_d + \tilde{F}(x), \\
G_\varepsilon(x) &:= \nabla H_L^\eta\Big(\frac{x}{\varepsilon}\Big) - P_L^\eta\Big(\frac{x}{\varepsilon}\Big) I_d + \tilde{G}(x),
\end{aligned}$$

with

$$\tilde{F}(x) := \sum_{z\in\Lambda} \Psi^\eta(x/\varepsilon + z) - S, \quad \tilde{G}(x) := \sum_{z\in\Lambda_L} \Psi^\eta(x/\varepsilon + z) - S.$$

Note that both F_ε and G_ε are divergence-free.

To handle the first term at the right-hand side of (5.23), we notice that

$$|\{z \in \Lambda \bigtriangleup \Lambda_L,\, K_L \cap B(-z, \eta) \neq \emptyset\}| = O(L^2) = O(\varepsilon^{-2}),$$

resulting in

$$\begin{aligned}
\int_{K_1} (R_\varepsilon - R_{\varepsilon,L}) \varepsilon^3 \nabla h_\varepsilon &\leqq C\Big(\varepsilon \int_{\mathbb{R}^3} |\Psi^\eta|^2\Big)^{1/2} \|\varepsilon^3 \nabla h_\varepsilon\|_{L^2(K_1)} \\
&\leqq C\varepsilon^{1/2} \|\varepsilon^3 \nabla h_\varepsilon\|_{L^2(K_1)}.
\end{aligned}$$

As regards the second term, one proceeds exactly as in Paragraph 5.1, replacing $\mathcal{O}$ by K_1: see the treatment of F_ε^2 and F_ε^3, defined in (5.13) and (5.15). One gets in this way that for all divergence-free $\varphi \in H^1(K_1)$,

$$|\int_{\partial K_1} F_\varepsilon n \cdot \varphi| \leqq \nu(\varepsilon)\|\nabla\varphi\|_{L^2(K_1)}, \quad \nu(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

As regards the last term, we take into account the periodicity of H_L^η and $\tilde{G}$ to write

$$\int_{\partial K_1} G_\varepsilon n \cdot \varepsilon^3 h_\varepsilon = \int_{\partial K_1} G_\varepsilon n \cdot \varepsilon^3 \bar{h}_\varepsilon^\eta \mathrm{d}x.$$

As $\int_{\partial K_1} \bar{h}_\varepsilon^\eta \cdot n = 0$, we can introduce a solution Φ_ε of

$$\mathrm{div}\Phi_\varepsilon = 0 \quad \text{in } K_1, \quad \Phi_\varepsilon|_{\partial K_1} = \varepsilon^3 \bar{h}_\varepsilon^\eta|_{\partial K_1}, \quad \|\Phi_\varepsilon\|_{H^1(K_1)} \leqq C\|\varepsilon^3 \bar{h}_\varepsilon^\eta|_{\partial K_1}\|_{H^{1/2}(\partial K_1)}.$$

Proceeding as in Paragraph 5.1 (replacing $\mathcal{O}$ by K_1), one can show that $\|\varepsilon^3 \bar{h}_\varepsilon^\eta\|_{H^{1/2}(\partial K_1)}$ goes to zero with ε, and so $\|\Phi^\varepsilon\|_{H^1(K_1)}$ goes to zero as well. Eventually, we write

$$\begin{aligned} |\int_{\partial K_1} G_\varepsilon n \cdot \varepsilon^3 \bar{h}_\varepsilon^\eta \mathrm{d}x| &= |\int_{K_1} G_\varepsilon \cdot \nabla\Phi_\varepsilon| \\ &= |\int_{K_1} \left(2D(H_L^\eta)(\cdot/\varepsilon) + \tilde{G}\right) \cdot \nabla\Phi_\varepsilon| \\ &\leqq C\left(\frac{1}{L^3}\|\nabla H_L^\eta\|_{L^2(K_L)}^2 + \|\Psi^\eta\|_{L^2}^2 + 1\right)^{1/2} \|\nabla\Phi_\varepsilon\|_{L^2} \\ &\leqq C'\|\nabla\Phi_\varepsilon\|_{L^2}. \end{aligned}$$

