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Decay Estimates of Gradient of a Generalized Oseen Evolution Operator Arising from Time-Dependent Rigid Motions in Exterior Domains

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Dedicated to Professor Yoshio Yamada on his 70th birthday

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Abstract

Let us consider the motion of a viscous incompressible fluid past a rotating rigid body in three dimensions, where the translational and angular velocities of the body are prescribed but time-dependent. In a reference frame attached to the body, we have the Navier–Stokes system with the drift and (one half of the) Coriolis terms in a fixed exterior domain. The existence of the evolution operator $T(t, s)$ in the space L^q generated by the linearized non-autonomous system was proved by Hansel and Rhandi (J Reine Angew Math 694:1–26, 2014) and the large time behavior of $T(t, s)f$ in L^r for $(t-s) \to \infty$ was then developed by Hishida (Math Ann 372:915–949, 2018) when f is taken from L^q with $q \leqq r$. The contribution of the present paper concerns such L^q-L^r decay estimates of $\nabla T(t, s)$ with optimal rates, which must be useful for the study of stability/attainability of the Navier–Stokes flow in several physically relevant situations. Our main theorem completely recovers the L^q-L^r estimates for the autonomous case (Stokes and Oseen semigroups, those semigroups with rotating effect) in three dimensional exterior domains, which were established by Hishida and Shibata (Arch Ration Mech Anal 193:339–421, 2009), Iwashita (Math Ann 285, 265–288, 1989), Kobayashi and Shibata (Math Ann 310:1–45, 1998), Maremonti and Solonnikov (Ann Sc Norm Super Pisa 24:395–449, 1997) and Shibata (in: Amann, Arendt, Hieber, Neubrander, Nicaise, von Below (eds) Functional analysis and evolution equations, the Günter Lumer volume. Birkhäuser, Basel, pp 595–611, 2008).

1. Introduction

This paper is the continuation of the previous study [34] on large time behavior of a generalized Oseen evolution operator $T(t, s)$, which is the solution operator

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$u(\cdot, s) = f \mapsto u(\cdot, t)$ to the initial value problem for the linear non-autonomous system

$$\begin{aligned} \partial_t u &= \Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u - \nabla p, \\ \operatorname{div} u &= 0, \\ u|_{\partial D} &= 0, \\ u &\to 0 \quad \text{as } |x| \to \infty, \\ u(\cdot, s) &= f, \end{aligned} \tag{1.1}$$

in $D \times (s, \infty)$, where D is an exterior domain in $\mathbb{R}^3$ with $C^{1,1}$-boundary ∂D, $\{u(x,t), p(x,t)\}$ with $u = (u_1, u_2, u_3)^\top$ is the pair of unknowns which are the velocity vector field and pressure of a viscous fluid, respectively, while the solenoidal vector field $f(x) = (f_1, f_2, f_3)^\top$ is a given initial velocity at initial time $s \geqq 0$ and $\{\eta(t), \omega(t)\} \in \mathbb{R}^{3\times 2}$ will be explained soon. Here and in what follows, $(\cdot)^\top$ stands for the transpose of vectors or matirices. Problem (1.1) is a linearized system for the Navier–Stokes problem modeling a viscous incompressible flow past an obstacle $\mathbb{R}^3 \setminus D$ (rigid body) that moves in a prescribed way. One usually makes a transformation of variables in order to reduce the problem to an equivalent one over the fixed domain in a frame attached to the obstacle, see Galdi [16] for details. Then the resulting system is (1.1) (with $s = 0$) in which the LHS of the equation of motion should be replaced by $\partial_t u + u \cdot \nabla u$ and the fluid velocity attains the rigid motion $\eta + \omega \times x$ (no-slip condition) at the boundary ∂D, where $\eta(t)$ and $\omega(t)$ respectively denote the translational and angular velocities of the rigid body (after the transformation mentioned above). This paper develops methods of analyzing the large time behavior of $T(t,s)$ for $(t-s) \to \infty$ when both translational and angular velocities are time-dependent. Our conditions on this dependence are

$$\eta,\ \omega \in C^\theta([0,\infty); \mathbb{R}^3) \cap L^\infty(0,\infty; \mathbb{R}^3) \tag{1.2}$$

with some $\theta \in (0,1)$, which are the same as in the previous study [34].

The well-posedness of (1.1), that is, generation of the evolution operator $\{T(t,s)\}_{t \geqq s \geqq 0}$ in the space L^q for $1 < q < \infty$ was successfully proved by Hansel and Rhandi [27] under the condition

$$\eta,\ \omega \in C^\theta_{loc}([0,\infty); \mathbb{R}^3) \tag{1.3}$$

with some $\theta \in (0,1)$. It is reasonable not to need the global behavior (1.2) just for the well-posedness of (1.1) and for regularity of the solution. They also derived a remarkable L^q-L^r smoothing action near the initial time, that is,

$$\|T(t,s)f\|_r \leqq C(t-s)^{-(3/q-3/r)/2}\|f\|_q \tag{1.4}$$

$$\|\nabla T(t,s)f\|_r \leqq C(t-s)^{-(3/q-3/r)/2-1/2}\|f\|_q \tag{1.5}$$

for $0 \leqq s < t \leqq \mathcal{T}$ and $1 < q \leqq r < \infty$ with some constant $C > 0$ that depends on $\mathcal{T} \in (0,\infty)$, where $\|\cdot\|_q$ denotes the norm of the space $L^q(D)$. Later on, the present author [34] developed the L^q-L^r decay estimate of $T(t,s)$, namely, (1.4) for all $t > s \geqq 0$ and $1 < q \leqq r < \infty$ with some constant $C > 0$ independent of (t, s). A duality argument is one of ingredients of the proof, so that the L^q-L^r estimate of the adjoint evolution operator $T(t, s)^*$ has been also deduced in [34] simultaneously with (1.4). Note that the adjoint $T(t, s)^*$ is the solution operator $v(\cdot, t) = g \mapsto v(\cdot, s)$ of the backward problem for the adjoint system subject to the final condition at $t > 0$, see (2.6) below. However, the decay estimate of $\nabla T(t, s)$ with optimal rate has remained open [34, Remark 2.1].

The purpose of the present paper is to develop the gradient estimate of the evolution operator for $(t - s) \to \infty$. Our main theorem (Theorem 2.1, particularly the first assertion) provides us with (1.5) for all $t > s \geqq 0$ and $1 < q \leqq r \leqq 3$. The rate of decay of $\nabla T(t, s)$ for the other case $1 < q \leqq r \in (3, \infty)$ is also discussed and it is given by $(t - s)^{-3/2q}$. In addition, we obtain the L^q-L^∞ decay estimate of $T(t, s)$ as well, that is, (1.4) with $1 < q < r = \infty$ for all $t > s \geqq 0$. Our theorem completely recovers the L^q-L^r estimates for the autonomous case developed by [38,43] (both for the Stokes semigroup $\eta = \omega = 0$), [7,8,40] (those three for the Oseen semigroup with constant $\eta \neq 0$, $\omega = 0$), [37] (semigroup with constant $\omega \neq 0$, $\eta = 0$) and [45] (semigroup with constants $\eta \neq 0$, $\omega \neq 0$). Therefore, analysis in this paper can be regarded as a unified approach not only to all the cases of uniform rigid motions but to several cases of time-dependent ones. Our result cannot be improved in general because Maremonti and Solonnikov [43] and the present author [30] observed that the rate of decay of $\nabla T(t, s)$ in our theorem is optimal when $\eta = \omega = 0$ (case of the Stokes semigroup). Nevertheless, there might be a chance of improvement when $\eta \neq 0$; for further discussion about the optimality, see Remark 2.1.

In view of the celebrated paper [39] by Kato, it is clear that we have several applications of the complete L^q-L^r estimates (1.4)–(1.5) for all $t > s \geqq 0$ obtained in this paper. In [34] (see also [35] for further development) the present author has proposed a new way of constructing a unique Navier–Stokes flow globally in time by use only of (1.4) combined with the energy relation (see [34, Lemma 5.1]), but the solution constructed in such a way possesses less information about the large time behavior; in fact, an improvement of Theorem 5.1 of [34] by using (1.5) with $r = 3$ for all $t > s \geqq 0$ is obvious. Since the same estimate for the adjoint $T(t, s)^*$ is available in the Lorentz spaces as well, see (2.25) in Theorem 2.2 below, we must have even more applications with the aid of interpolation technique developed by Yamazaki [53]. Once we have (2.25), his insight brings us the sharp estimate (2.26), which is quite useful to study the stability/attainability of several physically relevant background flows (not only steady flow but also time-dependent flows such as time-periodic one) being in the scale-critical Lorentz space $L^{3,\infty}$ (weak-L^3 space). This is indeed the case if, for instance, the obstacle is purely rotating or at rest without translation, where the optimality of the decay rate $|x|^{-1}$ for generic flow is interpreted in terms of asymptotic structure at infinity, see [9,10,41] and [33]. Several applications of our main theorems will be discussed elsewhere. Let us just mention, as one of them, a problem of attainablity of a (small) steady flow around a rigid body rotating from rest (that was raised by [31, Section 6]). This is called the starting problem and was proposed first by Finn [13] in the case when the rotation was replaced by translation of the body. Finn's problem was successfully solved by Galdi, Heywood and Shibata [19] by making use of the L^q-L^r estimate of the Oseen semigroup [40], see also [36] for further contributions, however, the same approach with the aid of the L^q-L^r estimate due to [37] no longer works for the question above because of unbounded coefficient $\omega \times x$ of the drift term. The right approach seems to be use of the results obtained here for the non-autonomous system, see [50]. Another application [20] of our theorems is the attainability of a time-periodic flow induced by time-periodic translation with zero average (oscillation like back and forth), whose existence has been recently proved by Galdi [18].

The proof of our main theorem consists of two stages: one is the so-called local energy decay estimates over a bounded domain $D \cap B_R$ (near the obstacle), see Propositions 6.1 and 6.2; the other is a decay estimate outside B_R (near inifinity), where B_R denotes the open ball centered at the origin with radius $R > 0$. Indeed this combination itself was adopted by several authors ([8,37,38,40] for 3D, [5,6,32,42] for 2D) for the autonomous case, but what is new is to deduce the former without spectral analysis. In fact, our assumption (1.2) is too general (without any specific structure such as time-periodicity) to carry out the spectral analysis. Note, however, that analysis of the resolvent near $\lambda = 0$ is the essential and hard step for the autonomous case in the literature above, where λ denotes the spectral parameter. We also refer the readers to a recent work [47] on the autonomous case by Shibata, who has developed even more in the resolvent side to furnish the L^q-L^r decay estimates. In this paper, (1.4) for all $t > s \geqq 0$ plays a role to obtain the local energy decay estimates (note that it is the opposite way to the argument in the literature mentioned above in which (1.4) was a conclusion of the local energy decay estimates), but such estimates of $\nabla T(t, s)$ are not enough since we have to control the behavior of the pressure at the other stage of deduction of decay estimates near infinity. The natural idea is to analyze the asymptotic behavior, both for $(t - s) \to \infty$ and for $(t - s) \to 0$, of the temporal derivative $\partial_t T(t, s)$ in the Sobolev space of order (-1) over the bounded domain $D \cap B_R$. To this end, we need to develop more analysis of regularity of the evolution operator $T(t, s)$, see Proposition 5.1, than the one done by Hansel and Rhandi [27]. Analysis of $\partial_t T(t, s)$ is in fact very nontrivial since the corresponding autonomous operator is no longer generator of an analytic semigroup in the space L^q unless $\omega = 0$, see [11,28,46] and the references therein, and it can be regarded as a substitution of Section 5 of [37] for the autonomous case (semigroup with constant $\omega \neq 0$), in which the authors made full use of precise behavior of parametrix of the resolvent with respect to the spectral parameter.

It is worthwhile summarizing the method developed in the present paper together with the previous study [34]. The clue at the beginning towards analysis of large time behavior of (1.1) would be

(i) L^q-L^r estimates (3.10) for the same system in the whole space;
(ii) energy relations [34, (2.15), (2.23)] for $T(t, s)$ and its adjoint;

both of which are clear because the equation in (1.1) is derived only from the transformation of variables concerning (i) and because the additional terms arising from this transformation are skew-symmetric concerning (ii). Those are fine, however, we would say that the only fine things for (1.1) are them. Note that, except for (ii), one does not have useful higher energy estimates (which play an important role in [43] for the Stokes semigroup) unless $\eta = \omega = 0$. In [34] some devices by use of the energy (ii) enable us to show the uniform boundedness of $T(t,s)$ and $T(t,s)^*$ in L^r with $r \in (2,\infty)$ by duality argument with the aid of (i) via cut-off procedure. With this at hand, the deduction of (1.4) for all $t > s \geqq 0$ can be reduced to computations of a differential inequality [34, Lemmas 4.1, 4.2]. Then, in this paper, (1.4) combined with a detailed analysis of $\partial_t T(t,s)$ leads us to (1.5) for all $t > s \geqq 0$ and $1 < q \leqq r \leqq 3$ as explained in the previous paragraph. To sum up, with the approach proposed in both papers, once we have (i) and (ii) above, we are able to deduce the large time behavior of $\nabla^j T(t,s)$ with $j = 0, 1$ in three dimensional exterior domains. In the more involved 2D case, however, the method developed in [34] unfortunately does not work well, see [34, Remark 4.1] for the difficulties. Concerning the L^q-L^r estimate for the autonomous case in 2D exterior domains, we refer to [5,6,43] (for the Stokes semigroup) and [32,42] (for the Oseen semigroup, where the latter is a significant refinement of the former). For the case of rotating obstacle, the desired decay property has still remained open in 2D even if $\omega \neq 0$ is a constant vector.

This paper is organized as follows: in the next section, after summarizing the knowledge from [27,34], we present the main theorems. We need further analysis of the same system in the whole space and the one in bounded domains, which are not covered by the literature. They are performed in Sections 3 and 4, respectively. By way of constructing the evolution operator due to [27], in Section 5, we develop more analysis of its regularity, in particular, smoothing rate as well as justification of the temporal derivative $\partial_t T(t,s) f$ for general solenoidal vector field f being in the space L^q. Local energy decay estimates of the evolution operator near the obstacle are established in Section 6. The final section is devoted to completion of the proof of the main theorems by showing the decay estimate of the evolution operator near spatial infinity.

2. Results

Let us begin with introducing notation. Given two vector fields u and v, we denote by $u \otimes v$ the matrix $(u_i v_j)$. Let $A = (A_{ij}(x))$ be a 3×3 matrix-valued function, then the vector field div A is defined by $(\mathrm{div}\, A)_i = \sum_j \partial_{x_j} A_{ij}$. By following this rule, the drift and Coriolis terms in (1.1) can be expressed as

$$(\eta + \omega \times x) \cdot \nabla u = \mathrm{div}\,\big[u \otimes (\eta + \omega \times x)\big], \qquad \omega \times u = \mathrm{div}\,\big[(\omega \times x) \otimes u\big],$$

the latter of which follows from $\mathrm{div}\, u = 0$. Those expressions appear in (3.16), (4.8) and (5.21) below.

Given a domain $G \subset \mathbb{R}^3$, $q \in [1,\infty]$ and integer $k \geqq 0$, the standard Lebesgue and Sobolev spaces are denoted by $L^q(G)$ and by $W^{k,q}(G)$. We abbreviate the norm $\|\cdot\|_{q,G} = \|\cdot\|_{L^q(G)}$ and even $\|\cdot\|_q = \|\cdot\|_{q,D}$, where D is the exterior domain under consideration with $C^{1,1}$-boundary ∂D.

Throughout this paper, we fix a number $R_0 > 0$ so large that

$$\mathbb{R}^3 \backslash D \subset B_{R_0}, \tag{2.1}$$

where B_R denotes the open ball centered at the origin with radius $R > 0$. We set $D_R = D \cap B_R$ for $R \in [R_0, \infty)$.

The class $C_0^\infty(G)$ consists of all C^∞ functions with compact support in G, then $W_0^{k,q}(G)$ denotes the completion of $C_0^\infty(G)$ in $W^{k,q}(G)$, where $k > 0$ is an integer. We set $W^{-1,q}(G) = W_0^{1,q'}(G)^*$, where $1/q' + 1/q = 1$ and $q \in (1, \infty)$. By $\langle \cdot, \cdot \rangle_G$ we denote various duality pairings over the domain G. In what follows we adopt the same symbols for denoting scalar and vector (even tensor) function spaces as long as there is no confusion.

