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High Frequency Limit for a Chain of Harmonic Oscillators with a Point Langevin Thermostat

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Abstract

We consider an infinite chain of coupled harmonic oscillators with a Langevin thermostat at the origin. In the high frequency limit, we establish the reflection-transmission coefficients for the wave energy for the scattering off the thermostat. To our surprise, even though the thermostat fluctuations are time-dependent, the scattering does not couple wave energy at various frequencies.

1. Introduction

Heat reservoirs with some given temperature T are usually modelled at the microscopic level by the Langevin stochastic dynamics, or by other random mechanisms such as the renewal of velocities at random times with Gaussian distributed velocities of variance T. This latter mechanism represents the interaction with an infinitely extended reservoir of independent particles in equilibrium at temperature T and uniform density.

When such reservoirs are in contact with the system boundary and if energy diffuses on the macroscopic space-time scale, then it is expected that a thermostat enforces a local equilibrium at the boundary at the temperature T. The situation is much less clear for kinetic (hyperbolic) space-time scales. For instance, if the bulk evolution is governed by a discrete nonlinear wave equation, then in the kinetic (high frequency) limit the wave number density is governed by a phonon Boltzmann equation [1,11]. If this system is coupled to a thermostat at the boundary, what are the appropriate macroscopic boundary conditions which have to be added to the kinetic equation?

To make a study feasible, we very much simplify the set-up. We consider an infinite one-dimensional chain of harmonic oscillators, characterized by its dispersion relation $\omega(k)$, and couple it with a single Langevin thermostat at the origin. An efficient way to localize the distribution of the energy at wave number k is to use the Wigner distribution. In a space-time hyperbolic rescaling, first ignoring the thermostat, the Wigner distribution converges to the solution $W(t,x,k)$ of a simple transport equation, namely phonons of wavenumber k have energy $\omega(k)$ and travel independently with group velocity $\omega'(k)/2\pi$. It will be proved that when the dispersion relation is unimodal, see Section 2 for the precise definition, in the scaling limit, the thermostat enforces the following reflection-transmission (and production) conditions at $x = 0$: phonons of wave number k are generated with rate $\mathfrak{g}(k)T$ and an incoming k-phonon is transmitted with probability $p_+(k)$, reflected with probability $p_-(k)$, and absorbed with probability $\mathfrak{g}(k)$, see formulas (2.28) below. These coefficients are positive, depend on $\omega(\cdot)$, and satisfy

$$p_+(k) + p_-(k) + \mathfrak{g}(k) = 1.$$

With such boundary conditions the stationary solution of the transport equation is the thermal equilibrium Wigner function $W(t,x,k) = T$.

The thermostat can be viewed as a “scatterer” of a time-varying strength: at the microscopic scale a wave incident on the thermostat would produce reflected and transmitted waves at all frequencies. It is remarkable that, after the scaling limit, the reflected and transmitted waves are of the same frequency as the incident wave, all other waves produced by the microscopic scattering are damped by oscillations in the macroscopic limit. The presence of oscillatory integrals, responsible for the damping mechanism, presents the main mathematical difficulty of the model. To deal with the issue we consider the high frequency limit of the Laplace transform of the Wigner distribution. The limit, see (2.33) below, can be decomposed into the parts that correspond to the production, transmission and reflection of a phonon. The calculation of the production term is relatively straightforward, see Section 4. In contrast, the computations related to the scattering terms are remarkably difficult, see Sections 5–9 for the proof. Moreover, the description of the limit is not intuitive and it is not clear to us how to obtain it by a simple heuristic argument.

The multimodal case, that we shall not consider here, can be also dealt with using the technique of the present paper. In this situation the level set of $\omega(k)$ has generically $2N$ points (we assume that ω is even) for some positive integer N. The macroscopic description of the system is as follows: a k-phonon arriving at interface with group velocity $\omega'(k) > 0$ is transmitted as a k'-phonon corresponding to the solutions of $\omega(k') = \omega(k)$, with a positive group velocity. The probabilities of transmission at a given k' can be computed explicitly in terms of the dispersion relation. On the other hand, it reflects as a k'-phonon corresponding to a solution of $\omega(k') = \omega(k)$, with a group velocity $\omega'(k') < 0$. The probability of absorption is the same as in the unimodal case.

Introducing a rarefied random scattering in the bulk, in the same fashion as in [1], leads to a similar transport equation with a linear scattering term, without modifying the transmission properties at the interface with the thermostat [8].

There are rather few results on the high frequency limits of the Wigner transform in the presence of boundaries, interfaces or sources. We mention [2–4,6,10] which, while highly non-trivial, are all ultimately based on essentially explicit computations of the Wigner transform near the interface. Our analysis also starts with computing the Wigner transform, but then passes to the limit in the resulting expression. The thermal production of phonons can be seen quite straightforwardly in this limit. However the scattering terms are much more difficult to handle and they constitute the major part of our work.

2. The Dynamics and the Main Result

The Infinite Chain of Harmonic Oscillators

We consider the evolution of an infinite particle system governed by the Hamiltonian

$$\mathcal{H}(\mathfrak{p}, \mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}} \mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}} \alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}. \tag{2.1}$$

Here, the particle label is $y \in \mathbb{Z}$, $(\mathfrak{p}_y, \mathfrak{q}_y)$ is the position and momentum of the y's particle, respectively, and $(\mathfrak{q}, \mathfrak{p}) = \{(\mathfrak{p}_y, \mathfrak{q}_y),\ y \in \mathbb{Z}\}$ denotes the entire configuration. The coupling coefficients α_y are assumed to have exponential decay and chosen such that the energy is bounded from below.

A stochastically perturbed version of this system was considered first in [1], where the long time behavior of the wave energy was analyzed, and then in [7], where the wave field itself was studied.

The stochasticity in [1,7] was introduced as a random exchange of momenta between particles at adjacent sites. Here, instead of random fluctuations "in the bulk", we couple the particle with label 0 to a Langevin thermostat at temperature T and with friction $\gamma > 0$. Then the Hamiltonian dynamics with stochastic source is governed by

$$\begin{aligned}
\dot{\mathfrak{q}}_y(t) &= \mathfrak{p}_y(t),\\
\mathrm{d}\mathfrak{p}_y(t) &= -(\alpha \star \mathfrak{q}(t))_y \mathrm{d}t + \big(-\gamma\mathfrak{p}_0(t)\mathrm{d}t + \sqrt{2\gamma T}\mathrm{d}w(t)\big)\delta_{0,y}, \quad y \in \mathbb{Z}.
\end{aligned} \tag{2.2}$$

Here, $\{w(t),\ t \geq 0\}$ is a standard Wiener process over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use the notation

$$(f \star g)_y = \sum_{y'\in\mathbb{Z}} f_{y-y'}g_{y'}$$

for the convolution of two functions on $\mathbb{Z}$.

It is convenient to introduce the complex wave function

$$\psi_y(t) := (\tilde{\omega} \star \mathfrak{q}(t))_y + i\mathfrak{p}_y(t), \tag{2.3}$$

where $\{\tilde{\omega}_y,\ y \in \mathbb{Z}\}$ is the inverse Fourier transform of the dispersion relation

$$\omega(k) := \sqrt{\hat{\alpha}(k)}. \tag{2.4}$$

Hence $|\psi_y(t)|^2$ is the local energy of the chain at time t. The Fourier transform of the wave function is given by

$$\hat{\psi}(t,k) := \omega(k)\hat{\mathfrak{q}}(t,k) + i\hat{\mathfrak{p}}(t,k), \tag{2.5}$$

so that

$$\hat{\mathfrak{p}}\,(t,k) = \frac{1}{2i}[\hat{\psi}(t,k) - \hat{\psi}^*(t,-k)], \quad \mathfrak{p}_0(t) = \int_{\mathbb{T}} \operatorname{Im}\hat{\psi}(t,k)\mathrm{d}k.$$

Using (2.2), it is easy to check that the wave function evolves according to

$$\mathrm{d}\hat{\psi}(t,k) = \big(-i\omega(k)\hat{\psi}(t,k) - i\gamma\mathfrak{p}_0(t)\big)\mathrm{d}t + i\sqrt{2\gamma T}\mathrm{d}w(t). \tag{2.6}$$

Above, the Fourier transform of $f_x \in l^2(\mathbb{Z})$ and the inverse Fourier transform of $\hat{f} \in L^2(\mathbb{T})$ are

$$\begin{aligned}\hat{f}(k) &= \sum_{x\in\mathbb{Z}} f_x \exp\{-2\pi i x k\}, \quad f_x\\ &= \int_{\mathbb{T}} \hat{f}(k)\exp\{2\pi i x k\}\mathrm{d}k, \quad x\in\mathbb{Z}, \quad k\in\mathbb{T}.\end{aligned} \tag{2.7}$$

