Arch. Rational Mech. Anal. 237 (2020) 1325–1382
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-020-01532-4

# *The Surface Diffusion Flow with Elasticity in Three Dimensions*

NICOLA FUSCO, VESA JULIN & MASSIMILIANO MORINI

*Communicated by* C. DE LELLIS

## Abstract

We establish the short-time existence of a smooth solution to the surface diffusion equation with an elastic term and without an additional curvature regularization in three space dimensions. We also prove the asymptotic stability of strictly stable stationary sets.

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## 1. Introduction

Morphological evolution of strained elastic solids, driven by stress and surface mass transport occurs in many physical systems. One instance is the heteroepitaxial growth of elastic films when a lattice mismatch between film and substrate is present. Another example is given by the phase separation in several small connected phases within a common elastic body, which takes place in certain alloys under specific temperature conditions. A third situation is represented by the nucleation and evolution of material voids inside a stressed elastic solid. From the mathematical point of view, such phenomena are related to a free energy functional, which is typically given by the sum of the stored elastic energy and the surface energy accounting for the surface tension along the interface between the phases. In this context the equilibria are identified with the local or global minimizers under a volume constraint of the aforementioned energy.

All these variational problems can be regarded as non-local *isoperimetric problems*, where the non-locality is given by the elastic term. They are very well studied in the physical and numerical literature, see for instance [26,29,40–42]. Concerning rigorous mathematical analysis, we refer to [6,8,10,17,21,25,28] for some existence, regularity and stability results related to a variational model describing the equilibrium configurations of two-dimensional epitaxially strained elastic films, and to [9,16] for results in three-dimensions. A hierarchy of variational principles to describe equilibrium shapes in the aforementioned contexts has been introduced in [30].

In what follows we consider the following prototypical energy:

$$\mathcal{J}(F) := \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F) : E(u_F)\,\mathrm{d}x + \mathcal{H}^2(\partial F). \tag{1.1}$$

The associated minimum problem under a volume constraint can be used to describe the equilibrium shapes of voids in elastically stressed solids (see for instance [41]). Here, the set $F \subset\subset \Omega$ represents the shape of the void that has formed within the elastic body $\Omega$ (an open subset of $\mathbb{R}^3$), $u_F$ stands for the equilibrium elastic displacement in $\Omega\setminus F$ subject to a prescribed boundary conditions $u_F = w_0$ on $\partial\Omega$ (see (2.12) below), $\mathbb{C}$ is the elasticity tensor of the (linearly) elastic material, $E(u_F) := (Du_F + D^T u_F)/2$ denotes the elastic strain of $u_F$, and $\mathcal{H}^2$ stands for the surface measure. The presence of a nontrivial Dirichlet boundary condition $u_F = w_0$ on $\partial\Omega$ is what causes the solid $\Omega\setminus F$ to be elastically stressed. We refer to [15,20] for related existence, regularity and stability results in two dimensions. See also [11] for a relaxation result valid in all dimensions for a variant of (1.1).

In this paper we study the morphological evolution of shapes towards equilibria of the functional (1.1), driven by stress and surface diffusion. Assuming that relaxation to equilibrium in the bulk occurs at a much faster time scale, see [38], we have, according to the Einstein–Nernst equation, that the evolution is governed by the following *volume preserving* law:

$$V_t = \Delta_{\partial F_t}\mu_t \qquad \text{on } \partial F_t, \tag{1.2}$$

where $V_t$ denotes the outer normal velocity of the evolving surface $\partial F_t$ at time $t$ and $\Delta_{\partial F_t}\mu_t$ stands for the Laplace–Beltrami operator acting on the chemical potential $\mu_t$ along $\partial F_t$. In turn, since $\mu_t$ is given by the *first variation* of the free-energy functional $\mathcal{J}$ evaluated at $F_t$ and taking into account (2.14) below, (1.2) reads as

$$V_t = \Delta_{\partial F_t}\big(H_{F_t} - Q(E(u_{F_t}))\big), \tag{1.3}$$

where $H_{F_t}$ is the sum of the principal curvatures of $\partial F_t$, with the orientation given by the outer normal, $u_{F_t}$ is the elastic equilibrium in $\Omega\setminus F_t$ subject to $u_{F_t} = w_0$ on

$\partial\Omega$ and $Q(E(u_{F_t})) := \frac{1}{2}\mathbb{C}E(u_{F_t}) : E(u_{F_t})$. Note that the last quantitity involves the traces of the gradient of the elastic equilibrium on the evolving boundary.

From the mathematical point of view, (1.3) is a fourth order geometric parabolic equation coupled with the elliptic Lamé system, which is solved time by time in the (evolving) bulk. Note also that when $w_0 = 0$ the elastic term vanishes and thus (1.3) reduces to the pure *surface diffusion flow*

$$V_t = \Delta_{\partial F_t} H_{F_t} \tag{1.4}$$

for evolving surfaces, studied in [19] (in the general $n$-dimensional case). Thus, we may also regard (1.3) as a sort of canonical nonlocal perturbation of (1.4) by an additive elastic contribution.

As observed already by Cahn and Taylor [14] for (1.4), the Equation (1.3) can be seen formally as the gradient flow of the energy functional $\mathcal{J}$ with respect to a suitable Riemannian metric of $H^{-1}$-type, see for instance [24, Remark 3.1].

Let us mention that in the physical literature a variant of the energy (1.1) with a *curvature regularization* term has also been considered, see [3,12,18,31,40,41]. This in turn leads to a variant of (1.3) with a sixth order regularization term. In particular, in [23] the regularized energy

$$\mathcal{J}_\varepsilon(F) := \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F) : E(u_F)\,\mathrm{d}x + \int_{\partial F}\Big(1 + \frac{\varepsilon}{p}|H_F|^p\Big)\,\mathrm{d}\mathcal{H}^2$$

and the associated evolution equation

$$\begin{aligned} V_t = \Delta_{\partial F_t}\Big[H_{F_t} - Q(E(u_{F_t})) - \varepsilon\Big(\Delta_{\partial F_t}(|H_{F_t}|^{p-2}H_{F_t}) \\ -|H_{F_t}|^{p-2}H_{F_t}\Big(\tfrac{p-1}{p}H_{F_t}^2 - 2K_{\partial F_t}\Big)\Big)\Big] \end{aligned} \tag{1.5}$$

are considered in the context of periodic graphs modeling the evolutions of epitaxially strained elastic films (see also [22] for the two-dimensional version of the same equation). Here $K_{\partial F_t}$ stands for the Gaussian curvature of $\partial F_t$, $\varepsilon > 0$ is a small parameter, and $p > 2$. The local-in-time existence and the asymptotic stability results proven in [23] (see also [22,39]) rely heavily on the presence of the curvature regularization, which makes the elastic contribution a lower order term easily controlled by the sixth order leading terms of the equation. In fact, all the estimates provided there are $\varepsilon$-dependent and degenerate as $\varepsilon \to 0^+$. This is not surprising as the nonlocal elastic term in (1.1) cannot be treated simply as a lower order perturbation of the perimeter, as shown by the fact that its presence may lead to formation of singularities in the static case (see [25] and references therein), and the numerical analysis in [41] suggests that in the evolutionary case the flow may form cusp-like singularities. Thus the case $\varepsilon = 0$ requires completely different methods.

A first breakthrough in this direction has been obtained in [24], where short time existence result for (1.3) was proved in the two-dimensional case. In [24] we also proved the asymptotic stability of strictly stable stationary sets. However, the techniques developed there cannot be applied to higher dimensions, as some of the crucial estimates rely on the fact that an $L^2$-bound of the curvature of the evolving curves provides uniform $C^{1,\alpha}$-bounds. This is of course no longer true in higher dimensions. Moreover, the higher dimensional case is of course much more involved from the geometric point of view.

In this paper we are able to address Equation (1.3) in the physical three-dimensional case and we establish short time existence and uniqueness of a solution starting from sufficiently regular initial sets, see Theorem 4.4. We highlight that Theorem 4.4 provides also quantitative estimates of the $k$-th order derivatives of the solution depending only on the $H^3$-norm of the initial datum, somewhat in the spirit of those proved in [32]. We also remark that in general one cannot expect global-in-time existence. Indeed, even when no elasticity is present, singularities such as pinching may develop in finite time, see for instance [27].

In the second main result of the paper we establish global-in-time existence and study the long-time behavior for a class of initial data: we show that *strictly stable stationary sets*, that is, sets $G$ that are stationary for the energy functional $\mathcal{J}$ and with positive second variation $\partial^2\mathcal{J}(G)$ are *exponentially stable* for the flow (1.3). More precisely, if the initial set $F_0$ is sufficiently close in $H^3$ to the strictly stable set $G$ and has the same volume, then the flow (1.3) starting from $F_0$ exists for all times and converges to $G$ exponentially fast in $C^k$ for every $k$ as $t \to +\infty$, see Theorem 5.1 for the precise statement.

A few comments on the proofs are in order. Concerning short-time existence, as in [24] our strategy is based on the natural idea of thinking of the elastic contribution $Q$ as a forcing term. More precisely, we set up a fixed point argument on the map $f \mapsto Q(E(u_{F_t^f}))$, where $F_t^f$ is the solution to the forced flow

$$V_t = \Delta_{\partial F_t}\big(H_{F_t} - f\big). \tag{1.6}$$

Major technical difficulties originate from the already mentioned fact that the non-local elastic term is not in general lower order with respect to the perimeter. One of the main technical breakthroughs obtained in the present paper is a new delicate elliptic estimate on the higher order derivatives of $Q(E(u_{F_t}))$ in terms of the higher order norms of the evolving boundaries $\partial F_t$, see Theorem 4.1. The crucial and somewhat surprising point of this result is the linear structure of the estimate, which allows us to show that the map $f \mapsto Q(E(u_{F_t^f}))$ is a contraction.

Concerning the asymptotic stability analysis, we adapt to the present situation the methods developed in [1] for the surface diffusion flow without elasticity (see also [24]). The rough idea is to look at the asymptotic behavior of the map

$$t \mapsto \int_{\partial F_t} \big|\nabla_{\partial F_t}\big(H_{F_t} - Q(E(u_{F_t}))\big|^2 \, \mathrm{d}\mathcal{H}^2,$$

where $\nabla_{\partial F_t}$ stands for the tangential gradient on $\partial F_t$, and to show that it is decreasing and that in fact it vanishes with exponential rate as $t \to +\infty$. A crucial role in this analysis is played by the energy identity proven in Proposition 5.3 and by the estimates on the flow provided by Theorem 4.4. Let us remark that such estimates allow us also to considerably simplify the arguments of [1] and to obtain stronger asymptotic convergence results.

This paper is organized as follows. In Section 2 we set up the problem, introduce the main notation and present some differential geometry preliminaries that will be useful in the subsequent analysis. We also collect several auxiliary results concerning the energy functional $\mathcal{J}$ in (1.1). In particular, we describe some properties of strictly stable stationary sets that are crucial for the asymptotic stability analysis carried out in Section 5. Section 3 is devoted to the study of (1.6), while the short-time existence theory for the flow (1.3) is addressed in Section 4. In Section 6 we briefly illustrate how to apply our main existence and asymptotic stability results in the case of evolving periodic graphs, that is in the geometric setting considered in [23]. In particular, in Theorem 6.1 we address the exponential asymptotic stability of *flat configurations*, thus extending to the evolutionary setting the results of [9]. In the final "Appendix" we collect the proofs of two technical lemmas and provide the derivation of the energy identity stated in Proposition 5.3.

From a technical point of view the three dimensions enter in a crucial way via the Sobolev embedding and affect the regularity of the space where the fixed point argument is set. It would be probably possible to extend the methods to higher dimensions at the expense of setting the problem in more regular spaces which in turn would require to differentiate the equation more and more as the dimension increases.

We conclude this introduction by mentioning that it would be interesting to investigate whether the flow (1.5) studied in [23] converge to (1.3) as $\varepsilon \to 0^+$. This issue could be probably addressed by adapting the methods developed in [7].

## 2. Preliminaries

### 2.1. Geometric Preliminaries

In this section we introduce notation related to Riemannian geometry. As an introduction to the topic we refer to [4,34]. Let $\Sigma \subset \mathbb{R}^n$ be a smooth $(n-1)$-dimensional compact hypersurface without boundary. Since $\Sigma$ is embedded in $\mathbb{R}^n$ it has a natural metric, denoted by $g$, induced by the Euclidean metric. We thus have a Riemannian manifold $(\Sigma, g)$ and we denote the inner product for vector fields $X, Y$ as $\langle X, Y\rangle$

$$\langle X, Y\rangle = g(X, Y) = g_{ij} X^i Y^j,$$

where the last expression is in local coordinates. Throughout the paper we adopt the Einstein summation convention. Similarly we define the inner product of covector fields $\omega, \eta$, which in local coordinates can be written as

$$\langle \omega, \eta\rangle = g^{ij} \omega_i \eta_j,$$

where $g^{ij}$ is the inverse matrix of $g_{ij}$. The inner product extends to $\binom{k}{0}$-tensor fields $T = T_{i_1 \cdots i_k}$ and $S = S_{j_1 \cdots j_k}$ as

$$\langle T, S\rangle = g^{i_1 j_1} \cdots g^{i_k j_k} T_{i_1 \cdots i_k} S_{j_1 \cdots j_k}.$$

The norm of a tensor $T$ is then $|T| = \sqrt{\langle T, T\rangle}$ and we have the inequality $\langle T, S\rangle \leqq |T||S|$. Given a $\binom{k}{0}$-tensor field $T$ we raise the first index by $T^{i_1}_{i_2\cdots i_k} = g^{i_1 l} T_{l\, i_2\cdots i_k}$ and thus we obtain a $\binom{k-1}{1}$-tensor field. We may thus write the above inner product as

$$\langle T, S\rangle = T^{j_1\cdots j_k} S_{j_1\cdots j_k}.$$

The trace of a $\binom{k}{0}$-tensor field $T$, with $k \geqq 2$, on the first two indeces is $\operatorname{tr} T = g^{jl} T_{jl\, i_3\cdots i_k}$.

We denote the Riemannian connection on $(\Sigma, g)$ by $\nabla$ and $\nabla^k T = \nabla_{i_1} \cdots \nabla_{i_k} T$ means the $k$-th covariant derivative of a tensor field $T$. There is a slight danger of confusion, since $\nabla^k f$ also denotes the $k$-th component of the gradient of a function $f$ defined by raising the index of $\nabla f$ as $\nabla^k f = g^{ki} \nabla_i f$. However, the meaning of $\nabla^k f$ will be clear from the context. We also recall that $\nabla$ is compatible with the metric $g$ which means that $\nabla g = 0$.

In local coordinates the components of the covariant derivative of a vector field $X = X^i$ and of a covector field $\omega = \omega_k$ are

$$\nabla_j X^i = \frac{\partial X^i}{\partial x^j} + \Gamma^i_{jk} X^k \qquad \text{and} \qquad \nabla_j \omega_k = \frac{\partial \omega_k}{\partial x^j} - \Gamma^l_{jk} \omega_l,$$

where $\Gamma^k_{ij}$ are the Christoffel symbols given in local coordinates by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

The covariant derivative of a $\binom{k}{l}$-tensor field $T = T^{j_1\cdots j_l}_{i_1\cdots i_k}$ is thus a $\binom{k+1}{l}$-tensor field which in local coordinates can be written as

$$\nabla_m T^{j_1\cdots j_l}_{i_1\cdots i_k} = \frac{\partial T^{j_1\cdots j_l}_{i_1\cdots i_k}}{\partial x^m} + \sum_{s=1}^{l} T^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k} \Gamma^{j_s}_{mp} - \sum_{s=1}^{k} T^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k} \Gamma^p_{m i_s}.$$

The divergence of a vector field $X^i$ is $\operatorname{div} X = \nabla_i X^i = \frac{\partial X^i}{\partial x^i} + \Gamma^i_{ik} X^k$ and the Laplace–Beltrami of a function $f$ is

$$\Delta f = \operatorname{div} \nabla f = \nabla_i \nabla^i f.$$

This can be written as the trace of the covariant Hessian $\nabla^2 f$ as

$$\Delta f = \operatorname{tr} \nabla^2 f = g^{ij} \nabla_i \nabla_j f.$$

We recall the divergence theorem for compact manifolds (without boundary), which states that for a vector field $X$ on $\Sigma$ it holds that

$$\int_\Sigma \operatorname{div} X \, d\mathcal{H}^{n-1} = 0.$$

This yields the integration by parts formula for a function $f$ and a vector field $X$

$$\int_\Sigma X^i \nabla_i f \, d\mathcal{H}^{n-1} = -\int_\Sigma f \operatorname{div} X \, d\mathcal{H}^{n-1}.$$

The integration by parts formula generalizes to any $\binom{k}{0}$-tensor field $T$ and $\binom{k+1}{0}$-tensor field $S$ as

$$\int_\Sigma \langle \nabla T, S\rangle \,\mathrm{d}\mathcal{H}^{n-1} = -\int_\Sigma \langle T, \operatorname{tr} \nabla S\rangle \,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.1}$$

where the trace is on the first two indeces of $\nabla S$.

The Riemann curvature endomorhpism is a $\binom{3}{1}$-tensor field $R^l_{ijk}$ defined such that for vector fields $X, Y, Z$ we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $\nabla_X$ is the covariant derivative in direction of $X$ and $\nabla_{[X,Y]}$ is the Lie bracket, see [4, Definition 2.15]. We adopt the convention to define the Riemann curvature tensor by lowering the index to the end, that is, $R_{ijkl} = g_{lm} R^m_{ijk}$. The commutation formula of the covariant derivatives for a vector field $X^k$ thus becomes

$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = g^{km} R_{ijlm} X^l, \tag{2.2}$$

and for a covector field $\omega_k$,

$$\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = -g^{ml} R_{ijkm} \omega_l.$$

Similar formulas hold for the commutation of higher order covariant derivatives. In particular, throughout the paper we will make repeated use of the fact that for any integer $k \geqq 3$ there exists a constant $C > 0$ such that

$$|\nabla_{i_1} \dots \nabla_{i_k} f - \nabla_{i_{\sigma(1)}} \dots \nabla_{i_{\sigma(k)}} f| \leqq C \sum_{l=1}^{k-2} |\nabla^l f| \tag{2.3}$$

for any choice of the indices $i_1, \dots, i_k$ and for any permutation $\sigma$ of $\{1, \dots, k\}$. We recall also that $\nabla_i \nabla_j f = \nabla_j \nabla_i f$ for any $i, j$.

Given a positive integer $k$ and $p \in [1, \infty]$ we denote by $W^{k,p}(\Sigma)$ the Sobolev space endowed with the norm

$$\|f\|_{W^{k,p}(\Sigma)} := \sum_{m=0}^{k} \left( \int_\Sigma |\nabla^m f|^p \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}},$$

when $p \in [1, \infty)$ and the obvious one when $p = \infty$. Here $\nabla^m f$ stands for the $m$-th covariant derivative of $f$. As customary, when $p = 2$ we shall always write $H^k$ instead of $W^{k,2}$. We further define the norms $\|f\|_{C^{k,\alpha}(\Sigma)}$, $\|f\|_{H^{k+1/2}(\Sigma)}$ and $\|f\|_{H^{-1/2}(\Sigma)}$ with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, in a standard way using the partition of unity. Then the standard embedding theorems for smooth domains hold also in these spaces. Moreover, we recall the following well known interpolation inequalities, see [35, Proposition 6.5] and [5, Theorem 3.70].

**Lemma 2.1.** *Let $\Sigma \subset \mathbb{R}^n$ be a smooth $(n-1)$-dimensional compact manifold without boundary. Let $l, m, k$ be integers such that $0 \leqq l < m$, $k \geqq 0$, $1 \leqq q, r \leqq \infty$. There exists a constant $C$ with the following property: for every smooth covariant tensor $T$ of order $k$, one has*

$$\|\nabla^l T\|_{L^p(\Sigma)} \leqq C\|T\|^{\vartheta}_{W^{m,r}(\Sigma)}\|T\|^{1-\vartheta}_{L^q(\Sigma)}, \tag{2.4}$$

*where*

$$\frac{1}{p} = \frac{l}{n-1} + \vartheta\Big(\frac{1}{r} - \frac{m}{n-1}\Big) + (1-\vartheta)\frac{1}{q}$$

*for all $\vartheta \in [l/m, 1)$ for which $p$ is nonnegative. Moreover, if $f$ is a smooth function then*

$$\|\nabla^l f\|_{L^p(\Sigma)} \leqq C\|\nabla^m f\|^{\vartheta}_{L^r(\Sigma)}\|f\|^{1-\vartheta}_{L^q(\Sigma)}$$

*for all $\vartheta \in [l/m, 1)$ for which $p$ is nonnegative, provided $l \geqq 1$.*

**Remark 2.2.** Note that (2.4) implies also that

$$\|\nabla^l T\|_{L^p(\Sigma)} \leqq C\|\nabla^m T\|^{\vartheta}_{L^r(\Sigma)}\|T\|^{1-\vartheta}_{L^q(\Sigma)} + C\|T\|_{L^{\max\{q,r\}}(\Sigma)}.$$

To see this it is enough to observe that $\|T\|_{W^{m,r}(\Sigma)} = \|T\|_{W^{m-1,r}(\Sigma)} + \|\nabla^m T\|_{L^r(\Sigma)}$ and that, in turn, for every $l = 1, \dots, m-1$ using (2.4) and Young's Inequality one gets

$$\|\nabla^l T\|_{L^r(\Sigma)} \leqq \varepsilon\|T\|_{W^{m,r}(\Sigma)} + C_\varepsilon\|T\|_{L^r(\Sigma)}.$$

We also recall that the Morrey's inequality implies

$$\|f\|_{C^{1,\alpha}(\Sigma)} \leqq C\|f\|_{W^{2,p}(\Sigma)}$$

for $p > n-1$ and $\alpha = 1 - (n-1)/p$.

We will also need the following result, (see the proof of [5, Theorem 4.19]).

**Lemma 2.3.** *Let $f$ be a smooth function on $\Sigma$ and let $k$ be a positive integer. There is a constant $C$, which depends on $k$ and $\Sigma$, such that*

$$\|\nabla^{2k} f\|^2_{L^2(\Sigma)} \leqq \int_\Sigma (\Delta^k f)^2 \,\mathrm{d}\mathcal{H}^{n-1} + C\|f\|^2_{H^{2k-1}(\Sigma)} \tag{2.5}$$

*and*

$$\|\nabla^{2k+1} f\|^2_{L^2(\Sigma)} \leqq \int_\Sigma |\nabla(\Delta^k f)|^2 \,\mathrm{d}\mathcal{H}^{n-1} + C\|f\|^2_{H^{2k}(\Sigma)}. \tag{2.6}$$

**Proof.** We only proof (2.5) in the cases $k = 1, 2$, since the higher order cases and (2.6) are analogous. Recall that Ricci tensor is given by $R_{jm} = g^{ik} R_{ijmk}$. Thus from (2.2), with $X$ equal to the covariant gradient of $f$ and taking $k = i$, we get

$$\nabla_i \nabla_j \nabla^i f - \nabla_j \Delta f = R_{jl} \nabla^l f.$$

We multiply the above equality by $\nabla^j f$ and use the integration by parts formula (2.1) to obtain

$$-\int_\Sigma \nabla_i \nabla^j f \, \nabla_j \nabla^i f \, \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma (\Delta f)^2 \, \mathrm{d}\mathcal{H}^{n-1} = \int_\Sigma R_{ij} \nabla^i f \, \nabla^j f \, \mathrm{d}\mathcal{H}^{n-1}.$$

This yields the claim since (recall that for any given function $f$, $\nabla_i \nabla_j f = \nabla_j \nabla_i f$)

$$\nabla_i \nabla^j f \, \nabla_j \nabla^i f = \nabla^i \nabla^j f \, \nabla_i \nabla_j f = |\nabla^2 f|^2.$$

The argument in the case $k = 2$ is similar but more technical. We have by the previous statement that

$$\int_\Sigma |\Delta^2 f|^2 \, \mathrm{d}\mathcal{H}^{n-1} \geqq \int_\Sigma |\nabla^2 \Delta f|^2 \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)}.$$

Hence, we need to prove that

$$\int_\Sigma |\nabla^2 \Delta f|^2 \, \mathrm{d}\mathcal{H}^{n-1} \geqq \int_\Sigma |\nabla^4 f|^2 \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)}. \tag{2.7}$$

First, by the integration by parts formula (2.1) we have

$$\begin{aligned}\int_\Sigma |\nabla^2 \Delta f|^2 \, \mathrm{d}\mathcal{H}^{n-1} &= \int_\Sigma (\nabla^i \nabla^j \nabla_k \nabla^k f)\, (\nabla_i \nabla_j \nabla^l \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} \\ &= -\int_\Sigma (\nabla_i \nabla^i \nabla^j \nabla_k \nabla^k f)\, (\nabla_j \nabla^l \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Then, using (2.3), we obtain

$$\begin{aligned}\int_\Sigma |\nabla^2 \Delta f|^2 \, \mathrm{d}\mathcal{H}^{n-1} &\geqq -\int_\Sigma (\nabla_k \nabla_i \nabla^i \nabla^j \nabla^k f)\, (\nabla_j \nabla^l \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)} \\ &= -\int_\Sigma (\nabla^i \nabla^j \nabla^k f)\, (\nabla_i \nabla_k \nabla_j \nabla^l \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)},\end{aligned}$$

where the last equality follows by integration by parts. We proceed using formula (2.3) again and integration by parts to deduce

$$\begin{aligned}\int_\Sigma |\nabla^2 \Delta f|^2 \, \mathrm{d}\mathcal{H}^{n-1} &\geqq -\int_\Sigma (\nabla^i \nabla^j \nabla^k f)\, (\nabla^l \nabla_i \nabla_j \nabla_k \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)} \\ &= -\int_\Sigma (\nabla_i \nabla^l \nabla^i \nabla^j \nabla^k f)\, (\nabla_j \nabla_k \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)} \\ &\geqq -\int_\Sigma (\nabla_i \nabla^i \nabla^j \nabla^k \nabla^l f)\, (\nabla_j \nabla_k \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)} \\ &= \int_\Sigma (\nabla^i \nabla^j \nabla^k \nabla^l f)\, (\nabla_i \nabla_j \nabla_k \nabla_l f) \, \mathrm{d}\mathcal{H}^{n-1} - C \|f\|^2_{H^3(\Sigma)}.\end{aligned}$$

Thus we have (2.7), since $(\nabla^i \nabla^j \nabla^k \nabla^l f)\, (\nabla_i \nabla_j \nabla_k \nabla_l f) = |\nabla^4 f|^2$. □

**Remark 2.4.** In the case $k = 1$ we have a more precise version of Lemma 2.3 for hypersurfaces. It is clear that the proof of Lemma 2.3 implies that

$$\int_\Sigma |\nabla^2 f|^2 \, d\mathcal{H}^{n-1} \leqq \int_\Sigma (\Delta f)^2 \, d\mathcal{H}^{n-1} + (\sqrt{n-1}+1) \int_\Sigma |B|^2 |\nabla f|^2 \, d\mathcal{H}^{n-1},$$

where $B$ denotes the (scalar) second fundamental form (see [34] for definition). This follows from the fact that we may estimate the Ricci curvature by $|\mathrm{Ric}| \leqq (\sqrt{n-1}+1)|B|^2$.

