



Global Well-Posedness of 3-D Anisotropic Navier–Stokes System with Small Unidirectional Derivative

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Abstract

In Liu and Zhang (Arch Ration Mech Anal 235:1405–1444, 2020), the authors proved that as long as the one-directional derivative of the initial velocity is sufficiently small in some scaling invariant spaces, then the classical Navier–Stokes system has a global unique solution. The goal of this paper is to extend this type of result to the 3-D anisotropic Navier–Stokes system (ANS) with only horizontal dissipation. More precisely, given initial data $u_0 = (u_0^h, u_0^3) \in \mathcal{B}^{0, \frac{1}{2}}$, (ANS) has a unique global solution provided that $|D_h|^{-1} \partial_3 u_0$ is sufficiently small in the scaling invariant space $\mathcal{B}^{0, \frac{1}{2}}$.

1. Introduction

In this paper, we investigate the global well-posedness of the following 3-D anisotropic Navier–Stokes system:

$$(ANS) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta_h u = -\nabla p, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$, u designates the velocity of the fluid and p the scalar pressure function which guarantees the divergence free condition of the velocity field.

Systems of this type appear in geophysical fluid dynamics (see for instance [5, 18]). In fact, meteorologists often model turbulent diffusion by using a viscosity of the form $-\mu_h \Delta_h - \mu_3 \partial_3^2$, where μ_h and μ_3 are empirical constants, and μ_3 is usually much smaller than μ_h . We refer to the book of PEDLOVSKY [18, Chap. 4], for a complete discussion about this model.

Considering that system (ANS) has only horizontal dissipation, it is reasonable to use functional spaces which distinguish horizontal derivatives from the vertical one, for instance, the anisotropic Sobolev space defined as follows:

Definition 1.1. For any (s, s') in \mathbb{R}^2 , the anisotropic Sobolev space $H^{s,s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution a such that

$$\|a\|_{H^{s,s'}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s'} |\widehat{a}(\xi)|^2 d\xi < \infty \quad \text{with } \xi_h = (\xi_1, \xi_2).$$

Mathematically, CHEMIN et al. [4] first studied the system (ANS). In particular, CHEMIN et al. [4] and IFTIME [13] proved that (ANS) is locally well-posed with initial data in $L^2 \cap H^{0, \frac{1}{2} + \varepsilon}$ for some $\varepsilon > 0$, and is globally well-posed if, in addition,

$$\|u_0\|_{L^2}^\varepsilon \|u_0\|_{H^{0, \frac{1}{2} + \varepsilon}}^{1-\varepsilon} \leq c \tag{1.1}$$

for some sufficiently small constant c .

Notice that just as the classical Navier–Stokes system

$$(NS) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

the system (ANS) has the following scaling invariant property:

$$u_\lambda(t, x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda u_0(\lambda x), \tag{1.2}$$

which means that if u is a solution of (ANS) with initial data u_0 on $[0, T]$, u_λ determined by (1.2) is also a solution of (ANS) with initial data $u_{0,\lambda}$ on $[0, T/\lambda^2]$.

It is easy to observe that the smallness condition (1.1) in [4] is scaling invariant under the scaling transformation (1.2), nevertheless, the norm of the space $H^{0, \frac{1}{2} + \varepsilon}$ is not. To work (ANS) with initial data in the critical spaces, we first recall the following anisotropic dyadic operators from [2]:

$$\begin{aligned} \Delta_k^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), & \Delta_\ell^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), \\ S_k^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\widehat{a}), & S_\ell^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}), \end{aligned} \tag{1.3}$$

where $\xi_h = (\xi_1, \xi_2)$, $\mathcal{F}a$ or \widehat{a} denotes the Fourier transform of a , while $\mathcal{F}^{-1}a$ designates the inverse Fourier transform of a , $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\begin{aligned} \operatorname{Supp} \varphi &\subset \left\{ \tau \in \mathbb{R} : \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1; \\ \operatorname{Supp} \chi &\subset \left\{ \tau \in \mathbb{R} : |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \forall \tau \in \mathbb{R}, \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1. \end{aligned}$$

Definition 1.2. We define $\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3)$ to be the set of homogenous tempered distribution a so that

$$\|a\|_{\mathcal{B}^{0, \frac{1}{2}}} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L^2(\mathbb{R}^3)} < \infty.$$

The above space was first introduced by IFTIMIE [12] to study the global well-posedness of the classical 3-D Navier–Stokes system with initial data in the anisotropic functional space. The second author [16] proved the local well-posedness of (ANS) with any solenoidal vector field $u_0 \in \mathcal{B}^{0, \frac{1}{2}}$ and also the global well-posedness with small initial data in $\mathcal{B}^{0, \frac{1}{2}}$. This result corresponds to Fujita–Kato’s theorem [11] for the classical Navier–Stokes system. Moreover, the authors [17, 19] proved the global well-posedness of (ANS) with initial data $u_0 = (u_0^h, u_0^3)$ satisfying that

$$\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp(C \|u_0^3\|_{\mathcal{B}^{0, \frac{1}{2}}}^4) \leq c_0 \tag{1.4}$$

for some c_0 sufficiently small.

Although the norm of $\mathcal{B}^{0, \frac{1}{2}}$ is scaling invariant under the the scaling transformation (1.2), yet we observe that the solenoidal vector field

$$u_0^\varepsilon(x) = \sin\left(\frac{x_1}{\varepsilon}\right) (0, -\partial_3\varphi, \partial_2\varphi) \tag{1.5}$$

is not small in the space $\mathcal{B}^{0, \frac{1}{2}}$ no matter how small ε is. In order to find a space so that the norm of $u_0^\varepsilon(x)$ given by (1.5) is small in this space for small ε , Chemin and the third author [8] introduced the following Besov–Sobolev type space with negative index:

Definition 1.3. We define the space $\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}$ to be the set of a homogenous tempered distribution a so that

$$\|a\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\left(\sum_{k=\ell-1}^{\infty} 2^{-k} \|\Delta_k^h \Delta_\ell^v a\|_{L_h^4(L_v^2)}^2 \right)^{\frac{1}{2}} + \|S_{\ell-1}^h \Delta_\ell^v a\|_{L^2} \right) < \infty.$$

Chemin and the third author [8] proved the global well-posedness of (ANS) with initial data being small in the space $\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}$. In particular, this result ensures the global well-posedness of (ANS) with initial data $u_0^\varepsilon(x)$ given by (1.5) as long as ε is sufficiently small. Furthermore the second and third authors [17] proved the global well-posedness of (ANS) provided that the initial data $u_0 = (u_0^h, u_0^3)$ satisfies that

$$\|u_0^h\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}} \exp(C \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}^4) \leq c_0 \tag{1.6}$$

for some c_0 sufficiently small. We remark that this result corresponds to Cannone, Meyer and Planchon’s result in [3] for the classical Navier–Stokes system, where the authors proved that if the initial data satisfies that

$$\|u_0\|_{\dot{B}_{p, \infty}^{-1+\frac{3}{p}}} \leq cv$$

for some p greater than 3 and some constant c small enough, then (NS) is globally well-posed. The end-point result in this direction is due to KOCH and TATARU [14] for initial data in the space of ∂BMO .

On the other hand, motivated by the study of the global well-posedness of the classical Navier–Stokes system with slowly varying initial data [6, 7, 9], the first and third authors proved the following theorem for (NS) in [15]:

Theorem 1.1. *Let $\delta \in]0, 1[$, $u_0 = (u_0^h, u_0^3) \in H^{\frac{1}{2}}(\mathbb{R}^3) \cap B_{2,1}^{0, \frac{1}{2}}(\mathbb{R}^3)$ with u_0^h belonging to $L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}_v; H^{-\delta}(\mathbb{R}_h^2)) \cap L^\infty(\mathbb{R}_v; H^3(\mathbb{R}_h^2))$. If we assume in addition that $\partial_3 u_0 \in H^{-\frac{1}{2}, 0}$, then there exists a small enough positive constant ε_0 such that if*

$$\|\partial_3 u_0\|_{H^{-\frac{1}{2}, 0}}^2 \exp\left(C(A_\delta(u_0^h) + B_\delta(u_0))\right) \leq \varepsilon_0, \tag{1.7}$$

(NS) has a unique global solution $u \in C(\mathbb{R}^+; H^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+; H^{\frac{3}{2}})$, where

$$\begin{aligned} A_\delta(u_0^h) &\stackrel{\text{def}}{=} \left(\frac{\|\nabla_h u_0^h\|_{L_v^\infty(L_h^2)}^2 \|u_0^h\|_{L_v^\infty(B_{2,\infty}^{-\delta})_h}^{\frac{2}{\delta}}}{\|u_0^h\|_{L_v^\infty(L_h^2)}^{\frac{2}{\delta}}} + \|u_0^h\|_{L_v^\infty(L_h^2)}^2 \right) \exp\left(C_\delta(1 + \|u_0^h\|_{L_v^\infty(L_h^2)}^4)\right), \\ \mathfrak{A}_\delta(u_0^h) &\stackrel{\text{def}}{=} \frac{\|u_0^h\|_{L_v^\infty(L_h^2)}^3 \|\nabla_h^3 u_0^h\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}}{\|\nabla_h u_0^h\|_{L_v^\infty(L_h^2)}^{\frac{3}{2}}} + A_\delta(u_0^h), \end{aligned} \tag{1.8}$$

and

$$B_\delta(u_0) \stackrel{\text{def}}{=} \|u_0^h\|_{B_{2,1}^{0, \frac{1}{2}}} \exp(C\mathfrak{A}_\delta(u_0^h)) + \|u_0\|_{B_{2,1}^{0, \frac{1}{2}}} \exp\left(\|u_0^h\|_{B_{2,1}^{0, \frac{1}{2}}} \exp(C\mathfrak{A}_\delta(u_0^h))\right) \tag{1.9}$$

are scaling invariant under the scaling transformation (1.2).

We remark that Theorem 1.1 ensures the global well-posedness of (NS) with initial data

$$u_0^\varepsilon(x) = (v_0^h + \varepsilon w_0^h, w_0^3)(x_h, \varepsilon x_3) \quad \text{with} \quad \text{div}_h v_0^h = 0 = \text{div} w_0 \tag{1.10}$$

for $\varepsilon \leq \varepsilon_0$, which was first proved in [6]. We mention that the proof of Theorem 1.1 requires a regularity criteria in [10], which can only be proved for the classical Navier–Stokes system so far.

Motivated by [15, 17, 19], here we are going to study the global well-posedness of (ANS) with initial data u_0 satisfying $\partial_3 u_0$ being sufficiently small in some critical spaces.

The main result of this paper is as follows:

Theorem 1.2. *Let Λ_h^{-1} be a Fourier multiplier with symbol $|\xi_h|^{-1}$, let $u_0 \in \mathcal{B}^{0, \frac{1}{2}}$ be a solenoidal vector field with $\Lambda_h^{-1} \partial_3 u_0 \in \mathcal{B}^{0, \frac{1}{2}}$. Then there exist some sufficiently small positive constant ε_0 and some universal positive constants L, M, N so that for $\mathfrak{A}_N(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}})$ given by (3.5) if*

$$\|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(L(1 + \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}^4) \exp(M\mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}))\right) \leq \varepsilon_0, \tag{1.11}$$

(ANS) has a unique global solution $u = v + e^{t\Delta_h} \begin{pmatrix} 0 \\ u_{0, \text{hh}}^3 \end{pmatrix}$ with $v \in C([0, \infty[; \mathcal{B}^{0, \frac{1}{2}})$ and $\nabla_h v \in L^2([0, \infty[; \mathcal{B}^{0, \frac{1}{2}})$, where $u_{0, \text{hh}}^3 \stackrel{\text{def}}{=} \sum_{k \geq \ell-1} \Delta_k^h \Delta_\ell^v u_0^3$.

We remark that all the norms of u^0 in (1.11) is scaling invariant under the scaling transformation (1.2). Especially for the term $\|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}}$, we do not know how to propagate this regularity for the solutions of 3-D Navier–Stokes system. In the sequel, we shall only propagate this regularity for the solutions of 2-D Navier–Stokes system with a parameter [(see (3.4) and (3.7)]. With regular initial data, we may write explicitly the constant $\mathfrak{A}_N(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}})$. For instance, we have

Corollary 1.1. *Let $u_0 \in L^2$ be a solenoidal vector field with $\partial_3 u_0 \in L^2$ and $\Lambda_h^{-1} \partial_3 u_0 \in \mathcal{B}^{0, \frac{1}{2}}$. Then there exist some sufficiently small positive constant ε_0 and some universal positive constants L, M so that if*

$$\|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(L(1 + \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}^4) \exp\left(\exp(M\|u_0^h\|_{L^2} \|\partial_3 u_0^h\|_{L^2})\right)\right) \leq \varepsilon_0, \tag{1.12}$$

(ANS) has a unique global solution u as in Theorem 1.2.

Remark 1.1. Several remarks are in order about Theorem 1.2:

(a) It follows from [8] that

$$\|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}} \lesssim \|u_0^3\|_{\mathcal{B}^{0, \frac{1}{2}}},$$

so that the smallness condition (1.11) and (1.12) can also be formulated as

$$\|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(L(1 + \|u_0^3\|_{\mathcal{B}^{0, \frac{1}{2}}}^4) \exp(M\mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}))\right) \leq \varepsilon_0, \tag{1.13}$$

and

$$\|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(L(1 + \|u_0^3\|_{\mathcal{B}^{0, \frac{1}{2}}}^4) \exp\left(\exp(M\|u_0^h\|_{L^2} \|\partial_3 u_0^h\|_{L^2})\right)\right) \leq \varepsilon_0. \tag{1.14}$$

(b) Due to $\text{div } u_0 = 0$, we find

$$\|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} = \|(\Lambda_h^{-1} \partial_3 u_0^h, -\Lambda_h^{-1} \text{div}_h u_0^h)\|_{\mathcal{B}^{0, \frac{1}{2}}}.$$

Therefore the smallness condition (1.11) is of a similar type as (1.4). Yet roughly speaking, (1.11) requires only $\partial_3 u_0^h$ and $\text{div}_h u_0^h$ to be small in some scaling invariant space, but without any restriction on $\text{curl}_h u_0^h$. Thus the smallness condition (1.11) is weaker than (1.4).

(c) Let w_0 be a smooth solenoidal vector field, we observe that the data

$$u_0^\varepsilon(x) = (\varepsilon(-\ln \varepsilon)^\delta w_0^h, (-\ln \varepsilon)^\delta w_0^3)(x_h, \varepsilon x_3) \quad \text{with } \delta \in]0, 1/4[$$

satisfies (1.4) for ε sufficiently small.

While since our smallness condition (1.14) does not have any restriction on $\text{curl } u_0^h$, for any smooth vector field v_0^h satisfying $\text{div}_h v_0^h = 0$, we find

$$u_0^\varepsilon(x) = (v_0^h + \varepsilon(-\ln \varepsilon)^\delta w_0^h, (-\ln \varepsilon)^\delta w_0^3)(x_h, \varepsilon x_3) \quad \text{with } \delta \in]0, 1/4[\quad (1.15)$$

satisfies (1.14) for any ε sufficiently small. Therefore Theorem 1.2 ensures the global well-posedness of (ANS) with initial data given by (1.15). Compared with (1.10), which corresponds to $\delta = 0$ in (1.15), this type of result is new even for the classical Navier–Stokes system.

(d) Given $\phi \in \mathcal{S}(\mathbb{R}^3)$, we deduce from Proposition 1.1 in [8] that

$$\|e^{ix_1/\varepsilon} \phi(x)\|_{B_4^{-\frac{1}{2}, \frac{1}{2}}} \leq C\varepsilon^{\frac{1}{2}}.$$

As a result, we find that for any $\delta \in]0, 1/4[$, the following class of initial data:

$$\begin{aligned} u_0^\varepsilon(x) &= (v^h, 0)(x_h, \varepsilon x_3) + (-\ln \varepsilon)^\delta \sin(x_1/\varepsilon) \\ & \quad \left(0, -\varepsilon^{\frac{1}{2}} \partial_3 \phi(x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}} \partial_2 \phi(x_h, \varepsilon x_3)\right), \end{aligned} \quad (1.16)$$

satisfies the smallness condition (1.13) for small enough ε , and hence the data given by (1.16) can also generate unique global solution of (ANS).