For a function $G(x,k)$, we denote by $\tilde{G}:\mathbb{R}\times\mathbb{Z}\to\mathbb{C}$, $\hat{G}:\mathbb{R}\times\mathbb{T}\to\mathbb{C}$ the Fourier transforms of G in the k and x variables, respectively,

$$\begin{aligned}\tilde{G}(x,y) &:= \int_{\mathbb{T}} e^{-2\pi i k y} G(x,k)\mathrm{d}k, \quad (x,y)\in\mathbb{R}\times\mathbb{Z},\\ \hat{G}(\eta,k) &:= \int_{\mathbb{R}} e^{-2\pi i \eta x} G(x,k)\mathrm{d}x, \quad (\eta,k)\in\mathbb{R}\times\mathbb{T}.\end{aligned}$$

The Initial Conditions For simplicity sake we restrict ourselves to initial configurations of finite energy. In addition, we assume that the initial energy density $|\psi_y|^2$ is finite per unit length on the macroscopic scale $x \sim \varepsilon y$, where $\varepsilon > 0$ is the scaling parameter. More precisely, given $\varepsilon > 0$, the initial wave function is distributed randomly, independent of the Langevin noise $w(\cdot)$, according to a probability measure μ_ε on $\ell^2(\mathbb{Z})$, and

$$\sup_{\varepsilon\in(0,1)} \sum_{y\in\mathbb{Z}} \varepsilon\langle|\psi_y|^2\rangle_{\mu_\varepsilon} = \sup_{\varepsilon\in(0,1)} \varepsilon\langle\|\hat{\psi}\|^2_{L^2(\mathbb{T})}\rangle_{\mu_\varepsilon} < \infty, \tag{2.8}$$

where $\langle\cdot\rangle_{\mu_\varepsilon}$ denotes the expectation with respect to μ_ε. We will also assume that

$$\langle\hat{\psi}(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon} = 0, \quad k,\ell\in\mathbb{T}. \tag{2.9}$$

Condition (2.9) can be replaced by $\langle\hat{\psi}(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon} \sim 0$, as $\varepsilon\to 0$ at the expense of some additional calculations that we prefer not to perform in this article.

An additional hypothesis concerning the initial configuration will be stated later on, see (2.18).

The Wigner Distribution To study the effect of the thermostat, we follow the evolution of the chain on the macroscopic time scale $t' \sim \varepsilon t$, and consider the rescaled wave function $\psi_y^{(\varepsilon)}(t) = \psi_y(t/\varepsilon)$. A convenient tool to analyse the energy density is the Wigner distribution (or Wigner transform) defined by its action on a test function $G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ as

$$\langle G, W^{(\varepsilon)}(t)\rangle := \frac{\varepsilon}{2} \sum_{y,y' \in \mathbb{Z}} \mathbb{E}_\varepsilon \left[\psi_y^{(\varepsilon)}(t) \left(\psi_{y'}^{(\varepsilon)}\right)^* (t)\right] \tilde{G}^*\Big(\varepsilon \frac{y+y'}{2}, y-y'\Big). \tag{2.10}$$

Here, $\mathbb{E}_\varepsilon$ is the expectation with respect to the product measure $\mu_\varepsilon \otimes \mathbb{P}$.

The Fourier transform of the Wigner distribution is

$$\begin{aligned}\widehat{W}_\varepsilon(t, \eta, k) :=\ & \frac{\varepsilon}{2}\mathbb{E}_\varepsilon \left[(\hat{\psi}^{(\varepsilon)})^*(t, k - \frac{\varepsilon\eta}{2})\hat{\psi}^{(\varepsilon)}(t, k + \frac{\varepsilon\eta}{2})\right],\\ & (t, \eta, k) \in [0, \infty) \times \mathbb{T}_{2/\varepsilon} \times \mathbb{T},\end{aligned} \tag{2.11}$$

so that

$$\langle G, W^{(\varepsilon)}(t)\rangle = \int_{\mathbb{T}\times\mathbb{R}} \widehat{W}_\varepsilon(t, \eta, k)\hat{G}^*(\eta, k)\mathrm{d}\eta\mathrm{d}k, \quad G \in \mathcal{S}(\mathbb{R} \times \mathbb{T}). \tag{2.12}$$

We use the notation $\mathbb{T}_a = [-a/2, a/2]$ for the torus of size $a > 0$, with identified endpoints.

A straightforward calculation shows that the macroscopic energy grows at most linearly in time,

$$\mathrm{d}\|\hat{\psi}^{(\varepsilon)}(t)\|^2_{L^2(\mathbb{T})} = \Big[-\frac{\gamma}{\varepsilon}[\mathfrak{p}_0^{(\varepsilon)}(t)]^2 + \frac{2\gamma T}{\varepsilon}\Big]\mathrm{d}t + \sqrt{\frac{2\gamma T}{\varepsilon}}\mathfrak{p}_0^{(\varepsilon)}(t)\mathrm{d}w(t), \tag{2.13}$$

with $\mathfrak{p}_0^{(\varepsilon)}(t) := \mathfrak{p}_0(t/\varepsilon)$. Thus, we have a uniform bound

$$\sup_{\varepsilon\in(0,1]} \varepsilon\mathbb{E}_\varepsilon\|\hat{\psi}^{(\varepsilon)}(t)\|^2_{L^2(\mathbb{T})} \le \sup_{\varepsilon\in(0,1]} \varepsilon\mathbb{E}_\varepsilon\|\hat{\psi}^{(\varepsilon)}(0)\|^2_{L^2(\mathbb{T})} + 2\gamma T t, \quad t \ge 0. \tag{2.14}$$

Let us denote by $\mathcal{A}$ the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ in the norm

$$\|G\|_{\mathcal{A}} := \int_{\mathbb{R}} \sup_{k\in\mathbb{T}} |\hat{G}(\eta, k)|\mathrm{d}\eta \tag{2.15}$$

and by $\mathcal{A}'$ its dual. We conclude from (2.14) that (see [5])

$$\sup_{t\in[0,\tau]} \|W^{(\varepsilon)}(t)\|_{\mathcal{A}'} < \infty, \text{ for each } \tau > 0, \tag{2.16}$$

hence $W^{(\varepsilon)}(\cdot)$ is sequentially weak-$\star$ compact over $(L^1([0, \tau]; \mathcal{A}))^\star$ for any $\tau > 0$. We will assume that the initial Wigner distribution

$$\widehat{W}_\varepsilon(\eta, k) := \widehat{W}_\varepsilon(0, \eta, k), \quad (\eta, k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T} \tag{2.17}$$

is a family that converges weakly in $\mathcal{A}'$ to a non-negative function $W_0 \in L^1(\mathbb{R} \times \mathbb{T}) \cap C(\mathbb{R} \times \mathbb{T})$. We will also assume that there exist $C, \kappa > 0$ such that

$$|\widehat{W}_\varepsilon(\eta, k)| \le C\varphi(\eta), \quad (\eta, k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T},\ \varepsilon \in (0, 1], \tag{2.18}$$

where

$$\varphi(\eta) := \frac{1}{(1 + \eta^2)^{3/2+\kappa}}. \tag{2.19}$$

Assumptions on the Dispersion Relation and Its Basic Properties We assume, as in [1], that α_y is a real-valued even function of $y \in \mathbb{Z}$, and there exists $C > 0$ so that

$$|\alpha_y| \le Ce^{-|y|/C}, \text{ for all } y \in \mathbb{Z},$$

thus $\hat{\alpha} \in C^\infty(\mathbb{T})$. We also assume that $\hat{\alpha}(k) > 0$ for $k \ne 0$, and if $\hat{\alpha}(0) = 0$ then $\hat{\alpha}''(0) > 0$, so that $\hat{\alpha}(k) = \sin^2(\pi k)\hat{\alpha}_0(k)$ for some strictly positive even function $\hat{\alpha}_0 \in C^\infty(\mathbb{T})$. It follows that the dispersion relation $\omega(k) = \sqrt{\hat{\alpha}(k)}$ is also an even and continuous function in $C^\infty(\mathbb{T} \setminus \{0\})$. We assume that ω is increasing on $[0, 1/2]$, and denote its unique minimum attained at $k = 0$ by $\omega_{\min} \ge 0$, its unique maximum, attained at $k = 1/2$, by $\omega_{\max}$, and the two branches of the inverse of $\omega(\cdot)$ as $\omega_- : [\omega_{\min}, \omega_{\max}] \to [-1/2, 0]$ and $\omega_+ : [\omega_{\min}, \omega_{\max}] \to [0, 1/2]$. They satisfy $\omega_- = -\omega_+$, $\omega_+(\omega_{\min}) = 0$, $\omega_+(\omega_{\max}) = 1/2$ and in the case $\omega \in C^\infty(\mathbb{T})$:

$$\omega'_\pm(w) = \pm(w - \omega_{\min})^{-1/2}\chi_*(w), \quad w - \omega_{\min} \ll 1, \tag{2.20}$$

and

$$\omega'_\pm(w) = \pm(\omega_{\max} - w)^{-1/2}\chi^*(w), \quad \omega_{\max} - w \ll 1, \tag{2.21}$$

with $\chi_*, \chi^* \in C^\infty(\mathbb{T})$ that are strictly positive. When ω is not differentiable at 0 (the acoustic case) instead of (2.20) we assume

$$\omega'_\pm(w) = \pm\chi_*(w), \quad w - \omega_{\min} \ll 1, \tag{2.22}$$

leaving condition (2.21) unchanged.

