Periodic Solutions of Symmetric Hamiltonian Systems

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Abstract

This paper is devoted to the study of periodic solutions of a Hamiltonian system $\dot{z}(t) = J\nabla H(z(t))$, where H is symmetric under an action of a compact Lie group. We are looking for periodic solutions in a neighborhood of non-isolated critical points of H which form orbits of the group action. We prove a Lyapunov-type theorem for symmetric Hamiltonian systems.

1. Introduction

Consider a first-order system

$$\dot{z}(t) = J\nabla H(z(t)) \tag{HS}$$

on $\mathbb{R}^{2N}$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix and $H : \mathbb{R}^{2N} \to \mathbb{R}$ is a Hamiltonian of the class C^2.

The existence of periodic orbits in Hamiltonian dynamics is an important and widely studied problem. In 1895 LYAPUNOV [20] proved his center theorem, i.e. the existence of a one-parameter family of periodic solutions of (HS) tending to a non-degenerate equilibrium. The next important result of WEINSTEIN [33] shows the existence of at least N geometrically distinct periodic solutions at any energy level of the Hamiltonian H. The further development of the Weinstein theorem was performed by MOSER [23]. In 1978 FADELL and RABINOWITZ [8] proved the lower bound for the number of small nontrivial solutions of (HS) depending on the period. See [27] for the general overview of the results up to 1982. The results of WEINSTEIN and MOSER were generalized by BARTSCH in 1997, [2]. The problem of the existence of periodic solutions of (HS) in a case of a degenerate equilibrium was also studied by SZULKIN [32] and DANCER with RYBICKI [6], who generalized the classical result of LYAPUNOV.

Suppose now that the compact Lie group Γ acts unitary on $\mathbb{R}^{2N}$ and $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ is a Γ-invariant potential i.e. $H(\gamma z) = H(z)$ for any $\gamma \in \Gamma$ and $z \in \mathbb{R}^{2N}$. The study of the existence of periodic solutions in this case was performed by MONTALDI et al. [22], who proved an equivariant version of the WEINSTEIN-MOSER theorem. In 1993 BARTSCH [1] generalized the theorem of MONTALDI, ROBERTS and STEWART for the wider class of a group actions which allowed him to generalize the result of FADELL and RABINOWITZ also. However, the authors mentioned above assumed that a critical point z_0 of H is a fixed point of the group action, i.e. the orbit of this action consists of one point. Then z_0 can be an isolated critical point.

We study a more general case. Assume that z_0 is a critical point of Hamiltonian H. Since H is Γ-invariant, $\Gamma(z_0) = \{\gamma z_0 : \gamma \in \Gamma\} \subset (\nabla H)^{-1}(0)$, i.e. the orbit $\Gamma(z_0)$ consists of critical points of H and, therefore, stationary solutions of the equation (HS); see Remark 2.4. Hence, if $\dim \Gamma_{z_0} < \dim \Gamma$ then the orbit is at least an one-dimensional manifold and, as a consequence, critical points are not isolated. Therefore the results mentioned above are not applicable. We are going to prove sufficient conditions for the existence of non-constant periodic solutions of an autonomous Hamiltonian system in the presence of symmetries of a compact Lie group the problem (HS) in any neighborhood of the orbit $\Gamma(z_0)$; see the main result Theorem 4.1 and Theorems 5.2, 5.3, 5.4.

This article is organized as follows: in Section 2 we recall some basic definitions and notions of group theory and equivariant topology. The equivariant Conley index which is a main tool of our reasoning is shortly defined in Section 2.3. Furhtermore, we recall the notion of a Euler ring: an equivariant Euler characteristic and its generalization (see Remarks 2.11, 2.12). In Theorem 2.16 we recall the very important theorem connecting an equivariant Euler characteristic, an equivariant Conley index and the idea of an orthogonal section introduced in the paper [24]. In Section 2.5 we formulate a so called equivariant splitting lemma—the theorem which allows us to simplify the study of Conley indexes up to the linear case in Lemma 4.7.

In Section 3 we parameterize the equation (HS) to study the solutions with constant period 2π in the equation (HS-P). Next we introduce an appropriate Sobolev space $\mathbb{E}$ and the action of the group $\Gamma \times S^1$ on it, and we define variational functional $\Phi : \mathbb{E} \to \mathbb{R}$ (see formulas (3.4), (3.5)) such that 2π-periodic solutions of the system (HS-P) are in bijective correspondence with S^1-orbits of critical points of Φ. In this way we begin to study the equation (3.7). Furthermore, we analyze the linear Hamiltonian system (HS-L); it is a base for the last step in the proof of the main result of the paper.

Section 4 is devoted to the formulation and the proof of the main result of this paper, Theorem 4.1. The notion of bifurcation theory is recalled in Definition 4.3 and in the nearby text. In Theorems 4.4 and 4.5 we formulate the necessary and sufficient condition for the existence of global bifurcation of solutions of the equation (3.7). The last part of this section is devoted to the proof of the change of equivariant Euler characteristics of equivariant Conley indexes, i.e. the formula (4.1). Firstly, we reduce our task to the space orthogonal to the orbit; see Lemma 4.6 and the text above them. Next, in Lemma 4.7 we reduce the problem to the linear case. To

study we prove Theorem 4.9. The flow of this work is summarized in Remark 4.10. Afterwards, the proof of Theorem 4.1 is finally done by the study of the minimal periods and the convergence of new solutions.

In the fifth section we reformulate the main result to make the assumptions easier to verify. The most friendly version of our result is the following theorem (see Theorem 5.4), where $m^+(A)$ denotes the positive Morse index of the symmetric matrix A:

Theorem 1.1. *Let* $H : \mathbb{R}^{2N} \to \mathbb{R}$ *be a* Γ-*invariant Hamiltonian of the class* C^2. *Let* z_0 *be a critical point of* H *such that* $\Gamma_{z_0} = \{e\}$ *and the orbit* $\Gamma(z_0)$ *is isolated in* $(\nabla H)^{-1}(0)$. *Assume that* $m^+(\nabla^2 H(z_0)) \neq N$ *and* $\deg(\nabla H_{|T_{z_0}^\perp \Gamma(z_0)}, B(z_0, \varepsilon), 0) \neq 0$ *for sufficiently small* ε. *Then there exists a connected family of non-stationary periodic solutions of the system* $\dot{z}(t) = J\nabla H(z(t))$ *emanating from the stationary solution* z_0 *such that periods (not necessarily minimal) of solutions in the small neighborhood of* z_0 *are close to* $2\pi/\beta_j$, *where* $i\beta_j$, $\beta_j > 0$, *is some eigenvalue of* $J\nabla^2 H(z_0)$.

For the two other versions see Theorems 5.2 and 5.3.

Furthermore, we show that the Lyapunov-type theorem of DANCER and RYBICKI (Theorem 5.5) is generalized by the main result of this paper: Theorem 4.1. In the last part of this section we reformulate the second-order Newtonian system to the Hamiltonian one. Then the two symmetric versions of the Lyapunov center theorem, Theorem 5.6 proven in [24] and Theorem 5.7 proven in [25], are also the consequences of the results proven in this paper.

The last section is devoted to AN APPLICATION of the abstract results of this paper. We study the existence of quasi-periodic motions of the satellite in a nearby of a geostationary orbit of an oblate spheroid. In order to do this we consider a gravitational motion in the rotating frame where the corresponding Hamiltonian is given by formula (6.2). It is $SO(2)$ invariant and possesses a critical point which represents the geostationary orbit in the original coordinates. Theorem 5.4 will be directly applied in this problem to prove the existence of trajectories with arbitrarily small deviations from the geostationary ones.

2. Preliminaries

In this section we recall the basic material on equivariant topology from [7], [17] and prove some preliminary results. Throughout this section G stands for a compact Lie group.

2.1. Groups and Their Representations

We denote by $\overline{\text{sub}}(G)$ the set of all closed subgroups of G. Two subgroups $H, H' \in \overline{\text{sub}}(G)$ are said to be conjugate in G if there is $g \in G$ such that $H = gH'g^{-1}$. The conjugacy is an equivalence relation on $\overline{\text{sub}}(G)$. The class of $H \in \overline{\text{sub}}(G)$ we denote by $(H)_G$ and the set of conjugacy classes will be denoted by $\overline{\text{sub}}[G]$.

If $x \in \mathbb{R}^n$ then $G(x) = \{gx : g \in G\}$ is called the orbit through x and a group $G_x = \{g \in G : gx = x\} \in \mathrm{sub}(G)$ is said to be the isotropy group of x. It is known that if $G(x_1) = G(x_2)$ then $(G_{x_1})_G = (G_{x_2})_G$ i.e. the isotropy groups of the elements of common orbit are conjugate. Moreover, the orbit $G(x)$ is a smooth G-manifold G-diffeomorphic to G/G_x. An open subset $\Omega \subset \mathbb{R}^n$ is said to be G-invariant if $G(x) \subset \Omega$ for every $x \in \Omega$.

Below we recall the notion of an admissible pair, which was introduced in [24], where one can find some examples and properties.

Definition 2.1. Fix $H \in \overline{\mathrm{sub}}(G)$. A pair (G, H) is said to be *admissible* if for any $K_1, K_2 \in \overline{\mathrm{sub}}(H)$ the following condition is satisfied: if $(K_1)_H \neq (K_2)_H$ then $(K_1)_G \neq (K_2)_G$.

Note that if Γ is a compact Lie group, then the pair $(\Gamma \times S^1, \{e\} \times S^1)$ is admissible; see Lemma 2.1 of [24]. This property will play a crucial role in the proof of the main result, Theorem 4.1.

Recall that a unitary group $U(N)$ is defined by

$$U(N) = Sp(2N, \mathbb{R}) \cap O(2N),$$

where

$$Sp(2N, \mathbb{R}) = \{A \in M_{2N \times 2N}(\mathbb{R}) : A^T J A = J\}$$

is a symplectic group and

$$O(2N, \mathbb{R}) = \{A \in M_{2N \times 2N}(\mathbb{R}) : A^T A = Id\}$$

is an orthogonal group. In particular, if $A \in U(N)$ then $JA = AJ$. Note that $U(N)$ is a compact subgroup of $GL(2N)$, $A \in U(N)$ implies $A^T = A^{-1} \in U(N)$ and $|\det A| = 1$.

Let $\rho : G \to U(N)$ be a continuous homomorphism. The space $\mathbb{R}^{2N}$ with the G-action defined by $G \times \mathbb{R}^{2N} \ni (g, x) \to \rho(g)x \in \mathbb{R}^{2N}$ is said to be a real, unitary representation of G which we write $\mathbb{V} = (\mathbb{R}^{2N}, \rho)$. To simplify notation we write gx instead of $\rho(g)x$ and $\mathbb{R}^{2N}$ instead of $\mathbb{V}$ if the homomorphism is given in general.

Two unitary representations of G, say $\mathbb{V} = (\mathbb{R}^{2N}, \rho)$, $\mathbb{V}' = (\mathbb{R}^{2N}, \rho')$, are equivalent (briefly $\mathbb{V} \approx_G \mathbb{V}'$) if there exists an equivariant linear isomorphism $L : \mathbb{V} \to \mathbb{V}'$ i.e. an isomorphism L satisfying $L(gx) = gL(x)$ for any $g \in G$, $x \in \mathbb{R}^{2N}$. Put $D(\mathbb{V}) = \{x \in \mathbb{V} : \|x\| \leq 1\}$, $S(\mathbb{V}) = \partial D(\mathbb{V})$, $S^{\mathbb{V}} = D(\mathbb{V})/S(\mathbb{V})$ and $B_{\mathbb{V}}(x_0, r) = \{x \in \mathbb{V} : \|x - x_0\| < r\}$. Since the representation $\mathbb{V}$ is orthogonal in particular, these sets are G-invariant if $G(x_0) = \{x_0\}$.

2.2. G-Equivariant Maps

Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be a unitary G-representation. Fix an open G-invariant subset $\Omega \subset \mathbb{V}$.

Definition 2.2. A map $\phi : \Omega \to \mathbb{R}$ of class C^k is called G-*invariant* C^k-*potential*, if $\phi(gx) = \phi(x)$ for every $g \in G$ and $x \in \Omega$. The set of G-invariant C^k-potentials will be denoted by $C_G^k(\Omega, \mathbb{R})$.

Definition 2.3. A map $\psi : \Omega \to \mathbb{V}$ of the class C^{k-1} is called G-*equivariant* C^{k-1}-*map*, if $\psi(gx) = g\psi(x)$ for every $g \in G$ and $x \in \Omega$. The set of G-equivariant C^{k-1}-maps will be denoted by $C_G^{k-1}(\Omega, \mathbb{V})$.

Fix $\varphi \in C_G^2(\Omega, \mathbb{R})$ and denote by $\nabla\varphi, \nabla^2\varphi$ the gradient and the Hessian of φ, respectively. For $x_0 \in \Omega$ denote by $m^-(\nabla^2\varphi(x_0))$ the Morse index of the Hessian of φ at x_0 i.e. the sum of the multiplicities of negative eigenvalues of the symmetric matrix $\nabla^2\varphi(x_0)$. Similarly, by the $m^+(\nabla^2\varphi(x_0))$ we denote the sum of the multiplicities of positive eigenvalues of $\nabla^2\varphi(x_0)$.

Remark 2.4. It is clear that if $\varphi \in C_G^k(\Omega, \mathbb{R})$, then $\nabla\varphi \in C_G^{k-1}(\Omega, \mathbb{V})$. Moreover, if $x_0 \in (\nabla\varphi)^{-1}(0)$, then $G(x_0) \subset (\nabla\varphi)^{-1}(0)$ i.e. the G-orbit of a critical point consists of critical points. If $\nabla\varphi(x_0) = 0$ then $\nabla\varphi(\cdot)$ is fixed on $G(x_0)$. That is why $T_{x_0}G(x_0) \subset \ker \nabla^2\varphi(x_0)$ and consequently $\dim \ker \nabla^2\varphi(x_0) \geqq \dim T_{x_0}G(x_0) = \dim G(x_0)$.