(e) Since all the results that work for the anisotropic Navier–Stokes system (ANS) should automatically do for the classical Navier–Stokes system (NS), Theorem 1.2 holds also for (NS).

Let us end this section with some notations that will be used throughout this paper.

Notations: Let A, B be two operators, we denote $[A; B] = AB - BA$, the commutator between A and B , for $a \lesssim b$, we means that there is a uniform constant C , which may be different in each occurrence, such that $a \leq Cb$. We shall denote by $(a|b)_{L^2}$ the $L^2(\mathbb{R}^3)$ inner product of a and b . $(d_j)_{j \in \mathbb{Z}}$ designates a generic elements on the unit sphere of $\ell^1(\mathbb{Z})$, i.e. $\sum_{j \in \mathbb{Z}} d_j = 1$. Finally, we denote $L_T^r(L_h^p(L_v^q))$ the space $L^r([0, T]; L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; L^q(\mathbb{R}_{x_3})))$, and $\nabla_h \stackrel{\text{def}}{=} (\partial_{x_1}, \partial_{x_2})$, $\text{div}_h = \partial_{x_1} + \partial_{x_2}$.

2. Littlewood–Paley Theory

In this section, we shall collect some basic facts on anisotropic Littlewood–Paley theory. We first recall the following anisotropic Bernstein inequalities from [8, 16]:

Lemma 2.1. *Let \mathbf{B}_h (resp. \mathbf{B}_v) a ball of \mathbb{R}_h^2 (resp. \mathbb{R}_v), and C_h (resp. C_v) a ring of \mathbb{R}_h^2 (resp. \mathbb{R}_v); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then it holds that*

$$\begin{aligned}
 \text{if } \text{Supp } \widehat{a} \subset 2^k \mathbf{B}_h &\Rightarrow \|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha| + \frac{2}{p_2} - \frac{2}{p_1})} \|a\|_{L_h^{p_2}(L_v^{q_1})}; \\
 \text{if } \text{Supp } \widehat{a} \subset 2^\ell \mathbf{B}_v &\Rightarrow \|\partial_{x_3}^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell(\beta + \frac{1}{q_2} - \frac{1}{q_1})} \|a\|_{L_h^{p_1}(L_v^{q_2})}; \\
 \text{if } \text{Supp } \widehat{a} \subset 2^k \mathcal{C}_h &\Rightarrow \|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})}; \\
 \text{if } \text{Supp } \widehat{a} \subset 2^\ell \mathcal{C}_v &\Rightarrow \|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N} \|\partial_{x_3}^N a\|_{L_h^{p_1}(L_v^{q_1})}.
 \end{aligned}$$

Definition 2.1. For any $p \in [1, \infty]$, let us define the Chemin–Lerner type norm

$$\|a\|_{\widetilde{L}_T^p(\mathcal{B}^{0, \frac{1}{2}})} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^\vee a\|_{L_T^p(L^2(\mathbb{R}^3))}.$$

In particular, we denote

$$\|a\|_{\mathcal{B}^{0, \frac{1}{2}}(T)} \stackrel{\text{def}}{=} \|a\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\nabla_h a\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}.$$

We remark that the inhomogeneous version of the anisotropic Sobolev space $H^{0,1}$ can be continuously imbedded into $\mathcal{B}^{0, \frac{1}{2}}$. Indeed for any integer N , we deduce from Lemma 2.1 that

$$\begin{aligned}
 \|a\|_{\mathcal{B}^{0, \frac{1}{2}}} &= \sum_{\ell \leq N} 2^{\frac{\ell}{2}} \|\Delta_\ell^\vee a\|_{L^2} + \sum_{\ell > N} 2^{\frac{\ell}{2}} \|\Delta_\ell^\vee a\|_{L^2} \\
 &\leq \sum_{\ell \leq N} 2^{\frac{\ell}{2}} \|\Delta_\ell^\vee a\|_{L^2} + \sum_{\ell > N} 2^{-\frac{\ell}{2}} \|\partial_3 \Delta_\ell^\vee a\|_{L^2} \\
 &\lesssim 2^{\frac{N}{2}} \|a\|_{L^2} + 2^{-\frac{N}{2}} \|\partial_3 a\|_{L^2}.
 \end{aligned}$$

Taking the integer N so that $2^N \sim \|\partial_3 a\|_{L^2} \|a\|_{L^2}^{-1}$ in the above inequality leads to

$$\|a\|_{\mathcal{B}^{0, \frac{1}{2}}} \lesssim \|a\|_{L^2}^{\frac{1}{2}} \|\partial_3 a\|_{L^2}^{\frac{1}{2}}. \tag{2.1}$$

Along the same lines, we have

$$\|a\|_{\widetilde{L}_T^p(\mathcal{B}^{0, \frac{1}{2}})} \lesssim \|a\|_{L_T^p(L^2)}^{\frac{1}{2}} \|\partial_3 a\|_{L_T^p(L^2)}^{\frac{1}{2}} \quad \forall p \in [1, \infty]. \tag{2.2}$$

To overcome the difficulty that one can not use Gronwall’s inequality in the Chemin–Lerner type norms, we recall the following time-weighted Chemin–Lerner norm from [17]:

Definition 2.2. Let $f(t) \in L_{\text{loc}}^1(\mathbb{R}_+)$, $f(t) \geq 0$. We define

$$\|a\|_{\widetilde{L}_{T,f}^2(\mathcal{B}^{0, \frac{1}{2}})} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\int_0^T f(t) \|\Delta_\ell^\vee a(t)\|_{L^2}^2 dt \right)^{\frac{1}{2}}.$$

In order to take into account functions with oscillations in the horizontal variables, we recall the following anisotropic Besov type space with negative indices from [8]:

Definition 2.3. For any $p \in [1, \infty]$, we define

$$\|a\|_{\tilde{L}^p(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\left(\sum_{k=\ell-1}^{\infty} 2^{-k} \|\Delta_k^h \Delta_\ell^v a\|_{L_T^p(L_h^4(L_v^2))}^2 \right)^{\frac{1}{2}} + \|S_{\ell-1}^h \Delta_\ell^v a\|_{L_T^p(L^2)} \right).$$

In particular, we denote

$$\|a\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(T)} \stackrel{\text{def}}{=} \|a\|_{\tilde{L}_T^\infty(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})} + \|\nabla_h a\|_{\tilde{L}_T^2(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})}.$$

In the sequel, for $a \in \mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}$, we shall frequently use the following decomposition:

$$a = a_{\text{h}} + a_{\text{hh}} \quad \text{with} \quad a_{\text{h}} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} S_{\ell-1}^h \Delta_\ell^v a \quad \text{and} \quad a_{\text{hh}} \stackrel{\text{def}}{=} \sum_{k \geq \ell-1} \Delta_k^h \Delta_\ell^v a. \quad (2.3)$$

Lemma 2.2. (Lemma 2.5 in [8]) For any $a \in \mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}$, it holds that

$$\|e^{t\Delta^h} a_{\text{hh}}\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(\infty)} \lesssim \|a\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}.$$

Definition 2.4. Let us define

$$\|a\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L_h^4(L_v^2)} \quad \text{and} \quad \|a\|_{\tilde{L}_t^4(\mathcal{B}_4^{0, \frac{1}{2}})} \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L_t^4(L_h^4(L_v^2))}.$$

In view of the 2-D interpolation inequality that

$$\|a\|_{L^4(\mathbb{R}^2)} \lesssim \|a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla_h a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \quad (2.4)$$

we find

$$\begin{aligned} \|a\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^2 &\lesssim \left(\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L^2}^{\frac{1}{2}} \|\Delta_\ell^v \nabla_h a\|_{L^2}^{\frac{1}{2}} \right)^2 \\ &\leq \left(\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L^2} \right) \left(\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v \nabla_h a\|_{L^2} \right) = \|a\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|\nabla_h a\|_{\mathcal{B}_4^{0, \frac{1}{2}}}. \end{aligned} \quad (2.5)$$

Similarly, we have

$$\|a\|_{\tilde{L}_t^4(\mathcal{B}_4^{0, \frac{1}{2}})}^2 \lesssim \|a\|_{\tilde{L}_t^\infty(\mathcal{B}_4^{0, \frac{1}{2}})} \|\nabla_h a\|_{\tilde{L}_t^2(\mathcal{B}_4^{0, \frac{1}{2}})}. \quad (2.6)$$

Before proceeding, let us recall Bony’s decomposition for the vertical variable from [1]:

$$ab = T_a^v b + R^v(a, b) \quad \text{with} \quad T_a^v b = \sum_{\ell \in \mathbb{Z}} S_{\ell-1}^v a \Delta_\ell^v b, \quad R^v(a, b) = \sum_{\ell \in \mathbb{Z}} \Delta_\ell^v a S_{\ell+2}^v b. \quad (2.7)$$

Sometimes we shall also use Bony’s decomposition for the horizontal variables.

Let us now apply the above basic facts on Littlewood–Paley theory to prove the following proposition:

Proposition 2.1. *For any $a \in \mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(T)$, it holds that*

$$\|a\|_{\tilde{L}_T^4(\mathcal{B}_4^{0, \frac{1}{2}})} \lesssim \|a\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(T)}. \tag{2.8}$$

Proof. In view of (2.3) and Definition 2.3, we get, by applying (2.6), that

$$\begin{aligned} \|a_{\text{hh}}\|_{\tilde{L}_T^4(\mathcal{B}_4^{0, \frac{1}{2}})} &\lesssim \|a_{\text{hh}}\|_{\tilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_{\text{h}} a_{\text{hh}}\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \\ &\lesssim \|a\|_{\tilde{L}_T^\infty(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_{\text{h}} a\|_{\tilde{L}_T^2(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned}$$

Then it remains to prove (2.8) for a_{hh} . Indeed in view of Definition 2.4, we write

$$\|a_{\text{hh}}\|_{\tilde{L}_T^4(\mathcal{B}_4^{0, \frac{1}{2}})} = \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|(\Delta_\ell^v a_{\text{hh}})^2\|_{L_T^2(L_h^2(L_v^1))}^{\frac{1}{2}}.$$

Applying Bony’s decomposition for the horizontal variables yields

$$(\Delta_\ell^v a_{\text{hh}})^2 = \sum_{k \in \mathbb{Z}} S_{k-1}^h \Delta_\ell^v a_{\text{hh}} \Delta_k^h \Delta_\ell^v a_{\text{hh}} + \sum_{k \in \mathbb{Z}} S_{k+2}^h \Delta_\ell^v a_{\text{hh}} \Delta_k^h \Delta_\ell^v a_{\text{hh}}. \tag{2.9}$$

We observe that

$$\begin{aligned} &\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \in \mathbb{Z}} \|S_{k-1}^h \Delta_\ell^v a_{\text{hh}} \Delta_k^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^2(L_h^2(L_v^1))} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \in \mathbb{Z}} 2^{-k} \|S_{k-1}^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^\infty(L_h^4(L_v^2))}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \in \mathbb{Z}} 2^k \|\Delta_k^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^2(L_h^4(L_v^2))}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \in \mathbb{Z}} 2^{-k} \|S_{k-1}^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^\infty(L_h^4(L_v^2))}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \|\nabla_{\text{h}} a_{\text{hh}}\|_{\tilde{L}_T^2(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})}. \end{aligned}$$

Whereas we get, by using Young’s inequality, that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^{-k} \|S_{k-1}^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^\infty(L_h^4(L_v^2))}^2 &= \sum_{k \in \mathbb{Z}} \left(\sum_{k' \leq k-2} 2^{-\frac{k-k'}{2}} 2^{-\frac{k'}{2}} \|\Delta_{k'}^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^\infty(L_h^4(L_v^2))} \right)^2 \\ &\leq \sum_{k \in \mathbb{Z}} 2^{-k} \|\Delta_k^h \Delta_\ell^v a_{\text{hh}}\|_{L_T^\infty(L_h^4(L_v^2))}^2. \end{aligned}$$

As a result, it turns out that

$$\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \in \mathbb{Z}} 2^{-k} \|S_{k-1}^h \Delta_\ell^v a_{hh}\|_{L_T^\infty(L_h^4(L_v^2))}^2 \right)^{\frac{1}{2}} \leq \|a\|_{\tilde{L}_T^\infty(\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}})},$$

and

$$\sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \in \mathbb{Z}} \|S_{k-1}^h \Delta_\ell^v a_{hh} \Delta_k^h \Delta_\ell^v a_{hh}\|_{L_T^2(L_h^2(L_v^1))}^2 \right)^{\frac{1}{2}} \lesssim \|a\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(T)}.$$

Along the same lines, we can prove that the second term in (2.9) shares the same estimate. This ensures that (2.8) holds for a_{hh} . We thus complete the proof of the proposition. \square

3. Sketch of the Proof

Motivated by the study of the global large solutions to the classical 3-D Navier–Stokes system with slowly varying initial data in one direction [6, 7, 9, 15], here we are going to decompose the solution of (ANS) as a sum of a solution to the two-dimensional Navier–Stokes system with a parameter and a solution to the three-dimensional perturbed anisotropic Navier–Stokes system. We point out that compared with the references [6, 7, 9, 15], here the 3-D solution to the perturbed anisotropic Navier–Stokes system will not be small. Indeed only its vertical component is not small. In order to deal with this part, we are going to appeal to the observation from [17, 19], where the authors proved the global well-posedness to 3-D anisotropic Navier–Stokes system with the horizontal components of the initial data being small [see the smallness conditions (1.4) and (1.6)].

For $u^h = (u^1, u^2)$, we first recall the two-dimensional Biot–Savart’s law:

$$u^h = u_{\text{curl}}^h + u_{\text{div}}^h \quad \text{with} \quad u_{\text{curl}}^h \stackrel{\text{def}}{=} \nabla_h^\perp \Delta_h^{-1}(\text{curl}_h u^h) \quad \text{and} \quad u_{\text{div}}^h \stackrel{\text{def}}{=} \nabla_h \Delta_h^{-1}(\text{div}_h u^h), \tag{3.1}$$

where $\text{curl}_h u^h \stackrel{\text{def}}{=}} \partial_1 u^2 - \partial_2 u^1$ and $\text{div}_h u^h \stackrel{\text{def}}{=} \partial_1 u^1 + \partial_2 u^2$.

In particular, let us decompose the horizontal components u_0^h of the initial velocity u_0 of (ANS) as the sum of $u_{0,\text{curl}}^h$ and $u_{0,\text{div}}^h$, and let us consider the following 2-D Navier–Stokes system with a parameter:

$$\begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \Delta_h \bar{u}^h = -\nabla_h \bar{p}, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \text{div}_h \bar{u}^h = 0, \\ \bar{u}^h|_{t=0} = \bar{u}_0^h = u_{0,\text{curl}}^h. \end{cases} \tag{3.2}$$

Concerning the system (3.2), we have the following a priori estimates:

Proposition 3.1. *Let $\bar{u}_0^h \in \mathcal{B}^{0, \frac{1}{2}}$ with $\Lambda_h^{-1} \partial_3 \bar{u}_0^h \in \mathcal{B}^{0, \frac{1}{2}}$. Then (3.2) has a unique global solution so that for any time $t > 0$, it holds that*

$$\|\bar{u}^h\|_{L_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\nabla_h \bar{u}^h\|_{L_t^2(\mathcal{B}^{0, \frac{1}{2}})} \leq C \mathfrak{A}_N(\|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}), \tag{3.3}$$

and

$$\|\Lambda_h^{-1} \partial_3 \bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \leq C \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(C \mathfrak{A}_N^4(\|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}})\right), \tag{3.4}$$

where

$$\begin{aligned} \bar{u}_{0,N}^h &\stackrel{\text{def}}{=} \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi_3| \leq \frac{1}{N}} \text{ or } |\xi_3| \geq N \mathcal{F}(\bar{u}^h)\right) \text{ and} \\ \mathfrak{A}_N(\|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}) &\stackrel{\text{def}}{=} N^{\frac{1}{2}} \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(C \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2\right) \\ &\quad + \|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(N^2 \exp\left(C \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2\right)\right), \end{aligned} \tag{3.5}$$

and N is taken so large that $\|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0, \frac{1}{2}}}$ is sufficiently small.