An important role in the analysis will be played by the function

$$J(t) = \int_{\mathbb{T}} \cos(\omega(k)t)\,\mathrm{d}k, \quad t \ge 0 \tag{2.23}$$

its Laplace transform

$$\tilde{J}(\lambda) := \int_0^\infty e^{-\lambda t} J(t)\mathrm{d}t = \int_{\mathbb{T}} \frac{\lambda}{\lambda^2 + \omega^2(k)}\mathrm{d}k, \quad \mathrm{Re}\,\lambda > 0, \tag{2.24}$$

and the function

$$\tilde{g}(\lambda) := (1 + \gamma\tilde{J}(\lambda))^{-1}. \tag{2.25}$$

Note that $\mathrm{Re}\,\tilde{J}(\lambda) > 0$ for $\lambda \in \mathbb{C}_+ := [\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0]$, therefore

$$|\tilde{g}(\lambda)| \le 1, \quad \lambda \in \mathbb{C}_+. \tag{2.26}$$

The function $\tilde{g}(\cdot)$ is analytic on $\mathbb{C}_+$ so, by the Fatou theorem, see e.g. p. 107 of [9], we know that

$$\nu(k) := \lim_{\varepsilon \to 0} \tilde{g}(\varepsilon - i\omega(k)) \tag{2.27}$$

exists a.e. in $\mathbb{T}$ and in any $L^p(\mathbb{T})$ for $p \in [1, \infty)$.

To state our main result we need some additional notation. Let us introduce the group velocity

$$\bar{\omega}'(k) := \omega'(k)/(2\pi)$$

and

$$\begin{aligned}
\wp(k) &:= \frac{\gamma\nu(k)}{2|\bar{\omega}'(k)|},\\
\mathfrak{g}(k) &:= \frac{\gamma|\nu(k)|^2}{|\bar{\omega}'(k)|},\\
p_+(k) &:= |1-\wp(k)|^2, \quad p_-(k) := |\wp(k)|^2.
\end{aligned} \tag{2.28}$$

We will show in Section 10 that

$$\operatorname{Re}\nu(k) = \left(1 + \frac{\gamma}{2|\bar{\omega}'(k)|}\right)|\nu(k)|^2. \tag{2.29}$$

It follows that

$$p_+(k) + p_-(k) = 1 - \mathfrak{g}(k) \le 1, \tag{2.30}$$

so that, in particular, we have

$$0 \le \mathfrak{g}(k) \le 1. \tag{2.31}$$

The Main Result

Our main result is as follows. For brevity, we use the notation $[0, x]$ both for $x < 0$ and $x > 0$, so as not to state the results separately for $\omega'(k) > 0$ and $\omega'(k) < 0$.

Theorem 2.1. *Suppose that the initial conditions and the dispersion relation satisfy the above assumptions. Then, for any $\tau > 0$ and $G \in L^1([0,\tau];\mathcal{A})$ we have*

$$\lim_{\varepsilon\to 0}\int_0^\tau \langle G(t), W_\varepsilon(t)\rangle \mathrm{d}t = \int_0^\tau \mathrm{d}t \int_{\mathbb{R}\times\mathbb{T}} G^*(t,x,k)W(t,x,k)\mathrm{d}x\mathrm{d}k, \tag{2.32}$$

where

$$\begin{aligned}
W(t,x,k) = {} & W_0\left(x - \bar{\omega}'(k)t, k\right) 1_{[0,\bar{\omega}'(k)t]^c}(x) + \mathfrak{g}(k)T\,1_{[0,\bar{\omega}'(k)t]}(x)\\
& + p_+(k)W_0\left(x - \bar{\omega}'(k)t, k\right) 1_{[0,\bar{\omega}'(k)t]}(x)\\
& + p_-(k)W_0\left(-x + \bar{\omega}'(k)t, -k\right) 1_{[0,\bar{\omega}'(k)t]}(x).
\end{aligned} \tag{2.33}$$

The limit dynamics has an obvious interpretation. The first term is the ballistic transport of those phonons which did not cross $\{x = 0\}$ up to time t. The second term in the right side of (2.33) describes the phonon production of the thermostat. The third and the fourth term correspond, respectively, to the transmission and reflection of the phonons at the boundary point $\{x = 0\}$. More precisely, $\mathfrak{g}(k)T$ is the phonon production rate, $p_-(k)$ is the probability of reflection, and $p_+(k)$ is the probability of transmission at $\{x = 0\}$. Notice that the phonons are absorbed by the thermostat with probability $1 - p_+(k) - p_-(k) = \mathfrak{g}(k)$. The scattering at the origin depends only on the friction coefficient γ. At zero temperature the production of phonons is turned off, while the scattering remains unmodified.

From (2.29) it follows that

$$\mathfrak{g}(k) = 2(\mathrm{Re}\ \nu(k) - |\nu(k)|^2),$$

and we also know that $\nu(k_0) = 0$ at the points where $\omega'(k_0) = 0$. This means that the thermostat does not generate phonons with zero velocity, which otherwise would have led to the accumulation of energy at the boundary.

Our main theorem is for the averaged Wigner distribution. In general, one expects a suitable law of large numbers for the quantity on the left of (2.32) with respect to $\mu_\varepsilon \otimes \mathbb{P}$, even if the definition (2.10) of the Wigner transform would not include the expectation $\mathbb{E}_\varepsilon$.

Our result can be written as a boundary value problem, which is a simple but useful exercise. First, $W(t, x, k)$ solves the homogeneous transport equation

$$\partial_t W(t, x, k) + \bar{\omega}'(k)\partial_x W(t, x, k) = 0, \tag{2.34}$$

away from the boundary point $\{x = 0\}$. Second, if we denote the right and left limits of W by

$$W_-(t, k) := W(t, 0^-, k), \quad W_+(t, k) := W(t, 0^+, k),$$

then at $\{x = 0\}$ the outgoing phonons are related to the incoming phonons as

$$W_+(t, k) = p_-(k)W_+(t, -k) + p_+(k)W_-(t, k) + \mathfrak{g}(k)T, \quad \text{for } 0 \le k \le 1/2, \tag{2.35}$$

and

$$W_-(t, k) = p_-(k)W_-(t, -k) + p_+(k)W_+(t, k) + \mathfrak{g}(k)T, \quad \text{for } -1/2 \le k \le 0. \tag{2.36}$$

By equipartition, the equilibrium Wigner distribution is given by

$$W(t, x, k) \equiv T,$$

which indeed satisfies (2.35)–(2.36), as one should expect.

In case the bulk is governed by a wave equation with a small nonlinearity, one would expect a nonlinear transport equation for the bulk, but the boundary terms would be dominated by the linear equation, hence of the form as written above.

It is interesting to consider what happens when the strength of the thermostat $\gamma \to +\infty$, so that the oscillations of the particle in contact with the thermostat are sped up by a factor of γ. Then, we have $\tilde{g}(\lambda) \sim \gamma^{-1}$, and $\nu(k) \sim \gamma^{-1}$, hence $\mathfrak{g}(k) \to 0$ (see (2.28)), and there is no phonon production or absorption by the thermostat as the particle at the thermostat moves "too incoherently". However, there is still non-trivial reflection and transmission at the interface.