Let X_1 and X_2 be two Banach spaces. Then $\mathcal{L}(X_1, X_2)$ stands for the Banach space consisting of all bounded linear operators from X_1 into X_2. We simply write $\mathcal{L}(X_1) = \mathcal{L}(X_1, X_1)$.

Consider the boundary value problem

$$\operatorname{div} w = f \quad \text{in } G, \qquad w|_{\partial G} = 0,$$

where G is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary ∂G. Let $1 < q < \infty$. Given $f \in L^q(G)$ with compatibility condition $\int_G f \, dx = 0$, there are a lot of solutions, some of which have already been found, see Galdi [17, Notes for Chapter III]. Among those solutions a particular one discovered by Bogovskii [2] is useful to recover the solenoidal condition in a cut-off procedure on account of some fine properties of his solution. The operator $f \mapsto$ his solution w, called the Bogovskii operator, is well defined as follows (for details, see [3,17]): there is a linear operator $\mathbb{B}_G : C_0^\infty(G) \to C_0^\infty(G)^3$ such that, for $1 < q < \infty$ and $k \geqq 0$ integers,

$$\|\nabla^{k+1} \mathbb{B}_G f\|_{q,G} \leqq C \|\nabla^k f\|_{q,G}, \tag{2.2}$$

with some $C = C(G, q, k) > 0$, which is invariant with respect to dilation of the domain G, and that

$$\operatorname{div} (\mathbb{B}_G f) = f \qquad \text{if} \quad \int_G f(x) \, dx = 0. \tag{2.3}$$

By continuity, $\mathbb{B}_G$ extends uniquely to a bounded operator from $W_0^{k,q}(G)$ to $W_0^{k+1,q}(G)^3$. In [23, Theorem 2.5] Geissert, Heck and Hieber proved that $\mathbb{B}_G$ can also extend to a bounded operator from $W^{1,q'}(G)^*$ to $L^q(G)^3$, that is,

$$\|\mathbb{B}_G f\|_{q,G} \leqq C \|f\|_{W^{1,q'}(G)^*}, \tag{2.4}$$

where $1/q' + 1/q = 1$. Note that this is not true from $W^{-1,q}(G)$ to $L^q(G)^3$, see Galdi [17, Chapter III], who nevertheless proved that

$$\|\mathbb{B}_G[\operatorname{div} F]\|_{q,G} \leqq C \|F\|_{q,G} \tag{2.5}$$

holds true for $F \in L^q(G)^3$ satisfying the vanishing normal trace condition $\nu \cdot F|_{\partial G} = 0$ as well as $\operatorname{div} F \in L^q(G)$ [17, Theorem III.3.4]. Instead of (2.4), one can employ (2.5) to discuss some delicate terms arising from cut-off procedures.

Let us introduce the solenoidal function space. Let $G \subset \mathbb{R}^3$ be one of the following domains; the exterior domain D under consideration, a bounded domain with $C^{1,1}$-boundary ∂G and the whole space $\mathbb{R}^3$. The class $C^\infty_{0,\sigma}(G)$ consists of all divergence-free vector fields being in $C^\infty_0(G)$. Let $1 < q < \infty$. By $L^q_\sigma(G)$ we denote the completion of $C^\infty_{0,\sigma}(G)$ in $L^q(G)$, then it is characterized as

$$L^q_\sigma(G) = \{u \in L^q(G);\ \mathrm{div}\, u = 0,\ \nu \cdot u|_{\partial G} = 0\},$$

where ν stands for the outer unit normal to ∂G and $\nu \cdot u$ is understood in the sense of normal trace on ∂G (this boundary condition is absent when $G = \mathbb{R}^3$). The space of L^q-vector fields admits the Helmholtz decomposition

$$L^q(G) = L^q_\sigma(G) \oplus \{\nabla p \in L^q(G);\ p \in L^q_{loc}(\overline{G})\},$$

which was proved by Fujiwara and Morimoto [15], Miyakawa [44] and Simader and Sohr [48]. By $P_G = P_{G,q} : L^q(G) \to L^q_\sigma(G)$, we denote the Fujita-Kato projection associated with the decompostion above. We then see that $P_G \in \mathcal{L}(W^{1,q}(G))$ as well as $P_G \in \mathcal{L}(L^q(G))$. Note the duality relation $(P_{G,q})^* = P_{G,q'}$ as well as $L^q_\sigma(G)^* = L^{q'}_\sigma(G)$, where $1/q' + 1/q = 1$. We simply write $P = P_D$ for the exterior domain D under consideration. Finally, we denote several positive constants by C, which may change from line to line.

We are in a position to introduce the generators which are related to (1.1) and to the backward problem for the adjoint system subject to the final condition at $t > 0$:

$$\begin{aligned} -\partial_s v &= \Delta v - (\eta(s) + \omega(s) \times y) \cdot \nabla v + \omega(s) \times v + \nabla\sigma, \\ \mathrm{div}\, v &= 0, \\ v|_{\partial D} &= 0, \\ v &\to 0 \quad \text{as } |y| \to \infty, \\ v(\cdot, t) &= g, \end{aligned} \tag{2.6}$$

in $D \times [0, t)$, where $\{v(y, s), \sigma(y, s)\}$ is the pair of unknowns. Let us define the operators $L_\pm(t)$ by

$$\begin{aligned} D_q(L_\pm(t)) &= \{u \in L^q_\sigma(D) \cap W^{1,q}_0(D) \cap W^{2,q}(D);\ (\omega(t) \times x) \cdot \nabla u \in L^q(D)\}, \\ L_\pm(t)u &= -P[\Delta u \pm (\eta(t) + \omega(t) \times x) \cdot \nabla u \mp \omega(t) \times u]. \end{aligned} \tag{2.7}$$

Then we have

$$\langle L_\pm(t)u, v\rangle_D = \langle u, L_\mp(t)v\rangle_D \tag{2.8}$$

for all $u \in D_q(L_\pm(t))$ and $v \in D_{q'}(L_\mp(t))$, see [34, (2.12)], where $1/q' + 1/q = 1$. Since the domain is time-dependent, as in Hansel and Rhandi [27], we need the regularity spaces

$$\begin{aligned} Y_q(D) &= \{u \in L^q_\sigma(D) \cap W^{1,q}_0(D) \cap W^{2,q}(D);\ |x|\nabla u \in L^q(D)\}, \\ Z_q(D) &= \{u \in L^q_\sigma(D) \cap W^{1,q}(D);\ |x|\nabla u \in L^q(D)\}, \end{aligned} \tag{2.9}$$

which are Banach spaces endowed with norms

$$\|u\|_{Y_q(D)} = \|u\|_{W^{2,q}(D)} + \||x|\nabla u\|_q, \quad \|u\|_{Z_q(D)} = \|u\|_{W^{1,q}(D)} + \||x|\nabla u\|_q,$$

respectively. Note that $Y_q(D) \subset D_q(L_\pm(t))$ for every $t \geqq 0$ and that, differently from [27], the homogeneous Dirichlet condition at ∂D is not involved in the space $Z_q(D)$. The reason why this modification is actually needed will be clarified in Section 5.

Hansel and Rhandi [27] proved the following:

Proposition 2.1. [27] *Suppose that η and ω fulfill (1.3) for some $\theta \in (0,1)$. Let $1 < q < \infty$. The operator family $\{L_+(t)\}_{t \geqq 0}$ generates an evolution operator $\{T(t,s)\}_{t \geqq s \geqq 0}$ on $L^q_\sigma(D)$ such that $T(t,s)$ is a bounded operator from $L^q_\sigma(D)$ into itself with the semigroup property*

$$T(t,\tau)T(\tau,s) = T(t,s) \quad (t \geqq \tau \geqq s \geqq 0); \qquad T(s,s) = I, \tag{2.10}$$

in $\mathcal{L}(L^q_\sigma(D))$ and that the map

$$\{t \geqq s \geqq 0\} \ni (t,s) \mapsto T(t,s)f \in L^q_\sigma(D)$$

is continuous for every $f \in L^q_\sigma(D)$. Furthermore, we have the following properties:

1. *Let $q \leqq r < \infty$. For each $\mathcal{T} \in (0,\infty)$ and $m \in (0,\infty)$, there is a constant $C = C(\mathcal{T}, m, q, r, \theta, D) > 0$ such that (1.4) and (1.5) hold for all (t,s) with $0 \leqq s < t \leqq \mathcal{T}$ and $f \in L^q_\sigma(D)$ whenever*

$$\sup_{0 \leqq t \leqq \mathcal{T}} (|\eta(t)| + |\omega(t)|) + \sup_{0 \leqq s < t \leqq \mathcal{T}} \frac{|\eta(t) - \eta(s)| + |\omega(t) - \omega(s)|}{(t-s)^\theta} \leqq m.$$

2. *Let $3/2 < q < \infty$ and fix $s \geqq 0$. For every $f \in Z_q(D)$ and $t \in (s,\infty)$, we have $T(t,s)f \in Y_q(D)$ and*

$$T(\cdot, s)f \in C^1((s,\infty); L^q_\sigma(D))$$

with

$$\partial_t T(t,s)f + L_+(t)T(t,s)f = 0, \qquad t \in (s,\infty), \tag{2.11}$$

in $L^q_\sigma(D)$.

3. *Fix $t > 0$. For every $f \in Y_q(D)$, we have*

$$T(t,\cdot)f \in C^1([0,t]; L^q_\sigma(D))$$

with

$$\partial_s T(t,s)f = T(t,s)L_+(s)f, \qquad s \in [0,t],$$

in $L^q_\sigma(D)$.

Among the assertions above, the second one tells us that $T(t,s)f$ provides a strong solution without assuming $f|_{\partial D} = 0$ nor $\nabla^2 f \in L^q(D)$. This is a slight improvement of the corresponding result in [27, Theorem 2.4 (b)], which claims the same for $f \in Y_q(D)$. The proof of this improvement only in the second assertion will be given in Section 5. The restriction $q \in (3/2, \infty)$ stems from Lemma 5.2 (and it seemed to be overlooked in [27]). Thus the corresponding part of Proposition 2.1 of [34] should be replaced by the second assertion above. Nevertheless, we observe that the semigroup property (2.10) in $\mathcal{L}(L^q_\sigma(D))$ holds still for every $q \in (1,\infty)$. In fact, given $f \in C^\infty_{0,\sigma}(D)$, it follows from the second and third assertions that $\partial_\tau\big(T(t,\tau)T(\tau,s)f\big) = 0$ in $L^q_\sigma(D)$ with $q \in (3/2,\infty)$, yielding $T(t,\tau)T(\tau,s)f = T(t,s)f$. Once we have that for all $f \in C^\infty_{0,\sigma}(D)$, a continuity argument leads to the same equality for all $f \in L^q_\sigma(D)$ with $q \in (1,\infty)$.

We should mention that the results obtained in the previous study [34] are still valid in spite of the restriction $q \in (3/2,\infty)$ above. Let $S(t,s)$ be the evolution operator generated by the backward problem

$$-\partial_s v(s) + L_-(s)v(s) = 0, \quad s \in [0,t); \qquad v(t) = g \tag{2.12}$$

in $L^q_\sigma(D)$, which corresponds to (2.6); it is given by

$$S(t,s) = \widetilde{T}(t-s, 0; t), \qquad t \geqq s \geqq 0, \tag{2.13}$$

where $\{\widetilde{T}(\tau,s;t)\}_{0\leqq s\leqq \tau\leqq t}$ is the evolution operator generated by the related initial value problem

$$\partial_\tau w(\tau) + L_-(t-\tau)w(\tau) = 0, \quad \tau \in (s,t]; \qquad w(s) = g, \tag{2.14}$$

see [34, Subsection 2.3]. For (2.14), note that $t > 0$ is just a parameter appearing in the coefficient of the equation. We then have the duality relation [34, Lemma 2.1]

$$T(t,s)^* = S(t,s), \qquad S(t,s)^* = T(t,s) \qquad \text{in } \mathcal{L}(L^q_\sigma(D)) \tag{2.15}$$

for $t \geqq s \geqq 0$, which plays an important role in [34]. In fact, given $f,\ g \in C^\infty_{0,\sigma}(D)$ (instead of $f \in Y_{q'}(D)$, $g \in Y_q(D)$ in the proof of [34, Lemma 2.1], where $1/q' + 1/q = 1$), we obtain

$$\langle T(t,s)f, g\rangle_D = \langle f, S(t,s)g\rangle_D \tag{2.16}$$

by computing $\partial_\tau \langle T(\tau,s)f, S(t,\tau)g\rangle_D = 0$ with use of (2.11) as well as

$$-\partial_s S(t,s)g + L_-(s)S(t,s)g = 0, \qquad s \in [0,t), \tag{2.17}$$

in $L^q_\sigma(D)$, where $\langle\cdot,\cdot\rangle_D$ should be understood for the pair of $L^{q'}_\sigma(D)$ and $L^q_\sigma(D)$ with $q \in (3/2,3)$. Once we have (2.16) for all $f,\ g \in C^\infty_{0,\sigma}(D)$, we have only to perform a continuity argument to justify (2.15) for every $q \in (1,\infty)$. In addition, as emphasized in Section 1, one of key ingredients in [34] is the energy relation which we certainly have since the second assertion of Proposition 2.1 is available in $L^2_\sigma(D)$. Finally, as described in [34, Section 4] for the proof of decay estimates, it suffices to carry out a cut-off procedure for fine initial velocities being in $C^\infty_{0,\sigma}(D)$, so that the restriction $q \in (3/2,\infty)$ does not cause any problem.

We recall the L^q-L^r estimates globally in time developed by the present author [34, Theorem 2.1, Proposition 3.1]. Let us introduce

$$|(\eta,\omega)|_0 = \sup_{t\geqq 0} (|\eta(t)| + |\omega(t)|),$$
$$|(\eta,\omega)|_\theta = \sup_{t>s\geqq 0} \frac{|\eta(t)-\eta(s)| + |\omega(t)-\omega(s)|}{(t-s)^\theta} \tag{2.18}$$

for $\theta \in (0,1)$ and

$$\Lambda(\tau_*) = \{(t,s);\ t > s \geqq 0,\ t-s \leqq \tau_*\} \tag{2.19}$$

for $\tau_* \in (0,\infty)$.

Proposition 2.2. [34] *Suppose that η and ω fulfill (1.2) for some $\theta \in (0,1)$. Let $1 < q \leqq r < \infty$. Then,*

1. *For each $m \in (0,\infty)$, there is a constant $C = C(m,q,r,\theta,D) > 0$ such that*

$$\|T(t,s)f\|_r \leqq C(t-s)^{-(3/q-3/r)/2}\|f\|_q,$$
$$\|T(t,s)^*g\|_r \leqq C(t-s)^{-(3/q-3/r)/2}\|g\|_q, \tag{2.20}$$

for all $t > s \geqq 0$ and $f,\ g \in L^q_\sigma(D)$ whenever

$$|(\eta,\omega)|_0 + |(\eta,\omega)|_\theta \leqq m \tag{2.21}$$

is satisfied.

2. *Given $\tau_* \in (0,\infty)$ and $m \in (0,\infty)$, let $\Lambda(\tau_*)$ be as in (2.19) and assume (2.21). Then there is a constant $C = C(\tau_*,m,q,r,\theta,D) > 0$ such that*

$$\|\nabla T(t,s)f\|_r \leqq C(t-s)^{-(3/q-3/r)/2-1/2}\|f\|_q,$$
$$\|\nabla T(t,s)^*g\|_r \leqq C(t-s)^{-(3/q-3/r)/2-1/2}\|g\|_q, \tag{2.22}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f,\ g \in L^q_\sigma(D)$.