Hence, we find

$$\varepsilon^6 \int_{K_1} |\nabla h|^2 \leqq C\left(\varepsilon + \nu(\varepsilon)^2 + \|\nabla\Phi_\varepsilon\|_{L^2}^2\right) \xrightarrow[\varepsilon\to 0]{} 0,$$

which concludes the proof. □

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A Proof of Lemma 2.4

For any open set U, we denote $⨍_U = \frac{1}{|U|}\int_U$. By (H2), we have

$$d := \frac{c}{4}N^{-1/3} \leqq \min_{i\neq j}\frac{|x_i - x_j|}{4}.$$

We write

$$A_i' = A_{i,1}' + A_{i,2}' + A_{i,3}',$$

with

$$A_{i,1}' = \sum_{j\neq i} ⨍_{B(x_j,d)} \Big(D(v[A_j])(x_i - x_j) - D(v[A_j])(x_i - x')\Big)\mathrm{d}x',$$

$$A_{i,2}' = \sum_{j\neq i} ⨍_{B(x_j,d)} \Big(D(v[A_j])(x_i - x') - ⨍_{B_i} D(v[A_j])(x - x')\mathrm{d}x\Big)\mathrm{d}x',$$

$$A_{i,3}' = \sum_{j\neq i} ⨍_{B(x_j,d)} ⨍_{B_i} D(v[A_j])(x - x')\mathrm{d}x\mathrm{d}x'.$$

Setting $y_i = N^{-1/3}x_i$, using that for $i \neq j$, $|y_i - y_j| \geqq \frac{1}{2}(c + |y_i - y_j|) \geqq c$,

$$|A_{i,1}'| \leqq Ca^3 \sum_{j\neq i}\frac{d}{|x_i - x_j|^4}|A_j| \leqq C'\phi \sum_j \frac{|A_j|}{(c + |y_i - y_j|)^4}.$$

From the inequality (2.35), applied with $a_{ij} = \frac{1}{(c+|y_i - y_j|)^4}$ and $b_j = A_j$, we deduce

$$\sum_i |A_{i,1}'|^q \ \leqq C\phi^q \sum_j |A_j|^q.$$

Similarly,

$$|A_{i,2}'| \leqq Ca^3 \sum_{j\neq i}\frac{a}{|x_i - x_j|^4}|A_j| \leqq C'\phi^{\frac{4}{3}} \sum_j \frac{|A_j|}{c + |y_i - y_j|^4}.$$

This leads to

$$\sum_i |A_{i,2}'|^q \ \leqq C\phi^{\frac{4q}{3}} \sum_j |A_j|^q.$$

The last term is the most difficult. We follow [21]. Let us remind ourselves that

$$v[A] = -\frac{5}{2}A : (x \otimes x)\frac{a^3 x}{|x|^5}.$$

Let $\chi_d(x) = \chi(x/d)$ a smooth function that is 0 in $B(0,d)$, 1 outside $B(0,2d)$. Introducing the function $F_A = \sum_j A_j 1_{B(x_j,d)}$, using that $d \leqq \min_{i\neq j} \frac{|x_i - x_j|}{4}$, we can write that

$$A'_{i,3} = \frac{1}{d^3}\int_{B_i}\int_{\mathbb{R}^3} \chi_d(x_i - x')\mathbf{K}(x-x')F_A(x')\mathrm{d}x'\mathrm{d}x,$$

where $\mathbf{K}(x)$ is an endomorphism of the space of symmetric matrices, defined by

$$\mathbf{K}(x)A = -\frac{5}{2}\Big(\frac{4\pi}{3}\Big)^{-2} D\Big(A : (x\otimes x)\frac{x}{|x|^5}\Big).$$