The paper is organized as follows: in Section 3, we define the Fourier-Laplace transform of the wave function and explain how the functions $J(t)$ and $\tilde{g}(\lambda)$ appear in this context. The Wigner transform can be decomposed into the ballistic part coming from the initial condition with no scattering, the thermostat production part (which is independent of the initial condition) and the scattering part. It is quite straightforward to analyze the former two terms and pass to the limit $\varepsilon \to 0$ in the corresponding expressions. Passage to the limit in the scattering term is much more difficult. It is outlined in Section 5, where one of the scattering terms is analyzed in Lemma 5.1, and the asymptotics for the other one is stated in Lemma 5.2. The scattering terms are put together in Section 5.3. The bulk of the remainder of the paper, Sections 6, 7 and 8, is essentially devoted to the proof of Lemma 5.2. The critical steps are outlined in Lemmas 7.1–7.4. Each of these statements is quite intuitive on the formal level but a rigorous justification is, unfortunately, rather lengthy and with little room to spare in the estimates. In Section 9, we remove an extra assumption that the initial condition is supported away from the non-propagating modes, made to simplify the proof. Finally, in Section 10 we prove relation (2.29).

3. The Laplace–Fourier Transform of the Wave Function and of the Wigner Distribution

In this section, we obtain an explicit expression for the Laplace-Fourier transform of the wave function. We use the mild formulation of (2.6):

$$\begin{aligned}\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k) - i\gamma \int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_0(s)\mathrm{d}s \\ + i\sqrt{2\gamma T}\int_0^t e^{-i\omega(k)(t-s)}\mathrm{d}w(t).\end{aligned} \tag{3.1}$$

Integrating both sides in the k-variable and taking the imaginary part in both sides, we obtain a closed equation for $\mathfrak{p}_0(t)$:

$$\mathfrak{p}_0(t) = \mathfrak{p}_0^0(t) - \gamma \int_0^t J(t-s)\mathfrak{p}_0(s)\mathrm{d}s + \sqrt{2\gamma T}\int_0^t J(t-s)\mathrm{d}w(s), \tag{3.2}$$

where $J(t)$ is given by (2.23) and

$$\mathfrak{p}_0^0(t) = \int_{\mathbb{T}} \mathrm{Im}\left(\hat{\psi}(0,k)e^{-i\omega(k)t}\right)\mathrm{d}k \tag{3.3}$$

is the momentum at $y = 0$ for the free evolution with $\gamma = 0$ (without the thermostat). Taking the Laplace transform

$$\tilde{\mathfrak{p}}_0(\lambda) = \int_0^{+\infty} e^{-\lambda t}\mathfrak{p}_0(t)\mathrm{d}t, \quad \operatorname{Re}\lambda > 0$$

in (3.2) we obtain

$$\tilde{\mathfrak{p}}_0(\lambda) = \tilde{g}(\lambda)\tilde{\mathfrak{p}}_0^0(\lambda) + \sqrt{2\gamma T}\,\tilde{g}(\lambda)\tilde{J}(\lambda)\tilde{w}(\lambda). \tag{3.4}$$

Here, $\tilde{g}(\lambda)$ is given by (2.25), and $\tilde{\mathfrak{p}}_0^0(\lambda)$ and $\tilde{J}(\lambda)$ are the Laplace transforms of $\mathfrak{p}_0^0(t)$ and $J(t)$, respectively, and $\tilde{w}(\lambda)$ is the Laplace transform of the Gaussian white noise.

It is a zero mean Gaussian process with the covariance

$$\mathbb{E}[\tilde{w}(\lambda_1)\tilde{w}(\lambda_2)] = \frac{1}{\lambda_1+\lambda_2}, \quad \operatorname{Re}\lambda_1,\ \operatorname{Re}\lambda_2 > 0. \tag{3.5}$$

Next, taking the Laplace transform of both sides of (3.1) and using (3.4), we arrive at an explicit formula for the Fourier-Laplace transform of $\psi_y(t)$:

$$\begin{aligned}\tilde{\psi}(\lambda,k) &= \frac{\hat{\psi}(0,k) - i\gamma\tilde{\mathfrak{p}}_0(\lambda) + i\sqrt{2\gamma T}\,\tilde{w}(\lambda)}{\lambda + i\omega(k)}\\ &= \frac{\hat{\psi}(0,k) - i\gamma\tilde{g}(\lambda)(\tilde{\mathfrak{p}}_0^0(\lambda) + \sqrt{2\gamma T}\,\tilde{J}(\lambda)\tilde{w}(\lambda)) + i\sqrt{2\gamma T}\,\tilde{w}(\lambda)}{\lambda + i\omega(k)}\\ &= \frac{\hat{\psi}(0,k) - i\gamma\tilde{g}(\lambda)\tilde{\mathfrak{p}}_0^0(\lambda) + i\tilde{g}(\lambda)\sqrt{2\gamma T}\,\tilde{w}(\lambda)}{\lambda + i\omega(k)}.\end{aligned} \tag{3.6}$$

Note that (3.6) implies, in particular, that, even at the zero temperature, and if the initial wave function is monochromatic, that is, $\hat{\psi}(0,k) = \delta_0(k-k_0)$ for some k_0, scattering at the thermostat generates various modes $k \neq k_0$, due to the damping at $y = 0$. This is a microscopic phenomenon not observed on the macroscopic level, as seen from the discussion following Theorem 2.1.

We will show below that $\tilde{g}(\lambda)$ is the Laplace transform of a signed locally finite measure $g(\mathrm{d}\tau)$. Then, the term $(\lambda + i\omega(k))^{-1}\tilde{g}(\lambda)\tilde{\mathfrak{p}}_0^0(\lambda)$, that appears in (3.6), is the Laplace transform of

$$\int_0^t \mathrm{d}s \int_0^{t-s} e^{-i\omega(k)(t-s-\tau)} g(\mathrm{d}\tau)\mathfrak{p}_0^0(s). \tag{3.7}$$

Now, the Laplace inversion of (3.6) gives an explicit expression for $\hat{\psi}(t,k)$:

$$\begin{aligned}\hat{\psi}(t,k) &= e^{-i\omega(k)t}\hat{\psi}(0,k) - i\gamma\int_0^t \mathrm{d}s\int_0^{t-s} e^{-i\omega(k)(t-s-\tau)}g(\mathrm{d}\tau)\mathfrak{p}_0^0(s)\\ &\quad + i\sqrt{2\gamma T}\int_0^t \mathrm{d}s\int_0^{t-s} e^{-i\omega(k)(t-s-\tau)}g(\mathrm{d}\tau)\mathrm{d}w(s)\\ &= e^{-i\omega(k)t}\hat{\psi}(0,k) - i\gamma\int_0^t \phi(t-s,k)\mathfrak{p}_0^0(s)\,\mathrm{d}s\\ &\quad + i\sqrt{2\gamma T}\int_0^t \phi(t-s,k)\,\mathrm{d}w(s),\end{aligned} \tag{3.8}$$

where

$$\phi(t,k)=\int_0^t e^{-i\omega(k)(t-\tau)}g(\mathrm{d}\tau). \tag{3.9}$$

Likewise, we conclude from (3.4) that

$$\mathfrak{p}_0(t)=\int_0^t \mathfrak{p}_0^0(t-s)g(\mathrm{d}s)+\sqrt{2\gamma T}\int_0^t \mathrm{d}w(s)\int_0^{t-s} J(t-s-\tau)g(\mathrm{d}\tau) \tag{3.10}$$

In order to understand how $g(\mathrm{d}\tau)$ looks like, note that a function $g_*(t)$ that has the Laplace transform

$$\tilde{g}_*(\lambda):=\tilde{g}(\lambda)-1=-\frac{\gamma\tilde{J}(\lambda)}{1+\gamma\tilde{J}(\lambda)},$$

is the solution of the Volterra equation

$$g_*(t)+\gamma J\star g_*(t)=-\gamma J(t),\quad t\geq 0. \tag{3.11}$$

Here, we denote by

$$f_1\star f_2(t)=\int_0^t f_1(t-s)f_2(s)\mathrm{d}s$$

the convolution of $f_1, f_2\in L^1_{loc}[0,+\infty)$. The solution g_* of (3.11) is given by the convolution series

$$g_*(t)=\sum_{n=1}^{+\infty}(-\gamma)^n J^{\star,n}(t). \tag{3.12}$$

Here, $J^{\star,n}(t)$ is the n-time convolution of J with itself. As $|J(t)|\leq 1$, we see that $g_*\in C^\infty[0,+\infty)$ and $|g_*(t)|\leq e^{\gamma t}$, $t\geq 0$. Then, we can represent $g(\mathrm{d}\tau)$ as

$$g(\mathrm{d}t)=\delta_0(\mathrm{d}t)+g_*(t)\mathrm{d}t,\quad t\geq 0. \tag{3.13}$$

Here, δ_0 is the Dirac distribution.