The proof of Proposition 3.1 will be presented in Section 4.

Remark 3.1. Under the assumptions that $\bar{u}_0^h \in L^2$ with $\partial_3 \bar{u}_0^h \in L^2$ and $\Lambda_h^{-1} \partial_3 \bar{u}_0^h \in \mathcal{B}^{0, \frac{1}{2}}$, we have the following alternative estimates for (3.3) and (3.4):

$$\|\bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\nabla_h \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \leq \|\bar{u}_0^h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \bar{u}_0^h\|_{L^2}^{\frac{1}{2}} \exp\left(C \|\bar{u}_0^h\|_{L^2} \|\partial_3 \bar{u}_0^h\|_{L^2}\right), \tag{3.6}$$

and

$$\|\Lambda_h^{-1} \partial_3 \bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \leq \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(\exp\left(C \|\bar{u}_0^h\|_{L^2} \|\partial_3 \bar{u}_0^h\|_{L^2}\right)\right). \tag{3.7}$$

We shall present the proof right after (4.7).

We notice that

$$v_0 \stackrel{\text{def}}{=} u_0 - (u_{0,\text{curl}}^h, 0) = (u_{0,\text{div}}^h, u_0^3), \tag{3.8}$$

which satisfies $\text{div } v_0 = 0$, and yet v_0 is not small according to our smallness condition (1.11).

Before proceeding, let us recall the main idea of the proof to Theorem 1.1 in [15]. The authors [15] first constructed (\bar{u}^h, \bar{p}) via the system (3.2). Then in order to get rid of the large part of the initial data v_0 , given by (3.8), the authors introduced a correction velocity, \tilde{u} , through the system

$$\begin{cases} \partial_t \tilde{u} + \bar{u}^h \cdot \nabla_h \tilde{u} - \Delta \tilde{u} = -\nabla \tilde{p}, \\ \text{div } \tilde{u} = 0, \\ \tilde{u}^h|_{t=0} = \tilde{u}_0^h = -\nabla_h \Delta_h^{-1}(\partial_3 u_0^3), \quad \tilde{u}^3|_{t=0} = \tilde{u}_0^3 = u_0^3. \end{cases} \tag{3.9}$$

With \bar{u}^h and \tilde{u} being determined respectively by the systems (3.2) and (3.9), the authors [15] decompose the solution (u, p) to the classical Navier–Stokes system (NS) as

$$u = \left(\bar{u}^h\right) + \tilde{u} + v, \quad p = \bar{p} + \tilde{p} + q. \tag{3.10}$$

The key estimate for v is as follows:

Proposition 3.2. *Let $u = (u^h, u^3) \in C([0, T^*]; H^{\frac{1}{2}}) \cap L^2([0, T^*]; H^{\frac{3}{2}})$ be a Fujita–Kato solution of (NS). We denote $\omega \stackrel{\text{def}}{=} \partial_1 v^2 - \partial_2 v^1$ and*

$$M(t) \stackrel{\text{def}}{=} \|\nabla v^3(t)\|_{H^{-\frac{1}{2},0}}^2 + \|\omega(t)\|_{H^{-\frac{1}{2},0}}^2, \quad N(t) \stackrel{\text{def}}{=} \|\nabla^2 v^3(t)\|_{H^{-\frac{1}{2},0}}^2 + \|\nabla \omega(t)\|_{H^{-\frac{1}{2},0}}^2. \tag{3.11}$$

Then under the assumption (1.7), there exists some positive constant η such that

$$\sup_{t \in [0, T^*[} \left(M(t) + \int_0^t N(t') dt' \right) \leq \eta. \tag{3.12}$$

Then in order to complete the proof of Theorem 1.1, the authors [15] invoked the following regularity criteria for the classical Navier–Stokes system:

Theorem 3.1. (Theorem 1.5 of [10]) *Let $u \in C([0, T^*]; H^{\frac{1}{2}}) \cap L^2([0, T^*]; H^{\frac{3}{2}})$ be a solution of (NS). If the maximal existence time T^* is finite, then for any (p_i, j) in $]1, \infty[^9$, one has*

$$\sum_{1 \leq i, j \leq 3} \int_0^{T^*} \|\partial_i u^j(t)\|_{B_{\infty, \infty}^{-2 + \frac{2}{p_i, j}}}^{p_i, j} dt = \infty. \tag{3.13}$$

We remark that Theorem 3.1 only works for the classical 3-D Navier–Stokes system. Therefore the above procedure to prove Theorem 1.1 cannot be applied to construct the global solutions to the 3-D anisotropic Navier–Stokes system.

On the other hand, we remark that the main observation in [17, 19] is that: by using $\operatorname{div} u = 0$, (ANS) can be equivalently reformulated as

$$(ANS) \quad \begin{cases} \partial_t u^h + u^h \cdot \nabla_h u^h + u^3 \partial_3 u^h - \Delta_h u^h = -\nabla_h p, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u^3 + u^h \cdot \nabla_h u^3 - u^3 \operatorname{div}_h u^h - \Delta_h u^3 = -\partial_3 p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = (u_0^h, u_0^3), \end{cases}$$

so that at least, seemingly, the u^3 equation is a linear one; this explains why there is no size restriction for u_0^3 in (1.4) and (1.6).

Motivated by [17, 19], for \bar{u}^h being determined by the systems (3.2), we decompose the solution u of (ANS) as $u = \begin{pmatrix} \bar{u}^h \\ 0 \end{pmatrix} + v$. It is easy to verify that the remainder term v satisfies

$$\begin{cases} \partial_t v^h + v \cdot \nabla v^h + \bar{u}^h \cdot \nabla_h v^h + v \cdot \nabla \bar{u}^h - \Delta_h v^h = -\nabla_h p + \nabla_h \bar{p}, \\ \partial_t v^3 + v^h \cdot \nabla_h v^3 - v^3 \operatorname{div}_h v^h + \bar{u}^h \cdot \nabla_h v^3 - \Delta_h v^3 = -\partial_3 p, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0 = (-\nabla_h \Delta_h^{-1}(\partial_3 u_0^3), u_0^3). \end{cases} \tag{3.14}$$

We notice that under the smallness condition (1.11), the horizontal components, v_0^h , are small in the critical space $\mathcal{B}^{0, \frac{1}{2}}$. Then the crucial ingredient used in the proof of Theorem 1.2 is that the horizontal components v^h of the remainder velocity keeps small for any positive time.

Due to the additional difficulty caused by the fact that u_0^3 belongs to the Sobolev–Besov type space with negative index, as in [8], we further decompose v^3 as

$$v^3 = v_F + w, \quad \text{where } v_F(t) \stackrel{\text{def}}{=} e^{t\Delta_h} u_{0,\text{hh}}^3 \quad \text{and} \quad u_{0,\text{hh}}^3 \stackrel{\text{def}}{=} \sum_{k \geq \ell-1} \Delta_k^h \Delta_\ell^v u_0^3. \tag{3.15}$$

Then w solves

$$\begin{cases} \partial_t w - \Delta_h w + v \cdot \nabla v^3 + \bar{u}^h \cdot \nabla_h v^3 = -\partial_3 p, \\ w|_{t=0} = u_{0,\text{hh}}^3 \stackrel{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} S_{\ell-1}^h \Delta_\ell^v u_0^3. \end{cases} \tag{3.16}$$

Proposition 3.3. *Let v be a smooth enough solution of (3.14) on $[0, T^*[$. Then there exists some positive constant C so that for any $t \in]0, T^*[$, we have*

$$\begin{aligned} & \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \left(\frac{5}{4} - C\|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}}\right) \|\nabla_h v^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \leq (\|v_0^h\|_{\mathcal{B}^{0,\frac{1}{2}}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}) \\ & \times \exp\left(C \int_0^t (\|w(t')\|_{\mathcal{B}^{0,\frac{1}{2}}}^2 \|\nabla_h w(t')\|_{\mathcal{B}^{0,\frac{1}{2}}}^2 + \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0,\frac{1}{2}}}^4 + \|v_F(t')\|_{\mathcal{B}_4^{0,\frac{1}{2}}}^4) dt'\right), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} & \left(\frac{5}{6} - C(\|v^h\|_{\mathcal{B}^{0,\frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}})\right) \|w\|_{\mathcal{B}^{0,\frac{1}{2}}(t)} \\ & \leq \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} + C\left(\|v^h\|_{\mathcal{B}^{0,\frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + \|v^h\|_{\mathcal{B}^{0,\frac{1}{2}}(t)}^2\right. \\ & \left. + (1 + \|v^h\|_{\mathcal{B}^{0,\frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(t)}\right) \exp\left(C\|\bar{u}^h\|_{L_t^4(\mathcal{B}_4^{0,\frac{1}{2}})}^4\right). \end{aligned} \tag{3.18}$$

The proof of the estimates (3.17) and (3.18) will be presented respectively in Sections 5 and 6. Now let us admit the above Propositions 3.1 and 3.3 temporarily, and continue our proof of Theorem 1.2.

Proof of Theorem 1.2. It is well-known that the existence of global solutions to a nonlinear partial differential equations can be obtained by first constructing the approximate solutions, and then performing uniform estimates and finally passing to the limit to such approximate solutions. For simplicity, here we just present the a priori estimates for smooth enough solutions of (ANS).

Let u be a smooth enough solution of (ANS) on $[0, T^*[$ with T^* being the maximal time of existence. Let \bar{u}^h and v be determined by (3.2) and (3.14), respectively. Thanks to (3.1) and Proposition 3.1, we first take L, M, N large enough and ε_0 small enough in (1.11) so that

$$\begin{aligned} & \|\Lambda_h^{-1} \partial_3 \bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \leq C\|\Lambda_h^{-1} \partial_3 u_0^h\|_{\mathcal{B}^{0,\frac{1}{2}}} \exp\left(C\mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0,\frac{1}{2}}})\right) \\ & \leq \frac{1}{16} \quad \text{for any } t > 0. \end{aligned} \tag{3.19}$$

We now define

$$T^* \stackrel{\text{def}}{=} \sup \left\{ t < T^*, \quad C \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \leq \frac{1}{16} \right\}. \tag{3.20}$$

Then, thanks to (3.19) and Proposition 3.3, for $t \leq T^*$, we find

$$\begin{aligned} \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} &\leq (\|\Lambda_h^{-1} \partial_3 u_0^3\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}) \\ &\times \exp\left(C \int_0^t (\|w(t')\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \|\nabla_h w(t')\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 + \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 + \|v_F(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4) dt'\right), \end{aligned} \tag{3.21}$$

and

$$\frac{1}{3} \|w\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \leq \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}} + C(1 + \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)}) \exp\left(C \|\bar{u}^h\|_{L_t^4(\mathcal{B}_4^{0, \frac{1}{2}})}^4\right). \tag{3.22}$$

It follows from Lemma 2.2 and Proposition 2.1 that

$$\|v_F\|_{L_t^4(\mathcal{B}_4^{0, \frac{1}{2}})} \lesssim \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \lesssim \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}},$$

whereas we deduce from (2.6) and Proposition 3.1 that

$$\begin{aligned} \|\bar{u}^h\|_{\tilde{L}_t^4(\mathcal{B}_4^{0, \frac{1}{2}})}^4 &\leq C \|\bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2 \|\nabla_h \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2 \\ &\leq C \mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}). \end{aligned}$$

By inserting the above two inequalities to (3.22) and using (3.3), we obtain that, for $t \leq T^*$,

$$\frac{1}{3} \|w\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \leq C(1 + \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}) \exp\left(C \mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}})\right). \tag{3.23}$$

Then we deduce that for $t \leq T^*$,

$$\begin{aligned} &\int_0^t (\|w(t')\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \|\nabla_h w(t')\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 + \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 + \|v_F(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4) dt' \\ &\leq \|w\|_{L_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2 \|\nabla_h w\|_{L_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2 + \|\bar{u}^h\|_{L_t^4(\mathcal{B}_4^{0, \frac{1}{2}})}^4 + \|v_F\|_{L_t^4(\mathcal{B}_4^{0, \frac{1}{2}})}^4 \\ &\leq C(1 + \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}})^4 \exp\left(C \mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}})\right). \end{aligned}$$

Inserting the above estimates into (3.21) gives

$$\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \leq \|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(C(1 + \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}})^4 \exp(C \mathfrak{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}))\right) \tag{3.24}$$

for $t \leqq T^*$. Therefore, if we take L, M, N large enough and ε_0 small enough in (1.11), we deduce from (3.24) that

$$C \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \leqq \frac{1}{32} \quad \text{for } t \leqq T^*. \tag{3.25}$$

(3.25) contradicts (3.20). This in turn shows that $T^* = T^*$. (3.23) along with (3.25) shows that $T^* = \infty$. Moreover, thanks to (3.15), we have $v \stackrel{\text{def}}{=} u - e^{t\Delta_h} \begin{pmatrix} 0 \\ u_{0, \text{hh}}^3 \end{pmatrix} \in C([0, \infty[; \mathcal{B}^{0, \frac{1}{2}})$ with $\nabla_h v \in L^2([0, \infty[; \mathcal{B}^{0, \frac{1}{2}})$. This completes the proof of our Theorem 1.2. \square

Proof of Corollary 1.1. Under the assumptions that $u_0^h \in L^2$ with $\partial_3 u_0^h \in L^2$ and $\Lambda_h^{-1} \partial_3 u_0^h \in \mathcal{B}^{0, \frac{1}{2}}$, we deduce from (3.1), (3.4) and (3.7) that

$$\begin{aligned} \|\bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\nabla_h \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} &\leqq \|u_0^h\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_0^h\|_{L^2}^{\frac{1}{2}} \exp(C \|u_0^h\|_{L^2} \|\partial_3 u_0^h\|_{L^2}), \\ \|\Lambda_h^{-1} \partial_3 \bar{u}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} &\leqq \|\Lambda_h^{-1} \partial_3 u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp(\exp(C \|u_0^h\|_{L^2} \|\partial_3 u_0^h\|_{L^2})). \end{aligned}$$

Then by repeating the argument from (3.19) to (3.24), we conclude the proof of Corollary 1.1. \square

4. Estimates of the 2-D Solution \bar{u}^h

The goal of this section is to present the proof of Proposition 3.1. Let us start the proof with the following lemma, which is in the spirit of Lemma 3.1 of [6]:

Lemma 4.1. *Let $a^h = (a^1, a^2)$ be a smooth enough solution of*

$$\begin{cases} \partial_t a^h + a^h \cdot \nabla_h a^h - \Delta_h a^h = -\nabla_h \pi, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div}_h a^h = 0, \\ a^h|_{t=0} = a_0^h. \end{cases} \tag{4.1}$$

Then for any $t > 0$ and any fixed $x_3 \in \mathbb{R}$, it holds that

$$\|a^h(t, \cdot, x_3)\|_{L_h^2}^2 + 2 \int_0^t \|\nabla_h a^h(t', \cdot, x_3)\|_{L_h^2}^2 dt' = \|a_0^h(\cdot, x_3)\|_{L_h^2}^2, \tag{4.2}$$

and

$$\begin{aligned} \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 + \int_0^t \|\nabla_h \partial_3 a^h(t', \cdot, x_3)\|_{L_h^2}^2 dt' \\ \leqq \|\partial_3 a_0^h(\cdot, x_3)\|_{L_h^2}^2 \exp(C \|a_0^h\|_{L_h^\infty(L_h^2)}^2). \end{aligned} \tag{4.3}$$

Proof. By taking L_h^2 inn-product of (4.1) with a^h and using $\operatorname{div}_h a^h = 0$, we obtain (4.2).

