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KAM Tori are No More than Sticky

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Abstract

When a Gevrey smooth perturbation is applied to a quasi-convex integrable Hamiltonian, it is known that the KAM invariant tori that survive are "sticky", that is doubly exponentially stable. We show by examples the optimality of this effective stability.

Contents

1. Introduction

We are interested in effective stability around invariant quasi-periodic tori of nearly integrable analytic or Gevrey regular Hamiltonian systems. Under generic non degeneracy assumptions on the integrable Hamiltonian, KAM theory (after Kolmogorv Arnold Moser) guarantees the existence of a large measure set of invariant quasi-periodic tori for the perturbed systems. The invariant tori given by KAM theory have Diophantine frequency vectors. To study the diffusion rate of orbits that start near these invariant tori, an important tool is the Birkhoff Normal Forms (or BNF) at an invariant torus, which introduces action-angle coordinates in which the system in small neighborhoods of an invariant Diophantine torus becomes integrable up to arbitrary high degrees in the Taylor series of the Hamiltonian (see for example [1] or [15]).

Exploiting the Diophantine property of the frequency vector of the invariant torus, it is possible to collect estimates in the successive BNFs and establish exponential stability of the torus, in the sense that nearby solutions remain close to the invariant torus for an interval of time which is exponentially large with respect to some power of the inverse of the distance r to the torus, a power that depends only on the Diophantine exponent τ of the torus in the case of real analytic Hamiltonians, and that involves additionally the degree of Gevrey smoothness in the case of Gevrey smooth Hamiltonians [13,14].

Combining BNF estimates with Nekhoroshev theory, Giorgilli and Morbidelli proved in [12] that for integrable Hamiltonians with a quasi-convex Hessian, the KAM tori of an analytic perturbation of the Hamiltonian are doubly exponentially stable: the exponential stability time $\exp\big(r^{-1/(\tau+1)}\big)$ is promoted to $\exp\big(\exp\big(r^{-1/(\tau+1)}\big)\big)$. Invariant quasi-periodic tori with this strong form of effective stability are termed *sticky*.

Stickiness of the invariant tori was later extended in [3] to a residual and prevalent set of integrable Hamiltonians and to the Gevrey category. It was proved there that generically, both in a topological and measure-theoretical sense, an invariant Lagrangian Diophantine torus of a Hamiltonian system is doubly exponentially stable. Also, for a residual and prevalent set of integrable Hamiltonians, for any small perturbation in Gevrey class, there is a set of almost full Lebesgue measure of KAM tori which are doubly exponentially stable.

Our aim here is to give examples showing that doubly exponential stability cannot in general be strengthened. Loosely stated, our main result is the following:

Theorem. *For arbitrary $N \geqq 3$, there exist quasi-convex Hamiltonian systems in N degrees of freedom that can be perturbed in the Gevrey smooth category so that most of the invariant tori of the perturbed system are no more than doubly exponentially stable.*

The exact statements will be given in Section 3. The diffusion mechanism we will use in our constructions is the so called Herman synchronized diffusion, which first appeared in [10] where the speed in Arnol'd diffusion is estimated for a class of nearly integrable system. In [10], completely integrable systems with twist are considered and it is shown that it is possible to construct perturbations of size ε in Gevrey class that have orbits diffusing in action at an exponential rate in inverse powers of ε. The diffusion rate is shown to be almost optimal due to the Nekhoroshev effective stability theory.

Our setting here is quite different, in this that the perturbative parameter is not an extra parameter ε but the *action variable itself* when viewed as the distance r of the diffusive orbit from the invariant torus. In this "singular perturbative setting", the nature of the construction is in fact expected to be different from [10] since the diffusion rate is at best doubly exponential in an inverse power of r, compared to the simple exponential one can achieve in the Arnol'd diffusion problem treated in [10].

The new difficulties that arise in the singular perturbative setting, as well as the novel constructions to overcome them will be commented in the next section where the heuristics of our construction are described in detail.

2. Description of the Construction

The construction of the diffusive flows is obtained by suspension from a perturbative construction in the discrete setting of symplectic maps on $M = (\mathbb{T} \times \mathbb{R})^n \simeq \mathbb{T}^n \times \mathbb{R}^n$, where $\mathbb{T}$ denotes the torus $\mathbb{R}/\mathbb{Z}$ and $n \geqq 2$.

We now explain the main ideas in the discrete construction in the case $n = 2$. We concentrate on the diffusion rate from the neighborhood of a single invariant torus. We will be dealing with perturbation of a product of two twist maps of the annulus $\mathbb{T} \times \mathbb{R}$, denote them by F_0 and G_0, $F_0(\theta_1, r_1) = (\theta_1 + \omega_1 + r_1 + \mathbb{Z}, r_1)$, $G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2 + \mathbb{Z}, r_2)$. Set $T_0 = F_0 \times G_0 \colon\ M \circlearrowleft$. Observe that T_0 has an invariant torus $\mathcal{T}_0 = \mathbb{T}^2 \times \{(0, 0)\}$, on which the restricted dynamics is the translation of vector $\omega = (\omega_1, \omega_2)$.

Let us explain how to perturb T_0 into a map T that is tangent to T_0 at $\mathcal{T}_0$ and that has pieces of orbits that diffuse away from a neighborhood of $\mathcal{T}_0$ at a doubly exponentially small speed. More precisely, we obtain a sequence $\rho_n \to 0$, points z_n such that $\mathrm{dist}(z_n, \mathcal{T}_0) < \rho_n$ and times Θ_n that are doubly exponentially large in $1/\rho_n$, such that $\mathrm{dist}(T^{\Theta_n}(z_n), \mathcal{T}_0)$ and $\mathrm{dist}(T^{-\Theta_n}(z_n), \mathcal{T}_0)$ are both doubly exponentially large in $1/\rho_n$. It will appear clearly from our diffusion mechanism that drifting away from $\mathcal{T}_0$ by an amount ρ_n, or by an amount that is doubly exponentially large in $1/\rho_n$, both require a doubly exponentially large time.

Herman Synchronized Diffusion

The diffusion mechanism we will use is the Herman synchronized diffusion that first appeared in [10]. Let us explain in some words what is the synchronized diffusion. It is based on the following mechanism of coupling of two twist maps of the annulus (the second one being integrable with linear twist): at exactly one point p of a well chosen periodic orbit of period q of the first twist map in $M_1 = \mathbb{T} \times \mathbb{R}$, the coupling consists of pushing the orbits in the second annulus up in $M_2 = \mathbb{T} \times \mathbb{R}$ on some fixed vertical Δ by an amount $1/q$ that sends an invariant curve whose rotation number is a multiple of $1/q$ exactly to another one having the same property (due to the linear twist property).

The dynamics of the q^{th} iterate of the coupled map on the line $\{p\} \times \Delta \subset M_1 \times M_2$ will thus drift at a linear speed : after q^2 iterates the point will have moved by 1 in the second action coordinate r_2, and after q^3 it will have moved by q. The diffusing orbits obtained this way are bi-asymptotic to infinity: their r_2-coordinates travel from $-\infty$ to $+\infty$ at average speed $1/q^2$.

For this mechanism to be implemented with a Gevrey regular small coupling of the two twist maps, it is necessary that the periodic point p be isolated from the rest of the points on its orbit by a distance σ that is greater than the inverse of some power of $\ln q$, since $1/q$ is the translation amount required from the coupling that must be exclusively localized around p. We call such periodic points "logarithmically" isolated.

Optimal Rates in Arnol'd Diffusion of [10]

In [10], only one periodic point is sufficient to have estimates on the Arnol'd diffusion rate in the nearly integrable system. In fact, in [10], a completely integrable twist map of the annulus such as F_0 is first perturbed to create a hyperbolic saddle point with a saddle connection (a pendulum). Near the separatrix of the pendulum, one can find periodic orbits of arbitrary high period q and isolation σ that is determined by the hyperbolicity of the saddle point. More precisely, with an ε perturbation of the integrable twist, the periodic orbits near the separatrix will then have an isolation of order $\varepsilon^{1/2}$ and choosing q exponentially large in the inverse of ε allows to use the coupling mechanism with a second completely integrable linear twist to obtain diffusive orbits at exponential rate in the inverse of ε.

Doubly Exponential Diffusion Rates in the Singular Perturbative Setting

In our singular perturbative setting, the main obstacles when one attempts to apply the synchronized diffusion mechanism are threefold : (1) the diffusion rate must be calculated from arbitrary small neighborhoods of the invariant torus $\mathcal{T}_0$, hence many perturbations and many diffusive orbits may enter into play as opposed to the single orbit of [10]; (2) each perturbation must not affect $\mathcal{T}_0$ and must allow for further perturbations; (3) the Diophantine property on the frequency of $\mathcal{T}_0$ imposes, due to averaging, strong restrictions on the period and the isolation properties of the periodic points that come near the invariant torus.

The main step to prepare for the coupling construction is to be able to perturb F_0 in order to get an annulus map F on the first factor $M_1 = \mathbb{T} \times \mathbb{R}$ that is tangent to F_0 at the circle $r = 0$ (we omit the subscript 1 for r_1 in this paragraph) and that has a sequence of periodic points p_n at distance r_n from the circle $r = 0$ and that are σ_n isolated from their orbits. Since we will work with perturbations of F_0 that are compactly supported away from $r = 0$, we cannot expect larger isolation σ_n than an exponentially small quantity in the inverse of r_n, and the precise exponent involved in this exponential is dictated by the Gevrey regularity α only. According to the above description of how the synchronized diffusion mechanism functions, the period q_n of the point p_n, that will also determine the order of the diffusion rate, should not be taken smaller than an exponential in the inverse of σ_n. Hence, the double exponential in $\frac{1}{r_n}$!!

Let us now suppose that a map F is constructed with such a sequence $p_n \in M_1$, and let us show how to obtain the coupling with the map G_0 which lives on the other factor $M_2 = \mathbb{T} \times \mathbb{R}$. The main idea is to couple F and G_0 separately at each periodic orbit with compactly supported Gevrey regular coupling functions. Indeed, while performing locally the couplings around the product of the orbit of p_n with M_2, we keep the direct product structure of F with G_0 in the products of smaller neighborhoods of the circle $\mathbb{T} \times \{0\} \subset M_1$ with M_2. Thus, the couplings that yield diffusive orbits involving the successive points p_n are done inductively without affecting each other.

How to Perturb the First Factor to Get a Sequence of Isolated Periodic Points

We turn now to the perturbative techniques that allow us to obtain F. We put together two tools. The first one allows us to perturb a periodic circular rotation of period P/Q while fine-tuning the rotation number so as to create circle diffeomorphisms with a σ-isolated periodic point p of arbitrary large period q, where σ is exponentially small in Q for large Q but otherwise independent of q. In particular, we can choose q exponentially large in $1/\sigma$ (doubly exponentially large in Q). The second tool is a trick due to M. Herman that allows us to embed the circle dynamics thus obtained inside the phase portrait of a perturbation of a linear twist map of the annulus M_1. In fact, the periodic map will appear in the neighborhood of the circle of period P/Q of the linear twist. The coupling mechanism will then yield orbits that diffuse at speed $1/q^2$ in $M_1 \times M_2$. Of course, to conclude we have to require that $1/Q$ be larger than the distance $r_Q = |\omega_1 - P/Q|$ from the circle $\mathbb{T} \times \{0\}$ to the periodic orbit of the linear twist near which the isolated periodic point p is embedded. At that stage, we could assume ω_1 irrational or Diophantine and then impose that $1/Q$ be of order $\sqrt{r_Q}$ or larger, depending on the Diophantine exponent of ω_1, with the hope to refine the estimates on σ and thus on the diffusing time. However, since we want to embed, using Herman's trick, the isolated periodic point in M_1 at distance r *without affecting* the circle $\mathbb{T} \times \{0\}$, we must accept the exponential smallness of the isolation parameter in M_1 to be dictated in the first place by the Gevrey regularity of our compactly supported perturbations, thus absorbing the potential gain stemming from arithmetics. This means that by our technique we cannot tackle the problem of matching the diffusion rate with the doubly exponential stability lower bounds obtained in [3,12], that are of the form $\exp\left(\exp\left(r^{-\frac{1}{\alpha(\tau+1)}}\right)\right)$, where α refers to the Gevrey regularity class and τ is the Diophantine exponent of the translation vector.

To emphasize the role that arithmetics should play in optimizing the diffusion speed we may ask the following question that is similar to the one raised in [6, Question 24] for elliptic fixed points:

Question 1. ive an example of an analytic or Gevrey smooth Hamiltonian that has a non-resonant invariant torus with positive definite twist that is not more than exponentially stable in time.

It follows from [2,3,12], that a super Liouville property must be required on the frequency vector of the invariant torus.

Questions on the Optimality of the Bounds, on Analytic Perturbations and on the Genericity of Doubly Exponential Diffusion

An interesting way to address the question of optimizing the bounds, as well as to aim at analytic constructions, is to look for a single analytic (or Gevrey smooth) perturbation of F_0 that yields a map $\tilde{F}$ that is tangent to F_0 at 0 to some fixed degree, and that has a sequence of periodic orbits p_n with isolation properties related to the arithmetics of ω_1 (for instance, with σ_n of order $e^{r_n^{-1/(\tau'+1)}}$, where τ' is such that ω_1

is not τ'-Diophantine). However, even if such a perturbation of F_0 is possible, it would still be a delicate task to perform an analytic coupling with G_0 since a single analytic intervention to couple the neighborhood of the orbit of any periodic point p_n with the second factor will affect the whole map everywhere and we cannot rely on the nice direct product structure of F with G_0 for further perturbations as we do in the Gevrey category. One should probably resort to the theory of normally hyperbolic invariant manifolds to say that some kind of product structure remains valid at the periodic orbits of the points p_n. Even then, however, the linear character of the twist of the second factor will definitely disappear, which will also bring extra difficulties.

Let us make a last remark concerning the analytic category. In fact, obtaining examples of analytic Hamiltonians having a topologically unstable invariant torus with positive definite twist at the torus is a hard task by itself, let alone the control of the diffusion time that is the object of our investigation here. Real analytic Hamiltonians with unstable invariant tori and elliptic fixed points (with arbitrary frequencies in the case of 4 degrees of freedom) were obtained in [4,5], but these examples do not have positive definite twist.

Finally, besides the analytic question and the question of optimizing the bounds, one can ask whether the upper bounds on the diffusion rates that we obtain in our examples are generic for KAM tori, or for invariant quasi-periodic Diophantine tori in general.

Plan of the Paper

Section 3 contains the main statements for symplectomorphisms and for flows. In Section 4, we state the main inductive step of the construction, that yields a diffusive segment of orbit for a perturbation of $F_0 \times G_0$ linked to one isolated periodic point that will be created on the first factor. In Section 4.2 we explain how the main inductive step is used to result in a diffusive invariant torus. In Section 4.1 we elaborate on this to get simultaneously a large measure set of invariant tori that are diffusive.

Sections 5 and 6 contain the proof of the main inductive step. Section 5 is devoted to the perturbation of F_0 in order to get the map F with isolated periodic points. In Section 5.1, we show how to perturb circular rotations to obtain a periodic orbit with the required isolation estimate and with arbitrary large period. In Section 5.2 we show how this periodic orbit can be imbedded in a perturbation of F_0. Section 6 introduces the coupling lemma of F with G_0 and shows how to use it to get diffusion using the isolated periodic point of F.

In Section 7 we provide the suspension trick that allows to transfer the results from the discrete case of symplectomorphisms to the continuous time context of Hamiltonian flows.

In the Appendix we collect and prove some necessary Gevrey estimates for maps and for flows that are used all along the paper.

3. Statements

3.1. Notations on Diffusive Dynamics and Gevrey Functions

We use the notation

$$E(\nu) = E_{C,\gamma}(\nu) := \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{for } \nu > 0 \tag{3.1}$$

for some choice of $C, \gamma > 0$ that we will explicit later. We will say informally that "$E(\nu)$ is doubly exponentially large for small ν". Notice that

$$E(\nu) \gg E(\lambda\nu)^{\mu} \text{ as } \nu \to 0, \qquad \text{for every } \lambda > 1 \text{and } \mu > 0. \tag{3.2}$$

Definition 3.1. Given a transformation T (or a flow) on a metric space (M, d) and $\nu > 0$, we say that:

- A point z of M is ν*-diffusive* if there exist an initial condition $\hat{z} \in M$ and a positive integer (or real) t such that $d(\hat{z}, z) \leqq \nu$, $t \leqq E(\nu)$ and $d(T^t\hat{z}, z) \geqq E(2\nu)$.
- A subset X of M is ν*-diffusive* if all points in X are ν-diffusive.
- A subset X of M is *diffusive* if there exists a sequence $\nu_n \to 0$ such that X is ν_n-diffusive for each n.

In the latter cases, we also say that T is ν*-diffusive on* X, resp. *diffusive on* X.