Observe that the existence of $g(\mathrm{d}t)$ with the above properties implies that

$$\int_0^t e^{i\omega(k)\tau}g(\mathrm{d}\tau)=e^{i\omega(k)t}\phi(t,k)\underset{t\to\infty}{\longrightarrow}\nu(k) \tag{3.14}$$

in the sense that for $\mathrm{Re}\lambda>0$ the limit defined by (2.27) implies that

$$\lim_{\varepsilon\to 0}\int_0^{+\infty} e^{-\lambda t}\int_0^{\varepsilon^{-1}t} e^{i\omega(k)\tau}g(\mathrm{d}\tau)=\frac{\nu(k)}{\lambda}. \tag{3.15}$$

We let

$$\widehat{w}_\varepsilon(\lambda,\eta,k):=\int_0^{+\infty} e^{-\lambda t}\widehat{W}_\varepsilon(t,\eta,k)\mathrm{d}t,\quad \mathrm{Re}\,\lambda>0 \tag{3.16}$$

be the Laplace-Fourier transform of the Wigner distribution defined in (2.10). The claim of Theorem 2.1 is equivalent to the following: for any test function $G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ we have

$$\lim_{\varepsilon \to 0+} \int_{\mathbb{R}\times\mathbb{T}} \hat{G}^*(\eta, k)\widehat{w}_\varepsilon(\lambda, \eta, k)\mathrm{d}\eta\mathrm{d}k = \int_{\mathbb{R}\times\mathbb{T}} \hat{G}^*(\eta, k)\widehat{w}(\lambda, \eta, k)\mathrm{d}\eta\mathrm{d}k, \tag{3.17}$$

where

$$\begin{aligned}\widehat{w}(\lambda, \eta, k) := {} & \frac{T|\bar{\omega}'(k)|\mathfrak{g}(k)}{\lambda(\lambda + i\omega'(k)\eta)} + \frac{\widehat{W}_0(\eta, k)}{\lambda + i\omega'(k)\eta} \\ & + \frac{|\bar{\omega}'(k)|(p_+(k) - 1)}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}} \frac{\widehat{W}_0(\eta', k)\mathrm{d}\eta'}{\lambda + i\omega'(k)\eta'} \\ & + \frac{|\bar{\omega}'(k)|p_-(k)}{\lambda + i\omega'(k)\eta} \int_{\mathbb{R}} \frac{\widehat{W}_0(\eta', -k)\mathrm{d}\eta'}{\lambda - i\omega'(k)\eta'},\end{aligned} \tag{3.18}$$

and $p_\pm(k)$ and $\mathfrak{g}(k)$ are given by (2.28). Indeed, (3.18) is nothing but the Fourier-Laplace transform of (2.33). The rest of the paper is devoted to the derivation of (3.18).

4. The Phonon Creation Term

Since the contribution to the energy given by the thermal term and the initial energy are completely separate, we can derive the first term in (3.18) assuming $\widehat{W}_0 = 0$. In this case $\hat{\psi}(0, k) = 0$ and (3.8) reduces to a stochastic integral:

$$\hat{\psi}(t, k) = i\sqrt{2\gamma T} \int_0^t \phi(t - s, k)\, \mathrm{d}w(s), \tag{4.1}$$

To shorten the notation, denote

$$\tilde{\phi}(t, k) = \int_0^t e^{i\omega(k)\tau} g(\mathrm{d}\tau) = e^{i\omega(k)t}\phi(t, k),$$

and

$$\delta_\varepsilon\omega(k, \eta) := \frac{1}{\varepsilon}\left[\omega\left(k + \frac{\varepsilon\eta}{2}\right) - \omega\left(k - \frac{\varepsilon\eta}{2}\right)\right]. \tag{4.2}$$

We can compute directly

$$\widehat{W}_\varepsilon(t, \eta, k) = \gamma T \int_0^t e^{-i\delta_\varepsilon\omega(k,\eta)s}\tilde{\phi}(s/\varepsilon, k + \varepsilon\eta/2)\tilde{\phi}^*(s/\varepsilon, k + \varepsilon\eta/2)\mathrm{d}s.$$

The Laplace transform of $\tilde{\phi}(\varepsilon^{-1}t, k)$ is given by $\lambda^{-1}\tilde{g}(\varepsilon\lambda - i\omega(k))$. Then we can compute directly the Laplace-Fourier tranform of the Wigner distribution and obtain

$$\begin{aligned}\widehat{w}_\varepsilon(\lambda, \eta, k) &= \gamma T \int_0^\infty \mathrm{d}t e^{-\lambda t} \int_0^t \mathrm{d}s e^{-i\delta_\varepsilon \omega(k,\eta)s} \tilde{\phi} \\ &\quad \left(\varepsilon^{-1}s, k + \frac{\varepsilon\eta}{2}\right) \tilde{\phi}^* \left(\varepsilon^{-1}s, k - \frac{\varepsilon\eta}{2}\right) \\ &= \frac{\gamma T}{\lambda} \int_0^\infty \mathrm{d}s e^{-(\lambda + i\delta_\varepsilon\omega(k,\eta))s} \tilde{\phi} \\ &\quad \left(\varepsilon^{-1}s, k + \frac{\varepsilon\eta}{2}\right) \tilde{\phi}^* \left(\varepsilon^{-1}s, k - \frac{\varepsilon\eta}{2}\right),\end{aligned} \tag{4.3}$$

and by using the inverse Laplace formula for the product of functions, we obtain, for $c > 0$,

$$\begin{aligned}\widehat{w}_\varepsilon(\lambda, \eta, k) &= \frac{\gamma T}{\lambda} \frac{1}{2\pi i} \lim_{\ell\to\infty} \\ &\int_{c-i\ell}^{c+i\ell} \frac{\tilde{g}(\varepsilon\sigma - i\omega(k + \frac{\varepsilon\eta}{2}))\tilde{g}^*(\varepsilon(\lambda + i\delta_\varepsilon\omega(k,\eta) - \sigma) - i\omega(k - \frac{\varepsilon\eta}{2}))}{\sigma\,(\lambda + i\delta_\varepsilon\omega(k,\eta) - \sigma)} \mathrm{d}\sigma.\end{aligned} \tag{4.4}$$

Since $\tilde{g}$ is bounded and $\mathrm{Re}\lambda > 0$, there is no problem in taking the limit as $\varepsilon \to 0$ obtaining

$$\frac{\gamma T |\nu(k)|^2}{\lambda\,(\lambda + i\omega'(k)\eta)}. \tag{4.5}$$

5. The Scattering Terms

If the thermal production at 0 was easy to prove, the scattering terms are much more challenging. Since the thermal part will not affect the scattering, we can set $T = 0$ and consider a non-zero initial energy.

We will first prove (3.18) under a stronger assumption than (2.18): we will assume no energy is concentrated around modes that have null velocity, more precisely that there exist $C, \delta > 0$ and $\kappa > 0$ such that

$$\begin{aligned}|\widehat{W}_\varepsilon(\eta, k)| &\le C\varphi(\eta)\chi\left(k - \frac{\varepsilon\eta}{2}\right)\chi\left(k + \frac{\varepsilon\eta}{2}\right), \\ &(\eta, k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T},\ \varepsilon \in (0, 1],\end{aligned} \tag{5.1}$$

here $\varphi(\cdot)$ is given by (2.19) and $\chi \in C(\mathbb{T})$ is non-negative and satisfies

$$\chi(k) \equiv 0 \text{ for } k \in L(\delta), \tag{5.2}$$

with

$$L(\delta) := [k : \mathrm{dist}(k, \Omega_*) < \delta] \tag{5.3}$$

and $\Omega_* := [k \in \mathbb{T} : \omega'(k) = 0] \subset \{0, 1/2\}$. The proof of Theorem 2.1 under the weaker assumption (2.18) is presented in Section 9 below.

We could have continued to compute $\widehat{w}_\varepsilon$ directly from the expression of the wave function, as we did for the termal part. We find it more practical to use the time evolution of $\widehat{W}_\varepsilon(t, \eta, k)$.