**Remark 2.5.** Using Lemma 2.1 we may write the statement of Lemma 2.3 in the following way: for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|f\|^2_{H^{2k}(\Sigma)} \leqq (1+\varepsilon) \int_\Sigma (\Delta^k f)^2 \, d\mathcal{H}^{n-1} + C_\varepsilon \|f\|^2_{L^2(\Sigma)}$$

and

$$\|f\|^2_{H^{2k+1}(\Sigma)} \leqq (1+\varepsilon) \int_\Sigma |\nabla(\Delta^k f)|^2 \, d\mathcal{H}^{n-1} + C_\varepsilon \|f\|^2_{L^2(\Sigma)}.$$

Indeed, this follows by the interpolation inequality together with standard Young's inequality

$$\begin{aligned} \|\nabla^l f\|_{L^2(\Sigma)} &\leqq C \|\nabla^h f\|^\theta_{L^2(\Sigma)} \|f\|^{1-\theta}_{L^2(\Sigma)} \\ &\leqq \varepsilon \|\nabla^h f\|_{L^2(\Sigma)} + C(\varepsilon) \|f\|_{L^2(\Sigma)} \end{aligned}$$

for every $1 \leqq l \leqq h-1$ and $\theta = \theta(h,l)$ is given by Lemma 2.1.

For clarity we denote the standard inner product between two vectors $x, y$ in $\mathbb{R}^n$ as $x \cdot y$ and the differential of the map $F : \mathbb{R}^n \to \mathbb{R}^m$ by $DF$ to distinguish them from the inner product on manifold and from the covariant derivative. There is, however, a possibility of confusion when we denote the divergence of a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ by $\operatorname{div} X$, since "div" also denotes the divergence of a vector field on manifold. We will denote the divergence of a vector field on the manifold $(\Sigma, g)$ by $\operatorname{div}_g$ and in $\mathbb{R}^n$ by $\operatorname{div}_{\mathbb{R}^n}$ if this is not clear from the context.

When the manifold $\Sigma$ is given by a boundary of a smooth bounded set $F \subset \mathbb{R}^n$ it has a natural orientation and we denote by $\nu_F$ the unit outer normal. In this case we may extend the definition of divergence on $\Sigma$ to vector fields which have values in $\mathbb{R}^n$. Let $X : U \to \mathbb{R}^n$ be a smooth vector field, where $U$ is an open neighborhood of $\Sigma$. We define the tangential divergence of $X$ on $\partial F$ by

$$\operatorname{div}_\tau X := \operatorname{div} X - \langle DX \nu_F, \nu_F \rangle.$$

The divergence theorem states that

$$\int_{\partial F} \operatorname{div}_\tau X \, d\mathcal{H}^{n-1} = \int_{\partial F} H_F \, (X \cdot \nu_F) \, d\mathcal{H}^{n-1},$$

where $H_F$ denotes the sum of the principal curvatures of $\partial F$. We denote the second fundamental form of $\partial F$ by $B_F$, which in our case is a symmetric $\binom{2}{0}$-tensor (or equivalently a symmetric matrix). Finally we may project a vector field $X : U \to \mathbb{R}^n$ to the tangent space of $\partial F$ by

$$X_\tau := X - (X \cdot \nu_F)\nu_F. \tag{2.8}$$

Then $X_\tau$ canonically defines a vector field on $(\partial F, g)$ and we denote by $\operatorname{div}_g X_\tau$ its divergence. For a given function $u : U \to \mathbb{R}$ we define the tangential gradient on $\Sigma = \partial F$ as the projection of its gradient $Du$

$$D_\tau u := (Du)_\tau. \tag{2.9}$$

The tangential gradient and the covariant gradient are canonically isomorphic. In particular, it holds

$$|\nabla u(x)|_g = |D_\tau u(x)| \qquad \text{for } x \in \Sigma, \tag{2.10}$$

where $|\cdot|_g$ denotes the norm given by the metric tensor $g$, and $|\cdot|$ is the length of a vector in $\mathbb{R}^n$.

### 2.2. The Energy Functional

In this section we introduce the energy functional that underlies the flow. We also introduce the proper notions of stationary points and stability that will be needed in the study of the long-time behavior of the flow. As explained in the introduction, the free energy functional is the sum of the perimeter and of a bulk elastic term. Throughout the paper $\Omega$ will denote a fixed bounded open set of $\mathbb{R}^3$ with Lipschitz boundary.

Concerning the elastic part, for $F \subset\subset \Omega$ and for an elastic displacement $u : \Omega \backslash F \to \mathbb{R}^3$ we denote by $E(u)$ the symmetric part of $Du$, that is, $E(u) := \frac{Du+(Du)^T}{2}$. In what follows, $\mathbb{C}$ stands for the *elasticity tensor* acting on $3 \times 3$-matrices, such that $\mathbb{C}A = \frac{1}{2}\mathbb{C}(A + A^T)$ and $\mathbb{C}A$ is symmetric for all $3 \times 3$-matrices $A$. Moreover, $\mathbb{C}A : A > 0$ if $A$ is symmetric and $A \neq 0$. Finally we shall denote by $Q(A) := \frac{1}{2}\mathbb{C}A : A$ the *elastic energy density*.

We are now ready to write the energy functional. For a fixed *boundary displacement* $w_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we set

$$\mathcal{J}(F) := \int_{\Omega\backslash F} Q(E(u_F))\,\mathrm{d}x + \mathcal{H}^2(\partial F), \tag{2.11}$$

where $u_F$ is the elastic equilibrium satisfying the Dirichlet boundary condition $w_0$ on a fixed relatively open subset $\partial_D\Omega \subseteq \partial\Omega$. More precisely, $u_F$ is the unique solution in $H^1(\Omega\backslash F; \mathbb{R}^3)$ of the following elliptic system:

$$\begin{cases} \operatorname{div} \mathbb{C}E(u_F) = 0 & \text{in } \Omega\backslash F, \\ \mathbb{C}E(u_F)[\nu_F] = 0 & \text{on } \partial F \cup (\partial\Omega\backslash\partial_D\Omega), \\ u_F = w_0 & \text{on } \partial_D\Omega. \end{cases} \tag{2.12}$$

Note that by the second condition for every $x \in \partial F$ the vector $\mathbb{C}E(u_F)(x)[e]$ belongs to the tangent space of $\partial F$ at $x$ for every vector $e$.

Next, we provide the first and the second variation formulas for (2.11). To this aim, for any vector field $X \in C_c^1(\mathbb{R}^3; \mathbb{R}^3)$, let $(\Phi_t)_{t\in(-1,1)}$ be the associated flow, that is the solution of

$$\begin{cases} \dfrac{\partial \Phi_t}{\partial t} = X(\Phi_t), \\ \Phi_0 = Id. \end{cases} \tag{2.13}$$

The first and the second variation of the functional (2.11) are stated in the following theorem. Recall that $H_F$ denotes the sum of the principal curvatures and $B_F$ the second fundamental form of $\partial F$. Sometimes, with a slight abuse of terminology, we will refer to $H_F$ as the mean curvature of $\partial F$.

**Theorem 2.6.** *Let $F \subset\subset \Omega$ be a smooth set, $X \in C_c^1(\Omega; \mathbb{R}^2)$ and let $(\Phi_t)_{t\in(-1,1)}$ be the associated flow as in (2.13). Set $\psi := X \cdot \nu_F$ on $\partial F$ and let $X_\tau$ be as in (2.8). Then,*

$$\frac{d}{\mathrm{d}t}\mathcal{J}(\Phi_t(F))\Big|_{t=0} = \int_{\partial F} (H_F - Q(E(u_F)))\psi \,\mathrm{d}\mathcal{H}^2. \tag{2.14}$$

*If in addition* $\operatorname{div}_{\mathbb{R}^n} X = 0$ *in a neighborhood of $\partial F$ we have*

$$\begin{aligned} \frac{d^2}{\mathrm{d}t^2}\mathcal{J}(\Phi_t(F))\Big|_{t=0} &= \int_{\partial F} |\nabla\psi|^2 - |B_F|^2\psi^2 \,\mathrm{d}\mathcal{H}^2 - 2\int_{\Omega\setminus F} Q(E(u_\psi))\,\mathrm{d}x \\ &\quad - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F)))\psi^2 \,\mathrm{d}\mathcal{H}^2 \\ &\quad - \int_{\partial F} (H_F - Q(E(u_F)))\operatorname{div}_g(\psi X_\tau)\,\mathrm{d}\mathcal{H}^2, \end{aligned} \tag{2.15}$$

*where the function $u_\psi$ is the unique solution in $H^1(\Omega\setminus F; \mathbb{R}^3)$, with $u_\psi = 0$ on $\partial_D\Omega$, of*

$$\int_{\Omega\setminus F} \mathbb{C}E(u_\psi) : E(\varphi)\,\mathrm{d}x = -\int_{\partial F} \operatorname{div}_g(\psi\,\mathbb{C}E(u_F))\cdot\varphi\,\mathrm{d}\mathcal{H}^2 \tag{2.16}$$

*for all $\varphi \in H^1(\Omega\setminus F; \mathbb{R}^2)$ such that $\varphi = 0$ on $\partial_D\Omega$.*

Formulas (2.14) and (2.15) have been derived in [9] when $F$ is the subgraph of a periodic function. The very same calculations apply to the more general situation considered here.

Throughout the paper we fix a smooth reference set $G \subset\subset \Omega$ and define the reference manifold as $(\Sigma, g)$, where $\Sigma = \partial G$ and $g$ is the metric induced by the Euclidean metric in $\mathbb{R}^3$. We denote the outer normal of $G$ simply by $\nu$. For every $\eta > 0$ we denote

$$\mathcal{N}_\eta(\Sigma) := \{x \in \mathbb{R}^3 : |d_G(x)| < \eta\},$$

where $d_G$ denotes the signed distance function of $G$. Denote also $\pi$ the orthogonal projection on the boundary of $G$. Since $G$ is smooth,

$$\text{there exists } \eta_0 > 0 \text{ such that } d_G \text{ and } \pi \text{ are smooth in } \mathcal{N}_{2\eta_0}(\Sigma). \tag{2.17}$$

We denote by $\mathfrak{h}^k_M(\Sigma)$ the following class of sets, whose boundary is a suitable normal graph over $\Sigma$. Precisely, for any integer $k \geqq 1$ and $M > 0$ we say

$$F \in \mathfrak{h}^k_M(\Sigma) \quad \text{if } \partial F = \{x + h_F(x)\nu(x) : x \in \Sigma\} \subset \mathcal{N}_{\eta_0}(\Sigma) \\ \text{with} \quad \|h_F\|_{H^k(\Sigma)} \leqq M. \tag{2.18}$$

In particular, by Morrey embedding any set in $\mathfrak{h}^3_M(\Sigma)$ is $C^{1,\alpha}$-diffeomorphic to the reference set $G$ for every $\alpha \in (0, 1)$. The space $\mathfrak{h}^{k,\alpha}_M(\Sigma)$, $\alpha \in (0, 1)$, is defined similarly in terms of the $C^{k,\alpha}$-norm of the function $h_F$.

Let $G_1, \ldots, G_m$ be the bounded open sets enclosed by the connected components $\Gamma_{G,1}, \ldots, \Gamma_{G,m}$ of the boundary $\partial G$. Note that the $G_i$'s are not in general the connected components of $G$ and it may happen that $G_i \subset G_j$ for some $i \neq j$. If $F \in \mathfrak{h}^3_M(\Sigma)$, then $F$ is $C^1$-diffeomorphic to $G$ and thus $\partial F$ has the same number $m$ of connected components $\Gamma_{F,1}, \ldots, \Gamma_{F,m}$, which can be numbered in such a way that

$$\Gamma_{F,i} = \{x + h_F(x)\nu(x) : x \in \Gamma_{G,i}\}, \tag{2.19}$$

for a suitable $h_F \in H^3(\Sigma)$. The boundaries $\Gamma_{F,i}$ then enclose the sets $F_i$, which in turn are diffeomorpic to $G_i$.

We are interested in area preserving variations, in the following sense:

**Definition 2.7.** Let $F \subset\subset \Omega$ be a smooth set. Given a vector field $X \in C^\infty_c(\Omega; \mathbb{R}^3)$, we say that the associated flow $(\Phi_t)_{t \in (-1,1)}$ is *admissible for* $F$ if there exists $\varepsilon_0 \in (0, 1)$ such that

$$|\Phi_t(F_i)| = |F_i| \quad \text{for } t \in (-\varepsilon_0, \varepsilon_0) \text{ and } i = 1, \ldots, m.$$

**Remark 2.8.** Note that if the flow associated with $X$ is admissible in the sense of the previous definition, then for $i = 1, \ldots, m$ we have

$$\int_{\Gamma_{F,i}} X \cdot \nu_F \, d\mathcal{H}^1 = 0.$$

In view of this remark it is convenient to introduce the space $\tilde{H}^1(\partial F)$ consisting of all functions $\psi \in H^1(\partial F)$ with zero average on each component of $\partial F$, that is,

$$\int_{\Gamma_{F,i}} \psi \, d\mathcal{H}^1 = 0 \qquad \text{for every } i = 1, \ldots, m.$$

Any admissible vector field $X$ thus defines a function $\psi \in \tilde{H}^1(\partial F)$. Conversely, given $\psi \in \tilde{H}^1(\partial F) \cap C^\infty(\partial F)$ it is possible to construct a sequence of vector fields $X_n \in C^\infty_c(\Omega; \mathbb{R}^2)$, with $\operatorname{div}_{\mathbb{R}^n} X_n = 0$ in a neighborhood of $\overline{F}$, such that $X_n \cdot \nu_F \to \psi$ in $C^1(\partial F)$, see [2, Proof of Corollary 3.4] for the details. Note that in particular the flows associated with $X_n$ are admissible.

**Definition 2.9.** Let $F \subset\subset \Omega$ be a set of class $C^2$. We say that $F$ is *stationary* if

$$\frac{d}{dt}\mathcal{J}(\Phi_t(F))\Big|_{t=0} = 0$$

for all admissible flows in the sense of Definition 2.7.

**Remark 2.10.** By Remark 2.8 and in view of (2.14) it follows that a set $F \subset\subset \Omega$ of class $C^2$ is stationary if and only if there exist constants $\lambda_1, \dots, \lambda_m$ such that

$$H_F - Q(E(u_F)) = \lambda_i \qquad \text{on } \Gamma_{F,i}$$

for every $i = 1, \dots, m$. Note that if $F$ is a sufficiently regular (local) minimizer of (2.11) under the constraint $|F| = const.$, then there exists a constant $\lambda$ such that

$$H_F - Q(E(u_F)) = \lambda \qquad \text{on } \partial F.$$

Thus, our notion of stationarity differs from the usual notion of criticality just recalled. Note that by a bootstrap argument it can be proved that a stationary set is smooth. In fact, it can be shown that it is even analytic, see [33]. Note that if $F$ is stationary, then the second variation formula (2.15) reduces to

$$\begin{aligned} \frac{d^2}{\mathrm{d}t^2}\mathcal{J}(\Phi_t(F))\Big|_{t=0} &= \int_{\partial F} |\nabla\psi|^2 - |B_F|^2\psi^2 \,\mathrm{d}\mathcal{H}^2 \\ &\quad - 2\int_{\Omega\setminus F} Q(E(u_\psi))\,\mathrm{d}x - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F)))\psi^2 \,\mathrm{d}\mathcal{H}^2, \end{aligned} \tag{2.20}$$

where we recall that $\psi = X \cdot \nu_F$ and $u_\psi$ is the function satisfying (2.16).

In view of (2.20), for any set $F \subset\subset \Omega$ of class $C^2$ it is convenient to introduce the quadratic form $\partial^2\mathcal{J}(F)$ defined on $\tilde{H}^1(\partial F)$ as

$$\begin{aligned} \partial^2\mathcal{J}(F)[\psi] := &\int_{\partial F} |\nabla\psi|^2 - |B_F|^2\psi^2 \,\mathrm{d}\mathcal{H}^2 \\ &- 2\int_{\Omega\setminus F} Q(E(u_\psi))\,\mathrm{d}x - \int_{\partial F} \partial_{\nu_F}(Q(E(u_F)))\psi^2 \,\mathrm{d}\mathcal{H}^2, \end{aligned} \tag{2.21}$$

where $u_\psi$ is the unique solution of (2.16) under the Dirichlet condition $u_\psi = 0$ on $\partial_D\Omega$. We may finally give the definition of stability for a stationary point.

**Definition 2.11.** Let $F \subset\subset \Omega$ be a stationary set in the sense of Definition 2.9. We say that $F$ is *strictly stable* if

$$\partial^2\mathcal{J}(F)[\psi] > 0 \qquad \text{for all } \psi \in \tilde{H}^1(\partial F)\setminus\{0\}. \tag{2.22}$$

It is not difficult to see that (2.22) is equivalent to the coercivity of $\partial^2\mathcal{J}(F)$ on $\tilde{H}^1(\partial F)$. More precisely, (2.22) holds if and only if there exists $c_0 > 0$ such that

$$\partial^2\mathcal{J}(F)[\psi] \geqq c_0\|\psi\|^2_{\tilde{H}^1(\partial F)} \qquad \text{for all } \psi \in \tilde{H}^1(\partial F); \tag{2.23}$$

see [9]. In turn the latter coercivity property is stable with respect to small $H^3$-perturbations. More precisely, we have

**Lemma 2.12.** *Assume that the reference set* $G \subset\subset \Omega$ *is a (smooth) strictly stable stationary set in the sense of Definition* 2.11. *Then, there exists* $\sigma_0 > 0$ *such that for all* $F \in \mathfrak{h}^3_{\sigma_0}(\Sigma)$, *defined in* (2.18), *we have*

$$\partial^2 \mathcal{J}(F)[\psi] \geqq \frac{c_0}{2}\|\psi\|^2_{\tilde{H}^1(\partial F)} \textit{ for all } \psi \in \tilde{H}^1(\partial F),$$

*where* $c_0$ *is the constant in* (2.23).

**Proof.** The proof follows the argument in [9, Proof of Theorem 5.2 and Lemma 5.3], where the case of $F$ being the subgraph of a periodic function is considered. Although the geometric framework here is more general, we may follow exactly the same line of argument up to the obvious changes due to the different setting. We note that in our case we may even simplify the aforementioned proof by taking advantage of the fact that $F \in \mathfrak{h}^3_{\sigma_0}(\Sigma)$ (while in [9] only $W^{2,p}$-bounds were assumed). Indeed, under this assumption we have that $u_F$ is of class $H^3$ in a neighborhood of $\Sigma$, with the norm estimated by a constant depending on $\sigma_0$ (see the proof of Theorem 4.1). In turn, $\partial_{\nu_F}(Q(E(u_F))) \in H^{\frac{1}{2}}(\partial F)$ with a bound depending on $\sigma_0$, which is a much stronger information than the boundedness in $H^{-\frac{1}{2}}(\partial F)$ proven in [9]. □

We conclude this section by showing that in a sufficiently small $H^3$-neighborhood of $G$ the stationary sets are isolated, once we fix the areas enclosed by the connected components of the boundary.

**Proposition 2.13.** *Assume that the reference set* $G \subset\subset \Omega$ *is a smooth strictly stable stationary set in the sense of Definition* 2.11 *and let* $\sigma_0$ *be the constant provided by Lemma* 2.12. *There exists* $\sigma_1 \in (0, \sigma_0)$ *with the following property: Let* $F_1$, $F_2 \in \mathfrak{h}^3_{\sigma_1}(\Sigma)$, *defined in* (2.18), *be stationary sets in the sense of Definition* 2.9 *and* (*with the same notation as in* (2.19)) *assume that* $|F_{1,i}| = |F_{2,i}|$ *for* $i = 1, \dots, m$. *Then* $F_1 = F_2$.

**Proof.** Let $F_1$ and $F_2$ be in $\mathfrak{h}^3_{\sigma_1}(\Sigma)$, with $\sigma_1 \in (0, \sigma_0)$ to be chosen, and denote the components defined in (2.19) by $F_{i,1}, \dots, F_{i,m}$ for $i = 1, 2$. We begin by constructing a vector field $X : \mathcal{N}_{\eta_0}(\Sigma) \to \mathbb{R}^3$ such that the associated flow $(\Phi_t)_{t\in([0,1])}$ is admissible is sense of Definition 2.8 and takes the set $F_1$ to $F_2$. More precisely, it holds $\Phi_0(F_1) = F_1$, $\Phi_1(F_1) = F_2$ and $|\Phi_t(F_{1,i})| = |F_{1,i}|$ for every $t \in [0, 1]$ and $i = 1, \dots, m$. The construction can be done as in [37, Proposition 3.4] (see also [24, Lemma 2.8]) in such a way that $|X(x)| \leqq 2|X(x) \cdot \nu_{F_t}(x)|$ for $x \in \partial F_t$ and for all $t \in [0, 1]$, and that

$$\partial F_t = \{x + h_{F_t}(x)\nu(x) : x \in \Sigma\} \qquad \text{with} \quad \|h_{F_t}\|_{H^3(\Sigma)} \leqq C\sigma_1 < \sigma_0,$$

where the last inequality holds provided that $\sigma_1$ is small enough with a constant $C$ depending only on $G$. Recalling (2.15), (2.21), using the Lemma 2.12 and by integrating by parts we get

$$\begin{aligned}\frac{d^2}{\mathrm{d}t^2}\mathcal{J}(\Phi_t(F_1)) &= \partial^2\mathcal{J}(F_t)[X\cdot\nu_{F_t}]\\ &\qquad - \int_{\partial F_t}(H_{F_t}-Q(E(u_{F_t})))\,\mathrm{div}_g((X\cdot\nu_{F_t})X_\tau)\,\mathrm{d}\mathcal{H}^2\\ &\geqq \frac{c_0}{2}\|X\cdot\nu_{F_t}\|^2_{H^1(\partial F_t)}\\ &\qquad + \int_{\partial F_t}\langle\nabla(H_{F_t}-Q(E(u_{F_t}))),(X\cdot\nu_{F_t})X_\tau\rangle\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

We denote $R_t := H_{F_t} - Q(E(u_{F_t}))$ and estimate the last term by (5.3), which we will show later in the proof of Theorem 5.1, to get that there exists $\theta \in (0,1)$ such that

$$\begin{aligned}\int_{\partial F_t}\langle\nabla R_t,(X\cdot\nu_{F_t})X_\tau\rangle\,\mathrm{d}\mathcal{H}^2 &\leqq \left(\int_{\partial F_t}|\nabla R_t|^2\,\mathrm{d}\mathcal{H}^2\right)^{1/2}\\ &\qquad\times\left(\int_{\partial F_t}|(X\cdot\nu_{F_t})X_\tau|^2\,\mathrm{d}\mathcal{H}^2\right)^{1/2}\\ &\leqq C\|h_{F_t}\|^{\theta/2}_{H^3(\Sigma)}\left(\int_{\partial F_t}|X\cdot\nu_{F_t}|^4\,\mathrm{d}\mathcal{H}^2\right)^{1/2}\\ &\leqq C\sigma_1^{\theta/2}\|X\cdot\nu_{F_t}\|^2_{L^4(\partial F_t)}.\end{aligned}$$

Therefore we have, by the Sobolev embedding, that

$$\begin{aligned}\frac{d^2}{\mathrm{d}t^2}\mathcal{J}(\Phi_t(F_1)) &\geqq \frac{c_0}{2}\|X\cdot\nu_{F_t}\|^2_{H^1(\partial F_t)} - C\sigma_1^{\theta/2}\|X\cdot\nu_{F_t}\|^2_{L^4(\partial F_t)}\\ &\geqq \frac{c_0}{2}\|X\cdot\nu_{F_t}\|^2_{H^1(\partial F_t)} - C\sigma_1^{\theta/2}\|X\cdot\nu_{F_t}\|^2_{H^1(\partial F_t)}\\ &\geqq \frac{c_0}{4}\|X\cdot\nu_{F_t}\|^2_{H^1(\partial F_t)},\end{aligned}$$

provided that $\sigma_1$ is small enough.