Our goal is to construct examples of diffusive dynamics in the context of near-integrable Hamiltonian systems and exact-symplectic maps. The requirement $d(T^t\hat{z}, z) \geqq E(2\nu)$ might seem exaggerated at first look, but, as mentioned earlier, there will be no essential difference in the order of magnitude of the time needed to diffuse by an amount ν or by an amount as large as $E(2\nu)$.

We will also use a variant of the above definition: we say that a point or a set is ν- *diffusive*-$*$ or *diffusive*-$*$ if the corresponding property holds with the function E replaced by

$$E^*(\nu) := E\big(\nu/|\ln\nu|\big) = \exp\big(\nu^{\gamma C\nu^{-\gamma}}\big) \quad \text{for } \nu \in (0, 1). \tag{3.3}$$

Notice that, as $\nu \to 0$, $E(\nu) \ll E^*(\nu) \ll \mathrm{e}^{\mathrm{e}^{C'\nu^{-\gamma'}}}$ for any $\gamma' > \gamma$ and $C' > 0$.

We will deal with Gevrey smooth functions and maps in several real variables. Periodicity may be required with respect to some of these variables, in which case we will consider that each of the corresponding variables is an angle, which lives in

$$\mathbb{T} := \mathbb{R}/\mathbb{Z}.$$

Recall that, given a real $\alpha \geqq 1$, Gevrey-α regularity is defined by the requirement that the partial derivatives exist at all (multi)orders ℓ and are bounded by $CM^{|\ell|}|\ell|!^{\alpha}$ for some C and M; when $\alpha = 1$ this simply means analyticity, but we shall take $\alpha > 1$ throughout the article. Upon fixing a real $L > 0$ which essentially stands for the inverse of the previous M, one can define a Banach algebra $G^{\alpha,L}(K) \subset C^{\infty}(K)$

when K is a Cartesian product of closed Euclidean balls and tori; the elements of $G^{\alpha,L}(K)$ are the "uniformly Gevrey-(α, L)" functions on K. In the non-compact case of a Cartesian product $\mathbb{R}^N \times K$ with K as above, we define

$$G^{\alpha,L}(\mathbb{R}^N \times K) \subset \mathcal{G}^{\alpha,L}(\mathbb{R}^N \times K) \subset C^\infty(\mathbb{R}^N \times K),$$

where the smaller space is a Banach algebra, with norm $\| \,.\, \|_{\alpha,L}$, consisting of uniformly Gevrey-(α, L) functions on $\mathbb{R}^N \times K$, while $\mathcal{G}^{\alpha,L}(\mathbb{R}^N \times K)$ is a complete metric space, with translation-invariant distance $d_{\alpha,L}$, obtained by covering $\mathbb{R}^N$ by an increasing sequence of closed balls and considering the Fréchet space structure accordingly; details are given in Appendix A.1.

3.2. Hamiltonian Flows

Let $n \geqq 2$. We work in $\mathbb{T}^n \times \mathbb{R}^n$, with coordinates $(\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$, or in $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$, with coordinates $(\theta_1, \ldots, \theta_n, \tau, r_1, \ldots, r_n, s)$. We use the standard symplectic structures $\sum_{j=1}^n \mathrm{d}\theta_j \wedge \mathrm{d}r_j$ or $\sum_{j=1}^n \mathrm{d}\theta_j \wedge \mathrm{d}r_j + \mathrm{d}\tau \wedge \mathrm{d}s$, so that it is equivalent to consider a non-autonomous Hamiltonian $h(\theta, r, t)$ on $\mathbb{T}^n \times \mathbb{R}^n$ which depends 1-periodically on the time t or a Hamiltonian of the form $H(\theta, \tau, r, s) = s + h(\theta, r, \tau)$ on $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$. Given an arbitrary $\omega \in \mathbb{R}^n$, we will be interested in non-autonomous 1-periodic perturbations of

$$h_0(r) := (\omega, r) + \tfrac{1}{2}(r, r) \tag{3.4}$$

or, equivalently, in certain autonomous perturbations of the integrable Hamiltonian

$$H_0(r, s) := s + h_0(r), \tag{3.5}$$

for which we denote by $\mathcal{T}_{(r,s)}$ the invariant torus $\mathbb{T}^{n+1} \times \{(r, s)\}$ associated with any $r \in \mathbb{R}^n$ and $s \in \mathbb{R}$ (it carries the quasi-periodic motion $\dot{\theta} = \omega + r$, $\dot{\tau} = 1$).

Theorem 3.1. *Let $\alpha > 1$ and $L > 0$ be real. For any $\varepsilon > 0$ there is $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ such that*

(1) $d_{\alpha,L}(h_0, h) < \varepsilon$,
(2) *the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and, for every $s \in \mathbb{R}$, the torus $\mathcal{T}_{(0,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ is invariant and diffusive for H. Here the exponent implied in (3.1) is $\gamma = \frac{1}{\alpha-1}$.*

Note that if ω is Diophantine then, for any h satisfying (1) of Theorem 3.1, we know from [3,12] that $\mathcal{T}_{(0,s)}$ is doubly exponentially stable (because H_0 is quasi-convex). More precisely, it holds that for any initial condition that is at distance ρ from $\mathcal{T}_{(0,s)}$, the orbit will stay within distance 2ρ from $\mathcal{T}_{(0,s)}$ during time $\exp\big(\exp\big(r^{-\frac{1}{\alpha(\tau+1)}}\big)\big)$, where τ is the Diophantine exponent. Theorem 3.1 shows that we cannot expect in general a stability better than doubly exponential. Observe that we do not recover however the factor $1/(1+\tau)$ in the exponent governing our lower bound on the diffusion time.

Note also that we know from [12] and [3] that, for H such that $d_{\alpha,L}(H_0, H) < \varepsilon$, we have a set of invariant tori that are doubly exponentially stable and fill a set whose complement has measure going to 0 as $\varepsilon \to 0$ (for r in the unit ball for example). Our next result gives an example where most of these tori are no more than doubly exponentially stable.

Theorem 3.2. *Let $\alpha > 1$ and $L > 0$ be real. For any $\varepsilon > 0$, there exist $h \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ and a closed set $X_\varepsilon \subset [0,1]$ with $\mathrm{Leb}(X_\varepsilon) \geqq 1-\varepsilon$, such that*

(1) *$d_{\alpha,L}(h_0, h) < \varepsilon$,*
(2) *the Hamiltonian vector field generated by $H := s + h(\theta, r, \tau)$ is complete and, for each $r \in (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}^{n-1}$ and $s \in \mathbb{R}$, the torus $\mathcal{T}_{(r,s)} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ is invariant and diffusive-$*$ for H. Here the exponent implied in (3.3) is $\gamma = \frac{1}{\alpha-1}$.*

Both theorems will be proved in Section 7 by suspension of analogous results which deal with exact-symplectic map and which we state in the next section. In fact, the union $\mathbb{T}^{n+1} \times (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}^n$ of all the tori mentioned in Theorem 3.2 will be shown to be itself diffusive-$*$ with exponent $\gamma = \frac{1}{\alpha-1}$.

As mentioned in Section 2, the unstable orbits which we will construct to prove our diffusiveness statements are in fact bi-asymptotic to infinity: we will see that their r_2-coordinates travel from $-\infty$ to $+\infty$.

3.3. Exact-Symplectic Maps

Let $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. Recall that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We set $M_1 := \mathbb{T} \times \mathbb{R}$ and $M_2 := \mathbb{T} \times \mathbb{R}$ and define $F_0 \colon M_1 \circlearrowleft$ and $G_0 \colon M_2 \circlearrowleft$ by

$$F_0(\theta_1, r_1) := (\theta_1 + \omega_1 + r_1 + \mathbb{Z}, r_1), \quad G_0(\theta_2, r_2) := (\theta_2 + \omega_2 + r_2 + \mathbb{Z}, r_2), \tag{3.6}$$

and we set

$$T_0 := F_0 \times G_0 \colon M_1 \times M_2 \circlearrowleft . \tag{3.7}$$

Using the identification $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$, we call $\mathcal{T}_0$ the torus $\mathbb{T}^2 \times \{(0,0)\}$. This torus is invariant by T_0 and the restricted dynamics on it is the translation of vector ω. More generally we set

$$\mathcal{T}_{(r_1,r_2)} := \mathbb{T}^2 \times \{(r_1, r_2)\} \quad \text{for any } (r_1, r_2) \in \mathbb{R}^2.$$

We say that a function is *flat* at a point or on a subset, if it vanishes together with all its partial derivatives of all orders there. Our first result for the discrete case is that one can find a Gevrey perturbation of the integrable twist map T_0 that is flat at the torus $\mathcal{T}_0$ (which thus stays invariant) and for which this invariant torus is diffusive.

From now on, when a function H on a symplectic manifold generates a complete Hamiltonian vector field, we denote by Φ^H the time-1 map of the flow (note that $t \mapsto \Phi^{tH}$ is then the continuous time flow generated by H). Thus, endowing $M_1 \times M_2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with the symplectic form $\mathrm{d}\theta_1 \wedge \mathrm{d}r_1 + \mathrm{d}\theta_2 \wedge \mathrm{d}r_2$, we can write

$$T_0 = \Phi^{\omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}(r_1^2 + r_2^2)}. \tag{3.8}$$

Theorem 3.3. *Let $\alpha > 1$ and $L > 0$ be real. For any $\varepsilon > 0$ there exist $u \in G^{\alpha,L}(M_1)$ and $v \in G^{\alpha,L}(M_1 \times M_2)$ such that*

(1) *u and v are flat for $r_1 = 0$,*
(2) $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,
(3) *$\mathcal{T}_0$ is invariant and diffusive for $T := \Phi^v \circ \Phi^u \circ T_0$, with exponent $\gamma = \frac{1}{\alpha-1}$ in (3.1).*

Here, when we write Φ^u with a function $u\colon M_1 \to \mathbb{R}$, we view u as defined on $M_1 \times M_2$ (and independent of the variables θ_2 and r_2) and thus mean $\Phi^u\colon M_1 \times M_2 \circlearrowleft$.

Our next result is a strengthening of Theorem 3.3, in which we find a perturbation that keeps invariant most of the tori of T_0 while insuring that they do become diffusive.

We will see that, since the construction of u and v is completely local, one can insure that in addition they are 1-periodic in r_1. If now $X \subset [0,1]$ is a closed set and if u and v are flat for $r_1 \in X$, then all the tori of the form $\mathcal{T}_{(r_1,r_2)}$, $r_1 \in X + \mathbb{Z}$, that are invariant by T_0, are also invariant by $T = \Phi^v \circ \Phi^u \circ T_0$ and carry the same translation dynamics of vector $\omega + (r_1, r_2)$. If moreover u and v are such that there are diffusive orbits for T on sufficiently dense scales in the neighborhood of the tori $\mathcal{T}_{(r_1,r_2)}$, $r_1 \in X + \mathbb{Z}$, then all these tori will be diffusive. This is the content of the next result, in which we also control the measure of the complement of the invariant tori (in bounded regions).

Theorem 3.4. *Let $\alpha > 1$ and $L > 0$ be real. For any $\varepsilon > 0$ there exist $u \in G^{\alpha,L}(M_1)$ and $v \in G^{\alpha,L}(M_1 \times M_2)$ that are 1-periodic in r_1, and a closed set $X_\varepsilon \subset [0,1]$ with $\mathrm{Leb}(X_\varepsilon) \geqq 1 - \varepsilon$, so that*

(1) *u and v are flat for $r_1 \in X_\varepsilon + \mathbb{Z}$.*
(2) $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$,
(3) *for each $(r_1, r_2) \in (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}$, the torus $\mathcal{T}_{(r_1,r_2)}$ is invariant and diffusive-$*$ for $T := \Phi^v \circ \Phi^u \circ T_0$, with exponent $\gamma = \frac{1}{\alpha-1}$ in (3.3).*

In fact, in (3), the union $\mathbb{T} \times (X_\varepsilon + \mathbb{Z}) \times M_2$ of all these tori will be shown to be itself diffusive-$*$ for T with exponent $\gamma = \frac{1}{\alpha-1}$.

Remark 3.1. As immediate corollaries, we get multidimensional versions of Theorems 3.3 and 3.4 , in $\mathbb{T}^n \times \mathbb{R}^n$ with any $n \geqq 3$, simply by taking direct product of the previous discrete systems with factors of the form $\Phi^{\omega_i r_i + \frac{1}{2} r_i^2}$, $i \geqq 3$: identifying $\mathbb{T}^n \times \mathbb{R}^n$ with $M_1 \times \cdots \times M_n$, where $M_i := \mathbb{T} \times \mathbb{R}$ for each i, and setting

$$T_0 := \Phi^{h_0}\colon\ \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$$

[with the same h_0 as in (3.4)—this is thus a generalization of (3.8)], the statements of Theorems 3.3 and 3.4 hold *verbatim* with this new interpretation of T_0 except that, in condition (3) of Theorem 3.4, $\mathcal{T}_{(r_1,r_2)}$ is to be replaced with $\mathbb{T}^n \times \{r\}$ for arbitrary $r \in (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}^n$ and the functions u and v are to be viewed as functions on $\mathbb{T}^n \times \mathbb{R}^n$.

Remark 3.2. To prove the above discrete-time diffusiveness statements, we will exhibit orbits $(T^k\hat{z})_{k\in\mathbb{Z}}$ which satisfy the first requirement of Definition 3.1, $d(T^t\hat{z}, z) \geqq E(2\nu)$ with a certain positive integer $t \leqq E(\nu)$, for smaller and smaller positive values of ν. We will see that, in fact, they even satisfy $d(T^{jt}\hat{z}, z) \geqq |j|E_{C,\gamma}(2\nu)$ for all $j \in \mathbb{Z}$ and are thus bi-asymptotic to infinity, and more precisely their r_2-coordinates grow linearly by an exact amount $1/q$ after q iterates, where $q = t^{1/3}$ is integer.

4. The Main Building Brick: Localized Diffusive Orbits

We now fix real numbers $\alpha > 1$ and $L > 0$ once for all. We also fix $\omega \in \mathbb{R}^2$ and work with $T_0 = \Phi^{\omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}(r_1^2 + r_2^2)} : M_1 \times M_2 \circlearrowleft$ as in Section 3.3.

To prove Theorems 3.3 and 3.4 , and then the continuous time versions of these, we will use the following building brick, where we use the notation

$$\mathcal{V}(r, \nu) := \mathbb{T} \times (r - \nu, r + \nu) \subset \mathbb{T} \times \mathbb{R} \quad \text{for any } r \in \mathbb{R} \text{ and } \nu > 0.$$

Proposition 4.1. *Let $\gamma := \frac{1}{\alpha - 1}$ and $b := \frac{1}{4}$. There exists $c = c(\alpha, L)$ such that, for any $\nu > 0$ small enough and $\bar{r} \in \mathbb{R}$, there exist $u \in G^{\alpha,L}(M_1)$ and $v \in G^{\alpha,L}(M_1 \times M_2)$ such that*

(1) *$u \equiv 0$ on $\mathcal{V}(\bar{r}, \nu)^c$ and $v \equiv 0$ on $\mathcal{V}(\bar{r}, \nu)^c \times M_2$,*
(2) *$\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqq \mathrm{e}^{-c\nu^{-\gamma}}$,*
(3) *the set $\mathcal{V}(\bar{r}, \nu) \times M_2$ is invariant and $(3\nu, \tau, \tau^b)$-diffusive for $T := \Phi^v \circ \Phi^u \circ T_0$, where $\tau := E_{3c\gamma,\gamma}(\nu)$.*

In Condition (3) of the statement, we have used a refinement of Definition 3.1: we say that a subset X of M is $(\tilde{\nu}, \tau, A)$-*diffusive for T* if, for every point z of X, there exist $\hat{z} \in M$ and t integer such that $d(\hat{z}, z) \leqq \tilde{\nu}$, $t \leqq \tau$ and $d(T^t\hat{z}, z) \geqq A$.

The proof of Proposition 4.1 is in Sections 5 and 6 . (The reader will see that our choice of $b = \frac{1}{4}$ is quite arbitrary; any positive number less than $\frac{1}{3}$ would do.)

4.1. Proof that Proposition 4.1 Implies Theorem 3.3

We are given $\varepsilon > 0$ and, without loss of generality, we can assume that ε is small enough so that we can apply Proposition 4.1 for every $n \geqq 1$ with the following values of ν and $\bar{r}$:

$$\nu_n := (c\gamma)^{1/\gamma} 10^{-n} \varepsilon, \quad \bar{r}^{(n)} := 2\nu_n.$$

We thus obtain $u_n \in G^{\alpha,L}(M_1)$, supported in $\mathcal{V}(\bar{r}^{(n)}, \nu_n)$, and $v_n \in G^{\alpha,L}(M_1 \times M_2)$, supported in $\mathcal{V}(\bar{r}^{(n)}, \nu_n) \times M_2$, which also satisfy Conditions (2) and (3) of Proposition 4.1. Observe that for any two different values of n the supports are disjoint (because $\bar{r}^{(n)} - \nu_n > \bar{r}^{(n+1)} + \nu_{n+1}$).