The point of the second assertion is that the constant $C > 0$ in (2.22) can be taken uniformly in (t,s) with $t-s \leqq \tau_*$. This must be the first step toward (2.22) for all $t > s \geqq 0$. It was not covered by [27] but shown by [34, Proposition 3.1] under the condition (1.2), however, only for $\nabla T(t,s)$. The same result for $\nabla T(t,s)^*$ follows from the one for $\nabla\widetilde{T}(\tau,s;t)$, which is the solution operator to (2.14) and can be constructed along the procedure adopted by [27], see also Section 5 of this paper. To this end, as clarified in [34, Subsections 3.1–3.3], it suffices to investigate the initial value problem for the same equation as in (2.14) over a bounded domain D_R with $R > 0$ large enough by following the Tanabe-Sobolevskii theory [51]. Taking a look at the generator $L_-(t-\tau)$ together with the condition (1.2), we observe that all the constants in several key estimates can be taken uniformly in (τ,s) with $\tau - s \leqq \tau_*$, see the proof of Lemma 3.2 of [34], which implies

$$\|\nabla\widetilde{T}(\tau,s;t)g\|_r \leqq C(\tau-s)^{-(3/q-3/r)/2-1/2}\|g\|_q$$

for all (τ, s) with $\tau - s \leqq \tau_*$ as well as $0 \leqq s < \tau \leqq t$ and $1 < q \leqq r < \infty$, where $C > 0$ depends on $\tau_* \in (0, t)$ but is independent of $t > 0$. By (2.13) and (2.15) we conclude that $\nabla T(t, s)^*$ also satisfies (2.22) for all $(t, s) \in \Lambda(\tau_*)$.

We are now in a position to present the main result of this paper.

Theorem 2.1. *Suppose that η and ω fulfill* (1.2) *for some $\theta \in (0, 1)$.*

1. *Let $1 < q \leqq r \leqq 3$. For each $m \in (0, \infty)$, there is a constant $C = C(m, q, r, \theta, D) > 0$ such that* (2.22) *holds for all $t > s \geqq 0$ and $f, g \in L^q_\sigma(D)$ whenever* (2.21) *is satisfied.*
2. *Let $1 < q \leqq r$ as well as $r \in (3, \infty)$. For each $m \in (0, \infty)$, there is a constant $C = C(m, q, r, \theta, D) > 0$ such that*

$$\begin{aligned} \|\nabla T(t, s) f\|_r &\leqq C(t-s)^{-3/2q} \|f\|_q, \\ \|\nabla T(t, s)^* g\|_r &\leqq C(t-s)^{-3/2q} \|g\|_q, \end{aligned} \tag{2.23}$$

for all (t, s) with

$$t - s > 2 \quad \text{as well as } 0 \leqq s < t$$

and $f, g \in L^q_\sigma(D)$ whenever (2.21) *is satisfied.*

3. *Let $1 < q < \infty$. For each $m \in (0, \infty)$, there is a constant $C = C(m, q, \theta, D) > 0$ such that* (2.20) *with $r = \infty$ holds true, that is,*

$$\begin{aligned} \|T(t, s) f\|_\infty &\leqq C(t-s)^{-3/2q} \|f\|_q, \\ \|T(t, s)^* g\|_\infty &\leqq C(t-s)^{-3/2q} \|g\|_q, \end{aligned} \tag{2.24}$$

for all $t > s \geqq 0$ and $f, g \in L^q_\sigma(D)$ whenever (2.21) *is satisfied.*

Remark 2.1. Maremonti and Solonnikov [43] first pointed out that the restriction $1 < q \leqq r \leqq 3 = n$ (space dimension) for the desired rate (2.22) of decay is optimal when $\eta = \omega = 0$. Later on, in this case of the Stokes semigroup, the present author [30] gave another proof of the optimality, where a key observation is that the issue is closely related to summability of the steady Stokes flow near spatial infinity. From this point of view, it is also conjectured by [30, Section 5] that the desired rate (1.5) of decay could be obtained for $1 < q \leqq r \leqq 6 = n(n+1)/(n-1)$ when the translation of the body is present, that is, $\eta \neq 0$. For the Stokes semigroup, the optimality of the rate (2.23) of decay was also proved by Maremonti and Solonnikov [43] in the sense that better rate $(t-s)^{-3/2q-\varepsilon}$ with some $\varepsilon > 0$ is impossible when $r > 3$.

Having several applications to the Navier–Stokes system in mind, we next provide useful estimates especially for the adjoint evolution operator. Let us introduce the Lorentz spaces which are usually defined as Banach spaces in terms of the average function of the rearrangement, see [1] for details. For simplicity, we just define the solenoidal Lorentz spaces by

$$L^{q,\rho}_\sigma(D) = \left(L^{q_0}_\sigma(D), L^{q_1}_\sigma(D)\right)_{\theta,\rho}$$

with

$$1 < q_0 < q < q_1 < \infty, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 1 \leqq \rho \leqq \infty,$$

where $(\cdot, \cdot)_{\theta,\rho}$ denotes the real interpolation functor. Then the RHS above is independent of choice of $\{q_0, q_1\}$, so that the space $L^{q,\rho}_\sigma(D)$, whose norm is denoted by $\|\cdot\|_{q,\rho}$, is well-defined. It is obvious by interpolation to obtain (2.20) and (2.22) for all $t > s \geqq 0$ in which the Lebesgue spaces are replaced by the Lorentz spaces except for (2.22) with $1 < q \leqq r = 3$. But we do need this end-point case for the adjoint evolution operator to study the large time behavior of the Navier–Stokes flow around a background flow (such as steady flow and time-periodic one) that decays with scale-critical rate at spatial infinity, see [37,53]. For completeness, it is worse while providing (2.25) below including the nontrivial case $r = 3$. Once we have (2.25), we can get (2.26) by following the argument developed by Yamazaki [53].

Theorem 2.2. *Suppose that η and ω fulfill* (1.2) *for some $\theta \in (0, 1)$. Let $1 < q \leqq r \leqq 3$ and $1 \leqq \rho < \infty$. Let $m \in (0, \infty)$ and assume* (2.21)*. Then there is a constant $C = C(m, q, r, \rho, \theta, D) > 0$ such that*

$$\|\nabla T(t,s)^* g\|_{r,\rho} \leqq C(t-s)^{-(3/q-3/r)/2-1/2}\|g\|_{q,\rho} \tag{2.25}$$

for all $t > s \geqq 0$ and $g \in L^{q,\rho}_\sigma(D)$. If, in particular $1/q - 1/r = 1/3$ as well as $1 < q < r \leqq 3$, then there is a constant $C = C(m, q, \theta, D) > 0$ such that

$$\int_0^t \|\nabla T(t,s)^* g\|_{r,1}\, ds \leqq C\|g\|_{q,1} \tag{2.26}$$

for all $t > 0$ and $g \in L^{q,1}_\sigma(D)$.

3. Whole space problem

In this section we consider the non-autonomous system

$$\begin{aligned} &\partial_t u = \Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u - \nabla p, \\ &\operatorname{div} u = 0 \end{aligned} \tag{3.1}$$

in $\mathbb{R}^3 \times (s, \infty)$ subject to

$$\begin{aligned} &u \to 0 \quad \text{as } |x| \to \infty, \\ &u(\cdot, s) = f. \end{aligned} \tag{3.2}$$

Indeed the system was studied by [4,21,25–27], but we have to supplement a couple of regularity properties: Lemma 3.1 on some smoothing actions and Lemma 3.2 on the time derivative for general $f \in L^q_\sigma(\mathbb{R}^3)$.

As long as f fulfills the compatibility condition div $f = 0$, we see that $\nabla p = 0$ within the class $\nabla p \in L^q(\mathbb{R}^3)$ and that the solution is just the heat semigroup in which a change of variables is made in an appropriate way, because

$$\text{div}\,[(\eta + \omega \times x) \cdot \nabla u - \omega \times u] = (\eta + \omega \times x) \cdot \nabla \text{div}\, u = 0. \tag{3.3}$$

In fact, the solution to (3.1)–(3.2) is explicitly described as

$$\begin{aligned} u(x,t) &= \big(U(t,s)f\big)(x) \\ &= \Phi(t,s)\left(e^{(t-s)\Delta} f\right)\left(\Phi(t,s)^\top\left(x + \int_s^t \Phi(t,\tau)\eta(\tau)\,\mathrm{d}\tau\right)\right), \end{aligned} \tag{3.4}$$

where

$$\big(e^{t\Delta} f\big)(x) = (4\pi t)^{-3/2}\left(e^{-|\cdot|^2/4t} * f\right)(x),$$

while 3×3 orthogonal matrix $\Phi(t,s)$ stands for the evolution operator for the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\varphi = -\omega \times \varphi$, see the literature above for details. By $\Gamma(x, y; t, s)$ we denote the fundamental solution, that is, the kernal matrix of (3.4):

$$u(x,t) = \int_{\mathbb{R}^3} \Gamma(x, y; t, s) f(y)\, dy.$$

Then the adjoint of $U(t,s)$ is given by

$$\big(U(t,s)^* g\big)(y) = \int_{\mathbb{R}^3} \Gamma(x, y; t, s)^\top g(x)\,\mathrm{d}x. \tag{3.5}$$

Given $t > 0$ (final time) and a suitable solenoidal vector field g (final data), the velocity $v(s) = U(t,s)^* g$ together with the trivial pressure gradient $\nabla\sigma = 0$ formally (even rigorously for fine g, see [34, third assertion of Lemma 3.1]) solves the backward system

$$\begin{aligned} &-\partial_s v = \Delta v - (\eta(s) + \omega(s) \times y) \cdot \nabla v + \omega(s) \times v + \nabla\sigma \\ &\text{div}\, v = 0, \end{aligned} \tag{3.6}$$

in $\mathbb{R}^3 \times [0, t)$ subject to

$$\begin{aligned} &v \to 0 \quad \text{as } |y| \to \infty, \\ &v(\cdot, t) = g. \end{aligned} \tag{3.7}$$

The initial value problem corresponding to (2.14) is given by

$$\begin{aligned} \partial_\tau w &= \Delta w - (\eta(t-\tau) + \omega(t-\tau) \times y) \cdot \nabla w + \omega(t-\tau) \times w + \nabla p_w, \\ \text{div}\, w &= 0, \\ w &\to 0 \quad \text{as } |y| \to \infty, \\ w(\cdot, s) &= g, \end{aligned} \tag{3.8}$$

in $\mathbb{R}^3 \times (s, t]$ (with $\nabla p_w = 0$ under the compatibility condition div $g = 0$), where $t > 0$ is just a parameter. The solution to (3.8) is described as

$$\begin{aligned} w(y, \tau) &= \big(\widetilde{U}(\tau, s; t)g\big)(y) \\ &= \Phi(t-\tau, t-s)\big(e^{(\tau-s)\Delta} g\big)(\cdots) \end{aligned} \tag{3.9}$$

with

$$(\cdots) = \Phi(t-\tau, t-s)^\top \left(y - \int_s^\tau \Phi(t-\tau, t-\sigma)\eta(t-\sigma)\, d\sigma \right)$$

where the orthogonal matrix $\Phi(\cdot,\cdot)$ is the same as in (3.4). It is verified that the relation

$$U(t,s)^* = \widetilde{U}(t-s, 0; t), \qquad t \geqq s \geqq 0,$$

recovers (3.5) as in (2.13).

Although we will provide the results (Lemmas 3.1, 3.2) only on the evolution operator $U(t,s)$, those for the adjoint $U(t,s)^*$ or $\widetilde{U}(\tau, s; t)$ are also available and will be needed to obtain the assertions for the adjoint $T(t,s)^*$.

Let $1 < q < \infty$. Correspondingly to the auxiliary spaces (2.9) for the exterior problem, let us introduce the Banach spaces

$$\begin{aligned} &Z_q(\mathbb{R}^3) = \{u \in L^q_\sigma(\mathbb{R}^3) \cap W^{1,q}(\mathbb{R}^3);\ |x|\nabla u \in L^q(\mathbb{R}^3)\}, \\ &Y_q(\mathbb{R}^3) = Z_q(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3), \end{aligned}$$

endowed with the corresponding norms to describe the regularity of the solution. We note that, under the condition (1.3) solely, the regularity deduced in the following lemma holds true subject to estimates (3.12)–(3.13) below for $0 \leqq s < t \leqq \mathcal{T}$ with $C > 0$ that depends on $\mathcal{T} \in (0, \infty)$. Nevertheless, for later use, we will show those estimates for $(t, s) \in \Lambda(\tau_*)$, see (2.19), under the additional assumption $\eta \in L^\infty(0, \infty; \mathbb{R}^3)$ [even under (1.2)].

Lemma 3.1. *Suppose that η and ω fulfill (1.3) for some $\theta \in (0, 1)$. Assume in addition that $\eta \in L^\infty(0, \infty; \mathbb{R}^3)$ for the second, third and fourth assertions below. Let $1 < q < \infty$. Then $\{U(t,s)\}_{t \geqq s \geqq 0}$ given by (3.4) defines an evolution operator on $L^q(\mathbb{R}^3)$ and on $L^q_\sigma(\mathbb{R}^3)$. Furthermore, we have the following properties:*

1. *Let $q \leqq r \leqq \infty$. For every integer $j \geqq 0$, there is a constant $c_j = c_j(q, r) > 0$, independent of η and ω, such that*

$$\begin{aligned} &\nabla^j U(\cdot, s) f \in C((s, \infty); L^r(\mathbb{R}^3)), \\ &\|\nabla^j U(t,s) f\|_{r,\mathbb{R}^3} \leqq c_j (t-s)^{-(3/q-3/r)/2 - j/2} \|f\|_{q,\mathbb{R}^3} \end{aligned} \tag{3.10}$$

for all $t > s \geqq 0$ and $f \in L^q(\mathbb{R}^3)$.

2. *Let* $q \leqq r < \infty$ *and* $m \in (0, \infty)$. *For every* $f \in Z_q(\mathbb{R}^3)$ *and* $t \in (s, \infty)$, *we have* $|x|\nabla U(t,s)f \in L^r(\mathbb{R}^3)$ *subject to*

$$\begin{aligned} &\| |x| \nabla U(t,s) f \|_{r,\mathbb{R}^3} \\ &\leqq C(t-s)^{-(3/q-3/r)/2} \| |x| \nabla f \|_{q,\mathbb{R}^3} \\ &\quad + C(t-s)^{-(3/q-3/r)/2+1/2} \{1 + m(t-s)^{1/2}\} \| \nabla f \|_{q,\mathbb{R}^3} \end{aligned} \tag{3.11}$$

for all $t > s \geqq 0$ *with some constant* $C = C(q,r) > 0$, *whenever* $|\eta|_0 := \sup_{t \geqq 0} |\eta(t)| \leqq m$.

3. *For every* $f \in Z_q(\mathbb{R}^3)$ *and* $t \in (s, \infty)$, *we have* $U(t,s)f \in Y_q(\mathbb{R}^3)$ *and*

$$u := U(\cdot, s) f \in C^1((s,\infty); L^q_\sigma(\mathbb{R}^3))$$

with (3.1)–(3.2) *in* $L^q_\sigma(\mathbb{R}^3)$. *Let* $\tau_* \in (0,\infty)$ *and* $m \in (0,\infty)$. *If in addition* (1.2) *is assumed, then there is a constant* $C = C(\tau_*, m, q) > 0$ *such that*

$$\|U(t,s)f\|_{Y_q(\mathbb{R}^3)} + \|\partial_t U(t,s) f\|_{q,\mathbb{R}^3} \leqq C(t-s)^{-1/2} \|f\|_{Z_q(\mathbb{R}^3)} \tag{3.12}$$

for all $(t,s) \in \Lambda(\tau_*)$ *and* $f \in Z_q(\mathbb{R}^3)$ *whenever* (2.21) *is satisfied, where* $\Lambda(\tau_*)$ *is given by* (2.19).

4. *Let* $q \leqq r < \infty$, $\tau_* \in (0,\infty)$ *and* $m \in (0,\infty)$. *For every* $f \in Z_q(\mathbb{R}^3)$ *and* $t \in (s,\infty)$, *we have* $U(t,s)f \in Z_r(\mathbb{R}^3)$ *subject to*

$$\|U(t,s)f\|_{Z_r(\mathbb{R}^3)} \leqq C(t-s)^{-(3/q-3/r)/2} \|f\|_{Z_q(\mathbb{R}^3)} \tag{3.13}$$

for all $(t,s) \in \Lambda(\tau_*)$ *with some constant* $C = C(\tau_*, m, q, r) > 0$ *whenever* $|\eta|_0 \leqq m$.

Proof. The first assertion follows from the corresponding properties of the heat semigroup. The third assertion is a slight improvement of the one in [26,27], but it follows from knowledge obtained there (see Proposition 3.1 (a) of [27]). The second and fourth assertions for the case $r > q$ are new and preparations for Lemma 5.4.