While by applying ∂_3 to (4.1) and then taking L_h^2 inner product of the resulting equation with $\partial_3 a^h$, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 + \|\nabla_h \partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 \\ & = -(\partial_3(a^h \cdot \nabla_h a^h)(t, \cdot, x_3) | \partial_3 a^h(t, \cdot, x_3))_{L_h^2}. \end{aligned} \tag{4.4}$$

Due to $\operatorname{div}_h a^h = 0$, we get, by applying (2.4), that

$$\begin{aligned} & |(\partial_3(a^h \cdot \nabla_h a^h)(t, \cdot, x_3) | \partial_3 a^h(t, \cdot, x_3))_{L_h^2}| \\ & = |((\partial_3 a^h \cdot \nabla_h a^h)(t, \cdot, x_3) | \partial_3 a^h(t, \cdot, x_3))_{L_h^2}| \\ & \leq \|\nabla_h a^h(t, \cdot, x_3)\|_{L_h^2} \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^4}^2 \\ & \leq C \|\nabla_h a^h(t, \cdot, x_3)\|_{L_h^2} \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2} \|\nabla_h \partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}. \end{aligned}$$

Applying Young’s inequality yields

$$\begin{aligned} & |(\partial_3(a^h \cdot \nabla_h a^h)(t, \cdot, x_3) | \partial_3 a^h(t, \cdot, x_3))_{L_h^2}| \\ & \leq \frac{1}{2} \|\nabla_h \partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 + C \|\nabla_h a^h(t, \cdot, x_3)\|_{L_h^2}^2 \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2. \end{aligned}$$

Inserting the above estimate into (4.4) gives

$$\begin{aligned} & \frac{d}{dt} \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 + \|\nabla_h \partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 \\ & \leq C \|\nabla_h a^h(t, \cdot, x_3)\|_{L_h^2}^2 \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2. \end{aligned}$$

Applying Gronwall’s inequality and using (4.2), we achieve

$$\begin{aligned} & \|\partial_3 a^h(t, \cdot, x_3)\|_{L_h^2}^2 + \int_0^t \|\nabla_h \partial_3 a^h(t', \cdot, x_3)\|_{L_h^2}^2 dt' \\ & \leq \|\partial_3 a_0^h(\cdot, x_3)\|_{L_h^2}^2 \exp\left(C \int_0^t \|\nabla_h a^h(t', \cdot, x_3)\|_{L_h^2}^2 dt'\right) \\ & \leq \|\partial_3 a_0^h(\cdot, x_3)\|_{L_h^2}^2 \exp(C \|a_0^h(\cdot, x_3)\|_{L_h^2}^2), \end{aligned}$$

which leads to (4.3). This completes the proof of this lemma. \square

Let us now present the proof of Proposition 3.1.

Proof of Proposition 3.1. For any positive integer N , and $\bar{u}_{0,N}^h$ being given by (3.5), we split the solution \bar{u}^h to (3.2) as

$$\bar{u}^h = \bar{u}_1^h + \bar{u}_2^h, \tag{4.5}$$

with \bar{u}_1^h and \bar{u}_2^h being determined, respectively, by

$$\begin{cases} \partial_t \bar{u}_1^h + \bar{u}_1^h \cdot \nabla_h \bar{u}_1^h - \Delta_h \bar{u}_1^h = -\nabla_h \bar{p}^{(1)}, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div}_h \bar{u}_1^h = 0, \\ \bar{u}_1^h|_{t=0} = \bar{u}_{1,0}^h \stackrel{\text{def}}{=} \bar{u}_0^h - \bar{u}_{0,N}^h, \end{cases} \quad (4.6)$$

and

$$\begin{cases} \partial_t \bar{u}_2^h + \operatorname{div}_h (\bar{u}_2^h \otimes \bar{u}_2^h + \bar{u}_1^h \otimes \bar{u}_2^h + \bar{u}_2^h \otimes \bar{u}_1^h) - \Delta_h \bar{u}_2^h = -\nabla_h \bar{p}^{(2)}, \\ \operatorname{div}_h \bar{u}_2^h = 0, \\ \bar{u}_2^h|_{t=0} = \bar{u}_{2,0}^h = \bar{u}_{0,N}^h. \end{cases} \quad (4.7)$$

Indeed for smoother initial data \bar{u}_0^h , we may write explicitly the constant $\mathfrak{A}_{N, \left(\|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}\right)}$ in (3.3). For instance, if $\bar{u}_0^h \in L^2$ with $\partial_3 \bar{u}_0^h \in L^2$ and $\Lambda_h^{-1} \partial_3 \bar{u}_0^h \in \mathcal{B}^{0, \frac{1}{2}}$, we deduce from Lemma 4.1 that

$$\begin{aligned} \|\bar{u}^h(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h \bar{u}^h(t')\|_{L^2}^2 dt' &= \|\bar{u}_0^h\|_{L^2}^2, \quad \text{and} \\ \|\partial_3 \bar{u}^h(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \partial_3 \bar{u}^h(t')\|_{L^2}^2 dt' &\leq \|\partial_3 \bar{u}_0^h\|_{L^2}^2 \exp(C \|\bar{u}_0^h\|_{L^\infty(L^2)}^2), \end{aligned}$$

which, together with (2.2) and

$$\|\bar{u}_0^h\|_{L^\infty(L_h^2)}^2 \leq \|\bar{u}_0^h\|_{L_h^2(L^\infty)}^2 \leq \|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \leq \|u_0^h\|_{L^2} \|\partial_3 u_0^h\|_{L^2},$$

ensures (3.6). By virtue of (3.6) and (4.22), we deduce (3.7).

In general, we first deduce from Lemma 4.1 that

$$\begin{aligned} \|\bar{u}_1^h(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h \bar{u}_1^h(t')\|_{L^2}^2 dt' &= \|\bar{u}_{1,0}^h\|_{L^2}^2 \lesssim N \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2, \quad \text{and} \\ \|\partial_3 \bar{u}_1^h(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \partial_3 \bar{u}_1^h(t')\|_{L^2}^2 dt' &\leq \|\partial_3 \bar{u}_{1,0}^h\|_{L^2}^2 \exp(C \|\bar{u}_{1,0}^h\|_{L^\infty(L^2)}^2) \\ &\lesssim N \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \exp(C \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2), \end{aligned}$$

which, together with (2.2), ensures that

$$\|\bar{u}_1^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\nabla_h \bar{u}_1^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \leq CN^{\frac{1}{2}} \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp(C \|\bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2). \quad (4.8)$$

Next we handle the estimate of \bar{u}_2^h . To do this, for any $\kappa > 0$, we denote

$$f^h(t) \stackrel{\text{def}}{=} \|\bar{u}_1^h(t)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \|\nabla_h \bar{u}_1^h(t)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \quad \text{and} \quad \bar{u}_{2,\kappa}^h(t) \stackrel{\text{def}}{=} \bar{u}_2^h(t) \exp\left(-\kappa \int_0^t f^h(t') dt'\right). \quad (4.9)$$

Then by multiplying $\exp\left(-\kappa \int_0^t f^h(t') dt'\right)$ to the \bar{u}_2^h equation in (4.7), we write

$$\begin{aligned} \partial_t \bar{u}_{2,\kappa}^h + \kappa f^h(t) \bar{u}_{2,\kappa}^h - \Delta_h \bar{u}_{2,\kappa}^h + \operatorname{div}_h (\bar{u}_2^h \otimes \bar{u}_{2,\kappa}^h + \bar{u}_1^h \otimes \bar{u}_{2,\kappa}^h \\ + \bar{u}_{2,\kappa}^h \otimes \bar{u}_1^h) = -\nabla_h \bar{p}_\kappa^{(2)}. \end{aligned}$$

Applying the operator Δ_ℓ^\vee to the above equation and taking L^2 inner product of the resulting equation with $\Delta_\ell^\vee \bar{u}_{2,\kappa}^h$, and then using integration by parts, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_\ell^\vee \bar{u}_{2,\kappa}^h(t)\|_{L^2}^2 + \kappa f^h(t) \|\Delta_\ell^\vee \bar{u}_{2,\kappa}^h(t)\|_{L^2}^2 + \|\Delta_\ell^\vee \nabla_h \bar{u}_{2,\kappa}^h\|_{L^2}^2 \\ &= -(\Delta_\ell^\vee(\bar{u}_2^h \cdot \nabla_h \bar{u}_{2,\kappa}^h) | \Delta_\ell^\vee \bar{u}_{2,\kappa}^h)_{L^2} + (\Delta_\ell^\vee(\bar{u}_1^h \otimes \bar{u}_{2,\kappa}^h + \bar{u}_{2,\kappa}^h \otimes \bar{u}_1^h) | \Delta_\ell^\vee \nabla_h \bar{u}_{2,\kappa}^h)_{L^2}. \end{aligned} \tag{4.10}$$

□

The estimate of the second line of (4.10) relies on the following lemma, whose proof will be postponed until the ‘‘Appendix A’’:

Lemma 4.2. *Let $a, b, c \in \mathcal{B}^{0, \frac{1}{2}}(T)$ and $\mathfrak{f}(t) \stackrel{def}{=} \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4$. Then for any smooth homogeneous Fourier multiplier, $A(D)$, of degree zero and any $\ell \in \mathbb{Z}$, it holds that*

$$\int_0^T |(\Delta_\ell^\vee A(D)(a \otimes b) | \Delta_\ell^\vee c)_{L^2}| dt \lesssim d_\ell^2 2^{-\ell} \|b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|c\|_{\tilde{L}_{T, \mathfrak{f}}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}, \tag{4.11}$$

$$\int_0^T |(\Delta_\ell^\vee A(D)(a \otimes b) | \Delta_\ell^\vee c)_{L^2}| dt \lesssim d_\ell^2 2^{-\ell} \|b\|_{\tilde{L}_{T, \mathfrak{f}}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}. \tag{4.12}$$

Moreover, for non-negative function $\mathfrak{g} \in L^\infty(0, T)$, one has

$$\begin{aligned} & \int_0^T |(\Delta_\ell^\vee A(D)(a \otimes b) | \Delta_\ell^\vee c)_{L^2}| \cdot \mathfrak{g}^2 dt \lesssim d_\ell^2 2^{-\ell} \|a\|_{\tilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\mathfrak{g} \nabla_h a\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \\ & \times \|\mathfrak{g} b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|c\|_{\tilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\mathfrak{g} \nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned} \tag{4.13}$$

By applying (4.13) with $a = c = \bar{u}_2^h$, $b = \nabla_h \bar{u}_2^h$ and $\mathfrak{g} = \exp(-\kappa \int_0^t f^h(t') dt')$, we get

$$\int_0^t |(\Delta_\ell^\vee(\bar{u}_2^h \cdot \nabla_h \bar{u}_{2,\kappa}^h) | \Delta_\ell^\vee \bar{u}_{2,\kappa}^h)_{L^2}| dt' \lesssim d_\ell^2 2^{-\ell} \|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2. \tag{4.14}$$

Whereas due to (2.5), one has

$$\|\bar{u}_1^h(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 \lesssim \|\bar{u}_1^h(t)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \|\nabla_h \bar{u}_1^h(t)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2.$$

By applying (4.12) with $a = \bar{u}_1^h$, $b = \bar{u}_{2,\kappa}^h$, $c = \nabla_h \bar{u}_{2,\kappa}^h$, we infer

$$\begin{aligned} & \int_0^t |(\Delta_\ell^\vee(\bar{u}_1^h \otimes \bar{u}_{2,\kappa}^h + \bar{u}_{2,\kappa}^h \otimes \bar{u}_1^h) | \Delta_\ell^\vee \nabla_h \bar{u}_{2,\kappa}^h)_{L^2}| dt' \\ & \lesssim d_\ell^2 2^{-\ell} \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_{t, f^h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{2}}. \end{aligned} \tag{4.15}$$

Then we get, by first integrating (4.10) over $[0, t]$ and inserting (4.14) and (4.15) into the resulting inequality, that

$$\begin{aligned} & \|\Delta_\ell^v \bar{u}_{2,\kappa}^h(t)\|_{L^2}^2 + 2\kappa \int_0^t f^h(t') \|\Delta_\ell^v \bar{u}_{2,\kappa}^h(t')\|_{L^2}^2 dt' + 2\|\Delta_\ell^v \nabla_h \bar{u}_{2,\kappa}^h\|_{L_t^2(L^2)}^2 \\ & \leq \|\Delta_\ell^v \bar{u}_{0,N}^h\|_{L^2}^2 + C d_\ell^2 2^{-\ell} \left(\|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2 \right. \\ & \quad \left. + \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_{t,f^h}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}} \right). \end{aligned}$$

Multiplying the above inequality by 2^ℓ and taking square root of the resulting inequality, and then summing up the inequalities for $\ell \in \mathbb{Z}$, we arrive at

$$\begin{aligned} & \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \sqrt{2\kappa} \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_{t,f^h}^2(\mathcal{B}^{0,\frac{1}{2}})} + \sqrt{2} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \\ & \leq \|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}} + C \left(\|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_{t,f^h}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{4}} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{4}} \right) \\ & \leq \|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}} + (\sqrt{2} - 1 + C \|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}}) \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + C \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_{t,f^h}^2(\mathcal{B}^{0,\frac{1}{2}})}. \end{aligned}$$

In particular, taking $2\kappa = C^2$ in the above inequality gives

$$\|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + (1 - C \|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}}) \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \leq \|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}}. \tag{4.16}$$

On the other hand, in view of (3.5), we can take N so large that

$$C \|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}}^{\frac{1}{2}} \leq \frac{1}{2}. \tag{4.17}$$

Then a standard continuity argument shows that, for any time $t > 0$, it holds that

$$\|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \frac{1}{2} \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \leq \|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}}. \tag{4.18}$$

Due to the definition of $\bar{u}_{2,\lambda}^h$ given by (4.9), one has

$$\begin{aligned} & \left(\|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \|\nabla_h \bar{u}_2^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \right) \exp\left(-\kappa \int_0^t f^h(t') dt'\right) \\ & \leq \|\bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \|\nabla_h \bar{u}_{2,\kappa}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}, \end{aligned}$$

which, together with (4.8) and (4.18), implies that

$$\begin{aligned} \|\bar{u}_2^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \|\nabla_h \bar{u}_2^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} & \leq 2\|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}} \exp\left(\kappa \int_0^t f^h(t') dt'\right) \\ & \leq 2\|\bar{u}_{0,N}^h\|_{\mathcal{B}^{0,\frac{1}{2}}} \exp\left(N^2 \exp(C \|\bar{u}_0^h\|_{\mathcal{B}^{0,\frac{1}{2}}})^2\right), \end{aligned} \tag{4.19}$$

By combining (4.8) with (4.19), we obtain (3.3).