2.3. Equivariant Conley Index

We denote by $\mathcal{F}_*(G)$ the category of finite pointed G-CW-complexes (see [7] for definition and examples), where morphisms are continuous G-equivariant maps preserving a base points; we denote by $\mathcal{F}_*[G]$ the set of G-homotopy types of elements of $\mathcal{F}_*(G)$, where $[\mathbb{X}]_G \in \mathcal{F}_*[G]$ (or $[\mathbb{X}]$ when no confusion can arise) denotes a G-homotopy type of the pointed G-CW complex $\mathbb{X} \in \mathcal{F}_*(G)$. If $\mathbb{X}$ is a G-CW-complex without a base point, then we denote by $\mathbb{X}^+$ a pointed G-CW-complex $\mathbb{X}^+ = \mathbb{X} \cup \{*\}$.

Now we briefly recall the definition of the equivariant version of the classical Conley index, see [1,4,9,10,30] for the details. Consider a finite-dimensional unitary G-representation $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ and $\mathcal{U} \subset \mathbb{V} \times \mathbb{R}$. Let $\eta : \mathcal{U} \to \mathbb{V}$ be a G-flow i.e. a flow which is equivariant under the G-action on $\mathbb{V}$. For G-invariant set $X \subset \mathbb{V}$ we denote by $Inv(X, \eta) = \{x \in X : \eta(x, t) \in X$ supposing $(x, t) \in \mathcal{U}\}$. Compact G-invariant set $S \subset \mathbb{V}$ is called isolated η-invariant set if $S \subset Inv(W, \eta) \subset \mathrm{int}W$ for some G-invariant set $W \subset \mathbb{V}$ which is called an η-isolating neighborhood.

Let S be isolated η-invariant set.

Definition 2.5. A pair of compact G-invariant sets $(\mathcal{N}, \mathcal{L})$, where $\mathcal{L} \subset \mathcal{N} \subset \mathbb{V}$, is called a G-*index pair* for the set S if

1. $\mathrm{cl}(\mathcal{N}/\mathcal{L})$ is an η-isolating neighborhood for S,
2. $\mathcal{L}$ is positive η-invariant in $\mathcal{N}$ i.e. if $x \in \mathcal{L}$ and $\eta(x, [0, t]) \subset \mathcal{N}$ then $\eta(x, [0, t]) \subset \mathcal{L}$,
3. $\mathcal{L}$ is the set of exit points from the set $\mathcal{N}$ i.e. if $x \in \mathcal{N}$ and $t_1 > 0$ are such that $\eta(x, t_1) \notin \mathcal{N}$ then there exist $t_0 \in [0, t_1)$ such that $\eta(x, [0, t_0]) \subset \mathcal{N}$ and $\eta(x, t_0) \in \mathcal{L}$.

It is known that for any isolated invariant set there exists a G-index pair contained in its isolating neighborhood. Moreover, for any two G-index pairs $(\mathcal{N}_1, \mathcal{L}_1)$, $(\mathcal{N}_2, \mathcal{L}_2)$ for the set S two G-homotopic types $[\mathcal{N}_1/\mathcal{L}_1]_G$, $[\mathcal{N}_2/\mathcal{L}_2]_G$ are equal. Therefore we are able to define the equivariant Conley index as the G-homotopic type of the pointed space

$$\mathcal{CI}_G(S, \eta) = [\mathcal{N}/\mathcal{L}, [\mathcal{L}]]_G.$$

Recall that $\mathcal{CI}_G(S, \eta) \in \mathcal{F}_*[G]$, see [10]. Definition of the classical Conley index (without G-action) coincides with the above construction with $G = \{e\}$.

Example 2.6. Consider $\mathbb{R}^n$ as an orthogonal representation of a group G. Let η be a flow generated by a gradient vector field $-\nabla F$, where $F \in C_G^2(\mathbb{R}^n, \mathbb{R})$, 0 is an isolated critical point of F and the hessian $\nabla^2 F(0)$ is an isomorphism. It is known that $\{0\}$ is an isolated η-invariant set and by Hartman-Grobman theorem the flow η is locally homeomorphic to the flow generated by the linearized vector field $y \mapsto -\nabla^2 F(0)y$. Denote by $\mathbb{R}^{n-}$ and $\mathbb{R}^{n+}$ the generalized eigenspaces of $\nabla^2 F(0)$ corresponding to the negative and positive eigenvalues. Then for sufficiently small $\varepsilon > 0$ we have $\mathcal{N} = D_{\mathbb{R}^{n-}}(0, \varepsilon) \times D_{\mathbb{R}^{n+}}(0, \varepsilon)$ and $\mathcal{L} = \partial D_{\mathbb{R}^{n-}}(0, \varepsilon) \times D_{\mathbb{R}^{n+}}(0, \varepsilon)$. Since the action of G is orthogonal, these sets are G-invariant. Moreover, the pairs $(\mathcal{N}, \mathcal{L})$ and $(D_{\mathbb{R}^{n-}}(0, \varepsilon), \partial D_{\mathbb{R}^{n-}}(0, \varepsilon))$ have the same homotopy type. Therefore $\mathcal{CI}_G(\{0\}, \eta) = [\mathcal{N}/\mathcal{L}, [\mathcal{L}]]_G = [D_{\mathbb{R}^{n-}}(0, \varepsilon)/\partial D_{\mathbb{R}^{n-}}(0, \varepsilon), [\partial D_{\mathbb{R}^{n-}}(0, \varepsilon)]]_G = [S^{\dim \mathbb{R}^{n-}}, *]_G = [S^{m^-(\nabla^2 F(0))}, *]_G$.

Note that an index pair $(\mathcal{N}, \mathcal{L})$ can be chosen without defining any isolated η-invariant set. Moreover, the Conley index is defined only by the sets $\mathcal{N}, \mathcal{L}$. Therefore, it is convenient to consider the Conley index of an isolating neighborhood $\mathcal{CI}_G(\mathcal{N}, \eta) = \mathcal{CI}_G(Inv(\mathcal{N}, \eta), \eta)$.

The most important properties of the Conley index are given in the following theorem:

Theorem 2.7. *Let $\mathcal{N}$ be an η-isolating neighborhood, where η is a G-flow. Then*

1. *if the index $\mathcal{CI}_G(\mathcal{N}, \eta)$ is nontrivial (i.e. is not a homotopy type of one G-fixed point) then $Inv(\mathcal{N}, \eta) \neq \emptyset$, i.e. there exists a complete η-trajectory contained in $\mathcal{N}$,*
2. *if $\{\eta_t\}_{t \in T}$ is a continuous family of G-flows and $\mathcal{N}$ is an η_t-isolated neighborhood for all $t \in T$, then the index $\mathcal{CI}_G(\mathcal{N}, \eta_t)$ is the same for all parameters $t \in T$.*

Below we present the infinite-dimensional extension of the equivariant Conley index to Hilbert spaces due to IZYDOREK [16]. The construction is similar to the developing of the Leray-Schauder degree from the Brouwer degree by finite-dimensional approximations. However, the index is not constant for sufficiently large approximations but only stabilize in the sense described below. Therefore, the construction requires the notation of equivariant spectra, see also [11,29]. Let $\xi = (\mathbb{V}_n)_{n=0}^{\infty}$ be a sequence of finite-dimensional orthogonal G-representations.

Definition 2.8. A pair $\mathcal{E}(\xi) = \left((\mathcal{E}_n)_{n=n(\mathcal{E}(\xi))}^\infty, (\varepsilon_n)_{n=n(\mathcal{E}(\xi))}^\infty\right)$, where $n(\mathcal{E}(\xi)) \in \mathbb{N}$, is called a G-spectrum of type ξ if

1. $\mathcal{E}_n \in \mathcal{F}_*(G)$ for $n \geq n(\mathcal{E}(\xi))$,
2. $\varepsilon_n \in Mor_G(S^{\mathbb{V}_n} \wedge \mathcal{E}_n, \mathcal{E}_{n+1})$ for $n \geq n(\mathcal{E}(\xi))$,
3. there exists $n_1(\mathcal{E}(\xi)) \geq n(\mathcal{E}(\xi))$ such that for $n > n_1(\mathcal{E}(\xi))$, ε_n is a G-homotopy equivalence.

The last property tells about some stabilization of the spectrum in the sense of a homotopy equivalence of spaces. The set of G-spectra of type ξ is denoted by $GS(\xi)$.

Definition 2.9. A G-map of G-spectra $\mathcal{E}(\xi), \mathcal{E}'(\xi)$ is a sequence of maps $f = (f_n)_{n=n_0}^\infty : \mathcal{E}(\xi) \to \mathcal{E}'(\xi)$, where $n_1 \geq \max(n_1(\mathcal{E}(\xi)), n_1(\mathcal{E}'(\xi)))$, such that

1. $f_n \in Mor_G(\mathcal{E}_n, \mathcal{E}'_n)$ for $n \geq n_1$,
2. G-maps $f_{n+1} \circ \varepsilon_n$ and $\varepsilon'_n \circ S^{\mathbb{V}_n} f_n$ are G-homotopic for every $n \geq n_1$, where $S^{\mathbb{V}} f_n$ denotes a suspension of f_n.

Two G-maps $f, g : \mathcal{E}(\xi) \to \mathcal{E}'(\xi)$ are G-homotopic if there exists $n_1 \geq n_0$ such that $f_n, g_n : \mathcal{E} \to \mathcal{E}'$ are G-homotopic for $n \geq n_1$. Following this definition in a natural way we understand a G-homotopy equivalence of two spectra $\mathcal{E}(\xi), \mathcal{E}'(\xi)$. The G-homotopy type of a G-spectrum $\mathcal{E}(\xi)$ will be denoted by $[\mathcal{E}(\xi)]_G$ (or shorter $[\mathcal{E}(\xi)]$) and the set of G-homotopy types of G-spectra by $[GS(\xi)]$ or simply $[GS]$ when ξ is fixed or is not known yet.

Remark 2.10. It follows from definition of G-spectrum (Definition 2.8) that the G-homotopy type $[\mathcal{E}(\xi)]$ of spectrum $\mathcal{E}(\xi) = \left((\mathcal{E}_n)_{n=n(\mathcal{E}(\xi))}^\infty, (\varepsilon_n)_{n=n(\mathcal{E}(\xi))}^\infty\right)$ depends only on the sequence $(\mathcal{E}_n)_{n=n_1(\mathcal{E}(\xi))}^\infty$.

We can consider G-spectra as a direct extension of G-CW-complexes if we consider a constant spectrum (each space is a given G-CW-complex).

Now we define a infinite-dimensional generalization of equivariant Conley index given by IZYDOREK [16] in the case we deal with. Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional orthogonal Hilbert representation of a compact Lie group G. Let $L : \mathbb{H} \to \mathbb{H}$ be a linear, bounded, self-adjoint and G-equivariant operator such that

(B.1) $\mathbb{H} = \overline{\bigoplus_{n=0}^\infty \mathbb{H}_n}$, where all subspaces $\mathbb{H}_n$ are mutually orthogonal G-representations of finite dimension,

(B.2) $H_0 = \ker L$ and $L(\mathbb{H}_n) = \mathbb{H}_n$ for all $n \geq 1$,

(B.3) 0 is not an accumulation point of $\sigma(L)$.

Put $\mathbb{H}^n := \bigoplus_{k=0}^n \mathbb{H}_k$ and denote by $P_n : \mathbb{H} \to \mathbb{H}^n$ the orthogonal projection onto $\mathbb{H}^n$. Moreover, denote by $\mathbb{H}_k^+$ the subspace of $\mathbb{H}_k$ corresponding to the positive part of spectrum of L. Consider a functional $\Phi : \mathbb{H} \to \mathbb{R}$ such that $\nabla\Phi(x) = Lx + \nabla K(x)$, where $\nabla K \in C_H^1(\mathbb{H}, \mathbb{H})$ is completely continuous. Denote by ϑ a G-$\mathcal{LS}$-flow, see Definition 2.1 of [16], generated by $\nabla\Phi$. Let $\mathcal{O}$ an isolating G-neighborhood for ϑ and put $\mathcal{N} = Inv(\mathcal{O}, \vartheta)$. Set $\xi = (\mathbb{H}_k^+)_{k=1}^\infty$. Let $\Phi_n : \mathbb{H}^n \to$

$\mathbb{R}$ be given by $\Phi_n = \Phi_{|\mathbb{H}^n}$ and ϑ_n denotes the G-flow generated by $\nabla \Phi_n$. Note that $\nabla \Phi_n(x) = Lx + P_n \circ \nabla K(x)$, i.e. it is the n-th approximation of the original flow. Choose sufficiently large n_0 such that for $n \geqq n_0$ the set $\mathcal{O}_n := \mathcal{O} \cap \mathbb{H}^n$ is an isolating G-neighborhood for the flow ϑ_n. Then the set $Inv(\mathcal{O}_n, \vartheta_n)$ admits a G-index pair (Y_n, Z_n) and we are able to define the Conley index $\mathcal{CI}_G(\mathcal{O}_n, \vartheta_n) = [Y_n/Z_n, [Z_n]] \in \mathcal{F}_*[G]$.

We define the spectrum $\mathcal{E}(\xi) := (Y_n/Z_n)_{n=n_0}^\infty$. Then the equivariant Conley index of $\mathcal{O}$ with respect to the flow ϑ is given by $\mathcal{CI}_G(\mathcal{O}, \vartheta) := [\mathcal{E}(\xi)] \in [GS]$.