As in the proof of (3.11) with $r = q$ by [26], we have

$$\begin{aligned} &|x| |\nabla (U(t,s)f)(x)| \\ &\leqq \int_{\mathbb{R}^3} (|x-y| + |y|) \frac{e^{-|x-y|^2/4(t-s)}}{\{4\pi(t-s)\}^{3/2}} \left| (\nabla f)\left(\Phi(t,s)^\top (y + h_{t,s}) \right) \right| dy \\ &=: I + J, \end{aligned}$$

where $h_{t,s} := \int_s^t \Phi(t,\tau)\eta(\tau)\, d\tau$. We then find that

$$\|I\|_{r,\mathbb{R}^3} \leqq C(t-s)^{-(3/q-3/r)/2+1/2} \|\nabla f\|_{q,\mathbb{R}^3}$$

and that

$$\begin{aligned} \|J\|_{r,\mathbb{R}^3} &\leqq C(t-s)^{-(3/q-3/r)/2} \left\| |\cdot| (\nabla f)\left(\Phi(t,s)^\top (\cdot + h_{t,s}) \right) \right\|_{q,\mathbb{R}^3} \\ &\leqq C(t-s)^{-(3/q-3/r)/2} \left\{ \| |\cdot| \nabla f\|_{q,\mathbb{R}^3} + |\eta|_0 (t-s) \|\nabla f\|_{q,\mathbb{R}^3} \right\}. \end{aligned}$$

They thus imply (3.11). It is easily seen that

$$\|\nabla^{j+1}U(t,s)f\|_{r,\mathbb{R}^3} \leqq C(t-s)^{-(3/q-3/r)/2-j/2}\|\nabla f\|_{q,\mathbb{R}^3}$$

for all $t > s \geqq 0$, $1 < q \leqq r < \infty$ and $j = 0, 1$, which together with (3.10)–(3.11) (and by using the Equation (3.1) for $\partial_t U(t,s)f$) leads to (3.12) as well as (3.13). The proof is complete. □

It is natural to expect that $U(t,s)f$ is a weak solution in a sense together with a reasonable estimate of $\partial_t U(t,s)f$ even if $f \in L^q_\sigma(\mathbb{R}^3)$ rather than $f \in Z_q(\mathbb{R}^3)$. The next lemma gives an affirmative answer. Indeed the assumption (1.3) is enough to obtain the assertion, but the constant in (3.15) below depends on $\mathcal{T} \in (0,\infty)$ for $0 \leqq s < t \leqq \mathcal{T}$. For later use, it is convenient to show the following form when assuming (1.2):

Lemma 3.2. *Suppose that η and ω fulfill (1.2) for some $\theta \in (0,1)$. Let $1 < q < \infty$ and $R > 0$. Given $f \in L^q_\sigma(\mathbb{R}^3)$ and $s \geqq 0$, we set $u(t) = U(t,s)f$. For each $\tau_* \in (0,\infty)$ and $m \in (0,\infty)$, there is a constant $C = C(\tau_*, m, q, R) > 0$ such that*

$$u \in C^1((s,\infty); W^{-1,q}(B_R)), \tag{3.14}$$

$$\|\partial_t U(t,s)f\|_{W^{-1,q}(B_R)} \leqq C(t-s)^{-1/2}\|f\|_{q,\mathbb{R}^3} \tag{3.15}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(\mathbb{R}^3)$ whenever (2.21) is satisfied, where $\Lambda(\tau_*)$ is given by (2.19). Furthermore, we have

$$\langle \partial_t u, \psi\rangle_{B_R} + \langle \nabla u + u \otimes (\eta + \omega \times x) - (\omega \times x) \otimes u, \nabla\psi\rangle_{B_R} = 0 \tag{3.16}$$

for all $t \in (s,\infty)$ and $\psi \in W^{1,q'}_0(B_R)^3$, where $1/q' + 1/q = 1$.

Proof. Given $f \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ and $s \geqq 0$, we set $u(t) = U(t,s)f$, which satisfies (3.16) for every $\psi \in C^\infty_0(B_R)^3$. From this together with (3.10) we see that

$$\begin{aligned}|\langle \partial_t u, \psi\rangle_{B_R}| &\leqq \left\{\|\nabla u\|_{q,\mathbb{R}^3} + m(1+2R)\|u\|_{q,\mathbb{R}^3}\right\}\|\nabla\psi\|_{q',B_R} \\ &\leqq C\left\{1 + m(1+2R)\sqrt{\tau_*}\right\}(t-s)^{-1/2}\|f\|_{q,\mathbb{R}^3}\|\nabla\psi\|_{q',B_R}\end{aligned}$$

as long as $t - s \leqq \tau_*$. We thus obtain (3.15) for $f \in C^\infty_{0,\sigma}(\mathbb{R}^3)$. Given $f \in L^q_\sigma(\mathbb{R}^3)$, we take $f_j \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ which converges to f as $j \to \infty$ in the norm $\|\cdot\|_{q,\mathbb{R}^3}$. Then $\partial_t U(t,s)f_j$ goes to some $W_R(t,s)f \in W^{-1,q}(B_R)$. Since the convergence is uniform with respect to t belonging to any compact interval in (s,∞), we have $W_R(\cdot,s)f \in C((s,\infty); W^{-1,q}(B_R))$. From this convergence with (3.10) we observe

$$U(t,s)f = U(s+\varepsilon,s)f + \int_{s+\varepsilon}^t W_R(\tau,s)f\,d\tau$$

in $W^{-1,q}(B_R)$, where $\varepsilon > 0$ is arbitrary. This implies (3.14) and $W_R(t,s)f$ coincides with $\partial_t U(t,s)f$ for every $R > 0$. Hence, we obtain (3.15). Equation (3.16) is easily verified by approximation procedure above. □

4. Interior Problem

This section is devoted to the study of the initial value problem for the non-autonomous system

$$\begin{aligned} \partial_t u &= \Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u - \nabla p, \\ \operatorname{div} u &= 0, \\ u|_{\partial D_R} &= 0, \\ u(\cdot, s) &= f, \end{aligned} \tag{4.1}$$

in $D_R \times (s, \infty)$ with $R \in [R_0, \infty)$ being fixed, where R_0 is as in (2.1). Let $1 < q < \infty$. Let us introduce the Stokes operator

$$\begin{aligned} D_q(A) &= L^q_\sigma(D_R) \cap W^{1,q}_0(D_R) \cap W^{2,q}(D_R), \\ Au &= -P_{D_R} \Delta u, \end{aligned}$$

and the operator

$$\begin{aligned} D_q(L_R(t)) &= D_q(A), \\ L_R(t)u &= -P_{D_R}[\Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u] \\ &= Au - (\eta(t) + \omega(t) \times x) \cdot \nabla u + \omega(t) \times u, \end{aligned}$$

where P_{D_R} denotes the Fujita–Kato projection associated with the Helmholtz decomposition [15], see Section 2. The last equality above follows from (3.3) and the fact that the normal trace of the drift term vanishes, see [34, (3.22)].

For the interior problem one can apply the general theory of parabolic evolution operators developed by Tanabe, see [51, Chapter 5], to find that $\{L_R(t)\}_{t \geqq 0}$ generates an evolution operator $\{V(t, s)\}_{t \geqq s \geqq 0}$ on $L^q_\sigma(D_R)$. For every $f \in L^q_\sigma(D_R)$, we know that $u(t) = V(t, s)f$ is of class

$$\begin{aligned} &u \in C^1((s, \infty); L^q_\sigma(D_R)) \cap C((s, \infty); D_q(A)) \cap C([s, \infty), L^q_\sigma(D_R)), \\ &\nabla p \in C((s, \infty); L^q(D_R)), \end{aligned} \tag{4.2}$$

and satisfies (4.1) in $L^q_\sigma(D_R)$. If, in addition, the pressure p is chosen such that $\int_{D_R} p \, dx = 0$ for each time t, then

$$p \in C((s, \infty); L^q(D_R)) \tag{4.3}$$

by the Poincaré inequality together with (4.2) for ∇p.

We start with the following lemma [27,34]:

Lemma 4.1. *Suppose that* η *and* ω *fulfill* (1.2) *for some* $\theta \in (0, 1)$. *Let* $1 < q \leqq r < \infty$. *For each* $\tau_* \in (0, \infty)$, $m \in (0, \infty)$ *and* $j = 0, 1$, *there are constants* $C_j = C_j(\tau_*, m, q, r, \theta, D_R) > 0$ *and* $C_2 = C_2(\tau_*, m, q, \theta, D_R) > 0$ *such that*

$$\|\nabla^j V(t, s)f\|_{r, D_R} \leqq C_j (t - s)^{-(3/q - 3/r)/2 - j/2} \|f\|_{q, D_R} \tag{4.4}$$

$$\|p(t)\|_{q, D_R} \leqq C_2 (t - s)^{-(1 + 1/q)/2} \|f\|_{q, D_R} \tag{4.5}$$

$$\|\partial_t V(t, s)f\|_{W^{-1,q}(D_R)} \leqq C_2 (t - s)^{-(1 + 1/q)/2} \|f\|_{q, D_R} \tag{4.6}$$

for all $(t,s)\in\Lambda(\tau_*)$ *and* $f\in L^q_\sigma(D_R)$ *whenever* (2.21) *is satisfied, where* $\Lambda(\tau_*)$ *is given by* (2.19). *Here,* $p(t)$ *denotes the pressure associsted with* $V(t,s)f$ *and it is singled out subject to the side condition* $\int_{D_R} p\,\mathrm{d}x=0$.

Proof. L^q-L^r estimate (4.4) was shown by [27] for $0\leqq s<t\leqq\mathcal{T}$ with $C_j>0$ that depends on $\mathcal{T}\in(0,\infty)$ under the condition (1.3). The present author [34, Lemma 3.2] verified that the constant C_j can be taken uniformly in (t,s) satisfying $t-s\leqq\tau_*$ as long as (1.2) is fulfilled. Set $u(t)=V(t,s)f$. Estimate (4.5) for the pressure was also proved by [34, Lemma 3.2] via

$$\|p(t)\|_{q,D_R}\leqq C\|\nabla^2u(t)\|_{q,D_R}^{1/q}\|\nabla u(t)\|_{q,D_R}^{1-1/q}+C\|\nabla u(t)\|_{q,D_R} \tag{4.7}$$

and it is a slight improvement of the one obtained by [27, Lemma 4.3]. The remarkable rate $(t-s)^{-(1+1/q)/2}$ for the pressure near the initial time was discovered first by [37] for the autonomous case (even for the Stokes system) and the proof relied on analysis of the resolvent. Estimate (4.6) immediately follows from

$$\begin{aligned}&\langle\partial_t u,\psi\rangle_{D_R}\\&=-\langle\nabla u+u\otimes(\eta+\omega\times x)-(\omega\times x)\otimes u,\nabla\psi\rangle_{D_R}+\langle p,\operatorname{div}\psi\rangle_{D_R}\end{aligned} \tag{4.8}$$

for every $\psi\in C_0^\infty(D_R)^3$ together with (4.4)–(4.5). □

We next deduce the asymptotic behavior of $V(t,s)f$ near $t=s$ in some Sobolev spaces when $f\in L^q_\sigma(D_R)\cap W^{1,q}(D_R)$. It should be emphasized that f does not satisfy the boundary condition $f|_{\partial D_R}=0$, and the reason why we have to discuss this case is related to the function space $Z_q(D)$, see (2.9), in which the boundary condition at ∂D is not involved. In fact, the following lemma plays a role in the proof of Lemma 5.3. Estimate (4.9) below should be compared with [27, Corollary 4.2], where less singular behavior $(t-s)^{-1/2}$ is deduced for $f\in L^q_\sigma(D_R)\cap W^{1,q}_0(D_R)$ satisfying $f|_{\partial D_R}=0$.

Lemma 4.2. *Suppose that* η *and* ω *fulfill* (1.2) *for some* $\theta\in(0,1)$. *Let* $1<q\leqq r<\infty$ *and* $\delta\in(0,1/2q)$. *For each* $\tau_*\in(0,\infty)$ *and* $m\in(0,\infty)$, *there are constants* $C_1=C_1(\tau_*,m,q,\delta,\theta,D_R)>0$ *and* $C_2=C_2(\tau_*,m,q,r,\delta,\theta,D_R)>0$ *such that*

$$\begin{aligned}&\|V(t,s)f\|_{W^{2,q}(D_R)}+\|\partial_tV(t,s)f\|_{q,D_R}+\|\nabla p(t)\|_{q,D_R}\\&\quad\leqq C_1(t-s)^{-1+\delta}\|f\|_{W^{1,q}(D_R)}\end{aligned} \tag{4.9}$$

$$\|p(t)\|_{q,D_R}\leqq C_1(t-s)^{-(1+1/q)/2+\delta}\|f\|_{W^{1,q}(D_R)} \tag{4.10}$$

$$\|V(t,s)f\|_{W^{1,r}(D_R)}\leqq C_2(t-s)^{-(3/q-3/r)/2-1/2+\delta}\|f\|_{W^{1,q}(D_R)} \tag{4.11}$$

for all $(t,s)\in\Lambda(\tau_*)$ *and* $f\in L^q_\sigma(D_R)\cap W^{1,q}(D_R)$ *whenever* (2.21) *is satisfied, where* $\Lambda(\tau_*)$ *is given by* (2.19). *Here,* $p(t)$ *denotes the pressure associated with* $V(t,s)f$ *and it is singled out subject to the side condition* $\int_{D_R}p\,\mathrm{d}x=0$.

Proof. As in the proof of [34, Lemma 3.2], there is a constant $k = k(m) > 0$ such that $k + L_R(t)$ is invertible in $L^q_\sigma(D_R)$ for all $t \geqq 0$ subject to

$$\sup_{t \geqq 0} \|(k + L_R(t))^{-1}\|_{\mathcal{L}(L^q_\sigma(D_R))} < \infty.$$

Indeed one can take even $k = 0$ by a compactness argument (see, for instance, [37, Section 3], [32, Section 5]), but this refinement is not needed here. We then know that

$$\|L_R(t)V(t,s)f\|_{q,D_R} \leqq C\|(k + L_R(s))f\|_{q,D_R} \leqq C\|f\|_{D_q(A)}, \qquad f \in D_q(A),$$

and

$$\|L_R(t)V(t,s)f\|_{q,D_R} \leqq C(t-s)^{-1}\|f\|_{q,D_R}, \qquad f \in L^q_\sigma(D_R),$$

for all $(t,s) \in \Lambda(\tau_*)$. In fact, the latter was shown in [34, (3.20)], while one verifies the former (particularly the first inequality) if one follows the argument of general theory [51, Chapter 5, Theorem 2.1] under the conditions (1.2) and (2.21).

By complex interpolation we have

$$\|L_R(t)V(t,s)f\|_{q,D_R} \leqq C(t-s)^{-1+\delta}\|f\|_{D_q(A^\delta)}$$

for all $(t,s) \in \Lambda(\tau_*)$ and

$$\begin{aligned} f \in D_q(A^\delta) &= [L^q_\sigma(D_R), D(A)]_\delta \\ &= L^q_\sigma(D_R) \cap [L^q(D_R), W^{1,q}_0(D_R) \cap W^{2,q}(D_R)]_\delta \end{aligned}$$

where $[\cdot,\cdot]_\delta$ stands for the complex interpolation functor and the characterization of $D_q(A^\delta)$ is due to Giga [24]. As a consequence, we get

$$\begin{aligned} &\|V(t,s)f\|_{W^{2,q}(D_R)} + \|\partial_t V(t,s)f\|_{q,D_R} + \|\nabla p(t)\|_{q,D_R} \\ &\leqq C\|L_R(t)V(t,s)f\|_{q,D_R} + C\|V(t,s)f\|_{q,D_R} \\ &\leqq C(t-s)^{-1+\delta}\|f\|_{D_q(A^\delta)} \end{aligned} \tag{4.12}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in D_q(A^\delta)$ provided $0 \leqq \delta \leqq 1$.

If in particular $\delta \in (0, 1/2q)$, then the space $D_q(A^\delta)$ does not involve the boundary condition, to be precise, $D_q(A^\delta) = L^q_\sigma(D_R) \cap H^{2\delta}_q(D_R)$, where $H^{2\delta}_q(D_R) := [L^q(D_R), W^{2,q}(D_R)]_\delta$ is the Bessel potential space, see Fujiwara [14, Section 2, Theorem 5] (this theorem asserts a characterization of some complex interpolation spaces). We thus have $L^q_\sigma(D_R) \cap W^{1,q}(D_R) \subset D_q(A^\delta)$ for $\delta \in (0, 1/2q)$ and, therefore, (4.12) leads us to (4.9).