On the other hand by the stationarity of $F_1$ and $F_2$ we have

$$\frac{d}{\mathrm{d}t}\mathcal{J}(\Phi_t(F_1))\Big|_{t=0} = \frac{d}{\mathrm{d}t}\mathcal{J}(\Phi_t(F_1))\Big|_{t=1} = 0.$$

This means that $\frac{d^2}{\mathrm{d}t^2}\mathcal{J}(\Phi_t(F_1)) = 0$ and therefore $X\cdot\nu_{F_t} = 0$ on $\partial F_t$ for all $t\in(0,1)$. Therefore $t\mapsto\Phi_t(F_1)$ is constant and $F_1 = F_2$. □

## 3. Short Time Existence for the Surface Diffusion with a Forcing Term

In the following we shall assume $n = 3$. Given a smooth function $f : \Sigma \times [0, +\infty) \to \mathbb{R}$ we shall consider the following forced surface diffusion equation

$$V_t = \Delta_{\partial F_t}(H_{F_t} + f(\cdot, t) \circ \pi) \tag{3.1}$$

where $V_t$ denotes the outer normal velocity of $\partial F_t$ and $\Delta_{\partial F_t}$ is the Laplace–Beltrami operator on $\partial F_t$ endowed with the metric induced by the Euclidean metric. Note that we consider a forcing term which time by time is constant along the normal directions to $\Sigma$. Although this class of forcing terms is not general, this choice is natural to obtain the existence of (1.3), where the nonlocal term is defined only on the evolving boundary (or, in fact, on $\Omega \setminus F_t$).

The goal in this section is to prove short time existence of a unique smooth solution of (3.1) starting from $F_0$ which is close to the reference set $G$. This will be done in Theorem 3.1.

### *3.1. The Flow in Coordinates*

Given a sufficiently smooth function $h : \Sigma \to (-\eta_0, \eta_0)$, where $\eta_0$ is introduced in (2.17), we denote by $F_h$ the bounded open set whose boundary is given by

$$\partial F_h = \{x + h(x)\nu(x) : x \in \Sigma\},$$

where $\nu$ is the outer unit normal to $\partial G$. Note that the projection $\pi|_{\partial F_h} : \partial F_h \to \Sigma$ is invertible and we denote by $\pi_{F_h}^{-1}$ its inverse. In this case we have $\pi_{F_h}^{-1}(x) = x + h(x)\nu(x)$.

We denote by $\nu$ the normal and by $k_1, k_2$ the principle curvatures of $\Sigma$, while $\tau_1, \tau_2$ denote the corresponding eigenvectors on the tangent plane. The exterior normal to $F_h$ at any point $\pi_{F_h}^{-1}(x)$ is

$$\begin{aligned}\nu_{F_h} \circ \pi_{F_h}^{-1} = \frac{1}{J}\big(&(1 + hk_1)(1 + hk_2)\nu \\ &-(1 + hk_2)\partial_{\tau_1} h\, \tau_1 - (1 + hk_1)\partial_{\tau_2} h\, \tau_2\big),\end{aligned} \tag{3.2}$$

where $J^2 = (1 + hk_1)^2(1 + hk_2)^2 + (1 + hk_1)^2(\partial_{\tau_1} h)^2 + (1 + hk_2)^2(\partial_{\tau_2} h)^2$. We recall (see [36, p. 21]) that the mean curvature $H_{F_h}$ of $\partial F_h$ can be written as

$$H_{F_h} \circ \pi_{F_h}^{-1} = -(\nu_{F_h} \circ \pi_{F_h}^{-1} \cdot \nu)\Delta h + P(x, h, \nabla h),$$

where $P$ is a smooth function such that $P(\cdot, 0, 0) = H_G$, the mean curvature of the boundary of $G$. We rewrite the above formula as

$$H_{F_h} \circ \pi_{F_h}^{-1} = -\Delta h + \langle A(x, h, \nabla h), \nabla^2 h\rangle + H_G + a(x, h, \nabla h), \tag{3.3}$$

where the tensor $A$ and the function $a$ are smooth and vanish when both $h$ and $\nabla h$ are 0.

Let us denote by $g_h$ the pull-back metric on $\Sigma$ induced by the diffeomorphism $\pi_{F_h}^{-1} : \Sigma \to \partial F_h$. Since the manifold $(\partial F_h, g)$ endowed with the Euclidean metric $g$ is isometric to $(\Sigma, g_h)$ then for every smooth function $f$ defined on $\Sigma$ we have

$$\big(\Delta_{\partial F_h}(f \circ \pi)\big) \circ \pi_{F_h}^{-1} = \Delta_{g_h} f$$

where $\Delta_{g_h}$ is the Laplace–Beltrami operator on $\Sigma$ with respect to the metric $g_h$. One can also check that (see [36, p. 21])

$$(g_h)_{ij} = g_{ij} + a_{ij}(\cdot, h, \nabla h),$$

where the functions $a_{ij}$ are smooth and vanish when both $h$ and $\nabla h$ vanish, and that we have the following expansion of the Christoffel symbols:

$$(\Gamma_{g_h})^i_{jk} = (\Gamma_g)^i_{jk} + a^i_{jk}(x, h, \nabla h) + b^{ilm}_{jk}(x, h, \nabla h)\frac{\partial^2 h}{\partial x_l \partial x_m}.$$

Above $b^{ilm}_{jk}$ is a smooth function and $a^i_{jk}$ is a smooth function which vanish when $h$ and $\nabla h$ vanish. We recall that the we may write the Laplace–Beltrami operator $\Delta_{g_h}$ as

$$\Delta_{g_h} f := (g_h)^{ij} \widetilde{\nabla}_i \widetilde{\nabla}_j f,$$

where $\widetilde{\nabla}_i \widetilde{\nabla}_j$ stands for the second order covariant derivatives with respect to $g_h$. Hence we get by the above formulas and after some straightforward calculations that

$$\begin{aligned}\Delta_{g_h} f = \Delta f &+ \langle A_1(x, h, \nabla h), \nabla^2 f\rangle + \langle A_2(x, h, \nabla h), \nabla f\rangle \\ &+ \langle B(x, h, \nabla h), (\nabla^2 h \otimes \nabla f)\rangle.\end{aligned} \tag{3.4}$$

Concerning the equation of interest, assume that a smooth flow $(F_t)_{t\in(0,T)}$ is a solution of (3.1) and that $\partial F_t$ can be written as

$$\partial F_t = \{x + h(x, t)\nu(x) : x \in \Sigma\}. \tag{3.5}$$

Then the normal velocity is given by $V_t = \partial_t h(\nu_{F_t} \cdot \nu)$. Therefore, combining (3.3) and (3.4) and after long but straightforward calculations, we may rewrite the Equation (3.1) as

$$\begin{aligned}\frac{\partial h}{\partial t} &= -\Delta^2 h + \langle A(x, h, \nabla h), \nabla^4 h\rangle \\ &\quad + J_1(x, h, \nabla h, \nabla^2 h, \nabla^3 h) + J_2(x, h, \nabla h, \nabla^2 h, \nabla f, \nabla^2 f),\end{aligned} \tag{3.6}$$

where as usual $A$ is a smooth 4th-order tensor depending on $(x, h, \nabla h)$ vanishing when both $h$ and $\nabla h$ vanish, $J_1$ is given by

$$\begin{aligned}J_1 = \langle B_1, (\nabla^3 h \otimes \nabla^2 h)\rangle + \langle B_2, \nabla^3 h\rangle + \langle B_3, (\nabla^2 h \otimes \nabla^2 h \otimes \nabla^2 h)\rangle \\ + \langle B_4, (\nabla^2 h \otimes \nabla^2 h)\rangle + \langle B_5, \nabla^2 h\rangle + b_6,\end{aligned} \tag{3.7}$$

and $J_2$ is of the form

$$J_2 = \Delta f + \langle A_1, \nabla^2 f \rangle + \langle A_2, \nabla f \rangle + \langle B, (\nabla^2 h \otimes \nabla f) \rangle. \tag{3.8}$$

Here and throughout the paper we denote by $A$ (possibly with a subscript) a smooth tensor-valued function depending on $(x, h, \nabla h)$ and vanishing at $(x, 0, 0)$, while $B$ (possibly with a subscript) stands for a smooth tensor-valued function depending on $(x, h, \nabla h)$. We replace capital letters $A$ and $B$ with $a$ and $b$, respectively, in case of scalar valued functions.

### *3.2. Short Time Existence and Uniqueness*

Let us fix an initial set $F_0 \in \mathfrak{h}^3_{K_0}(\Sigma)$ which is close to $G$. Finding a solution of (3.1) for a short time with intial set $F_0$ is equivalent to finding a solution $h$ of (3.6) with initial datum $h(\cdot, 0) = h_{F_0} =: h_0$. This is the goal of this section and the result is stated in the following theorem:

**Theorem 3.1.** *Let $f : \Sigma \times [0, +\infty) \to \mathbb{R}$ be a smooth function. Given $\delta_0 > 0$ and $K_0 > 1$, there exist $\varepsilon_0$, $T_0 \in (0, 1)$ with the following property: if $F_0 \in \mathfrak{h}^3_{K_0}(\Sigma)$, defined in (2.18), if*

$$\sup_{0 \leqq t \leqq T_0} \|f(\cdot, t)\|_{L^\infty(\Sigma)} + \int_0^{T_0} \|f(\cdot, t)\|^2_{H^3(\Sigma)} \, \mathrm{d}t \leqq K_0, \tag{3.9}$$

*and $\|h_0\|_{L^2(\Sigma)} < \varepsilon_0$, where $h_0 := h_{F_0}$, then the Equation (3.1) has a unique smooth solution $(F_t)$ of the form (3.5) with $h \in C^\infty(0, T_0; C^\infty(\Sigma)) \cap H^1(0, T_0; H^1(\Sigma))$ and*

$$\sup_{0 \leqq t \leqq T_0} \|h(\cdot, t)\|_{L^2(\Sigma)} \leqq \delta_0. \tag{3.10}$$

*Moreover, for every integer $k \geqq 0$ there exist constants $C_k, q_k > 0$, independent of $\delta_0$ and $K_0$, such that*

$$\begin{aligned} &\sup_{0 \leqq t \leqq T} t^k \|h(\cdot, t)\|^2_{H^{2k+3}(\Sigma)} + \int_0^T t^k \|h(\cdot, t)\|^2_{H^{2k+5}(\Sigma)} \, \mathrm{d}t \\ &\qquad \leqq C_k \Big( \|h_0\|^2_{H^3(\Sigma)} + \int_0^T \big(1 + \|f\|^{q_k}_{L^\infty(\Sigma)} + \sum_{i=0}^k t^i \|f(\cdot, t)\|^2_{H^{2i+3}(\Sigma)}\big) \, \mathrm{d}t \Big), \end{aligned} \tag{3.11}$$

*for every $T \leqq T_0$.*

The proof of Theorem 3.1 is based on a fixed point argument in a carefully chosen function space and to this aim we need two lemmas. In the first one we estimate the derivatives of the nonlinear terms in (3.6).

**Proposition 3.2.** *Let $h$ and $f$ be of class $C^\infty(\Sigma)$. For every integer $k \geqq 1$ there exist $\tilde{C}_k > 0$ and $p_k \geqq 2$ such that given $M_0 > 0$ there is $\sigma_0 > 0$ with the property that if*

$$\|h\|^2_{H^3(\Sigma)} \leqq M_0 \quad \text{and} \quad \|h\|_{L^2(\Sigma)} \leqq \sigma_0$$

*then*

$$\int_\Sigma |\nabla^k(\langle A, \nabla^4 h\rangle)|^2 + |\nabla^k J_1|^2 + |\nabla^k J_2|^2 \,\mathrm{d}\mathcal{H}^2 \leqq \frac{1}{4}\int_\Sigma |\nabla^{k+4}h|^2 \,\mathrm{d}\mathcal{H}^2$$
$$+\tilde{C}_k\left(1 + \|f\|^{p_k}_{L^\infty(\Sigma)} + \int_\Sigma |\nabla^{k+2} f|^2 \,\mathrm{d}\mathcal{H}^2\right),$$

*where $A$, $J_1$, and $J_2$ are as in (3.6)–(3.8).*

**Proof.** Recall that $A(x, h, \nabla h)$ vanishes at $(x, 0, 0)$ and thus given $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that if $\|h\|_{C^1(\Sigma)} \leqq \delta$, then by Leibniz formula

$$|\nabla^k(\langle A, \nabla^4 h\rangle)|^2 \leqq \varepsilon|\nabla^{k+4}h|^2 + C\sum_{i=1}^{k} |\nabla^i(A(x, h, \nabla h))|^2 |\nabla^{k+4-i}h|^2.$$

On the other hand, the assumptions on $h$ together with standard interpolation imply that $\|h\|_{C^1} \leqq \delta$ and $\|h\|_{W^{2,4}} \leqq 1$ when $\sigma_0$ is chosen small (depending on $M_0$). It turns out to be convenient to set $w := \nabla h$. Since $\|w\|_\infty \leqq \delta < 1$, one may check that

$$\begin{aligned}
&\sum_{i=1}^{k} |\nabla^i(A(x, h, \nabla h)|^2 |\nabla^{k+4-i}h|^2 \\
&\quad\leqq C\sum_{i=1}^{k} |\nabla^{k+3-i} w|^2 \\
&\qquad + C\sum_{i=1}^{k} \sum_{\substack{1\leqq j_1\leqq \ldots \leqq j_{m-1}\leqq i \\ j_1+\cdots+j_{m-1}\leqq i \\ m\geqq 2}} |\nabla^{j_1} w|^2 \cdots |\nabla^{j_{m-1}} w|^2 |\nabla^{k+3-i} w|^2 \\
&\quad\leqq C\sum_{i=1}^{k} |\nabla^{k+3-i} w|^2 + C \sum_{\substack{1\leqq j_1\leqq \ldots \leqq j_m\leqq k+2 \\ j_1+\cdots+j_m\leqq k+3 \\ m\geqq 2}} |\nabla^{j_1} w|^2 \cdots |\nabla^{j_m} w|^2.
\end{aligned}$$

Then, by Hölder's inequality, we obtain

$$\begin{aligned}
\int_\Sigma |\nabla^k(\langle A, \nabla^4 h\rangle)|^2 \,\mathrm{d}\mathcal{H}^2 \leqq& \int_\Sigma \Big(\varepsilon|\nabla^{k+3} w|^2 + C\sum_{i=1}^{k} |\nabla^{k+3-i} w|^2\Big) \,\mathrm{d}\mathcal{H}^2 \\
&+ C \sum_{\substack{1\leqq j_1\leqq \ldots \leqq j_m\leqq k+2 \\ j_1+\cdots+j_m\leqq k+3 \\ m\geqq 2}} \|\nabla^{j_1} w\|^2_{\frac{2(k+3)}{j_l}} \cdots \|\nabla^{j_m} w\|^2_{\frac{2(k+3)}{j_m}}.
\end{aligned}$$

Observe that for every $l = 1, \ldots, m-1$, it holds by the interpolation Lemma 2.1

$$\|\nabla^{j_l} w\|_{\frac{2(k+3)}{j_l}} \leqq C\|w\|_{H^{k+3}}^{\theta_l}\|w\|_{\infty}^{1-\theta_l},$$

where $\theta_l = \frac{j_l}{k+3}$. To treat the last derivative we use a different interpolation:

$$\|\nabla^{j_m} w\|_{\frac{2(k+3)}{j_m}} \leqq C\|w\|_{H^{k+3}}^{\theta_m}\|\nabla w\|_{4}^{1-\theta_m},$$

where $\theta_m = \frac{2j_m(k+2)}{(2k+3)(k+3)} - \frac{1}{2k+3} < \frac{j_m}{k+3}$ (recall that $3 \leqq j_m < k+3$). Therefore, recalling that $\|w\|_\infty, \|\nabla w\|_4 \leqq 1$, we get

$$\begin{aligned}\int_\Sigma |\nabla^k(\langle A, \nabla^4 h\rangle)|^2 \,\mathrm{d}\mathcal{H}^2 \leqq& \int_\Sigma \big(\varepsilon|\nabla^{k+4}h|^2 + C\sum_{i=1}^{k}|\nabla^{k+4-i}h|^2\big)\,\mathrm{d}\mathcal{H}^2\\ &+ C\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq k+2\\ j_1+\cdots+j_m\leqq k+3\\ m\geqq 2}}\prod_{l=1}^{m}\|w\|_{H^{k+3}}^{2\theta_l}.\end{aligned}$$

Observe that for every choice of $j_1, \ldots, j_m$ the sum of the corresponding $\theta_l$ satisfies

$$\sum_{l=1}^{m}\theta_l < \sum_{l=1}^{m}\frac{j_l}{k+3} \leqq 1.$$

Therefore by Young's inequality, by Remark 2.2, and recalling that $\|w\|_\infty \leqq 1$, we conclude from the above inequality that

$$\int_\Sigma |\nabla^k(\langle A, \nabla^4 h\rangle)|^2 \,\mathrm{d}\mathcal{H}^2 \leqq \frac{1}{20}\int_\Sigma |\nabla^{k+4}h|^2\,\mathrm{d}\mathcal{H}^2 + \tilde{C}_k. \tag{3.12}$$

Using again $\|w\|_\infty \leqq 1$, we have that

$$|\nabla^k J_1| \leqq C\sum_{i=1}^{k}|\nabla^{k+3-i}w| + C\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq 2+k\\ j_1+\cdots+j_m\leqq 3+k\\ m\geqq 2}}|\nabla^{j_1}w|\ldots|\nabla^{j_m}w|.$$

Therefore, arguing exactly as above, we have

$$\int_\Sigma |\nabla^k J_1|^2\,\mathrm{d}\mathcal{H}^2 \leqq \frac{1}{20}\int_\Sigma |\nabla^{k+4}h|^2\,\mathrm{d}\mathcal{H}^2 + \tilde{C}_k. \tag{3.13}$$

In order to control the derivatives of $J_2$ we need a slightly different argument, because we need to separate the terms involving $f$ and $h$ from each other. We recall (3.8) and begin by estimating

$$
\begin{aligned}
|\nabla^k(\Delta f + \langle A_1, \nabla^2 f\rangle)| &\leqq C\sum_{l=0}^{k} |\nabla^{l+2} f| \\
&+C\sum_{i=1}^{k}\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq i\\ j_1+\cdots+j_m\leqq i\\ m\geqq 1}} |\nabla^{j_1}w|\ldots|\nabla^{j_m}w||\nabla^{k+2-i}f|.
\end{aligned}
$$

Therefore, using interpolation as above,

$$
\begin{aligned}
\int_\Sigma |\nabla^k(\Delta f + \langle A_1, \nabla^2 f\rangle)|^2\,\mathrm{d}\mathcal{H}^2 &\leqq C\Big(\|f\|_\infty^2 + \|\nabla^{k+2}f\|_2^2\Big) \\
&+ C\sum_{i=1}^{k}\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq i\\ j_1+\cdots+j_m\leqq i\\ m\geqq 1}} \prod_{l=1}^{m} \|\nabla^{j_l}w\|^2_{\frac{2(2+k)}{j_l}}\|\nabla^{k+2-i}f\|^2_{\frac{2(2+k)}{2+k-i}} \\
&\leqq C\Big(\|f\|_\infty^2 + \|\nabla^{k+2}f\|_2^2\Big) \\
&+ C\sum_{i=1}^{k}\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq i\\ j_1+\cdots+j_m\leqq i\\ m\geqq 1}} \prod_{l=1}^{m} \|w\|^{2\theta(j_l)}_{H^{k+3}}\|w\|_\infty^{2(1-\theta(j_l))}\|\nabla^{2+k}f\|_2^{\frac{2(k+2-i)}{k+2}}\|f\|_\infty^{\frac{2i}{k+2}},
\end{aligned}
$$

where $\theta(j_l) := \frac{j_l(k+1)}{(k+2)^2}$. Observe that since $j_1 + \cdots + j_m \leqq i$,

$$
\sum_{l=1}^{m}\Big(2\theta(j_l) + 2\frac{(2+k-i)}{k+2}\Big) \leqq \frac{2[(2+k)^2 - i]}{(2+k)^2} < 2.
$$

Therefore, using Young's inequality, we may conclude that

$$
\begin{aligned}
\int_\Sigma |\nabla^k(\Delta f + \langle A_1, \nabla^2 f\rangle)|^2\,\mathrm{d}\mathcal{H}^2 &\leqq \frac{1}{20}\|\nabla^{k+4}h\|_2^2 \\
&+\tilde{C}_k\Big(1 + \|f\|_\infty^{p_k} + \|\nabla^{k+2}f\|_2^2\Big).
\end{aligned} \tag{3.14}
$$

A similar argument, whose details are left to the reader, shows that

$$
\begin{aligned}
\int_\Sigma |\nabla^k\langle A_2, \nabla f\rangle + \langle B, (\nabla^2 h\otimes\nabla f)\rangle|^2\,\mathrm{d}\mathcal{H}^2 &\leqq \frac{1}{20}\|\nabla^{k+4}h\|_2^2 \\
&+\tilde{C}_k\Big(1 + \|f\|_\infty^{p_k} + \|\nabla^{k+2}f\|_2^2\Big).
\end{aligned}
$$

The conclusion then follows by combining this inequality with (3.12), (3.13), and (3.14). □

In the second lemma we "linearize" the terms $J_1$ and $J_2$ in the Equation (3.6). The argument is similar to the previous one and therefore we postpone its proof until the "Appendix".

**Lemma 3.3.** *Let* $T \in (0,1)$ *and let* $h_1, h_2, f : \Sigma \times (0,T) \to \mathbb{R}$ *be smooth functions such that*

$$\sup_{0 \leqq t \leqq T} \|h_i(\cdot,t)\|^2_{H^3(\Sigma)} + \int_0^T \int_\Sigma |\nabla^5 h_i|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t \leqq M_0,$$

*and*

$$\sup_{0 \leqq t \leqq T} \|f(\cdot,t)\|_{L^\infty(\Sigma)} + \int_0^T \int_\Sigma |\nabla^3 f|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t \leqq K_0.$$

*Then, there exists* $\theta \in (0,1)$ *with the following property: for any* $\varepsilon > 0$ *there exist* $C = C(\varepsilon, K_0, M_0) > 0$ *and* $\delta = \delta(\varepsilon, M_0) > 0$ *such that if* $\sup_{0 \leqq t \leqq T} \|h_i(\cdot,t)\|_{L^2(\Sigma)} \leqq \delta$, $i = 1, 2$, *then*

$$\int_0^T \int_\Sigma |J_{h_2} - J_{h_1}|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t \leqq \varepsilon \int_0^T \int_\Sigma |\nabla^4 h_2 - \nabla^4 h_1|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t$$
$$+ C T^\theta \sup_{0 \leqq t \leqq T} \|h_2(\cdot,t) - h_1(\cdot,t)\|^2_{H^2(\Sigma)},$$

*where* $J_h$ *is defined as in* (3.17).

**Proof of Theorem 3.1.** Given $K_0$, let us define the set $\mathcal{S}$ of functions in $C^\infty(0, T_0; C^\infty(\Sigma)) \cap H^1(0, T_0; H^1(\Sigma))$, which satisfy

$$\sup_{0 \leqq t \leqq T_0} \|h(\cdot,t)\|_{L^2(\Sigma)} \leqq \sigma_0,$$
$$\sup_{0 \leqq t \leqq T_0} \|h(\cdot,t)\|^2_{H^3(\Sigma)} + \int_0^{T_0} \|h(\cdot,t)\|^2_{H^5(\Sigma)} \leqq M_0, \tag{3.15}$$

where the constants $M_0$ and $\sigma_0$ will be chosen later. We also define a subclass $\mathcal{S}' \subset \mathcal{S}$ of functions which satisfy the additional requirement (3.11), where the constants $C_k$ and $q_k$ will again be chosen later. The goal is to obtain a solution of (3.6) in $\mathcal{S}'$ which is unique in $\mathcal{S}$.

We begin by assuming that $h_0$ is smooth with $\|h_0\|_{H^3(\Sigma)} < K_0$ and $2\|h_0\|_{L^2(\Sigma)} \leqq \sigma_0$. We now define a map $\mathscr{L} : \mathcal{S} \to C^\infty(0, T_0; C^\infty(\Sigma))$ by setting $\mathscr{L}(h) := \tilde{h}$, where $\tilde{h} : \Sigma \times [0,\infty) \to \mathbb{R}$ is the solution of

$$\begin{cases} \dfrac{\partial \tilde{h}}{\partial t} = -\Delta^2 \tilde{h} + J_h(x,t) \\ \tilde{h}(\cdot,0) = h_0 \end{cases} \tag{3.16}$$

and where we have set

$$J_h(x,t) := \langle A(x,h,\nabla h), \nabla^4 h\rangle + J_1(x,h,\nabla h,\nabla^2 h,\nabla^3 h)$$
$$+ J_2(x,h,\nabla h,\nabla^2 h,\nabla f,\nabla^2 f) \tag{3.17}$$

with $A$, $J_1$, $J_2$ as in (3.6).