We will write a vector field and an isolated invariant set or a flow and and isolating neighborhood synonymously i.e. $\mathcal{CI}_G(\mathcal{N}, \nabla \Phi) \equiv \mathcal{CI}_G(\mathcal{O}, \vartheta)$. The equivariant Conley index defined above inherits the properties of the finite-dimensional Conley index described in Theorem 2.7. The second of this properties (known as a continuation) provides the suitable local bifurcation theorem. Since we are going to prove the existence of a connected branch of solutions we need to apply a bifurcation theorem in some degree theory, therefore in the next section we briefly introduce an equivariant Euler characteristic.

2.4. Equivariant Euler Characteristic

Let $(U(G), +, \star))$ be the Euler ring of G, see [7] for the definition and more details. Let us briefly recall that the Euler ring $U(G)$ is commutative, generated by $\chi_G(G/H^+)$, where $(H) \in \overline{\text{sub}}[G]$ with the unit $\mathbb{I}_{U(G)} = \chi_G(G/G^+)$, where $\chi_G : \mathcal{F}_*[G] \to U(G)$ is the universal additive invariant for finite pointed G-CW-complexes known as the equivariant Euler characteristic.

Remark 2.11. Below we present some properties of the Euler characteristic $\chi_G(\cdot)$.

– For $\mathbb{X}, \mathbb{Y} \in \mathcal{F}_*(G)$ we have: $\chi_G(\mathbb{X}) + \chi_G(\mathbb{Y}) = \chi_G(\mathbb{X} \vee \mathbb{Y})$ and $\chi_G(\mathbb{X}) \star \chi_G(\mathbb{Y}) = \chi_G(\mathbb{X} \wedge \mathbb{Y})$.
– If $\mathbb{W}$ is a G-representation then $\chi_G(S^{\mathbb{W}})$ is an invertible element of $U(G)$, see [5].
– If G is connected and $\mathbb{V}, \mathbb{V}'$ are G-representations such that $\dim \mathbb{V} > \dim \mathbb{V}'$ but $\mathbb{V} \not\approx_G \mathbb{V}' \oplus \mathbb{W}$, where $\mathbb{W}$ is even-dimensional trivial G-representations then

$$\chi_G(S^{\mathbb{V}}) \neq \chi_G(S^{\mathbb{V}'}).$$

For the proof of this fact see Lemma 3.4 in [18].

For the trivial group $G = \{e\}$ the equivariant Euler characteristic $\chi_{\{e\}}$ is the well-known Euler characteristic and $U(\{e\}) = \mathbb{Z}$.

There is a natural extension of the equivariant Euler characteristic for finite pointed G-CW-complexes to the category of G-equivariant spectra due to GOŁĘBIEWSKA and RYBICKI [14]. Since a spectrum does not have to be constant from some point but only stabilizes, to define some element of Euler ring as an equivariant Euler characteristic of a spectrum we need to utilize this kind of stabilization.

Let $\xi = (\mathbb{V}_n)_{n=0}^{\infty}$ and put $\mathbb{V}^n = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_n$, for $n \geq 0$. Recall that due to Remark 2.11 an element $\chi_G(S^{\mathbb{V}^n})$ is invertible in the Euler ring $U(G)$ and define a map $\Upsilon_G : [GS(\xi)] \to U(G)$ by the following formula

$$\Upsilon_G([\mathcal{E}(\xi)]) = \lim_{n \to \infty} \left(\chi_G \left(S^{\mathbb{V}^{n-1}} \right)^{-1} \star \chi_G(\mathcal{E}_n) \right). \tag{2.1}$$

Remark 2.12. It was shown in [14] that Υ_G is well-defined. In fact,

$$\Upsilon_G([\mathcal{E}(\xi)]) = \chi_G \left(S^{\mathbb{V}^{n_1(\mathcal{E})-1}} \right)^{-1} \star \chi_G(\mathcal{E}_{n_1(\mathcal{E})}), \tag{2.2}$$

where $n_1(\mathcal{E}) = n_1(\mathcal{E}(\xi))$ comes from Definition 2.8.

Remark 2.13. Note that a finite pointed G-CW-complex $\mathbb{X}$ can be considered as a *constant* spectrum $\mathcal{E}(\xi)$, where $\mathcal{E}_n = \mathbb{X}$ for all $n \geq 0$ and ξ is a sequence of trivial, one-point representations. Then $\Upsilon_G([\mathbb{X}]) = \Upsilon_G([\mathcal{E}(\xi)]) = \mathbb{I}_{U(G)}^{-1} \star \chi_G([\mathbb{X}]) = \chi_G([\mathbb{X}])$. Therefore we can treat $\mathcal{CI}_G$ and Υ_G as natural extensions of $\mathcal{CI}_G$ and χ_G, respectively.

In theorem 3.5 of [14] we find a very important formula connecting an equivariant Conley index, an equivariant Euler characteristic and a degree for equivariant gradient maps, defined in [13].

Theorem 2.14. *Denote by η a local G-$\mathcal{LS}$-flow generated by $-\nabla\Phi$. Let $\mathcal{O}$ be an isolated η-invariant G-set. Then*

$$\Upsilon_G\left(\mathcal{CI}_G(\mathcal{O}, \eta) \right) = \nabla_G - deg(\nabla\Phi, \mathcal{O}).$$

By the above result and Theorem 3.1 of [14] we obtain the following product formula:

Theorem 2.15. *If $\mathcal{N}_1$, $\mathcal{N}_2$ are isolated G-invariant sets for the local G-$\mathcal{LS}$ flows generated by $\nabla\Psi_1$ and $\nabla\Psi_2$, respectively, then*

$$\Upsilon_G\left(\mathcal{CI}_G\left(\mathcal{N}_1 \times \mathcal{N}_2, (\nabla\Psi_1, \nabla\Psi_2) \right) \right) = \Upsilon_G\left(\mathcal{CI}_G(\mathcal{N}_1, \nabla\Psi_1) \right) \star \Upsilon_G\left(\mathcal{CI}_G(\mathcal{N}_2, \nabla\Psi_2) \right).$$

The next theorem is one of the most important fact in our reasoning. It allows us to simplify the distinguishing of the infinite-dimensional equivariant Conley indexes, significantly. In the view of theorem 2.14 and good properties of the degree for equivariant gradient maps it will provide the existence of global bifurcation in the proof of the main result.

Let $\mathbb{H} = \overline{\bigoplus_{n=0}^{\infty} \mathbb{H}_n}$ be a representation of the compact Lie group G. Consider two functionals $\varphi_1, \varphi_2 \in C_G^2(\mathbb{H}, \mathbb{R})$ such that $\nabla\varphi_i = Lx + \nabla K_i(x)$, where $\nabla K_i \in C_G^1(\mathbb{H}, \mathbb{H})$ is completely continuous for $i = 1, 2$, which satisfy the conditions (B.1)–(B.3) described previously in Section 2.3. Note that $T_x^{\perp}G(x)$, a space orthogonal to the orbit, is a representation of the isotropy group G_x and if φ is G-invariant then $\varphi_{|T_x^{\perp}G(x)}$ is G_x-invariant.

Theorem 2.16. ([25], Theorem 2.4.3) *Let $G(x_1)$, $G(x_2)$ be isolated orbits of critical points of the potentials φ_1 and φ_2, respectively. Moreover, assume that $G_{x_1} = G_{x_2}(= H)$. If the pair (G, H) is admissible and $\Upsilon_H(\mathbb{CI}_H(\{x_1\}, -\nabla\phi_1)) \neq \Upsilon_H(\mathbb{CI}_H(\{x_2\}, -\nabla\phi_2)) \in U(H)$ where $\phi_i = \varphi_{i\,|T_{x_i}^{\perp}G(x_i)}$, then*

$$\Upsilon_G(\mathbb{CI}_G(G(x_1), -\nabla\varphi_1)) \neq \Upsilon_G(\mathbb{CI}_G(G(x_2), -\nabla\varphi_2)) \in U(G).$$

The proof of the theorem above is based on a concept of smash product over group. One can find more details in [25], especially Definition 2.4.2, Theorem 2.4.1 and Theorem 2.4.2.

2.5. Equivariant Splitting Lemma

Let K be a compact Lie group and let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be an orthogonal Hilbert representation of K with an invariant scalar product $\langle \cdot, \cdot \rangle$. Assume additionally that $\dim \mathbb{V}^K < \infty$. Here and subsequently, $\Omega \subset \mathbb{V}$ stands for an open and invariant subset of $\mathbb{V}$ such that $0 \in \Omega$.

Consider a functional $\Psi \in C_K^2(\Omega, \mathbb{R})$ given by the formula

$$\Psi(x) = \frac{1}{2}\langle Ax, x \rangle + \zeta(x), \tag{2.3}$$

which satisfies the following assumptions:

(F.1) $A : \mathbb{V} \to \mathbb{V}$ is a K-equivariant self-adjoint linear Fredholm operator,
(F.2) $\ker A \subset \mathbb{V}^K$,
(F.3) $\nabla\zeta : \mathbb{V} \to \mathbb{V}$ is a K-equivariant, compact operator,
(F.4) $\nabla\zeta(0) = 0$ and $||\nabla^2\zeta(0)|| = 0$,
(F.5) $0 \in \Omega$ is an isolated critical point of Ψ.

Denote by $\ker A$ and $\operatorname{im} A$ the kernel and the image of $\nabla^2\Psi(0) = A$, respectively. Notice that both, $\ker A$ and $\operatorname{im} A$, are orthogonal representations of K. Moreover, $\ker A$ is finite dimensional and trivial representation of K. Since A is self-adjoint, $\mathbb{V} = \ker A \oplus \operatorname{im} A$. Put $x = (u, v)$, where $u \in \ker A$ and $v \in \operatorname{im} A$.

The following theorem (known as the *splitting lemma*) proves the existence of equivariant homotopy which allows us to study the product (splitted) flow $(\nabla\varphi(u), Av)$, where $u \in \ker A$, $v \in \operatorname{im} A$ instead of the general $\Psi(x) = \frac{1}{2}\langle Ax, x \rangle + \zeta(x)$ (the proof of this theorem one can find in [25] (Theorem 2.5.2)):

Theorem 2.17. *Suppose that a functional $\Psi \in C_K^2(\Omega, \mathbb{R})$ is defined by formula (2.3) and satisfies assumptions (F.1)–(F.5). Then, there exists $\varepsilon_0 > 0$ and K-equivariant gradient homotopy $\nabla\mathcal{H} : (B_{\varepsilon_0}(\ker A) \times B_{\varepsilon_0}(\operatorname{im} A)) \times [0, 1] \to \mathbb{V}$ satisfying the following conditions:*

1. *$\nabla\mathcal{H}((u, v), t) = Av - \nabla\xi_t(u, v)$, for $t \in [0, 1]$, where $\nabla\xi_t = \nabla\xi(\cdot, t)$ and $\nabla\xi : \mathbb{V} \times [0, 1] \to \mathbb{V}$ is compact and K–equivariant;*
2. *$(\nabla\mathcal{H})^{-1}(0) \cap (B_{\varepsilon_0}(\ker A) \times B_{\varepsilon_0}(\operatorname{im} A)) \times [0, 1] = \{0\} \times [0, 1]$ i.e. 0 is an isolated critical point of $\nabla\mathcal{H}(\cdot, t)$ for any $t \in [0, 1]$;*

3. $\nabla \mathcal{H}((u, v), 0) = \nabla \Psi(u, v)$;

4. *There exists an K–equivariant, gradient mapping $\nabla \varphi : B_{\varepsilon_0}(\ker A) \to \ker A$ such that $\nabla \mathcal{H}((u, v), 1) = (\nabla \varphi(u), Av)$, for all $(u, v) \in B_{\varepsilon_0}(\ker A) \times B_{\varepsilon_0}(\operatorname{im} A)$.*

Remark 2.18. The homotopy $\mathcal{H}$ is given by

$$\mathcal{H}((u, v), t) = \frac{1}{2}\langle Av, v \rangle + \frac{1}{2}t(2 - t)\langle Aw(u), w(u) \rangle$$
$$+ t\zeta(u, w(u)) + (1 - t)\zeta(u, v + tw(u)).$$

Moreover, from the proof of Theorem 2.17 it follows that the potential $\varphi : B_{\varepsilon_0}(\ker A) \to \mathbb{R}$ is given by $\varphi(u) = \Psi(u, w(u))$, where $w : B_{\varepsilon}(\ker A) \to \operatorname{im} A \cap \mathbb{V}^K$ is K-equivariant; see Remark 2.5.1 in [25].

Remark 2.19. Note that we don't assume that $\ker A \neq \{0\}$. In the case of trivial kernel the homotopy given in Theorem 2.17 provides a linearization of functional.

3. Variational Formulation for Hamiltonian Systems

Recall that we are interested in the existence of periodic solutions with any period of the system (HS). In order to find them we are going to study 2π-periodic solutions of the parameterized system

$$\dot{z}(t) = \lambda J \nabla H(z(t)), \tag{HS-P}$$

which is in one-to-one correspondence to $2\pi\lambda$-periodic solutions of the system (HS).

To prove the existence of solutions of the Hamiltonian system (HS-P) we are going to the study critical points of a corresponding functional.