A straightforward computation starting from (2.6) and (2.11) shows that the Wigner transform obeys, for $T = 0$, an evolution equation

$$\begin{aligned}\partial_t \widehat{W}_\varepsilon(t, \eta, k) = &-i\delta_\varepsilon\omega(k, \eta)\widehat{W}_\varepsilon(t, \eta, k)\\ &+i\gamma\left\{\int_{\mathbb{T}} \mathbb{E}_\varepsilon\left[\hat{\psi}^{(\varepsilon)}\left(t, k + \frac{\varepsilon\eta}{2}\right)(\hat{\mathfrak{p}}^{(\varepsilon)})^*(t, k')\right]\mathrm{d}k'\right.\\ &\left.-\int_{\mathbb{T}} \mathbb{E}_\varepsilon\left[(\hat{\psi}^{(\varepsilon)})^*\left(t, k - \frac{\varepsilon\eta}{2}\right)\hat{\mathfrak{p}}^{(\varepsilon)}(t, k')\right]\mathrm{d}k'\right\}. \end{aligned} \tag{5.4}$$

Performing the Laplace transform in both sides of (5.4), we obtain

$$(\lambda + i\delta_\varepsilon\omega(k, \eta))\, w_\varepsilon(\lambda, \eta, k) = \widehat{W}_\varepsilon(\eta, k) - \frac{\gamma}{2}\left[\mathfrak{d}_\varepsilon\left(\lambda, k - \frac{\varepsilon\eta}{2}\right) + \mathfrak{d}^\star_\varepsilon\left(\lambda, k + \frac{\varepsilon\eta}{2}\right)\right], \tag{5.5}$$

where

$$\mathfrak{d}_\varepsilon(\lambda, k) := i\int_0^{+\infty} e^{-\lambda t}\mathbb{E}_\varepsilon\left[(\hat{\psi}^{(\varepsilon)})^*(t, k)\,\mathfrak{p}_0^{(\varepsilon)}(t)\right]\mathrm{d}t. \tag{5.6}$$

As $\delta_\varepsilon\omega(k, \eta) \to \omega'(k)\eta$ as $\varepsilon \to 0$, to get (3.18), we need to understand the limit of $\mathfrak{d}_\varepsilon(\lambda, k)$. Using (3.8) for $T = 0$, we may write

$$\mathfrak{d}_\varepsilon(\lambda, k) = \mathfrak{d}^1_\varepsilon(\lambda, k) + \mathfrak{d}^2_\varepsilon(\lambda, k).$$

Here, $\mathfrak{d}^j_\varepsilon(\lambda, k)$, $j = 1, 2$ are the Laplace transforms of $I_\varepsilon(t/\varepsilon)$, $II_\varepsilon(t/\varepsilon)$, where

$$\begin{aligned} I_\varepsilon(t, k) &:= ie^{i\omega(k)t}\int_0^t \left\langle \mathfrak{p}_0^0(t-s)\hat{\psi}^*(k)\right\rangle_{\mu_\varepsilon} g(\mathrm{d}s),\\ II_\varepsilon(t, k) &:= -\gamma\int_0^t g\left(\mathrm{d}s'\right)\int_0^t \phi^*(t-s, k)\,\langle\mathfrak{p}_0^0(s)\mathfrak{p}_0^0(t-s')\rangle_{\mu_\varepsilon}\mathrm{d}s. \end{aligned} \tag{5.7}$$

Now, with (3.17) in mind, we can introduce

$$\mathfrak{L}_\varepsilon(\lambda) := \int_{\mathbb{R}\times\mathbb{T}} \hat{G}^*(\eta, k)\widehat{w}_\varepsilon(\lambda, \eta, k)\mathrm{d}\eta\mathrm{d}k = \mathfrak{L}^\varepsilon_{init}(\lambda) + \mathfrak{L}^\varepsilon_{scat}(\lambda). \tag{5.8}$$

The first term in the right side is

$$\mathfrak{L}^\varepsilon_{init}(\lambda) := \int_{\mathbb{R}\times\mathbb{T}} \hat{G}^*(\eta, k)\frac{\widehat{W}_\varepsilon(\eta, k)}{\lambda + i\delta_\varepsilon\omega(k, \eta)}\mathrm{d}\eta\mathrm{d}k. \tag{5.9}$$

The scattering term in the right side of (5.8) is

$$\mathfrak{L}^\varepsilon_{scat}(\lambda) = \mathfrak{L}^\varepsilon_{scat,1} + \mathfrak{L}^\varepsilon_{scat,2}, \tag{5.10}$$

with

$$\mathfrak{L}^{\varepsilon}_{scat,j} := -\frac{\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}} \frac{\hat{G}^*(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k,\eta)} \left[\mathfrak{d}^j_\varepsilon\left(\lambda, k-\frac{\varepsilon\eta}{2}\right) + (\mathfrak{d}^j_\varepsilon)^\star\left(\lambda, k+\frac{\varepsilon\eta}{2}\right)\right] \mathrm{d}\eta\mathrm{d}k, \quad j=1,2. \tag{5.11}$$

The Ballistic Term

Thanks to the assumption that $\widehat{W}_\varepsilon(\eta,k)$ converges weakly in $\mathcal{A}'$ to $W_0 \in L^1(\mathbb{R}\times\mathbb{T})\cap C(\mathbb{R}\times\mathbb{T})$, we can easily show that

$$\mathfrak{L}^{\varepsilon}_{init}(\lambda) := \int_{\mathbb{R}\times\mathbb{T}} \hat{G}^*(\eta,k)\frac{\widehat{W}_\varepsilon(\eta,k)}{\lambda + i\delta_\varepsilon\omega(k,\eta)}\mathrm{d}\eta\mathrm{d}k \to \int_{\mathbb{R}\times\mathbb{T}} \hat{G}^*(\eta,k)\frac{\widehat{W}_0(\eta,k)}{\lambda + i\omega'(k)\eta}\mathrm{d}\eta\mathrm{d}k, \text{ as } \varepsilon\to 0, \tag{5.12}$$

which is the second term in the right side of (3.18).

5.1. The Limit of the First Scattering Term

Here, we use the notation

$$\begin{aligned} \delta^+_\varepsilon\omega(k,\eta) &:= \delta_\varepsilon\omega\Big(k+\frac{\varepsilon\eta}{2}\Big) = \frac{1}{\varepsilon}\Big[\omega(k+\varepsilon\eta)-\omega(k)\Big],\\ \delta^-_\varepsilon\omega(k,\eta) &:= \delta_\varepsilon\omega\Big(k-\frac{\varepsilon\eta}{2}\Big) = \frac{1}{\varepsilon}\Big[\omega(k)-\omega(k-\varepsilon\eta)\Big]. \end{aligned} \tag{5.13}$$

We now compute the limit of $\mathfrak{L}^{\varepsilon}_{scat,1}(\lambda)$ in (5.10) that we can re-write, after a simple change of variables as

$$\mathfrak{L}^{\varepsilon}_{scat,1}(\lambda) = -\frac{\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}}\Big[\frac{\hat{G}^*(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}\mathfrak{d}^1_\varepsilon(\lambda,k) + \frac{\hat{G}^*(\eta,k-\varepsilon\eta/2)}{\lambda+i\delta^-_\varepsilon\omega(k,\eta)}\left(\mathfrak{d}^1_\varepsilon\right)^*(\lambda,k)\Big]\mathrm{d}\eta\mathrm{d}k.$$

We will show the following:

Lemma 5.1. *For any test function $G\in\mathcal{S}(\mathbb{R}\times\mathbb{T})$ and $\lambda>0$ we have*

$$\lim_{\varepsilon\to 0+}\mathfrak{L}^{\varepsilon}_{scat,1}(\lambda) = -\gamma\int_{\mathbb{R}\times\mathbb{T}}\mathrm{Re}[\nu(k)]\frac{\widehat{W}_0(\eta',k)}{\lambda+i\omega'(k)\eta'}\left\{\int_{\mathbb{R}}\frac{G^*(\eta,k)}{\lambda+i\omega'(k)\eta}\mathrm{d}\eta\right\}\mathrm{d}k\mathrm{d}\eta'. \tag{5.14}$$

Proof. From (5.7) and (3.3) we get

$$I_\varepsilon(t,k) = \frac{1}{2}\int_0^t g\,(\mathrm{d}s)\int_{\mathbb{T}}\Big\{\langle\hat\psi^\star(k)\hat\psi(\ell)\rangle_{\mu_\varepsilon}e^{i(\omega(k)-\omega(\ell))t+i\omega(\ell)s} - \langle\hat\psi^\star(k)\hat\psi^\star(\ell)\rangle_{\mu_\varepsilon}e^{i(\omega(k)+\omega(\ell))t-i\omega(\ell)s}\Big\}\,\mathrm{d}\ell. \tag{5.15}$$