We note that the set $\mathcal{S}'$ is nonempty when the constants $C_k$ are chosen properly. To see this consider the solution $\bar{h}$ of

$$\begin{cases} \dfrac{\partial \bar{h}}{\partial t} = -\Delta^2 \bar{h} \\ \bar{h}(\cdot, 0) = h_0. \end{cases} \tag{3.18}$$

By classical regularity estimates $\bar{h}$ is smooth and satisfies $\sup_{0 \leqq t \leqq 1} \|\bar{h}(\cdot, t)\|_{L^2(\Sigma)} \leqq \|h_0\|_{L^2(\Sigma)}$ and

$$\sup_{0 \leqq t \leqq 1} t^k \|h(\cdot, t)\|^2_{H^{2k+3}(\Sigma)} + \int_0^1 t^k \|h(\cdot, t)\|^2_{H^{2k+5}(\Sigma)} \, \mathrm{d}t \leqq C'_k \, \|h_0\|^2_{H^3(\Sigma)}$$

for all integers $k \geqq 0$, and therefore $\bar{h} \in \mathcal{S}'$ provided that we choose $M_0$ sufficiently large. We remark that in Steps 1 and 2 below we give an argument which can be applied to prove the above estimate.

**Step 1:** In this step we prove that if $h \in \mathcal{S}$ then $\tilde{h} = \mathscr{L}(h) \in \mathcal{S}$ for a suitable choice of $M_0$, $\sigma_0$ and $T_0$.

To prove this we multiply (3.16) by $\Delta^3 \tilde{h}$. Integrating by parts both sides we get

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int_\Sigma |\nabla(\Delta \tilde{h})|^2 \, \mathrm{d}\mathcal{H}^2 &= -\int_\Sigma \frac{\partial \tilde{h}}{\partial t} \Delta^3 \tilde{h} \, \mathrm{d}\mathcal{H}^2 \\ &= \int_\Sigma (\Delta^2 \tilde{h} - J_h) \Delta^3 \tilde{h} \, \mathrm{d}\mathcal{H}^2 \\ &= \int_\Sigma \big( -|\nabla(\Delta^2 \tilde{h})|^2 + \langle \nabla J_h, \nabla(\Delta^2 \tilde{h}) \rangle \big) \, \mathrm{d}\mathcal{H}^2. \end{aligned}$$

By Proposition 3.2 it follows that if $\sigma_0$ is sufficiently small, then by Young inequality,

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int_\Sigma |\nabla(\Delta \tilde{h})|^2 \, \mathrm{d}\mathcal{H}^2 &\leqq -\frac{1}{2} \int_\Sigma |\nabla(\Delta^2 \tilde{h})|^2 \, \mathrm{d}\mathcal{H}^2 + \frac{1}{2} \int_\Sigma |\nabla J_h|^2 \, \mathrm{d}\mathcal{H}^2 \\ &\leqq -\frac{1}{2} \int_\Sigma |\nabla(\Delta^2 \tilde{h})|^2 \, \mathrm{d}\mathcal{H}^2 + \frac{3}{8} \int_\Sigma |\nabla^5 h|^2 \, \mathrm{d}\mathcal{H}^2 \\ &\quad + \frac{3}{2} \tilde{C}_1 \bigg( 1 + \|f\|^{q_0}_{L^\infty(\Sigma)} + \int_\Sigma |\nabla^3 f|^2 \, \mathrm{d}\mathcal{H}^2 \bigg), \end{aligned}$$

where $q_0 = p_1$ and $\tilde{C}_1$ are from the Proposition 3.2. Integrate this over $(0, t)$ with $t \leqq T_0$, where $T_0$ will be chosen later, and get

$$\begin{aligned} &\int_\Sigma |\nabla(\Delta \tilde{h}(\cdot, t))|^2 \, \mathrm{d}\mathcal{H}^2 - \int_\Sigma |\nabla(\Delta h_0)|^2 \, \mathrm{d}\mathcal{H}^2 + \int_0^t \int_\Sigma |\nabla(\Delta^2 \tilde{h})|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}s \\ &\quad \leqq \frac{3}{4} \int_0^{T_0} \int_\Sigma |\nabla^5 h(\cdot, t)|^2 \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}t \\ &\qquad + 3\tilde{C}_1 \int_0^{T_0} \bigg( 1 + \|f(\cdot, t)\|^{q_0}_{L^\infty(\Sigma)} + \int_\Sigma |\nabla^3 f(\cdot, t)|^2 \, \mathrm{d}\mathcal{H}^2 \bigg) \mathrm{d}t. \end{aligned} \tag{3.19}$$

From this estimate, from the fact that $h$ satisfies (3.15), $f$ satisfies (3.9), $\|h_0\|_{H^3(\Sigma)} < K_0$ and using Remark 2.5 (with a sufficiently small $\varepsilon$) we obtain

$$\begin{aligned}
&\sup_{0\leqq t\leqq T_0} \|\tilde{h}(\cdot,t)\|_{H^3} + \int_0^{T_0} \|\tilde{h}(\cdot,t)\|^2_{H^5}\,\mathrm{d}t \\
&\quad\leqq C \sup_{0\leqq t\leqq T_0} \|\tilde{h}(\cdot,t)\|_{L^2} + K_0^2 + \frac{4}{5}M_0 \\
&\quad\quad +4\tilde{C}_1((T_0 + T_0K_0^{q_0}) + K_0).
\end{aligned}\tag{3.20}$$

In order to estimate the $L^2$-norm of $\tilde{h}$, we multiply the Equation (3.16) by $\tilde{h}$. Recalling (3.17), using the interpolation Lemma 2.1 to estimate the derivatives of $h$ in terms of $\|\nabla^5 h\|_2$ and $\|\nabla^3 h\|_2$ and the derivatives of $f$ in terms of $\|\nabla^3 f\|_2$ and $\|f\|_\infty$ and then using the $H^3$-bound on $h$, we get

$$\begin{aligned}
&\int_\Sigma \frac{\partial\tilde{h}}{\partial t}\tilde{h}\,\mathrm{d}\mathcal{H}^2 = -\int_\Sigma \Delta^2\tilde{h}\,\tilde{h}\,\mathrm{d}\mathcal{H}^2 + \int_\Sigma J_h\tilde{h}\,\mathrm{d}\mathcal{H}^2 \\
&\quad\leqq \int_\Sigma (-|\Delta\tilde{h}|^2 + \frac{\tilde{h}^2}{\eta})\,\mathrm{d}\mathcal{H}^2 + \eta\int_\Sigma J_h^2\,\mathrm{d}\mathcal{H}^2 \\
&\quad\leqq \frac{1}{\eta}\int_\Sigma \tilde{h}^2\,\mathrm{d}\mathcal{H}^2 \\
&\quad\quad + C\eta\int_\Sigma \Big(1 + |\nabla^4 h|^2 + (1+|\nabla^2 h|^2)|\nabla^3 h|^2 + |\nabla^2 h|^6 \\
&\quad\quad +(1+|\nabla^2 h|^2)(|\nabla f|^2 + |\nabla^2 f|^2)\Big)\,\mathrm{d}\mathcal{H}^2 \\
&\quad\leqq \frac{1}{\eta}\int_\Sigma \tilde{h}^2\,\mathrm{d}\mathcal{H}^2 + C\eta\Big(1 + \|f\|^2_{L^\infty} + \int_\Sigma (|\nabla^5 h|^2 + |\nabla^3 f|^2)\,\mathrm{d}\mathcal{H}^2\Big),
\end{aligned}\tag{3.21}$$

for some $C > 0$ depending on $M_0$ and $K_0$. Integrating this over $(0,t)$ and using the fact that $h$ satisfies (3.15) and $f$ satisfies (3.9) yields that

$$\begin{aligned}
\frac{1}{2}\int_\Sigma \tilde{h}(\cdot,t)^2\,\mathrm{d}\mathcal{H}^2 - \frac{1}{2}\int_\Sigma h_0^2\,\mathrm{d}\mathcal{H}^2 &\leqq \frac{T_0}{\eta}\sup_{0\leqq t\leqq T_0} \|\tilde{h}(\cdot,t)\|^2_{L^2(\Sigma)} \\
&\quad + \tilde{C}\eta\Big(T_0 + T_0K_0^2 + M_0 + K_0\Big).
\end{aligned}$$

Hence, recalling that $\|h_0\|_{L^2(\Sigma)} \leqq \frac{\sigma_0}{2}$ we have

$$\sup_{0\leqq t\leqq T} \|\tilde{h}(\cdot,t)\|^2_{L^2} \leqq \frac{\sigma_0^2}{4} + \frac{2T_0}{\eta}\sup_{0\leqq t\leqq T_0} \|\tilde{h}(\cdot,t)\|^2_{L^2} + 2\tilde{C}\eta\Big(T_0 + T_0K_0^2 + M_0 + K_0\Big).$$

From this inequality, choosing $\eta$ and $T_0$ sufficiently small (depending on $M_0$ and $K_0$) we conclude that

$$\sup_{0\leqq t\leqq T_0} \|\tilde{h}(\cdot,t)\|_{L^2(\Sigma)} \leqq \sigma_0.$$

In turn, since $\sigma_0 \leqq 1$, we may choose $M_0$ sufficiently large (depending on $K_0$) and $T_0$ smaller if needed to deduce that from (3.20) that

$$\sup_{0\leqq t\leqq T_0} \|\tilde{h}(\cdot,t)\|_{H^3(\Sigma)} + \int_0^{T_0} \|\tilde{h}(\cdot,t)\|^2_{H^5(\Sigma)}\,\mathrm{d}t \leqq M_0.$$

This concludes the proof of the fact that $\tilde{h} = \mathscr{L}(h)$ satisfies (3.15) and thus belongs to $\mathcal{S}$.

**Step 2:** Let us now prove that if $h \in \mathcal{S}'$ then $\tilde{h} = \mathscr{L}(h) \in \mathcal{S}'$, that is, it satisfies (3.11) with $h$ replaced by $\tilde{h}$. We begin by observing that the case $k = 0$ can be proven by a similar argument as the one used in Step 1, by combining (3.19), (3.21) and replacing $T_0$ by $T \leqq T_0$. We proceed by induction and assume that (3.11) holds for $k-1$ and prove it for $k$. We argue similarly as in the previous step and multiply the Equation (3.16) by $\Delta^{2k+3}\tilde{h}$, and after integrating by parts the left-hand side $(2k+3)$-times and the right-hand side $(2k+1)$-times and using Proposition 3.2 with $k$ replaced by $2k+1$ we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}\int_\Sigma |\nabla(\Delta^{k+1}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2 &\leqq -\frac{1}{2}\int_\Sigma |\nabla(\Delta^{k+2}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2 + \frac{1}{2}\int_\Sigma |\nabla^{2k+1}J_h|^2\,\mathrm{d}\mathcal{H}^2\\ &\leqq -\frac{1}{2}\int_\Sigma |\nabla(\Delta^{k+2}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2 + \frac{3}{8}\int_\Sigma |\nabla^{2k+5}h|^2\,\mathrm{d}\mathcal{H}^2\\ &\quad + \frac{3}{2}\tilde{C}_{2k+1}\Big(1 + \|f(\cdot,t)\|^{p_{2k+1}}_{L^\infty} + \int_\Sigma |\nabla^{2k+3}f|^2\,\mathrm{d}\mathcal{H}^2\Big).\end{aligned}$$

From this estimate we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\Big(t^k\int_\Sigma |\nabla(\Delta^{k+1}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2\Big) &\leqq k\,t^{k-1}\int_\Sigma |\nabla(\Delta^{k+1}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2\\ &\quad - t^k\int_\Sigma |\nabla(\Delta^{k+2}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2\\ &\quad + \frac{3}{4}t^k\int_\Sigma |\nabla^{2k+5}h|^2\,\mathrm{d}\mathcal{H}^2\\ &\quad + 3\tilde{C}_{2k+1}t^k\Big(1 + \|f(\cdot,t)\|^{p_{2k+1}}_{L^\infty}\\ &\quad + \int_\Sigma |\nabla^{2k+3}f|^2\,\mathrm{d}\mathcal{H}^2\Big).\end{aligned}$$

Integrating this inequality over $(0,t)$ for $t \leqq T$ yields

$$\begin{aligned}&\sup_{0\leqq t\leqq T} t^k\int_\Sigma |\nabla(\Delta^{k+1}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2 + \int_0^T t^k\int_\Sigma |\nabla(\Delta^{k+2}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\ &\leqq k\int_0^T t^{k-1}\int_\Sigma |\nabla^{2k+3}\tilde{h}|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\end{aligned}$$

$$+\int_0^T t^k\Big(\frac{3}{4}\int_\Sigma |\nabla^{2k+5}h|^2\,\mathrm{d}\mathcal{H}^2+3\tilde{C}_{2k+1}(1+\|f(\cdot,t)\|_{L^\infty}^{p_{2k+1}}$$
$$+\int_\Sigma |\nabla^{2k+3}f|^2\,\mathrm{d}\mathcal{H}^2)\Big)\mathrm{d}t.$$

By using the fact that $\tilde{h}$ satisfies (3.11) with $k-1$, and $h$ satisfies (3.11) we deduce

$$\sup_{0\leqq t\leqq T} t^k\|\nabla(\Delta^{k+1}\tilde{h})\|_{L^2(\Sigma)}^2+\int_0^T t^k\int_\Sigma |\nabla(\Delta^{k+2}\tilde{h})|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t$$
$$\leqq (kC_{k-1}+\frac{3}{4}C_k+3\tilde{C}_{2k+1})\Big(\int_0^{T_0}(\|h_0\|_{H^3(\Sigma)}^2+3+3\|f(\cdot,t)\|_{L^\infty(\Sigma)}^{q_k}$$
$$+\sum_{i=0}^k t^i\|f(\cdot,t)\|_{H^{2i+3}}^2)\,\mathrm{d}t\Big)$$

when we choose $q_k\geqq\max\{q_{k-1},p_{2k+1}\}$. Using the fact that $\sup_{0\leqq t\leqq T_0}\|\tilde{h}(\cdot,t)\|_{L^2}\leqq\sigma_0$ and by Remark 2.5, we obtain the estimate (3.11) for $\tilde{h}$ by choosing $C_k$ large enough.

**Step 3:** In this step we prove that the map $\mathscr{L}$ introduced in the previous step is a contraction with respect to a suitable norm, provided that $\sigma_0$ and $T_0$ are chosen sufficiently small.

To this aim, let $h_1,h_2\in\mathcal{S}$ and let $\tilde{h}_1,\tilde{h}_2\in\mathcal{S}$ be the corresponding solutions of (3.16). Multiplying the equation satisfied by $\tilde{h}_i$ by $\Delta^2(\tilde{h}_2-\tilde{h}_1)$, subtracting and integrating by parts we get

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Sigma|\Delta(\tilde{h}_2(\cdot,t)-\tilde{h}_1(\cdot,t))|^2\,\mathrm{d}\mathcal{H}^2$$
$$=-\int_\Sigma|\Delta^2(\tilde{h}_2-\tilde{h}_1)(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2$$
$$+\int_\Sigma\Delta^2(\tilde{h}_2-\tilde{h}_1)(\cdot,t)(J_{h_2}(\cdot,t)-J_{h_1}(\cdot,t))\,\mathrm{d}\mathcal{H}^2$$
$$\leqq-\frac{1}{2}\int_\Sigma|\Delta^2(\tilde{h}_2-\tilde{h}_1)(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2$$
$$+\frac{1}{2}\int_\Sigma|J_{h_2}(\cdot,t)-J_{h_1}(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2.$$

Fix $\varepsilon>0$ small. By choosing $\sigma_0$ smaller in (3.15) if needed, we may integrate the above inequality over $(0,t)$, with $t<T_0$, and use Remark 2.5 and Lemma 3.3 to obtain

$$
\begin{aligned}
&\|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)\|^2_{H^2(\Sigma)}+\int_0^{T_0}\int_\Sigma|\nabla^4(\tilde h_2-\tilde h_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\quad\leqq C\|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)\|^2_{L^2(\Sigma)}+C\int_0^{T_0}\int_\Sigma|\tilde h_2-\tilde h_1|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\qquad+\varepsilon\int_0^{T_0}\int_\Sigma|\nabla^4(h_2-h_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\qquad+CT_0^\theta\sup_{0\leqq t\leqq T_0}\|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{H^2(\Sigma)}\\
&\quad\leqq C\sup_{0\leqq t\leqq T_0}\|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)\|^2_{L^2(\Sigma)}\\
&\qquad+\varepsilon\int_0^{T_0}\int_\Sigma|\nabla^4(h_2-h_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\qquad+CT_0^\theta\sup_{0\leqq t\leqq T_0}\|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{H^2(\Sigma)}.
\end{aligned}
\tag{3.22}
$$

Next we have to estimate the first term on the right-hand side. To this aim we multiply the equations satisfied by $\tilde h_1$ and $\tilde h_2$ by $\tilde h_2-\tilde h_1$, subtract and get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_\Sigma|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2\\
&\quad=\int_\Sigma(\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t))\frac{\partial}{\partial t}(\tilde h_2-\tilde h_1)(\cdot,t)\,\mathrm{d}\mathcal{H}^2\\
&\quad=-\int_\Sigma(\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t))\Delta^2(\tilde h_2-\tilde h_1)(\cdot,t)\,\mathrm{d}\mathcal{H}^2\\
&\qquad+\int_\Sigma(\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t))(J_{h_2}(\cdot,t)-J_{h_1}(\cdot,t))\,\mathrm{d}\mathcal{H}^2\\
&\quad\leqq-\int_\Sigma|\Delta(\tilde h_2-\tilde h_1)(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2+\frac{1}{2}\int_\Sigma|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2\\
&\qquad+\frac{1}{2}\int_\Sigma|J_{h_2}(\cdot,t)-J_{h_1}(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2.
\end{aligned}
$$

Integrating over $(0,t)$, with $t<T_0$, and using again Lemma 3.3 we get

$$
\begin{aligned}
&\int_\Sigma|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)|^2\,\mathrm{d}\mathcal{H}^2\leqq T_0\sup_{0\leqq t\leqq T_0}\|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)\|^2_{L^2(\Sigma)}\\
&\quad+\varepsilon\int_0^{T_0}\int_\Sigma|\nabla^4h_1-\nabla^4h_2|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t+CT_0^\theta\sup_{0\leqq t\leqq T_0}\|h_1(\cdot,t)-h_2(\cdot,t)\|^2_{H^2(\Sigma)},
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\sup_{0\leqq t\leqq T_0}\|\tilde h_2(\cdot,t)-\tilde h_1(\cdot,t)\|^2_{L^2(\Sigma)}\leqq 2\varepsilon\int_0^{T_0}\int_\Sigma|\nabla^4h_1-\nabla^4h_2|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\quad+2CT_0^\theta\sup_{0\leqq t\leqq T_0}\|h_1(\cdot,t)-h_2(\cdot,t)\|^2_{H^2(\Sigma)},
\end{aligned}
\tag{3.23}
$$

provided that $T_0 \leqq \frac{1}{2}$. Combining (3.22) and (3.23), and taking $\varepsilon$ small and $T_0$ smaller if needed, we deduce that

$$\sup_{0\leqq t\leqq T_0} \|\tilde{h}_2(\cdot,t)-\tilde{h}_1(\cdot,t)\|^2_{H^2(\Sigma)} + \int_0^{T_0}\int_\Sigma |\nabla^4(\tilde{h}_2-\tilde{h}_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t$$
$$\leqq \frac{1}{2}\bigg(\sup_{0\leqq t\leqq T_0}\|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{H^2(\Sigma)}$$
$$+\int_0^{T_0}\int_\Sigma |\nabla^4(h_2-h_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\bigg). \tag{3.24}$$

**Step 4.** (*Conclusion*) We may proceed with a standard argument, by recursively setting $h_1 = \bar{h}$, with $\bar{h}$ defined as in (3.18), and $h_n := \mathcal{L}(h_{n-1})$ and for every $n \geqq 2$. From (3.24) we have that there exists $h$ such that $h_n \to h$ in $L^\infty(0,T_0;H^2(\Sigma)) \cap L^2(0,T_0;H^4(\Sigma))$. Moreover, from Step 1 and Step 2 we have also that $h_n \rightharpoonup h$ weakly in $H^1_{loc}(0,T;H^k(\Sigma))$ and that $h$ satisfies (3.10) and (3.11). Using these convergences one can easily pass to the limit in the equations satisfied by the $h_n$'s to conclude that $h$ is a solution of (3.1). We remark that the smoothness of $h$ in time follows from the equation and from the regularity in space of $h$. Note that the smoothness assumption on $h_0$ can be removed by a standard approximation argument. Finally, the uniqueness follows from the same argument used to prove (3.24). □

## 4. Short Time Existence for the Surface Diffusion Flow with Elasticity

Here we will prove the existence of the flow

$$V_t = \Delta_{\partial F_t}(H_{F_t} - Q(E(u_{F_t}))), \tag{4.1}$$

where $u_{F_t}$ is the minimizer of the elastic energy, that is the solution to (2.12), with $F$ replaced bu $F_t$.

The most crucial point for the proof of the short time existence of (4.1), is to prove sharp regularity estimates for $u_F$ up to the boundary $\partial F$ in terms of regularity of $\partial F$. We prove this in the theorem below.

**Theorem 4.1.** *Let $K > 0$, $\alpha \in (0,1)$, and let $k \geqq 3$ be an integer. There exists $C_k = C_k(K) > 0$ such that if $h \in H^k(\Sigma)$ and $F_h \in \mathfrak{h}^{1,\alpha}_K(\Sigma)$, defined as in (2.18), then*

$$\|Q(E(u_{F_h}))\circ\pi^{-1}_{F_h}\|_{H^{k-\frac{3}{2}}(\Sigma)} \leqq C_k(\|h\|_{H^k(\Sigma)}+1). \tag{4.2}$$

*Moreover if $h_1$, $h_2 \in H^3(\Sigma)$ and $F_{h_i} \in \mathfrak{h}^3_K(\Sigma)$ for $i = 1, 2$, then there exists $C = C(K) > 0$ such that*

$$\|u_{F_{h_2}}\circ\pi^{-1}_{F_{h_2}} - u_{F_{h_1}}\circ\pi^{-1}_{F_{h_1}}\|_{H^{3/2}(\Sigma)} \leqq C\|h_2-h_1\|_{H^2(\Sigma)}. \tag{4.3}$$

**Proof.** We begin by proving (4.2). By standard approximation argument we may assume that $h$ is smooth, which implies that $u_{F_h}$ is smooth up to the boundary $\partial F_h$.

We consider a diffeomorphism $\Phi_h : \Omega\backslash G \to \Omega\backslash F_h$ such that

$$\Phi_h(x) = x + h(\pi(x))\nu(\pi(x))$$

in $\mathcal{N}^+_{\eta_0}(G)$, where for any $\sigma > 0$ $\mathcal{N}^+_\sigma(G) = \{x \in \Omega\backslash G : d_G \leqq \sigma\}$ is the one-sided neighborhood of $\Sigma$. Note that we may construct $\Phi_h$ such that $\|\Phi_h - I\|_{H^k(\Omega\backslash G)} + \|\Phi_h^{-1} - I\|_{H^k(\Omega\backslash G)} \leqq C\|h\|_{H^k(\Sigma)}$.

Let us fix $x_0 \in \Sigma$. There exists a smooth diffeomorphism $\Phi$ from a neighborhood $U \subset\subset \Omega$ of $x_0$ to a ball $B_{2R}$ which straightens the boundary such that $\Phi(U\backslash G) = B^+_{2R} = B_{2R} \cap \{x_3 > 0\}$. Setting $v = u_{F_h} \circ \Phi_h \circ \Phi^{-1}$ and $\bar{h} := h \circ \pi \circ \Phi^{-1}$, $v$ is a solution of a system of the form

$$\int_{B^+_{2R}} \mathbb{A}(x, \bar{h}, D\bar{h})Dv : D\varphi \, \mathrm{d}x = 0 \tag{4.4}$$

for all $\varphi \in C^\infty(B^+_{2R}; \mathbb{R}^3)$ vanishing on $\partial B_{2R} \cap \{x_3 > 0\}$, where the tensor $\mathbb{A}$ is smooth. In particular, by using the explicit definition of $\bar{h}$ and Lemma 7.1 it holds $\|\bar{h}\|_{H^k(B^+_{2R})} \leqq C(k)(1 + \|h\|_{H^k(\Sigma)})$ for every $k \in \mathbb{N}$. Moreover, by using Korn's inequality, one may check that $\mathbb{A}$ is elliptic in the sense that

$$\int_{B^+_{2R}} \mathbb{A}(x, \bar{h}, D\bar{h})D\varphi : D\varphi \, \mathrm{d}x \geqq c_0 \int_{B^+_{2R}} |D\varphi|^2 \, \mathrm{d}x \tag{4.5}$$

for all $\varphi \in C^\infty(B^+_{2R}; \mathbb{R}^3)$ vanishing on $\partial B_{2R} \cap \{x_3 > 0\}$.

We now start differentiating the equation in the tangential directions so to estimate the tangential derivatives. Then we will use the equation to extract from these estimates also information and the normal and the mixed derivatives.