We next observe

$$\|V(t,s)f\|_{W^{1+j,q}(D_R)} \leqq C(t-s)^{-j/2}\|f\|_{W^{1,q}(D_R)} \tag{4.13}$$

for all $(t,s) \in \Lambda(\tau_*)$, $f \in L^q_\sigma(D_R) \cap W^{1,q}_0(D_R)$ and $j = 0,\ 1$. In [27, Corollary 4.2] Hansel and Rhandi proved (4.13) for such data satisfying $f|_{\partial D_R} = 0$ and $0 \leqq s < t \leqq \mathcal{T}$ with $C > 0$ that depends on $\mathcal{T} \in (0,\infty)$ under the condition

(1.3), however, we need to show that the constant $C > 0$ can be taken uniformly in $(t, s) \in \Lambda(\tau_*)$ as long as (1.2) is fulfilled. In fact, using (4.12) with $\delta = 1/2$, we find $(4.13)_{j=1}$ since we know from [14,24] that $D_q(A^{1/2}) = L^q_\sigma(D_R) \cap W^{1,q}_0(D_R)$. We also have (4.12) with $\delta = 1$ as well as $(4.4)_{j=0}$ with $r = q$, which implies $(4.13)_{j=0}$ by interpolation. The interpolation argument once more by use of $(4.13)_{j=0}$ and (4.4) with $r = q$ yields

$$\|V(t,s)f\|_{W^{1,q}(D_R)} \leqq C(t-s)^{-1/2+\delta}\|f\|_{H^{2\delta}_q(D_R)}$$

for all $(t, s) \in \Lambda(\tau_*)$ and

$$f \in [L^q_\sigma(D_R), L^q_\sigma(D_R) \cap W^{1,q}_0(D_R)]_{2\delta} = L^q_\sigma(D_R) \cap H^{2\delta}_q(D_R)$$

provided $\delta \in (0, 1/2q)$, where the last equality follows from the reiteration theorem for the complex interpolation [1] combined with the Fujiwara theorem [14] employed above; thereby, we infer

$$\|V(t,s)f\|_{W^{1,q}(D_R)} \leqq C(t-s)^{-1/2+\delta}\|f\|_{W^{1,q}(D_R)} \tag{4.14}$$

for all $(t, s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(D_R) \cap W^{1,q}(D_R)$. This together with (4.9) concludes (4.10) by virtue of (4.7).

It turns out that

$$\|V(t,s)g\|_{W^{1,r}(D_R)} \leqq C(t-s)^{-(3/q-3/r)/2}\|g\|_{W^{1,q}(D_R)} \tag{4.15}$$

for all $(t, s) \in \Lambda(\tau_*)$ and $g \in L^q_\sigma(D_R) \cap W^{1,q}_0(D_R)$, where $1 < q \leqq r < \infty$. In fact, this follows from (4.13) together with the Gagliardo-Nirenberg inequality provided that $3/q - 3/r \leqq 1$. If r is not close to q, then one has only to use the semigroup property. Note that $g = V((s+t)/2, s)f$ fulfills the boundary condition $g|_{\partial D_R} = 0$ so that $g \in L^q_\sigma(D_R) \cap W^{1,q}_0(D_R)$ even though $f \in L^q_\sigma(D_R) \cap W^{1,q}(D_R)$. Hence, by the semigroup property, (4.14) and (4.15) imply (4.11). The proof is complete. □

5. Regularity of the Evolution Operator

Some regularity properties as well as construction of the evolution operator $T(t, s)$ were proved by Hansel and Rhandi [27], nevertheless, we need more analysis, especially,

– The smoothing effect of $T(t, s) : Z_q(D) \to Y_q(D)$ when $3/2 < q < \infty$;
– The smoothing effect of $T(t, s) : Z_q(D) \to Z_r(D)$ when $3/2 < q < r < \infty$;
– The justification of $\partial_t T(t, s)f$ in $W^{-1,q}(D_R)$ for $f \in L^q_\sigma(D)$ when $1 < q < \infty$;

which are not covered by [27], where $Y_q(D)$ and $Z_q(D)$ are defined by (2.9). We will also show the second assertion of Proposition 2.1, that is related to the first issue above since it slightly improves the corresponding result of [27]. The restriction $q > \frac{3}{2} = \frac{n}{n-1}$ (n denotes the space dimension) stems from Lemma 5.2 below on some weighted estimate of the Fujita-Kato projection. The third issue above is quite important to proceed to analysis of large time behavior of $T(t, s)$.

Let us recall the idea of [27] for construction of a parametrix of the evolution operator by use of evolution operators in the whole space $\mathbb{R}^3$ and in the bounded domain D_{R_0+6}, where R_0 is as in (2.1). We fix three cut-off functions

$$\begin{aligned}
&\phi \in C_0^\infty(B_{R_0+4}), \qquad \phi = 1 \quad \text{in } B_{R_0+3},\\
&\phi_0 \in C_0^\infty(B_{R_0+2}), \qquad \phi = 1 \quad \text{in } B_{R_0+1},\\
&\phi_1 \in C_0^\infty(B_{R_0+6}), \qquad \phi = 1 \quad \text{in } B_{R_0+5},
\end{aligned}$$

and set

$$\begin{aligned}
&A = \{R_0 + 2 < |x| < R_0 + 4\}, \quad A_0 = \{R_0 < |x| < R_0 + 2\},\\
&A_1 = \{R_0 + 4 < |x| < R_0 + 6\}.
\end{aligned}$$

By $\mathbb{B} = \mathbb{B}_A$, $\mathbb{B}_0 = \mathbb{B}_{A_0}$ and $\mathbb{B}_1 = \mathbb{B}_{A_1}$ we denote the Bogovskii operators, see (2.3), in the bounded domains A, A_0 and A_1, respectively. Given $f \in L^q_\sigma(D)$, $1 < q < \infty$, let us set

$$\begin{aligned}
&f_0 = (1 - \phi_0)f + \mathbb{B}_0[f \cdot \nabla\phi_0] \in L^q_\sigma(\mathbb{R}^3),\\
&f_1 = \phi_1 f - \mathbb{B}_1[f \cdot \nabla\phi_1] \in L^q_\sigma(D_{R_0+6}),
\end{aligned}$$

where f_0 is understood as its extension to $\mathbb{R}^3$ by putting zero outside D, then we see from (2.2) that

$$\begin{aligned}
&\|f_0\|_{q,\mathbb{R}^3} + \|f_1\|_{q,D_{R_0+6}} \leqq C\|f\|_q,\\
&\|\nabla f_0\|_{q,\mathbb{R}^3} + \|\nabla f_1\|_{q,D_{R_0+6}} \leqq C\|f\|_{W^{1,q}(D)},\\
&\||x|\nabla f_0\|_{q,\mathbb{R}^3} \leqq C\||x|\nabla f\|_q + C\|f\|_q,
\end{aligned} \tag{5.1}$$

where $\nabla f \in L^q(D)$ is additionally assumed for $(5.1)_2$ and even $|x|\nabla f \in L^q(D)$ is assumed for $(5.1)_3$. Thus, $f_0 \in Z_q(\mathbb{R}^3)$ follows from $f \in Z_q(D)$.

It is reasonable to start with

$$W(t,s)f = (1-\phi)U(t,s)f_0 + \phi V(t,s)f_1 + \mathbb{B}[(U(t,s)f_0 - V(t,s)f_1)\cdot\nabla\phi] \tag{5.2}$$

as a fine approximation of the evolution operator, where $U(t,s)$ is the evolution operator for the whole space problem (Section 3) and $V(t,s)$ is the one for the interior problem (Section 4) over D_{R_0+6}. Note that $W(s,s)f = f$. In what follows, let us fix $\tau_* \in (0,\infty)$ as well as $m \in (0,\infty)$, and suppose (2.21). By (3.10), (4.4) and $(5.1)_1$ together with (2.2), we easily observe

$$\|\nabla^j W(t,s)f\|_q \leqq C(t-s)^{-j/2}\|f\|_q \tag{5.3}$$

for all $(t,s) \in \Lambda(\tau_*)$, $j = 0, 1$ and $f \in L^q_\sigma(D)$ with $C = C(\tau_*, m, q, \theta, D) > 0$. By p_1 we denote the pressure associated with $V(t,s)f_1$ for the interior problem over D_{R_0+6}, and it is singled out subject to the side condition $\int_{D_{R_0+6}} p_1\,dx = 0$. Then the pair of

$$u := W(t,s)f, \qquad p := \phi p_1$$

should obey

$$\begin{aligned}\partial_t u &= \Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u - \nabla p - K(t,s)f,\\ \operatorname{div} u &= 0,\\ u|_{\partial D} &= 0,\\ u &\to 0 \quad \text{as } |x| \to \infty,\\ u(\cdot, s) &= f,\end{aligned}$$

in $D \times (s, \infty)$ (the equation is actually understood in $L^q(D)$ for $f \in Z_q(D)$) with

$$\begin{aligned}&K(t,s)f\\ &= -2\nabla\phi \cdot \nabla(Uf_0 - Vf_1) - \{\Delta\phi + (\eta + \omega \times x) \cdot \nabla\phi\}(Uf_0 - Vf_1)\\ &\quad - (\nabla\phi)p_1 - \mathbb{B}[(\partial_t Uf_0 - \partial_t Vf_1) \cdot \nabla\phi] + \Delta\mathbb{B}[(Uf_0 - Vf_1) \cdot \nabla\phi]\\ &\quad + (\eta + \omega \times x) \cdot \nabla\mathbb{B}[(Uf_0 - Vf_1) \cdot \nabla\phi] - \omega \times \mathbb{B}[(Uf_0 - Vf_1) \cdot \nabla\phi],\end{aligned} \tag{5.4}$$

where we abbreviate $Uf_0 = U(t,s)f_0$ and $Vf_1 = V(t,s)f_1$. As in [27, (5.3)], it follows from (2.2), (2.4), (3.10), (3.14), (3.15), (4.2), (4.3), Lemma 4.1 and $(5.1)_1$ that

$$\begin{aligned}&PK(\cdot, s)f \in C((s,\infty); L^q_\sigma(D)),\\ &\|PK(t,s)f\|_q \leqq C(t-s)^{-(1+1/q)/2}\|f\|_q,\end{aligned} \tag{5.5}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(D)$ with some $C = C(\tau_*, m, q, \theta, D) > 0$, where $\Lambda(\tau_*)$ is given by (2.19).

The approach adopted by [27] is somewhat similar to the one for construction of parabolic evolution operators, see [51, Chapter 5], although the first approximation (5.2) is completely different from general theory. In fact, the idea of [27] is to solve the integral equation

$$T(t,s)f = W(t,s)f + \int_s^t T(t,\tau)PK(\tau,s)f\, d\tau. \tag{5.6}$$

To this end, consider the iteration scheme

$$\begin{aligned}T_0(t,s)f &= W(t,s)f,\\ T_{j+1}(t,s)f &= \int_s^t T_j(t,\tau)PK(\tau,s)f\, d\tau \qquad (j = 0, 1, 2, \cdots).\end{aligned} \tag{5.7}$$

One can expect that (5.9) below provides a solution as long as it is convergent. The argument of [27] is based on the next lemma on iterated convolutions, see [22, Lemma 4.6], [26, Lemma 3.3] and [27, Lemma 5.2] (the same idea was essentially employed in [51, Chapter 5, Sections 2 and 3], too). In those literature the operator families are parametrized by (t,s) with $0 \leqq s < t \leqq \mathcal{T}$ for fixed $\mathcal{T} \in (0,\infty)$, but we need to discuss the ones parametrized by $(t,s) \in \Lambda(\tau_*)$, see (2.19), and what is important is that the constant in (5.8) below can be taken uniformly in $(t,s) \in \Lambda(\tau_*)$. This is easily verified by following the proof in the literature above.

Lemma 5.1. [22,26,27] *Let X_1 and X_2 be two Banach spaces, and fix $\tau_* \in (0, \infty)$. Suppose that there are constants $\alpha,\ \beta \in [0, 1)$ and $\kappa > 0$ such that*

$$\{A_0(t, s);\ (t, s) \in \Lambda(\tau_*)\} \subset \mathcal{L}(X_1, X_2), \qquad \{Q(t, s);\ (t, s) \in \Lambda(\tau_*)\} \subset \mathcal{L}(X_1)$$

with

$$\|A_0(t, s)\|_{\mathcal{L}(X_1,X_2)} \leqq \kappa(t - s)^{-\alpha}, \qquad \|Q(t, s)\|_{\mathcal{L}(X_1)} \leqq \kappa(t - s)^{-\beta}$$

for all $(t, s) \in \Lambda(\tau_*)$. For $f \in X_1$ and $(t, s) \in \Lambda(\tau_*)$, define a sequence $\{A_j(t, s) f\}_{j=0}^{\infty} \subset X_2$ by

$$A_{j+1}(t, s) f = \int_s^t A_j(t, \tau) Q(\tau, s) f \, \mathrm{d}\tau \qquad (j = 0, 1, 2, \cdots).$$

Then

$$A(t, s) f := \sum_{j=0}^{\infty} A_j(t, s) f \qquad \text{in } X_2$$

converges absolutely and uniformly in $(t, s) \in \Lambda(\tau_*)$ with $t - s \geqq \varepsilon$ for every $\varepsilon \in (0, \tau_*)$. Moreover, there is a constant $C = C(\tau_*, \kappa, \alpha, \beta) > 0$ such that

$$\|A(t, s) f\|_{X_2} \leqq \sum_{j=0}^{\infty} \|A_j(t, s) f\|_{X_2} \leqq C(t - s)^{-\alpha} \|f\|_{X_1} \tag{5.8}$$

for all $(t, s) \in \Lambda(\tau_*)$ and $f \in X_1$. If in particular $\alpha = 0$, then the convergence of the series above is uniform in $(t, s) \in \overline{\Lambda(\tau_*)} = \{(t, s);\ 0 \leqq s \leqq t,\ t - s \leqq \tau_*\}$.

With (5.3) and (5.5) at hand, Hansel and Rhandi [27] applied Lemma 5.1 with

$$A_0 = W, \quad Q = PK, \quad X_1 = X_2 = L^q_\sigma(D), \quad \alpha = 0, \quad \beta = \frac{1}{2}\left(1 + \frac{1}{q}\right)$$

to (5.7) and succeeded in construction of the evolution operator

$$T(t, s) f := \sum_{j=0}^{\infty} T_j(t, s) f, \tag{5.9}$$

which solves (5.6). This was quite successful. In order to show that $T(t, s)$ leaves $Y_q(D)$ invariant, they first intended to prove $T(t, s) Z_{q,0}(D) \subset Z_{q,0}(D)$, where $Z_{q,0}(D) = \{f \in Z_q(D);\ f|_{\partial D} = 0\}$, see (2.9). Note that $Z_{q,0}(D)$ is denoted by Z in their paper, see [27, p. 17]. To this end, they applied Lemma 5.1 with $X_1 = X_2 = Z_{q,0}(D)$ as well as $A_0 = W$ and $Q = PK$, however, $PK(t, s) f$ cannot always belong to $Z_{q,0}(D)$ because $PK(t, s) f$ does not satisfy the homogeneous Dirichlet boundary condition at ∂D no matter how fine f is. Indeed this is unfortunately an oversight of [27], but their argument can be corrected in the following way.

The idea of correction is to replace $Z_{q,0}(D)$ by $Z_q(D)$, which does not involve the homogeneous Dirichlet boundary condition, and to employ the following weighted estimate of the Fujita-Kato projection. For the weighted estimate, one needs the restriction $q \in (3/2, \infty)$, however, this is not an obstacle for later argumant. See [28, Proposition 4.3] for similar consideration in the case $q = 2$. Note that the next lemma holds true even for $g \in W^{1,q}(D)$ (without boundary condition) with $|x|\nabla g \in L^q(D)$ if the second term of the RHS of (5.10) is replaced by $\|\nabla g\|_q$. Since we will use this lemma only with $g = K(t, s)f$, see (5.4), it is given in the following form:

Lemma 5.2. *Let* $3/2 < q < \infty$. *Then there is a constant* $C = C(q, D) > 0$ *such that*

$$\||x|\nabla Pg\|_q \leqq C(\||x|\nabla g\|_q + \|\mathrm{div}\, g\|_q + \|g\|_q) \tag{5.10}$$

for all $g \in W_0^{1,q}(D)^3$ *with* $|x|\nabla g \in L^q(D)^{3\times 3}$.