Define the Sobolev space of 2π-periodic $\mathbb{R}^{2N}$-valued functions

$$\mathbb{E} := \left\{ z(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt) : a_i, b_i \in \mathbb{R}^{2N}, \sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty \right\}.$$

Then $\mathbb{E} = \overline{\mathbb{E}_0 \oplus \bigoplus_{k=1}^{\infty} \mathbb{E}_k}$ where $\mathbb{E}_0 = \mathbb{R}^{2N}$ is a subspace of constant functions and $\mathbb{E}_k = \{a \cos(kt) + b \sin(kt) : a, b \in \mathbb{R}^{2N}\}$. Moreover $\mathbb{E}_k = \mathbb{E}_k^- \oplus \mathbb{E}_k^+$ for $k \geq 1$, where

$$\mathbb{E}_k^- = \{a \cos(kt) + Ja \sin(kt) : a \in \mathbb{R}^{2N}\},$$
$$\mathbb{E}_k^+ = \{a \cos(kt) - Ja \sin(kt) : a \in \mathbb{R}^{2N}\}.$$

The space $\mathbb{E}$ with inner product given by

$$\langle z, \tilde{z} \rangle_{\mathbb{E}} := 2\pi a_0 \cdot \tilde{a}_0 + \pi \sum_{k=1}^{\infty} k(a_k \cdot \tilde{a}_k + b_k \cdot \tilde{b}_k), \tag{3.1}$$

where $\cdot$ denotes the standard scalar product, is a Hilbert space usually denoted by $\mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N})$. Since we consider $\mathbb{R}^{2N}$ as a unitary representation of the compact Lie group Γ, $\mathbb{E}$ is a unitary $G = \Gamma \times S^1$-representation with the action given by

$$G \times \mathbb{E} \ni ((\gamma, e^{i\theta}), z(t)) \to \gamma z(t + \theta) \tag{3.2}$$

and $\mathbb{E}_k$ is a unitary G-representation for any $k \geqq 1$. Indeed,

$$a_k \cos(k(t + \theta)) + b_k \sin(k(t + \theta))$$
$$= \begin{bmatrix} \cos(k\theta) \cdot Id_{2N} & \sin(k\theta) \cdot Id_{2N} \\ -\sin(k\theta) \cdot Id_{2N} & \cos(k\theta) \cdot Id_{2N} \end{bmatrix} \begin{bmatrix} a_k \cos(kt) \\ b_k \sin(kt) \end{bmatrix},$$

and therefore the action proposed in (3.2) is given on $\mathbb{E}_k$ by the product of unitary matrices $\begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$ and $\begin{bmatrix} \cos(k\theta) \cdot Id_{2N} & \sin(k\theta) \cdot Id_{2N} \\ -\sin(k\theta) \cdot Id_{2N} & \cos(k\theta) \cdot Id_{2N} \end{bmatrix}$.

Remark 3.1. Since we are going to study the Hamiltonian system (HS-P) is a neighborhood of the orbit of critical points $\Gamma(z_0)$, without loss of generality we can assume that Hamiltonian H satisfies the following growth restriction:

$$|\nabla H(z)| \leqq a_1 + a_2 |z|^s \quad \text{for some } a_1, a_2 > 0, \quad s \in [1, \infty). \tag{3.3}$$

Indeed, we may choose $\bar{H}$ such that $\nabla \bar{H}$ is bounded (i.e. $s = 1$) and $\bar{H}(z) = H(z)$ in a neighborhood of the orbit $\Gamma(z_0)$.

It is known (see [21]) that periodic solutions of the system (HS-P) are in one to one correspondence with S^1-orbits of critical points of a potential $\Phi : \mathbb{E} \times (0, \infty) \to \mathbb{R}$ of a class C^1 defined by

$$\Phi(z, \lambda) = \frac{1}{2} \langle Lz, z \rangle_{\mathbb{E}} + K_\lambda(z), \tag{3.4}$$

where

$$\langle Lz, z \rangle_{\mathbb{E}} = \int_0^{2\pi} J\dot{z}(t) \cdot z(t) \, dt, \qquad K_\lambda(z) = \int_0^{2\pi} \lambda H(z(t)) \, dt. \tag{3.5}$$

Note that $\Phi(\cdot, \lambda)$ acts on the subspace of constant functions $\mathbb{E}_0$ as $\Phi_{|\mathbb{E}_0 \times \mathbb{R}}(z, \lambda) = 2\pi\lambda H(z)$. Moreover, L is given explicit on $z(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt)$ by

$$(Lz)(t) = \sum_{k=1}^{\infty} Jb_k \cos(kt) - Ja_k \sin(kt); \tag{3.6}$$

see [12], the formula (3.3).

Since we consider $\mathbb{R}^{2N}$ as a unitary representation of a group Γ and H is Γ-invariant, the potential Φ is Γ-invariant. Moreover, it is S^1-invariant since it acts on 2π-periodic functions.

Recall that since the Hamiltonian H is Γ-invariant, the solutions of the system (HS-P) form Γ-orbits i.e. if z_0 is a solution on (HS-P) then γz_0 solves (HS-P) for any $\gamma \in \Gamma$. Therefore we are going to study $G = \Gamma \times S^1$-orbits of critical points

of the corresponding G-invariant potential Φ i.e. we are interested in solutions of the system

$$\nabla_z \Phi(z, \lambda) = 0. \tag{3.7}$$

Note that $\nabla_z \Phi(z, \lambda) = Lz + \nabla K_\lambda(z)$, L is a linear, self-adjoint and G-equivariant operator and $\nabla K_\lambda(z)$ is completely continuous. Since $\ker L = \mathbb{E}_0$ and $L_{|\mathbb{E}_k^\pm} = \pm Id$, the conditions (B.1)–(B.3) given on the page 6 are satisfied.

Let $z_0 \in (\nabla H)^{-1}(0)$ and consider a linear Hamiltonian system

$$\dot{z}(t) = \lambda J A(z(t) - z_0), \tag{HS-L}$$

which has a form of (HS-P) with $H(z) = \frac{1}{2} A(z - z_0) \cdot (z - z_0)$. The variational potential has the form $\Phi_L(z, \lambda) = \frac{1}{2} \langle Lz, z \rangle_\mathbb{E} + \frac{1}{2} \langle B_\lambda(z - z_0), z - z_0 \rangle_\mathbb{E}$, where

$$
\begin{aligned}
\langle B_\lambda z, z' \rangle_\mathbb{E} &= \int_0^{2\pi} \lambda A z(t) \cdot z'(t) \, dt \\
&= \int_0^{2\pi} \left((\lambda A) a_0 + \sum_{k=1}^\infty (\lambda A) a_k \cos(kt) + (\lambda A) b_k \sin(kt) \right) \\
&\quad \cdot \left(a_0' + \sum_{k=1}^\infty a_k' \cos(kt) + b_k' \sin(kt) \right) \\
&= 2\pi (\lambda A) a_0 \cdot a_0' + \pi \sum_{k=1}^\infty (\lambda A) a_k \cdot a_k' + (\lambda A) b_k b_k'.
\end{aligned}
\tag{3.8}
$$

Taking into account the scalar product in $\mathbb{E}$ given by (3.1) and the formula (3.8), we obtain

$$B_\lambda(z - z_0) = (\lambda A)(a_0 - z_0) + \sum_{k=1}^\infty \frac{\lambda}{k} A a_k \cos(kt) + \frac{\lambda}{k} A b_k \sin(kt),$$

AND, AS a consequence,

$$\nabla_z \Phi_L(z, \lambda) = (L + B_\lambda)(z - z_0) = \lambda A(a_0 - z_0) + \sum_{k=1}^\infty \left(\frac{\lambda}{k} A a_k + J b_k \right) \cos(kt)$$

$$+ \left(\frac{\lambda}{k} A b_k - J a_k \right) \sin(kt).$$

This means that $\nabla \Phi_L(z, \lambda)$ acts on $\mathbb{E}_k = \{a \cos(kt) + b \sin(kt) \, : \, a, b \in \mathbb{R}^{2N}\}$ for $k \geqq 1$ as a linear map

$$T_{k,\lambda}(A) = \begin{bmatrix} -\frac{\lambda}{k} A & -J \\ J & -\frac{\lambda}{k} A \end{bmatrix}. \tag{3.9}$$

Lemma 3.2. *The linear equation* (HS-L) *possesses a non-constant 2π-periodic solution if and only if $T_{k,\lambda}(A)$ is singular for some $k \geqq 1$ and it holds true if $\lambda = \frac{k}{\beta_r}$ where $i\beta_r \in \sigma(JA)$.*

Proof. Let $(z, \lambda) = (a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt), \lambda) \neq (const., \lambda)$ be a critical point of Φ_L and let k be such that $|a_k|^2 + |b_k|^2 \neq 0$. Then, in particular, $T_{k,\lambda}(A)(a_k, b_k)^T = 0$ i.e. $T_{k,\lambda}(A)$ has a nontrivial kernel.

It is easy to see that equation $T_{k,\lambda}(A)(a_k, b_k)^T = 0$ has the form

$$\begin{cases} -\dfrac{\lambda}{k} A a_k = J b_k \\ \quad J a_k = \dfrac{\lambda}{k} A b_k \end{cases},$$

which implies that $JA(a_k - ib_k) = \frac{k}{\lambda}(b_k + ia_k) = \frac{ki}{\lambda}(a_k - ib_k)$ i.e. $\frac{ki}{\lambda} \in \sigma(JA)$. $\square$

4. Main Result

In this section we prove our main result of this paper i.e. the global bifurcation of periodic solutions of the system (HS) in the most general version. We emphasize our assumptions:

(A1) $H : \mathbb{R}^{2N} \to \mathbb{R}$ is a Γ-invariant Hamiltonian of the class C^2,
(A2) $z_0 \in \mathbb{R}^{2N}$ is a critical point of H such that the isotropy group Γ_{z_0} is trivial,
(A3) the orbit $\Gamma(z_0)$ is isolated in $(\nabla H)^{-1}(0)$,
(A4) $\pm i\beta_1, \ldots, \pm i\beta_m, 0 < \beta_m < \ldots < \beta_1, m \geqq 1$ are the purely imaginary eigenvalues of $J\nabla^2 H(z_0)$,
(A5) $\deg(\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)}, B(z_0, \varepsilon), 0) \neq 0$ for sufficiently small ε,
(A6) β_{j_0} is such that $\beta_j / \beta_{j_0} \notin \mathbb{N}$ for all $j \neq j_0$
(A7) $m^-\left(T_{1,\lambda}(\nabla^2 H(z_0))\right)$ changes at $\lambda = \frac{1}{\beta_{j_0}}$ when λ varies.

Theorem 4.1. *Under the assumptions (A1)–(A7) there exists a connected family of non-stationary periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$ emanating from the stationary solution z_0 (i.e. with amplitudes tending to 0) such that minimal periods of solutions in a small neighborhood of z_0 are close to $2\pi/\beta_{j_0}$.*

Remark 4.2. The assumption **(A7)** is very general and laborious to verify. We will change and simplify them in some specific cases. However, it does not follow directly from the structure of a Hamiltonian system in general situation as we obtained in a study of Newtonian systems, see [24], the proof of Lemma 4.1.

Let $z_0 \in \mathbb{R}^{2N}$ be a critical point of the Hamiltonian H such that the assumptions **(A1)–(A4)** are satisfied. From now we study variational reformulation (3.7) of the parameterized Hamiltonian system (HS-P). Then z_0 is a constant functions which solves the equation (3.7) for any $\lambda \in (0, \infty)$ and the orbit $G(z_0) = \Gamma(z_0)$ consists of solutions of the equation (3.7). Therefore we put $\mathcal{T} = G(z_0) \times (0, \infty)$ for the

family of trivial solutions of the equation (3.7) and $\mathcal{N} = \{(z, \lambda) \in \mathbb{E} \times (0, +\infty) \setminus \mathcal{T} : \nabla_z \Phi(z, \lambda) = 0\}$ is called a family of non-trivial solutions.

Denote by $\mathcal{C}(z_0, \lambda_0)$ a connected component of the set $\overline{\mathcal{N}}$ which contains the set $\{z_0\} \times \{\lambda_0\}$.

Definition 4.3. We say that the orbit $G(z_0) \times \{\lambda_0\}$ is an orbit of *global bifurcation* of solutions of the equation (3.7) if the set $\mathcal{C}(z_0, \lambda_0)$ is unbounded in $\mathbb{E} \times (0, \infty)$ or $(\mathcal{C}(z_0, \lambda_0) \cap \mathcal{T}) \setminus (G(z_0) \times \lambda_0) \neq \emptyset$ i.e. $\mathcal{C}(z_0, \lambda_0)$ coincide with the trivial family outside the orbit $G(z_0) \times \{\lambda_0\}$.

The definition above does not depend on the choice of $z \in G(z_0)$. Indeed, if $z_1 = g_1 z_0$ then, taking into account an equivariancy of the equation (3.7), we obtain $\mathcal{C}(z_1, \lambda_0) = g_1 \mathcal{C}(z_0, \lambda_0)$ i.e. the connected component of $\{z_1\} \times \{\lambda_0\}$ satisfies the same conditions as the connected component of $\{z_0\} \times \{\lambda_0\}$. In other words, global bifurcation from the orbit $G(z_0) \times \{\lambda_0\}$ provides the existence of solutions emanating from any point of the orbit. In fact, using the equivariant method we obtain the existence the bifurcation of the G-orbits of solutions. However, we are working with the bifurcation of single solutions (not orbits) to connect the main result of the paper to the original theorem of Lyapunov directly.

Note that the definition of global bifurcation implies that the set $\mathcal{C}(z_0, \lambda_0)$ is not empty; i.e. there is a family of solutions of the equation (3.7) emanating from the orbit $G(z_0) \times \{\lambda_0\}$ at the point $\{z_0\} \times \{\lambda_0\}$. Therefore, to prove Theorem 4.1 we have to show the existence of global bifurcation from the orbit $G(z_0) \times \{\lambda_0\}$ and to control the bifurcation level $\{\lambda_0\}$ to determine periods of bifurcating solutions. Finally, since the existence of bifurcation provides the convergence in the norm of Sobolev space $\mathbb{E} = \mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N})$, we have to prove that new periodic solutions tend to $\{z_0\}$ in the L^∞-norm.