Let us fix $k \geqq 3$ and a multi-index $\beta = (\beta_1, \beta_2, 0)$, with $\beta_1 + \beta_2 = k - 1$. By differentiating the Equation (4.4) in the $\beta$-directions we have

$$\int_{B^+_{2R}} D^\beta(\mathbb{A}(x, \bar{h}, D\bar{h})Dv) : D\varphi \, \mathrm{d}x = 0. \tag{4.6}$$

Let $\eta \in C^\infty_0(B_{2R})$ be a standard cut-off function such that $\eta \equiv 1$ in $B_R$ and $0 \leqq \eta \leqq 1$. By choosing $\varphi = D^\beta v\eta^2$ as a test function in (4.6) and by expanding the term $D^\beta((\mathbb{A}(x, \bar{h}, D\bar{h})Dv)$ by Leibniz formula we deduce

$$\begin{aligned}\int_{B^+_{2R}} (\mathbb{A}(x, \bar{h}, D\bar{h})DD^\beta v) : DD^\beta v\eta^2 \, \mathrm{d}x &\leqq 2\int_{B^+_{2R}} |\mathbb{A}(x, \bar{h}, D\bar{h})||DD^\beta v||D\eta|\eta|D^\beta v| \, \mathrm{d}x \\ &+ C\sum_{i=1}^{k-1} \int_{B^+_{2R}} |D^i\mathbb{A}(x, \bar{h}, D\bar{h})||D^{k-i}v|(|DD^\beta v|\eta^2 + |D^\beta v||D\eta|\eta) \, \mathrm{d}x.\end{aligned}$$

Moreover, by the ellipticity condition (4.5) we have

$$\begin{aligned}\frac{c_0}{2}\int_{B_{2R}^+}|D(D^\beta v)|^2\eta^2\,\mathrm{d}x &\leqq c_0\int_{B_{2R}^+}|D(D^\beta v\eta)|^2\,\mathrm{d}x\\ &\quad+c_0\int_{B_{2R}^+}|D^\beta v|^2|D\eta|^2\,\mathrm{d}x\\ &\leqq \int_{B_{2R}^+}(\mathbb{A}(x,\bar h,D\bar h)D(D^\beta v\eta)):D(D^\beta v\eta)\,\mathrm{d}x\\ &\quad+c_0\int_{B_R^+}|D^\beta v|^2|D\eta|^2\,\mathrm{d}x\\ &\leqq \int_{B_{2R}^+}(\mathbb{A}(x,\bar h,D\bar h)DD^\beta v):DD^\beta v\eta^2\,\mathrm{d}x\\ &\quad+C\int_{B_{2R}^+}(|DD^\beta v||D\eta|\eta|D^\beta v|\\ &\qquad+|D^\beta v|^2|D\eta|^2)\,\mathrm{d}x,\end{aligned}$$

where in the last inequality we have used fact that $\|\bar h\|_{C^{1,\alpha}}\leqq C$, which in turn implies that $\mathbb{A}(x,\bar h,D\bar h)$ is bounded. Combining the previous estimates and using Young's inequality we obtain, recalling that $\eta=1$ on $B_R^+$,

$$\begin{aligned}\int_{B_R^+}|D(D^\beta v)|^2\,\mathrm{d}x &\leqq C\int_{B_{2R}^+}|D^{k-1}v|^2\,\mathrm{d}x\\ &+C\sum_{i=1}^{k-1}\int_{B_{2R}^+}|D^i\mathbb{A}(x,\bar h,D\bar h)|^2|D^{k-i}v|^2\,\mathrm{d}x.\end{aligned}\tag{4.7}$$

We denote $w=D\bar h$ and estimate by the Leibniz formula that

$$\begin{aligned}\sum_{i=1}^{k-1}|D^i\mathbb{A}(x,\bar h,D\bar h)|^2|D^{k-i}v|^2 &\leqq C\sum_{i=1}^{k-1}|D^{k-i}v|^2\\ &+C\sum_{i=1}^{k-1}\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq i\\ j_1+\cdots+j_m\leqq i\\ m\geqq 1}}|D^{j_1}w|^2\cdots|D^{j_m}w|^2|D^{k-i}v|^2.\end{aligned}$$

Then, by Hölder's inequality, we get

$$\begin{aligned}\sum_{i=1}^{k-1}\int_{B_{2R}^+}|D^i\mathbb{A}(x,\bar h,D\bar h)|^2|D^{k-i}v|^2\,\mathrm{d}x &\leqq C\|v\|^2_{H^{k-1}(B_{2R}^+)}\\ &+C\sum_{i=1}^{k-1}\sum_{\substack{1\leqq j_1\leqq\ldots\leqq j_m\leqq i\\ j_1+\cdots+j_m\leqq i\\ m\geqq 1}}\|D^{j_1}w\|^2_{\frac{2(k-1)}{j_1}}\ldots\|D^{j_m}w\|^2_{\frac{2(k-1)}{j_m}}\|D^{k-i}v\|^2_{\frac{2(k-1)}{k-i-1}},\end{aligned}$$

where all the norms in the last line are evaluated in $B^+_{2R}$. Note that if $i = k-1$ then in the last term it is understood that $\|D^{k-i}v\|_{\frac{k-1}{k-1-i}} = \|Dv\|_{L^\infty}$. Note that by standard Schauder estimates the assumption $\|h\|_{C^{1,\alpha}(\Sigma)} \leqq K$ implies that $\|Dv\|_{L^\infty(B^+_{2R})} \leqq C$. We use Lemma 2.1 to estimate

$$\|D^{j_l}w\|_{\frac{2(k-1)}{j_l}} \leqq C\|w\|^{\theta(j_l)}_{H^{k-1}}\|w\|^{1-\theta(j_l)}_{L^\infty} \leqq C\|w\|^{\theta(j_l)}_{H^{k-1}}$$

for $\theta(j_l) := \frac{j_l}{k-1}$. By the same lemma we also have

$$\|D^{k-i}v\|_{\frac{2(k-1)}{k-i-1}} \leqq C\|v\|^{\theta}_{H^k}\|Dv\|^{1-\theta}_{L^\infty} \leqq C\|v\|^{\theta}_{H^k}$$

for $\theta = \frac{k-i-1}{k-1}$. Since $\theta(j_1)+\cdots+\theta(j_m) \leqq \frac{i}{k-1}$, from (4.7) and from the previous estimate we have, by Young's inequality,

$$\begin{aligned}\int_{B^+_R}|D(D^\beta v)|^2\,\mathrm{d}x &\leqq C\|v\|^2_{H^{k-1}(B^+_{2R})} + C\sum_{i=1}^{k-1}(\|w\|^{\frac{2i}{k-1}}_{H^{k-1}(B^+_{2R})}+1)\|v\|^{\frac{2(k-i-1)}{k-1}}_{H^k(B^+_{2R})}\\ &\leqq \varepsilon\|D^kv\|^2_{L^2(B^+_{2R})} + C\|v\|^2_{H^{k-1}(B^+_{2R})} + C(1+\|h\|^2_{H^k(\Sigma)}).\end{aligned}$$

In order to control the remaining derivatives we use the Equation (4.4) in the strong form

$$\operatorname{div}(\mathbb{A}(x,\bar h, D\bar h)Dv) = 0.$$

Indeed, observe that we have estimated all the derivatives of the type $D^\beta(Dv)$, where $\beta = (\beta_1,\beta_2,0)$, with $\beta_1+\beta_2 = k-1$. Using these estimates and differentiating the equation $k-2$ times with respect to the horizontal directions and once in the vertical direction, we may estimate $D^\beta(D_{x_3x_3}v)$ for all $\beta = (\beta_1,\beta_2,0)$, with $\beta_1+\beta_2 = k-2$, by using an interpolation argument as before to control the lower order derivatives. Then we proceed by induction by differentiating the equation $k-3$ times with respect to the horizontal directions and twice in the vertical direction, and so on, until we differentiate the equation $k-1$ times only in the vertical direction. As a result we obtain

$$\int_{B^+_R}|D^kv|^2\,\mathrm{d}x \leqq \varepsilon\|D^kv\|^2_{L^2(B^+_{2R})} + C\|v\|^2_{H^{k-1}(B^+_{2R})} + C(1+\|h\|^2_{H^k(\Sigma)}).$$

The previous estimate holds at every point on $\partial F_h$. Thus we may cover $\mathcal{N}^+_{\sigma_1}(F_h)$, with $\sigma_1 < \frac{\eta_0}{2}$, by a finite union of balls and use the previous estimate in every ball of the covering. Precisely, we go back to the original map, set $u = u_{F_h}\circ\Phi_h$ for simplicity, use Lemma 7.1 and conclude that there are $0 < \sigma_1 < \sigma_2$ such that

$$\begin{aligned}\int_{\mathcal{N}^+_{\sigma_1}}|D^ku|^2\,\mathrm{d}x &\leqq C\varepsilon\int_{\mathcal{N}^+_{\sigma_2}}|D^ku|^2\,\mathrm{d}x + C\|u\|^2_{H^{k-1}(\mathcal{N}^+_{\sigma_2})} + C(1+\|h\|^2_{H^k(\Sigma)})\\ &\leqq 2C\varepsilon\int_{\mathcal{N}^+_{\sigma_2}}|D^ku|^2\,\mathrm{d}x + C\|u\|^2_{L^2(\mathcal{N}^+_{\sigma_2})} + C(1+\|h\|^2_{H^k(\Sigma)}),\end{aligned}$$

where the last inequality follows from standard interpolation inequality. Choosing $\varepsilon$ small we obtain

$$\int_{\mathcal{N}^+_{\sigma_1}} |D^k u|^2 \,\mathrm{d}x \leqq 2 \int_{\mathcal{N}^+_{\sigma_2} \setminus \mathcal{N}^+_{\sigma_1}} |D^k u|^2 \,\mathrm{d}x + C\|u\|^2_{L^2(\mathcal{N}^+_{\sigma_2})} + C(1 + \|h\|^2_{H^k(\Sigma)}).$$

By standard interior regularity it holds that

$$\int_{\mathcal{N}^+_{\sigma_2} \setminus \mathcal{N}^+_{\sigma_1}} |D^k u|^2 \,\mathrm{d}x \leqq C\|u_{F_h}\|^2_{L^2(\Omega \setminus F_h)}.$$

From the two previous inequalities and by standard interpolation we have that

$$\|u\|_{H^k(\mathcal{N}^+_{\sigma_1})} \leqq C(1 + \|u\|_{L^2(\mathcal{N}^+_{\sigma_1})} + \|h\|_{H^k(\Sigma)}).$$

By the minimality and by Poincaré inequality we have that $\|u_{F_h}\|_{L^2(\Omega \setminus F_h)}$ is bounded by the boundary value $w_0$. Using the last part of Lemma 7.1, we have from the above inequality that

$$\|Q(E(u_{F_h})) \circ \Phi_h\|_{H^{k-1}(\mathcal{N}^+_{\sigma_1})} \leqq C(1 + \|h\|_{H^k(\Sigma)}).$$

From this inequality the first claim follows by the trace theorem.

As for the second part of the lemma, let $\Phi_i$ be a diffeomorphism constructed as above from $\Omega \setminus G$ to $\Omega \setminus F_{h_i}$. Note that, since $h_1$ and $h_2$ are bounded in $C^{1,\alpha}$, we may construct the $\Phi_i$'s in such a way that

$$\|\Phi_2 - \Phi_1\|_{H^1(\Omega \setminus G)} \leqq C\|h_2 - h_1\|_{H^1(\Sigma)}.$$

As before we fix $x_0 \in \Sigma$ and denote as before by $\Phi$ the diffeomorphism that straightens $\Sigma$. Setting $v_i = u_{F_{h_i}} \circ \Phi_i \circ \Phi^{-1}$ and $\bar{h}_i = h_i \circ \pi \circ \Phi$, we have that

$$\int_{B^+_{2R}} \mathbb{A}(x, \bar{h}_i, D\bar{h}_i) Dv_i : D\varphi \,\mathrm{d}x = 0$$

for all $\varphi \in C^\infty(B^+_{2R}; \mathbb{R}^3)$ vanishing on $\partial B_{2R} \cap \{x_3 > 0\}$, where $\mathbb{A}$ is the same tensor as before.

Differentiating the equations in the $x_j$-direction, $j = 1, 2$, and subtracting the two resulting equations we obtain

$$\begin{aligned}
&\int_{B^+_{2R}} \mathbb{A}(x, \bar{h}_2, D\bar{h}_2) D(D_j(v_2 - v_1)) : D\varphi \,\mathrm{d}x \\
&\quad = -\int_{B^+_{2R}} D_j(\mathbb{A}(x, \bar{h}_2, D\bar{h}_2)) D(v_2 - v_1) : D\varphi \,\mathrm{d}x \\
&\qquad - \int_{B^+_{2R}} [\mathbb{A}(x, \bar{h}_2, D\bar{h}_2) - \mathbb{A}(x, \bar{h}_1, D\bar{h}_1)] DD_j v_1 : D\varphi \,\mathrm{d}x \\
&\qquad - \int_{B^+_{2R}} D_j[\mathbb{A}(x, \bar{h}_2, D\bar{h}_2) - \mathbb{A}(x, \bar{h}_1, D\bar{h}_1)] Dv_1 : D\varphi \,\mathrm{d}x.
\end{aligned}$$

We choose $\varphi = D_j(v_2 - v_1)\eta^2$ as a test function to get an inequality similar to (4.7) with $v$ replaced by $v_2 - v_1$, from which we obtain

$$\begin{aligned}\int_{B_R^+} |D(D_j(v_2 - v_1))|^2 \, \mathrm{d}x \leqq C \int_{B_{2R}^+} &(1 + |D^2\bar{h}_2|^2 + |D^2\bar{h}_1|^2)|Dv_2 - Dv_1|^2 \, \mathrm{d}x \\ &+ C \int_{B_{2R}^+} (|\bar{h}_2 - \bar{h}_1|^2 + |D\bar{h}_2 - D\bar{h}_1|^2 \\ &+ |D^2\bar{h}_2 - D^2\bar{h}_1|^2)|Dv_1|^2 \, \mathrm{d}x \\ &+ C \int_{B_{2R}^+} (|\bar{h}_2 - \bar{h}_1|^2 + |D\bar{h}_2 - D\bar{h}_1|^2)|D^2 v_1|^2 \, \mathrm{d}x.\end{aligned}$$

Recall first that as before $\|Dv_1\|_{L^\infty} \leqq C$. Moreover, we assume that $\|h_i\|_{H^3(\Sigma)} \leqq K$ and therefore by the proof of the first statement we conclude that $\|v_i\|_{H^3(B_{2R}^+)} \leqq C$. Using interpolation we get

$$\begin{aligned}\int_{B_{2R}^+} |D^2\bar{h}_1|^2 |Dv_2 - Dv_1|^2 \, \mathrm{d}x &\leqq \|D^2\bar{h}_1\|_{L^4}^2 \, \|Dv_2 - Dv_1\|_{L^4}^2 \\ &\leqq C\|\bar{h}_1\|_{H^3}^2 \, \|v_2 - v_1\|_{H^2}^{\frac{3}{2}} \|v_2 - v_1\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

In the same way, using interpolation and the fact that $\|v_1\|_{H^3(B_{2R}^+)}$ is bounded, we may estimate the remaining two integrals on the right hand side by $C\|\bar{h}_2 - \bar{h}_1\|_{H^2(B_{2R}^+)}^2$, with a constant $C$ depending only on $\|v_1\|_{H^3(B_{2R}^+)}$, hence on $\|h_1\|_{H^3(\Sigma)}$. Then, using the equation to estimate $D_{33}(v_2 - v_1)$, we get, for any $\varepsilon \in (0, 1)$,

$$\begin{aligned}\int_{B_R^+} |D^2(v_2 - v_1)|^2 &\leqq C\|v_2 - v_1\|_{H^2(B_{2R}^+)}^{\frac{3}{2}} \|v_2 - v_1\|_{L^2(B_{2R}^+)}^{\frac{1}{2}} \\ &\quad + C\|\bar{h}_2 - \bar{h}_1\|_{H^2(B_{2R}^+)}^2 \\ &\leqq \varepsilon \int_{B_{2R}^+} |D^2(v_2 - v_1)|^2 \\ &\quad + C \int_{B_{2R}^+} |v_2 - v_1|^2 + C\|h_2 - h_1\|_{H^2(\Sigma)}^2.\end{aligned}$$

Using a simple covering argument as before, going back to the original functions and arguing as above we get

$$\begin{aligned}&\|D^2(u_{F_{h_2}} \circ \Phi_{h_2} - u_{F_{h_1}} \circ \Phi_{h_1})\|_{L^2(\mathcal{N}_{\sigma_1}^+)} \\ &\quad \leqq C\|u_{F_{h_2}} \circ \Phi_{h_2} - u_{F_{h_1}} \circ \Phi_{h_1}\|_{L^2(\mathcal{N}_{\sigma_2}^+)} + C\|h_2 - h_1\|_{H^2(\Sigma)}.\end{aligned}$$

Observe now that writing down the equations satisfied by $u_{F_{h_i}} \circ \Phi_{h_i}$ in $\Omega \backslash G$ and using as an admissible test function $\varphi = u_{F_{h_1}} \circ \Phi_{h_1} - u_{F_{h_2}} \circ \Phi_{h_2}$, one may check that

$$\begin{aligned}&\|D(u_{F_{h_1}} \circ \Phi_{h_1} - u_{F_{h_2}} \circ \Phi_{h_2})\|_{L^2(\Omega \backslash G)} \\ &\quad \leqq C\|\Phi_1 - \Phi_2\|_{H^1(\Omega \backslash G)} \leqq C\|h_1 - h_2\|_{H^1(\Sigma)}.\end{aligned}$$

The conclusion follows from this estimate and from the previous one by the Poincaré inequality. □

**Remark 4.2.** Let $h_{F_i}$ and $u_{F_i}$ for $i = 1, 2$ be as in Theorem 4.1. The inequality at the end of the proof of the theorem implies that

$$\|u_{F_{h_2}} \circ \pi^{-1}_{F_{h_2}} - u_{F_{h_1}} \circ \pi^{-1}_{F_{h_1}}\|_{H^{1/2}(\Sigma)} \leqq C\|h_2 - h_1\|_{H^1(\Sigma)}.$$

Moreover, if in addition to the assumptions of the second part of Theorem 4.1 we know also that $\|h_i\|_{C^1(\Sigma)}$ is sufficiently small for $i = 1, 2$, then the proof of the inequality (4.3) also gives the estimate

$$\|(Du_{F_{h_2}}) \circ \pi^{-1}_{F_{h_2}} - (Du_{F_{h_1}}) \circ \pi^{-1}_{F_{h_1}}\|_{L^2(\Sigma)} \leqq C\|h_2 - h_1\|_{H^2(\Sigma)}.$$

Let us consider the smooth flow $(F_t)_{t\in(0,T_0)}$ with initial set $F_0$, which is a solution of (3.1) with smooth forcing term $f : \Sigma \times [0, T_0) \to \mathbb{R}$. Here $T_0$ is the existence time provided by Theorem 3.1. For every given time $t \in (0, T_0)$ we consider the elastic equilibrium $u_t$ in $\Omega\setminus F_t$ defined in (2.12) and we use the regularity estimates from Theorem 4.1 to establish

**Lemma 4.3.** *Let $K_0 > 1$ be such that $||Q(E(u_G))||_{L^\infty(\Sigma)} < K_0/4$. There exist $T > 0$ and $\tilde{\varepsilon} > 0$ with the following property: if $\|h_0\|_{H^3(\Sigma)} < K_0$, and $\|h_0\|_{L^2(\Sigma)} < \tilde{\varepsilon}$, and $f$ is a smooth function satisfying (3.9) then the solution of (3.1), with initial datum $h_0$, provided by Theorem 3.1 exists for the time interval $(0, T)$ and it holds that*

$$\sup_{0\leqq t\leqq T} \|Q(E(u_t)) \circ \pi^{-1}_{F_t}\|_{L^\infty(\Sigma)} + \int_0^T \|Q(E(u_t)) \circ \pi^{-1}_{F_t}\|^2_{H^3(\Sigma)}\, \mathrm{d}t \leqq K_0. \tag{4.8}$$

*Moreover, for every $k \in \mathbb{N}$ there exists $C'_k(K_0) > 0$ such that*

$$\sum_{i=0}^{k} \int_0^T t^i \|Q(E(u_t)) \circ \pi^{-1}_{F_t}\|^2_{H^{2i+3}(\Sigma)}\, \mathrm{d}t \leqq \frac{1}{2}\bigg(C'_k(K_0) + \sum_{i=0}^{k} \int_0^T t^i \|f(\cdot, t)\|^2_{H^{2i+3}(\Sigma)}\, \mathrm{d}t\bigg). \tag{4.9}$$

**Proof.** We begin by proving (4.8). Let us fix $\alpha \in (0, 1)$. Given $\delta_0 > 0$ to be chosen later and taking $\tilde{\varepsilon}$ equal to the corresponding $\varepsilon_0$, let $h(\cdot, t)$ be the solution defined on $(0, T_0)$, provided by Theorem 3.1. Note that from (3.10) and (3.11) we have $\sup_{0\leqq t\leqq T_0} \|h(\cdot, t)\|_{H^3} \leqq C(K_0)$ and $\sup_{0\leqq t\leqq T_0} \|h(\cdot, t)\|_{L^2} \leqq \delta_0$. In turn, by interpolation $\sup_{0\leqq t\leqq T_0} \|h(\cdot, t)\|_{C^{1,\alpha}} \leqq C\delta_0^\theta < 1$ for some $\theta \in (0, 1)$. Recall also that by choosing $\tilde{\varepsilon}$ small we can make $\delta_0$ as small as we wish. Observing that the coefficients of the equation solved by $u_t \circ \pi^{-1}_{F_t}$ are close in $C^{0,\alpha}$ to the ones of the equation solved by $u_G$, by standard elliptic estimates we have that

$$\sup_{0\leqq t\leqq T_0} \|u_t \circ \pi^{-1}_{F_t} - u_G\|_{C^{1,\alpha}(\Sigma)} \leqq \omega(\delta_0),$$

and $\omega(\delta_0) \to 0$ as $\delta_0 \to 0$. In turn, we conclude that for every $t \in (0, T_0)$ it holds that

$$\begin{aligned}\|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty} &\leqq \|Q(E(u_t)) - Q(E(u_G)) \circ \pi_{F_t}^{-1}\|_{L^\infty}\\ &\quad + \|Q(E(u_G)) \circ \pi_{F_t}^{-1}\|_{L^\infty} \leqq \frac{K_0}{3},\end{aligned}$$

provided $\tilde{\varepsilon}$ (and thus $\delta_0$) is small enough.

Concerning the second term on the left-hand side of (4.8), we have by a well-known interpolation result and by (4.2) for $k = 5$ from Theorem 4.1

$$\begin{aligned}&\int_0^T \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^3(\Sigma)}^2 \,\mathrm{d}t\\ &\leqq C \int_0^T \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{\frac{7}{2}}(\Sigma)}^{2\theta} \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty(\Sigma)}^{2(1-\theta)} \,\mathrm{d}t\\ &\leqq C \int_0^T (1 + \|h(\cdot, t)\|_{H^5(\Sigma)}^{2\theta}) K_0^{2(1-\theta)} \,\mathrm{d}t\\ &\leqq \eta \int_0^T \|h(\cdot, t)\|_{H^5(\Sigma)}^2 \,\mathrm{d}t + C_\eta K_0^2 T\\ &\leqq \eta C \left( K_0^2 + \int_0^T \left(1 + \|f(\cdot, t)\|_{L^\infty(\Sigma)}^{q_0} + \|f(\cdot, t)\|_{H^3(\Sigma)}^2\right) \mathrm{d}t \right) + C_\eta K_0^2 T\\ &\leqq \eta C \left( K_0^2 + T + T K_0^{q_0} + K_0 \right) + C_\eta K_0^2 T,\end{aligned}$$

where the second last inequality follows from (3.11). The inequality (4.8) follows by choosing $\eta$ and $T \leqq T_0$ sufficiently small.

The inequality (4.9) follows by a similar argument. For all $i = 1, \ldots, k$ we have again by interpolation and by (4.2) that

$$\begin{aligned}&\int_0^T t^i \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{2i+3}(\Sigma)}^2 \,\mathrm{d}t\\ &\leqq C \int_0^T t^i \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{H^{2i+\frac{7}{2}}(\Sigma)}^{2\theta} \|Q(E(u_t)) \circ \pi_{F_t}^{-1}\|_{L^\infty(\Sigma)}^{2(1-\theta)} \,\mathrm{d}t\\ &\leqq C_k \int_0^T t^i (1 + \|h(\cdot, t)\|_{H^{2i+5}}^{2\theta}) K_0^{2(1-\theta)} \,\mathrm{d}t\\ &\leqq \eta \int_0^T t^i \|h(\cdot, t)\|_{H^{2i+5}}^2 \,\mathrm{d}t + C_{k,\eta} K_0^2 T.\end{aligned}$$

The conclusion then follows by estimating the last integral by means of (3.11) and choosing $\eta$ sufficiently small and $C_k'(K_0)$ sufficiently large. □

**Theorem 4.4.** *Let $K_0 > 1$ be such that $||Q(E(u_G))||_{L^\infty(\Sigma)} < K_0/4$ and fix $\delta_0 > 0$. There exist $T \in (0, 1)$ and $\varepsilon_1 \in (0, 1)$ with the following property: if $F_0 \in \mathfrak{h}_{K_0}^3(\Sigma)$, defined in (2.18), with $\|h_0\|_{L^2(\Sigma)} < \varepsilon_1$ then there exists a unique solution $h$ to*

(4.1) *in* $H^1(0, T; H^1(\Sigma)) \cap L^\infty(0, T; H^3(\Sigma))$. *Moreover, the solution belongs to* $H^1_{loc}(0, T; H^k(\Sigma))$ *for every* $k \geqq 1$ *and it holds that*

$$\sup_{0\leqq t\leqq T} \|h(\cdot, t)\|_{L^2(\Sigma)} < \delta_0 \tag{4.10}$$

*and*

$$\sup_{0\leqq t\leqq T} t^k \|h(\cdot, t)\|^2_{H^{2k+3}(\Sigma)} + \int_0^T t^k \|h(\cdot, t)\|^2_{H^{2k+5}(\Sigma)}\,\mathrm{d}t \leqq C(k, K_0). \tag{4.11}$$

**Proof.** We divide the proof into three steps.
**Step 1.** Let $K_0$, $T$ be as in Lemma 4.3. Let $\mathcal{S}$ be the set of functions in $C^\infty(0, T; C^\infty(\Sigma))$ that satisfy

$$\sup_{0\leqq t\leqq T} \|f(\cdot, t)\|_{L^\infty(\Sigma)} + \int_0^T \|f(\cdot, t)\|^2_{H^3(\Sigma)}\,\mathrm{d}t \leqq K_0$$

and

$$\sum_{i=0}^{k} \int_0^T \left(t^i \|f(\cdot, t)\|^2_{H^{2i+3}(\Sigma)}\right) \mathrm{d}t \leqq C'_k(K_0)$$

for every $k \in \mathbb{N}$, where $C'_k(K_0)$ are the constants from (4.9). We define a map $\mathscr{L} : \mathcal{S} \to \mathcal{S}$ as $\mathscr{L}(f)(\cdot, t) := -Q(E(u_t)) \circ \pi^{-1}_{F_t}$ for all $t \in (0, T)$, where $F_t$ is the solution of (3.1) with initial datum $h_0$ and forcing term $f$, and where $u_t$ stands for $u_{F_t}$, that is for the elastic equilibrium in $\Omega\backslash F_t$. Lemma 4.3 implies that the map $\mathscr{L} : \mathcal{S} \to \mathcal{S}$ is well defined, provided that $\varepsilon_1 \leqq \tilde{\varepsilon}$. Note also that $\mathcal{S}$ is clearly nonempty as the zero function belongs to $\mathcal{S}$.