Since

$$\mathrm{e}^x > x \quad \text{and} \quad \mathrm{e}^{-x} < x^{-1} \quad \text{for all } x > 0, \tag{4.1}$$

we have

$$e^{-c\nu_n^{-\gamma}} = \left(e^{-c\gamma\nu_n^{-\gamma}}\right)^{1/\gamma} < \left(c\gamma\nu_n^{-\gamma}\right)^{-1/\gamma} = (c\gamma)^{-1/\gamma}\nu_n, \tag{4.2}$$

hence the formulas $u := \sum_{n=1}^{\infty} u_n$ and $v := \sum_{n=1}^{\infty} v_n$ define functions $u \in G^{\alpha,L}(M_1)$ and $v \in G^{\alpha,L}(M_1 \times M_2)$ with $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqq \sum e^{-c\nu_n^{-\gamma}} < \varepsilon$. Moreover, since the u_n's, v_n's and all their partial derivatives vanish for $r_1 = 0$, the same is true for u and v. The functions u and v thus satisfy properties (1)–(2) of Theorem 3.3.

We claim that $T = \Phi^v \circ \Phi^u \circ T_0$ also satisfies property (3) of Theorem 3.3. Indeed, the disjointness of the supports implies that T coincides with $\Phi^{v_n} \circ \Phi^{u_n} \circ T_0$ on $\mathcal{V}(\bar{r}^{(n)}, \nu_n) \times M_2$. Now, for each $z \in \mathcal{T}_0$ and $n \geqq 1$, we can pick $\bar{z} \in \mathcal{V}(\bar{r}^{(n)}, \nu_n) \times M_2$ such that $d(\bar{z}, z) < 2\nu_n$, and then, by (3) of Proposition 4.1, we can find $\hat{z} \in \mathcal{V}(\bar{r}^{(n)}, \nu_n) \times M_2$ and $t \leqq \tau_n := E_{3c\gamma,\gamma}(\nu_n)$ such that $d(\hat{z}, \bar{z}) \leqq 3\nu_n$ and $d(T^t\hat{z}, \bar{z}) \geqq \tau_n^b$. We get $d(\hat{z}, z) \leqq 5\nu_n$ and $\tau_n = E_{C,\gamma}(5\nu_n)$ if we use $C := 3c\gamma \cdot 5^\gamma$ in the definition (3.1) of the doubly exponentially large function $E_{C,\gamma}$. Then, $d(T^t\hat{z}, z) > \tau_n^b - 2\nu_n > \frac{1}{2}\tau_n^b \gg E_{C,\gamma}(10\nu_n)$ by (3.2), hence T is $5\nu_n$-diffusive on $\mathcal{T}_0$ for every n large enough.

4.2. *Proof that Proposition 4.1 Implies Theorem 3.4*

We are given $\varepsilon > 0$ and, without loss of generality, we can assume $\varepsilon \leqq 1$. Let $\gamma := \frac{1}{\alpha-1}$ and let c be as in Proposition 4.1.

Here is a definition that we will use from now on: we say that a subset Y of a metric space X is ν-*dense* if, for every $z \in X$, there exists $\bar{z} \in Y$ such that $d(z, \bar{z}) \leqq \nu$.

(**a**) We first define a fast increasing sequence of integers by

$$\begin{aligned} &N_1 := \lceil * \rceil \exp(4\kappa/\varepsilon), \quad N_i := N_{i-1} \\ &\lceil * \rceil \exp\left(\exp\left(\tilde{C}(N_{i-1}\ln N_{i-1})^\gamma\right)\right) \quad \text{for } i \geqq 2, \end{aligned} \tag{4.3}$$

where $\kappa := \max\{1, (c\gamma)^{-1/\gamma}\}$ and $\tilde{C} := \max\{6c\gamma, 1/\gamma\}$. We also set

$$\tilde{\nu}_i := \frac{1}{N_i}, \qquad \nu_i := \frac{\tilde{\nu}_i}{|\ln \tilde{\nu}_i|} = \frac{1}{N_i \ln N_i}, \qquad \tau_i := E_{3c\gamma,\gamma}(\nu_i).$$

According to Lemma B.1 in the appendix, one has

$$2\nu_i N_i = \frac{2}{\ln N_i} \leqq 2^{-2i+1}\varepsilon/\kappa < 1. \tag{4.4}$$

(**b**) We now construct a sequence $(Y_i)_{i\geqq 1}$ of mutually disjoint subsets of $\mathbb{R}/\mathbb{Z}$ such that, for each $i \geqq 1$,

(i) Y_i is a disjoint union of at most N_i open arcs $Y^{(i,j)}$,
(ii) each of these open arcs can be written $Y^{(i,j)} = (r^{(i,j)} - \nu_i, r^{(i,j)} + \nu_i) \bmod \mathbb{Z}$, with $r^{(i,j)} \in [0, 1)$,
(iii) Y_i is $\tilde{\nu}_i$-dense in $(\mathbb{R}/\mathbb{Z}) - \bigsqcup_{1\leqq i'\leqq i-1} Y_{i'}$,

(iv) $(\mathbb{R}/\mathbb{Z}) - \bigsqcup_{1 \leqq i' \leqq i} Y_{i'}$ is a disjoint union of N_i closed arcs of equal length.

To do this, we start with $r^{(1,j)} := \frac{j-1}{N_1}$ for $j = 1, \dots, N_1$ and define the arcs $Y^{(1,j)}$ and the set Y_1 by (i)–(ii) (the disjointness requirement results from $2\nu_1 N_1 < 1$). We then go on by induction and suppose that, for a given $i \geqq 1$, (i)–(iv) hold for $i' = 1, \dots, i$. As a consequence of (i)–(ii) and (4.4), we have

$$\mathrm{Leb}(Y_{i'}) \leqq 2\nu_{i'} N_{i'} \leqq 2^{-i'}\varepsilon/\kappa \leqq 2^{-i'} \quad \text{for } i' = 1, \dots, i. \tag{4.5}$$

We observe that $M_{i+1} := \frac{N_{i+1}}{N_i}$ is an integer $\geqq 3$. Inside each closed arc mentioned in (iv), we can place $M_{i+1} - 1$ disjoint open arcs of length $2\nu_{i+1}$ so that the complement is made of M_{i+1} closed intervals of equal length. Indeed, on the one hand $2\nu_{i+1}(M_{i+1} - 1) < 2\nu_{i+1}N_{i+1}/N_i < 2^{-i-1}/N_i$, on the other hand, the common length of the closed arcs of (iv) is

$$\mu_i = \frac{1}{N_i}\Big(1 - \sum_{i'=1}^{i} \mathrm{Leb}(Y_{i'})\Big) > 2^{-i-1}/N_i$$

by (4.5). Labelling all the new open arcs as $Y^{(i+1,j)}$, where j runs through a set of $N_i(M_{i+1} - 1) < N_{i+1}$ indices, and calling Y_{i+1} their union, we get the desired properties (note that Y_{i+1} is $\frac{\mu_i}{M_{i+1}}$-dense in each closed arc mentioned in (iv), and $\frac{\mu_i}{M_{i+1}} < \frac{1}{N_i M_{i+1}} = \tilde{\nu}_{i+1}$).

(**c**) Now, for each i and j, we apply Proposition 4.1 with $\bar{r} = r^{(i,j)}$ just constructed and $\nu = \nu_i$ (assuming ε small enough so that the ν_i's are small enough to allow us to do so). We obtain Gevrey functions $u^{(i,j)}$ and $v^{(i,j)}$ supported in $\mathcal{V}(r^{(i,j)}, \nu_i)$ and $\mathcal{V}(r^{(i,j)}, \nu_i) \times M_2$, with

$$\|u^{(i,j)}\|_{\alpha,L} + \|v^{(i,j)}\|_{\alpha,L} \leqq \xi_i, \quad \text{where } \xi_i := \mathrm{e}^{-c\nu_i^{-\gamma}} < \kappa\nu_i$$

(incorporating (4.2)), so that $\mathcal{V}(r^{(i,j)}, \nu_i) \times M_2$ is invariant and $(3\nu_i, \tau_i, \tau_i^b)$-diffusive for $\Phi^{v^{(i,j)}} \circ \Phi^{u^{(i,j)}} \circ T_0$. We set $u_{\mathrm{per}}^{(i,j)}(\theta_1, r_1) := \sum_{k\in\mathbb{Z}} u^{(i,j)}(\theta_1, r_1 + k)$ and $v_{\mathrm{per}}^{(i,j)}(\theta_1, r_1, \theta_2, r_2) := \sum_{k\in\mathbb{Z}} v^{(i,j)}(\theta_1, r_1 + k, \theta_2, r_2)$ so as to get functions which are 1-periodic in r_1 and have the same Gevrey norms.

Consider the finite sums $u_i := \sum_j u_{\mathrm{per}}^{(i,j)}$ and $v_i := \sum_j v_{\mathrm{per}}^{(i,j)}$ for each $i \geqq 1$: the disjointness of the supports implies that

$$T_i := \Phi^{v_i} \circ \Phi^{u_i} \circ T_0 \text{ is } (3\nu_i, \tau_i, \tau_i^b)\text{-diffusive on } \mathbb{T} \times \tilde{Y}_i \times M_2, \tag{4.6}$$

where $\tilde{Y}_i$ is the lift of Y_i in $\mathbb{R}$. Finally, let $u := \sum_i u_i$ and $v := \sum_i v_i$. These are well-defined Gevrey functions which are 1-periodic in r_1 because

$$\|u_i\|_{\alpha,L} + \|v_i\|_{\alpha,L} \leqq N_i\xi_i < N_i\kappa\nu_i = \frac{\kappa}{\ln N_i} \leqq 2^{-i-1}\varepsilon \quad \text{for each } i \geqq 1 \tag{4.7}$$

(by (4.4)), hence the series over i are convergent in Gevrey norm, with $\|u\|_{\alpha,L} + \|v\|_{\alpha,L} < \varepsilon$.

(**d**) We claim that $T := \Phi^v \circ \Phi^u \circ T_0$ satisfies Theorem 3.4 with

$$X_\varepsilon := \Big(\mathbb{R} - \bigsqcup_{i \geqq 1} \tilde{Y}_i\Big) \cap (0,1).$$

To prove this claim, firstly we note that $\operatorname{Leb}\big([0,1] - X_\varepsilon\big) = \sum_i \operatorname{Leb}(Y_i) \leqq \varepsilon$ by (4.5), and X_ε is closed because $\tilde{Y}_1$ contains 0 and 1.

Secondly the functions u and v vanish with all their partial derivatives for $r_1 \in X_\varepsilon$ because the functions $u_{\mathrm{per}}^{(i,j)}$ and $v_{\mathrm{per}}^{(i,j)}$ and all their partial derivatives do.

Lastly, by virtue of the previous point, for each $(r_1, r_2) \in (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}$ the torus $\mathcal{T}_{(r_1,r_2)}$ is invariant for T, thus it only remains for us to show that the union $\mathbb{T} \times (X_\varepsilon + \mathbb{Z}) \times M_2$ of these tori is diffusive-$*$ with exponent $\gamma = \frac{1}{\alpha - 1}$. In view of the definition of X_ε, it is sufficient to show that, for i large enough,

$$Z_i := M_1 \times M_2 \\ - \bigsqcup_{1 \leqq i' \leqq i-1} \mathbb{T} \times \tilde{Y}_{i'} \times M_2 \text{ is } 2\tilde{\nu}_i\text{-diffusive-}* \text{ for } T \text{ with exponent } \gamma. \tag{4.8}$$

We will prove this with the constant $C := 3^{\gamma+1} c\gamma$ in the definitions (3.1) and (3.3) of the functions $E = E_{C,\gamma}$ and $E^* = E^*_{C,\gamma}$, as a consequence of the fact that, for i large enough,

$$Z_i \text{ is } (2\tilde{\nu}_i, \tau_i, \tfrac{1}{2}\tau_i^b) - \text{diffusive for } T. \tag{4.9}$$

Indeed, $\tau_i = E_{3c\gamma,\gamma}(\nu_i) = E(3\nu_i) = E\big(\frac{3\tilde{\nu}_i}{|\ln \tilde{\nu}_i|}\big) \ll E\big(\frac{2\tilde{\nu}_i}{|\ln \tilde{\nu}_i| - \ln 2}\big) = E^*(2\tilde{\nu}_i)$ by (3.2), and $\frac{1}{2}\tau_i^b \gg E\big(\frac{4\tilde{\nu}_i}{|\ln \tilde{\nu}_i| - \ln 4}\big) = E^*(4\tilde{\nu}_i)$ still by (3.2), hence (4.9) implies (4.8).

(**e**) To prove (4.9), let $z \in Z_i$. We set $W_i := \mathbb{T} \times \tilde{Y}_i \times M_2$. By property (iii) of the construction of Y_i, we can find $\bar{z} \in W_i$ such that $d(\bar{z}, z) \leqq \tilde{\nu}_i$, and (4.6) then yields $\hat{z} \in W_i$ and $t \leqq \tau_i$ such that $d(\hat{z}, \bar{z}) \leqq 3\nu_i$ and $d(T_i^t \hat{z}, \bar{z}) \geqq \tau_i^b$. We have $|\ln \tilde{\nu}_i| = \ln N_i > 3$ by virtue of (4.4), hence $3\nu_i < \tilde{\nu}_i$ and $d(\hat{z}, z) \leqq 2\tilde{\nu}_i$. Since $u_1, \dots, u_{i-1}, v_1, \dots, v_{i-1}$ and all their partial derivatives vanish on W_i, the restriction of T to W_i coincides with that of

$$\tilde{T}_i := \Phi^{v_i + g_i} \circ \Phi^{u_i + f_i} \circ T_0, \quad \text{where} \quad f_i := \sum_{i' > i} u_{i'}, \quad g_i := \sum_{i' > i} v_{i'}. \tag{4.10}$$

Comparing the definition T_i in (4.6) and that of $\tilde{T}_i$ in (4.10), we observe that the first t points on the T_i-orbit of $\hat{z}$ are close to the first t points on its T-orbit (which is the same as its $\tilde{T}_i$-orbit). More precisely, (4.7) yields

$$\|u_i\|_{\alpha,L} + \|f_i\|_{\alpha,L} + \|v_i\|_{\alpha,L} + \|g_i\|_{\alpha,L} \leqq \sum_{i' \geqq i} N_{i'} \xi_{i'} \leqq 2^{-i}\varepsilon,$$

hence we can use Corollary A.3 from the appendix for i large enough, obtaining

$$d\big(T_i^t(\hat{z}), \tilde{T}_i^t(\hat{z})\big) \leqq 3^{\tau_i} C_1\big(\|f_i\|_{\alpha,L} + \|g_i\|_{\alpha,L}\big)$$

from (A.11), where $C_1 = C_1(\alpha, L)$, but

$$\|f_i\|_{\alpha,L} + \|g_i\|_{\alpha,L} \leqq \sum_{i' \geqq i+1} N_{i'}\xi_{i'} \leqq 2N_{i+1}\xi_{i+1} \leqq 2 \cdot 3^{-\tau_i}$$

by the first part of (4.7) and (B.2)–(B.3), whence $d(T_i^t \hat{z}, \tilde{T}_i^t \hat{z}) \leqq 2C_1 \leqq \frac{1}{2}\tau_i^b - d(\bar{z}, z)$ and thus $d(T^t \hat{z}, z) = d(\tilde{T}_i^t \hat{z}, z) \geqq \frac{1}{2}\tau_i^b$ for i large enough, and we conclude that z is $(2\tilde{\nu}_i, \tau_i, \frac{1}{2}\tau_i^b)$-diffusive for T. The proof of (4.9) is thus complete.

5. Isolated Periodic Points for Twist Maps of the Annulus

Definition 5.1. Given a real $\sigma > 0$ and a discrete dynamical system in a metric space, we say that a periodic point is σ*-isolated* if it lies at a distance $\geqq \sigma$ of the rest of its orbit.

The goal of this section is to prove the following statement, which will be instrumental in the proof of Proposition 4.1, where a perturbation of F_0 is obtained that has an isolated periodic point in M_1.