Using assumption (2.9) and (5.15), we conclude that

$$\begin{aligned}\mathfrak{d}^1_\varepsilon(\lambda,k) &= \frac{1}{2}\int_{\mathbb{T}}\varepsilon\langle\hat\psi^\star(k)\hat\psi(\ell)\rangle_{\mu_\varepsilon}\mathrm{d}\ell\int_0^{+\infty}\exp\{i\omega(\ell)s\}\,g(\mathrm{d}s)\\ &\quad\int_s^{+\infty}e^{-\lambda\varepsilon t}\exp\{i(\omega(k)-\omega(\ell))t\}\,\mathrm{d}t\\ &= \frac{1}{2}\int_{\mathbb{T}}\frac{\varepsilon\langle\hat\psi^\star(k)\hat\psi(\ell)\rangle_{\mu_\varepsilon}\mathrm{d}\ell}{\lambda\varepsilon+i(\omega(\ell)-\omega(k))}\int_0^{+\infty}g(\mathrm{d}s)e^{-\lambda\varepsilon s}\exp\{i\omega(k)s\}\,\mathrm{d}s\\ &= \int_{\mathbb{T}}\frac{(\varepsilon/2)\langle\hat\psi^\star(k)\hat\psi(\ell)\rangle_{\mu_\varepsilon}\mathrm{d}\ell}{\lambda\varepsilon+i(\omega(\ell)-\omega(k))}\tilde g(\varepsilon\lambda-i\omega(k)).\end{aligned}$$

For any test function $G\in\mathcal{S}(\mathbb{T}\times\mathbb{R})$ we can write, therefore, (cf. (5.13))

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{T}}\frac{\hat G^\star(\eta,k+\varepsilon\eta/2)\mathfrak{d}^1_\varepsilon(\lambda,k)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}\mathrm{d}k\mathrm{d}\eta\\ &\quad=\int_{\mathbb{R}\times\mathbb{T}}\frac{\hat G^\star(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}\mathrm{d}k\mathrm{d}\eta\\ &\qquad\left\{\int_{\mathbb{T}}\frac{(\varepsilon/2)\langle\hat\psi^\star(k)\hat\psi(\ell)\rangle_{\mu_\varepsilon}\tilde g(\varepsilon\lambda-i\omega(k))}{\lambda\varepsilon+i(\omega(\ell)-\omega(k))}\mathrm{d}\ell\right\}.\end{aligned} \tag{5.16}$$

Changing variables $k:=k'-\varepsilon\eta'/2$, $\ell:=k'+\varepsilon\eta'/2$ the right hand side of (5.16) can be rewritten in the form

$$\int_{\mathbb{R}}\mathrm{d}\eta\left\{\int_{T_\varepsilon}\frac{\widehat W_\varepsilon(\eta',k')\tilde g(\varepsilon\lambda-i\omega(k'-\varepsilon\eta'/2))\hat G^*(\eta,k'+\varepsilon\eta/2-\varepsilon\eta'/2)}{[\lambda+i\delta^+_\varepsilon\omega(k'-\varepsilon\eta'/2,\eta)][\lambda+i\varepsilon^{-1}(\omega(k'+\varepsilon\eta'/2)-\omega(k'-\varepsilon\eta'/2))]}\mathrm{d}k'\mathrm{d}\eta'\right\}. \tag{5.17}$$

Here, $T_\varepsilon\subset\mathbb{T}_{2/\varepsilon}\times\mathbb{T}$ is the image of $\mathbb{T}^2$ under the inverse map $k':=(\ell+k)/2$, $\eta':=(\ell-k)/\varepsilon$. Note that

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{\tilde g(\varepsilon\lambda-i\omega(k'-\varepsilon\eta'/2))\hat G^\star(\eta,k'+\varepsilon\eta/2-\varepsilon\eta'/2)}{[\lambda+i\delta^+_\varepsilon\omega(k'-\varepsilon\eta'/2,\eta)][\lambda+i\varepsilon^{-1}(\omega(k'+\varepsilon\eta'/2)-\omega(k'-\varepsilon\eta'/2))]}\\ &\quad=\frac{\nu(k')\hat G^\star(\eta,k')}{[\lambda+i\omega'(k')\eta][\lambda+i\omega'(k')\eta']}\end{aligned}$$

a.e. in (η,η',k'). Using bounds (5.1) and (2.26) we can argue, via the dominated convergence theorem that the limit of (5.17), as $\varepsilon\to0$, is the same as that of

$$\int_{\mathbb{R}^2\times\mathbb{T}}\frac{\widehat W_\varepsilon(\eta',k')\nu(k')\hat G^*(\eta,k')\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}k'}{[\lambda+i\omega'(k')\eta][\lambda+i\omega'(k')\eta']}. \tag{5.18}$$

Summarizing, the above argument proves that

$$\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}} \frac{\hat{G}^*(\eta, k+\varepsilon\eta/2)\mathfrak{d}^1_\varepsilon(\lambda,k)}{\lambda + i\delta^+_\varepsilon\omega(k,\eta)}\mathrm{d}k\mathrm{d}\eta$$
$$= \int_{\mathbb{R}\times\mathbb{T}} \frac{\nu(k)\widehat{W}_0(\eta',k)}{\lambda+i\omega'(k)\eta'}\left\{\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)}{\lambda+i\omega'(k)\eta}\mathrm{d}\eta\right\}\mathrm{d}\eta'\mathrm{d}k \tag{5.19}$$

for any test function $G\in\mathcal{S}(\mathbb{T}\times\mathbb{R})$. Similarly, we have

$$\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}} \frac{\hat{G}^*(\eta, k-\varepsilon\eta/2)(\mathfrak{d}^1_\varepsilon)^*(\lambda,k)}{\lambda + i\delta^-_\varepsilon\omega(k,\eta)}\mathrm{d}k\mathrm{d}\eta$$
$$= \int_{\mathbb{R}\times\mathbb{T}} \frac{\nu^*(k)\widehat{W}^*_0(\eta',k)}{\lambda-i\omega'(k)\eta'}\left\{\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)}{\lambda+i\omega'(k)\eta}\mathrm{d}\eta\right\}\mathrm{d}\eta'\mathrm{d}k. \tag{5.20}$$

As $\widehat{W}^*_\varepsilon(\eta,k)=\widehat{W}_\varepsilon(-\eta,k)$, we conclude that (5.14) holds. □

5.2. Asymptotics of the Second Scattering Term

Let us split $\mathfrak{L}^\varepsilon_{scat,2}(\lambda)$ as

$$\mathfrak{L}^\varepsilon_{scat,2}(\lambda) = -\frac{\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}}\left[\mathfrak{d}^2_\varepsilon\left(\lambda, k-\frac{\varepsilon\eta}{2}\right)+(\mathfrak{d}^2_\varepsilon)^*\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]$$
$$\frac{\hat{G}^*(\eta,k)}{\lambda+i\delta_\varepsilon\omega(k,\eta)}\mathrm{d}\eta\mathrm{d}k$$
$$= -\frac{\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}}\left[\mathfrak{d}^2_\varepsilon(\lambda,k)\frac{\hat{G}^*(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}+(\mathfrak{d}^2_\varepsilon)^*(\lambda,k)\right.$$
$$\left.\frac{\hat{G}^*(\eta,k-\varepsilon\eta/2)}{\lambda+i\delta^-_\varepsilon\omega(k,\eta)}\right]\mathrm{d}\eta\mathrm{d}k$$
$$= \mathfrak{L}^\varepsilon_{scat,21}(\lambda)+\mathfrak{L}^\varepsilon_{scat,22}(\lambda), \tag{5.21}$$

with the two terms corresponding to writing

$$\mathfrak{d}^2_\varepsilon = \mathrm{Re}\mathfrak{d}^2_\varepsilon + i\mathrm{Im}\mathfrak{d}^2_\varepsilon. \tag{5.22}$$

We recall that

$$\mathfrak{d}^2_\varepsilon(\lambda,k) = \varepsilon\int_0^{+\infty}e^{-\lambda\varepsilon t}II_\varepsilon(t,k)\,\mathrm{d}t$$
$$= -\gamma\varepsilon\int_0^{+\infty}e^{-\lambda\varepsilon t}\mathrm{d}t\left\{\int_0^t e^{i\omega(k)(t-s)}\left\langle g\star\mathfrak{p}^0_0(s)g\star\mathfrak{p}^0_0(t)\right\rangle_{\mu_\varepsilon}\right\}\mathrm{d}s.$$

We will prove the following:

Lemma 5.2. *For any* $\lambda > 0$ *and* $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ *we have*

$$\lim_{\varepsilon\to 0}\mathfrak{L}^{\varepsilon}_{scat,2}(\lambda) = \frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{g}(k)\widehat{W}(\eta',k)\mathrm{d}\eta'\mathrm{d}k}{\lambda+i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta} + \frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{g}(k)\widehat{W}(\eta',-k)\mathrm{d}\eta'\mathrm{d}k}{\lambda-i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta}. \quad (5.23)$$