Put $\Lambda = \{\frac{k}{\beta_j} : k \in \mathbb{N}, i\beta_j \in \sigma(J\nabla_z^2 H(z_0))\}$. In the theorem below we prove the necessary condition for the existence of bifurcation from the orbit $G(z_0) \times \{\lambda_0\}$.

Theorem 4.4. *(Necessary condition) If $G(z_0) \times \{\lambda_0\}$ is an orbit of global bifurcation of solutions of the equation (3.7) then $\ker \nabla_z^2 \Phi(z_0, \lambda_0) \cap \overline{\bigoplus_{k=1}^\infty \mathbb{E}_k} \neq \emptyset$ i.e. $\lambda_0 \in \Lambda$.*

Proof. By a reasoning given in the proof of Theorem 3.2.1 in [25] we obtain $\ker \nabla_z^2 \Phi(z_0, \lambda_0) \cap \overline{\bigoplus_{k=1}^\infty \mathbb{E}_k} \neq \emptyset$. To complete the proof we have to prove that it implies $\lambda_0 \in \Lambda$. The study of the kernel of $\nabla_z^2 \Phi(z_0, \lambda_0)$ is equivalent to the study of the linearized system (HS-L) where $A = \nabla_z^2 H(z_0)$. Therefore, by Lemma 3.2 we obtain the thesis. $\square$

Choose λ_0 such that the necessary condition and assumptions **(A6), (A7)** are satisfied i.e. $\lambda_0 = \frac{1}{\beta_{j_0}} \in \Lambda$ and put $\lambda_\pm = \frac{1 \pm \varepsilon}{\beta_{j_0}}$ such that $\lambda_\pm > 0$ and $[\lambda_-, \lambda_+] \cap \Lambda = \{\lambda_0\}$. To prove the existence of global bifurcation we are going to apply the following theorem:

Theorem 4.5. *(Sufficient condition). Under the assumptions above, if*

$$\Upsilon_G\left(\mathcal{C}\mathcal{I}_G\left(G(z_0), -\nabla\Phi(\cdot, \lambda_+)\right)\right) \neq \Upsilon_G\left(\mathcal{C}\mathcal{I}_G\left(G(z_0), -\nabla\Phi(\cdot, \lambda_-)\right)\right), \quad (4.1)$$

then $G(z_0) \times \{\lambda_0\}$ is an orbit of a global bifurcation.

Proof. The theorem above follows directly from the relation

$$\Upsilon_G(\mathcal{CJ}_G(X, f) = \nabla_G - deg(f, \mathrm{Int}X)$$

(see [3], Theorem 3.10) and from a global bifurcation theorem for equivariant gradient degree (see [13], Theorem 3.3). □

Define $\mathbb{H} \subset \mathbb{E}$ by $\mathbb{H} = T_{z_0}^{\perp} G(z_0)$. Recall that the space perpendicular to the orbit at z_0 is an G_{z_0}-representation. Since z_0 is a constant function and by the assumption **(A2)** $G_{z_0} = \{e\} \times S^1$, $\mathbb{H}$ is an unitary S^1-representation.

Put $\Psi_{\pm} : \mathbb{H} \to \mathbb{R}$ by $\Psi_{\pm}(z) = \Phi(z, \lambda_{\pm})$. Note that since $\mathbb{H}$ is an S^1-representation, the potenatial $\Psi_{\pm}$ is S^1-invariant. Moreover, z_0 is an isolated critical point of $\Psi_{\pm}$. Since $G(z_0) = \Gamma(z_0) \subset \mathbb{E}_0$ we have the following decomposition:

$$\mathbb{H} = T_{z_0}^{\perp}\Gamma(z_0) \oplus \overline{\bigoplus_{k=1}^{\infty} \mathbb{E}_k}.$$

In order to prove the main result of this paper we prove the existence of global bifurcation from the orbit $G(z_0) \times \{\lambda_0\}$; i.e. we need to prove formula (4.1). In the theorem below we simplify this formula to the study of potentials defined on the orthogonal section $T_{z_0}^{\perp} G(z_0)$.

Lemma 4.6. *Under the above assumptions if*

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla\Psi_+)\right) \neq \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla\Psi_-)\right), \tag{4.2}$$

then

$$\Upsilon_G\left(\mathcal{CJ}_G(G(z_0), -\nabla\Phi(\cdot, \lambda_+))\right) \neq \Upsilon_G\left(\mathcal{CJ}_G(G(z_0), -\nabla\Phi(\cdot, \lambda_-))\right). \tag{4.3}$$

Proof. Since the pair $(\Gamma \times S^1, \{e\} \times S^1)$ is admissible (because S^1 is abelian), see Definition 2.1, and both $\nabla\Phi(\cdot, \lambda_-)$, $\nabla\Phi(\cdot, \lambda_+)$ are in the form of a compact perturbation of the same linear operator L, we can apply Theorem 2.16 to obtain the thesis directly. □

From now our goal is to prove formula (4.2). The next step is to transform a problem into the study of Conley indexes with simpler structure of flows.

We define $\widetilde{H} : T_{z_0}^{\perp}\Gamma(z_0) \to \mathbb{R}$ by $\widetilde{H}(z) = H(z + z_0)$ and $\widetilde{\Psi}_{\pm} : \mathbb{H} \to \mathbb{R}$ by $\widetilde{\Psi}_{\pm}(z) = \Psi_{\pm}(z+z_0)$. Since $\widetilde{\Psi}_{\pm|\mathbb{E}_0} = 2\pi\lambda_{\pm}\widetilde{H}$ and the orbits $G(z_0) \times \{\lambda_+\}$, $G(z_0) \times \{\lambda_-\}$ do not satisfy the necessary condition for the existence of bifurcation we obtain $\ker \nabla^2\widetilde{\Psi}_{\pm|}(0) = \ker \nabla^2\widetilde{H}(0)$ so the kernel is independent on $\lambda_{\pm}$. Since $\nabla^2\widetilde{\Psi}_{\pm}(0)$ is self-adjoint we are able to decompose

$$\mathbb{H} = \mathcal{N} \oplus \mathcal{R} = \ker \nabla^2\widetilde{\Psi}_{\pm}(0) \oplus \mathrm{im} \ \nabla^2\widetilde{\Psi}_{\pm}(0)$$

independently on λ. We further decompose $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_\infty$, where $\mathcal{R}_0 = \mathcal{R} \cap \mathbb{E}_0 \subset \mathbb{H}^{S^1}$ and $\mathcal{R}_\infty = \overline{\bigoplus_{k=1}^{\infty} \mathbb{E}_k} \subset \mathcal{R} \subset \mathbb{H}$.

Note that the S^1-invariant potential $\Pi_\pm : \mathcal{R}_\infty \to \mathbb{R}$ of the linear vector field $\nabla^2 \Psi_{\pm|\mathcal{R}_\infty}(z_0)(z - z_0)$ is defined by $\Pi_\pm(z) = \dfrac{1}{2}\langle \nabla^2 \Psi_{\pm|\mathcal{R}_\infty}(z_0)(z - z_0), z - z_0\rangle_{\mathbb{E}}$.

The next theorem simplifies the proof of formula (4.2) to the study of Conley indexes of linear vector fields. In order to prove it we apply the splitting lemma (Theorem 2.17).

Lemma 4.7. *Under the above assumptions the formula* (4.2) *holds true if and only if*

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_+\right)\right) \neq \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_-\right)\right). \tag{4.4}$$

Proof. It is clear that by the properties of Conley index we have

$$\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla \Psi_\pm) = \mathcal{CJ}_{S^1}(\{0\}, -\nabla \widetilde{\Psi}_\pm).$$

Since we are going to apply splitting lemma (Theorem 2.17), now we verify that $\widetilde{\Psi}_\pm$ satisfies conditions (F.1)–(F.5) given on the page 8 with $K = S^1$, $A = \nabla^2 \widetilde{\Psi}_\pm(0)$ and $\zeta_\pm(z) = \widetilde{\Psi}_\pm(z) - \langle \nabla^2 \widetilde{\Psi}_\pm(0)z, z\rangle_{\mathbb{E}}$. Thus,

(F.1) Since $\widetilde{\Psi}_\pm$ is S^1-invariant (it is the invariant Ψ translated by $z_0 \in \mathbb{E}^{S^1}$) its hessian is S^1-equivariant. Moreover, a hessian is a self-adjoint operator. By Theorem 4.4 $\ker \nabla^2 \widetilde{\Psi}_\pm \subset \mathbb{E}_0 \cap \mathbb{H}$ is finite dimensional, since $\lambda_\pm \notin \Lambda$.

(F.2) In a fashion similar to as above, $\ker \nabla^2 \widetilde{\Psi}_\pm \subset \mathbb{E}_0 \cap \mathbb{H} = \mathbb{H}^{S^1}$

(F.3) Since $\nabla \zeta_\pm(z) = \nabla \widetilde{\Psi}_\pm(z) - \nabla^2 \widetilde{\Psi}_\pm(0)z$ and both summands are compact and S^1-equivariant, $\nabla \zeta_\pm$ is also compact and S^1-equivariant.

(F.4) It is obvious due to formula given in (F.3).

(F.5) Since $\lambda_\pm \notin \Lambda$, i.e. the orbits $G(z_0) \times \{\lambda_\pm\}$ do not satisfy the necessary conditions for the existence of bifurcations, the orbit $G(z_0)$ is isolated in the set $(\nabla \Phi(\cdot, \lambda_\pm))^{-1}(0)$. Therefore $0 \in \mathbb{H}$ is an isolated critical point of $\widetilde{\Psi}_\pm$.

Applying Theorem 2.17 (splitting lemma) and Theorem 2.15 (product formula), we obtain

$$\begin{aligned}
\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla \widetilde{\Psi}_\pm)\right) &= \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla \varphi_\pm)\right) \\
&\quad \star \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla^2 \widetilde{\Psi}_\pm(0)_{|\mathcal{R}})\right),
\end{aligned}$$

where $0 = (0, 0) \in \mathcal{N} \oplus \mathcal{R}$, $\varphi_\pm : B_{\varepsilon_0}(\mathcal{N}) \to \mathbb{R}$, $\varphi_\pm(u) = \widetilde{\Psi}_\pm(u, w(u))$ and $\nabla \varphi_\pm(u)$ is S^1-equivariant.

Since $\mathcal{R}_0$ is an invariant space of the linear map $\nabla^2 \widetilde{\Psi}_\pm(0)$ we are able to decompose the linear flow to obtain

$$\begin{aligned}
\Upsilon_{S^1}&\left(\mathcal{CJ}_{S^1}(\{(0, 0)\}, -\nabla^2 \widetilde{\Psi}_\pm(0)_{|\mathcal{R}})\right) \\
&= \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla^2 \widetilde{\Psi}_\pm(0)_{|\mathcal{R}_0})\right) \\
&\quad \star \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla^2 \widetilde{\Psi}_\pm(0)_{|\mathcal{R}_\infty})\right),
\end{aligned}$$

and combining the flows given on the $\mathbb{E}_0 \cap \mathbb{H}$, we finally obtain

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla\widetilde{\Psi}_{\pm})\right)$$
$$= \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{(0,0)\}, (-\nabla\varphi_{\pm}, -\nabla^2\widetilde{\Psi}_{\pm}(0)_{|\mathcal{R}_0}))\right)$$
$$\star \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla^2\widetilde{\Psi}_{\pm}(0)_{|\mathcal{R}_\infty})\right). \tag{4.5}$$

If we study the homotopy $\mathcal{H}$ (see Theorem 2.17 and Remark 2.18) acting on the subspace of constant function $\mathbb{E}_0$, we obtain

$$\nabla\mathcal{H}_{|\mathbb{E}_0}((u,v), 1) = (-\nabla\varphi_{\pm}(u), -\nabla^2\widetilde{\Psi}_{\pm}(0)_{|\mathcal{R}_0}(v)),$$
$$\nabla\mathcal{H}_{|\mathbb{E}_0}((u,v), 0) = \nabla\widetilde{\Psi}_{\pm|\mathbb{E}_0}(u,v) = \widetilde{L}_{|\mathbb{E}_0}(u,v) + \nabla\widetilde{K}_{\lambda_{\pm}|\mathbb{E}_0}(u,v)$$
$$= 2\pi\lambda_{\pm}\nabla\widetilde{H}(u,v).$$

By the homotopy invariance of the Conley index, and since λ_-, λ_+ are both positive, we have

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{(0,0)\}, (-\nabla\varphi_{\pm}, -\nabla^2\widetilde{\Psi}_{\pm}(0)_{|\mathcal{R}_0}))\right)$$
$$= \Upsilon_{S^1}(\mathcal{CJ}_{S^1}(\{0\}, -\nabla\widetilde{H})) = \Upsilon_{S^1}(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)})).$$

Note that the space $\mathbb{E}_0$ is finite-dimensional and consists of constant functions (elements invariant on S^1 action), therefore

$$\Upsilon_{S^1}(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)})) = \chi_{S^1}(\mathcal{CI}_{S^1}(\{z_0\}, -\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)}))$$
$$= \chi(\mathcal{CI}(\{z_0\}, -\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)}))$$
$$= \deg(\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)}, B(z_0, \varepsilon)) \cdot \mathbb{I} \in U(S^1) \tag{4.6}$$

for sufficiently small $\varepsilon > 0$, where the last equality follows from Poincaré-Hopf theorem (see [31]). Since

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla^2\widetilde{\Psi}_{\pm}(0)_{|\mathcal{R}_\infty})\right) = \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla\Pi_{\pm})\right),$$

we finally have that

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{0\}, -\nabla\widetilde{\Psi}_{\pm})\right) = \deg(\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)}, B(z_0, \varepsilon))$$
$$\cdot\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla\Pi_{\pm})\right). \tag{4.7}$$

By the assumption **(A5)** $\deg(\nabla H_{|T_{z_0}^{\perp}\Gamma(z_0)}, B(z_0, \varepsilon), 0) \neq 0$ and due to equation (4.7) we obtain that the formula (4.2) is equivalent to

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla\Pi_+)\right) \neq \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}(\{z_0\}, -\nabla\Pi_-)\right),$$

and the proof is completed. $\square$

To verify formula (4.4) we are going to study the equivariant Conley index and equivariant Euler characteristic by definitions. Note that the vector field $-\nabla \Pi_{\pm}:$ $\mathbb{R}_{\infty} \to \mathbb{R}_{\infty}$ is linear and the decomposition $\mathcal{R}_{\infty} = \overline{\bigoplus_{k=1}^{\infty} \mathbb{E}_k}$ satisfies conditions (B.1)–(B.3) given on the page 6. Recall that $\lambda_0 = \frac{1}{\beta_{j_0}} \in \Lambda$ where $i\beta_{j_0} \in \sigma(J\nabla^2 H(z_0))$ and $\lambda_{\pm} = \frac{1+\varepsilon}{\beta_{j_0}}$ is such that $[\lambda_-, \lambda_+] \cap \Lambda = \{\lambda_0\}$.