Proposition 5.1. *Let* $\gamma := \frac{1}{\alpha - 1}$. *There exists* $c = c(\alpha, L)$ *with the following property: if we are given real numbers* ν *and* σ *with* $\nu > 0$ *small enough and* $0 < \sigma \leqq \exp(-2c\nu^{-\gamma})$, *then, for any integer* $\ell \geqq 6/\nu$ *and for any* $\bar{r} \in \mathbb{R}$, *there exists* $u \in G^{\alpha,L}(M_1)$ *such that*

(1) $u \equiv 0$ *on* $\mathcal{V}(\bar{r}, \nu)^c$,
(2) $\|u\|_{\alpha,L} \leqq \frac{1}{2}\exp(-c\nu^{-\gamma})$,
(3) $F := \Phi^u \circ F_0 \colon M_1 \circlearrowleft$ *has a* σ*-isolated periodic point* $z_1 \in \mathcal{V}(\bar{r}, 3\nu/4)$ *of period* $q \in [\ell, 3\ell/\nu]$,
(4) *the set* $\{F^s(z_1) \mid s \in \mathbb{N},\ 2/\nu \leqq s \leqq 6/\nu\}$ *is* 2ν*-dense in* $\mathcal{V}(\bar{r}, \nu)$.

The interest of the statement is that, although σ is required to be exponentially small, it can be kept independent of ℓ, even if we choose ℓ, and thus q, doubly exponentially large.

The proof will start with the construction of a circle map with an isolated periodic point.

5.1. Circle Diffeomorphisms with Isolated Periodic Points

We start by constructing a circle diffeomorphism with an isolated periodic point. For any $Q \in \mathbb{N}^*$ we denote by Δ_Q a function in $C^\infty(\mathbb{T})$ satisfying

$$\Delta_Q \text{ is } \tfrac{1}{Q}\text{-periodic}, \quad 0 \leqq \Delta_Q \leqq 1, \quad \Delta_Q(0) = 0, \quad \Delta_Q \equiv 1 \text{on} \big[\tfrac{1}{4Q}, \tfrac{3}{4Q}\big]. \tag{5.1}$$

Since $\alpha > 1$, every space $G^{\alpha,L'}(\mathbb{T})$ contains such a function, and one can choose it so that

$$\|\Delta_Q\|_{\alpha,L'} \leqq L'^{\alpha} \exp(\tilde{c}\, Q^{\gamma}), \tag{5.2}$$

with $\tilde{c} = \tilde{c}(\alpha, L')$ independent of Q according to Lemma A.5 in the appendix (take for example $\Delta_Q(\theta) := \sum_{j=1}^{Q} \eta_{2Q}\big(\frac{2j-1}{2Q} + \theta\big)$ and $\tilde{c} := L'^{-\alpha} + \frac{1}{\gamma} + 2^\gamma c_1(\alpha, L')$ with the notation of Lemma A.5, using the inequalities $L'^{-\alpha} \leqq \mathrm{e}^{L'^{-\alpha}}$ and $1 \leqq Q \leqq \mathrm{e}^{\frac{1}{\gamma}Q^\gamma}$).

Proposition 5.2. *Let $Q \in \mathbb{N}^*$ and $P \in \mathbb{Z}$ be coprime and let $\sigma \in \big(0, \frac{1}{\max(-\Delta_Q')}\big)$. Then, for every $\ell \in \mathbb{N}^*$, there exist an integer $q = q_\ell(Q, P)$ and a real $\delta = \delta_\ell(\sigma, Q, P)$ such that*

- *$\ell \leqq q \leqq \ell Q$ and $0 < \delta \leqq \frac{1}{\ell Q}$,*
- *the point $\frac{1}{2Q} + \mathbb{Z}$ is periodic of period q and σ-isolated for the circle map*

$$\theta \in \mathbb{T} \mapsto f_{\sigma,\delta}(\theta) := \theta + \frac{P}{Q} + \delta + \sigma\Delta_Q(\theta) \mod \mathbb{Z}. \tag{5.3}$$

If, moreover, $\ell \geqq 2Q$, then the set $\{f^s_{\sigma,\delta}(\frac{1}{2Q} + \mathbb{Z}) \mid s \in \mathbb{N},\ \ Q \leqq s \leqq 2Q - 1\}$ is $\frac{1}{Q}$-dense in $\mathbb{T}$.

Proof. (**a**) We define $q = q_\ell(Q, P)$ by writing

$$\frac{\ell P + 1}{\ell Q} = \frac{p}{q} \quad \text{with } q \in \mathbb{N}^* \text{and } p \in \mathbb{Z} \text{ coprime.}$$

Since 1 is the only common divisor of ℓ and $\ell P + 1$, we must have

$$\ell P + 1 = pD, \quad Q = q'D, \quad q = \ell q' \tag{5.4}$$

with $D \in \mathbb{N}^*$ and $p \wedge q' = 1$, hence $\ell \leqq q \leqq \ell Q$.

The condition $0 < \sigma < \frac{1}{\max(-\Delta_Q')}$ ensures that $1 + \sigma\Delta_Q'$ stays positive hence, for every $\delta \in \mathbb{R}$, the formula

$$F_{\sigma,\delta}(x) := x + \frac{P}{Q} + \delta + \sigma\Delta_Q(x)$$

defines an increasing diffeomorphism of $\mathbb{R}$ such that $F_{\sigma,\delta}(x + 1) = F_{\sigma,\delta}(x) + 1$. Formula (5.3) then defines a diffeomorphism $f_{\sigma,\delta}$ of $\mathbb{T}$, a lift of which is $F_{\sigma,\delta}$. We will tune δ so as to get the rotation number of $f_{\sigma,\delta}$ equal to p/q.

(**b**) To study the dynamics of $F_{\sigma,\delta}$ and particularly the orbit of $x_0 := \frac{1}{2Q}$, we perform the change of variable $X = Qx$ and set $X_0 := \frac{1}{2}$ and

$$\begin{aligned}
G_{\sigma,\delta}(X) &:= QF_{\sigma,\delta}(X/Q) = X + P + \delta Q + \sigma Q\Delta_Q(X/Q),\\
\tilde{G}_{\sigma,\delta}(X) &:= X + \delta Q + \sigma Q\Delta_Q(X/Q).
\end{aligned}$$

Note that also $\tilde{G}_{\sigma,\delta}$ is an increasing diffeomorphism of $\mathbb{R}$ for each $\delta \in \mathbb{R}$. For every $\ell \in \mathbb{N}^*$ we have

$$\tilde{G}^\ell_{\sigma,0}(X_0) < X_0 + 1 = \tilde{G}^\ell_{0,\frac{1}{\ell Q}}(X_0) \leqq \tilde{G}^\ell_{\sigma,\frac{1}{\ell Q}}(X_0)$$

(the left inequality holds because $X_0 < 1$ and 1 is a fixed point of $\tilde{G}_{\sigma,0}$; the right inequality holds because $\sigma Q \Delta_Q(x) \geqq 0$ for all x). Therefore, since $\delta \mapsto \tilde{G}^{\ell}_{\sigma,\delta}(X_0)$ is continuous and increasing, we can define $\delta := \delta_\ell(\sigma, Q, P)$ as the unique solution of the equation

$$\tilde{G}^{\ell}_{\sigma,\delta}(X_0) = X_0 + 1,$$

and we know that $0 < \delta_\ell(\sigma, Q, P) \leqq \frac{1}{\ell Q}$.

(**c**) We now fix δ to be this value $\delta_\ell(\sigma, Q, P)$ and check that it satifies the desired properties. First, notice that

$$4\sigma Q < 1 \tag{5.5}$$

(because $\Delta_Q(3/4Q) = 1$ and $\Delta_Q(1) = 0$, hence the mean value theorem implies $\max(-\Delta'_Q) \geqq 4Q$). Let us denote the full orbits of $X_0 = \frac{1}{2}$ under $G_{\sigma,\delta}$ and $\tilde{G}_{\sigma,\delta}$ by

$$X_j := G^j_{\sigma,\delta}(X_0), \quad \tilde{X}_j := \tilde{G}^j_{\sigma,\delta}(X_0), \qquad j \in \mathbb{Z}.$$

We have $X_0 = \tilde{X}_0 < \tilde{X}_1 < \cdots < \tilde{X}_{\ell-1} < \tilde{X}_\ell = X_0 + 1$. In fact,

$$X_0 + \sigma Q < \tilde{X}_1 < \cdots < \tilde{X}_{\ell-1} < X_0 + 1 - \sigma Q. \tag{5.6}$$

Indeed, $\Delta_Q(X_0/Q) = 1$ hence $\tilde{X}_1 = X_0 + \sigma Q + \delta Q$, and either $\tilde{X}_{\ell-1} \geqq X_0 + 1 - \frac{1}{4}$, in which case $X_0 + 1 = \tilde{G}_{\sigma,\delta}(\tilde{X}_{\ell-1}) = \tilde{X}_{\ell-1} + \sigma Q + \delta Q > \tilde{X}_{\ell-1} + \sigma Q$, or $\tilde{X}_{\ell-1} < X_0 + 1 - \frac{1}{4} < X_0 + 1 - \sigma Q$ by (5.5).

Since Δ_Q is $\frac{1}{Q}$-periodic, we have $\tilde{G}_{\sigma,\delta}(X + 1) = \tilde{G}_{\sigma,\delta}(X) + 1$ and the pattern (5.6) repeats 1-periodically: for every $m \in \mathbb{Z}$, $\tilde{X}_{\ell m} = X_0 + m$ and

$$X_0 + m + \sigma Q < \tilde{X}_{\ell m+s} < X_0 + m + 1 - \sigma Q \quad \text{for } s = 1, \ldots, \ell - 1.$$

Since Δ_Q is $\frac{1}{Q}$-periodic, we have $G^j_{\sigma,\delta}(X) = \tilde{G}^j_{\sigma,\delta}(X) + jP$ for every $j \in \mathbb{Z}$, hence $X_j = \tilde{X}_j + jP$ and, for every $m \in \mathbb{Z}$,

$$X_{\ell m} = X_0 + m(\ell P + 1) = X_0 + mpD \tag{5.7}$$

$$X_0 + M_{m,s} + \sigma Q < X_{\ell m+s} < X_0 + M_{m,s} + 1 - \sigma Q \quad \text{for } s = 1, \ldots, \ell - 1, \tag{5.8}$$

with $M_{m,s} := m(\ell P + 1) + sP$.

(**d**) Going back to the variable $x = X/Q$, we see that the orbit $(x_j)_{j \in \mathbb{Z}}$ of x_0 under $F_{\sigma,\delta}$ satisfies

$$x_{\ell m} = x_0 + \frac{mp}{q'} \quad \text{and} \quad x_0 + \frac{M_{m,s}}{Q} + \sigma < x_{\ell m+s} < x_0 + \frac{M_{m,s} + 1}{Q} - \sigma \tag{5.9}$$

for $m \in \mathbb{Z}$ and $1 \leqq s < \ell$ (thanks to (5.4) and (5.7)–(5.8)). In particular, $x_q = x_0 + p$, hence it induces a q-periodic orbit of type p/q for $f_{\sigma,\delta}$. The σ-isolation property amounts to

$$\operatorname{dist}(x_j, x_0 + \mathbb{Z}) \geqq \sigma \quad \text{for } 1 \leqq j < q = \ell q'.$$

This holds because either $j = \ell m$ with $1 \leqq m < q'$ and the first part of (5.9) entails $x_{\ell m} - x_0 \in \mathbb{Q} - \mathbb{Z}$ with $\operatorname{dist}(x_{\ell m} - x_0, \mathbb{Z}) \geqq \frac{1}{q'} \geqq \frac{1}{Q} > 4\sigma$ by (5.5), or $j = \ell m + s$ with $1 \leqq s < \ell$ and the second part of (5.9) yields $x_j - x_0 \in (\frac{M_{m,s}}{Q} + \sigma, \frac{M_{m,s}+1}{Q} - \sigma)$, but $(\frac{M_{m,s}}{Q}, \frac{M_{m,s}+1}{Q}) \cap \mathbb{Z} = \emptyset$, hence $\operatorname{dist}(x_j - x_0, \mathbb{Z}) > \sigma$.

(**e**) We now suppose $\ell \geqq 2Q$ and prove the $\frac{1}{Q}$-density statement.

If $P = 0$, then $Q = 1$ (because of the assumption $P \wedge Q = 1$) and there is nothing to prove. We thus suppose $P \neq 0$. Using the second part of (5.9) with $1 \leqq s \leqq 2Q - 1 < \ell$ and $m = 0$, since $M_{0,s} = sP$, we get

$$x_0 + \frac{sP}{Q} < x_s < x_0 + \frac{sP}{Q} + \frac{1}{Q} \quad \text{for } s = Q, \dots, 2Q - 1. \tag{5.10}$$

Since $P \wedge Q = 1$, the Q arcs $\left[x_0 + \frac{sP}{Q}, x_0 + \frac{sP}{Q} + \frac{1}{Q}\right)$ mod $\mathbb{Z}$ are mutually disjoint and cover $\mathbb{T}$; each of them has length $\frac{1}{Q}$ and, according to (5.10), contains a point of $\{x_s + \mathbb{Z} \mid s = Q, \dots, 2Q - 1\} = \{f^s_{\sigma,\delta}(x_0 + \mathbb{Z}) \mid s = Q, \dots, 2Q - 1\}$. This set is thus $\frac{1}{Q}$-dense in $\mathbb{T}$. □

5.2. Herman Imbedding Trick of Circle Diffeomorphisms into Twist Maps

We will have to imbed the circle dynamics of Proposition 5.2 into the annulus *via* a perturbation of the twist map F_0. For this we will use the celebrated technique introduced by Herman in [8] to imbed circle dynamics as restricted dynamics on an invariant graph by a twist map.

Recall that trivial examples of symplectic maps of the annulus are given by

$$\Phi^h(\theta, r) = \big(\theta + h'(r) + \mathbb{Z}, r\big), \qquad \Phi^w(\theta, r) = \big(\theta, r - w'(\theta)\big), \tag{5.11}$$

where the function $h = h(r)$ does not depend on the angle $\theta \in \mathbb{T}$, and the function $w = w(\theta)$ does not depend on the action r. In particular,

$$h(r) \equiv \omega_1 r + \tfrac{1}{2} r^2 \;\Rightarrow\; \Phi^h = F_0. \tag{5.12}$$

Proposition 5.3. *Suppose that we are given a circle diffeomorphism of the form*

$$\theta \in \mathbb{T} \mapsto f(\theta) = \theta + h'\big(\hat{r} + \varepsilon(\theta)\big) \mod \mathbb{Z},$$

where $h = h(r)$ and $\varepsilon = \varepsilon(\theta)$ are smooth functions and $\hat{r} \in \mathbb{R}$. Then the equation

$$-w' = \varepsilon - \varepsilon \circ f^{-1}, \tag{5.13}$$

determines a smooth function $w = w(\theta)$ up to an additive constant, and the annulus map

$$\Phi^w \circ \Phi^h\colon\; (\theta, r) \mapsto \big(\theta + h'(r) + \mathbb{Z}, r - w'(\theta + h'(r))\big) \tag{5.14}$$

leaves invariant the graph $\big\{\big(\theta,\hat r+\varepsilon(\theta)\big)\mid\theta\in\mathbb{T}\big\}$, *with induced dynamics* $\theta\mapsto f(\theta)$ *on it.*

Proof. Let us check that the right-hand side of (5.13) has zero mean value:

$$\langle\varepsilon\circ f^{-1}\rangle=\int_{\mathbb{T}}\varepsilon\big(f^{-1}(\tilde\theta)\big)\,\mathrm{d}\tilde\theta=\int_{\mathbb{T}}\varepsilon(\theta)f'(\theta)\,\mathrm{d}\theta\quad\text{and}$$
$$f'(\theta)=1+\frac{\mathrm{d}}{\mathrm{d}\theta}\big[h'\big(\hat r+\varepsilon(\theta)\big)\big],$$

hence $\langle\varepsilon\rangle-\langle\varepsilon\circ f^{-1}\rangle=-\int_{\mathbb{T}}\varepsilon(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\big[h'\big(\hat r+\varepsilon(\theta)\big)\big]\,\mathrm{d}\theta=\int_{\mathbb{T}}\varepsilon'(\theta)h'\big(\hat r+\varepsilon(\theta)\big)\,\mathrm{d}\theta=0$, since this is the mean value of the derivative of the periodic function $h\big(\hat r+\varepsilon(\theta)\big)$. Consequently, any primitive of $\varepsilon-\varepsilon\circ f^{-1}$ induces a smooth function on $\mathbb{T}$.

Now, consider an arbitrary point $(\theta,r)=\big(\theta,\hat r+\varepsilon(\theta)\big)$ on the graph mentioned in the statement. Its image by $\Phi^w\circ\Phi^h$ is $(\theta_1,r_1):=\big(\theta+h'(r)+\mathbb{Z},r-w'(\theta_1)\big)$. We have

$$\theta_1=\theta+h'\big(\hat r+\varepsilon(\theta)\big)+\mathbb{Z}=f(\theta)$$

and $r_1=\hat r+\varepsilon(\theta)-w'(\theta_1)=\hat r+\varepsilon\circ f^{-1}(\theta_1)-w'(\theta_1)=\hat r+\varepsilon(\theta_1)$. □

Let us have a look at the solutions of (5.13) in the case of Gevrey data, with h as in (5.12), that is $h'(r)\equiv\omega_1+r$.