We then split $A'_{i,3} = M_i + N_i$, with

$$\begin{aligned} M_i &= \frac{1}{d^3}\int_{B_i}\int_{\mathbb{R}^3} \chi_d(x - x')\mathbf{K}(x-x')F_A(x')\mathrm{d}x'\mathrm{d}x, \\ N_i &= \frac{1}{d^3}\int_{B_i}\int_{\mathbb{R}^3} (\chi_d(x_i - x') - \chi_d(x-x'))\mathbf{K}(x-x')F_A(x')\mathrm{d}x'\mathrm{d}x. \end{aligned}$$

By Hölder inequality,

$$|M_i|^q \leqq \frac{1}{d^{3q}} a^{\frac{3q}{p}} \|\big(\chi_d\mathbf{K}\big)\star F_A\|^q_{L^q(B_i)},$$

and so

$$\sum_i |M_i|^q \leqq \frac{1}{d^{3q}} a^{\frac{3q}{p}} \|\big(\chi_\mathrm{d}\mathbf{K}\big)\star F_A\|^q_{L^q(\mathbb{R}^3)}.$$

The kernel $\chi_d\mathbf{K}$ enters the framework of the Calderón–Zygmund theorem, see for instance [31, Chapters 4 and 5]: for all $1 < q < +\infty$, the operator $\big(\chi_\mathrm{d}\mathbf{K}\big)\star$ is continuous from $L^q(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$, with

$$\|\big(\chi_\mathrm{d}\mathbf{K}\big)\star\|_{\mathcal{L}(L^q,L^q)} \leqq C_q.$$

We stress that the constant C_q depends only on q, and not on d, as can be seen from the rescaling $x' := x'/d$. It follows that

$$\sum_i |M_i|^q \leqq \frac{C}{d^{3q}} a^{\frac{3q}{p}} \|F_A\|^q_{L^q(\mathbb{R}^3)}.$$

As the balls $B(x_j,d)$ are disjoint, $|\sum A_j 1_{B(x_j,d)}|^q = \sum |A_j|^q 1_{B(x_j,d)}$, so that $\|F_A\|^q_{L^q(\mathbb{R}^3)} = \frac{4\pi}{3}\sum |A_j|^q d^3$, and

$$\sum_i |M_i|^q \leqq C'\Big(\frac{a}{d}\Big)^{\frac{3q}{p}} \sum_i |A_i|^q \leqq C\phi^{\frac{q}{p}}\sum_i |A_i|^q.$$

To bound N_i, we notice that for all $x\in B_i$, the support of $x' \to \chi_d(x_i - x') - \chi_d(x-x')$ is included in

$$\Big(B(x_i,2d)\cup B(x,2d)\Big)\setminus\Big(B(x,d)\cap B(x_i,d)\Big) \subset B(x,2d+a)\setminus B(x,d-a)$$

(remark that by definition of ϕ, a is less than d for ϕ small enough). We get

$$|N_i|^q \leqq \frac{1}{d^{3q}} a^{\frac{3q}{p}} \| \left| 1_{B(0,2d+a)\setminus B(0,d-a)} \mathbf{K} \right| \star \left| F_A \right| \|^q_{L^q(B_i)},$$

so that

$$\begin{aligned}\sum_i |N_i|^q &\leqq \frac{C}{d^{3q}} a^{\frac{3q}{p}} \| \left| 1_{B(0,2d+a)\setminus B(0,d-a)} |x|^{-3} \right| \star \left| F_A \right| \|^q_{L^q(\mathbb{R}^3)} \\ &\leqq \frac{C'}{d^{3q}} a^{\frac{3q}{p}} \left| \ln\left(\frac{2d+a}{d-a}\right) \right|^q \| F_A \|^q_{L^q(\mathbb{R}^3)} \\ &\leqq C'' \left(\frac{a}{d}\right)^{\frac{3q}{p}} \sum_i |A_i|^q \leqq \mathcal{C} \phi^{\frac{q}{p}} \sum_i |A_i|^q,\end{aligned}$$

using that, for $\phi \ll 1$, $a \ll d$ and $\left| \ln\left(\frac{2d+a}{d-a}\right) \right|$ is bounded by an absolute constant.

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