Remark 4.8. Note that the linearization of the variational functional Φ for the parameterized Hamiltonian system (HS-P) is equal to variational functional for the linearized system (HS-L) (we remove high order tenses in both cases). Therefore the action of the linear vector field $\nabla \Pi_{\pm} : \mathbb{R}_{\infty} \to \mathbb{R}_{\infty}$ is given on $\mathbb{E}_k$ by

$$T_{k,\lambda}(A) = \begin{bmatrix} -\frac{\lambda}{k}A & -J \\ J & -\frac{\lambda}{k}A \end{bmatrix}, \tag{4.8}$$

where $A = \nabla^2 \widetilde{H}(0) = \nabla^2 H(z_0)$. For $k \to \infty$ we have $T_{k,\lambda}(A) \to \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}$ i.e. $m^-(T_{k,\lambda}) = 2N$ for k big enough.

Theorem 4.9. *Under the assumptions (A1)–(A7) of Theorem* 4.1,

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_+\right)\right) \neq \Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_-\right)\right). \tag{4.9}$$

Proof. Since $T_{k,\lambda}$ is singular iff $\lambda = \frac{k}{\beta_j}$ (see Lemma 3.2), $[\lambda_-, \lambda_+] \cap \Lambda = \{\lambda_0\} = \{\frac{1}{\beta_{j_0}}\}$ and $\frac{1}{\beta_{j_0}} \neq \frac{k}{\beta_j}$ for any $k \in \mathbb{N}$ and $\beta_j \neq \beta_{j_0}$ (see assumption (**A6**)), matrices $T_{k,\lambda}$ for $k \geq 2$ are nonsingular if λ varies. Therefore the spectral decomposition of $\mathbb{E}_k$ for $k \geq 2$ given by $-\nabla^2 \Pi_{\pm}$ does not depend on $\lambda_{\pm}$, i.e.

$$\mathbb{E}_k = \mathbb{E}_{k,-} \oplus \mathbb{E}_{k,+},$$

but

$$\mathbb{E}_1 = \mathbb{E}_{1,\lambda_{\pm},-} \oplus \mathbb{E}_{1,\lambda_{\pm},+}.$$

As a consequence the spectra $\mathcal{E}_-, \mathcal{E}_+$ whose homotopy types are Conley indexes $\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_-\right)$, $\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_+\right)$ are of the same type $\xi = (\mathbb{E}_{k,+})_{k=2}^{\infty}$. Define $\mathbb{P}_n = \bigoplus_{k=2}^{n} \mathbb{E}_{k,+}$.

Put $\mathcal{R}_n = \bigoplus_{k=1}^{n} \mathbb{E}_k$ and consider $\Pi_{\pm}^n = \Pi_{\pm|\mathcal{R}_n} : \mathcal{R}_n \to \mathbb{R}$. By Remark 2.12 we obtain

$$\Upsilon_{S^1}\left(\mathcal{CJ}_{S^1}\left(\{z_0\}, -\nabla \Pi_{\pm}\right)\right) = \left(\chi_{S^1}\left(S^{\mathbb{P}_{n-1}}\right)\right)^{-1}$$
$$\star \chi_{S^1}\left(\mathcal{CI}_{S^1}\left(\{z_0\}, -\nabla \Pi_{\pm}^n\right)\right) \tag{4.10}$$

for n large enough. Now, to prove formula (4.9), it is enough to show that

$$\chi_{S^1}\left(\mathcal{CI}_{S^1}\left(\{z_0\}, -\nabla \Pi_-^n\right)\right) \neq \chi_{S^1}\left(\mathcal{CI}_{S^1}\left(\{z_0\}, -\nabla \Pi_+^n\right)\right).$$

Since $-\nabla \Pi_{\pm}^n$ is a linear isomorphism, Conley indexes are very simple, i.e.

$$\mathcal{CI}_{S^1}\left(\{z_0\}, -\nabla \Pi_{\pm}^n\right) = S^{\mathbb{E}_{1,\lambda_{\pm},+} \oplus \mathbb{P}_n} = S^{\mathbb{E}_{1,\lambda_{\pm},+}} \wedge S^{\mathbb{P}_n}. \tag{4.11}$$

By the assumption **(A7)**, we have

$$\dim \mathbb{E}_{1,\lambda_-,+} = m^- \left(T_{1,\lambda_-}(\nabla^2 H(z_0)) \right) \neq m^- \left(T_{1,\lambda_+}(\nabla^2 H(z_0)) \right)$$
$$= \dim \mathbb{E}_{1,\lambda_+,+}.$$

Since $\mathbb{E}_1$ is a non-trivial S^1-representation, by Remark 2.11 we obtain that

$$\chi_{S^1} \left(S^{\mathbb{E}_{1,\lambda_-,+}} \right) \neq \chi_{S^1} \left(S^{\mathbb{E}_{1,\lambda_+,+}} \right). \tag{4.12}$$

Combining formulas (4.11) and (4.12), we finally obtain

$$\chi_{S^1} \left(\mathcal{CI}_{S^1} \left(\{z_0\}, -\nabla \Pi^n_- \right) \right) = \chi_{S^1} \left(S^{\mathbb{E}_{1,\lambda_-,+}} \right) \star \chi_{S^1} \left(S^{\mathbb{P}_n} \right)$$
$$\neq \chi_{S^1} \left(S^{\mathbb{E}_{1,\lambda_+,+}} \right) \star \chi_{S^1} \left(S^{\mathbb{P}_n} \right)$$
$$= \chi_{S^1} \left(\mathcal{CI}_{S^1} \left(\{z_0\}, -\nabla \Pi^n_+ \right) \right),$$

which completes the proof. $\square$

Remark 4.10. Lets summarize the work we have already done.

1. By the change of variables we translate the equation (HS) into (HS-P).
2. We formulate the equation (HS-P) as a variational problem (3.7).
3. We apply the equivariant Conley index and the equivariant Euler characteristic to provide the existence of global bifurcation of solutions of the equation (3.7) from the orbit $G(z_0) \times \{\lambda_0\}$. From now we are going to prove formula (4.1) i.e. the change of the equivariant gradient degree at the level λ_0 satisfying the necessary condition.
4. To study the change of equivariant the Conley index of the orbit we apply the method of the orthogonal section, reducing the problem to formula (4.2).
5. Applying the equivariant splitting lemma and the assumption **(A5)**, we reduce formula (4.2) to the linear case, i.e., to formula (4.4).
6. Finally we prove formula (4.4), computing equivariant Conley index by the definition.

Proof of Theorem 4.1. Applying bifurcation theory to the variational potential $\Phi : \mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N}) \times (0, \infty) \to \mathbb{R}$ (as is summarized in the above remark) we have proved the existence of a family of critical points of Φ emanating from $\{z_0\} \times \{\lambda_0\} \in \mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N}) \times (0, \infty)$ in the norm of $\mathbb{H}^{1/2}(S^1, \mathbb{R}^{2N})$. Now we prove that corresponding periodic solutions of Hamiltonian system tend to $z_0 \in \mathbb{R}^{2N}$ in L^∞-norm, i.e. the amplitudes of these solutions are tending to zero.

Let $z(t) = a_0 + \sum_{k=1}^\infty a_k \cos(kt) + b_k \sin(kt)$ be a solution of (HS-P) for λ close to λ_0. Firstly,

$$||z - z_0||^2_{L^2} = 2\pi |a_0 - z_0|^2 + \pi \sum_{k=1}^\infty (|a_k|^2 + |b_k|^2)$$
$$\leqq 2\pi |a_0 - z_0|^2 + \pi \sum_{k=1}^\infty k(|a_k|^2 + |b_k|^2) = ||z - z_0||^2_{\mathbb{H}^{1/2}}.$$

Under the condition (3.3), the map $z(t) \to \nabla H(z(t))$ is continuous from $L^2(S^1)$ to $L^2(S^1)$ (see Proposition B.1 in [28]). Let $\varepsilon > 0$ and choose $0 < \delta < \varepsilon$ such that $||z - z_0||_{L^2} \leqq ||z - z_0||_{\mathbb{H}^{1/2}} < \delta$ implies that $||\nabla H(z)||_{L^2} = ||\nabla H(z) - \nabla H(z_0)||_{L^2} < \varepsilon$. Since z is a solution of (HS-P), we obtain

$$||(z - z_0)'||_{L^2} = ||\dot{z}||_{L^2} = ||\lambda \nabla H(z)||_{L^2} \leqq \lambda \varepsilon.$$

Applying Sobolev inequality (see Proposition 1.1 in [21]), we obtain

$$||z - z_0||_{L^\infty}^2 \leqq c||z - z_0||_{\mathbb{H}^1}^2 = c \left(||z - z_0||_{L^2}^2 + ||(z - z_0)'||_{L^2}^2 \right) \leqq c(1 + \lambda^2)\varepsilon^2.$$

Since λ is bounded in the neighborhood of λ_0 the convergence of solutions z to z_0 in the norm of $\mathbb{H}^1$ implies the convergence in L^∞.

To finish the proof we have to study the minimal periods of new periodic solutions. These were obtained by the bifurcation from the orbit $G(z_0) \times \{\lambda_0\}$, where the parameter λ_0 comes from the change of variables and describes the period of solutions. More precisely, we have already obtained the connected branch of solutions of the system (HS) emanating from the stationary solutions z_0 with periods close to $2\pi\lambda_0 = \frac{2\pi}{\beta_{j_0}}$.

By the non-resonance condition for eigenvalues (i.e $\beta_j/\beta_{j_0} \notin \mathbb{N}$ for all $j \neq j_0$), we obtain $\frac{\lambda_0}{r} \notin \Lambda = \{\frac{k}{\beta_r} : k \in \mathbb{N}, i\beta_r \in \sigma(J\nabla_z^2 H(z_0))\}$ for any $r \in \mathbb{N}$, and therefore, in the view of Theorem 4.4, there are no $\frac{2\pi\lambda_0}{r}$-periodic non-stationary solutions in a neighborhood of the orbit $\Gamma(z_0)$. Hence we can consider periods tending to $\frac{2\pi}{\beta_{j_0}}$ as minimal and the proof of Theorem 4.1 is completed. $\square$

Remark 4.11. The assumption (A6) was used only in the proof of Theorem 4.9, i.e. in the last step of the proof of our main theorem. We are able to remove this assumption, but then in the proof of Theorem 4.9 we need to study λ-depending decompositions of not only $\mathbb{E}_1$ but any $\mathbb{E}_k$ such that $\frac{1}{\beta_{j_0}} \neq \frac{k}{\beta_j}$ for some j. It will cause a complicated notation and the proof will be less readable. However, the change of dim $\mathbb{E}_{k,\lambda,+}$ when λ varies we will obtain in the same way as for $k = 1$. Note that assumption (A6) is always satisfied for $j_0 = 1$ since β_1 is the maximum of β_i.

5. Corollaries

In this section we study how it is possible to modify assumption (A7). Moreover, we show that the results of this paper are generalizations of some versions of the Lyapunov center theorem.

The following theorem was proven by SZULKIN ([32], Proposition 3.6):

Theorem 5.1. *Suppose that A is symmetric and $i\beta_j$, $\beta_j > 0$, is an eigenvalue of JA. Let E_j be the eigenspace of JA in $\mathbb{C}^{2N}$ corresponding to $i\beta_j$ and Z_j the invariant subspace of JA in $\mathbb{R}^{2N}$ corresponding to $\pm i\beta_j$. Then $m^-(T_{1,\lambda}(A))$ changes at $\lambda = 1/\beta_j$ if and only if the following two equivalent conditions are satisfied:*

1. $m^-(A_{|Z_j}) \neq m^+(A_{|Z_j})$,
2. $m^-(-i J_{|E_j}) \neq m^+(-i J_{|E_j})$.

Due to the theorem above we are able to formulate new versions of assumption **(A7)**:

(A7.1) $m^-(\nabla^2 H(z_0)_{|Z_{j_0}}) \neq m^+(\nabla^2 H(z_0)_{|Z_{j_0}})$,
(A7.2) $m^-(-i J_{|E_{j_0}}) \neq m^+(-i J_{|E_{j_0}})$,

where E_j is the eigenspace of $J\nabla^2 H(z_0)$ in $\mathbb{C}^{2N}$ corresponding to $i\beta_j$ and Z_j the invariant subspace of $J\nabla^2 H(z_0)$ in $\mathbb{R}^{2N}$ corresponding to $\pm i\beta_j$.