Lemma 5.4. *Let* $L'>L$. *Suppose*

$$f(\theta)=\theta+\omega_1+\hat r+\varepsilon(\theta)\mod\mathbb{Z}\quad\textit{for all }\theta\in\mathbb{T},\quad\textit{and}\quad\varepsilon=\delta+\varepsilon_*,$$

where $\hat r,\delta\in\mathbb{R}$ *and* $\varepsilon_*\in G^{\alpha,L'}(\mathbb{T})$ *satisfies* $\|\varepsilon_*\|_{\alpha,L'}\leqq\varepsilon_{\mathrm{i}}$ *with* $\varepsilon_{\mathrm{i}}=\varepsilon_{\mathrm{i}}(\alpha,L,L')$ *as in Lemma* A.4. *Then* f *is a circle diffeomorphism and Equation* (5.13) *has a solution* w *such that*

$$\|w\|_{\alpha,L}\leqq(1+2L^\alpha)\|\varepsilon_*\|_{\alpha,L'}.\tag{5.15}$$

Proof. We can write $f=(\mathrm{Id}+\omega)\circ(\mathrm{Id}+\varepsilon_*)$ with $\omega:=\omega_1+\hat r+\delta$. By Lemma A.4, we obtain that $\mathrm{Id}+\varepsilon_*$ is a diffeomorphism of $\mathbb{R}$, which (because of periodicity) can be viewed as the lift of a circle diffeomorphism. Hence f is a circle diffeomorphism and the right-hand side of (5.13) is

$$g:=\varepsilon-\varepsilon\circ f^{-1}=\varepsilon_*-\varepsilon_*\circ f^{-1}=\varepsilon_*-\varepsilon_*\circ(\mathrm{Id}+\varepsilon_*)^{-1}\circ(\mathrm{Id}-\omega).$$

Lemma A.4 yields $\|\varepsilon_*\circ(\mathrm{Id}+\varepsilon_*)^{-1}\|_{\alpha,L}\leqq\|\varepsilon_*\|_{\alpha,L'}$, which easily implies $\|g\|_{\alpha,L}\leqq 2\|\varepsilon_*\|_{\alpha,L'}$.

Now, we already know that g has zero mean value, and a solution to (5.13) can be defined by the formula

$$w(\theta+\mathbb{Z})\equiv-\int_0^\theta g(\theta_1)\,\mathrm{d}\theta_1\quad\text{for }\theta\in(-\tfrac12,\tfrac12].$$

One has $\|w\|_{C^0(\mathbb{T})}\leqq\frac12\|g\|_{C^0(\mathbb{T})}$ and, for each $k\geqq0$,

$$\frac{L^{(k+1)\alpha}}{(k+1)!^\alpha}\|w^{(k+1)}\|_{C^0(\mathbb{T})}=\frac{L^{(k+1)\alpha}}{(k+1)!^\alpha}\|g^{(k)}\|_{C^0(\mathbb{T})}\leqq L^\alpha\cdot\frac{L^{k\alpha}}{k!^\alpha}\|g^{(k)}\|_{C^0(\mathbb{T})},$$

whence $\|w\|_{\alpha,L}\leqq(\frac12+L^\alpha)\|g\|_{\alpha,L}$, and the conclusion follows. □

5.3. Proof of Proposition 5.1

(**a**) We start with an elementary fact.

Lemma 5.5. *Suppose x and ν are real, with $\nu > 0$ small enough. Then there exist $Q \in \mathbb{N}^*$ and $P \in \mathbb{Z}$ coprime such that $|x - P/Q| < \nu/2$ and $1 < \frac{2}{\nu} < Q < \frac{3}{\nu}$.*

Proof. For $\nu > 0$ small enough, thanks to the Prime Number Theorem, we can pick a prime number Q in $(2\nu^{-1}, 3\nu^{-1})$ (for example $Q = p_k$ with $k := \lfloor * \rfloor 5\nu^{-1}/2| \ln \nu|$, where $p_k \sim k \ln k$ is the kth prime number). The interval $(Qx - \frac{Q\nu}{2}, Qx + \frac{Q\nu}{2})$ has length > 2, hence it contains at least two consecutive integers, one of which is not a multiple of Q and can be taken as P. $\square$

We now define

$$L' := 2L, \qquad c := \max\{3^{\gamma+1}\tilde{c}, 2^{\gamma+1}c_1\},$$

with $\tilde{c} = \tilde{c}(\alpha, L')$ as in (5.2) and $c_1 = c_1(\alpha, L)$ as in Lemma A.5, and suppose that we are given $\nu, \sigma, \ell, \bar{r}$ as in the statement of Proposition 5.1, with ν small enough so as to be able to apply Lemma 5.5.

Applying Lemma 5.5 with $x = \omega_1 + \bar{r}$, we get a rational P/Q such that $P \wedge Q = 1$ and

$$1 < 2/\nu < Q < 3/\nu \quad \text{and} \quad \frac{P}{Q} = \omega_1 + \hat{r} \quad \text{with} \quad |\hat{r} - \bar{r}| < \nu/2.$$

(**b**) Let us choose a function $\Delta = \Delta_Q \in G^{\alpha,L'}(\mathbb{T})$ satisfying (5.1)–(5.2) and apply Proposition 5.2. We can do so since $2c \geqq 3^\gamma \tilde{c}$, hence $0 < \sigma \leqq \mathrm{e}^{-2c\nu^{-\gamma}} < \mathrm{e}^{-\tilde{c}Q^\gamma} \leqq \frac{1}{\max(-\Delta')}$ by (5.2). We get an integer q and a real δ satisfying

$$q \in [\ell, \ell Q] \subset [\ell, 3\ell/\nu], \qquad 0 < \delta \leqq \frac{1}{\ell Q} < \frac{\nu}{2\ell} \leqq \frac{\nu^2}{12},$$

so that $\frac{1}{2Q} + \mathbb{Z}$ is a σ-isolated periodic point of period q for the circle map f defined by

$$\theta \in \mathbb{T} \mapsto f(\theta) := \theta + \frac{P}{Q} + \delta + \varepsilon_*(\theta) \mod \mathbb{Z} \qquad \text{with } \varepsilon_* := \sigma\Delta.$$

Moreover, since $\ell \geqq 6/\nu \geqq 2Q$ and $\frac{1}{Q} < \frac{\nu}{2}$, the set $\{f^s(\frac{1}{2Q} + \mathbb{Z}) \mid s \in \mathbb{N},\ 2/\nu \leqq s \leqq 6/\nu\}$ is $\frac{\nu}{2}$-dense in $\mathbb{T}$.

(**c**) We are in the situation of Lemma 5.4, with

$$\|\varepsilon_*\|_{\alpha,L'} \leqq \sigma L'^\alpha \mathrm{e}^{\tilde{c}\,Q^\gamma} \leqq L'^\alpha \mathrm{e}^{-(2c-3^\gamma\tilde{c})\nu^{-\gamma}} \leqq L'^\alpha \mathrm{e}^{-\frac{5c}{3}\nu^{-\gamma}},$$

which is less than $\varepsilon_{\mathrm{i}}(\alpha, L, L')$ for ν small enough, thus Equation (5.13) with $\varepsilon := \delta + \varepsilon_*$ and $h' = \mathrm{Id} + \omega_1$ has a solution $w \in G^{\alpha,L}(\mathbb{T})$ such that

$$\|w\|_{\alpha,L} \leqq (1 + 2L^\alpha)\|\varepsilon_*\|_{\alpha,L'} \leqq (1 + 2L^\alpha)L'^\alpha \mathrm{e}^{-\frac{5c}{3}\nu^{-\gamma}}.$$

Proposition 5.3 now tells us that the annulus map $\tilde{F} := \Phi^w \circ F_0 \colon M_1 \circlearrowleft$ leaves invariant the graph $\mathcal{G} := \{(\theta, \hat{r} + \varepsilon(\theta)) \mid \theta \in \mathbb{T}\}$, with $f \colon \mathbb{T} \circlearrowleft$ as induced dynamics on $\mathcal{G}$. In particular,

$$z_1 := \left(\tfrac{1}{2Q} + \mathbb{Z}, \hat{r} + \varepsilon\left(\tfrac{1}{2Q}\right)\right) \in \mathbb{T} \times \mathbb{R}$$

is a σ-isolated periodic point of period q for $\tilde{F}$, with all its orbit contained in $\mathcal{G}$. Notice that, for ν small enough, $\|\varepsilon\|_{C^0(\mathbb{T})} \leqq \delta + \|\varepsilon_*\|_{C^0(\mathbb{T})} \leqq \nu^2/12 + L'^{\alpha} \mathrm{e}^{-\frac{5c}{3}\nu^{-\gamma}} \leqq \nu/4$, hence $\mathcal{G} \subset \mathcal{V}(\hat{r}, \nu/4) \subset \mathcal{V}(\bar{r}, 3\nu/4)$ and $\{\tilde{F}^s(z_1) \mid s \in \mathbb{N},\ 2/\nu \leqq s \leqq 6/\nu\}$ is 2ν-dense in $\mathcal{V}(\bar{r}, \nu)$.

(**d**) The only shortcoming of the perturbation Φ^w is that it is not supported on the strip $\mathcal{V}(\bar{r}, \nu)$, but this is easy to remedy: we will multiply w by a function which vanishes outside $\mathcal{V}(\bar{r}, \nu)$ without modifying the dynamics in $\mathcal{V}(\hat{r}, \nu/4)$. Note that

$$\mathcal{V}(\hat{r}, \nu/4) \subset \mathcal{V}(\hat{r}, \nu/2) \subset \mathcal{V}(\bar{r}, \nu).$$

Let us thus pick $\eta \in G^{\alpha,L}(\mathbb{R})$ such that $\eta(r) = 1$ for all $r \in [\hat{r} - \nu/4, \hat{r} + \nu/4]$ and $\eta(r) = 0$ whenever $|r - \hat{r}| \geqq \nu/2$. According to Lemma A.5, we can achieve $\|\eta\|_{\alpha,L} \leqq \exp\left(2^{\gamma} c_1 \nu^{-\gamma}\right)$ (using, in fact, a non-periodic version of Lemma A.5, with $p = \frac{2}{\nu}$). We now set

$$u(\theta, r) := \eta(r) w(\theta) \quad \text{for all } (\theta, r) \in \mathbb{T} \times \mathbb{R}.$$

One can check that u satisfies conditions (1) and (2) of Proposition 5.1 for ν small enough, because then $\|w\|_{\alpha,L} \leqq \frac{1}{2} \mathrm{e}^{-\frac{3c}{2}\nu^{-\gamma}}$, while $\|\eta\|_{\alpha,L} \leqq \mathrm{e}^{\frac{c}{2}\nu^{-\gamma}}$. Since $F := \Phi^u \circ F_0$ and $\tilde{F}$ coincide on $\mathcal{G}$ (in fact on all of $\mathcal{V}(\hat{r}, \nu/4)$), requirements (3) and (4) are also fulfilled.

6. Coupling Lemma and Synchronized Diffusion *à la Herman*

6.1. Coupling Lemma

The following “coupling lemma” due to M. Herman was already used in [9–11] to construct examples of unstable near-integrable Hamiltonian flows.

Lemma 6.1. *Let M and M' be symplectic manifolds. Suppose we are given two maps, $F \colon M \circlearrowleft$ and $G \colon M' \circlearrowleft$, and two Hamiltonian functions $f \colon M \to \mathbb{R}$ and $g \colon M' \to \mathbb{R}$ which generate complete vector fields and define time-1 maps Φ^f and Φ^g. Suppose moreover that $z_* \in M$ is F-periodic, of period q, and that*

$$f(z_*) = 1, \qquad \mathrm{d}f(z_*) = 0 \tag{6.1}$$

$$f(F^s(z_*)) = 0, \qquad \mathrm{d}f(F^s(z_*)) = 0 \quad \textit{for } 1 \leqq s \leqq q - 1. \tag{6.2}$$

Then $f \otimes g$ generates a complete Hamiltonian vector field and the maps

$$T := \Phi^{f \otimes g} \circ (F \times G) \colon M \times M' \circlearrowleft \quad \textit{and} \quad \psi := \Phi^g \circ G^q \colon M' \circlearrowleft$$

satisfy

$$T^{nq+s}(z_*, z') = \big(F^s(z_*), G^s \circ \psi^n(z')\big) \tag{6.3}$$

for all $z' \in M'$ *and* $n, s \in \mathbb{Z}$ *such that* $0 \leqq s \leqq q-1$.

We have denoted by $f \otimes g$ the function $(z, z') \mapsto f(z)g(z')$, and by $F \times G$ the map $(z, z') \mapsto (F(z), G(z'))$.

Proof. See [10]. □

6.2. *Proof of Proposition 4.1*

(**a**) Given $\bar{r} \in \mathbb{R}$ and $\nu > 0$ small enough, we apply Proposition 5.1 with

$$\sigma := \mathrm{e}^{-2c\nu^{-\gamma}}$$

(where $c = c(\alpha, L)$ is provided by Proposition 5.1) and an integer $\ell \geqq 6/\nu$ that we will specify later.

We thus get a function $u \in G^{\alpha,L}(M_1)$ and a map $F = \Phi^u \circ F_0\colon M_1 \circlearrowleft$ satisfying properties (1)–(4) of Proposition 5.1. We call $z_1^{(0)}$ the σ-isolated periodic point mentioned in property (4), the period of which is an integer $q \in [\ell, 3\ell/\nu]$.

Let $f := \eta_{z_1^{(0)},\sigma}$ be defined by Lemma A.6. Observe that f, F and $z_* = z_1^{(0)}$ satisfy conditions (6.1)–(6.2) of Lemma 6.1 because $z_1^{(0)}$ is a σ-isolated periodic point.

(**b**) We now define $g\colon M_2 \to \mathbb{R}$ by the formula $g(r_2, \theta_2) = -\frac{1}{2\pi q}\sin(2\pi\theta_2)$. According to (5.11), we have $\Phi^g(\theta_2, r_2) = \big(\theta_2, r_2 + \frac{1}{q}\cos(2\pi\theta_2)\big)$ for all $(\theta_2, r_2) \in M_2$. In particular,

$$\Phi^g(0+\mathbb{Z}, r_2) = \big(0+\mathbb{Z}, r_2 + \tfrac{1}{q}\big) \quad \text{for all } r_2 \in \mathbb{R}.$$

On the other hand, (3.6) gives $G_0^s(\theta_2, r_2) = \big(\theta_2 + s(\omega_2 + r_2) + \mathbb{Z}, r_2\big)$ for all $s \in \mathbb{Z}$. Therefore

$$\psi := \Phi^g \circ G_0^q$$

satisfies $\psi^n(0+\mathbb{Z}, -\omega_2) = (0+\mathbb{Z}, -\omega_2 + \frac{n}{q})$ for all $n \in \mathbb{Z}$, whence

$$G_0^s \circ \psi^n\big(z_2^{(0,0)}\big) = z_2^{(n,s)} \quad \text{with } z_2^{(n,s)} := \Big(\frac{sn}{q} + \mathbb{Z}, -\omega_2 + \frac{n}{q}\Big) \quad \text{for all } n, s \in \mathbb{Z}. \tag{6.4}$$

(**c**) Let $v := f \otimes g$. We now apply Lemma 6.1 with the above functions f and g and the maps F and G_0, taking $z_* = z_1^{(0)}$. In view of (6.3)–(6.4), we get

$$T^{nq+s}\big(z_1^{(0)}, z_2^{(0,0)}\big) = \big(F^s(z_1^{(0)}), z_2^{(n,s)}\big) \quad \text{for all } n, s \in \mathbb{Z} \text{ such that } 0 \leqq s \leqq q-1, \tag{6.5}$$

with $T = \Phi^v \circ (F \times G_0) = \Phi^v \circ \Phi^u \circ (F_0 \times G_0)$.

Notice that, for ν small enough, we have $\sigma < \nu/4$, hence $f \equiv 0$ on $\mathcal{V}(\bar{r}, \nu)^c$, thus v satisfies condition (1) of Proposition 4.1, and we already knew that u also did.

We have $\|u\|_{\alpha,L} \leqq \frac{1}{2}\exp(-c\nu^{-\gamma})$ with $c = c(\alpha, L)$ stemming from Proposition 5.1. We can also achieve $\|v\|_{\alpha,L} \leqq \frac{1}{2}\exp(-c\nu^{-\gamma})$ by choosing appropriately ℓ. Indeed, calling K the Gevrey-(α, L) norm of the function $\theta \mapsto \frac{1}{2\pi}\sin(2\pi\theta)$ and using $q \geqq \ell$ and (A.15), we get $\|v\|_{\alpha,L} \leqq \frac{K}{q}\|\eta_{z,\nu}\|_{\alpha,L} \leqq \frac{K}{\ell}\exp(c_2\sigma^{-\gamma})$, where $c_2 = c_2(\alpha, L)$ stems from Lemma A.6. Therefore, the condition

$$q \geqq \ell \geqq L := 2K\mathrm{e}^{c\nu^{-\gamma}}\mathrm{e}^{c_2\sigma^{-\gamma}} \tag{6.6}$$

is sufficient to ensure that u and v satisfy condition (2) of Proposition 4.1. We will fine-tune our choice of ℓ later, when considering the "diffusion speed" of the T-orbit described by (6.5).