We will show that $\mathscr{L} : \mathcal{S} \to \mathcal{S}$ is a contraction with respect to a suitable norm.
**Step 2.** Fix $\mu \in (0, 1)$. Let $f_1$ and $f_2$ be two smooth functions in $\mathcal{S}$ and let $h_1$ and $h_2$ be the corresponding solutions of (3.6) with intial datum $h_0$. The goal in this step is to show that it holds that

$$\begin{aligned}\int_0^T \int_\Sigma (h_2(\cdot, t) - h_1(\cdot, t))^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t \\ \leqq \mu \int_0^T \int_\Sigma (f_2(\cdot, t) - f_1(\cdot, t))^2\mathrm{d}\mathcal{H}^2\mathrm{d}t,\end{aligned} \tag{4.12}$$

possibly by decreasing the time $T$ in a way independent of $f_1$ and $f_2$. We recall that by Theorem 3.1 we have that

$$\sup_{0\leqq t\leqq T} \|h(\cdot, t)\|_{L^2(\Sigma)} \leqq \delta_0 \qquad \text{and} \qquad \sup_{0\leqq t\leqq T} \|h(\cdot, t)\|_{H^3(\Sigma)} \leqq C(K_0),$$

provided that $\varepsilon_1 < \varepsilon_0$. By interpolation these imply that $\sup_{0 \leqq t \leqq T} \|h(\cdot,t)\|_{C^{1,\alpha}(\Sigma)} \leqq C\delta_0^\theta < 1$ for some $\theta \in (0,1)$. In turn, by standard Schauder estimates the corresponding elastic equilibria in $F_{h(\cdot,t)}$ are uniformly bounded in $C^{1,\alpha}$ up to the boundary, that is, $\sup_{0 \leqq t \leqq T} \|u_t \circ \pi_{F_t}^{-1}\|_{C^{1,\alpha}(\Sigma)} \leqq C$. We will use these facts repeatedly in the proof.

We denote by $F_{t,i}$ the set related to $h_i(\cdot,t)$ with $\partial F_{t,i} = \{x + h_i(x,t)\nu(x) : x \in \Sigma\}$. We multiply (3.1) for $i = 1,2$ by $\big((h_2 - h_1)\circ\pi\big)\big((J_i \circ \pi)\nu_{F_{t,i}} \cdot (\nu\circ\pi)\big)^{-1}$, where $J_i$ stands for the tangential Jacobian on $\Sigma$ of the map $x \mapsto x + h_i(x)\nu(x)$ and $\pi$ for the projection on $\Sigma$. We then get

$$\begin{aligned}\int_{\partial F_{t,i}} &(\partial_t h_i(\cdot,t)\circ\pi)\frac{(h_1-h_2)\circ\pi}{J_i\circ\pi}\,\mathrm{d}\mathcal{H}^2\\ &= \int_{\partial F_{t,i}} \Delta_{\partial F_{t,i}}[H_{\partial F_{t,i}} + f_i(\cdot,t)\circ\pi)]\big((h_2-h_1)\circ\pi\big)\\ &\quad\big((J_i\circ\pi)\nu_{F_{t,i}}\cdot(\nu\circ\pi)\big)^{-1}\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

Recall that, denoting by $\partial_{\tau_1}h_i$ and $\partial_{\tau_2}h_i$ the tangential derivatives of $h_i$ in the directions of the principal curvatures, we have

$$J_i = \sqrt{(1+h_ik_1)^2(1+h_ik_2)^2 + (1+h_ik_1)^2(\partial_{\tau_1}h_i)^2 + (1+h_ik_2)^2(\partial_{\tau_2}h_i)^2},$$

where $k_1, k_2$ are the principal curvatures of $\Sigma$. Therefore we have by the formula for the outer normal (3.2) that

$$\big((J_i\circ\pi)\nu_{F_{t,i}}\cdot(\nu\circ\pi)\big)^{-1} = \frac{1}{(1+h_ik_1)(1+h_ik_2)}\circ\pi =: R(\cdot,h_i)\circ\pi.$$

By integrating by parts we get

$$\begin{aligned}\int_{\partial F_{t,i}} &(\partial_t h_i(\cdot,t)\circ\pi)\frac{(h_1-h_2)\circ\pi}{J_i\circ\pi}\,\mathrm{d}\mathcal{H}^2\\ &= \int_{\partial F_{t,i}} (H_{\partial F_{t,i}} + f_i(\cdot,t)\circ\pi)\Delta_{\partial F_{t,i}}[(h_1-h_2)\circ\pi\; R(\cdot,h_i)\circ\pi]\,\mathrm{d}\mathcal{H}^2\end{aligned}$$

Rewriting the integrals above on $\Sigma$ and subtracting, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}&\int_\Sigma (h_2-h_1)^2\,\mathrm{d}\mathcal{H}^2\\ &= \int_\Sigma \big(J_2 H_{\partial F_{t,2}}\circ\pi_{F_{t,2}}^{-1} - J_1 H_{\partial F_{t,1}}\circ\pi_{F_{t,1}}^{-1}\\ &\quad + J_2 f_2 - J_1 f_1\big)\Delta_{\partial F_{t,2}}[(h_2-h_1)\circ\pi\, R(\cdot,h_2)\circ\pi]\circ\pi_{F_{t,2}}^{-1}\,\mathrm{d}\mathcal{H}^2\\ &\quad + \int_\Sigma J_1(H_{\partial F_{t,1}}\circ\pi_{F_{t,1}}^{-1} + f_1)\big(\Delta_{\partial F_{t,2}}[(h_2-h_1)\circ\pi\, R(\cdot,h_2)\circ\pi]\circ\pi_{F_{t,2}}^{-1}\\ &\quad - \Delta_{\partial F_{t,1}}[(h_2-h_1)\circ\pi\, R(\cdot,h_1)\circ\pi]\circ\pi_{F_{t,1}}^{-1}\big)\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

We recall (3.3) and (3.4), where the coefficients $A$, $A_1$ and $A_2$ vanish as $(h, \nabla h) = 0$. We recall also that $\|h_i(\cdot, t)\|_{C^{1,\alpha}}$ is small uniformly in time and that $f_i$ are uniformly bounded with respect to time. After straighforward calculations we have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Sigma (h_2 - h_1)^2 \,\mathrm{d}\mathcal{H} + \frac{1}{2}\int_\Sigma |\Delta(h_2 - h_1)|^2 \,\mathrm{d}\mathcal{H}^2 \leqq \varepsilon \int_\Sigma |\nabla^2(h_2 - h_1)|^2 \,\mathrm{d}\mathcal{H}^2 \\
&\quad + C\int_\Sigma (1 + |\nabla^2 h_1| + |\nabla^2 h_2|)(|h_2 - h_1| + |\nabla(h_2 - h_1)|)\cdot \\
&\quad \cdot (|h_2 - h_1| + |\nabla(h_2 - h_1)| + |\nabla^2(h_2 - h_1)|)\,\mathrm{d}\mathcal{H}^2 \\
&\quad + C\int_\Sigma |f_2 - f_1| \Big((1 + |\nabla^2 h_1| + |\nabla^2 h_2|)(|h_2 - h_1| \\
&\quad + |\nabla(h_2 - h_1)|) + |\nabla^2(h_2 - h_1)|\Big)\,\mathrm{d}\mathcal{H}^2 \\
&\quad + C\int_\Sigma (1 + |\nabla^2 h_1|^2 + |\nabla^2 h_2|^2)(|h_2 - h_1|^2 + |\nabla(h_2 - h_1)|^2)\,\mathrm{d}\mathcal{H}^2 =: RHS.
\end{aligned}$$

Using Young's Inequality we obtain

$$\begin{aligned}
RHS &\leqq \varepsilon \int_\Sigma |\nabla^2(h_2 - h_1)|^2 \,\mathrm{d}\mathcal{H}^2 \\
&\quad + C\int_\Sigma (1 + |\nabla^2 h_1|^2 + |\nabla^2 h_2|^2)(|h_2 - h_1|^2 \\
&\quad + |\nabla(h_2 - h_1)|^2)\,\mathrm{d}\mathcal{H}^2 + C\int_\Sigma |f_2 - f_1|^2 \,\mathrm{d}\mathcal{H}^2.
\end{aligned}$$

Observe now that by interpolation, by controlling the second derivatives of $h_i$ with the $H^3$-norms, and using the fact that $\|h(\cdot, t)\|_{H^3}$ is bounded uniformly with respect to time we have

$$\begin{aligned}
&\int_\Sigma (1 + |\nabla^2 h_1|^2 + |\nabla^2 h_2|^2)(|h_2 - h_1|^2 \\
&\qquad + |\nabla(h_2 - h_1)|^2)\,\mathrm{d}\mathcal{H}^2 \\
&\leqq C(1 + \|\nabla^2 h_1\|_{L^4}^2 + \|\nabla^2 h_2\|_{L^4}^2)\|h_2 - h_1\|_{W^{1,4}}^2 \\
&\leqq C\|h_2 - h_1\|_{H^2}^{\frac{3}{2}}\|h_2 - h_1\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

From the previous inequalities we get

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_\Sigma (h_2 - h_1)^2 \,\mathrm{d}\mathcal{H}^2 \leqq{}& -\frac{1}{2}\int_\Sigma |\Delta(h_2 - h_1)|^2 \,\mathrm{d}\mathcal{H}^2 \\
&+ \varepsilon \int_\Sigma |\nabla^2(h_2 - h_1)|^2 \,\mathrm{d}\mathcal{H}^2 \\
&+ C_\varepsilon \int_\Sigma (|\nabla(h_2 - h_1)|^2 \\
&+ (h_2 - h_1)^2 + (f_2 - f_1)^2)\,\mathrm{d}\mathcal{H}^2.
\end{aligned}$$

Using now Remark 2.5 we in turn obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Sigma (h_2-h_1)^2\,\mathrm{d}\mathcal{H}^2+\frac{1}{4}\int_\Sigma |\nabla^2(h_2-h_1)|^2\,\mathrm{d}\mathcal{H}^2\\ \leqq C\int_\Sigma (|h_2-h_1|^2+(f_2-f_1)^2)\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

Integrating this with respect to time over $(0,t)$, with $t\in(0,T)$, we have

$$\begin{aligned}&\int_\Sigma (h_2(\cdot,t)-h_1(\cdot,t))^2\,\mathrm{d}\mathcal{H}^2+\frac{1}{2}\int_0^t\int_\Sigma |\nabla^2(h_2(\cdot,s)-h_1(\cdot,s))|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}s\\ &\quad\leqq C\int_0^t\int_\Sigma (|h_2(\cdot,s)-h_1(\cdot,s)|^2+(f_2(\cdot,s)-f_1(\cdot,s))^2)\,\mathrm{d}\mathcal{H}^2\mathrm{d}s. \qquad (4.13)\end{aligned}$$

Integrating the above inequality with respect to time over $(0,T)$ we obtain (4.12) when $T$ is sufficiently small.

**Step 3.** Here we finally prove that the map $\mathscr{L}:\mathcal{S}\to\mathcal{S}$ is a contraction with respect to the $L^2(0,T;L^2(\Sigma))$-norm. To be more precise, let $f_1$ and $f_2$ be two functions in $\mathcal{S}$ and $h_1$ and $h_2$ the corresponding solutions of (3.1). For simplicity we denote the elastic equilibrium for $F_{h_i}$ as $u_i(\cdot,t):=u_{F_{t,i}}$, for $i=1,2$. Then $\mathscr{L}(f_i)=-Q(E(u_i))\circ\pi_{F_{t,i}}^{-1}$ and our goal is to show

$$\begin{aligned}&\int_0^T \|Q(E(u_2(\cdot,t)))\circ\pi_{F_{t,2}}^{-1}-Q(E(u_1(\cdot,t)))\circ\pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)}^2\mathrm{d}t\\ &\quad\leqq \frac{1}{2}\int_0^T \|f_2(\cdot,t)-f_1(\cdot,t)\|_{L^2(\Sigma)}^2\,\mathrm{d}t. \qquad (4.14)\end{aligned}$$

Let us fix $t\in(0,T)$. We begin by proving that given $\varepsilon>0$, if $\delta_0$ is small enough we have

$$\begin{aligned}&\|Q(E(u_2(\cdot,t)))\circ\pi_{F_{t,2}}^{-1}-Q(E(u_1(\cdot,t)))\circ\pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)}\\ &\quad\leqq C\|\nabla(u_2(\cdot,t)\circ\pi_{F_{t,2}}^{-1})-\nabla(u_1(\cdot,t)\circ\pi_{F_{t,1}}^{-1})\|_{L^2(\Sigma)} \qquad (4.15)\\ &\qquad+\varepsilon\|\nabla^2(h_2(\cdot,t)-h_1(\cdot,t))\|_{L^2(\Sigma)}^2+C\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^1(\Sigma)}.\end{aligned}$$

To shorten the notation we denote $U_i:=Du_i\circ\pi_{F_{t,i}}^{-1}$, $\nu_i=\nu_{F_{t,i}}\circ\pi_{F_{t,i}}^{-1}$ and $h_i=h_i(\cdot,t)$ for $i=1,2$. Recall that $Q(E(u_i(\cdot,t)))\circ\pi_{F_{t,i}}^{-1}=\frac{1}{2}\mathbb{C}U_i:U_i$. We may thus write

$$\begin{aligned}&\|Q(E(u_2(\cdot,t)))\circ\pi_{F_{t,2}}^{-1}-Q(E(u_1(\cdot,t)))\circ\pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)}\\ &\quad=\frac{1}{2}\|\mathbb{C}(U_2+U_1):(U_2-U_1)\|_{L^2(\Sigma)}.\end{aligned}$$

We estimate this simply as

$$\begin{aligned}&\|\mathbb{C}(U_2+U_1):(U_2-U_1)\|_{L^2(\Sigma)}\\ &\quad\leqq\|\mathbb{C}(U_2+U_1):\big((U_2-U_1)(I-\nu\otimes\nu)\big)\|_{L^2(\Sigma)} \qquad (4.16)\\ &\qquad+\|\mathbb{C}(U_2+U_1):\big((U_2-U_1)\,(\nu\otimes\nu)\big)\|_{L^2(\Sigma)}.\end{aligned}$$

Note that by the second condition in (2.12) it holds $\mathbb{C}U_i[\nu_i] = \mathbb{C}E(u_i) \circ \pi_{F_{t,i}}^{-1}[\nu_i] = 0$ on $\Sigma$. We use this equality to estimate the last term in (4.16) by

$$\begin{aligned}
&\|\mathbb{C}(U_2 + U_1) : \big((U_2 - U_1)\,(\nu \otimes \nu)\big)\|_{L^2(\Sigma)} \\
&\quad \leqq \|\mathbb{C}(U_2 + U_1) : \big((U_2 - U_1)\,(\nu \otimes (\nu - \nu_2))\big)\|_{L^2(\Sigma)} \\
&\qquad + \|\mathbb{C}U_1 : \big((U_2 - U_1)\,(\nu \otimes \nu_2)\big)\|_{L^2(\Sigma)} \\
&\quad = \|\mathbb{C}(U_2 + U_1) : \big((U_2 - U_1)\,(\nu \otimes (\nu - \nu_2))\big)\|_{L^2(\Sigma)} \\
&\qquad + \|\mathbb{C}U_1 : \big((U_2 - U_1)\,(\nu \otimes (\nu_2 - \nu_1))\big)\|_{L^2(\Sigma)}.
\end{aligned}$$

Using the expression (3.2) for the normal $\nu_2$ and the uniform $C^{1,\alpha}$-bound for $h_i$ we deduce that $\|\nu - \nu_2\|_{L^\infty(\Sigma)} \leqq C\delta_0^\theta$ and $\|\nu_2 - \nu_1\|_{L^2(\Sigma)} \leqq C\|h_2 - h_1\|_{H^1(\Sigma)}$. Moreover, by the $C^{1,\alpha}$-bound for $u_i$ we have that $\|U_i\|_{L^\infty} \leqq C$ and by the second inequality in Remark 4.2 it holds $\|U_2 - U_1\|_{L^2(\Sigma)} \leqq C\|h_2 - h_1\|_{H^2(\Sigma)}$. Therefore we may estimate the above inequality as

$$\begin{aligned}
&\|\mathbb{C}(U_2 + U_1) : \big((U_2 - U_1)(\nu \otimes \nu)\big)\|_{L^2(\Sigma)} \\
&\quad \leqq \varepsilon\|h_2 - h_1\|_{H^2(\Sigma)} + C\|h_2 - h_1\|_{H^1(\Sigma)}.
\end{aligned}$$

Thus we deduce by (4.16) that

$$\begin{aligned}
&\|\mathbb{C}(U_2 + U_1) : (U_2 - U_1)\|_{L^2(\Sigma)} \\
&\quad \leqq \|\mathbb{C}(U_2 + U_1) : \big((U_2 - U_1)(I - \nu \otimes \nu)\big)\|_{L^2(\Sigma)} \\
&\qquad + \varepsilon\|h_2 - h_1\|_{H^2(\Sigma)} + C\|h_2 - h_1\|_{H^1(\Sigma)}.
\end{aligned}$$

The inequality (4.15) then follows from (2.10) as

$$\begin{aligned}
&\|\mathbb{C}(U_2 + U_1) : \big((U_2 - U_1)(I - \nu \otimes \nu)\big)\|_{L^2(\Sigma)} \\
&\quad = \|\mathbb{C}(U_2 + U_1) : \big((Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})(I - \nu \otimes \nu)\big)\|_{L^2(\Sigma)} \\
&\quad \leqq C\|\big(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1}\big)_\tau\|_{L^2(\Sigma)} \\
&\quad \leqq C\|\big[(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1})D\pi_{F_{t,2}}^{-1} - (Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})D\pi_{F_{t,1}}^{-1}\big]_\tau\|_{L^2(\Sigma)} \\
&\qquad + C\|\big[(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1})(D\pi_{F_{t,2}}^{-1} - D\pi_{F_{t,1}}^{-1})\big]_\tau\|_{L^2(\Sigma)} \\
&\qquad + C\|\big[(Du_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1} - Du_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})(I - D\pi_{F_{t,1}}^{-1})\big]_\tau\|_{L^2(\Sigma)} \\
&\quad \leqq C\|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{L^2(\Sigma)} \\
&\qquad + C\|h_2 - h_1\|_{H^1(\Sigma)} + \varepsilon\|h_2 - h_1\|_{H^2(\Sigma)},
\end{aligned}$$

where in the last inequality we used the second estimate in Remark 4.2 and the fact that the $C^1$-norm of $h_1$ is small.

We proceed by using (4.15) and interpolation to deduce

$$\begin{aligned}
&\|Q(E(u_2(\cdot, t))) \circ \pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot, t))) \circ \pi_{F_{t,1}}^{-1}\|_{L^2(\Sigma)} \\
&\quad \leqq C\|\nabla(u_2(\cdot, t) \circ \pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot, t) \circ \pi_{F_{t,1}}^{-1})\|_{H^{\frac{1}{2}}(\Sigma)}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\times \|\nabla(u_2(\cdot,t)\circ\pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot,t)\circ\pi_{F_{t,1}}^{-1})\|_{H^{-\frac{1}{2}}(\Sigma)}^{\frac{1}{2}}\\
&+\varepsilon\|h_2-h_1\|_{H^2(\Sigma)} + C\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^1(\Sigma)}.
\end{aligned}
$$

By the estimate (4.3) in Theorem 4.1 we have

$$
\begin{aligned}
&\|\nabla(u_2(\cdot,t)\circ\pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot,t)\circ\pi_{F_{t,1}}^{-1})\|_{H^{\frac{1}{2}}(\Sigma)}\\
&\quad\leqq \|u_2(\cdot,t)\circ\pi_{F_{t,2}}^{-1} - u_1(\cdot,t)\circ\pi_{F_{t,1}}^{-1}\|_{H^{3/2}(\Sigma)} \leqq C\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2}.
\end{aligned}
$$

Moreover by using the well-known inequality $\|\nabla g\|_{H^{-\frac{1}{2}}(\Sigma)} \leqq C\|g\|_{H^{\frac{1}{2}}(\Sigma)}$ and Remark 4.2 we have

$$
\begin{aligned}
&\|\nabla(u_2(\cdot,t)\circ\pi_{F_{t,2}}^{-1}) - \nabla(u_1(\cdot,t)\circ\pi_{F_{t,1}}^{-1})\|_{H^{-\frac{1}{2}}(\Sigma)}\\
&\quad\leqq C\|u_2(\cdot,t)\circ\pi_{F_{t,2}}^{-1} - u_1(\cdot,t)\circ\pi_{F_{t,1}}^{-1}\|_{H^{\frac{1}{2}}(\Sigma)} \leqq C\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^1(\Sigma)}.
\end{aligned}
$$

Collecting the previous three inequalities, using standard interpolation,

$$
\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^1(\Sigma)} \leqq C\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2(\Sigma)}^{1/2}\|h_2(\cdot,t)-h_1(\cdot,t)\|_{L^2(\Sigma)}^{1/2},
$$

and by Young's inequality we obtain

$$
\begin{aligned}
&\|Q(E(u_2(\cdot,t)))\circ\pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot,t)))\circ\pi_{F_{t,1}}^{-1}\|_{L^2}^2\\
&\quad\leqq 2\varepsilon\|\nabla^2(h_2(\cdot,t)-h_1(\cdot,t))\|_{L^2}^2 + C_\varepsilon\|h_2(\cdot,t)-h_1(\cdot,t)\|_{L^2}^2.
\end{aligned}
$$

Integrating the previous inequality over $(0,T)$ and using (4.12) and (4.13), we obtain

$$
\begin{aligned}
&\int_0^T \|Q(E(u_2(\cdot,t)))\circ\pi_{F_{t,2}}^{-1} - Q(E(u_1(\cdot,t)))\circ\pi_{F_{t,1}}^{-1}\|_{L^2}^2\mathrm{d}t\\
&\quad\leqq \big((C_\varepsilon+\varepsilon C)\mu+\varepsilon C\big)\int_0^T \|f_2(\cdot,s)-f_1(\cdot,s)\|_{L^2}^2\,\mathrm{d}\mathcal{H}^1\mathrm{d}s\\
&\quad\leqq \frac{1}{2}\int_0^T \|f_2(\cdot,s)-f_1(\cdot,s)\|_{L^2}^2\,\mathrm{d}\mathcal{H}^1\mathrm{d}s,
\end{aligned}
$$

provided that $\varepsilon$ and then $\mu$ are chosen sufficiently small. This proves (4.14) and we conclude that $\mathscr{L}:\mathcal{S}\to\mathcal{S}$ is a contraction with respect to the $L^2(0,T;L^2(\Sigma))$-norm.