Proof. Consider the Neumann problem

$$-\Delta w = \mathrm{div}\, g \quad \text{in } D, \qquad \frac{\partial w}{\partial \nu}\Big|_{\partial D} = -\nu \cdot g|_{\partial D} = 0,$$

where ν stands for the outer unit normal to ∂D. It then suffices to show

$$\||x|\nabla^2 w\|_q \leqq C(\||x|(\mathrm{div}\, g)\|_q + \|\mathrm{div}\, g\|_q + \|g\|_q), \tag{5.11}$$

which implies (5.10) since $Pg = g + \nabla w$. We fix $L \in (R_0, \infty)$ and take a cut-off function $\phi \in C_0^\infty(D_L)$ such that $\phi = 1$ in B_{R_0}, where R_0 is as in (2.1). We choose a solution w satisfying $\int_{D_L} w \, \mathrm{d}x = 0$, so that

$$\|w\|_{q,D_L} \leqq C\|\nabla w\|_{q,D_L} \leqq C\|\nabla w\|_q \leqq C\|g\|_q, \tag{5.12}$$

where the last inequality is due to [44,48]. Then ϕw obeys

$$-\Delta(\phi w) = \phi(\mathrm{div}\, g) - 2\nabla\phi \cdot \nabla w - (\Delta\phi)w \quad \text{in } D_L, \qquad \nu \cdot \nabla(\phi w)|_{\partial D_L} = 0,$$

which leads to

$$\|\nabla^2(\phi w)\|_{q,D_L} \leqq C\|\mathrm{div}\, g\|_q + C\|w\|_{W^{1,q}(D_L)}, \tag{5.13}$$

where, this time, ν denotes the outer unit normal to ∂D_L. On the other hand, $(1-\phi)w$ obeys

$$-\Delta\{(1-\phi)w\} = (1-\phi)(\mathrm{div}\, g) + 2\nabla\phi \cdot \nabla w + (\Delta\phi)w =: h \quad \text{in } \mathbb{R}^3.$$

By $\mathcal{R} = \nabla(-\Delta)^{-1/2}$ we denote the Riesz transform, then we know

$$\||x|\mathcal{R}h\|_{q,\mathbb{R}^3} \leqq C\||x|h\|_{q,\mathbb{R}^3}$$

from the Muckenhoupt theory for singular integrals as long as $\frac{n}{n-1} = \frac{3}{2} < q < \infty$; in fact, for such q, the weight $|x|^q$ belongs to the Muckenhoupt class $\mathcal{A}_q(\mathbb{R}^3)$, see Farwig and Sohr [12, Section 2], Stein [49, Chapter V], Torchinsky [52, Chapter IX] for details. We thus obtain

$$\begin{aligned}\||x|\nabla^2\{(1-\phi)w\}\|_{q,\mathbb{R}^3} &= \||x|(\mathcal{R}\otimes\mathcal{R})h\|_{q,\mathbb{R}^3} \leqq C\||x|h\|_{q,\mathbb{R}^3} \\ &\leqq C\||x|(\mathrm{div}\, g)\|_q + C\|w\|_{W^{1,q}(D_L)}\end{aligned} \tag{5.14}$$

for $3/2 < q < \infty$. We collect (5.12), (5.13) and (5.14) to conclude (5.11). □

Since the functions being in our class $Z_q(D)$ do not satisfy the Dirichlet boundary condition, we have to replace [27, (5.4)] by (5.16) of the following lemma. The smoothing rate $(t-s)^{-1+\delta}$ below stems from (4.9) for the interior problem.

Lemma 5.3. *Suppose that η and ω fulfill (1.2) for some $\theta \in (0,1)$. Let $1 < q < \infty$ and $\delta \in (0, 1/2q)$. Given $\tau_* \in (0,\infty)$ and $m \in (0,\infty)$, let $\Lambda(\tau_*)$ be as in (2.19) and assume (2.21). Then,*

1. *There is a constant $C = C(\tau_*, m, q, \delta, \theta, D) > 0$ such that, for every $f \in Z_q(D)$ and $t \in (s,\infty)$, we have $W(t,s)f \in Y_q(D)$ subject to*

$$\|W(t,s)f\|_{Y_q(D)} \leqq C(t-s)^{-1+\delta}\|f\|_{Z_q(D)} \tag{5.15}$$

for all $(t,s) \in \Lambda(\tau_*)$.

2. *There is a constant $C = C(\tau_*, m, q, \delta, \theta, D) > 0$ such that*

$$\|K(t,s)f\|_{W^{1,q}(D)} \leqq C(t-s)^{-1+\delta}\|f\|_{Z_q(D)} \tag{5.16}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(D)$. If in particular $q \in (3/2,\infty)$, then there is a constant $C = C(\tau_*, m, q, \delta, \theta, D) > 0$ such that

$$\|PK(t,s)f\|_{Z_q(D)} \leqq C(t-s)^{-1+\delta}\|f\|_{Z_q(D)} \tag{5.17}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(D)$.

Proof. We collect (2.2), (3.10), (3.12), (4.4), (4.9), (4.10) and (5.1) to obtain (5.15) and (5.16). Since $K(t,s)f \in W_0^{1,q}(D)$ with $|x|\nabla K(t,s)f \in L^q(D)$, one can use (5.10) to obtain (5.17). □

Proof of the second assertion of Proposition 2.1. Let $3/2 < q < \infty$. In view of (5.7), (5.15) and (5.17) one can apply Lemma 5.1 with

$$A_0 = W, \quad Q = PK, \quad X_1 = Z_q(D), \quad X_2 = Y_q(D), \quad \alpha = \beta = 1-\delta$$

to see that $T(t,s)f \in Y_q(D)$ with

$$\|T(t,s)f\|_{Y_q(D)} \leqq C(t-s)^{-1+\delta}\|f\|_{Z_q(D)} \tag{5.18}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(D)$. Note that [27, (5.9)] is now replaced by (5.18). The proof of the other parts by [27] is correct and there is no need to repeat it. Here, the assertion has been proved under the condition (1.2) in order to deduce all the estimates with constants uniformly in $(t,s) \in \Lambda(\tau_*)$; in fact, such estimates are needed for Proposition 2.2. But one can show Proposition 2.1 under the same condition (1.3) as in [27] subject to the corresponding estimates for $0 \leqq s < t \leqq \mathcal{T}$, where $\mathcal{T} \in (0,\infty)$ is arbitrarily fixed. □

The following lemma on smoothing effect in the framework of the space $Z_q(D)$ is needed in the proof of Proposition 6.1.

Lemma 5.4. *Suppose that η and ω fulfill (1.2) for some $\theta \in (0,1)$. Let $3/2 < q \leqq r < \infty$. For every $f \in Z_q(D)$ and $t \in (s,\infty)$, we have $T(t,s)f \in Z_r(D)$.*

Proof. Let $\tau_* \in (0,\infty)$ and $\delta \in (0, 1/2q)$. By (2.2), (3.13) and (4.11) together with (5.1) we find

$$\|W(t,s)f\|_{Z_r(D)} \leqq C(t-s)^{-(3/q-3/r)/2-1/2+\delta}\|f\|_{Z_q(D)}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(D)$ even if $1 < q \leqq r < \infty$. By virtue of this combined with (5.17), we apply Lemma 5.1 with

$$A_0 = W, \quad Q = PK, \quad X_1 = Z_q(D), \quad X_2 = Z_r(D),$$
$$\alpha = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1}{2} - \delta, \quad \beta = 1 - \delta$$

to get the conclusion subject to

$$\|T(t,s)f\|_{Z_r(D)} \leqq C(t-s)^{-\alpha}\|f\|_{Z_q(D)}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(D)$ provided $\alpha < 1$ as well as $q \in (3/2,\infty)$. The condition $\alpha < 1$ with some $\delta \in (0, 1/2q)$ is always accomplished for every $r \in [q,\infty)$ when $q \geqq 2$. Otherwise ($3/2 < q < 2$), one needs a restriction that r is not too large. In this latter case, $T(t,s)f \in Z_r(D)$ for $r \in (q,2]$ is always possible and then we have only to use the semigroup property to obtain $T(t,s)f \in Z_r(D)$ even for $r \in (2,\infty)$ as follows:

$$\begin{aligned}\|T(t,s)f\|_{Z_r(D)} &\leqq C(t-s)^{-(3/2-3/r)/2-1/2+\widetilde{\delta}}\|T((t+s)/2,s)f\|_{Z_2(D)} \\ &\leqq C(t-s)^{-(3/q-3/r)/2-1+\widetilde{\delta}+\delta}\|f\|_{Z_q(D)}\end{aligned}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in Z_q(D)$, where $\max\{1/4 - 3/2r,\ 0\} < \widetilde{\delta} < 1/4$ and $\delta \in (0, 1/2q)$. The proof is complete. □

The following result justifies the derivative with respect to time variable with values in $W^{-1,q}(D_R)$ for general data being in $L^q_\sigma(D)$. This is indeed a key observation in the present paper and can be regarded as a substitution of [37, Theorem 5.1] for autonomous case. Here, a bounded domain D_R can be independent of D_{R_0+6} in which the solution $V(t,s)f_1$ was found in constructing the parametrix (5.2).

Proposition 5.1. *Suppose that η and ω fulfill (1.2) for some $\theta \in (0,1)$. Let $1 < q < \infty$ and $R \in (R_0+1,\infty)$, where R_0 is as in (2.1). Given $f \in L^q_\sigma(D)$, we set $u(t) = T(t,s)f$. Given $\tau_* \in (0,\infty)$ and $m \in (0,\infty)$, let $\Lambda(\tau_*)$ be as in (2.19) and assume (2.21). Then,*

1. *There is a constant $C = C(\tau_*, m, q, R, \theta, D) > 0$ such that*

$$u \in C^1((s,\infty); W^{-1,q}(D_R)), \tag{5.19}$$
$$\|\partial_t T(t,s)f\|_{W^{-1,q}(D_R)} \leqq C(t-s)^{-(1+1/q)/2}\|f\|_q \tag{5.20}$$

for all $(t,s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(D)$. Furthermore, we have the pressure $p(t)$ subject to $\int_{D_R} p\,dx = 0$ such that the pair $\{u,p\}$ satisfies

$$\begin{aligned}\langle \partial_t u, \psi\rangle_{D_R} + \langle \nabla u + u \otimes (\eta + \omega \times x) - (\omega \times x) \otimes u, \nabla\psi\rangle_{D_R} \\ - \langle p, div\ \psi\rangle_{D_R} = 0\end{aligned} \tag{5.21}$$

for all $t \in (s, \infty)$ *and* $\psi \in W_0^{1,q'}(D_R)^3$*, where* $1/q' + 1/q = 1$*, that*

$$\|p(t)\|_{q,D_R} \leqq C\|\partial_t u(t)\|_{W^{-1,q}(D_R)} + C\|u(t)\|_{W^{1,q}(D_R)} \tag{5.22}$$

for all $t \in (s, \infty)$ *with a constant* $C = C(m, q, R, D) > 0$ *and that*

$$\|p(t)\|_{q,D_R} \leqq C(t-s)^{-(1+1/q)/2}\|f\|_q \tag{5.23}$$

for all $(t, s) \in \Lambda(\tau_*)$ *with a constant* $C = C(\tau_*, m, q, R, \theta, D) > 0$*, where both constants above are independent of* $f \in L^q_\sigma(D)$*.*

2. *If in particular* $q \in (3/2, \infty)$ *and* $f \in Z_q(D)$*, then there is a constant* $C = C(\tau_*, m, q, R, \theta, D) > 0$ *such that*

$$\|L_+(t)T(t,s)f\|_{W^{-1,q}(D_R)} \leqq C(t-s)^{-(1+1/q)/2}\|f\|_q \tag{5.24}$$

for all $(t, s) \in \Lambda(\tau_*)$*.*

Proof. From Lemma 3.2, (4.2) and Lemma 4.1 we infer that

$$W(\cdot, s)f \in C^1((s,\infty); W^{-1,q}(D_R))$$

with

$$\|\partial_t W(t,s)f\|_{W^{-1,q}(D_R)} \leqq C(t-s)^{-(1+1/q)/2}\|f\|_q$$

for all $(t, s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(D)$. Here, notice that

$$\partial_t \mathbb{B}[(U(t,s)f)\cdot\nabla\phi] = \mathbb{B}[(\partial_t U(t,s)f)\cdot\nabla\phi]$$

holds even in $L^q(D_R)$, which follows from (2.4) and (3.14). Starting from $W(t, s)f$ together with (5.5), we use (5.7) to show by induction that

$$T_j(\cdot, s) \in C^1((s,\infty); W^{-1,q}(D_R))$$

for every j with

$$\begin{aligned}
\partial_t T_0(t,s)f &= \partial_t W(t,s)f, \\
\partial_t T_1(t,s)f &= PK(t,s)f + \int_s^t \partial_t W(t,\tau)PK(\tau,s)f\,\mathrm{d}\tau, \\
\partial_t T_{j+1}(t,s)f &= \int_s^t \partial_t T_j(t,\tau)PK(\tau,s)f\,\mathrm{d}\tau \qquad (j = 1, 2, \cdots).
\end{aligned}$$

and that

$$\|\partial_t T_j(t,s)f\|_{W^{-1,q}(D_R)} \leqq \mu_j(t-s)^{-(1+1/q)/2}\|f\|_q \tag{5.25}$$

for all $(t, s) \in \Lambda(\tau_*)$ and $f \in L^q_\sigma(D)$ with

$$\mu_j = \mu_j(\tau_*, m, q, R, \theta, D) = \frac{c_0 c_1^j}{\Gamma((1-\alpha)j)} \qquad (j = 1, 2, \cdots)$$

where $\alpha = (1 + 1/q)/2$, $\Gamma(\cdot)$ denotes the Gamma function, and positive constants c_0, c_1 are independent of j, so that $\sum_{j=1}^{\infty} \mu_j < \infty$. This can be verified along the same way as in the proof of Lemma 5.1, see [26, Lemma 3.3], [51, Chapter 5, Section 2]. Hence, for each $s \geqq 0$, the series $\sum_{j=0}^{\infty} \partial_t T_j(t,s)f$ converges in $W^{-1,q}(D_R)$ uniformly with respect to $t \in [s+\varepsilon, s+\tau_*]$ for every $\varepsilon \in (0, \tau_*)$. We thus conclude (5.19) with

$$\partial_t T(t,s)f = \sum_{j=0}^{\infty} \partial_t T_j(t,s)f$$

in $W^{-1,q}(D_R)$, which yields (5.20). This combined with the second assertion of Proposition 2.1 implies (5.24) as well. Formally, the result obtained here is observed by applying Lemma 5.1 with

$$A_0 = \partial_t T_1, \quad Q = PK, \quad X_1 = L^q_\sigma(D), \quad X_2 = W^{-1,q}(D_R),$$
$$\alpha = \beta = \frac{1}{2}\left(1 + \frac{1}{q}\right),$$

however, the differentiability of $T_j(t,s)f$ with respect to t is verified simultaneously with (5.25); thus, we should take the way explained above.

Suppose $f \in C^\infty_{0,\sigma}(D)$ and set $u(t) = T(t,s)f$. By $p(t)$ we denote the associated pressure which is singled out such that $\int_{D_R} p\,\mathrm{d}x = 0$. Combining the Equation (1.1) with

$$\|p(t)\|_{q,D_R} \leqq C\|\nabla p(t)\|_{W^{-1,q}(D_R)}$$

(see, for instance, [29, Remark 4.1] for its proof with the aid of (2.2)), we find (5.22) for $f \in C^\infty_{0,\sigma}(D)$ as well as $p \in C((s,\infty); L^q(D_R))$. Thus, (5.23) follows from (5.20) together with the second assertion of Proposition 2.2 when $f \in C^\infty_{0,\sigma}(D)$. We next take general $f \in L^q_\sigma(D)$, then by approximation we get the function $p_R \in C((s,\infty); L^q(D_R))$ which together with $u(t) = T(t,s)f$ enjoys (5.21) as well as the same estimates (5.22)–(5.23) and $\int_{D_R} p_R\,\mathrm{d}x = 0$. In this way, for every integer $k > 0$, we obtain the pressure p_{R+k} over D_{R+k} satisfying $\int_{D_{R+k}} p_{R+k}\,\mathrm{d}x = 0$, however, we see from (5.21) that

$$\langle p_{R+k}(t) - p_{R+j}(t), \operatorname{div} \psi\rangle_{D_{R+j}} = 0$$

for every $\psi \in C^\infty_0(D_{R+j})^3$ and $k > j \geqq 0$. Consequently, $p_{R+k}(x,t) - p_R(x,t) = c_k(t)$ almost everywhere D_R with some $c_k(t)$ independent of $x \in D_R$. Let us define

$$p(x,t) = \begin{cases} p_R(x,t), & x \in D_R, \\ p_{R+k}(x,t) - c_k(t), & x \in D_{R+k} \setminus D_{R+k-1} \quad (k = 1, 2, \cdots), \end{cases}$$

which is the desired pressure over D satisfying

$$p \in C((s,\infty); L^q(D_R)), \qquad \int_{D_R} p\,\mathrm{d}x = 0$$

as well as (5.21)–(5.23) for all $f \in L^q_\sigma(D)$. □

Analysis in this section can be also carried out for the evolution operator $\widetilde{T}(\tau, s; t)$ generated by the initial value problem (2.14) with use of $\widetilde{U}(\tau, s; t)$ given by (3.9) and the corresponding evolution operator in the bounded domain D_{R_0+6}. Although the latter one is not explicitly given, we do have it by the Tanabe-Sobolevskii theory [51, Chapter 5] and it possesses the same properties as described in Section 4. All the constants in several key estimates can be independent of t and taken uniformly in (τ, s) with $\tau - s \leqq \tau_*$ as well as $0 \leqq s < \tau \leqq t$. In view of the relations (2.13) and (2.15), the corresponding results for the adjoint $T(t, s)^*$, especially (6.3) and (6.10) in the next section, are available.