Note that if $\nabla^2 H(z_0)_{|Z_{j_0}}$ is a definite matrix, then the condition **(A7.1)** is satisfied. Therefore we put that

(A7.3) $\nabla^2 H(z_0)_{|Z_{j_0}}$ is definite.

Theorem 5.2. *Under the assumptions (A1)–(A6) and one of the conditions (A7.1)–(A7.3) there exists a connected family of non-stationary periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$ emanating from the stationary solution z_0 such that minimal periods of solutions in the small neighborhood of z_0 are close to $2\pi/\beta_{j_0}$.*

If we are not interested in the minimal period of new solutions but only in the study of its existence, the assumptions can be modified. The computation of invariant subspaces Z_j we can change to the study of general invariant subspace of $J\nabla^2 H(z_0)$ associated to all the eigenvalues of the form $\pm i\beta_k$. Denoting this subspace by Z we formulate a new condition.

(A7.4) $\nabla^2 H(z_0)_{|Z}$ is definite.

Under this condition the assumption **(A7.3)** is satisfied for some eigenvalue of $J\nabla^2 H(z_0)$ and we do not know it precisely. Therefore we exclude assumption **(A6)**. In the theorem below we prove the existence of periodic solutions of the system (HS) without information about their minimal periods. According to the reasoning above, it is clear that Theorem 5.3 is a direct consequence of Theorem 4.1.

Theorem 5.3. *Under the assumptions (A1)–(A5) and (A7.4) there exists a connected family of non-stationary periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$ emanating from the stationary solution z_0 such that periods (not necessarily minimal) of solutions in the small neighborhood of z_0 are close to $2\pi/\beta_j$ where $i\beta_j$, $\beta_j > 0$, is some eigenvalue of $J\nabla^2 H(z_0)$.*

Looking on the $T_{1,\lambda}(\nabla^2 H(z_0))$ from the other point of view, we see that

$$T_{1,\lambda}(\nabla^2 H(z_0)) \to \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix} \quad \text{and} \quad m^-(T_{1,\lambda}(\nabla^2 H(z_0))) \to 2N$$

for $\lambda \to 0$,

$$\frac{1}{\lambda} T_{1,\lambda}(\nabla^2 H(z_0)) \to \begin{bmatrix} -\nabla^2 H(z_0) & 0 \\ 0 & -\nabla^2 H(z_0) \end{bmatrix} \quad \text{for } \lambda \to \infty$$

and $m^-(T_{1,\lambda}(\nabla^2 H(z_0))) = m^-(\frac{1}{\lambda} T_{1,\lambda}(\nabla^2 H(z_0))) \to 2m^+(\nabla^2 H(z_0))$

for $\lambda \to \infty$.

Therefore if $m^+(\nabla^2 H(z_0)) \neq N$, then $m^-(T_{1,\lambda}(\nabla^2 H(z_0)))$ changes at some $\lambda \in (0, \infty)$. Recall that the levels λ where it can change is Λ (see Lemma 3.2). Therefore the change of $m^-(T_{1,\lambda}(\nabla^2 H(z_0)))$ implies the existence of purely imaginary eigenvalue of $J\nabla^2 H(z_0)$. As a consequence we can propose a new condition:

(A7.5) $m^+(\nabla^2 H(z_0)) \neq N$,

and we are able to formulate the next theorem without assumption **(A4)**.

Theorem 5.4. *Under the assumptions (A1),(A2),(A3),(A5) and (A7.5) there exists a connected family of non-stationary periodic solutions of the system $\dot{z}(t) = J\nabla H(z(t))$ emanating from the stationary solution z_0 such that periods (not necessarily minimal) of solutions in the small neighborhood of z_0 are close to $2\pi/\beta_j$, where $i\beta_j$, $\beta_j > 0$, is some eigenvalue of $J\nabla^2 H(z_0)$.*

Below we present a way in where the theorems presented above generalize classical Lyapunov center theorem and an analogous theorem for Hamiltonian systems that has been proved by DANCER and RYBICKI [6]. Moreover, two symmetric versions of the Lyapunov center theorem proposed in [24] and [25] are generalized in this paper.

Theorem 5.5. ([6], Theorem 3.3. (reformulated)) *Consider an equation $\dot{z}(t) = J\nabla H(z(t))$, where $H : \mathbb{R}^{2N} \to \mathbb{R}$ is of the class C^2. Let $z_0 \in \mathbb{R}^{2N}$ be an isolated critical point of H. Let $i\beta_0$ $\beta_0 > 0$ be an eigenvalue of $J\nabla^2 H(z_0)$. If $\deg(\nabla H, B(z_0, \varepsilon), 0) \neq 0$ for sufficiently small ε and $m^-\left(T_{1,\lambda}(\nabla^2 H(z_0))\right)$ changes at $\lambda = \frac{1}{\beta_0}$ when λ varies, then there exists a connected family of periodic solution of the Hamiltonian system emanating from z_0.*

Proof. This theorem follows directly from Theorem 4.1 if we consider trivial group $\Gamma = \{e\}$. In this case $T_{z_0}\Gamma(z_0) = \mathbb{R}^{2N} + z_0$. $\square$

Consider a Newtonian (second-order) system

$$\ddot{q}(t) = -\nabla U(q(t)), \tag{NS}$$

where $U : \mathbb{R}^N \to \mathbb{R}$ is Γ-invariant potential of the class C^2, Γ acts orthogonally on $\mathbb{R}^N$ and $q_0 \in (\nabla U)^{-1}(0)$. If we substitute $r = \dot{q}$ we can reformulate the second-order system (NS) to the first-order system

$$\begin{cases} \dot{q}(t) = r(t), \\ \dot{r}(t) = -\nabla U(q(t)), \end{cases} \tag{5.1}$$

which can be considered as a Hamiltonian system with $H : \mathbb{R}^{2N} \to \mathbb{R}$ defined by $H(z) = H(q, r) = \frac{1}{2}r^2 + U(q)$. An action of Γ on $\mathbb{R}^{2N}$ induced by action on $\mathbb{R}^N$ is diagonal i.e $\gamma(q, r) \to (\gamma q, \gamma r)$. It is easy to verify that this action is symplectic, so Γ acts unitary on $\mathbb{R}^{2N}$. Moreover, $z_0 = (q_0, r_0) = (q_0, \dot{q}_0) = (q_0, 0)$ (since we consider q_0 as a constant function) is a critical point of H. We see that $J\nabla^2 H(z_0) = \begin{bmatrix} 0 & \nabla^2 U(q_0) \\ -I & 0 \end{bmatrix}$. The easy block–form of the matrix $J\nabla^2 H(z_0)$ lets us to observe

a bijective correspondence between positive eigenvalues of $\nabla^2 U(q_0)$ and the pairs of purely imaginary eigenvalues of $J\nabla^2 H(z_0)$. In fact, if $\beta^2 \in \sigma(\nabla^2 U(q_0))$ then $\pm i\beta \in \sigma(J\nabla^2 H(z_0))$. Taking into account the above reasoning, the following theorems are consequences of Theorem 4.1:

Theorem 5.6. (Symmetric Lyapunov center theorem, [24]). *Let $U : \Omega \to \mathbb{R}$ be a Γ-invariant potential of the class C^2 and $q_0 \in \Omega$. Assume that*

1. *q_0 is a critical points of the potential U,*
2. *$\dim \ker \nabla^2 U(q_0) = \dim \Gamma(q_0)$,*
3. *the isotropy group Γ_{q_0} is trivial,*
4. *$\sigma(\nabla^2 U(q_0)) \cap (0, +\infty) = \{\beta_1^2, \ldots, \beta_m^2\}$ and $m \geqq 1$.*

Then for any β_{j_0} such that $\beta_j/\beta_{j_0} \notin \mathbb{N}$ for all $j \neq j_0$, there exists a sequence $(q_k(t))$ of periodic solutions of the system (NS) with minimal period tending to $2\pi/\beta_{j_0}$ such that in any open neighborhood of the orbit $\Gamma(q_0)$ there is an element of the sequence $(q_k(t))$.

Theorem 5.7. (Symmetric Lyapunov center theorem for minimal orbit, [25]). *Let $U : \Omega \to \mathbb{R}$ be a Γ-invariant potential of the class C^2 and $q_0 \in \Omega$. Assume that*

1. *q_0 is a minimum of potential U,*
2. *the orbit $\Gamma(q_0)$ is isolated in $(\nabla U)^{-1}(0)$,*
3. *the isotropy group Γ_{q_0} is trivial,*
4. *$\sigma(\nabla^2 U(q_0)) \cap (0, +\infty) = \{\beta_1^2, \ldots, \beta_m^2\}$, $\beta_1 > \beta_2 > \ldots > \beta_m > 0$ and $m \geq 1$.*

Then for any β_{j_0} such that $\beta_j/\beta_{j_0} \notin \mathbb{N}$ for $j \neq j_0$ there exists a sequence $(q_k(t))$ of periodic solutions of the system (NS) with a sequence of minimal periods (T_k) such that $\mathrm{dist}(\Gamma(q_0), q_k([0, T_k])) \to 0$ and $T_k \to 2\pi/\beta_{j_0}$ as $k \to \infty$.

Proof. Note that the assumptions **(A1)**–**(A4)** and **(A6)** are satisfied directly due to statements of the theorems above.

Firstly, we check that the assumption **(A7)** is always satisfied for Newtonian systems. Since $\nabla^2 H(z_0) = \begin{bmatrix} \nabla^2 U(q_0) & 0 \\ 0 & I \end{bmatrix}$ and the matrix $\nabla^2 U(q_0)$ is orthogonally diagonalizable (say by $D \in O(N)$) then the symplectic matrix $\bar{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ diago-

nalize the hessian $\nabla^2 H(z_0)$ and we are able to simplify the form of $T_{1,\lambda}(\nabla^2 H(z_0))$ as follows:

$$\begin{bmatrix} \bar{D} & 0 \\ 0 & \bar{D} \end{bmatrix} \cdot \begin{bmatrix} -\lambda \nabla^2 H(z_0) & -J \\ J & -\lambda \nabla^2 H(z_0) \end{bmatrix} \cdot \begin{bmatrix} \bar{D}^T & 0 \\ 0 & \bar{D}^T \end{bmatrix}$$
$$= \begin{bmatrix} -\lambda \bar{D}\nabla^2 H(z_0)\bar{D}^T & -J \\ J & -\lambda \bar{D}\nabla^2 H(z_0)\bar{D}^T \end{bmatrix},$$

where $\bar{D}\nabla^2 H(z_0)\bar{D}^T = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_N, 1, \ldots, 1)$ and $\eta_1, \eta_2, \ldots, \eta_N$ are the eigenvalues of $\nabla^2 U(q_0)$ (not necessarily different). Further, we apply the permutation of the basis $(1, 2, \ldots, 4N) \to (1, N+1, 2N+1, 3N+1, 2, N+2, 2N+$

$2, 3N+2, \ldots, N, 2N, 3N, 4N$) to transform our matrix to diag $(A_1, A_2, \ldots, A_N)$,

where $A_i = \begin{bmatrix} -\lambda\eta_i & 0 & 0 & 1 \\ 0 & -\lambda & -1 & 0 \\ 0 & -1 & -\lambda\eta_i & 0 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}$. The characteristic polynomial of the matrix

A_i has the form $W_{A_i}(t) = ((\lambda\eta_i + t)(\lambda + t) - 1)^2$ and has 4 negative roots for $\lambda^2 = \lambda_+^2 = \frac{1+\varepsilon}{\eta_i}$ and 2 negative roots for $\lambda^2 = \lambda_-^2 = \frac{1-\varepsilon}{\eta_i}$, where η_i is positive. Therefore $m^-(T_{1,\lambda_+}(\nabla^2 H(z_0)) = m^-(T_{1,\lambda_-}(\nabla^2 H(z_0))) + 2mult(\eta_i)$ for sufficiently small ε, so the assumption **(A7)** is satisfied automatically for any positive eigenvalue β_i^2 of $\nabla^2 U(q_0)$ when we study Newtonian system translated into the Hamiltonian one.

To complete the proofs of theorems we have to verify assumption **(A5)** in both cases.

- In Theorem 5.6 we assume that the orbit is non-degenerate i.e. dim ker $\nabla^2 U(q_0)$ = dim $\Gamma(q_0)$ and $T_{q_0}\Gamma(q_0) = \ker \nabla^2 U(q_0)$. Therefore z_0 is non-degenerate critical point of $\nabla H_{|T_{z_0}^\perp \Gamma(z_0)}$ i.e. $\nabla^2 H_{|T_{z_0}^\perp \Gamma(z_0)}$ is an isomorphism. In such case $\deg(\nabla H_{|T_{z_0}^\perp \Gamma(z_0)}, B(z_0, \varepsilon), 0) = \pm 1$.
- In Theorem 5.6 we assume that the orbit $\Gamma(q_0)$ consists of minima of U and is isolated in critical points of U. Therefore $z_0 = (q_0, 0)$ is an isolated minimum of $H_{|T_{z_0}^\perp \Gamma(z_0)}$. However, it is known that the Brouwer degree of minimum equals 1 (see [26]).

Theorems 5.6 and 5.7 are given with the original thesis, but in fact they are directly related to the thesis of Theorem 4.1; see the remarks below Definition 4.3. □

6. An Application

In this section we apply our abstract results to the study of the quasi-periodic motions of a satellite near the geostationary orbit of an oblate spheroid with rotational symmetry. Note that the Earth is flattened and therefore the study of gravitation potential of such bodies has crucial role in the design of missions of satellites.

A gravitational potential of an oblate spheroid has a general form

$$U_G(r, \theta) = -G\frac{E}{r}\left(1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)\right),$$

where r is a distance from the center of mass of the spheroid, θ—deviation from the axis of rotation, G—gravitational constant, E is the mass of the spheroid and R is its equatorial radius, (J_n) is the sequence of coefficients realted to the spherical harmonics and P_n denotes the n-th Legendre polynomial, see [19] for the details. In the case of axial symmetry the dominating term is J_2, so called dynamical form-factor, which is directly related to the body's flattening and for the oblate body J_2 is positive. For the Earth $J_2 = 1.0826359 \cdot 10^{-3}$.