**Step 4.** (*Conclusion*) We may proceed with a standard argument, by recursively setting $f_1=0$, $f_n:=\mathcal{L}(f_{n-1})$ and for every $n\geqq 1$ letting $h_n$ be the solution to (3.1) with $f$ replaced by $f_n$. From Step 2 and Step 3 we have that there exist $f$ and $h$ such that $f_n\to f$ and $h_n\to h$ in $L^2(0,T;L^2(\Sigma))$. Moreover, using (4.9) and (3.11), we conclude easily that for every $n\geqq 1$ the functions $h_n$ satisfy (4.10) and (4.11) for every $k\in\mathbb{N}$, with constants depending only on $k$ and $K_0$. Thus, we have that $h_n\rightharpoonup h$ weakly in $H^1(0,T;H^1(\Sigma))\cap L^\infty(0,T;H^3(\Sigma))$. Moreover using the equation satisfied by $h_n$ and (3.11) we also have that $\partial_t h_n$ is bounded in $L^2_{loc}(0,T;H^k(\Sigma))$ for every $k\in\mathbb{N}$. Therefore we have that $h_n\rightharpoonup h$ weakly in $H^1_{loc}(0, T; H^k(\Sigma))$ and thus strongly in $L^2_{loc}(0, T; H^k(\Sigma))$ and that $h$ satisfies (4.10) and (4.11). Using these convergences one can easily pass to the limit in the equations satisfied by the $h_n$'s to conclude that $h$ is a solution of (4.1). The uniqueness follows from the same argument used in Step 2 and Step 3. □

## 5. Asymptotic Stability

In this section we study the flow when the initial set is close to a smooth strictly stable stationary set $G$, which will be our reference set, that is, we set $\Sigma = \partial G$. Throughout this section we denote

$$R_t := H_{F_t} - Q(E(u_{F_t})).$$

Moreover, in what follows we shall drop the subscript $\partial F_t$ (and similar) in all the covariant differential operators, when no danger of confusion arises. Here is the main result:

**Theorem 5.1.** *Let $G \subset\subset \Omega$ be a regular strictly stable stationary set in the sense of Definition 2.11. There exists $\delta > 0$ such that if $F_0 \in \mathfrak{h}^3_\delta(\Sigma)$, then the unique solution $(F_t)_{t>0}$ of the flow (4.1) with intial datum $F_0$ is defined for all times $t > 0$.*

*Moreover $F_t \to F_\infty$ exponentially fast, where $F_\infty$ is the unique stationary set near $G$ such that $|F_{\infty,i}| = |F_{0,i}|$ for $i = 1, \dots, m$. In particular, if $|F_{0,i}| = |G_i|$ for $i = 1, \dots, m$, then $F_t \to G$ exponentially fast. Here $G_i$ denote the open bounded sets enclosed by the components $\Gamma_{G,1}, \dots, \Gamma_{G,m}$ of $\partial G$, $F_{\infty,i}$ and $F_{0,i}$ are diffeomorphic to $G_i$, and $\partial F_{0,i}$ and $\partial F_{\infty,i}$ are the components of $\partial F_0$ and $\partial F_\infty$ respectively.*

**Remark 5.2.** By exponential convergence of $F_t$ to $F_\infty$ we mean precisely the following: writing $\partial F_t := \{x + \tilde{h}(x, t)\nu_{F_\infty}(x) : x \in \partial F_\infty\}$, we have that for every $k \in \mathbb{N}$ there exists $c_k > 0$ and $C_k > 1$ such that

$$\|\tilde{h}(\cdot, t)\|_{C^k(\partial F_\infty)} \leqq C_k e^{-c_k t}$$

for $t \geqq 1$.

The proof of stability is based on the following energy identity:

**Proposition 5.3.** *Let $(F_t)_{t\in[0,T)}$ be the solution of (4.1) provided by Theorem 4.4. Then the function*

$$t \mapsto \int_{\partial F_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2$$

*is absolutely continuous and for almost every $t \in (0, T)$ we have the following energy identity*

$$\frac{d}{\mathrm{d}t}\left(\int_{\partial F_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2\right) = -2\partial^2 \mathcal{J}(F_t)[\Delta R_t]$$
$$-2\int_{\partial F_t} B_{F_t}[\nabla R_t, \nabla R_t]\,(\Delta R_t)\,\mathrm{d}\mathcal{H}^2 + \int_{\partial F_t} H_{F_t}|\nabla R_t|^2\,(\Delta R_t)\,\mathrm{d}\mathcal{H}^2, \quad (5.1)$$

*where $\partial^2\mathcal{J}(F_t)$ is defined as in (2.21) and $B_{F_t}[\cdot, \cdot]$ denotes the fundamental form of $\partial F_t$.*

The proof of the proposition is similar to [24, Proposition 4.3] (see also [1, Lemma 4.4]) and therefore we shift it to the "Appendix".

In order to control the two last terms in (5.1) we need the following interpolation result on the evolving boundaries. The proof of the next lemma is precisely the same as [1, Lemma 4.7] and therefore we omit it.

**Lemma 5.4.** *If $F \subset U$ is such that $\partial F = \{x + h_F(x)\nu(x) : x \in \Sigma\}$ with $\|h_F\|_{C^{1,\alpha}(\Sigma)} \leqq M$, then for every smooth function $f \in C^\infty(\partial F)$ it holds that*

$$\int_{\partial F} |B_F||\nabla f|^2|\Delta f| \, d\mathcal{H}^2 \leqq C \left(1 + \|H_F\|^3_{L^6(\partial F)}\right) \|\nabla \Delta f\|^2_{L^2(\partial F)} \|\nabla f\|_{L^2(\partial F)}.$$

*The constant $C$ depends only on $M$ and $\Sigma$.*

We are now ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** For any set $F \in \mathfrak{h}^3_1(\Sigma)$ consider

$$D(F) := \int_{F \Delta G} \operatorname{dist}(x, \Sigma) \, dx$$

and note that

$$\frac{1}{C}\|h_F\|^2_{L^2(\partial G)} \leqq D(F) \leqq C\|h_F\|^2_{L^2(\partial G)} \tag{5.2}$$

for a constant depending only on $G$. Moreover, we define

$$R_F := H_F - Q(E(u_F)),$$

which is defined on $\partial F$.

**Step 1.** (*Preliminary estimates*) In this step we show that if $F \in \mathfrak{h}^3_1(\Sigma)$ and $\|h_F\|_{C^1(\Sigma)} \leqq \delta$ for $\delta$ sufficiently small, then it holds that

$$\frac{1}{C}\,\|h_F\|^{1/\theta}_{H^3(\Sigma)} \leqq D(F) + \int_{\partial F} |\nabla R_F|^2 \, d\mathcal{H}^2 \leqq C\,\|h_F\|^\theta_{H^3(\Sigma)}. \tag{5.3}$$

for some $\theta \in (0, 1)$ and some constant $C > 1$.

We begin by proving the first inequality. We use interpolation, (4.2) and the second inequality in Remark 4.2 to deduce that

$$\begin{aligned}
&\|\nabla\big(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\big)\|_{L^2(\Sigma)} \\
&\quad \leqq C\|Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\|^{\theta'}_{H^{\frac{3}{2}}(\Sigma)} \\
&\qquad \times \|Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\|^{1-\theta'}_{L^2(\Sigma)} \\
&\quad \leqq (C + \|Q(E(u_F)) \circ \pi_F^{-1}\|^{\theta'}_{H^{3/2}(\Sigma)})\|(Du_F) \circ \pi_F^{-1} - Du_G\|^{1-\theta'}_{L^2(\Sigma)} \\
&\quad \leqq (C + \|h_F\|^{\theta'}_{H^3(\Sigma)})\|h_F\|^{1-\theta'}_{H^2(\Sigma)} \\
&\quad \leqq C\|h_F\|^{1-\theta'}_{H^2(\Sigma)}
\end{aligned} \tag{5.4}$$

for $\theta' \in (0, 1)$. Since $G$ is a stationary set it holds $\nabla R_G = 0$ on $\Sigma$. Therefore we conclude by the above inequality that

$$\begin{aligned}
&\|\nabla\big(H_F \circ \pi_F^{-1} - H_G\big)\|^2_{L^2(\partial F)} \\
&\quad \leqq 2\int_\Sigma |\nabla(R_F \circ \pi_F^{-1})|^2 \, \mathrm{d}\mathcal{H}^2 + 2\|\nabla\big(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\big)\|^2_{L^2(\Sigma)} \\
&\quad \leqq 2C\int_{\partial F} |\nabla R_F|^2 \, \mathrm{d}\mathcal{H}^2 + C\|h_F\|^{2(1-\theta')}_{H^2(\Sigma)}.
\end{aligned}$$

We use (2.6), (3.3) and the fact that $\|h_F\|_{C^1(\Sigma)} \leqq \delta$ to deduce with straightforward calculations

$$\|h_F\|^2_{H^3(\Sigma)} \leqq C\|\nabla(H_F \circ \pi_F^{-1} - H_G)\|^2_{L^2(\Sigma)} + C\|h_F\|^2_{H^2(\Sigma)}.$$

Therefore, from the two previous inequalities and by interpolation we obtain that

$$\begin{aligned}
\|h_F\|^2_{H^3(\Sigma)} &\leqq C\int_{\partial F} |\nabla R_F|^2 \, \mathrm{d}\mathcal{H}^2 + C\|h_F\|^{2(1-\theta')}_{H^2} + C\|h_F\|^2_{H^2} \\
&\leqq C\int_{\partial F} |\nabla R_F|^2 \, \mathrm{d}\mathcal{H}^2 + \frac{1}{2}\|h_F\|^2_{H^3} + C\|h_F\|^{\theta''}_{L^2}
\end{aligned}$$

for a suitable $\theta'' \in (0, 1)$. The first inequality in (5.3) then follows from the previous estimate and from (5.2), recalling that since $\|h_F\|_{H^3(\Sigma)} \leqq 1$, we also have $\|\nabla R_F\|_{L^2(\partial F)} \leqq C$.

To prove the second inequality in (5.3) we argue similarly as above and use (3.3) to conclude that

$$\|\nabla(H_F \circ \pi_F^{-1} - H_G)\|^2_{L^2(\Sigma)} \leqq C\,\|h_F\|^2_{H^3(\Sigma)}.$$

Moreover, by (5.4), we have that

$$\|\nabla\big(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\big)\|_{L^2(\Sigma)} \leqq C\,\|h_F\|^{1-\theta'}_{H^2(\Sigma)}$$

for $\theta' \in (0, 1)$. Therefore since $G$ is a critical set we obtain

$$\begin{aligned}
\int_{\partial F} |\nabla R_F|^2 \, \mathrm{d}\mathcal{H}^2 &\leqq C\int_\Sigma |\nabla(R_F \circ \pi_F^{-1} - R_G)|^2 \, \mathrm{d}\mathcal{H}^2 \\
&\leqq C\int_\Sigma |\nabla(H_F \circ \pi_F^{-1} - H_G)|^2 \, \mathrm{d}\mathcal{H}^2 \\
&\quad + C\int_\Sigma |\nabla\big(Q(E(u_F)) \circ \pi_F^{-1} - Q(E(u_G))\big)|^2 \, \mathrm{d}\mathcal{H}^2 \\
&\leqq C\|h_F\|^2_{H^3(\Sigma)} + C\|h_F\|^{2(1-\theta')}_{H^2(\Sigma)} \leqq C\|h_F\|^{\theta}_{H^3(\Sigma)}.
\end{aligned}$$

Hence, we have (5.3).

**Step 2.** (*Global existence*) Let us assume that the initial set $F_0$ is in $\mathfrak{h}^3_\delta(\Sigma)$ with $\delta < \varepsilon_1$, where $\varepsilon_1 \in (0, 1)$ is the constant provided by Theorem 4.4 corresponding to the choice $\delta_0 = 1$, $K_0 = \max\{2, 5\|Q(E(u_G))\|_{L^\infty(\Sigma)}\}$. Then the flow $(F_t)_{t\in[0,T)}$ starting from $F_0$ which is a solution of (4.1) exists for a time interval $(0, T)$, with

$T$ bounded from below by a positive constant which depends only $G$. Let $\sigma > 0$ be a small number which will be chosen later. Note that by (5.3) and by continuity we have

$$D(F_t) + \int_{\partial F_t} |\nabla R_t|^2 \, d\mathcal{H}^2 \leqq C\|h(\cdot,t)\|^{\theta}_{H^3(\Sigma)} \leqq C\delta^{\theta} < \sigma \tag{5.5}$$

for some time interval $(0, T')$, where the last inequality holds provided that $\delta$ is small enough. Note that by (5.3) it follows that

$$\|h(\cdot,t)\|_{H^3(\Sigma)} < C\sigma^{\theta} < \min\{\varepsilon_1, \sigma_1\} \qquad \text{for every } t \in (0, T'), \tag{5.6}$$

when $\sigma$ is small enough, where $\sigma_1$ is the constant provided by Proposition 2.13. In particular, we conclude from Theorem 4.4 that as long as the flow $(F_t)_{t\in(0,T)}$ satisfies (5.5) it is well defined. In other words, if $(0, T^*)$ is the maximal time of existence and if it satisfies (5.5) for every $t \in (0, T^*)$, then $T^* = \infty$, that is, the flow exists for all times.

Let us denote by $[0, T')$ the maximal time interval where the flow satisfies (5.5). We claim that if $\|h_0\|_{H^3(\Sigma)} < \delta$ for $\delta$ small enough, then the flow satisfies (5.5) for every $t \in (0, T^*)$ and thus $T^* = T' = +\infty$.

We start by recalling that by Lemma 2.12 and (5.6), since $\sigma_1 < \sigma_0$, we have

$$\partial^2 \mathcal{J}(F_t)[\Delta R_t] \geqq \frac{c_0}{2}\|\Delta R_t\|^2_{H^1(\partial F_t)} \qquad \text{for every } t \in (0, T').$$

Thus, from the energy identity (5.1), using also Lemma 5.4 and again (5.5), we may estimate

$$\begin{aligned} \frac{d}{dt}\int_{\partial F_t} |\nabla R_t|^2 \, d\mathcal{H}^2 &\leqq -2\partial^2\mathcal{J}(F_t)[\Delta R_t] + C\int_{\partial F_t} |B_{F_t}||\nabla R_t|^2 \, |\Delta R_t| \, d\mathcal{H}^2 \\ &\leqq -c_0\|\Delta R_t\|^2_{H^1(\partial F_t)} \\ &\quad + C\,(1 + \|H_{F_t}\|^3_{L^6(\partial F_t)})\|\nabla \Delta R_t\|^2_{L^2(\partial F_t)} \, \|\nabla R_t\|_{L^2(\partial F_t)} \\ &\leqq -c_0\|\Delta R_t\|^2_{H^1(\partial F_t)} + C\sqrt{\sigma}\, \|\nabla \Delta R_t\|^2_{L^2(\partial F_t)} \\ &\leqq -\frac{c_0}{2}\|\Delta R_t\|^2_{H^1(\partial F_t)}, \end{aligned} \tag{5.7}$$

where the last inequality holds by taking $\sigma$ smaller if needed.

Next we show that

$$\|\nabla R_t\|_{L^2(\partial F_t)} \leqq C\|\Delta R_t\|_{L^2(\partial F_t)} \tag{5.8}$$

for some constant which depends on $\Sigma$. Let us fix a component of $\partial F_t$ and denote it by $\Gamma_t$. Since $F_t$ is diffeomorphic to $G$ we denote the component of $\Sigma$ diffeomorphic to $\Gamma_t$ by $\Gamma$. Since $\Gamma$ is smooth, compact and connected Riemannian manifold we conclude by [5, Theorem 3.67] that the Poincaré inequality holds on $\Gamma$, that is, for every $\varphi \in C^{\infty}(\Gamma)$ with $\int_{\Gamma} \varphi \, d\mathcal{H}^2 = 0$ it holds

$$\|\varphi\|_{L^2(\Gamma)} \leqq C\|\nabla\varphi\|_{L^2(\Gamma)}.$$

Therefore since $\Gamma_t = \Phi_t(\Gamma)$ with $\Phi_t(x) = x + h(x,t)\nu(x)$ and $\|h(\cdot,t)\|_{C^{1,\alpha}} \leqq C$ the Poincaré inequality holds also on $\Gamma_t$. In particular, we have

$$\|R_t - \bar{R}_t\|_{L^2(\Gamma_t)} \leqq C\|\nabla R_t\|_{L^2(\Gamma_t)},$$

where $\bar{R}_t$ denotes the average of $R_t$ on $\Gamma_t$ and the constant depends on $\Sigma$. Then by integration by parts we get

$$\begin{aligned}\int_{\Gamma_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2 &= -\int_{\Gamma_t} (R_t - \bar{R}_t)\Delta R_t \, \mathrm{d}\mathcal{H}^2 \\ &\leqq \|R_t - \bar{R}_t\|_{L^2(\Gamma_t)}\|\Delta R_t\|_{L^2(\Gamma_t)} \leqq C\|\nabla R_t\|_{L^2(\Gamma_t)}\|\Delta R_t\|_{L^2(\Gamma_t)}.\end{aligned}$$

We obtain (5.8) by repeating the above argument for every component of $\partial F_t$.

By (5.7) and (5.8) we conclude that

$$\frac{d}{\mathrm{d}t}\int_{\partial F_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2 \leqq -c\int_{\partial F_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2$$

for every $t \in (0, T')$. Integrating this over $(0,t)$, using (5.3) and $\|h_0\|_{H^3(\Sigma)} \leqq \delta$, yields

$$\int_{\partial F_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2 \leqq Ce^{-ct}\delta^\theta. \tag{5.9}$$

On the other hand by differentiating $D(F_t)$ with respect to time and using the same calculations as in [24, Lemma 3.3] we get

$$\begin{aligned}\frac{d}{\mathrm{d}t}D(F_t) &= \int_{\partial F_t} d_G \, \Delta R_t \, \mathrm{d}\mathcal{H}^2 = -\int_{\partial F_t} \langle \nabla d_G, \nabla R_t\rangle \, \mathrm{d}\mathcal{H}^2 \\ &\leqq \mathcal{H}^2(\partial F_t)^{1/2}\left(\int_{\partial F_t} |\nabla R_t|^2 \, \mathrm{d}\mathcal{H}^2\right)^{1/2} \leqq Ce^{-\frac{c}{2}t}\delta^{\frac{\theta}{2}}.\end{aligned}$$

Integrating this over $(0,t)$, using (5.2) and $\|h_0\|_{H^3(\Sigma)} \leqq \delta$, yields

$$D(F_t) \leqq D(F_0) + Ce^{-\frac{c}{2}t}\delta^{\frac{\theta}{2}} \leqq C\delta^2 + Ce^{-\frac{c}{2}t}\delta^{\frac{\theta}{2}} < \sigma \tag{5.10}$$

when $\delta$ is chosen small enough. Hence, we have that (5.5) holds for the whole life span of the flow $(0, T^*)$ and by the previous discussion this implies that $T^* = \infty$.

**Step 3.** (*Convergence*) Combining (5.3) and (5.5) we have that $\sup_{t>0}\|h(\cdot,t)\|_{H^3(\Sigma)} \leqq C\sigma^\theta$. Therefore there exists a subsequence such that

$$h(\cdot, t_m) \to h_\infty(\cdot) \qquad \text{in } H^2(\Sigma).$$

We denote the target set by $F_\infty$, that is, $\partial F_\infty = \{x + h_\infty(x)\nu(x) : x \in \Sigma\}$. By (5.9) we deduce that $\nabla R_{F_\infty} = 0$, that is, $F_\infty$ is a stationary set. We will show that $F_t \to F_\infty$ exponentially fast.

To this end we define

$$D_\infty(F) := \int_{F\Delta F_\infty} \mathrm{dist}\,(x, F_\infty)\,\mathrm{d}x.$$

Repeating the calculations leading to (5.10) we get

$$\begin{aligned}\left|\frac{d}{\mathrm{d}t}D_\infty(F_t)\right| &= \left|\int_{\partial F_t} d_{F_\infty}\,\Delta R_t\,\mathrm{d}\mathcal{H}^2\right|\\ &\leqq \mathcal{H}^2(\partial F_t)^{1/2}\left(\int_{\partial F_t}|\nabla R_t|^2\,\mathrm{d}\mathcal{H}^2\right)^{1/2} \leqq Ce^{-\frac{c}{2}t}\delta^{\frac{\theta}{2}},\end{aligned}$$

where the last inequality follows from (5.9). This implies that $\lim_{t\to\infty} D_\infty(F_t)$ exists and the choice of $F_\infty$ implies that $D_\infty(F_t)\to 0$. Therefore integrating the above inequality over $(t,\infty)$ we get

$$D_\infty(F_t)\leqq Ce^{-\frac{c}{2}t}\delta^{\frac{\theta}{2}}$$

for every $t>0$. We change the reference set from $\Sigma=\partial G$ to $\partial F_\infty$ and write $\partial F_t=\{x+\tilde h(x,t)\nu_{F_\infty}(x) : x\in\partial F_\infty\}$. Then by inequality (5.2), with $\partial G$ replaced by $\partial F_\infty$, and by the above inequality we have

$$\|\tilde h(\cdot,t)\|_{L^2(\partial F_\infty)}\leqq Ce^{-\frac{c}{4}t}\delta^{\frac{\theta}{4}}.$$

Moreover, since $\|h(\cdot,t)\|_{H^3(\Sigma)}\leqq C\sigma^\theta$ for all $t>0$ then also $\|\tilde h(\cdot,t)\|_{H^3(\partial F_\infty)}\leqq C$ for all $t>0$. By Theorem 4.4 we conclude that $\|\tilde h(\cdot,t)\|_{H^{2k+3}(\partial F_\infty)}\leqq C(k,\sigma)$ for all $t\geqq 1$ and for every $k\in\mathbb{N}$. Thus we deduce by interpolation that

$$\|\tilde h(\cdot,t)\|_{C^k(\partial F_\infty)}\leqq C_k e^{-c_k t}\qquad\text{for all } t\geqq 1$$

for some constants $c_k>0$ and $C_k>1$ depending on $k$ and $K_0$.

To conclude the proof, for every $t\in[0,+\infty]$ denote by $(\Gamma_{F_t,i})_{i=1,\dots,m}$ the connected components of $\partial F_t$, numbered according to (2.19). Denote also by $F_{t,i}$ the bounded open set enclosed by $\Gamma_{F_t,i}$ and recall that the flow preserves the volume of each $F_{t,i}$. Indeed,

$$\frac{d}{\mathrm{d}s}|F_{t+s,i}|_{|s=0}=\int_{\Gamma_{F_t,i}}V_t\,\mathrm{d}\mathcal{H}^2=\int_{\Gamma_{F_t,i}}\Delta R_t\,\mathrm{d}\mathcal{H}^2=0.$$

Thus, recalling (5.6) and Proposition 2.13, we may conclude that $F_\infty$ is the unique stationary set in $\mathfrak{h}^3_{\sigma_1}(\partial G)$ such that $|F_{\infty,i}|=|F_{0,i}|$ for $i=1,\dots,m$.

□

## 6. Evolution of Epitaxially Strained Elastic Films

In this section we briefly describe how our main results read in the context of evolving periodic graphs.

In this framework, given a (sufficiently regular) non-negative function $h:\mathbb{R}^2\to[0,+\infty)$, 1-periodic with respect to both variables $x_1,x_2$, the free energy associated with it reads as

$$\mathcal{J}(h):=\int_{\Omega_h}Q(E(u_h))\,\mathrm{d}x+\mathcal{H}^2(\Gamma_h),\tag{6.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^2$, $\Gamma_h$, $\Omega_h$ denote the graph and the subgraph of $h$, respectively, over the periodic cell, that is,

$$\Omega_h := \{(x_1, x_2, x_3) \in (0,1)^2 \times \mathbb{R} : 0 < x_3 < h(x_1, x_2)\},$$
$$\Gamma_h := \{(x_1, x_2, x_3) \in (0,1)^2 \times \mathbb{R} : x_3 = h(x_1, x_2)\},$$

and $u_h$ is the elastic equilibrium in $\Omega_h$, namely the solution of the elliptic system

$$\begin{cases} \operatorname{div} \mathbb{C}E(u_h) = 0 & \text{in } \Omega_h, \\ \mathbb{C}E(u_h)[\nu_{\Omega_h}] = 0 & \text{on } \Gamma_h, \\ Du_h(\cdot, x_3) & \text{is } 1 - periodic, \\ u(x_1, x_2, 0) = e_0(x_1, x_2, 0), \end{cases} \tag{6.2}$$

for a suitable fixed constant $e_0 \neq 0$. The above energy relates to a variational model for epitaxial growth, see the introduction. Precisely, the graph $\Gamma_h$ describes the (free) profile of the elastic film, which occupies the region $\Omega_h$ and is grown on a (rigid) and much thicker substrate, while the *mismatch strain* constant $e_0$ appearing in the Dirichlet condition for $u_h$ at the interface $\{x_1 = 0\}$ between film and substrate measures the mismatch between the characteristic atomic distances in the lattices of the two materials. In this framework, the (local) minimizers of (6.1) under an area constraint on $\Omega_h$ describe the equilibrium configurations of epitaxially strained elastic films, see [21–23,25] and the references therein.

In the context of periodic graphs, given an initial 1-periodic profile $h_0 \in H^3_{loc}(\mathbb{R}^2)$ (in short $h_0 \in H^3_{\mathrm{per}}\big((0,1)^2\big)$), we look for a local-in-time solution $h(\cdot, t)$ of the following problem:

$$\begin{cases} \frac{1}{J_t}\partial_t h = \Delta_{\Gamma_t}(H_t + Q(E(u_t))) & \text{on } \Gamma_t \text{ and for all } t \in (0,T), \\ h(\cdot, t) \text{ is 1-periodic} & \text{for all } t \in (0,T), \\ h(\cdot, 0) = h_0, \end{cases} \tag{6.3}$$

where $J_t := \sqrt{1 + |Dh(\cdot, t)|^2}$, $u_t$ stands for the solution of (6.2), with $\Omega_{h_t}$ in place of $\Omega_h$, we wrote $\Gamma_t$ instead of $\Gamma_{h_t}$, and $H_t$ denotes the mean curvature of $\Gamma_t$. Note that in the first equation of (6.3) we have $+Q(E(u_t))$ instead of $-Q(E(u_t))$. This is due to the fact that in (6.1) the vector $\nu_{\Omega_h}$ now points outwards with respect to the elastic body.

Although the setting is a bit different from that of the previous sections, the short-time existence theory of Section 4 clearly extends also to the present situation, with the same arguments. In this way we improve upon the results of [23] at least in the case of isotropic surface energy density. We assume that the solution is periodic for all times. If we had uniqueness in unbounded domains for this equation, the periodicity would be implied by the periodic initial conditions.