It remains to prove (3.4). In order to do, this for any $\gamma > 0$, we denote

$$g^h(t) \stackrel{\text{def}}{=} \|\bar{u}^h(t)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \|\nabla_h \bar{u}^h(t)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 \quad \text{and} \quad \bar{u}_\gamma^h(t) \stackrel{\text{def}}{=} \bar{u}^h(t) \exp\left(-\gamma \int_0^t g^h(t') dt'\right). \quad (4.20)$$

Then, by multiplying $\exp\left(-\gamma \int_0^t g^h(t') dt'\right)$ to the \bar{u}^h equation in (3.2), we write

$$\partial_t \bar{u}_\gamma^h + \gamma g^h(t) \bar{u}_\gamma^h - \Delta_h \bar{u}_\gamma^h + \bar{u}^h \cdot \nabla_h \bar{u}_\gamma^h = -\nabla_h \bar{p}_\gamma.$$

Applying the operator $\Delta_\ell^\vee \Lambda_h^{-1} \partial_3$ to the above equation and then taking L^2 inner product of the resulting equation with $\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h(t)\|_{L^2}^2 + \gamma g^h(t) \|\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h(t)\|_{L^2}^2 + \|\Delta_\ell^\vee \nabla_h \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{L^2}^2 \\ &= -(\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 (\bar{u}^h \cdot \nabla_h \bar{u}_\gamma^h) | \Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h)_{L^2} \\ &= -(\Delta_\ell^\vee \Lambda_h^{-1} \operatorname{div}_h (\bar{u}^h \otimes \partial_3 \bar{u}_\gamma^h + \partial_3 \bar{u}_\gamma^h \otimes \bar{u}^h) | \Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h)_{L^2}. \end{aligned} \quad (4.21)$$

Noting that $\Lambda_h^{-1} \operatorname{div}_h$ is a bounded Fourier multiplier, we get, by using (4.11) with $a = \bar{u}^h$, $b = \partial_3 \bar{u}_\gamma^h$ and $c = \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h$, that

$$\begin{aligned} & \int_0^t |(\Delta_\ell^\vee \Lambda_h^{-1} \operatorname{div}_h (\bar{u}^h \otimes \partial_3 \bar{u}_\gamma^h + \partial_3 \bar{u}_\gamma^h \otimes \bar{u}^h) | \Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h)_{L^2}| dt' \\ & \lesssim d_\ell^2 2^{-\ell} \|\partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{2}} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_{t, g^h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned}$$

By integrating (4.21) over $[0, t]$ and then inserting the above estimate into the resulting inequality, we find

$$\begin{aligned} & \|\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h(t)\|_{L^2}^2 + 2\gamma \int_0^t g^h(t') \|\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h(t')\|_{L^2}^2 dt' + 2\|\Delta_\ell^\vee \partial_3 \bar{u}_\gamma^h\|_{L_t^2(L^2)}^2 \\ & \leq \|\Delta_\ell^\vee \Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{L^2}^2 + Cd_\ell^2 2^{-\ell} \|\partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{2}} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_{t, g^h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned}$$

Multiplying the above inequality by 2^ℓ and taking square root of the resulting inequality, and then summing up the inequalities for $\ell \in \mathbb{Z}$, we arrive at

$$\begin{aligned} & \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \sqrt{2\gamma} \|\Lambda_h^{-1} \partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_{t, f^h}^2(\mathcal{B}^{0, \frac{1}{2}})} + \sqrt{2} \|\partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \\ & \leq \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} + C \|\partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{4}} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_{t, g^h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{4}} \\ & \leq \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} + (\sqrt{2} - 1) \|\partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} + C \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_{t, g^h}^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

In particular, taking $2\gamma = C^2$ in the above inequality gives

$$\|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{u}_\gamma^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \leq \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}.$$

Then a similar derivation from (4.18) to (4.19) leads to

$$\|\Lambda_h^{-1} \partial_3 \bar{u}^h\|_{\tilde{L}_t^\infty(B^{0, \frac{1}{2}})} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(B^{0, \frac{1}{2}})} \leq \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{B^{0, \frac{1}{2}}} \exp\left(\gamma \int_0^t g^h(t') dt'\right), \tag{4.22}$$

which together with (3.3), ensures (3.4). This completes the proof of this proposition. \square

5. The Estimate of the Horizontal Components v^h

The goal of this section is to present the proof of (3.17), namely, we are going to deal with the estimate to the horizontal components of the remainder velocity determined by (3.14).

In order to do this, let u be a smooth enough solution of (ANS) on $[0, T^*[,$ let \bar{u}^h, v_F and w be determined respectively by (3.2), (3.15) and (3.16), for any constant $\lambda > 0,$ we denote

$$v_\lambda^h(t) \stackrel{\text{def}}{=} v^h(t) \exp\left(-\lambda \int_0^t f(t') dt'\right) \quad \text{with} \tag{5.1}$$

$$f(t) \stackrel{\text{def}}{=} \|w(t)\|_{B^{0, \frac{1}{2}}}^2 \|\nabla_h w(t)\|_{B^{0, \frac{1}{2}}}^2 + \|\bar{u}^h(t)\|_{B_4^{0, \frac{1}{2}}}^4 + \|v_F(t)\|_{B_4^{0, \frac{1}{2}}}^4,$$

and similar notations for $\bar{u}_\lambda^h, p_\lambda, \bar{p}_\lambda$ and $v_{\lambda/2}^h.$

By multiplying $\exp\left(-\lambda \int_0^t f(t') dt'\right)$ to the v^h equation of (3.14), we get

$$\partial_t v_\lambda^h + \lambda f(t) v_\lambda^h + v \cdot \nabla v_\lambda^h + \bar{u}^h \cdot \nabla_h v_\lambda^h + v_\lambda \cdot \nabla \bar{u}^h - \Delta_h v_\lambda^h = -\nabla_h p_\lambda + \nabla_h \bar{p}_\lambda.$$

Applying Δ_ℓ^v to the above equation and taking L^2 inner product of the resulting equation with $\Delta_\ell^v v_\lambda^h,$ and then integrating the equality over $[0, t],$ we obtain

$$\frac{1}{2} \|\Delta_\ell^v v_\lambda^h(t)\|_{L^2}^2 + \lambda \int_0^t f(t') \|\Delta_\ell^v v_\lambda^h\|_{L^2}^2 dt' + \int_0^t \|\nabla_h \Delta_\ell^v v_\lambda^h\|_{L^2}^2 dt' = \frac{1}{2} \|\Delta_\ell^v v_0^h\|_{L^2}^2 - \sum_{i=1}^6 I_i, \tag{5.2}$$

where

$$I_1 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(\bar{u}^h \cdot \nabla_h v_\lambda^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt', \quad I_2 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(v^h \cdot \nabla_h v_\lambda^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt',$$

$$I_3 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(v_\lambda^h \cdot \nabla_h \bar{u}^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt', \quad I_4 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(v^3 \partial_3 \bar{u}_\lambda^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt',$$

$$I_5 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(v^3 \partial_3 v_\lambda^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt', \quad I_6 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v \nabla_h(p_\lambda - \bar{p}_\lambda) | \Delta_\ell^v v_\lambda^h)_{L^2} dt'.$$

We mention that since our system (3.14) has only horizontal dissipation, it is reasonable to distinguish the terms above with horizontal derivatives from the ones with vertical derivative. Next let us handle the above term by term.

• The estimates of I_1 to I_4 .

We first get, by using (4.11) with $a = \bar{u}^h$, $b = \nabla_h v_\lambda^h$ and $c = v_\lambda^h$, that

$$|I_1| \lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}. \tag{5.3}$$

Applying (4.13) with $a = v^h$, $b = \nabla_h v^h$, $c = v^h$ and $\mathfrak{g}(t) = \exp(-\lambda \int_0^t f(t') dt')$ yields

$$|I_2| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2. \tag{5.4}$$

To handle I_3 , by using integration by parts, we write

$$I_3 = - \int_0^t (\Delta_\ell^\vee(\operatorname{div}_h v_\lambda^h \cdot \bar{u}^h) | \Delta_\ell^\vee v_\lambda^h)_{L^2} dt' - \int_0^t (\Delta_\ell^\vee(\bar{u}^h \otimes v_\lambda^h) | \Delta_\ell^\vee \nabla_h v_\lambda^h)_{L^2} dt'.$$

Applying (4.11) with $a = \bar{u}^h$, $b = \operatorname{div}_h v_\lambda^h$ and $c = v_\lambda^h$ gives

$$\left| \int_0^t (\Delta_\ell^\vee(\operatorname{div}_h v_\lambda^h \cdot \bar{u}^h) | \Delta_\ell^\vee v_\lambda^h)_{L^2} dt' \right| \lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}.$$

Whereas applying (4.12) with $a = \bar{u}^h$, $b = v_\lambda^h$ and $c = \nabla_h v_\lambda^h$ yields

$$\left| \int_0^t (\Delta_\ell^\vee(\bar{u}^h \otimes v_\lambda^h) | \Delta_\ell^\vee \nabla_h v_\lambda^h)_{L^2} dt' \right| \lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}.$$

As a result, it turns out that

$$|I_3| \lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}. \tag{5.5}$$

While by applying (4.11) with $a = v^3$, $b = \partial_3 \bar{u}_\lambda^h$, $c = v_\lambda^h$, and using the fact that

$$\|v^3(t)\|_{\mathcal{B}_4^{0,\frac{1}{2}}} \leq \|v_F(t)\|_{\mathcal{B}_4^{0,\frac{1}{2}}} + \|w(t)\|_{\mathcal{B}^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h w(t)\|_{\mathcal{B}^{0,\frac{1}{2}}}^{\frac{1}{2}},$$

we find

$$|I_4| \lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}. \tag{5.6}$$

• The estimates of I_5 .

The estimate of I_5 is much more complicated, since there is no vertical dissipation in (ANS). To overcome this difficulty, we first use Bony’s decomposition in vertical variable (2.7) to write

$$I_5 = \int_0^t (\Delta_\ell^\vee(T_{v^3}^\vee \partial_3 v_\lambda^h + R^\vee(v^3, \partial_3 v_\lambda^h)) | \Delta_\ell^\vee v_\lambda^h)_{L^2} dt' \stackrel{\text{def}}{=} I_5^T + I_5^R.$$

Following [8, 16], we get, by using a standard commutator's process, that

$$\begin{aligned} I_5^T &= \sum_{|\ell' - \ell| \leq 5} \left(\int_0^t ([\Delta_\ell^v; S_{\ell'-1}^v v^3] \Delta_{\ell'}^v \partial_3 v_\lambda^h | \Delta_\ell^v v_\lambda^h)_{L^2} dt' \right. \\ &\quad \left. + \int_0^t ((S_{\ell'-1}^v v^3 - S_{\ell-1}^v v^3) \Delta_\ell^v \Delta_{\ell'}^v \partial_3 v_\lambda^h | \Delta_\ell^v v_\lambda^h)_{L^2} dt' \right) \\ &\quad + \int_0^t (S_{\ell-1}^v v^3 \Delta_\ell^v \partial_3 v_\lambda^h | \Delta_\ell^v v_\lambda^h)_{L^2} dt' \stackrel{\text{def}}{=} I_5^{T,1} + I_5^{T,2} + I_5^{T,3}. \end{aligned}$$

By applying the commutator's estimate (see Lemma 2.97 in [2]), we find

$$\begin{aligned} |I_5^{T,1}| &\leq \sum_{|\ell' - \ell| \leq 5} \|[\Delta_\ell^v; S_{\ell'-1}^v v^3] \Delta_{\ell'}^v \partial_3 v_\lambda^h\|_{L_t^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} \|\Delta_\ell^v v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))} \\ &\lesssim \sum_{|\ell' - \ell| \leq 5} 2^{-\ell} \|\partial_3 S_{\ell'-1}^v v^3\|_{L_t^2(L_h^2(L_v^\infty))} \|\Delta_{\ell'}^v \partial_3 v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))} \|\Delta_\ell^v v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))}. \end{aligned}$$

Due to $\partial_3 v^3 = -\text{div}_h v^h$, we get, by applying (2.4), that

$$\begin{aligned} |I_5^{T,1}| &\lesssim \sum_{|\ell' - \ell| \leq 5} 2^{-\ell} \|S_{\ell'-1}^v \text{div}_h v_\lambda^h\|_{L_t^2(L_h^2(L_v^\infty))} 2^{\ell'} \|\Delta_{\ell'}^v v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))} \|\Delta_\ell^v v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))} \\ &\lesssim \sum_{|\ell' - \ell| \leq 5} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \|\Delta_{\ell'}^v v^h\|_{L_t^{\frac{1}{2}}(L^2)} \|\nabla_h \Delta_{\ell'}^v v_\lambda^h\|_{L_t^{\frac{1}{2}}(L^2)} \\ &\quad \times \|\Delta_\ell^v v^h\|_{L_t^\infty(L^2)} \|\nabla_h \Delta_\ell^v v_\lambda^h\|_{L_t^2(L^2)} \\ &\lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2. \end{aligned}$$

Next, since the support to the Fourier transform of $\sum_{|\ell' - \ell| \leq 5} (S_{\ell'-1}^v v^3 - S_{\ell-1}^v v^3)$ is contained in $\mathbb{R}^2 \times \cup_{|\ell' - \ell| \leq 5} 2^{\ell'} C_v$, we get, by applying Lemma 2.1, that

$$\begin{aligned} |I_5^{T,2}| &\lesssim \sum_{|\ell' - \ell| \leq 5} 2^{-\ell} \|\partial_3 (S_{\ell'-1}^v v_\lambda^3 - S_{\ell-1}^v v_\lambda^3)\|_{L_t^2(L_h^2(L_v^\infty))} \\ &\quad \|\Delta_{\ell'}^v \partial_3 v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))} \|\Delta_\ell^v v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))}, \end{aligned}$$

from which we infer

$$|I_5^{T,2}| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2.$$

Finally, by using integration by parts and $\partial_3 v^3 = -\text{div}_h v^h$ again, we find that

$$\begin{aligned} |I_5^{T,3}| &= \frac{1}{2} \left| \int_0^t \int_{\mathbb{R}^3} S_{\ell-1}^v \partial_3 v_\lambda^3 \cdot |\Delta_\ell^v v_\lambda^h|^2 dx dt' \right| \\ &\lesssim \|S_{\ell-1}^v \text{div}_h v_\lambda^h\|_{L_t^2(L_h^2(L_v^\infty))} \|\Delta_\ell^v v_\lambda^h\|_{L_t^4(L_h^4(L_v^2))}^2 \\ &\lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2. \end{aligned}$$

As a result, it turns out that

$$|I_5^T| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2. \quad (5.7)$$

On the other hand, by applying Lemma 2.1 once again, we find that

$$\begin{aligned} |I_5^R| &\lesssim \sum_{\ell' \geq \ell-4} \|\Delta_{\ell'}^v v_\lambda^3\|_{L_t^2(L^2)} 2^{\ell'} \|S_{\ell'+2}^v v_{\lambda/2}^h\|_{L_t^4(L_h^4(L_v^\infty))} \|\Delta_{\ell'}^v v_{\lambda/2}^h\|_{L_t^4(L_h^4(L_v^2))} \\ &\lesssim \sum_{\ell' \geq \ell-4} \|\partial_3 \Delta_{\ell'}^v v_\lambda^3\|_{L_t^2(L^2)} \|S_{\ell'+2}^v v_{\lambda/2}^h\|_{L_t^4(L_h^4(L_v^\infty))} \|\Delta_{\ell'}^v v_{\lambda/2}^h\|_{L_t^4(L_h^4(L_v^2))}. \end{aligned}$$

Observing that

$$\begin{aligned} \|\partial_3 \Delta_{\ell'}^v v_\lambda^3\|_{L_t^2(L^2)} &\lesssim d_{\ell'} 2^{-\frac{\ell'}{2}} \|\operatorname{div}_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}, \\ \|S_{\ell'+2}^v v_{\lambda/2}^h\|_{L_t^4(L_h^4(L_v^\infty))} &\lesssim \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}, \end{aligned}$$

we infer

$$|I_5^R| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2,$$

which, together with (5.7), ensures that

$$|I_5| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^2. \tag{5.8}$$

• The estimates of I_6 .