We are going to study the motions under approximate potential

$$U(r, \theta) = -\frac{GE}{r}\left(1 - \frac{J_2 R^2}{r^2} P_2(\cos \theta)\right).$$

By the choose of the units we may assume $GE = 1$. Moreover, $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$ and by the change of coordinates to the axially symmetric cylindrical ones we obtain

$$V(r, z) = -\frac{1}{d}\left(1 - \frac{c}{d^2}\left(3\frac{z^2}{d^2} - 1\right)\right) = -\frac{1}{d} - \frac{c}{d^3} + \frac{3cz^2}{d^5}, \qquad (6.1)$$

where $c = \frac{1}{2}R^2 J_2 > 0$, $d = \sqrt{r^2 + z^2}$.

Assume that axially symmetric and oblate planet is rotating with an angular velocity ω. We study the move of the satellite in the gravity field of this planet without influence of other bodies. Denote by q_1, q_2, q_3 coordinates of the satellite in a frame rotating with an angular velocity ω (the frame fixed with planet), where the axis q_3 is the axis of rotation and symmetry of the planet and denote by p_1, p_2, p_3 the corresponding momenta. The equation of motion is generated by the Hamiltonian H of the form:

$$H(q_1, q_2, q_3, p_1, p_2, p_3) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \omega(q_1 p_2 - q_2 p_1)$$
$$+ V(r, q_3), \qquad (6.2)$$

where $r = \sqrt{q_1^2 + q_2^2}$ and V is given in (6.1), see [15]. Note that this Hamiltonian is S^1-invariant where the symplectic action is given by

$$S^1 \times \mathbb{R}^6 \ni (e^{i\theta}, (q_1, q_2, q_3, p_1, p_2, p_3))$$
$$\rightarrow \left(\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}\right).$$

Non-zero equilibria of the Hamiltonian system $\dot{z}(t) = J\nabla H(z(t))$ describe a motion of a satellite along geostationary orbits. We apply Theorem 5.4 to prove the existence of periodic solutions in a nearby of any equilibrium. Since the coordinates frame is rotating, we obtain the quasi-periodic motions of the satellite in a neighborhood of the geostationary orbit. We are interested in geostationary circular orbit so we assume $r > 0$

Firstly, we have to find critical points of H:

$$\begin{cases} H'_{q_1}(q_1, q_2, q_3, p_1, p_2, p_3) = \omega p_2 + V'_r(r, q_3)\frac{q_1}{r}, \\ H'_{q_2}(q_1, q_2, q_3, p_1, p_2, p_3) = -\omega p_1 + V'_r(r, q_3)\frac{q_2}{r}, \\ H'_{q_3}(q_1, q_2, q_3, p_1, p_2, p_3) = V'_z(r, q_3), \\ H'_{p_1}(q_1, q_2, q_3, p_1, p_2, p_3) = p_1 - \omega q_2, \\ H'_{p_2}(q_1, q_2, q_3, p_1, p_2, p_3) = p_2 + \omega q_1, \\ H'_{p_3}(q_1, q_2, q_3, p_1, p_2, p_3) = p_3. \end{cases}$$

Therefore, critical points of H need to satisfy

$$\begin{cases} \left(r\omega^2 - V_r'(r, q_3)\right) q_1 = 0, \\ \left(r\omega^2 - V_r'(r, q_3)\right) q_2 = 0, \\ V_z'(r, q_3) = 0, \\ p_1 = \omega q_2, \\ p_2 = -\omega q_1, \\ p_3 = 0. \end{cases}$$

Since $r = \sqrt{q_1^2 + q_2^2} > 0$, by the first two equations we have

$$r\omega^2 = V_r'(r, q_3) = \frac{r}{d^3} + \frac{3cr}{d^5} - \frac{15cq_3^2 r}{d^7} \Rightarrow \omega^2 d^7 = d^4 + 3cd^2 - 15cq_3^2. \quad (6.3)$$

Further, by the third equation,

$$0 = V_z'(r, q_3) = \frac{q_3}{d^3} + \frac{9cq_3}{d^5} - \frac{15cq_3^3}{d^7} = \frac{q_3}{d^7}\left(d^4 + 9cd^2 - 15cq_3^2\right)$$
$$= \frac{q_3}{d^7}\left(w^2 d^7 + 6cd^2\right).$$

Since $\omega, c, d > 0$, we obtain $q_3 = 0$. As a consequence, the equation (6.3) has the form

$$\omega^2 d^5 - d^2 - 3c = 0.$$

Since $c > 0$, by Descartes' rule of signs there exists exactly one positive root of this equation, say d_0. This means that there exist one $SO(2)$ orbit of critical points of H i.e. $SO(2)(Q)$ where $Q = (d_0, 0, 0, 0, -\omega d_0, 0)$. The point Q from this orbit is chosen such that $r = \sqrt{q_1^2 + q_2^2} = q_1 = d_0$. This orbit is obviously isolated in $(\nabla H)^{-1}(0)$. To apply theorem 5.4 we compute the Hessian $\nabla^2 H(Q)$. We have

$$\nabla^2 H(Q) = \begin{bmatrix} V_{r,r}''(d_0, 0) & 0 & V_{r,z}''(d_0, 0) & 0 & \omega & 0 \\ 0 & \omega^2 & 0 & -\omega & 0 & 0 \\ V_{r,z}''(d_0, 0) & 0 & V_{z,z}''(d_0, 0) & 0 & 0 & 0 \\ 0 & -\omega & 0 & 1 & 0 & 0 \\ \omega & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} V_{r,r}''(d_0, 0) &= -\frac{2}{d_0^3} - \frac{12c}{d_0^5} < 0, \\ V_{z,z}''(d_0, 0) &= \frac{1}{d_0^3} + \frac{9c}{d_0^5} > 0, \\ V_{r,z}''(d_0, 0) &= 0. \end{aligned} \quad (6.4)$$

The Hessian is obviously degenerate (see Remark 2.4). One can see that it possesses eigenvalues $1 + \omega^2$ (with eigenvector $[0, 1, 0, -1/\omega, 0, 0]^T$) and 1 (with eigenvector $[0, 0, 0, 0, 0, 1]^T$). Denote be $\lambda_1, \lambda_2, \lambda_3$ the other three eigenvalues. If we compute the characteristic polynomial $w(t)$ of $\nabla^2 H(Q)$ its coefficient of the

term t (which is the additive inverse of the product of eigenvalues different from the one zero-eigenvalue we have already know) equals

$$-(1 + \omega^2)\left(-\omega^2 V_{z,z}''(d_0, 0) + V_{r,r}''(d_0, 0)V_{z,z}''(d_0, 0) - (V_{r,z}''(d_0, 0))^2\right),$$

and substituting formulas (6.4), we obtain

$$\lambda_1\lambda_2\lambda_3 = \left(\frac{1}{d_0^3} + \frac{9c}{d_0^5}\right)\left(-\omega^2 - \frac{2}{d_0^3} - \frac{12c}{d_0^5}\right) < 0.$$

Hence one or three of λ_i are negative. Therefore the Hessian $\nabla^2 H(Q)$ has two or four positive eigevalues. This means that the assumption **(A7.5)** is satisfied. Moreover, the kernel of this Hessian is one-dimensional which provides that the orbit $SO(2)(Q)$ is non-degenerate. Therefore the assumption **(A5)** is also satisfied (see the reasoning in the last paragraph of the previous section on the page 19). To summarize, all assumptions of Theorem 5.4 are satisfied. This provides the existence of periodic solutions in a nearby of an equilibrium Q in the rotating frame. These solutions correspond to a motion in a neighborhood of the geostationary orbit.

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References

1. BARTSCH, T.: *Topological Methods for Variational Problems with Symmetries, Lecture Notes in Mathematics*, vol. 1560. Springer, Berlin 1993
2. BARTSCH, T.: A generalization of the Weinstein–Moser theorems on periodic orbits of a Hamiltonian system near an equilibrium. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14**(6), 691–718, 1997
3. BŁASZCZYK, Z., GOŁEBIEWSKA, A., RYBICKI, Sa: Conley index in Hilbert spaces versus the generalized topological degree. *Adv. Differ. Equ.* **22**(11–12), 963–982, 2017
4. BOLSINOV, A.V., BORISOV, A.V., MAMAEV, I.S.: Bifurcation analysis and the Conley index in mechanics. *Regul. Chaotic Dyn.* **17**(5), 451–478, 2012. Reprinted from ıt Russian J. Nonlinear Dyn. **7**(3), 649–681 (2011)

5. DANCER, E.N., GEBA, K., RYBICKI, S.: Classification of homotopy classes of equivariant gradient maps. *Fund. Math.* **185**(1), 1–18, 2005
6. DANCER, E.N., RYBICKI, S.: A note on periodic solutions of autonomous Hamiltonian systems emanating from degenerate stationary solutions. *Differ. Integral Equ.* **12**(2), 147–160, 1999
7. tom DIECK, T.: *Transformation Groups, De Gruyter Studies in Mathematics*, vol. 8. Walter de Gruyter & Co., Berlin 1987
8. FADELL, E.R., RABINOWITZ, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.* **45**(2), 139–174, 1978
9. FLOER, A.: A refinement of the Conley index and an application to the stability of hyperbolic invariant sets. *Ergod. Theory Dyn. Syst.* **7**(1), 93–103, 1987
10. GEBA, K.: Degree for gradient equivariant maps and equivariant Conley index. In: *Topological nonlinear analysis, II (Frascati, 1995), Progress in Nonlinear Differential Equations Applications.*, vol. 27, pp. 247–272. Birkhäuser Boston, Boston, MA 1997
11. GEBA, K., IZYDOREK, M., PRUSZKO, A.: The Conley index in Hilbert spaces and its applications. *Studia Math.* **134**(3), 217–233, 1999
12. GOŁĘBIEWSKA, A.: Periodic solutions of asymptotically linear autonomous Hamiltonian systems. *J. Math. Anal. Appl.* **400**(1), 254–265, 2013
13. GOŁĘBIEWSKA, A., RYBICKI, S.: Global bifurcations of critical orbits of *G*-invariant strongly indefinite functionals. *Nonlinear Anal.* **74**(5), 1823–1834, 2011
14. GOŁĘBIEWSKA, A., RYBICKI, S.: Equivariant Conley index versus degree for equivariant gradient maps. *Discrete Contin. Dyn. Syst. Ser. S* **6**(4), 985–997, 2013
15. HAND, L.N., FINCH, J.D.: *Analytical Mechanics*. Cambridge University Press, Cambridge 1998
16. IZYDOREK, M.: Equivariant Conley index in Hilbert spaces and applications to strongly indefinite problems. *Nonlinear Anal.* **51**(1, Ser. A: Theory Methods), 33–66, 2002
17. KAWAKUBO, K.: *The Theory of Transformation Groups*. The Clarendon Press, Oxford University Press, New York 1991
18. LÓPEZ GARZA, G., RYBICKI, S.: Equivariant bifurcation index. *Nonlinear Anal.* **73**(9), 2779–2791, 2010
19. LOWRIE, W.: *Fundamentals of Geophysics*. Cambridge University Press, New York 2007
20. LYAPUNOV, A.M.: Problème général de la stabilité du mouvement. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2)* **9**, 203–474, 1907. Russian original in 1895
21. MAWHIN, J., WILLEM, M.: *Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences*, vol. 74. Springer, New York 1989
22. MONTALDI, J.A., ROBERTS, R.M., STEWART, I.N.: Periodic solutions near equilibria of symmetric Hamiltonian systems. *Philos. Trans. R. Soc. Lond. Ser. A* **325**(1584), 237–293, 1988
23. MOSER, J.: Periodic orbits near an equilibrium and a theorem by Alan Weinstein. *Commun. Pure Appl. Math.* **29**(6), 724–747, 1976
24. PÉREZ-CHAVELA, E., RYBICKI, S., STRZELECKI, D.: Symmetric Liapunov center theorem. *Calc. Var. Partial Differ. Equ.* **56**(2), 23, 2017. https://doi.org/10.1007/s00526-017-1120-1. Art. 26
25. PÉREZ-CHAVELA, E., RYBICKI, S., STRZELECKI, D.: Symmetric Liapunov center theorem for minimal orbit. *J. Differ. Equ.* **265**(3), 752–778, 2018. https://doi.org/10.1016/j.jde.2018.03.009
26. RABINOWITZ, P.H.: A note on topological degree for potential operators. *J. Math. Anal. Appl.* **51**(2), 483–492, 1975
27. RABINOWITZ, P.H.: Periodic solutions of Hamiltonian systems: a survey. *SIAM J. Math. Anal.* **13**(3), 343–352, 1982
28. RABINOWITZ, P.H.: Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics, vol. 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI 1986

29. Rybakowski, K.P.: On the homotopy index for infinite-dimensional semiflows. *Trans. Am. Math. Soc*. **269**(2), 351–382, 1982
30. Smoller, J., Wasserman, A.G.: Bifurcation and symmetry-breaking. *Invent. Math*. **100**(1), 63–95, 1990
31. Srzednicki, R.: On rest points of dynamical systems. *Fundam. Math*. **126**(1), 69–81, 1985
32. Szulkin, A.: Bifurcation for strongly indefinite functionals and a Liapunov type theorem for Hamiltonian systems. *Differ. Integral Equ*. **7**(1), 217–234, 1994
33. Weinstein, A.: Normal modes for nonlinear Hamiltonian systems. *Invent. Math*. **20**, 47–57, 1973

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