(**d**) We note that $\mathcal{V}(\bar{r}, \nu) \times M_2$ is invariant by T because it is invariant by T_0 as well as by Φ^{tu} and Φ^{tv} for all $t \in \mathbb{R}$ in view of the condition on the supports of u and v. Let $b := \frac{1}{4}$. To get condition (3) of Proposition 4.1 and thus complete its proof, we will require

Lemma 6.2. *The set* $\left\{\left(F^s(z_1^{(0)}), z_2^{(n,s)}\right) \mid n, s \in \mathbb{Z},\ \frac{2}{\nu} \leqq s \leqq \frac{6}{\nu}\right\}$ *is* 3ν*-dense in* $\mathcal{V}(\bar{r}, \nu) \times M_2$ *if* (6.6) *holds and* ν *is small enough.*

Taking Lemma 6.2 for granted, we now show how to choose ℓ so as to make $\mathcal{V}(\bar{r}, \nu) \times M_2$ $\left(3\nu, \tau, \tau^b\right)$-diffusive for T with $\tau := E_{3c\gamma,\gamma}(\nu)$.

Given arbitrary $z \in \mathcal{V}(\bar{r}, \nu) \times M_2$, Lemma 6.2 yields n and s integer such that $\hat{z} := \left(F^s(z_1^{(0)}), z_2^{(n,s)}\right) = T^{nq+s}\left(z_1^{(0)}, z_2^{(0,0)}\right)$ is 3ν-close to z. For any $m \geqq 1$, comparing the last coordinates of $\hat{z}$ and $T^{mq}(\hat{z}) = \left(F^s(z_1^{(0)}), z_2^{(n+m,s)}\right)$, namely $-\omega_2 + \frac{n}{q}$ and $-\omega_2 + \frac{n+m}{q}$, we see that $d(T^{mq}(\hat{z}), \hat{z}) \geqq m/q$, hence

$$d(T^{q^3}(\hat{z}), z) \geqq q - 3\nu \quad \text{with } \ell \leqq q \leqq \frac{3\ell}{\nu}. \tag{6.7}$$

Let $\mu := \mathrm{e}^{c\nu^{-\gamma}} = \sigma^{-1/2}$. The number L of (6.6) is $2K\mu\,\mathrm{e}^{c_2\mu^{2\gamma}} \leqq \mu^2\mathrm{e}^{c_2\mu^{2\gamma}} < \mathrm{e}^{(c_2+\frac{1}{\gamma})\mu^{2\gamma}}$ and $(c_2 + \frac{1}{\gamma})\mu^{2\gamma} \leqq b\mu^{3\gamma}$ provided ν is small enough, and then

$$L < \mathrm{e}^{b\,\mathrm{e}^{C\nu^{-\gamma}}} \quad \text{with } C := 3c\gamma. \tag{6.8}$$

Since $b < \frac{1}{3}$, we can satisfy (6.6) by choosing $\ell := \lfloor \frac{\nu}{3}\mathrm{e}^{\frac{1}{3}\mathrm{e}^{C\nu^{-\gamma}}} \rfloor$ and we then have $q^3 \leqq (3\ell/\nu)^3 \leqq \mathrm{e}^{\mathrm{e}^{C\nu^{-\gamma}}} = E_{C,\gamma}(\nu)$.

On the other hand, $q - 3\nu \geqq \ell - 3\nu \geqq \mathrm{e}^{b\,\mathrm{e}^{C\nu^{-\gamma}}} = E_{C,\gamma}(\nu)^b$ for ν small enough. We thus get property (3) of Proposition 4.1 from (6.7). The proof of that Proposition is thus complete up to the proof of Lemma 6.2.

6.3. *Proof of Lemma 6.2*

We keep the notations and assumptions of Section 6.2 and give ourselves an arbitrary $z = (z_1, z_2) \in \mathcal{V}(\bar{r}, \nu) \times M_2$. We look for integers n and s such that $d\big(\big(F^s(z_1^{(0)}), z_2^{(n,s)}\big), (z_1, z_2)\big) \leqq 3\nu$ and $\frac{2}{\nu} \leqq s \leqq \frac{6}{\nu}$.

By property (4) of Proposition 5.1, we can choose the integer s so that

$$d\big(F^s(z_1^{(0)}), z_1\big) \leqq 2\nu, \qquad \tfrac{2}{\nu} \leqq s \leqq \tfrac{6}{\nu}.$$

On the other hand, writing $z_2 = (\theta_2 + \mathbb{Z}, r_2)$ with $\theta_2, r_2 \in \mathbb{Z}$, we see from (6.4) that the last coordinate of $z_2^{(n,s)}$ will be ν-close to r_2 if and only if $|n - q(\omega_2 + r_2)| \leqq \nu q$. Let us denote by n_* the integer nearest to $q(\omega_2 + r_2)$, thus $|n_* - q(\omega_2 + r_2)| \leqq \frac{1}{2}$. To conclude, it is sufficient to take n of the form $n = n_* + m$ with m integer such that

$$|m| \leqq \nu q - \tfrac{1}{2} \quad \text{and} \quad \operatorname{dist}\left(\frac{s(n_* + m)}{q}, \theta_2 + \mathbb{Z}\right) \leqq \nu. \tag{6.9}$$

Indeed, we will then have $d\big(\big(F^s(z_1^{(0)}), z_2^{(n,s)}\big), (z_1, z_2)\big) \leqq \sqrt{4\nu^2 + \nu^2 + \nu^2} < 3\nu$.

The second part of (6.9) is equivalent to $\operatorname{dist}\left(m, \dfrac{q\theta_2}{s} - n_* + \dfrac{q}{s}\mathbb{Z}\right) \leqq \dfrac{\nu q}{s}$. Let

$$I := \Big[-(\nu q - \tfrac{1}{2}), \nu q - \tfrac{1}{2}\Big], \quad J := \Big[x_2 - \frac{\nu q}{s}, x_2 + \frac{\nu q}{s}\Big] \text{ with } x_2 = \frac{q\theta_2}{s} - n_*.$$

The whole of (6.9) is thus equivalent to

$$m \in I \quad \text{and} \quad m \in \frac{kq}{s} + J \text{ for some } k \in \mathbb{Z}. \tag{6.10}$$

Now, $\frac{kq}{s} + J \subset I$ is equivalent to

$$|I| \geqq |J| \quad \text{and} \quad -\tfrac{1}{2}\big(|I| - |J|\big) \leqq \frac{kq}{s} + x_2 \leqq \tfrac{1}{2}\big(|I| - |J|\big). \tag{6.11}$$

Since $\Delta := |I| - |J| = 2\nu q - 1 - 2\nu q/s$, (6.11) amounts to $\Delta \geqq 0$ and k belonging to the interval $\big[-\frac{sx_2}{q} - \frac{s\Delta}{2q}, -\frac{sx_2}{q} + \frac{s\Delta}{2q}\big]$, which has length $s\Delta/q = 2\nu s - 2\nu - s/q \geqq 4 - 2\nu - 6/(\nu L)$ (using $q \geqq L$). That length is $\geqq 1$ for ν small enough and the interval then contains at least one integer k_*. Diminishing ν if necessary, we then have $|\frac{k_* q}{s} + J| = 2\nu q/s \geqq \nu^2 L/3 \geqq 1$, hence we can find $m \in (\frac{k_* q}{s} + J) \cap \mathbb{Z} \subset I \cap \mathbb{Z}$, thus solution to (6.10) or, equivalently, to (6.9).

7. Continuous Time

In Sections 4–6, we have proved Theorems 3.3 and 3.4 , providing examples of discrete systems of the form $\Phi^v \circ \Phi^u \circ \Phi^{h_0} \colon \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ with diffusive invariant tori (using Remark 3.1). We will now deduce Theorems 3.1 and 3.2 by a “suspension” device adapted from [10].

Definition 7.1. Given an exact symplectic map $T: \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$, we call suspension of T any non-autonomous Hamiltonian which depends 1-periodically on time

$$h: \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T} \to \mathbb{R},$$

for which the flow map between the times $t = 0$ and $t = 1$ exists and coincides with T.

Lemma 7.1. *Let $h_0: r \in \mathbb{R}^n \mapsto \langle \omega, r \rangle + \frac{1}{2}\langle r, r \rangle$ as in Section 3.2. Suppose that u and v are C^∞ functions on $\mathbb{T}^n \times \mathbb{R}^n$ which generate complete Hamiltonian vector fields. Let $\psi, \chi \in C^\infty([0, 1])$ be such that*

$$\int_0^1 \psi(t)\,\mathrm{d}t = \int_0^1 \chi(t)\,\mathrm{d}t = 1, \quad \operatorname{supp}(\psi) \subset \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \quad \operatorname{supp}(\chi) \subset \left[\tfrac{2}{3}, 1\right]. \quad (7.1)$$

Then the formula

$$\begin{aligned} h(\theta, r, t) := h_0(r) + \psi(t) u\big(\theta + (1-t)(\omega + r) \\ + \mathbb{Z}^n, r\big) + \chi(t) v\big(\theta + (1-t)(\omega + r) + \mathbb{Z}^n, r\big) \end{aligned} \quad (7.2)$$

defines a function on $\mathbb{T}^n \times \mathbb{R}^n \times [0, 1]$ which uniquely extends by periodicity to a C^∞ function on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}$ and, when viewed as a non-autonomous time-periodic Hamiltonian, is a suspension of $\Phi^v \circ \Phi^u \circ \Phi^{h_0}$.

Proof. Let $\varphi \in C^\infty([0, 1])$ have support $\subset [0, \frac{1}{3}]$ and $\int_0^1 \varphi(t)\,\mathrm{d}t = 1$. We observe that (7.2) entails, for all $(\theta, r, t) \in \mathbb{T}^n \times \mathbb{R}^n \times [0, 1]$,

$$\begin{aligned} h(\theta, r, t) = h_0(r) + \psi(t) u\big(\theta + \tilde{\varphi}(t)(\omega + r) + \mathbb{Z}^n, r\big) \\ + \chi(t) v\big(\theta + \tilde{\varphi}(t)(\omega + r) + \mathbb{Z}^n, r\big), \end{aligned} \quad (7.3)$$

where $\tilde{\varphi}(t) := \int_0^t \big(\varphi(t') - 1\big)\mathrm{d}t'$ extends to a 1-periodic C^∞ function. This takes care of the first statement.

Let $z_0 \in \mathbb{T}^n \times \mathbb{R}^n$ and let $z(t) = \big(\theta(t), r(t)\big)$ denote the maximal solution of the initial value problem $\frac{\mathrm{d}z}{\mathrm{d}t} = X_h(z, t)$, $z(0) = z_0$. Defining $\theta^*(t) := \theta(t) + \tilde{\varphi}(t)\big(\omega + r(t)\big)$ and $z^*(t) := \big(\theta^*(t), r(t)\big)$, we compute

$$\begin{aligned} \tfrac{\mathrm{d}z^*}{\mathrm{d}t}(t) &= \varphi(t) X_{h_0}\big(z^*(t)\big) \quad \text{for } t \in \left[0, \tfrac{1}{3}\right], \\ \tfrac{\mathrm{d}z^*}{\mathrm{d}t}(t) &= \psi(t) X_u\big(z^*(t)\big) \quad \text{for } t \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \\ \tfrac{\mathrm{d}z^*}{\mathrm{d}t}(t) &= \chi(t) X_v\big(z^*(t)\big) \quad \text{for } t \in \left[\tfrac{2}{3}, 1\right]. \end{aligned}$$

The flow map of X_h between the times $t = 0$ and $t = \frac{1}{3}$ is thus a reparametrization of the flow of X_{h_0}: $t \in \left[0, \frac{1}{3}\right] \Rightarrow z^*(t) = \Phi^{h_0}\big(\int_0^t \varphi(t')\,\mathrm{d}t'\big)$, whence $z^*\big(\frac{1}{3}\big) = \Phi^{h_0}(z_0)$ since $\int_0^{1/3} \varphi(t')\,\mathrm{d}t' = 1$. Similarly, $z^*\big(\frac{2}{3}\big) = \Phi^u\big(z^*\big(\frac{1}{3}\big)\big)$ since $\int_{1/3}^{2/3} \psi(t')\,\mathrm{d}t' = 1$, and $z^*(1) = \Phi^v\big(z^*\big(\frac{2}{3}\big)\big)$, since $\int_{2/3}^1 \chi(t')\,\mathrm{d}t' = 1$. We thus get $z^*(1) = \Phi^v \circ \Phi^u \circ \Phi^{h_0}(z_0)$, which yields the desired result because $z^*(1) = z(1)$. $\square$

Notice that, if $T = \Phi^v \circ \Phi^u \circ \Phi^{h_0}$ satisfies properties (1) and (3) of Theorem 3.3, resp. Theorem 3.4, then *any suspension of T of the form* (7.3) *satisfies property (2) of Theorem* 3.1, resp. *Theorem* 3.2. The invariance of the torus $\mathcal{T}_{(r,s)}$ with $r = 0$, resp. $r \in (X_\varepsilon + \mathbb{Z}) \times \mathbb{R}^{n-1}$, stems from the vanishing of $\partial_\theta h$ and $\partial_t h$ for $r_1 = 0$, resp. $r_1 \in X_\varepsilon + \mathbb{Z}$.

Lemma 7.2. *Consider the mapping*

$$\mathcal{S}_\omega \colon\ (\theta, r, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T} \mapsto \big(\theta + (1-t)\omega + \mathbb{Z}^n, r\big) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Let $\alpha \geqq 1$ and $\Lambda > 0$ be real, and $\Lambda_1 \geqq \Lambda\Big(1 + \max\limits_{1 \leqq i \leqq n} |\omega_i|^{1/\alpha}\Big)$. Then

$$w \in G^{\alpha,\Lambda_1}(\mathbb{T}^n \times \mathbb{R}^n) \ \Rightarrow\ w \circ \mathcal{S}_\omega \in G^{\alpha,\Lambda}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$$
$$\textit{and } \|w \circ \mathcal{S}_\omega\|_{\alpha,\Lambda} \leqq \|w\|_{\alpha,\Lambda_1}.$$

Proof. One can check that, for every $(p, q, s) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}$,

$$\partial_\theta^p \partial_r^q \partial_t^s (w \circ \mathcal{S}_\omega) = (-1)^s \sum_{m \in \mathbb{N}^n \text{ s.t. } |m| = s} \omega_1^{m_1} \cdots \omega_n^{m_n} (\partial_\theta^{p+m} \partial_r^q w) \circ \mathcal{S}_\omega,$$

whence, with the notation $\Omega := \max\limits_{1 \leqq i \leqq n} |\omega_i|$,

$$\begin{aligned} \|w \circ \mathcal{S}_\omega\|_{\alpha,\Lambda} &\leqq \sum_{p,q,m \in \mathbb{N}^n} \frac{\Omega^{|m|} \Lambda^{|p+q+m|\alpha}}{p!^\alpha q!^\alpha |m|!^\alpha} \|\partial_\theta^{p+m} \partial_r^q w\|_{C^0(\mathbb{T}^n \times \mathbb{R}^n)} \\ &= \sum_{\ell, q \in \mathbb{N}^n} A_\ell \frac{\Lambda^{|\ell+q|\alpha}}{q!^\alpha} \|\partial_\theta^\ell \partial_r^q w\|_{C^0(\mathbb{T}^n \times \mathbb{R}^n)} \quad \text{with } A_\ell \\ &:= \sum_{p,m \in \mathbb{N}^n \text{ s.t. } p+m=\ell} \frac{\Omega^{|m|}}{p!^\alpha |m|!^\alpha}. \end{aligned}$$

Now, $A_\ell \leqq \sum\limits_{p+m=\ell} \frac{\Omega^{|m|}}{p!^\alpha m!^\alpha} \leqq \Big(\sum\limits_{p+m=\ell} \frac{\Omega^{|m|/\alpha}}{p! m!}\Big)^\alpha = \frac{1}{\ell!^\alpha}(1 + \Omega^{1/\alpha})^{|\ell|\alpha}$, hence

$$\begin{aligned} \|w \circ \mathcal{S}_\omega\|_{\alpha,\Lambda} &\leqq \sum_{\ell, q \in \mathbb{N}^n} \frac{(1 + \Omega^{1/\alpha})^{|\ell|\alpha} \Lambda^{|\ell+q|\alpha}}{\ell!^\alpha q!^\alpha} \|\partial_\theta^\ell \partial_r^q w\|_{C^0} \\ &\leqq \frac{\Lambda_1^{|\ell+q|\alpha}}{\ell!^\alpha q!^\alpha} \|\partial_\theta^\ell \partial_r^q w\|_{C^0} = \|w\|_{\alpha, L_1}. \end{aligned}$$

□

Lemma 7.3. *Consider an interval $I \subset [0, 1]$ and the mapping*

$$\mathcal{R}_I \colon\ (\theta, r, t) \in \mathbb{T}^n \times \mathbb{R}^n \times I \mapsto \big(\theta + (1-t)r + \mathbb{Z}^n, r, t\big) \in \mathbb{T}^n \times \mathbb{R}^n \times I.$$

Let $\alpha \geqq 1$ and $\Lambda > 0$ be real, and $\Lambda \geqq L \max\big\{2^{3/\alpha}, 2^{1/\alpha} L\big\}$. Then

$$\begin{aligned} &w \in G^{\alpha,\Lambda}(\mathbb{T}^n \times \mathbb{R}^n \times I) \\ &\quad \Rightarrow\ w \circ \mathcal{R}_I \in \mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times I) \textit{ and } d_{\alpha,L}(0, w \circ \mathcal{R}_I) \leqq \|w\|_{\alpha,\Lambda}. \end{aligned}$$

Proof. Let $L_j := 2^{-\frac{j-1}{\alpha}} L$ and $R_j := 2^j$ for each $j \in \mathbb{N}^*$, as in Appendix A.1. We also set $\mathcal{K}_j := \mathbb{T}^n \times \overline{B}_{R_j} \times I$, so $\mathcal{G}^{\alpha,L}(\mathbb{T}^n \times \mathbb{R}^n \times I) = \bigcap_{j \geqq 1} G^{\alpha,L_j}(\mathcal{K}_j)$.