We first get, by taking the space divergence operators, div and div_h , to (ANS) and (3.2) respectively, that

$$-\Delta p = \operatorname{div}(u \cdot \nabla u) \quad \text{and} \quad -\Delta_h \bar{p} = \operatorname{div}_h(\bar{u} \cdot \nabla_h \bar{u}), \tag{5.9}$$

so that thanks to the fact that

$$u = (u^h, u^3) = (\bar{u}^h, 0) + (v^h, v^3),$$

we write

$$\begin{aligned} \nabla_h p - \nabla_h \bar{p} &= \nabla_h (-\Delta)^{-1} \operatorname{div}_h(v \cdot \nabla u^h + \bar{u}^h \cdot \nabla_h v^h) \\ &\quad + \nabla_h (-\Delta)^{-1} \partial_3(u \cdot \nabla v^3) \\ &\quad + \nabla_h ((-\Delta)^{-1} - (-\Delta_h)^{-1}) \operatorname{div}_h \operatorname{div}_h(\bar{u}^h \otimes \bar{u}^h). \end{aligned}$$

Accordingly, we decompose I_6 as

$$I_6 = I_{6,1} + I_{6,2} + I_{6,3} + I_{6,4},$$

where

$$I_{6,1} = \int_0^t (\Delta_\ell^v \nabla_h (-\Delta)^{-1} \operatorname{div}_h(\bar{u}^h \cdot \nabla_h v_\lambda^h + v^h \cdot \nabla_h v_\lambda^h + v_\lambda \cdot \nabla \bar{u}^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt',$$

$$I_{6,2} = \int_0^t (\Delta_\ell^v \nabla_h (-\Delta)^{-1} \operatorname{div}_h(v^3 \partial_3 v_\lambda^h) | \Delta_\ell^v v_\lambda^h)_{L^2} dt',$$

$$I_{6,3} = \int_0^t (\Delta_\ell^v \nabla_h (-\Delta)^{-1} \partial_3(v_\lambda \cdot \nabla v^3 + \bar{u}^h \cdot \nabla_h v_\lambda^3) | \Delta_\ell^v v_\lambda^h)_{L^2} dt',$$

$$I_{6,4} = \sum_{i=1}^2 \sum_{j=1}^2 \int_0^t (\Delta_\ell^v \nabla_h ((-\Delta)^{-1} - (-\Delta_h)^{-1}) \partial_i \partial_j(\bar{u}^i \bar{u}_\lambda^j) | \Delta_\ell^v v_\lambda^h)_{L^2} dt'.$$

Noticing that $\nabla_h(-\Delta)^{-1}\operatorname{div}_h$ is a bounded Fourier multiplier. Then along the same line to the estimate of I_1 to I_4 , we achieve

$$\begin{aligned} |I_{6,1}| &\lesssim d_\ell^2 2^{-\ell} \left(\|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2 \right. \\ &\quad \left. + \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \left(\|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \right) \right). \end{aligned} \tag{5.10}$$

However, $I_{6,2}$ can not be handled along the same line to that of I_5 , since the symbol of the operator $\nabla_h(-\Delta)^{-1}\operatorname{div}_h$ depends not only on ξ_3 , but also on ξ_h , which makes it impossible for us to deal with the commutator’s estimate. Fortunately, the appearance of the operator $(-\Delta)^{-1}$ can absorb the vertical derivative. Indeed, by using integration by parts, and the divergence-free condition of v , we write

$$\begin{aligned} I_{6,2} &= \int_0^t \left(\Delta_\ell^v \nabla_h(-\Delta)^{-1} \operatorname{div}_h (\partial_3(v^3 v_\lambda^h) - \partial_3 v^3 \cdot v_\lambda^h) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt' \\ &= - \int_0^t \left(\Delta_\ell^v \nabla_h(-\Delta)^{-1} \partial_3(v^3 v_\lambda^h) \mid \Delta_\ell^v \nabla_h v_\lambda^h \right)_{L^2} dt' \\ &\quad + \int_0^t \left(\Delta_\ell^v \nabla_h(-\Delta)^{-1} \operatorname{div}_h (\operatorname{div}_h v^h \cdot v_\lambda^h) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt'. \end{aligned}$$

Since both $\nabla_h(-\Delta)^{-1}\partial_3$ and $\nabla_h(-\Delta)^{-1}\operatorname{div}_h$ are bounded Fourier multiplier, we get, by applying Lemma 4.2, that

$$|I_{6,2}| \lesssim d_\ell^2 2^{-\ell} \left(\|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}} + \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2 \right). \tag{5.11}$$

To handle $I_{6,3}$, we use $\operatorname{div} v = \operatorname{div}_h \bar{u}^h = 0$ to write

$$\begin{aligned} I_{6,3} &= \int_0^t \left(\Delta_\ell^v \nabla_h(-\Delta)^{-1} \partial_3 \operatorname{div} (v_\lambda v^3) + \Delta_\ell^v \nabla_h(-\Delta)^{-1} \partial_3 \operatorname{div}_h (\bar{u}^h v_\lambda^3) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt' \\ &= \int_0^t \left(\nabla_h(-\Delta)^{-1} \Delta_\ell^v (\operatorname{div}_h (v^3 \partial_3 v_\lambda^h + v^h \partial_3 v_\lambda^3) + 2\partial_3(v^3 \partial_3 v_\lambda^3)) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt' \\ &\quad + \int_0^t \left(\nabla_h(-\Delta)^{-1} \operatorname{div}_h \Delta_\ell^v (v^3 \partial_3 \bar{u}_\lambda^h + \bar{u}^h \partial_3 v_\lambda^3) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt' \\ &= \int_0^t \left(\nabla_h(-\Delta)^{-1} \Delta_\ell^v (\operatorname{div}_h (v^3 \partial_3 v_\lambda^h - v^h \operatorname{div}_h v_\lambda^h) - 2\partial_3(v^3 \operatorname{div}_h v_\lambda^h)) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt' \\ &\quad + \int_0^t \left(\nabla_h(-\Delta)^{-1} \operatorname{div}_h \Delta_\ell^v (v^3 \partial_3 \bar{u}_\lambda^h - \bar{u}^h \operatorname{div}_h v_\lambda^h) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} dt'. \end{aligned}$$

Applying (4.11) with $A(D) = \nabla_h(-\Delta)^{-1}\partial_3$, $a = v^3$, $b = \operatorname{div}_h v_\lambda^h$ and $c = v_\lambda^h$ yields

$$\begin{aligned} &\int_0^t \left| \left(\nabla_h(-\Delta)^{-1} \partial_3 \Delta_\ell^v (v^3 \operatorname{div}_h v_\lambda^h) \mid \Delta_\ell^v v_\lambda^h \right)_{L^2} \right| dt' \\ &\lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}. \end{aligned}$$

The remaining terms in $I_{6,3}$ can be handled along the same lines as to those of $I_{6,1}$ and $I_{6,2}$. As a consequence, we obtain

$$\begin{aligned} |I_{6,3}| &\lesssim d_\ell^2 2^{-\ell} \left(\|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2 \right. \\ &\quad \left. + \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \left(\|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \right) \right). \end{aligned} \quad (5.12)$$

To deal with $I_{6,4}$, it is crucial to observe that

$$\Delta_\ell^\vee \nabla_h ((-\Delta)^{-1} - (-\Delta_h)^{-1}) \partial_i \partial_j (\bar{u}^i \bar{u}_\lambda^j) = \Delta_\ell^\vee \nabla_h \partial_3^2 (-\Delta)^{-1} (-\Delta_h)^{-1} \partial_i \partial_j (\bar{u}^i \bar{u}_\lambda^j).$$

Then due to the fact that $\sum_{i,j=1}^2 \nabla_h \partial_3 (-\Delta)^{-1} (-\Delta_h)^{-1} \partial_i \partial_j$ is a bounded Fourier multiplier, we get, by applying (4.11) with $a = \bar{u}^h$, $b = \partial_3 \bar{u}_\lambda^h$, $c = v_\lambda^h$, that

$$\begin{aligned} |I_{6,4}| &\leq 2 \sum_{i=1}^2 \sum_{j=1}^3 \int_0^t |(\Delta_\ell^\vee \nabla_h \partial_3 (-\Delta)^{-1} (-\Delta_h)^{-1} \partial_i \partial_j (\bar{u}^i \partial_3 \bar{u}_\lambda^j) | \Delta_\ell^\vee v_\lambda^h |_{L^2} dt' \\ &\lesssim d_\ell^2 2^{-\ell} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}. \end{aligned} \quad (5.13)$$

By summing up (5.10–5.13), we arrive at

$$\begin{aligned} |I_6| &\lesssim d_\ell^2 2^{-\ell} \left(\|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2 \right. \\ &\quad \left. + \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \left(\|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \right) \right). \end{aligned} \quad (5.14)$$

Now we are in a position to complete the proof of (3.17).

Proof of (3.17). By inserting the estimates (5.3–5.6), (5.8) and (5.14) into (5.2), we achieve

$$\begin{aligned} &\frac{1}{2} \|\Delta_\ell^\vee v_\lambda^h(t)\|_{L^2}^2 + \lambda \int_0^t f(t') \|\Delta_\ell^\vee v_\lambda^h(t')\|_{L^2}^2 dt' + \int_0^t \|\nabla_h \Delta_\ell^\vee v_\lambda^h(t')\|_{L^2}^2 dt' \\ &\leq \frac{1}{2} \|\Delta_\ell^\vee v_0^h\|_{L^2}^2 + C d_\ell^2 2^{-\ell} \left(\|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^2 \right. \\ &\quad \left. + \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \left(\|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}} \right) \right). \end{aligned}$$

Multiplying the above inequality by $2^{\ell+1}$ and taking square root of the resulting inequality, and then summing up the inequalities over \mathbb{Z} , we find that

$$\begin{aligned} &\|v_\lambda^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \sqrt{2\lambda} \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})} + \sqrt{2} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \\ &\leq \|v_0^h\|_{\mathcal{B}^{0,\frac{1}{2}}} + C \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \\ &\quad + C \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{4}} \left(\|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{4}} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{4}} \right). \end{aligned} \quad (5.15)$$

It follows from Young’s inequality that

$$\begin{aligned} & C \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{4}} \left(\|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{4}} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{4}} \right) \\ & \leq \frac{1}{10} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} + C \|v_\lambda^h\|_{\tilde{L}_{t,f}^2(\mathcal{B}^{0,\frac{1}{2}})}. \end{aligned}$$

Inserting the above inequality into (5.15) and taking λ so that $\sqrt{2\lambda} = C$, we obtain

$$\begin{aligned} \|v_\lambda^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})} + \frac{5}{4} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} & \leq \|v_0^h\|_{\mathcal{B}^{0,\frac{1}{2}}} + \|\partial_3 \bar{u}_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})} \\ & \quad + C \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\lambda^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}, \end{aligned}$$

which, together with the following consequence of (5.1):

$$\|a\|_{\tilde{L}_t^p(\mathcal{B}^{0,\frac{1}{2}})} \exp\left(-\lambda \int_0^t f(t') dt'\right) \leq \|a_\lambda\|_{\tilde{L}_t^p(\mathcal{B}^{0,\frac{1}{2}})} \quad \text{for } p = 2 \text{ or } \infty,$$

gives rise to (3.17). \square

6. The Estimate of the Vertical Component v^3

The purpose of this section is to present the proof of (3.18). Compared with [17], where the third component of the velocity field can be estimated in the standard Besov spaces, here, due to the additional terms like $\bar{u}^h \cdot \nabla_h v$ that appears in (3.14), we will have to use the weighted Chemin–Lerner norms once again. Indeed for any constant $\mu > 0$, we denote

$$w_\mu(t) \stackrel{\text{def}}{=} w(t)\bar{g}(t) \quad \text{with} \quad \bar{g}(t) \stackrel{\text{def}}{=} \exp\left(-\mu \int_0^t \bar{h}(t') dt'\right) \quad \text{and} \quad \bar{h}(t) \stackrel{\text{def}}{=} \|\bar{u}^h(t)\|_{\mathcal{B}_4^{0,\frac{1}{2}}}^4, \tag{6.1}$$

and similar notations for v_μ , \bar{u}_μ^h , and p_μ .

By multiplying $\bar{g}(t)$ to (3.16), we write

$$\partial_t w_\mu + \mu \bar{h}(t) w_\mu - \Delta_h w_\mu + v \cdot \nabla v_\mu^3 + \bar{u}^h \cdot \nabla_h v_\mu^3 = -\partial_3 p_\mu.$$

By applying Δ_ℓ^v to the above equation and taking L^2 inner product of the resulting equation with $\Delta_\ell^v w_\mu$, and then integrating the equality over $[0, t]$, we obtain

$$\frac{1}{2} \|\Delta_\ell^v w_\mu(t)\|_{L^2}^2 + \mu \|\sqrt{\bar{h}} \Delta_\ell^v w_\mu\|_{L_t^2(L^2)}^2 + \|\nabla_h \Delta_\ell^v w_\mu\|_{L_t^2(L^2)}^2 = \frac{1}{2} \|\Delta_\ell^v u_{0,\text{th}}^3\|_{L^2}^2 - \sum_{i=1}^6 \Pi_i, \tag{6.2}$$

where

$$\Pi_1 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(\bar{u}^h \cdot \nabla_h w_\mu) | \Delta_\ell^v w_\mu)_{L^2} dt', \quad \Pi_2 \stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^v(v^h \cdot \nabla_h w_\mu) | \Delta_\ell^v w_\mu)_{L^2} dt',$$

$$\begin{aligned} \Pi_3 &\stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^\vee(v_\mu^h \cdot \nabla_h v_F) | \Delta_\ell^\vee w_\mu)_{L^2} dt', & \Pi_4 &\stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^\vee(\bar{u}_\mu^h \cdot \nabla_h v_F) | \Delta_\ell^\vee w_\mu)_{L^2} dt', \\ \Pi_5 &\stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^\vee(v^3 \partial_3 v_\mu^3) | \Delta_\ell^\vee w_\mu)_{L^2} dt', & \Pi_6 &\stackrel{\text{def}}{=} \int_0^t (\Delta_\ell^\vee \partial_3 p_\mu | \Delta_\ell^\vee w_\mu)_{L^2} dt'. \end{aligned}$$

Let us handle the above term by term.

• The estimates of Π_1 and Π_2

We first get, by applying (4.11) with $a = \bar{u}^h$, $b = \nabla_h w_\mu$ and $c = w_\mu$, that

$$|\Pi_1| \lesssim d_\ell^2 2^{-\ell} \|w_\mu\|_{\tilde{L}_{t,h}^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}, \tag{6.3}$$

whereas by applying a modified version of (4.13) with $a = v^h$, $b = \nabla_h w_\mu$, $c = w_\mu$ and $\mathfrak{g}(t) = \exp(-\mu \int_0^t \tilde{h}(t') dt')$, we find

$$|\Pi_2| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v^h\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|w_\mu\|_{\tilde{L}_t^\infty(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0,\frac{1}{2}})}^{\frac{3}{2}}. \tag{6.4}$$

• The estimate of Π_3

The estimate of Π_3 relies on the following lemma, the proof of which will be postponed until the ‘‘Appendix A’’:

Lemma 6.1. *Let $a, c \in \mathcal{B}^{0,\frac{1}{2}}(T)$ and $b \in \mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)$. Then for any smooth homogeneous Fourier multiplier, $A(D)$, of degree zero and any $\ell \in \mathbb{Z}$, it holds that*

$$\int_0^T |(A(D)\Delta_\ell^\vee(a \otimes b) | \Delta_\ell^\vee c)_{L^2}| dt' \lesssim d_\ell^2 2^{-\ell} \|a\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})}, \tag{6.5}$$

and

$$\int_0^T |(A(D)\Delta_\ell^\vee(a \otimes b) | \Delta_\ell^\vee c)_{L^2}| dt' \lesssim d_\ell^2 2^{-\ell} \|a\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})} \|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \|c\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}. \tag{6.6}$$

Remark 6.1. Indeed the proof of Lemma 6.1 shows that $\|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)}$ in (6.5) and (6.6) can be replaced by $\|b\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}$.