Also the stability analysis of Section 5 applies without any essential changes, thus showing that strictly stable stationary 1-periodic configurations are exponentially stable in the sense of Theorem 5.1.

A particular class of critical configurations to which our stability theorem applies are the flat configurations, that is, in the case of constants profiles $h \equiv d$, provided that $d > 0$ is sufficiently small. Indeed in [9, Proposition 7.3] it is shown that if $d$ is sufficiently small then the flat configuration $h \equiv d$ is strictly stable for the functional $\mathcal{J}$. Therefore, we may state the following theorem:

**Theorem 6.1.** *There exists $d_0 > 0$ with the following property: Let $d \in (0, d_0)$. Then, there exists $\delta > 0$ such that if*

$$\|h_0 - d\|_{H^3((0,1)^2)} \leqq \delta \quad \text{and} \quad \int_{(0,1)^2} h_0 \, \mathrm{d}x = d,$$

*then the unique solution $h(\cdot, t)$ of (6.3) exists for all $t > 0$ and for every integer $k \geqq 1$ we have*

$$\|h(\cdot, t) - d\|_{C^k([0,1]^2)} \leqq C_k \mathrm{e}^{-c_k t} \quad \text{for all } t > 1$$

*and for suitable positive constants $C_k, c_k$.*

*Acknowledgements.* The research of N.F. has been funded by PRIN Project 2015PA5MP7. The research of V.J. was supported by the Academy of Finland Grant 314227. N.F. and M.M. are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM.

## 7. Appendix: Technical Lemmas

In this appendix we collect a few technical results and we give the proof of Lemma 3.3 and of Proposition 5.3.

**Lemma 7.1.** *Let $\Sigma$ be an $m$-dimensional smooth compact manifold in $\mathbb{R}^n$ and let $k \geqq 1$. If $f, g \in H^k(\Sigma) \cap L^\infty(\Sigma)$, then $fg \in H^k(\Sigma)$ and $\|fg\|_{H^k(\Sigma)} \leqq C\big(\|f\|_{H^k(\Sigma)}\|g\|_{L^\infty(\Sigma)} + \|g\|_{H^k(\Sigma)}\|f\|_{L^\infty(\Sigma)}\big)$. Moreover, if $A \in C^\infty(\mathbb{R})$ then $A(f) \in H^k(\Sigma)$ and $\|A(f)\|_{H^k(\Sigma)} \leqq C(1 + \|f\|_{H^k(\Sigma)})$ where the constant depends on $A$ and on $\|f\|_{L^\infty(\Sigma)}$.*
*If $U \subset \mathbb{R}^m$ is an open set $\Phi : \overline{U} \to \Phi(\overline{U}) \subset \Sigma$ is a diffeomorphism of class $H^k \cap C^1$, $k \geqq 1$, and $f \in H^k(\Phi(U)) \cap C^1(\Phi(\overline{U}))$, then $\|f \circ \Phi\|_{H^k(U)} \leqq C(\|Df\|_\infty, \|D\Phi\|_\infty)(\|f\|_{H^k} + \|\Phi\|_{H^k})$.*
*Moreover, if $k \geqq 3$, $f \in H^{k-1}(\Phi(U))$, then $\|f \circ \Phi\|_{H^{k-1}(U)} \leqq C(\|f\|_\infty, \|D\Phi\|_\infty)$ $(\|f\|_{H^{k-1}} + \|\Phi\|_{H^k})$.*

**Proof.** The first two statements of the lemma are classical, see for instance [43, Propositions 3.7 and 3.9]. The third one can be proven by a similar argument. We leave the details for the reader. □

We now prove Lemma 3.3.

**Proof of Lemma 3.3.** First, recall (3.17) and observe that from the assumption on $h_i$ we have $\sup_{0\leqq t\leqq T}\|h_i(\cdot,t)\|_{C^{1,\alpha}(\Sigma)}\leqq C\delta^{\theta'}$ for a suitable $C>0$ and $\theta'\in(0,1)$. We begin by estimating, for $\varepsilon>0$,

$$\begin{aligned}
&\int_0^T\int_\Sigma |\langle A(x,h_2,\nabla h_2),\nabla^4 h_2\rangle-\langle A(x,h_1,\nabla h_1),\nabla^4 h_1\rangle|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\quad\leqq 2\int_0^T\int_\Sigma |A(x,h_2,\nabla h_2)|^2|\nabla^4 h_2-\nabla^4 h_1|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\qquad+2\int_0^T\int_\Sigma |\nabla^4 h_1|^2|A(x,h_2,\nabla h_2)-A(x,h_1,\nabla h_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t \qquad (7.1)\\
&\quad\leqq \varepsilon\int_0^T\int_\Sigma |\nabla^4 h_2-\nabla^4 h_1|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\qquad+C\int_0^T\int_\Sigma |\nabla^4 h_1|^2(|h_2-h_1|^2+|\nabla h_2-\nabla h_1|^2)\,\mathrm{d}\mathcal{H}^2\mathrm{d}t.
\end{aligned}$$

To estimate the last term, we use the Sobolev inequality and the interpolation Lemma 2.1, and have

$$\begin{aligned}
&\int_0^T\int_\Sigma |\nabla^4 h_2|^2(|h_2-h_1|^2+|\nabla h_2-\nabla h_1|^2)\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\quad\leqq C\int_0^T \|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{W^{1,4}}\|\nabla^4 h_2(\cdot,t)\|^2_{L^4}\,\mathrm{d}t\\
&\quad\leqq C\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{H^2}\int_0^T\|h_2(\cdot,t)\|^{\frac{5}{3}}_{H^5}\|\nabla h_2(\cdot,t)\|^{\frac{1}{3}}_{L^\infty}\,\mathrm{d}t \qquad (7.2)\\
&\quad\leqq C\delta^{\frac{\theta'}{3}}\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{H^2}T^{\frac{1}{6}}\left(\int_0^T\|h_2(\cdot,t)\|^2_{H^5}\,\mathrm{d}t\right)^{\frac{5}{6}}\\
&\quad\leqq C(M_0)T^{\frac{1}{6}}\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|^2_{H^2}.
\end{aligned}$$

Concerning the estimate of,

$$\int_0^T\int_\Sigma |J_1(x,h_2,\nabla h_2,\nabla^2 h_2,\nabla^3 h_2)-J_1(x,h_1,\nabla h_1,\nabla^2 h_1,\nabla^3 h_1)|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t$$

we observe that

$$\begin{aligned}
&\int_0^T\int_\Sigma |\langle B_1(x,h_2,\nabla h_2),\nabla^3 h_2\otimes\nabla^2 h_2\rangle\\
&\qquad-\langle B_1(x,h_1,\nabla h_1),\nabla^3 h_1\otimes\nabla^2 h_1\rangle|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\quad\leqq C\int_0^T\int_\Sigma |B_1(x,h_2,\nabla h_2)-B_1(x,h_1,\nabla h_1)|^2|\nabla^3 h_2\otimes\nabla^2 h_2|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\qquad+C\int_0^T\int_\Sigma |B_1(x,h_1,\nabla h_1)|^2|\nabla^3 h_2-\nabla^3 h_1|^2|\nabla^2 h_2|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^T\int_\Sigma |B_1(x,h_1,\nabla h_1)|^2|\nabla^2 h_2-\nabla^2 h_1|^2|\nabla^3 h_1|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&\leqq C\int_0^T\int_\Sigma (|h_2-h_1|^2+|\nabla h_2-\nabla h_1|^2)|\nabla^3 h_2|^2|\nabla^2 h_2|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&+ C\int_0^T\int_\Sigma |\nabla^2 h_2-\nabla^2 h_1|^2|\nabla^3 h_1|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t\\
&+ C\int_0^T\int_\Sigma |\nabla^3 h_2-\nabla^3 h_1|^2|\nabla^2 h_2|^2\,\mathrm{d}\mathcal{H}^2\mathrm{d}t =: I_1+I_2+I_3.
\end{aligned}
$$

By a simple interpolation argument, we have

$$
\begin{aligned}
I_3 &\leqq \int_0^T \|\nabla^3 h_2-\nabla^3 h_1\|_{L^4}^2\|\nabla^2 h_2\|_{L^4}^2\,\mathrm{d}t\\
&\leqq CM_0\int_0^T \|h_1-h_2\|_{H^4}^{\frac{3}{2}}\|\nabla^2 h_2-\nabla^2 h_1\|_{L^2}^{\frac{1}{2}}\\
&\leqq \varepsilon\int_0^T \|\nabla^4 h_2-\nabla^4 h_1\|_{L^2}^2\,\mathrm{d}t\\
&\quad + C_\varepsilon(M_0)T\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &\leqq \int_0^T \|\nabla^2 h_2-\nabla^2 h_1\|_{L^4}^2\|\nabla^3 h_1\|_{L^4}^2\,\mathrm{d}t\\
&\leqq C\int_0^T \|h_2-h_1\|_{H^4}^{\frac{1}{2}}\|h_2-h_1\|_{H^2}^{\frac{3}{2}}\|h_1\|_{H^5}^{\frac{1}{2}}\|\nabla^3 h_1\|_{L^2}^{\frac{3}{2}}\,\mathrm{d}t\\
&\leqq \varepsilon\int_0^T \|\nabla^4 h_2-\nabla^4 h_1\|_{L^2}^2\,\mathrm{d}t\\
&\quad + C_\varepsilon(M_0)\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2}^2\int_0^T 1+\|h_1\|_{H^5}^{\frac{2}{3}}\,\mathrm{d}t\\
&\leqq \varepsilon\int_0^T \|\nabla^4 h_2-\nabla^4 h_1\|_{L^2}^2\,\mathrm{d}t\\
&\quad + C_\varepsilon(M_0)T^{\frac{2}{3}}\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2}^2.
\end{aligned}
$$

Finally, arguing similarly as above,

$$
\begin{aligned}
I_1 &\leqq \int_0^T \|h_1-h_2\|_{W^{1,6}}^2\|\nabla^3 h_2\|_{L^6}^2\|\nabla^2 h_2\|_{L^6}^2\,\mathrm{d}t\\
&\leqq CM_0\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2}^2\int_0^T \|h_2\|_{H^5}^{\frac{2}{3}}\|\nabla^3 h_2\|_{L^2}^{\frac{4}{3}}\,\mathrm{d}t\\
&\leqq C(M_0)T^{\frac{2}{3}}\sup_{0\leqq t\leqq T}\|h_2(\cdot,t)-h_1(\cdot,t)\|_{H^2}^2.
\end{aligned}
$$

Since the difference of the remaining terms in $J_1$ can be treated in a similar (in fact easier) way, we conclude that

$$\begin{aligned}\int_0^T \int_\Sigma &|J_1(x, h_2, \nabla h_2, \nabla^2 h_2, \nabla^3 h_2 \\ &- J_1(x, h_1, \nabla h_1, \nabla^2 h_1, \nabla^3 h_1))|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t \\ &\leqq \varepsilon \int_0^T \|\nabla^4 h_2(\cdot, t) - \nabla^4 h_1(\cdot, t)\|_2^2 \, \mathrm{d}t \\ &+ C_\varepsilon(M_0) T^\theta \sup_{0 \leqq t \leqq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 . \end{aligned} \tag{7.3}$$

We are left to show that

$$\begin{aligned}\int_0^T \int_\Sigma &|J_2(x, h_2, \nabla h_2, \nabla^2 h_2, \nabla f, \nabla^2 f \\ &- J_2(x, h_1, \nabla h_1, \nabla^2 h_1, \nabla f, \nabla^2 f))|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t \\ &\leqq \varepsilon \int_0^T \|\nabla^4 h_1(\cdot, t) - \nabla^4 h_2(\cdot, t)\|_2^2 \, \mathrm{d}t \\ &+ C_\varepsilon(M_0, K_0) T^\theta \sup_{0 \leqq t \leqq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 . \end{aligned} \tag{7.4}$$

As before we only prove the estimate for

$$I_4 := \int_0^T \int_\Sigma |\langle A_1(x, h_2, \nabla h_2) - A_1(x, h_1, \nabla h_1), \nabla^2 f \rangle|^2 \, \mathrm{d}\mathcal{H}^2 \mathrm{d}t,$$

the other terms being similar (or easier). Using Lemma 2.1 once again we have

$$\begin{aligned} I_4 &\leqq \int_0^T \|h_2 - h_1\|_{W^{1,4}}^2 \|\nabla^2 f\|_{L^4}^2 \, \mathrm{d}t \\ &\leqq C \sup_{0 \leqq t \leqq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T \|\nabla^3 f\|_2^{\frac{3}{2}} \|f\|_{L^\infty}^{\frac{1}{2}} \, \mathrm{d}t \\ &\leqq C K_0^{\frac{1}{2}} \sup_{0 \leqq t \leqq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 \int_0^T \|\nabla^3 f\|_{L^2}^{\frac{3}{2}} \, \mathrm{d}t \\ &\leqq C K_0^{\frac{5}{4}} T^{\frac{1}{4}} \sup_{0 \leqq t \leqq T} \|h_2(\cdot, t) - h_1(\cdot, t)\|_{H^2}^2 . \end{aligned}$$

The conclusion then follows by collecting (7.1)–(7.4). □

Finally we give the proof of Proposition 5.3.

**Proof of Proposition 5.3.** The proof is similar to the proof of [24, Lemma 3.3]. For this reason we adopt the same notation as there and extend every function on $\partial F_t$ using the signed distance function $d_{F_t}$. In particular, the normal $\nu_t = \nu_{F_t}$, the second fundamental form $B_t = B_{F_t}$ and the mean curvature $H_t = H_{F_t}$ are extended to a tubular neighborhood of $\partial F_t$. Recall that $D_\tau$ denotes the tangential gradient defined in (2.9) and $\operatorname{div}_\tau$ denotes the tangential divergence, which is defined as $\operatorname{div}_\tau X = \operatorname{div} X - (DX\nu_t)\cdot\nu_t$. The Laplace–Beltrami operator on $F_t$ can be written as $\Delta v = \operatorname{div}_\tau(D_\tau v)$, the second fundamental form as $B_t = D_\tau \nu_t$ and the mean curvature as $H_t = \operatorname{div}_\tau \nu_t$.

The regularity properties of $h$ stated in Theorem 4.4 imply that for every integer $k \geqq 1$ $\nabla^k h \in H^1_{loc}(0, T; L^2(\Sigma))$. Therefore, in what follows all the time derivatives are well defined almost everywhere. In turn, this allows us to differentiate $u_t := u_{F_t}$ with respect to time. More precisely, setting $\dot{u}_t := \frac{\partial u_{t+s}}{\partial s}\big|_{s=0}$, we can argue as in [9, Theorem 4.1] to conclude that $\dot{u}$ solves that

$$\int_{\Omega\setminus F_t} \mathbb{C}E(\dot{u}_t) : E(\varphi)\,\mathrm{d}x = -\int_{\partial F_t} \operatorname{div}_\tau(\Delta R_t\,\mathbb{C}E(u_t))\cdot\varphi\,\mathrm{d}\mathcal{H}^2 \tag{7.5}$$

for all $\varphi \in H^1(\Omega\setminus F_t;\mathbb{R}^3)$ such that $\varphi = 0$ on $\partial_D\Omega$. Note also that $\dot{u}_t = 0$ on $\partial_D\Omega$.

Let us fix time $t > 0$. To continue we observe that, by redefining the velocity field $X$ assosiated with the flow (4.1) if needed (in a time interval centered at $t$), we may assume that $X_t$ has only a normal component on $\partial F_t$; that is,

$$X_t = (X_t\cdot\nu_t)\nu_t = (\Delta R_t)\nu_t \qquad \text{on } \partial F_t.$$

Since we extended all the geometric quantities by means of the gradient of the signed distance from $F_t$ we have the following equality (see [13]):

$$\dot{\nu}_t = -D_\tau(X_t\cdot\nu_t) = -D_\tau(\Delta R_t) \qquad \text{on } \partial F_t.$$

This implies (see the proof of [1, Eq. (5.15)])

$$\dot{H}_t := \frac{\partial}{\partial s}H_{t+s}\big|_{s=0} = -\Delta^2 R_t \qquad \text{on } \partial F_t. \tag{7.6}$$

Moreover we have (see [13])

$$\partial_{\nu_t} H_t = -|B_t|^2 \qquad \text{on } \partial F_t. \tag{7.7}$$

Denoting by $D_{\tau_{t+s}}$ the tangential gradient on $\partial F_{t+s}$ and by $J_\tau\Phi_s$ the tangential Jacobian of $\Phi_s$, we have

$$\begin{aligned}
&\frac{d}{\mathrm{d}s}\left(\frac{1}{2}\int_{\partial F_{t+s}} |D_{\tau_{t+s}} R_{t+s}|^2\,\mathrm{d}\mathcal{H}^2\right)\Big|_{s=0}\\
&\quad= \frac{d}{\mathrm{d}s}\left(\frac{1}{2}\int_{\partial F_t} (|D_{\tau_{t+s}} R_{t+s}|^2\circ\Phi_s)\,J_\tau\Phi_s\,\mathrm{d}\mathcal{H}^2\right)\Big|_{s=0}\\
&\quad= \frac{1}{2}\int_{\partial F_t} |D_\tau R_t|^2 \operatorname{div}_\tau(\Delta R_t\,\nu_t)\,\mathrm{d}\mathcal{H}^2\\
&\qquad+ \int_{\partial F_t} D_\tau R_t\cdot\frac{\partial}{\partial s}\left(D_{\tau_{t+s}} R_{t+s}\circ\Phi_s\right)\Big|_{s=0}\mathrm{d}\mathcal{H}^2\\
&\quad= \frac{1}{2}\int_{\partial F_t} H_t|D_\tau R_t|^2\Delta R_t\,\mathrm{d}\mathcal{H}^2\\
&\qquad+ \int_{\partial F_t} D_\tau R_t\cdot\frac{\partial}{\partial s}\left(D_{\tau_{t+s}} R_{t+s}\circ\Phi_s\right)\Big|_{s=0}\mathrm{d}\mathcal{H}^2
\end{aligned} \tag{7.8}$$

We write the last term as

$$D_{\tau_{t+s}} R_{t+s} \circ \Phi_s = \big[I - \nu_{t+s} \circ \Phi_s \otimes \nu_{t+s} \circ \Phi_s\big] DR_{t+s} \circ \Phi_s$$

and get (recall $\dot{\Phi} = X_t = (\Delta R_t)\nu_t$)

$$\begin{aligned}\frac{\partial}{\partial s}\big(D_{\tau_{t+s}} R_{t+s} \circ \Phi_s\big)\Big|_{s=0} &= [I - \nu_t \otimes \nu_t]\,(D\dot{R}_t + D^2 R_t X_t) \\ &\quad + (-\dot{\nu}_t \otimes \nu_t - \nu_t \otimes \dot{\nu}_t) DR_t \\ &= D_\tau \dot{R}_t + \Delta R_t \Big((I - \nu_t \otimes \nu_t) D^2 R_t\Big)[\nu_t] \\ &\quad + (DR_t \cdot \nu_t)\, D_\tau \Delta R_t - (DR_t \cdot \dot{\nu}_t)\nu_t.\end{aligned}$$

Note that $D_\tau(DR_t \cdot \nu_t) = B_t D_\tau R_t + \big((I - \nu_t \otimes \nu_t) D^2 R_t\big)[\nu_t]$. Thus we have

$$\begin{aligned}D_\tau R_t \cdot \frac{\partial}{\partial s}\big(D_{\tau_{t+s}} R_{t+s} \circ \Phi_s\big)\Big|_{s=0} &= (D_\tau R_t \cdot D_\tau \dot{R}_t) \\ &\quad - \Delta R_t (B_t[D_\tau R, D_\tau R_t]) \\ &\quad + \Delta R_t \big(D_\tau R \cdot D_\tau(DR_t \cdot \nu_t)\big) \\ &\quad + (D_\tau R_t \cdot D_\tau \Delta R_t)\,(DR_t \cdot \nu_t).\end{aligned}$$

Therefore by integrating by parts the first and the third terms we obtain

$$\begin{aligned}&\int_{\partial F_t} D_\tau R_t \cdot \frac{\partial}{\partial s}\big(D_{\tau_{t+s}} R_{t+s} \circ \Phi_s\big)\Big|_{s=0} \,\mathrm{d}\mathcal{H}^2 \\ &= \int_{\partial F_t} (D_\tau R_t \cdot D_\tau \dot{R}_t) - \Delta R_t \big(B_t[D_\tau R, D_\tau R_t]\big)\,\mathrm{d}\mathcal{H}^2 \\ &\quad + \int_{\partial F_t} \Delta R_t \big(D_\tau R \cdot D_\tau(DR_t \cdot \nu_t)\big) + (D_\tau R_t \cdot D_\tau \Delta R_t)\,(DR_t \cdot \nu_t)\,\mathrm{d}\mathcal{H}^2 \\ &= \int_{\partial F_t} -\Delta R_t\, \dot{R}_t - \Delta R_t \big(B_t[D_\tau R, D_\tau R_t]\big)\,\mathrm{d}\mathcal{H}^2 \\ &\quad + \int_{\partial F_t} -(DR_t \cdot \nu_t)\, \operatorname{div}_\tau(\Delta R_t D_\tau R_t) + (D_\tau R_t \cdot D_\tau \Delta R_t)\,(DR_t \cdot \nu_t)\,\mathrm{d}\mathcal{H}^2 \\ &= \int_{\partial F_t} -\Delta R_t\, \dot{R}_t - (DR_t \cdot \nu_t)\,(\Delta R_t)^2 - \Delta R_t \big(B_t[D_\tau R, D_\tau R_t]\big)\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

Let us denote $u_t = u_{F_t}$ and $\dot{u}_t = \frac{\partial}{\partial t} u_t$. By (7.6) it holds that

$$\dot{R}_t = \dot{H}_t + \frac{\partial}{\partial t} Q(E(u_t)) = -\Delta^2 R_t + \mathbb{C}E(\dot{u}_t) : E(u_t),$$

and by (7.7) we have

$$(DR_t, \nu_t) = \partial_{\nu_t} H_t + \partial_{\nu_t} Q(E(u_t)) = -|B_t|^2 + \partial_{\nu_t} Q(E(u_t)).$$

Therefore we get

$$\begin{aligned}\int_{\partial F_t} & D_\tau R_t \cdot \frac{\partial}{\partial s}\left(D_{\tau_{t+s}} R_{t+s} \circ \Phi_s\right)\Big|_{s=0} \mathrm{d}\mathcal{H}^2 \\ &= \int_{\partial F_t} \Delta R_t\, \Delta^2 R_t - \mathbb{C}E(\dot{u}_t) : E(u_t)\Delta R_t \,\mathrm{d}\mathcal{H}^2 \\ &\quad + \int_{\partial F_t} |B_t|^2 (\Delta R_t)^2 - \partial_{\nu_t} Q(E(u_t))\,(\Delta R_t)^2 \\ &\quad - \Delta R_t \big(B_t[D_\tau R, D_\tau R_t]\big)\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

Observe now that using the second equation in (2.12) and (7.5) we have

$$\begin{aligned}\int_{\partial F_t} \mathbb{C}E(\dot{u}_t) : E(u_t)\Delta R_t \,\mathrm{d}\mathcal{H}^2 &= \int_{\partial F_t} \mathbb{C}E(u_t) : D(\dot{u}_t)\Delta R_t \,\mathrm{d}\mathcal{H}^2 \\ &= \int_{\partial F_t} \mathbb{C}E(u_t) : D_\tau(\dot{u}_t)\Delta R_t \,\mathrm{d}\mathcal{H}^2 \\ &= -\int_{\partial F_t} \mathrm{div}_\tau(\Delta R_t \mathbb{C}E(u_t)) \cdot \dot{u}_t \\ &= 2\int_{\Omega\setminus F_t} Q(E(\dot{u}_t))\,\mathrm{d}x.\end{aligned}$$

Collecting the previous three identities we then get

$$\begin{aligned}\int_{\partial F_t} & D_\tau R_t \cdot \frac{\partial}{\partial s}\left(D_{\tau_{t+s}} R_{t+s} \circ \Phi_s\right)\Big|_{s=0} \mathrm{d}\mathcal{H}^2 \\ &= -\int_{\partial F_t} |\nabla \Delta_\tau R_t|^2 + 2Q(E(\dot{u}_t))\Delta R_t \,\mathrm{d}\mathcal{H}^2 \\ &\quad + \int_{\partial F_t} |B|^2 (\Delta R_t)^2 - \partial_{\nu_t} Q(E(u_t))\,(\Delta R_t)^2 - B_t[\nabla R_t, \nabla R_t]\,\Delta R_t \,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

We notice that the first four terms coincide with $-\partial^2 J(F_t)[\Delta R_t]$ (see (2.21)). Thus, combining the last identity with (7.8), we obtain (5.1). □

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Nicola Fusco
Dipartimento di Matematica e Applicazioni,
Università di Napoli Federico II,
Naples,
Italy.
e-mail: n.fusco@unina.it

and

Vesa Julin
Matematiikan ja Tilastotieteen Laitos,
Jyväskylän Yliopisto,
Jyväskylä,
Finland.
e-mail: vesa.julin@jyu.fi

and

Massimiliano Morini
Dipartimento di Scienze Matematiche Fisiche e Informatiche,
Università di Parma,
Parma,
Italy.
e-mail: massimiliano.morini@unipr.it

(*Received April 12, 2019 / Accepted April 24, 2020*)