A simple adaptation of [10, Remark A.1] shows that, if $\phi \in G^{\alpha,L_j}(I)$ and

$$L_j^\alpha + (R_j + L_j^\alpha)\|\phi\|_{\alpha,L_j,I} - R_j\|\phi\|_{C^0(I)} \leqq \Lambda^\alpha, \tag{7.4}$$

then the composition with the mapping

$$\mathcal{R}\colon\ (\theta, r, t) \in \mathbb{T}^n \times \mathbb{R}^n \times I \mapsto \big(\theta + \phi(t)r + \mathbb{Z}^n, r, t\big) \in \mathbb{T}^n \times \mathbb{R}^n \times I$$

has the property

$$w \in G^{\alpha,\Lambda}(\mathcal{K}_j) \ \Rightarrow\ w \circ \mathcal{R} \in G^{\alpha,L_j}(\mathcal{K}_j) \text{ and } \|w \circ \mathcal{R}\|_{\alpha,L_j,\mathcal{K}_j} \leqq \|w\|_{\alpha,\Lambda}. \tag{7.5}$$

Taking $\phi(t) := 1 - t$, since our interval I is $\subset [0, 1]$, we have $\|\phi\|_{C^0(I)} \leqq 1$ and $\|\phi\|_{\alpha,L_j,I} = \|\phi\|_{C^0(I)} + L_j^\alpha$, hence the left-hand side of (7.4) equals

$$\begin{aligned}&L_j^\alpha + R_j L_j^\alpha + L_j^\alpha\big(\|\phi\|_{C^0(I)} + L_j^\alpha\big)\\ &\leqq L_j^\alpha(L_j^\alpha + R_j + 2) \leqq L^{2\alpha} + 4L^\alpha \leqq \tfrac12\Lambda^\alpha + \tfrac12\Lambda^\alpha\end{aligned}$$

and (7.5) allows us to conclude, in view of (A.4). □

Proof of Theorems 3.1 and 3.2

In both cases, we are given $\alpha > 1$, $L > 0$ and $\varepsilon > 0$. Let

$$\Lambda_1 := \Lambda\Big(1 + \max_{1\leqq i\leqq n} |\omega_i|^{1/\alpha}\Big), \quad \Lambda := L \max\big\{2^{3/\alpha}, 2^{1/\alpha}L\big\}.$$

Since $\alpha > 1$, we can pick $\psi, \chi \in G^{\alpha,\Lambda}(\mathbb{T})$ satisfying (7.1).

Let us apply the multidimensional version of Theorem 3.3 or Theorem 3.4 (*cf.* Remark 3.1) with parameters Λ_1 instead of L and

$$\varepsilon_1 := \frac{\varepsilon}{\max\big\{1, \|\psi\|_{\alpha,\Lambda}, \|\chi\|_{\alpha,\Lambda}\big\}}$$

instead of ε. We thus get $u, v \in G^{\alpha,\Lambda_1}(\mathbb{T}^n \times \mathbb{R}^n)$ and, in the second case, $X_{\varepsilon_1} \subset [0, 1]$, such that $\|u\|_{\alpha,\Lambda_1} + \|v\|_{\alpha,\Lambda_1} < \varepsilon_1$ and any suspension of $\Phi^v \circ \Phi^u \circ \Phi^{h_0}$ satisfies property (2) of Theorem 3.1, resp. Theorem 3.2.

By Lemma 7.1, we can choose the suspension to be

$$h := h_0 + \tilde u \circ \mathcal{R}_{[\frac13,\frac23]} + \tilde v \circ \mathcal{R}_{[\frac23,1]}$$

with

$$\tilde u(\theta, r, t) := \psi(t)(u \circ \mathcal{S}_\omega)(\theta, r, t), \quad \tilde v(\theta, r, t) := \chi(t)(v \circ \mathcal{S}_\omega)(\theta, r, t).$$

Lemma 7.2 and (A.2) give

$$\|\tilde u\|_{\alpha,\Lambda} \leqq \|\psi\|_{\alpha,\Lambda}\|u\|_{\alpha,\Lambda_1}, \quad \|\tilde v\|_{\alpha,\Lambda} \leqq \|\chi\|_{\alpha,\Lambda}\|v\|_{\alpha,\Lambda_1},$$

whence $\|\tilde u\|_{\alpha,\Lambda} + \|\tilde v\|_{\alpha,\Lambda} < \varepsilon$. Then, since the distance $d_{\alpha,L}$ is translation-invariant,

$$d_{\alpha,L}(h_0, h) \leqq d_{\alpha,L}\big(0, \tilde u \circ \mathcal{R}_{[\frac13,\frac23]}\big) + d_{\alpha,L}\big(0, \tilde v \circ \mathcal{R}_{[\frac23,1]}\big) \leqq \|\tilde u\|_{\alpha,\Lambda} + \|\tilde v\|_{\alpha,\Lambda}$$

by Lemma 7.3, and we are done.

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Appendix A: Gevrey Estimates

We fix real numbers $\alpha \geqq 1$ and $L > 0$.

A.1 Gevrey Functions and Gevrey Maps

Here we adapt definitions and facts taken from [7,10,11].

The Banach Algebra $G^{\alpha,L}(\mathbb{R}^M \times K)$ of Uniformly Gevrey-(α, L) Functions
Let $N \geqq 1$ be integer. We will deal with real functions of N variables defined on $\mathbb{R}^M \times K$, where $M \geqq 0$ and $K \subset \mathbb{R}^{N-M}$ is a Cartesian product of closed Euclidean balls and tori.
We define the uniformly Gevrey-(α, L) functions on $\mathbb{R}^M \times K$ by

$$G^{\alpha,L}(\mathbb{R}^M \times K) := \{f \in C^\infty(\mathbb{R}^M \times K) \mid \|f\|_{\alpha,L} < \infty\},$$
$$\|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^M \times K)}. \tag{A.1}$$

We have used the standard notations $|\ell| = \ell_1 + \cdots + \ell_N$, $\ell! = \ell_1! \dots \ell_N!$, $\partial^\ell = \partial_{x_1}^{\ell_1} \dots \partial_{x_N}^{\ell_N}$, and $\mathbb{N} := \{0, 1, 2, \dots\}$. The space $G^{\alpha,L}(\mathbb{R}^M \times K)$ turns out to be a Banach algebra, with

$$\|fg\|_{\alpha,L} \leqq \|f\|_{\alpha,L} \|g\|_{\alpha,L} \tag{A.2}$$

for all f and g, and there are "Cauchy–Gevrey inequalities": if $0 < L_0 < L$, then

$$\sum_{m \in \mathbb{N}^N;\ |m|=p} \|\partial^m f\|_{\alpha,L_0} \leqq \frac{p!^\alpha}{(L - L_0)^{p\alpha}} \|f\|_{\alpha,L} \quad \text{for all } p \in \mathbb{N}. \tag{A.3}$$

When necessary, we use the notation $\| \,.\, \|_{\alpha,L,\mathbb{R}^M \times K}$ instead of $\| \,.\, \|_{\alpha,L}$ to keep track of the domain to which the norm relates.

The Metric Space $\mathcal{G}^{\alpha,L}(\mathbb{R}^M \times K)$ When $M \geqq 1$, instead of restricting ourselves to uniformly Gevrey-(α, L) functions on $\mathbb{R}^M \times K$, we may cover the factor $\mathbb{R}^M$ by an increasing sequence of closed balls and consider a Fréchet space accordingly. For technical reasons, we choose the sequences

$$L_j := 2^{-\frac{j-1}{\alpha}} L, \quad R_j := 2^j \qquad \text{for } j \in \mathbb{N}^*,$$

and set

$$\begin{aligned} \mathcal{G}^{\alpha,L}(\mathbb{R}^M \times K) &:= \bigcap_{j \geqq 1} G^{\alpha,L_j}\big(\overline{B}_{R_j} \times K\big), \\ & \qquad d_{\alpha,L}(f,g) := \sum_{j \geqq 1} 2^{-j} \min\big\{1, \|g - f\|_{\alpha,L_j,\overline{B}_{R_j} \times K}\big\}. \end{aligned} \tag{A.4}$$

Clearly, $G^{\alpha,L}(\mathbb{R}^M \times K) \subset \mathcal{G}^{\alpha,L}(\mathbb{R}^M \times K)$ but the inclusion is strict, and the larger space is a complete metric space for the distance $d_{\alpha,L}$.
This construction is needed in Section 7 only. In the rest of this appendix, we focus on uniformly Gevrey functions and maps on $\mathbb{R}^N$ (with $M = N$ and no factor K).

Composition with Uniformly Gevrey-(α, L) Maps For $N \geqq 1$ integer, we define

$$\begin{aligned} G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N) &:= \{F \in C^\infty(\mathbb{R}^N, \mathbb{R}^N) \mid \|F\|_{\alpha,L} < \infty\}, \\ \|F\|_{\alpha,L} &:= \|F_{[1]}\|_{\alpha,L} + \cdots + \|F_{[N]}\|_{\alpha,L}. \end{aligned} \tag{A.5}$$

This is a Banach space.
We also define

$$\mathcal{N}^*_{\alpha,L}(f) := \sum_{\ell \in \mathbb{N}^N \smallsetminus \{0\}} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^N)},$$

so that $\|f\|_{\alpha,L} = \|f\|_{C^0(\mathbb{R}^N)} + \mathcal{N}^*_{\alpha,L}(f)$.

Lemma A.0. *Let $L_0 \in (0, L)$. There exists $\varepsilon_{\mathrm{c}} = \varepsilon_{\mathrm{c}}(\alpha, L, L_0, N)$ such that, for any $f \in G^{\alpha,L}(\mathbb{R}^N)$ and $F = (F_{[1]}, \ldots, F_{[N]}) \in G^{\alpha,L_0}(\mathbb{R}^N, \mathbb{R}^N)$, if*

$$\mathcal{N}^*_{\alpha,L_0}(F_{[1]}), \ldots, \mathcal{N}^*_{\alpha,L_0}(F_{[N]}) \leqq \varepsilon_{\mathrm{c}},$$

then $f \circ (\mathrm{Id} + F) \in G^{\alpha,L_0}(\mathbb{R}^N)$ and $\|f \circ (\mathrm{Id} + F)\|_{\alpha,L_0} \leqq \|f\|_{\alpha,L}$.

The proof is in Appendix A of [7].

A.2 Comparison Estimates for Gevrey Flows

In Section 4.2, we use comparison estimates for the flows of two nearby Gevrey Hamiltonian systems. We prove them here, building upon some facts which are proved in [7] about the flows of Gevrey vector fields.

Lemma A.1. (General case) *Suppose that* $0 < L_0 < L$ *and* $N \geqq 1$. *Then there exists* $\varepsilon_{\mathrm{f}} = \varepsilon_{\mathrm{f}}(\alpha, L, L_0, N) > 0$ *such that, for every vector field* $X \in G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)$ *with* $\|X\|_{\alpha,L} \leqq \varepsilon_{\mathrm{f}}$, *the time*-1 *map* Φ *of the flow generated by* X *satisfies that*

$$\|\Phi - \mathrm{Id}\,\|_{\alpha,L_0} \leqq \|X\|_{\alpha,L}, \tag{A.6}$$

and, if we are given another vector field $\tilde{X} \in G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)$ *with* $\|\tilde{X}\|_{\alpha,L} \leqq \varepsilon_{\mathrm{f}}$, *then its time*-1 *map* $\tilde{\Phi}$ *satisfies*

$$\|\tilde{\Phi} - \Phi\|_{\alpha,L_0} \leqq 2\|\tilde{X} - X\|_{\alpha,L}. \tag{A.7}$$

Proof. The first part of the statement is exactly Part (i) of Lemma A.1 from [7]. There, the flow $t \in [0,1] \mapsto \Phi(t)$ was obtained by considering the functional $\xi \mapsto \mathcal{F}(\xi)$ defined by

$$\mathcal{F}(\xi)(t) := \int_0^t X \circ \big(\mathrm{Id} + \xi(\tau)\big)\,\mathrm{d}\tau.$$

Using an auxiliary $L' \in (L_0, L)$ and Lemma A.0, it was shown that, if $\|X\|_{\alpha,L} \leqq \varepsilon_{\mathrm{f}}$ small enough, then $\mathcal{F}$ maps into itself

$$\mathcal{B} := \{\, \xi \in C^0\big([0,1], G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)\big) \mid \|\xi\| \leqq \|X\|_{\alpha,L} \,\}$$

(which is a closed ball in a Banach space) and has a unique fixed point, none other than $\xi^*(t) := \Phi(t) - \mathrm{Id}$.

In that proof, $\mathcal{F}$ was shown to be K-Lipschitz, with $K := \max_{i,j} \|\partial_{x_j} X_{[i]}\|_{\alpha,L'}$. We can ensure $K \leqq \frac{1}{2}$ by diminishing ε_{f} if necessary and using (A.3). Then, for *any* $\xi_0 \in \mathcal{B}$, the fixed point ξ^* is the limit of the sequence of iterates $(\mathcal{F}^k(\xi_0))_{k\in\mathbb{N}}$ and $\|\xi^* - \xi_0\| \leqq 2\|\mathcal{F}(\xi_0) - \xi_0\|$.