Let us admit this lemma temporarily, and continue our estimate of Π_3 . By using integration by parts, we write

$$\Pi_3 = - \int_0^t (\Delta_\ell^\vee(\text{div}_h v_\mu^h \cdot v_F) | \Delta_\ell^\vee w_\mu)_{L^2} dt' - \int_0^t (\Delta_\ell^\vee(v_\mu^h \otimes v_F) | \Delta_\ell^\vee \nabla_h w_\mu)_{L^2} dt'. \tag{6.7}$$

Applying (6.6) with $a = \text{div}_h v_\mu^h$, $b = v_F$ and $c = w_\mu$ yields

$$\begin{aligned} & \left| \int_0^t (\Delta_\ell^v (\operatorname{div}_h v_\mu^h \cdot v_F) | \Delta_\ell^v w_\mu)_{L^2} dt' \right| \\ & \lesssim d_\ell^2 2^{-\ell} \|\nabla_h v_\mu^h\|_{\tilde{L}_t^2(\mathcal{B}_4^{0, \frac{1}{2}})} \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}, \end{aligned} \tag{6.8}$$

whereas by applying (6.5) with $a = v_\mu^h$, $b = v_F$ and $c = \nabla_h w_\mu$, we obtain

$$\begin{aligned} & \left| \int_0^t (\Delta_\ell^v (v_\mu^h \otimes v_F) | \Delta_\ell^v \nabla_h w_\mu)_{L^2} dt' \right| \\ & \lesssim d_\ell^2 2^{-\ell} \|v_\mu^h\|_{\tilde{L}_t^4(\mathcal{B}_4^{0, \frac{1}{2}})} \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

Inserting the above two estimates into (6.7) and using (2.6), we achieve

$$|\Pi_3| \lesssim d_\ell^2 2^{-\ell} \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \|v_\mu^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}. \tag{6.9}$$

• The estimate of Π_4

Due to $\operatorname{div}_h \bar{u}^h = 0$, by using integration by parts, we write

$$\begin{aligned} \Pi_4 &= \int_0^t (\Delta_\ell^v \operatorname{div}_h (\bar{u}^h v_F) | \Delta_\ell^v w_\mu)_{L^2} \bar{g}(t') dt' \\ &= - \int_0^t (\Delta_\ell^v (\bar{u}^h v_F) | \Delta_\ell^v \nabla_h w_\mu)_{L^2} \bar{g}(t') dt'. \end{aligned}$$

By applying Bony’s decomposition (2.7), we get

$$\Pi_4 = - \int_0^t (\Delta_\ell^v (T_{\bar{u}^h}^v v_F + R^v(\bar{u}^h, v_F)) | \Delta_\ell^v \nabla_h w_\mu)_{L^2} \bar{g}(t') dt'.$$

We first observe that

$$\begin{aligned} & \int_0^t |(\Delta_\ell^v (R^v(\bar{u}^h, v_F)) | \Delta_\ell^v \nabla_h w_\mu)_{L^2}| \bar{g}(t') dt' \\ & \lesssim \sum_{\ell' \geq \ell - N_0} \int_0^t \bar{g}(t') \|\Delta_{\ell'}^v \bar{u}^h(t')\|_{L_h^4(L_v^\infty)} \|S_{\ell'+2}^v v_F(t')\|_{L_h^4(L_v^\infty)} \|\Delta_\ell^v \nabla_h w_\mu(t')\|_{L^2} dt' \\ & \lesssim \sum_{\ell' \geq \ell - N_0} 2^{-\frac{\ell'}{2}} \int_0^t d_{\ell'}(t') \bar{g}(t') \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|v_F(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|\Delta_\ell^v \nabla_h w_\mu(t')\|_{L^2} dt' \\ & \lesssim \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \int_0^t \bar{g}(t') \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|v_F(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|\Delta_\ell^v \nabla_h w_\mu(t')\|_{L^2} dt', \end{aligned}$$

and applying Hölder’s inequality and Proposition 2.1 gives

$$\begin{aligned} & \int_0^t |(\Delta_\ell^v (R^v(\bar{u}^h, v_F)) | \Delta_\ell^v \nabla_h w_\mu)_{L^2}| \bar{g}(t') dt' \\ & \lesssim \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \left(\int_0^t \bar{g}^4(t') \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 dt' \right)^{\frac{1}{4}} \|v_F\|_{L_t^4(\mathcal{B}_4^{0, \frac{1}{2}})} \|\Delta_\ell^v \nabla_h w_\mu\|_{L_t^2(L^2)} \end{aligned}$$

$$\lesssim \mu^{-\frac{1}{4}} d_\ell^2 2^{-\ell} \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \|\nabla_h w_\mu\|_{L_t^2(\mathcal{B}^{0, \frac{1}{2}})}.$$

Along the same lines, we find

$$\begin{aligned} & \int_0^t |(\Delta_\ell^v(T_{\bar{u}^h}^v v_F) | \Delta_\ell^v \nabla_h w_\mu)_{L^2}| \bar{g}(t') dt' \\ & \lesssim \sum_{|\ell' - \ell| \leq 5} \int_0^t \bar{g}(t') \|S_{\ell' - 1}^v \bar{u}^h(t')\|_{L_h^4(L_v^\infty)} \|\Delta_{\ell'}^v v_F(t')\|_{L_h^4(L_v^2)} \|\Delta_{\ell'}^v \nabla_h w_\mu(t')\|_{L^2} dt' \\ & \lesssim \sum_{|\ell' - \ell| \leq 5} \int_0^t \bar{g}(t') \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|\Delta_{\ell'}^v v_F(t')\|_{L_h^4(L_v^2)} \|\Delta_{\ell'}^v \nabla_h w_\mu(t')\|_{L^2} dt' \\ & \lesssim \sum_{|\ell' - \ell| \leq 5} \left(\int_0^t \bar{g}^4(t') \|\bar{u}^h(t')\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 dt' \right)^{\frac{1}{4}} \|\Delta_{\ell'}^v v_F\|_{L_t^4(L_h^4(L_v^2))} \|\Delta_{\ell'}^v \nabla_h w_\mu\|_{L_t^2(L^2)} \\ & \lesssim \mu^{-\frac{1}{4}} d_\ell^2 2^{-\ell} \|v_F\|_{\tilde{L}_t^4(\mathcal{B}_4^{0, \frac{1}{2}})} \|\nabla_h w_\mu\|_{L_t^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

As a result, it turns out that

$$|\Pi_4| \lesssim \mu^{-\frac{1}{4}} d_\ell^2 2^{-\ell} \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}. \tag{6.10}$$

• The estimates of Π_5

Due to $\partial_3 v^3 = -\operatorname{div}_h v^h$ and $v^3 = w + v_F$, we write

$$\begin{aligned} \Pi_5 &= \int_0^t (\Delta_\ell^v(-v^3 \operatorname{div}_h v^h) | \Delta_\ell^v w_\mu)_{L^2} dt' \\ &= - \int_0^t (\Delta_\ell^v(v_F \operatorname{div}_h v^h + w_\mu \operatorname{div}_h v^h) | \Delta_\ell^v w_\mu)_{L^2}. \end{aligned}$$

Then applying (6.6) gives rise to

$$\begin{aligned} |\Pi_5| &\lesssim d_\ell^2 2^{-\ell} \|\nabla_h v^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \left(\|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \\ &\lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \left(\|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}. \end{aligned} \tag{6.11}$$

• The estimates of Π_6

The estimate of Π_6 can be handled similarly as Π_6 . Indeed in view of (5.9), we write

$$\begin{aligned} \partial_3 p &= \partial_3(-\Delta)^{-1} \operatorname{div}_h (v^h \cdot \nabla_h v^h + \bar{u}^h \cdot \nabla_h v^h + v^h \cdot \nabla_h \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h \\ &\quad + v^3 \partial_3 \bar{u}^h + v^3 \partial_3 v^h) + \partial_3^2(-\Delta)^{-1} (v \cdot \nabla v^3 + \bar{u}^h \cdot \nabla_h v^3). \end{aligned}$$

Accordingly, we have the decomposition $\Pi_6 = \sum_{i=1}^5 \Pi_{6,i}$ with

$$\Pi_{6,1} = \int_0^t (\Delta_\ell^v \partial_3(-\Delta)^{-1} \operatorname{div}_h (v^h \cdot \nabla_h v_\mu^h + \bar{u}^h \cdot \nabla_h v_\mu^h + v_\mu^h \cdot \nabla_h \bar{u}^h) | \Delta_\ell^v w_\mu)_{L^2} dt',$$

$$\begin{aligned} \Pi_{6,2} &= \int_0^t (\Delta_\ell^\vee \partial_3 (-\Delta)^{-1} \operatorname{div}_h (v^3 \partial_3 v_\mu^h) \mid \Delta_\ell^\vee w_\mu)_{L^2} dt', \\ \Pi_{6,3} &= \int_0^t (\Delta_\ell^\vee \partial_3 (-\Delta)^{-1} \operatorname{div}_h (v_\mu^3 \partial_3 \bar{u}^h) \mid \Delta_\ell^\vee w_\mu)_{L^2} dt', \\ \Pi_{6,4} &= \int_0^t (\Delta_\ell^\vee \partial_3^2 (-\Delta)^{-1} (v \cdot \nabla v_\mu^3 + \bar{u}^h \cdot \nabla_h v_\mu^3) \mid \Delta_\ell^\vee w_\mu)_{L^2} dt', \\ \Pi_{6,5} &= \sum_{i=1}^2 \sum_{j=1}^2 \int_0^t (2\Delta_\ell^\vee (-\Delta)^{-1} \partial_i \partial_j (\bar{u}^i \partial_3 \bar{u}^j) \mid \Delta_\ell^\vee w_\mu)_{L^2} dt'. \end{aligned}$$

It is easy to observe from the estimate of $I_{6,1}$ that

$$\begin{aligned} |\Pi_{6,1}| &\lesssim d_\ell^2 2^{-\ell} \left(\|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{2}} \|w_\mu\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \right. \\ &\quad \left. + \|w_\mu\|_{\tilde{L}_{t,h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v_\mu^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \right). \end{aligned} \quad (6.12)$$

Mean while, by using $\partial_3 v^3 = -\operatorname{div}_h v^h$ and integration by parts, we write

$$\begin{aligned} \Pi_{6,2} &= \int_0^t (\Delta_\ell^\vee \partial_3 (-\Delta)^{-1} \operatorname{div}_h [\partial_3 (v^3 v_\mu^h) - v_\mu^h \partial_3 v^3] \mid \Delta_\ell^\vee w_\mu)_{L^2} dt' \\ &= - \int_0^t (\Delta_\ell^\vee (-\Delta)^{-1} \partial_3^2 (v^3 v_\mu^h) \mid \Delta_\ell^\vee \nabla_h w_\mu)_{L^2} dt' \\ &\quad + \int_0^t (\Delta_\ell^\vee \partial_3 (-\Delta)^{-1} \operatorname{div}_h (v_\mu^h \operatorname{div}_h v^h) \mid \Delta_\ell^\vee w_\mu)_{L^2} dt' \stackrel{\text{def}}{=} \Pi_{6,2}^a + \Pi_{6,2}^b. \end{aligned}$$

It follows from (6.5) and $v^3 = v_F + w$ that

$$|\Pi_{6,2}^a| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \left(\|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)},$$

whereas by using a modified version of (4.13), we infer

$$|\Pi_{6,2}^b| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{2}} \|w_\mu\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h w_\mu\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}.$$

Therefore, we obtain

$$|\Pi_{6,2}| \lesssim d_\ell^2 2^{-\ell} \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \left(\|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}, \quad (6.13)$$

whereas applying (6.6) with $a = \partial_3 \bar{u}^h$, $b = v_\mu^3$ and $c = w_\mu$ leads to

$$|\Pi_{6,3}| \lesssim d_\ell^2 2^{-\ell} \|\partial_3 \bar{u}^h\|_{\tilde{L}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \left(\|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}. \quad (6.14)$$

On the other hand, again due to $\operatorname{div} v = 0$, we write

$$\begin{aligned} \Pi_{6,4} = & \int_0^t \left(\Delta_\ell^\vee \partial_3^2 (-\Delta)^{-1} (v^h \cdot \nabla_h w_\mu + v_\mu^h \cdot \nabla_h v_F + v^3 \partial_3 v_\mu^3 \right. \\ & \left. + \bar{u}^h \cdot \nabla_h w_\mu + \bar{u}^h \cdot \nabla_h v_F) \mid \Delta_\ell^\vee w_\mu \right)_{L^2} dt'. \end{aligned}$$

Noticing that $(-\Delta)^{-1} \partial_3^2$ is a bounded Fourier operator, we observe that $\Pi_{6,4}$ shares the same estimate as $\sum_{i=1}^5 \Pi_i$ given before, that is,

$$\begin{aligned} |\Pi_{6,4}| \lesssim & d_\ell^2 2^{-\ell} \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^2 + \|w_\mu\|_{\tilde{L}_{i,h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{3}{2}} \right. \\ & \left. + \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \left(\mu^{-\frac{1}{4}} + \|v_\mu^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right). \end{aligned} \tag{6.15}$$

Finally since $(-\Delta)^{-1} \partial_i \partial_j$ is a bounded Fourier operator, we get, by applying (4.11) with $a = \bar{u}^h$, $b = \partial_3 \bar{u}_\mu^h$, $c = w_\mu$, that

$$|\Pi_{6,5}| \lesssim d_\ell^2 2^{-\ell} \|w_\mu\|_{\tilde{L}_{i,h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h w_\mu\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\partial_3 \bar{u}_\mu^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}. \tag{6.16}$$

By summing (6.12–6.16), we arrive at

$$\begin{aligned} |\Pi_6| \lesssim & d_\ell^2 2^{-\ell} \left((\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}) \|w_\mu\|_{\tilde{L}_{i,h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} \right. \\ & + (\mu^{-\frac{1}{4}} + \|\partial_3 \bar{u}_\mu^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \\ & \left. + (\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^2 + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^2 \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right). \end{aligned} \tag{6.17}$$

Let us now complete the proof of (3.18).

Proof of (3.18). By inserting the estimates (6.3), (6.4), (6.9–6.11) and (6.17) into (6.2), and then multiplying $2^{\ell+1}$ to the resulting inequality, and finally taking square root and then summing up the resulting inequalities over \mathbb{Z} , we obtain

$$\begin{aligned} & \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \sqrt{2\mu} \|w_\mu\|_{\tilde{L}_{i,h}^2(\mathcal{B}^{0, \frac{1}{2}})} \\ & \leq \|u_{0,\text{lh}}^3\|_{\mathcal{B}^{0, \frac{1}{2}}} + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \\ & + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + (\mu^{-\frac{1}{8}} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} \\ & + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} \right) \|w_\mu\|_{\tilde{L}_{i,h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{4}} \|\nabla_h w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{4}}. \end{aligned}$$

Applying Young’s inequality gives

$$\begin{aligned} & C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + (\mu^{-\frac{1}{8}} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} \\ & \leq \frac{1}{12} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + C \left(\mu^{-\frac{1}{4}} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{L}_i^2(\mathcal{B}^{0, \frac{1}{2}})} \right) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} + C \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^2, \end{aligned}$$

and

$$\begin{aligned}
 & C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} + \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} \right) \|w_\mu\|_{\tilde{\mathcal{L}}_{t,h}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{4}} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{4}} \\
 & \leq \frac{1}{12} \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + C \|w_\mu\|_{\tilde{\mathcal{L}}_{t,h}^2(\mathcal{B}^{0, \frac{1}{2}})} + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \right).
 \end{aligned}$$

As a result, we have

$$\begin{aligned}
 & \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \sqrt{2\mu} \|w_\mu\|_{\tilde{\mathcal{L}}_{t,h}^2(\mathcal{B}^{0, \frac{1}{2}})} \leq \|u_{0,\text{lh}}^3\|_{\mathcal{B}^{0, \frac{1}{2}}} + C \|w_\mu\|_{\tilde{\mathcal{L}}_{t,h}^2(\mathcal{B}^{0, \frac{1}{2}})} \\
 & + \left(\frac{1}{6} + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \right) \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \right. \\
 & \left. + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^2 + (\mu^{-\frac{1}{4}} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})}) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)} \right).
 \end{aligned}$$

Taking μ in the above inequality so that $\sqrt{2\mu} = C$ gives rise to

$$\begin{aligned}
 & \left(\frac{5}{6} - C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^{\frac{1}{2}} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \right) \right) \|w_\mu\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} \\
 & \leq \|u_{0,\text{lh}}^3\|_{\mathcal{B}^{0, \frac{1}{2}}} + C \left(\|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})} + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)}^2 \right) \tag{6.18} \\
 & + \left(1 + \|v^h\|_{\mathcal{B}^{0, \frac{1}{2}}(t)} + \|\partial_3 \bar{u}^h\|_{\tilde{\mathcal{L}}_t^2(\mathcal{B}^{0, \frac{1}{2}})} \right) \|v_F\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}(t)}.
 \end{aligned}$$

On the other hand, in view of the definition of $u_{0,\text{lh}}^3$, it holds for any $\ell \in \mathbb{Z}$ that

$$\|\Delta_\ell^v u_{0,\text{lh}}^3\|_{L^2} \lesssim \sum_{|j-\ell| \leq 1} \|S_{j-1}^h \Delta_j^v u_0^3\|_{L^2} \lesssim d_\ell 2^{-\frac{\ell}{2}} \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}},$$

which indicates that

$$\|u_{0,\text{lh}}^3\|_{\mathcal{B}^{0, \frac{1}{2}}} \lesssim \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}.$$

Inserting the above estimate into (6.18) and repeating the argument from (4.18) to (4.19), we conclude the proof of (3.18). \square

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Appendix A. The Proof of Lemmas 4.2 and 6.1

In this section, we present the proof of Lemmas 4.2 and 6.1.