6. Local Energy Decay of the Evolution Operator

In this section we deduce local energy decay estimates of the evolution operator: Proposition 6.1 for initial velocity with bounded support and Proposition 6.2 for general data. The former is a step to get the latter. In Proposition 6.1 we have a bit less sharp rate of decay than the desired one $(t-s)^{-3/2}$, but this does not cause any problem. If we took the same way for general data as in the proof of Proposition 6.1, we would obtain less decay rate $(t - s)^{-3/2q+\varepsilon}$ than the one in Proposition 6.2. This never implies Theorem 2.1, and thus we should take the following way:

Proposition 6.1. *Suppose that η and ω fulfill (1.2) for some $\theta \in (0, 1)$. Let $R \in (R_0 + 1, \infty)$, where R_0 is as in (2.1). Let $\varepsilon > 0$ be arbitrarily small. Then,*

1. *Let $1 < q < \infty$. For each $m \in (0, \infty)$, there is a constant $C = C(m, \varepsilon, q, R, \theta, D) > 0$ such that*

$$\begin{aligned} \|T(t, s)f\|_{W^{1,q}(D_R)} &\leqq C(t-s)^{-3/2+\varepsilon}\|f\|_q, \\ \|T(t, s)^*g\|_{W^{1,q}(D_R)} &\leqq C(t-s)^{-3/2+\varepsilon}\|g\|_q, \end{aligned} \tag{6.1}$$

for all (t, s) with

$$t - s > 2 \quad \textit{as well as } 0 \leqq s < t$$

and $f,\ g \in L^q_\sigma(D)$ with

$$f(x) = 0, \quad g(x) = 0 \quad \text{almost everywhere } \mathbb{R}^3 \setminus B_{3R_0}$$

whenever (2.21) is satisfied.

2. *Let $3/2 < q < \infty$. For each $m \in (0, \infty)$, there is a constant $C = C(m, \varepsilon, q, R, \theta, D) > 0$ such that*

$$\begin{aligned} \|\partial_t T(t, s)f\|_{W^{-1,q}(D_R)} &\leqq C(t-s)^{-3/2+\varepsilon}\|f\|_q, \\ \|\partial_s T(t, s)^*g\|_{W^{-1,q}(D_R)} &\leqq C(t-s)^{-3/2+\varepsilon}\|g\|_q, \end{aligned} \tag{6.2}$$

for all (t, s) with

$$t - s > 2 \quad \textit{as well as } 0 \leqq s < t$$

and $f,\ g \in L^q_\sigma(D) \cap W^{1,q}(D)$ with

$$f(x) = 0, \quad g(x) = 0 \quad \text{almost everywhere } \mathbb{R}^3 \setminus B_{3R_0}$$

whenever (2.21) is satisfied.

Proof. Given $\varepsilon > 0$ arbitrarily small as well as $q \in (1, \infty)$, let us take p_0 and q_0 such that $1 < p_0 < q < q_0 < \infty$ and $(3/p_0 - 3/q_0)/2 = 3/2 - \varepsilon$.

Set $u(t) = T(t, s)f$ and suppose $t - s > 2$. By both assertions in Proposition 2.2 we find

$$\|\nabla T(t, t-1)u(t-1)\|_{q_0, D_R} \leqq C\|u(t-1)\|_{q_0} \leqq C(t-s-1)^{-3/2+\varepsilon}\|f\|_{p_0}$$

which implies (6.1) for $\nabla T(t, s)f$. As for $T(t, s)f$ itself (without derivative), the argument is straightforward without using semigroup property.

To show the second assertion for $\partial_t u(t)$, we note that $f \in Z_q(D)$ and, thereby, $\partial_t u(t) = -L_+(t)u(t)$ provided $q > 3/2$, see Proposition 2.1. By Lemma 5.4 we know that $T(t-1, s)f \in Z_{q_0}(D)$ for every $q_0 \in (q, \infty)$ and $t \in (s+1, \infty)$. It then follows from (2.20) with (5.24) that

$$\begin{aligned}\|L_+(t)T(t, t-1)u(t-1)\|_{W^{-1,q_0}(D_R)} &\leqq C\|u(t-1)\|_{q_0}\\ &\leqq C(t-s-1)^{-3/2+\varepsilon}\|f\|_{p_0},\end{aligned}$$

which proves (6.2) for $\partial_t T(t, s)f$.

Set $v(s) = T(t, s)^* g$, then we have $v(s) = T(s+1, s)^* v(s+1)$ by the backward semigroup property. We then take the same way as above; to be sure, we just describe several lines only for (6.2). As mentioned at the end of the previous section, we have

$$\|L_-(s)T(t, s)^* g\|_{W^{-1,q}(D_R)} \leqq C(t-s)^{-(1+1/q)/2}\|g\|_q \tag{6.3}$$

for all $(t, s) \in \Lambda(\tau_*)$ and $g \in Z_q(D)$ with $q \in (3/2, \infty)$, which corresponds to (5.24) for $T(t, s)$. Furthermore, similarly to Lemma 5.4, we have

$$v(s+1) = T(t, s+1)^* g = \widetilde{T}(t-s-1, 0; t)g \in Z_{q_0}(D)$$

for every $q_0 \in (q, \infty)$, see (2.13). Therefore, we combine (6.3) with (2.20) to obtain

$$\begin{aligned}\|L_-(s)T(s+1, s)^* v(s+1)\|_{W^{-1,q_0}(D_R)} &\leqq C\|v(s+1)\|_{q_0}\\ &\leqq C(t-s-1)^{-3/2+\varepsilon}\|g\|_{p_0}\end{aligned}$$

which leads to (6.2) for $\partial_s T(t, s)^* g$. □

Let us proceed to the second stage of the local energy decay properties, in which we intend to estimate the evolution operator still over the bounded domain D_R near the boundary for general data being in $f \in L^q_\sigma(D)$.

Proposition 6.2. *Suppose that η and ω fulfill (1.2) for some $\theta \in (0, 1)$. Let $R \in (R_0 + 1, \infty)$, where R_0 is as in (2.1). Let $1 < q < \infty$. For each $m \in (0, \infty)$, there is a constant $C = C(m, q, R, \theta, D) > 0$ such that*

$$\begin{aligned}\|T(t, s)f\|_{W^{1,q}(D_R)} + \|\partial_t T(t, s)f\|_{W^{-1,q}(D_R)} &\leqq C(t-s)^{-3/2q}\|f\|_q,\\ \|T(t, s)^* g\|_{W^{1,q}(D_R)} + \|\partial_s T(t, s)^* g\|_{W^{-1,q}(D_R)} &\leqq C(t-s)^{-3/2q}\|g\|_q,\end{aligned} \tag{6.4}$$

for all (t, s) *with*

$$t - s > 2 \quad \text{as well as } 0 \leqq s < t$$

and $f,\ g \in L^q_\sigma(D)$ *whenever* (2.21) *is satisfied. Here, the temporal derivatives are understood as in Proposition* 5.1.

Proof. By (2.20), (2.22) and (5.20) it suffices to prove (6.4) for all $f,\ g \in C^\infty_{0,\sigma}(D)$. Concerning the temporal derivatives $\partial_t T(t,s)f$ and $\partial_s T(t,s)^* g$, it is also sufficient to show the assertion for $q \in (3/2, \infty)$; in fact, once we have that for such q (for instance, $q = 3$), (2.20) yields

$$\begin{aligned}\|\partial_t T(t,s)f\|_{W^{-1,q}(D_R)} &\leqq C\|L_+(t)T(t,s)f\|_{W^{-1,3}(D_R)} \\ &\leqq C(t-s)^{-1/2}\|T((t+s)/2, s)f\|_3 \\ &\leqq C(t-s)^{-3/2q}\|f\|_q,\end{aligned}$$

even if $q \in (1, 3/2]$.

As in the previous study [34, Section 4], given $f \in C^\infty_{0,\sigma}(D) \subset C^\infty_{0,\sigma}(\mathbb{R}^3)$, we regard the solution $T(t,s)f$ as the perturbation from a modification of the $\mathbb{R}^3$-flow $U(t,s)f$ as follows:

$$T(t,s)f = (1-\phi)U(t,s)f + \mathbb{B}[(U(t,s)f)\cdot\nabla\phi] + v(t),$$

where $v(t)$ denotes the perturbation, $\phi \in C^\infty_0(B_{3R_0})$ is a cut-off function satisfying $\phi = 1$ on B_{2R_0} and $\mathbb{B} = \mathbb{B}_{A_{R_0}}$ is the Bogovskii operator on the domain $A_{R_0} = B_{3R_0}\setminus\overline{B_{R_0}}$, see (2.3). From (2.2) and L^q-L^∞ estimate (3.10) (together with the Equation (3.1) for $\partial_t U(t,s)f$), it follows that $(1-\phi)U(t,s)f + \mathbb{B}[(U(t,s)f)\cdot\nabla\phi]$ and its temporal derivative (even in $L^q(D_R)$) possess the desired decay rate $(t-s)^{-3/2q}$. Our task is thus to estimate

$$v(t) = T(t,s)\widetilde{f} + \int_s^t T(t,\tau)F(\tau)\,\mathrm{d}\tau \tag{6.5}$$

and

$$\partial_t v(t) = \partial_t T(t,s)\widetilde{f} + F(t) + \int_s^t \partial_t T(t,\tau)F(\tau)\,\mathrm{d}\tau \tag{6.6}$$

where $\widetilde{f} = \phi f - \mathbb{B}[f\cdot\nabla\phi]$ and

$$\begin{aligned}F(x,t) = &-2\nabla\phi\cdot\nabla U(t,s)f - [\Delta\phi + (\eta(t)+\omega(t)\times x)\cdot\nabla\phi]U(t,s)f \\ &- \mathbb{B}[(\partial_t U(t,s)f)\cdot\nabla\phi] + \Delta\mathbb{B}[(U(t,s)f)\cdot\nabla\phi] \\ &+ (\eta(t)+\omega(t)\times x)\cdot\nabla\mathbb{B}[(U(t,s)f)\cdot\nabla\phi] \\ &- \omega(t)\times\mathbb{B}[(U(t,s)f)\cdot\nabla\phi].\end{aligned}$$

The forcing term F fulfills, (see [34, (4.2)])

$$\|F(t)\|_q \leqq C(m+1)\|f\|_q \begin{cases}(t-s)^{-1/2}, & 0 < t-s < 1,\\ (t-s)^{-3/2q}, & t-s \geqq 1,\end{cases} \tag{6.7}$$

as well as div $F = \Delta p = 0$ (so that $PF = F$) which follows at once from the equation that $\{v, p\}$ obeys, where p is the pressure associated with $T(t,s)f$. Given $q \in (1, \infty)$, let us take $\varepsilon > 0$ so small that $3/2 - \varepsilon > 3/2q$. Suppose $t - s > 2$. By Proposition 6.1 with such ε and by (6.7) it is seen that $T(t,s)\widetilde{f}$ and $\partial_t T(t,s)\widetilde{f} + F(t)$ satisfy the desired decay property.

Let us consider the last terms of (6.5)–(6.6). Concerning the latter one for the temporal derivative we can apply (6.2) since $F \in L^q_\sigma(D) \cap W^{1,q}(D)$ with $F(x,t) = 0$ almost everywhere $|x| \geqq 3R_0$ (note that estimate of ∇F is not needed). It follows from Propositions 6.1, 5.1 and 2.2 together with (6.7) that

$$\|T(t,\tau)F(\tau)\|_{W^{1,q}(D_R)} \leqq C(m+1)\|f\|_q\, \alpha(\tau),$$
$$\|\partial_t T(t,\tau)F(\tau)\|_{W^{-1,q}(D_R)} \leqq C(m+1)\|f\|_q\, \beta(\tau),$$

with

$$\alpha(\tau) = (t-\tau)^{-1/2}(1+t-\tau)^{-1+\varepsilon}(\tau - s)^{-1/2}(1+\tau-s)^{-3/2q+1/2},$$
$$\beta(\tau) = (t-\tau)^{-(1+1/q)/2}(1+t-\tau)^{-1+1/2q+\varepsilon}(\tau - s)^{-1/2}(1+\tau-s)^{-3/2q+1/2},$$

for $\tau \in (s,t)$. Then we see that

$$\int_s^{(s+t)/2} \alpha(\tau)\, \mathrm{d}\tau \leqq C(t-s)^{-3/2+\varepsilon} \begin{cases} 1, & q < 3/2, \\ \log(t-s), & q = 3/2, \\ (t-s)^{1-3/2q}, & q > 3/2, \end{cases}$$

as well as

$$\int_{(s+t)/2}^t \alpha(\tau)\, \mathrm{d}\tau \leqq C(t-s)^{-3/2q}$$

and that the same estimates as above hold for $\beta(\tau)$, too. We have completed the proof of $(6.4)_1$.