Now, suppose we also have $\|\tilde{X}\|_{\alpha,L} \leqq \varepsilon_{\mathrm{f}}$. The time-$t$ map of $\tilde{X}$ is thus $\tilde{\Phi}(t) = \mathrm{Id} + \tilde{\xi}^*(t)$, with $\tilde{\xi}^*$ fixed point of $\tilde{\mathcal{F}}\colon\ \mathcal{B} \circlearrowleft$. Lemma A.0 yields

$$\|\tilde{\mathcal{F}}(\xi) - \mathcal{F}(\xi)\| = \Big\|\int_0^t (\tilde{X} - X) \circ \big(\mathrm{Id} + \xi(\tau)\big)\,\mathrm{d}\tau\Big\| \leqq \|\tilde{X} - X\|_{\alpha,L} \quad \text{for any } \xi \in \mathcal{B},$$

thus we can compare the fixed points ξ^* and $\tilde{\xi}^*$ by writing the former as the limit of the sequence $(\mathcal{F}^k(\xi_0))_{k\in\mathbb{N}}$ with $\xi_0 := \tilde{\xi}^*$; we get

$$\|\xi^* - \tilde{\xi}^*\| \leqq 2\|\mathcal{F}(\tilde{\xi}^*) - \tilde{\xi}^*\| = 2\|\mathcal{F}(\tilde{\xi}^*) - \tilde{\mathcal{F}}(\tilde{\xi}^*)\| \leqq 2\|\tilde{X} - X\|_{\alpha,L},$$

which yields the desired result. □

Lemma A.2. (Hamiltonian case) *Suppose that* $0 < L_0 < L$ *and* $n \geqq 1$. *Then there exist* $\varepsilon_{\mathrm{H}}, C_0 > 0$ *such that, for every* $u \in G^{\alpha,L}(\mathbb{R}^{2n})$ *with* $\|u\|_{\alpha,L} \leqq \varepsilon_{\mathrm{H}}$,

$$\|\Phi^u - \mathrm{Id}\,\|_{\alpha,L_0} \leqq C_0\|u\|_{\alpha,L}, \tag{A.8}$$

and, given another $\tilde{u} \in G^{\alpha,L}(\mathbb{R}^{2n})$ *with* $\|\tilde{u}\|_{\alpha,L} \leqq \varepsilon_{\mathrm{H}}$,

$$\|\Phi^{\tilde{u}} - \Phi^u\|_{\alpha,L_0} \leqq C_0\|\tilde{u} - u\|_{\alpha,L}. \tag{A.9}$$

Proof. Let $L' := (L_0+L')/2$. Any $u \in G^{\alpha,L}(\mathbb{R}^{2n})$ generates a Hamiltonian vector field X_u which, according to (A.3) with $p = 1$, satisfies

$$\|X_u\|_{\alpha,L'} = \sum_{m\in\mathbb{N}^{2n};\ |m|=1} \|\partial^m u\|_{\alpha,L'} \leqq (L-L')^{-\alpha}\|u\|_{\alpha,L}.$$

Similarly, $\|X_{\tilde{u}} - X_u\|_{\alpha,L'} \leqq (L-L')^{-\alpha}\|\tilde{u} - u\|_{\alpha,L}$. Thus, with $\varepsilon_{\mathrm{H}} := (L-L')^{\alpha}\varepsilon_{\mathrm{f}}(\alpha, L', L_0, 2n)$ and $C_0 := 2(L-L')^{-\alpha}$, we get

$$\begin{aligned}&\|u\|_{\alpha,L}, \|\tilde{u}\|_{\alpha,L} \leqq \varepsilon_{\mathrm{H}} \;\Rightarrow\; \|\Phi^u - \mathrm{Id}\,\|_{\alpha,L_0} \leqq \tfrac{1}{2}C_0\|u\|_{\alpha,L}\\ &\text{and } \|\Phi^{\tilde{u}} - \Phi^u\|_{\alpha,L_0} \leqq C_0\|\tilde{u} - u\|_{\alpha,L}.\end{aligned}$$

□

Corollary A.3. *(Iteration of maps of the form* $\Phi^v \circ \Phi^u \circ T_0$*) Suppose that* $n \geqq 1$. *Then there exist* $\varepsilon_{\mathrm{d}}, C_1 > 0$ *such that, for every* $u, v, \tilde{u}, \tilde{v} \in G^{\alpha,L}(\mathbb{R}^{2n})$ *such that*

$$\|u\|_{\alpha,L} + \|v\|_{\alpha,L} \leqq \varepsilon_{\mathrm{d}}, \quad \|\tilde{u}\|_{\alpha,L} + \|\tilde{v}\|_{\alpha,L} \leqq \varepsilon_{\mathrm{d}} \tag{A.10}$$

and for every $z \in \mathbb{R}^{2n}$, *the orbits of* z *under the maps* $T := \Phi^v \circ \Phi^u \circ T_0$ *and* $\tilde{T} := \Phi^{\tilde{v}} \circ \Phi^{\tilde{u}} \circ T_0$ *satisfy*

$$\operatorname{dist}\big(T^k(z), \tilde{T}^k(z)\big) \leqq 3^k C_1\big(\|\tilde{u} - u\|_{\alpha,L} + \|\tilde{v} - v\|_{\alpha,L}\big) \quad \textit{for all } k \in \mathbb{N}. \tag{A.11}$$

Proof. For any $u, v, \tilde{u}, \tilde{v} \in G^{\alpha,L}(\mathbb{R}^{2n})$ and $z, \tilde{z} \in \mathbb{R}^{2n}$, the maps $T := \Phi^v \circ \Phi^u \circ T_0$ and $\tilde{T} := \Phi^{\tilde{v}} \circ \Phi^{\tilde{u}} \circ T_0$ satisfy

$$\begin{aligned}\operatorname{dist}\big(T(z), \tilde{T}(z)\big) &\leqq \operatorname{dist}\big(\Phi^v(\Phi^u(T_0(z))), \Phi^v(\Phi^u(T_0(\tilde{z})))\big)\\ &\quad + \operatorname{dist}\big(\Phi^v(\Phi^u(T_0(\tilde{z}))), \Phi^v(\Phi^{\tilde{u}}(T_0(\tilde{z})))\big)\\ &\quad + \operatorname{dist}\big(\Phi^v(\Phi^{\tilde{u}}(T_0(\tilde{z}))), \Phi^{\tilde{v}}(\Phi^{\tilde{u}}(T_0(\tilde{z})))\big)\\ &\leqq (\operatorname{Lip}\Phi^v)(\operatorname{Lip}\Phi^u)(\operatorname{Lip}T_0)\operatorname{dist}(\tilde{z}, z) + (\operatorname{Lip}\Phi^v)\|\Phi^{\tilde{u}} - \Phi^u\|_{C^0(\mathbb{R}^{2n})}\\ &\quad + \|\Phi^{\tilde{v}} - \Phi^v\|_{C^0(\mathbb{R}^{2n})}.\end{aligned}$$

On the one hand, $\operatorname{Lip}T_0 = 2$. On the other hand, for any $L_0 > 0$, the Lipschitz constant of a map Ψ such that $\Psi - \mathrm{Id} \in G^{\alpha,L_0}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is bounded by $1 + \operatorname{Lip}(\Psi - \mathrm{Id}) \leqq 1 + L_0^{-\alpha}\|\Psi - \mathrm{Id}\,\|_{\alpha,L_0}$ (using the mean value inequality, (A.1) and (A.5)). Applying Lemma A.2 with $L_0 = L/2$, we can thus choose ε_{d} so that assumption (A.10) entails

$$\operatorname{Lip} \Phi^u, \operatorname{Lip} \Phi^v \leqq 1 + 2^\alpha L^{-\alpha} C_0 \varepsilon_{\mathrm{d}} \leqq (3/2)^{1/2}$$

and

$$\|\Phi^{\tilde{u}} - \Phi^u\|_{\alpha,L_0} \leqq C_0 \|\tilde{u} - u\|_{\alpha,L}, \quad \|\Phi^{\tilde{v}} - \Phi^v\|_{\alpha,L_0} \leqq C_0 \|\tilde{v} - v\|_{\alpha,L},$$

whence $\operatorname{dist}\big(T(z), \tilde{T}(z)\big) \leqq 3 \operatorname{dist}(\tilde{z}, z) + \eta$ with $\eta := (3/2)^{1/2} C_0 \big(\|\tilde{u} - u\|_{\alpha,L} + \|\tilde{v} - v\|_{\alpha,L}\big)$. Iterating this, we get $\operatorname{dist}\big(T^k(z), \tilde{T}^k(z)\big) \leqq 3^k (\operatorname{dist}(\tilde{z}, z) + \frac{1}{2}\eta) - \frac{1}{2}\eta$ for all $k \in \mathbb{N}$, thus we can conclude by choosing $C_1 := \frac{1}{2}(3/2)^{1/2} C_0$. □

A.3 A Gevrey Inversion Result

In Section 5.2, we use the following

Lemma A.4. *Suppose $L < L_1$. Then there exists $\varepsilon_{\mathrm{i}} = \varepsilon_{\mathrm{i}}(\alpha, L, L_1)$ such that, for every $\varepsilon \in G^{\alpha,L_1}(\mathbb{R})$, if $\|\varepsilon\|_{\alpha,L_1} \leqq \varepsilon_{\mathrm{i}}$, then $\mathrm{Id} + \varepsilon$ is a diffeomorphism of $\mathbb{R}$ and*

$$(\mathrm{Id} + \varepsilon)^{-1} = \mathrm{Id} + \tilde{\varepsilon} \quad \textit{with} \quad \|\tilde{\varepsilon}\|_{\alpha,L} \leqq \|\varepsilon\|_{\alpha,L_1}, \tag{A.12}$$

$$\| * \|g \circ (\mathrm{Id} + \varepsilon)^{-1}{}_{\alpha,L} \leqq \|g\|_{\alpha,L_1} \quad \textit{for any } g \in G^{\alpha,L_1}(\mathbb{R}). \tag{A.13}$$

Proof. Let $L' := (L + L')/2$. We use Lemma A.0 and define

$$\varepsilon_{\mathrm{i}} := \min\big\{\tfrac{1}{2}(L_1 - L')^\alpha, \varepsilon_{\mathrm{c}}(\alpha, L', L, 1), \varepsilon_{\mathrm{c}}(\alpha, L_1, L, 1)\big\}.$$

Given $\varepsilon \in G^{\alpha,L_1}(\mathbb{R})$ such that $\|\varepsilon\|_{\alpha,L_1} \leqq \varepsilon_{\mathrm{i}}$, the functional

$$\mathcal{F}\colon\ f \in \mathcal{B} \mapsto -\varepsilon \circ (\mathrm{Id} + f), \quad \text{where } \mathcal{B} := \{f \in G^{\alpha,L}(\mathbb{R}) \mid \|f\|_{\alpha,L} \leqq \|\varepsilon\|_{\alpha,L_1}\},$$

is well defined (because $\|\varepsilon\|_{\alpha,L_1} \leqq \varepsilon_{\mathrm{c}}(\alpha, L', L, 1)$ and $\varepsilon \in G^{\alpha,L'}(\mathbb{R})$), maps $\mathcal{B}$ into itself (we even have $\|\mathcal{F}(f)\| \leqq \|\varepsilon\|_{\alpha,L'}$), and is K-Lipschitz with $K := \|\varepsilon'\|_{\alpha,L'}$ (using also (A.2) and the mean value inequality). But (A.3) yields $\|\varepsilon'\|_{\alpha,L'} \leqq (L_1 - L')^{-\alpha} \|\varepsilon\|_{\alpha,L_1} \leqq \frac{1}{2}$, which implies that $\mathcal{F}$ is a contraction, and also that $\mathrm{Id} + \varepsilon$ is a diffeomorphism of $\mathbb{R}$ (since its derivative stays $\geqq 1/2$). The unique fixed point $\tilde{\varepsilon}$ of $\mathcal{F}$ in $\mathcal{B}$ is $(\mathrm{Id} + \varepsilon)^{-1} - \mathrm{Id}$, which yields $\|\tilde{\varepsilon}\|_{\alpha,L} \leqq \|\varepsilon\|_{\alpha,L_1} \leqq \varepsilon_{\mathrm{c}}(\alpha, L_1, L, 1)$ and hence (A.13) by another application of Lemma A.0.

□

A.4 Gevrey Functions with Small Support

From now on we suppose $\alpha > 1$. We quote, without proof, Lemma 3.3 of [11]:

Lemma A.5. *There exists a real $c_1 = c_1(\alpha, L) > 0$ such that, for each real $p > 2$, the space $G^{\alpha,L}(\mathbb{T})$ contains a function η_p which takes its values in $[0, 1]$ and satisfies*

$$-\frac{1}{2p} \leqq \theta \leqq \frac{1}{2p} \ \Rightarrow\ \eta_p(\theta + \mathbb{Z}) = 1, \qquad \frac{1}{p} \leqq \theta \leqq 1 - \frac{1}{p} \ \Rightarrow\ \eta_p(\theta + \mathbb{Z}) = 0$$

and

$$\|\eta_p\|_{\alpha,L} \leqq \exp\left(c_1\, p^{\frac{1}{\alpha-1}}\right). \tag{A.14}$$

The proof can be found in [11, p. 1633]. This easily implies

Lemma A.6. *There exists a real $c_2 = c_2(\alpha, L) > 0$ such that, for any $z \in \mathbb{T} \times \mathbb{R}$ and $\nu > 0$, there is a function $\eta_{z,\nu} \in G^{\alpha,L}(\mathbb{T} \times \mathbb{R})$ which takes its values in $[0, 1]$ and satisfies*

$$\eta_{z,\nu} \equiv 1 \text{ on } B(z, \nu/2), \qquad \eta_{z,\nu} \equiv 0 on B(z, \nu)^c$$

and

$$\|\eta_{z,\nu}\|_{\alpha,L} \leqq \exp(c_2 \nu^{-\frac{1}{\alpha-1}}). \tag{A.15}$$

Here, for arbitrary $\tilde{\nu} > 0$, we have denoted by $B(z, \tilde{\nu})$ the closed ball relative to $\|\,.\,\|_\infty$ centred at z with radius $\tilde{\nu}$.

Appendix B: Some Estimates on Doubly Exponentially Growing Sequences

According to (4.3), the increasing sequence $(N_i)_{i \geqq 1}$ is defined by

$$N_1 := \lceil * \rceil \exp(4\kappa/\varepsilon), \qquad N_i := N_{i-1} \lceil * \rceil \exp\left(\exp\left(\tilde{C}(N_{i-1} \ln N_{i-1})^\gamma\right)\right) \ \text{ for } i \geqq 2,$$

where $0 < \varepsilon \leqq 1$, $\kappa \geqq 1$ and $\tilde{C} := \max\{6c\gamma, 1/\gamma\}$, with $c, \gamma > 0$. Here, we show a few inequalities which are used in Section 4.2. Recall that $\nu_i := \frac{1}{N_i \ln N_i}$ and $\xi_i := \mathrm{e}^{-c\nu_i^{-\gamma}}$.

Lemma B.1. *One has*

$$\ln N_i \geqq 4^i \kappa/\varepsilon \qquad \textit{for every } i \geqq 1, \tag{B.1}$$

$$N_{i+1}\xi_{i+1} \leqq \tfrac{1}{2} N_i \xi_i \qquad \textit{for } i \textit{ large enough}, \tag{B.2}$$

$$N_{i+1}\xi_{i+1} \leqq 3^{-E_{3c\gamma,\gamma}(\nu_i)} \qquad \textit{for } i \textit{ large enough}. \tag{B.3}$$

Proof. We have $\ln(N_1) \geqq 4\kappa/\varepsilon$ and, by virtue of (4.1), $N_1 \geqq 4\kappa/\varepsilon \geqq 4$. Now, for $i \geqq 2$, since $\gamma \tilde{C} \geqq 1$, we have

$$\begin{aligned} \ln N_i \geqq \exp\left(\tilde{C}(N_{i-1} \ln N_{i-1})^\gamma\right) &= \left[\exp\left(\gamma \tilde{C}(N_{i-1} \ln N_{i-1})^\gamma\right)\right]^{1/\gamma} \\ &\geqq \left[\exp(N_{i-1} \ln N_{i-1})^\gamma\right]^{1/\gamma} \end{aligned}$$

and (4.1) yields $\ln(N_i) \geqq N_{i-1} \ln N_{i-1} \geqq 4 \ln N_{i-1}$, whence (B.1) follows.
We have $\ln \frac{1}{N_i \xi_i} = c(N_i \ln N_i)^\gamma - \ln N_i$ and, since $\ln(N_i) \ll (N_i \ln N_i)^\gamma$,

$$c\Lambda_i^\gamma \geqq \ln \frac{1}{N_i \xi_i} \geqq c(\Lambda_i/\sqrt{3})^\gamma \quad \text{for } i \text{ large enough,} \quad \text{where } \Lambda_i := N_i \ln N_i = 1/\nu_i.$$

Inequality (B.2), being equivalent to

$$\ln \frac{1}{N_{i+1}\xi_{i+1}} \geqq \ln \frac{1}{N_i \xi_i} + \ln 2 \quad \text{for } i \text{ large enough,}$$

thus results from $(\Lambda_{i+1}/\sqrt{3})^\gamma \geqq \Lambda_i^\gamma + \frac{\ln 2}{c}$ (which holds for i large enough because $N_{i+1} \geqq 3N_i$, hence $\Lambda_{i+1} = N_{i+1} \ln N_{i+1} > 3\Lambda_i$).
Let $C := 3c\gamma$. Inequality (B.3), being equivalent to

$$\ln \frac{1}{N_{i+1}\xi_{i+1}} \geqq (\ln 3) E_{C,\gamma}(1/\Lambda_i) \quad \text{for } i \text{ large enough,}$$

results from $\Lambda_{i+1}^\gamma \geqq \frac{3^{\gamma/2} \ln 3}{c} E_{C,\gamma}(1/\Lambda_i)$, which holds since $E_{C,\gamma}(1/\Lambda_i) = \lceil * \rceil \exp\big(\exp(C\Lambda_i^\gamma)\big)$ and

$$\Lambda_{i+1}^\gamma = N_i^\gamma (\ln N_{i+1})^\gamma \lceil * \rceil \exp\big(\gamma \exp(\tilde{C}\Lambda_i^\gamma)\big), \quad N_i^\gamma (\ln N_{i+1})^\gamma \geqq \frac{3^{\gamma/2} \ln 3}{c}$$

and $\gamma \exp(\tilde{C}\Lambda_i^\gamma) \geqq \exp(C\Lambda_i^\gamma)$ for i large enough since $\tilde{C} > C$. $\square$

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