Proof of Lemma 4.2. By applying Bony’s decomposition in the vertical variable (2.7) to $a \otimes b$, we write

$$\begin{aligned} \int_0^T (\Delta_\ell^v A(D)(a \otimes b) | \Delta_\ell^v c)_{L^2} dt &= Q_1 + Q_2 \quad \text{with} \\ Q_1 &\stackrel{\text{def}}{=} \int_0^T (\Delta_\ell^v A(D)(T_a^v b) | \Delta_\ell^v c)_{L^2} dt = \int_0^T (\Delta_\ell^v (T_a^v b) | A(D) \Delta_\ell^v c)_{L^2} dt \quad \text{and} \\ Q_2 &\stackrel{\text{def}}{=} \int_0^T (\Delta_\ell^v A(D)R^v(a, b) | \Delta_\ell^v c)_{L^2} dt = \int_0^T (\Delta_\ell^v R^v(a, b) | A(D) \Delta_\ell^v c)_{L^2} dt. \end{aligned} \tag{A.1}$$

Considering the support properties to the Fourier transform of the terms in $T_a^v b$, and noting that $A(D)$ is a smooth homogeneous Fourier multiplier of degree zero, we find

$$\begin{aligned} |Q_1| &\leq \int_0^T \|\Delta_\ell^v (T_a^v b)\|_{L_h^{\frac{4}{3}}(L_v^2)} \|A(D) \Delta_\ell^v c\|_{L_h^4(L_v^2)} dt \\ &\lesssim \sum_{|\ell' - \ell| \leq 5} \int_0^T \|S_{\ell' - 1}^v a\|_{L_h^4(L_v^\infty)} \|\Delta_{\ell'}^v b\|_{L^2} \|A(D) \Delta_\ell^v c\|_{L^2}^{\frac{1}{2}} \|\nabla_h A(D) \Delta_\ell^v c\|_{L^2}^{\frac{1}{2}} dt \\ &\lesssim \sum_{|\ell' - \ell| \leq 5} \left(\int_0^T \|S_{\ell' - 1}^v a(t)\|_{L_h^4(L_v^\infty)}^4 \|\Delta_{\ell'}^v c(t)\|_{L^2}^2 dt \right)^{\frac{1}{4}} \|\Delta_{\ell'}^v b\|_{L_T^2(L^2)} \|\nabla_h \Delta_\ell^v c\|_{L_T^2(L^2)}^{\frac{1}{2}}. \end{aligned}$$

It follows from Lemma 2.1 and Definition 2.4 that

$$\begin{aligned} \|S_{\ell' - 1}^v a(t)\|_{L_h^4(L_v^\infty)} &\leq \sum_{j \leq \ell' - 2} \|\Delta_j^v a(t)\|_{L_h^4(L_v^\infty)} \\ &\lesssim \sum_{j \leq \ell' - 2} 2^{\frac{j}{2}} \|\Delta_j^v a(t)\|_{L_h^4(L_v^2)} \lesssim \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}. \end{aligned}$$

This together with Definition 2.2 ensures that

$$|Q_1| \lesssim d_\ell^2 2^{-\ell} \|c\|_{\tilde{L}_{T,1}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \tag{A.2}$$

Along the same lines, we get, by applying (2.5), that

$$\begin{aligned} |Q_{1, \mathfrak{g}}| &\stackrel{\text{def}}{=} \int_0^T |(\Delta_\ell^v (T_a^v b) | A(D) \Delta_\ell^v c)_{L^2}| \mathfrak{g}^2 dt \\ &\lesssim \sum_{|\ell' - \ell| \leq 5} \|\sqrt{\mathfrak{g}} S_{\ell' - 1}^v a\|_{L_T^4(L_h^4(L_v^\infty))} \|\mathfrak{g} \Delta_{\ell'}^v b\|_{L_T^2(L^2)} \|\Delta_\ell^v c\|_{L_T^\infty(L^2)}^{\frac{1}{2}} \|\mathfrak{g} \nabla_h \Delta_\ell^v c\|_{L_T^2(L^2)}^{\frac{1}{2}} \\ &\lesssim d_\ell^2 2^{-\ell} \|a\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\mathfrak{g} \nabla_h a\|_{L_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\mathfrak{g} b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|c\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\mathfrak{g} \nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned} \tag{A.3}$$

On the other hand, once again considering the support properties to the Fourier transform of the terms in $R^\nu(a, b)$, we find

$$\begin{aligned} |Q_2| &\leq \int_0^T \|\Delta_\ell^\nu R^\nu(a, b)\|_{L_h^4(L_v^2)} \|A(D)\Delta_\ell^\nu c\|_{L_h^4(L_v^2)} dt \\ &\lesssim \sum_{\ell' \geq \ell - N_0} \int_0^T \|\Delta_{\ell'}^\nu a\|_{L_h^4(L_v^2)} \|S_{\ell'+2}^\nu b\|_{L_h^2(L_v^\infty)} \|A(D)\Delta_\ell^\nu c\|_{L^2}^{\frac{1}{2}} \|\nabla_h A(D)\Delta_\ell^\nu c\|_{L^2}^{\frac{1}{2}} dt \\ &\lesssim \sum_{\ell' \geq \ell - N_0} 2^{-\frac{\ell'}{2}} \int_0^T d_{\ell'}(t) \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|b(t)\|_{L_h^2(L_v^\infty)} \|\Delta_\ell^\nu c(t)\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta_\ell^\nu c(t)\|_{L^2}^{\frac{1}{2}} dt \\ &\lesssim \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|b(t)\|_{L_h^2(L_v^\infty)} \|\Delta_\ell^\nu c(t)\|_{L^2}^{\frac{1}{2}} \|\nabla_h \Delta_\ell^\nu c(t)\|_{L^2}^{\frac{1}{2}} dt. \end{aligned}$$

It follows from Lemma 2.1 however that

$$\|b\|_{L_T^2(L_h^2(L_v^\infty))} \lesssim \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^\nu b\|_{L_T^2(L^2)} \leq \|b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}.$$

As a result, by virtue of Definition 2.2, we obtain

$$\begin{aligned} |Q_2| &\lesssim \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \left(\int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 \|\Delta_\ell^\nu c(t)\|_{L^2}^2 dt \right)^{\frac{1}{4}} \|\nabla_h \Delta_\ell^\nu c\|_{L_T^2(L^2)}^{\frac{1}{2}} \|b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \\ &\lesssim \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \left(d_\ell 2^{-\frac{\ell}{2}} \|c\|_{\tilde{L}_{T,f}^2(\mathcal{B}^{0, \frac{1}{2}})} \right)^{\frac{1}{2}} (d_\ell 2^{-\frac{\ell}{2}} \|\nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})})^{\frac{1}{2}} \|b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \\ &\lesssim d_\ell^2 2^{-\ell} \|c\|_{\tilde{L}_{T,f}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned} \tag{A.4}$$

Similarly, thanks to (2.5), one has

$$\begin{aligned} |Q_{2,g}| &\stackrel{\text{def}}{=} \int_0^T |(\Delta_\ell^\nu R^\nu(a, b) | A(D)\Delta_\ell^\nu c)_{L^2}| g^2 dt \\ &\lesssim \sum_{\ell' \geq \ell - N_0} \|\sqrt{g} \Delta_{\ell'}^\nu a\|_{L_T^4(L_h^4(L_v^2))} \|g S_{\ell'+2}^\nu b\|_{L_T^2(L_h^2(L_v^\infty))} (\|\Delta_\ell^\nu c\|_{L_T^2(L^2)} \|g \nabla_h \Delta_\ell^\nu c\|_{L_T^2(L^2)})^{\frac{1}{2}} \\ &\lesssim d_\ell^2 2^{-\ell} \|a\|_{\tilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|g \nabla_h a\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|g b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|c\|_{\tilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|g \nabla_h c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned} \tag{A.5}$$

Combining (A.2) with (A.4) gives (4.11), and (4.13) follows from (A.3) and (A.5). It remains to prove (4.12). Similarly to the proof of (A.2), we write

$$\begin{aligned} |Q_1| &\lesssim \sum_{|\ell' - \ell| \leq 5} \int_0^T \|S_{\ell'-1}^\nu a\|_{L_h^4(L_v^\infty)} \|\Delta_{\ell'}^\nu b\|_{L_h^4(L_v^2)} \|A(D)\Delta_\ell^\nu c\|_{L^2} dt \\ &\lesssim \sum_{|\ell' - \ell| \leq 5} \int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|\Delta_{\ell'}^\nu b(t)\|_{L^2}^{\frac{1}{2}} \|\Delta_{\ell'}^\nu \nabla_h b(t)\|_{L^2}^{\frac{1}{2}} \|\Delta_\ell^\nu c(t)\|_{L^2} dt \end{aligned}$$

$$\lesssim \sum_{|\ell' - \ell| \leq 5} \left(\int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 \|\Delta_{\ell'}^{\vee} b(t)\|_{L^2}^2 dt \right)^{\frac{1}{4}} \|\Delta_{\ell'}^{\vee} \nabla_{\text{h}} b\|_{L_T^2(L^2)}^{\frac{1}{2}} \|\Delta_{\ell}^{\vee} c\|_{L_T^2(L^2)},$$

from which, along with Definition 2.2, we infer

$$\begin{aligned} |Q_1| &\lesssim d_{\ell} 2^{-\frac{\ell}{2}} \sum_{|\ell' - \ell| \leq 5} d_{\ell'} 2^{-\frac{\ell'}{2}} \|b\|_{\tilde{L}_{T, f}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_{\text{h}} b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \\ &\lesssim d_{\ell}^2 2^{-\ell} \|b\|_{\tilde{L}_{T, f}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_{\text{h}} b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned} \tag{A.6}$$

We deduce from Definition 2.4 that

$$\begin{aligned} |Q_2| &\lesssim \sum_{\ell' \geq \ell - N_0} \int_0^T \|\Delta_{\ell'}^{\vee} a\|_{L_{\text{h}}^4(L_{\vee}^2)} \|S_{\ell'+2}^{\vee} b\|_{L_{\text{h}}^4(L_{\vee}^{\infty})} \|A(D) \Delta_{\ell}^{\vee} c\|_{L^2} dt \\ &\lesssim \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}} \|b(t)\|_{L_{\text{h}}^4(L_{\vee}^{\infty})} \|\Delta_{\ell}^{\vee} c(t)\|_{L^2} dt \\ &\lesssim d_{\ell} 2^{-\frac{\ell}{2}} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \sum_{\ell' \geq \ell - 4} d_{\ell'} 2^{-\frac{\ell'}{2}} \left(\int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^2 \|b(t)\|_{L_{\text{h}}^4(L_{\vee}^{\infty})}^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

whereas we get, by applying the triangle inequality and Lemma 2.1, that

$$\begin{aligned} &\left(\int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^2 \|b(t)\|_{L_{\text{h}}^4(L_{\vee}^{\infty})}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^2 \|\Delta_{\ell} b(t)\|_{L^2} \|\nabla_{\text{h}} \Delta_{\ell} b(t)\|_{L^2} dt \right)^{\frac{1}{2}} \\ &\lesssim \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\int_0^T \|a(t)\|_{\mathcal{B}_4^{0, \frac{1}{2}}}^4 \|\Delta_{\ell} b(t)\|_{L^2}^2 dt \right)^{\frac{1}{4}} \|\nabla_{\text{h}} \Delta_{\ell} b\|_{L_T^2(L^2)}^{\frac{1}{2}} \\ &\lesssim \|b\|_{\tilde{L}_{T, f}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_{\text{h}} b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}}. \end{aligned}$$

This in turn shows that

$$|Q_2| \lesssim d_{\ell}^2 2^{-\ell} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|b\|_{\tilde{L}_{T, f}^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_{\text{h}} b\|_{\tilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}},$$

which, together with (A.6), ensures (4.12). This completes the proof of Lemma 4.2. \square

Proof of Lemma 6.1. Let Q_1 be given by (A.1). We first get, by a similar derivation of (A.2), that

$$\begin{aligned} |Q_1| &\lesssim \sum_{|\ell'-\ell|\leq 5} \|S_{\ell'-1}^{\vee} a\|_{L_T^4(L_h^4(L_{\nabla}^{\infty}))} \|\Delta_{\ell'}^{\vee} b\|_{L_T^4(L_h^4(L_{\nabla}^2))} \|A(D)\Delta_{\ell}^{\vee} c\|_{L_T^2(L^2)} \\ &\lesssim d_{\ell} 2^{-\frac{\ell}{2}} \sum_{|\ell'-\ell|\leq 5} d_{\ell'} 2^{-\frac{\ell'}{2}} \|a\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|b\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})}, \end{aligned}$$

which, together with Proposition 2.1, implies that

$$|Q_1| \lesssim d_{\ell}^2 2^{-\ell} \|a\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})}. \quad (\text{A.7})$$

For Q_2 given by (A.1), we get, by a similar derivation of (A.4), that

$$\begin{aligned} |Q_2| &\lesssim \sum_{\ell' \geq \ell - N_0} \|\Delta_{\ell'}^{\vee} a\|_{L_T^4(L_h^4(L_{\nabla}^2))} \|S_{\ell'+2}^{\vee} b\|_{L_T^4(L_h^4(L_{\nabla}^{\infty}))} \|A(D)\Delta_{\ell}^{\vee} c\|_{L_T^2(L^2)} \\ &\lesssim d_{\ell} 2^{-\frac{\ell}{2}} \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \|a\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|b\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})}, \end{aligned}$$

from which, with Proposition 2.1, we infer

$$|Q_2| \lesssim d_{\ell}^2 2^{-\ell} \|a\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|b\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(T)} \|c\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})}.$$

This, together with (A.1) and (A.7), ensures (6.5).

The inequality (6.6) can be proved similarly. As a matter of fact, we observe that

$$\begin{aligned} |Q_1| &\lesssim \sum_{|\ell'-\ell|\leq 5} \|S_{\ell'-1}^{\vee} a\|_{L_T^2(L_h^2(L_{\nabla}^{\infty}))} \|\Delta_{\ell'}^{\vee} b\|_{L_T^4(L_h^4(L_{\nabla}^2))} \|A(D)\Delta_{\ell}^{\vee} c\|_{L_T^4(L_h^4(L_{\nabla}^2))} \\ &\lesssim \sum_{|\ell'-\ell|\leq 5} \|S_{\ell'-1}^{\vee} a\|_{L_T^2(L_h^2(L_{\nabla}^{\infty}))} \|\Delta_{\ell'}^{\vee} b\|_{L_T^4(L_h^4(L_{\nabla}^2))} \|\Delta_{\ell}^{\vee} c\|_{L_T^{\infty}(L^2)}^{\frac{1}{2}} \|\Delta_{\ell}^{\vee} \nabla_{\text{h}} c\|_{L_T^2(L^2)}^{\frac{1}{2}} \\ &\lesssim d_{\ell} 2^{-\frac{\ell}{2}} \sum_{|\ell'-\ell|\leq 5} d_{\ell'} 2^{-\frac{\ell'}{2}} \|a\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})} \|b\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|c\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}, \end{aligned}$$

and

$$\begin{aligned} |Q_2| &\lesssim \sum_{\ell' \geq \ell - N_0} \|\Delta_{\ell'}^{\vee} a\|_{L_T^2(L^2)} \|S_{\ell'+2}^{\vee} b\|_{L_T^4(L_h^4(L_{\nabla}^{\infty}))} \|A(D)\Delta_{\ell}^{\vee} c\|_{L_T^4(L_h^4(L_{\nabla}^2))} \\ &\lesssim d_{\ell} 2^{-\frac{\ell}{2}} \sum_{\ell' \geq \ell - N_0} d_{\ell'} 2^{-\frac{\ell'}{2}} \|a\|_{\tilde{L}_T^2(\mathcal{B}^{0,\frac{1}{2}})} \|b\|_{\tilde{L}_T^4(\mathcal{B}_4^{0,\frac{1}{2}})} \|c\|_{\mathcal{B}^{0,\frac{1}{2}}(T)}. \end{aligned}$$

Then (6.6) follows from Proposition 2.1. This completes the proof of this lemma. \square

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