It remains to discuss the adjoint $T(t,s)^*$. Given $g \in C^\infty_{0,\sigma}(D)$, we describe the solution $T(t,s)^*g$ in the form

$$T(t,s)^*g = (1-\phi)U(t,s)^*g + \mathbb{B}[(U(t,s)^*g)\cdot\nabla\phi] + u(s),$$

where ϕ and $\mathbb{B}$ are the same as before, while $U(t,s)^*$ is the evolution operator for the backward problem (3.6)–(3.7) in the whole space and the first two terms above possess the decay rate $(t-s)^{-3/2q}$. Given vector field $\psi \in C_0^\infty(D)^3$, we know from Lemma 5.2 that $P\psi \in Z_q(D)$ for every $q \in (3/2, \infty)$, which implies (2.11) with $f = P\psi$ for such q. With this at hand, as in [34, (4.17)], we utilize (2.8) and (2.15) to compute

$$\begin{aligned} &\partial_\tau \langle P\psi, T(\tau,s)^*u(\tau)\rangle_D \\ &\quad = \partial_\tau \langle T(\tau,s)P\psi, u(\tau)\rangle_D \\ &\quad = \langle T(\tau,s)P\psi, \partial_\tau u(\tau)\rangle_D - \langle L_+(\tau)T(\tau,s)P\psi, u(\tau)\rangle_D \\ &\quad = \langle T(\tau,s)P\psi, \partial_\tau u(\tau) - L_-(\tau)u(\tau)\rangle_D. \end{aligned}$$

This implies the Duhamel formula in the weak form

$$\langle\psi, u(s)\rangle_D = \langle\psi, T(t,s)^*\widetilde{g}\rangle_D + \int_s^t \langle\psi, T(\tau,s)^*G(\tau)\rangle_D \,\mathrm{d}\tau \tag{6.8}$$

for all $\psi \in C_0^\infty(D)^3$ on account of $Pu(s) = u(s)$. Here, $\widetilde{g} = \phi g - \mathbb{B}[g\cdot\nabla\phi]$ and

$$\begin{aligned} G(y,s) = &-2\nabla\phi\cdot\nabla U(t,s)^*g - [\Delta\phi - (\eta(s)+\omega(s)\times y)\cdot\nabla\phi]U(t,s)^*g \\ &+ \mathbb{B}[(\partial_s U(t,s)^*g)\cdot\nabla\phi] + \Delta\mathbb{B}[(U(t,s)^*g)\cdot\nabla\phi] \\ &- (\eta(s)+\omega(s)\times y)\cdot\nabla\mathbb{B}[(U(t,s)^*g)\cdot\nabla\phi] \\ &+ \omega(s)\times\mathbb{B}[(U(t,s)^*g)\cdot\nabla\phi], \end{aligned}$$

both of which are solenoidal. It follows from (6.8) that

$$\langle\psi, \partial_s u(s)\rangle_D = \langle\psi, \partial_s T(t,s)^*\widetilde{g}\rangle_D - \langle\psi, G(s)\rangle_D + \int_s^t \langle\psi, \partial_s T(\tau,s)^*G(\tau)\rangle_D \,\mathrm{d}\tau \tag{6.9}$$

for all $\psi \in C_0^\infty(D)^3$. Since we intend to derive estimates over D_R, let us consider the test functions $\psi \in C_0^\infty(D_R)^3$ in (6.8)–(6.9). Suppose $t-s>2$. We know that $\|G(s)\|_q$ enjoys exactly the same estimate as in (6.7), see [34, (4.16)]. By use of this combined with Propositions 6.1, 2.2 and

$$\|\partial_s T(t,s)^*g\|_{W^{-1,q}(D_R)} \leqq C(t-s)^{-(1+1/q)/2}\|g\|_q \tag{6.10}$$

for all $(t,s)\in\Lambda(\tau_*)$ and $g\in L^q_\sigma(D)$, which corresponds to (5.20) for $T(t,s)$, we find the desired estimates for $\|u(s)\|_{q,D_R}$ and $\|\partial_s u(s)\|_{W^{-1,q}(D_R)}$, in which computations are essentially the same as those for the last terms of (6.5)–(6.6) although we employ the duality. One can get the desired estimate of $\|\nabla u(s)\|_{q,D_R}$ as well by taking test functions of the form $\psi = \operatorname{div}\Psi$ with $\Psi\in C_0^\infty(D_R)^{3\times 3}$ and then by adopting the same argument as above after integration by parts in (6.8). The proof is complete. □

As a corollary to Proposition 6.2 as well as Proposition 5.1, one can derive the following asymptotic behavior of the pressures associated with $T(t,s)f$ and $T(t,s)^*g$ (this plays an important role in the next section):

Corollary 6.1. *Suppose that η and ω fulfill (1.2) for some $\theta\in(0,1)$. Let $R\in(R_0+1,\infty)$, where R_0 is as in (2.1). Let $1<q<\infty$. Given $f\in L^q_\sigma(D)$, we denote by $p(t)$ the pressure associated with $T(t,s)f$ subject to $\int_{D_R} p\,\mathrm{d}x = 0$, which is determined by Proposition 5.1. Let $\phi\in C_0^\infty(B_R)$ satisfy $\phi = 1$ in B_{R_0+1}, and $\mathbb{B} = \mathbb{B}_{A_R}$ the Bogovskii operator on the bounded domain $A_R = \{R_0<|x|<R\}$, see (2.3). Then, for each $m\in(0,\infty)$, there is a constant $C = C(m,q,R,\theta,D)>0$ such that*

$$\begin{aligned} &\|p(t)\|_{q,D_R} + \|\mathbb{B}[(\partial_t T(t,s)f)\cdot\nabla\phi]\|_{q,A_R} \\ &\leqq C\|f\|_q \begin{cases} (t-s)^{-(1+1/q)/2}, & 0<t-s\leqq 2, \\ (t-s)^{-3/2q}, & t-s>2, \end{cases} \end{aligned} \tag{6.11}$$

for all $f \in L^q_\sigma(D)$ *whenever* (2.21) *is satisfied. Here, the temporal derivative is understood as in Proposition* 5.1. *The same assertion holds true for* $T(t,s)^*g$ *with* $g \in L^q_\sigma(D)$ *and the associated pressure as well.*

Proof. Estimate (6.11) near $t = s$ for the pressure was already obtained in (5.23). By (2.4) we have

$$\begin{aligned}\|\mathbb{B}[(\partial_t T(t,s)f)\cdot\nabla\phi]\|_{q,A_R} &\leqq C\|(\partial_t T(t,s)f)\cdot\nabla\phi\|_{W^{1,q'}(A_R)^*}\\ &\leqq C\|\partial_t T(t,s)f\|_{W^{-1,q}(D_R)},\end{aligned}$$

which together with (5.22) implies that (6.11) follows from (6.4) for large $(t-s)$ as well as (5.20) for small $(t-s)$. □

Another corollary to Proposition 6.2 is the L^∞-estimate.

Corollary 6.2. *Suppose that* η *and* ω *fulfill* (1.2) *for some* $\theta \in (0,1)$. *Let* $R \in (R_0+1,\infty)$, *where* R_0 *is as in* (2.1). *Let* $1 < q < \infty$. *For each* $m \in (0,\infty)$, *there is a constant* $C = C(m,q,R,\theta,D) > 0$ *such that*

$$\begin{aligned}\|T(t,s)f\|_{\infty,D_R} &\leqq C(t-s)^{-3/2q}\|f\|_q,\\ \|T(t,s)^*g\|_{\infty,D_R} &\leqq C(t-s)^{-3/2q}\|g\|_q,\end{aligned}\tag{6.12}$$

for all (t,s) *with*

$$t-s>2 \quad \textit{as well as } 0 \leqq s < t$$

and $f,\ g \in L^q_\sigma(D)$ *whenever* (2.21) *is satisfied.*

Proof. L^∞-estimate follows directly from (6.4) together with the Sobolev embedding when $q > 3$. If $q \leqq 3$, then we have

$$\|T(t,s)f\|_{\infty,D_R} \leqq C(t-s)^{-1/4}\|T((t+s)/2,s)f\|_6,$$

which leads to (6.12) by the first assertion of Proposition 2.2. □

7. Proof of the Main Theorems

In the final section we complete the proof of the main results on decay estimates of gradient of the evolution operator $T(t,s)$ and its adjoint $T(t,s)^*$ as well as L^∞-decay estimates.

Proof of Theorem 2.1. Let $1 < q < \infty$ ($q > 3/2$ for L^∞-estimates) and fix $R \in (R_0+1,\infty)$, where R_0 is as in (2.1). It then suffices to prove

$$\begin{aligned}\|\nabla T(t,s)f\|_{q,\mathbb{R}^3\setminus B_R} &\leqq C(t-s)^{-\min\{1/2,\,3/2q\}}\|f\|_q,\\ \|T(t,s)f\|_{\infty,\mathbb{R}^3\setminus B_R} &\leqq C(t-s)^{-3/2q}\|f\|_q,\end{aligned}\tag{7.1}$$

and

$$\|\nabla T(t,s)^* g\|_{q,\mathbb{R}^3\setminus B_R} \leqq C(t-s)^{-\min\{1/2,\,3/2q\}}\|g\|_q,$$
$$\|T(t,s)^* g\|_{\infty,\mathbb{R}^3\setminus B_R} \leqq C(t-s)^{-3/2q}\|g\|_q, \tag{7.2}$$

for all (t,s) with $t-s>2$ as well as $0 \leqq s < t$ and $f,\ g \in C_{0,\sigma}^\infty(D)$. From this combined with (6.4), (6.12), Proposition 2.2 and the semigroup property we conclude Theorem 2.1. Note that (2.24) for $t-s \leqq 2$ follows from Proposition 2.2 together with an embedding relation and that (2.24) for $q>3/2$ yields (2.24) even for $q \leqq 3/2$ on account of the semigroup property and (2.20).

Let us take a cut-off function $\phi \in C_0^\infty(B_R)$ and the Bogovskii operator $\mathbb{B}=\mathbb{B}_{A_R}$ as in Corollary 6.1. Given $f \in C_{0,\sigma}^\infty(D)$, we denote by $p(t)$ the pressure associated with the velocity $T(t,s)f$ such that $\int_{D_R} p\,\mathrm{d}x = 0$. Set

$$v(t) = (1-\phi)T(t,s)f + \mathbb{B}[(T(t,s)f)\cdot\nabla\phi], \qquad p_v(t) = (1-\phi)p(t). \tag{7.3}$$

Since $v(t) = T(t,s)f$ in $\mathbb{R}^3\setminus B_R$, let us consider $\|\nabla v(t)\|_{q,\mathbb{R}^3}$ and $\|v(t)\|_{\infty,\mathbb{R}^3}$ by using

$$v(t) = U(t,s)\widetilde{f} + \int_s^t U(t,\tau)P_{\mathbb{R}^3}H(\tau)\,\mathrm{d}\tau \tag{7.4}$$

where $P_{\mathbb{R}^3} = I + \mathcal{R}\otimes\mathcal{R}$ is the Fujita-Kato projection in the whole space, $\widetilde{f} = (1-\phi)f + \mathbb{B}[f\cdot\nabla\phi] \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ and

$$\begin{aligned} H(x,t) = {} & 2\nabla\phi\cdot\nabla T(t,s)f + \{\Delta\phi + (\eta+\omega\times x)\cdot\nabla\phi\}T(t,s)f \\ & - \Delta\mathbb{B}[(T(t,s)f)\cdot\nabla\phi] - (\eta+\omega\times x)\cdot\nabla\mathbb{B}[(T(t,s)f)\cdot\nabla\phi] \\ & + \omega\times\mathbb{B}[(T(t,s)f)\cdot\nabla\phi] \\ & + \mathbb{B}[(\partial_t T(t,s)f)\cdot\nabla\phi] - (\nabla\phi)p. \end{aligned}$$

Among several terms of which H consists, the last two terms are always delicate in cut-off procedures, but we have Corollary 6.1 and that is why we have made effort to analyze $\partial_t T(t,s)$ in Propositions 6.1 and 6.2, while the other terms are harmless. Clearly, $H=0$ for $|x| \geqq R$, and it is seen from (6.4), (6.11) and the second assertion of Proposition 2.2 that

$$\|H(t)\|_{r,\mathbb{R}^3} \leqq C(m+1)\|f\|_q \begin{cases} (t-s)^{-(1+1/q)/2}, & 0<t-s\leqq 2, \\ (t-s)^{-3/2q}, & t-s>2 \end{cases} \tag{7.5}$$

for every $r \in (1,q]$.

Suppose $t-s>2$. By (3.10) the first term $U(t,s)\widetilde{f}$ of (7.4) satisfies the desired estimate. Let us consider the second term of (7.4). To this end, we combine (7.5) with (3.10) to observe

$$\|\nabla U(t,\tau)P_{\mathbb{R}^3}H(\tau)\|_{q,\mathbb{R}^3} \leqq C(m+1)\|f\|_q\,\widetilde{\alpha}(\tau),$$
$$\|U(t,\tau)P_{\mathbb{R}^3}H(\tau)\|_{\infty,\mathbb{R}^3} \leqq C(m+1)\|f\|_q\,\widetilde{\beta}(\tau),$$

with

$$\widetilde{\alpha}(\tau) = (t-\tau)^{-1/2}(1+t-\tau)^{-(3/r-3/q)/2}(\tau-s)^{-(1+1/q)/2}(1+\tau-s)^{-1/q+1/2},$$
$$\widetilde{\beta}(\tau) = (t-\tau)^{-3/2q}(1+t-\tau)^{-(3/r-3/q)/2}(\tau-s)^{-(1+1/q)/2}(1+\tau-s)^{-1/q+1/2}$$

for $\tau \in (s,t)$, where $r \in (1,q]$ will be soon chosen appropriately. Then we have

$$\int_s^{(s+t)/2} \widetilde{\alpha}(\tau)\,\mathrm{d}\tau \leqq C(t-s)^{-(3/r-3/q)/2-1/2} \begin{cases} 1, & q < 3/2, \\ \log(t-s), & q = 3/2, \\ (t-s)^{1-3/2q}, & q > 3/2 \end{cases}$$

By a suitable choice of $r \in (1,q]$, that is,

$$r = q < 3/2, \qquad r < 3/2 = q, \qquad r \leqq 3/2 < q,$$

we find

$$\int_s^{(s+t)/2} \widetilde{\alpha}(\tau)\,\mathrm{d}\tau \leqq C(t-s)^{-1/2}$$

for every $q \in (1,\infty)$. On the other hand, we observe

$$\begin{aligned}\int_{(s+t)/2}^{t} \widetilde{\alpha}(\tau)\,\mathrm{d}\tau &\leqq C \begin{cases} (t-s)^{-3/2q+1/2}, & q \leqq 3/2, \\ (t-s)^{-3/2q}, & q > 3/2, \end{cases} \\ &\leqq C \begin{cases} (t-s)^{-1/2}, & q \leqq 3, \\ (t-s)^{-3/2q}, & q > 3, \end{cases}\end{aligned}$$

where r is chosen to be close to 1 in such a way that $1/r > 1/q + 1/3$ for the case $q > 3/2$, while it is enough to choose $r = q$ for the other case $q \leqq 3/2$. Summing up all computations above, we are led to the gradient estimate in (7.1). L^∞-estimate is discussed similarly by use of $\widetilde{\beta}(\tau)$ above as long as $q > 3/2$.

Given $g \in C_{0,\sigma}^\infty(D)$, we next consider $T(t,s)^*g$ together with the associated pressure $\sigma(s)$ such that $\int_{D_R} \sigma\, dy = 0$, see (2.6). As in (7.3), we set

$$u(s) = (1-\phi)T(t,s)^*g + \mathbb{B}[(T(t,s)^*g)\cdot\nabla\phi], \qquad \sigma_u(s) = (1-\phi)\sigma(s). \tag{7.6}$$

The same argument as above with use of the adjoint $U(t,s)^*$ being the solution operator to the backward system (3.6)–(3.7) in the whole space implies (7.2). The proof of Theorem 2.1 is thus complete. □

Let us close the paper with a brief description of the proof of Theorem 2.2.

Proof of Theorem 2.2. Given $g \in C_{0,\sigma}^\infty(D)$, as in the last part of the proof of Theorem 2.1, we still consider the strong solution $T(t,s)^*g$ and single out the associated pressure $\sigma(s)$ satisfying the side condition $\int_{D_R} \sigma\, dy = 0$.

Toward (2.25) with $r = 3$ (the most important case for us), as was discussed in [37, Section 8] for the autonomous case, the real interpolation is performed at the level of (6.4) and (6.11) for the adjoint $T(t,s)^*$ [as well as (2.20) and (2.22)] to find that

$$\|\nabla^j T(t,s)^* g\|_{L^{q,\rho}(D_R)} \leqq C\|g\|_{q,\rho} \begin{cases} (t-s)^{-j/2}, & 0 < t-s \leqq 2, \\ (t-s)^{-3/2q}, & t-s > 2, \end{cases} \tag{7.7}$$

with $j = 0,\ 1$ and that

$$\begin{aligned} &\|\sigma(s)\|_{L^{q,\rho}(D_R)} + \|\mathbb{B}[(\partial_s T(t,s)^* g) \cdot \nabla\phi]\|_{L^{q,\rho}(A_R)} \\ &\leqq C\|g\|_{q,\rho} \begin{cases} (t-s)^{-(1+1/q)/2}, & 0 < t-s \leqq 2, \\ (t-s)^{-3/2q}, & t-s > 2, \end{cases} \end{aligned} \tag{7.8}$$

where $1 < q < \infty$, $1 \leqq \rho \leqq \infty$ and $g \in L^{q,\rho}_\sigma(D)$. We then proceed to the final step in this section to obtain $(7.2)_1$ in which L^q-norm is now replaced by $L^{q,\rho}$-norm. To this end, we consider $u(s)$ given by (7.6) and have only to estimate $\|\nabla u(s)\|_{L^{q,\rho}(\mathbb{R}^3)}$ by making use of $L^{q,\rho}$-$L^{r,\rho}$ estimates of $\nabla U(t,s)^*$ and the estimate of the Bogovskii operator $\mathbb{B} = \mathbb{B}_{A_R}$ in $L^{q,\rho}(A_R)$, which follows from (2.2) by interpolation, as well as (7.7)–(7.8). The argument ends up with continuity and that is why the case $\rho = \infty$ is missing in (2.25).

Finally, following Yamazaki [53], we perform the real interpolation for the sublinear operator: $g \mapsto \|\nabla T(t,\cdot)^* g\|_{r,1}$ (for fixed $t > 0$ and $r \in (3/2, 3]$) to conclude (2.26). The proof is complete. □

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