The conclusion of this lemma is the consequence of the following two limits

$$\lim_{\varepsilon\to 0+}\mathfrak{L}^{\varepsilon}_{scat,21}(\lambda) = \frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{g}(k)\widehat{W}(\eta',k)\mathrm{d}\eta'\mathrm{d}k}{\lambda+i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta} + \frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{g}(k)\widehat{W}(\eta',-k)\mathrm{d}\eta'\mathrm{d}k}{\lambda-i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta}, \quad (5.24)$$

and

$$\lim_{\varepsilon\to 0+}\mathfrak{L}^{\varepsilon}_{scat,22}(\lambda) = 0. \quad (5.25)$$

5.3. The Limit of the Full Scattering Term

Putting together the results of Lemmas 5.1 and 5.2, we see that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathfrak{L}^{\varepsilon}_{scat}(\lambda) &= -\gamma\int_{\mathbb{R}\times\mathbb{T}}\mathrm{Re}[\nu(k)]\frac{\widehat{W}_{\varepsilon}(\eta,k)}{\lambda+i\omega'(k)\eta}\left\{\int_{\mathbb{R}}\frac{G^*(\eta,k)}{\lambda+i\omega'(k)\eta}\mathrm{d}\eta\right\}\mathrm{d}k\mathrm{d}\eta' \\ &\quad+\frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{g}(k)\widehat{W}(\eta',k)\mathrm{d}\eta'\mathrm{d}k}{\lambda+i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta} \\ &\quad+\frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{g}(k)\widehat{W}(\eta',-k)\mathrm{d}\eta'\mathrm{d}k}{\lambda-i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta} \\ &= \int[\mathcal{W}_{tr}(\eta,k)+\mathcal{W}_{ref}(\eta,k)]\hat{G}^*(\eta,k)\frac{|\bar{\omega}'(k)|\mathrm{d}\eta\mathrm{d}k}{\lambda+i\omega'(k)\eta}\mathrm{d}\eta\mathrm{d}k, \end{aligned}\quad (5.26)$$

with the transmission term

$$\begin{aligned}\mathcal{W}_{tr}(\eta,k) &= \frac{\gamma}{|\bar{\omega}'(k)|}\Big[-\mathrm{Re}[\nu(k)]+\frac{\mathfrak{g}(k)}{4}\Big]\int\frac{\widehat{W}(\eta',k)\mathrm{d}\eta'\mathrm{d}k}{\lambda+i\omega'(k)\eta'} \\ &= (p_+(k)-1)\int\frac{\widehat{W}(\eta',k)\mathrm{d}\eta'\mathrm{d}k}{\lambda+i\omega'(k)\eta'}.\end{aligned}\quad (5.27)$$

We used (2.29) in the last step. The other term in (5.26), corresponding to reflection, is

$$\mathcal{W}_{ref}(\eta, k) = \frac{\gamma \mathfrak{g}(k)}{4|\bar{\omega}'(k)|} \int_{\mathbb{R}\times\mathbb{T}} \frac{\widehat{W}(\eta', -k)\mathrm{d}\eta'\mathrm{d}k}{\lambda - i\omega'(k)\eta'}$$
$$= p_-(k) \int_{\mathbb{R}\times\mathbb{T}} \frac{\widehat{W}(\eta', -k)\mathrm{d}\eta'\mathrm{d}k}{\lambda - i\omega'(k)\eta'}. \tag{5.28}$$

Combining the scattering terms in (5.26)–(5.28), together with the ballistic term in (5.12) , we get (3.18). Thus, the proof of Theorem 2.1 is reduced to the computation in Lemma 5.2.

6. The Proof of Lemma 5.2: The Limit of $\mathfrak{L}^{\varepsilon}_{scat,21}(\lambda)$

We now turn to the proof of Lemma 5.2. In this section, we begin the rather long and technical computation leading to (5.24).

A Calculation of $\mathrm{Re}\,\mathfrak{d}^2_\varepsilon$

Recall that $\mathfrak{L}^{\varepsilon}_{scat,21}(\lambda)$ comes from the contribution to $\mathfrak{L}^{\varepsilon}_{scat,2}(\lambda)$ that appears from $\mathrm{Re}\,\mathfrak{d}^2_\varepsilon$. Our first task, therefore, is to compute $\mathrm{Re}\,\mathfrak{d}^2_\varepsilon$. We have

$$\begin{aligned}
&2\mathrm{Re}\,\mathfrak{d}^2_\varepsilon(\lambda, k)\\
&= -2\gamma\varepsilon \left\langle \int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t \left\{\int_0^t \cos(\omega(k)s)(g \star \mathfrak{p}_0^0)(s)\mathrm{d}s\right\} \cos(\omega(k)t)(g \star \mathfrak{p}_0^0)(t)\right\rangle_{\mu_\varepsilon}\\
&\quad -2\gamma\varepsilon \left\langle \int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t \left\{\int_0^t \sin(\omega(k)s)(g \star \mathfrak{p}_0^0)(s)\mathrm{d}s\right\} \sin(\omega(k)t)(g \star \mathfrak{p}_0^0)(t)\right\rangle_{\mu_\varepsilon}\\
&= -\gamma\varepsilon \left\langle \int_0^{+\infty} e^{-\lambda\varepsilon t}\frac{\mathrm{d}}{\mathrm{d}t} \left\{\left[\int_0^t \cos(\omega(k)s) g * \mathfrak{p}_0^0(s)\mathrm{d}s\right]^2\right.\right.\\
&\quad \left.\left. +\left[\int_0^t \sin(\omega(k)s) g * \mathfrak{p}_0^0(s)\mathrm{d}s\right]^2\right\}\mathrm{d}t\right\rangle_{\mu_\varepsilon}.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
2\mathrm{Re}\,\mathfrak{d}^2_\varepsilon(\lambda, k) &= -\gamma\varepsilon^2\lambda \int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t \left\langle \left\{\int_0^t \cos(\omega(k)s)(g \star \mathfrak{p}_0^0)(s)\mathrm{d}s\right\}^2\right\rangle_{\mu_\varepsilon}\\
&\quad -\gamma\varepsilon^2\lambda \int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t \left\langle \left\{\int_0^t \sin(\omega(k)s)(g \star \mathfrak{p}_0^0)(s)\mathrm{d}s\right\}^2\right\rangle_{\mu_\varepsilon}\\
&:= C_\varepsilon(\lambda, k) + S_\varepsilon(\lambda, k).
\end{aligned} \tag{6.1}$$

The first term in the right side is

$$C_\varepsilon(\lambda, k) = -\gamma\varepsilon^2\lambda \int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t \int_0^t \int_0^t \mathrm{d}s\mathrm{d}s' \cos(\omega(k)s)\cos(\omega(k)s')\langle g * \mathfrak{p}_0^0(s) g * \mathfrak{p}_0^0(s')\rangle_{\mu_\varepsilon}. \tag{6.2}$$

Using (2.9) and (3.3) gives

$$
\begin{aligned}
&\varepsilon\langle (g\star \mathfrak{p}_0^0)(s)(g\star \mathfrak{p}_0^0)(s')\rangle_{\mu_\varepsilon}\\
&\quad=\frac{1}{4}\int_0^s\int_0^{s'} g(\mathrm{d}\tau)g(d\tau')\int_{\mathbb{T}^2}\mathrm{d}\ell\mathrm{d}\ell' e^{-i\omega(\ell)(s-\tau)}e^{i\omega(\ell')(s'-\tau')}\varepsilon\langle\hat\psi(\ell)\hat\psi^*(\ell')\rangle_{\mu_\varepsilon}\\
&\qquad+\frac{1}{4}\int_0^s\int_0^{s'} g(\mathrm{d}\tau)g(d\tau')\int_{\mathbb{T}^2}\mathrm{d}\ell\mathrm{d}\ell' e^{i\omega(\ell)(s-\tau)}e^{-i\omega(\ell')(s'-\tau')}\varepsilon\langle\hat\psi^*(\ell)\hat\psi(\ell')\rangle_{\mu_\varepsilon}.
\end{aligned}
\tag{6.3}
$$

Now, symmetry implies that the two terms above make an identical contribution to $C_\varepsilon(\lambda, t)$, hence

$$
\begin{aligned}
C_\varepsilon(\lambda,k) &= -\frac{\gamma}{2}\varepsilon\lambda\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t\int_0^t\\
&\qquad\int_0^t \mathrm{d}s\mathrm{d}s'\cos(\omega(k)s)\cos(\omega(k)s')\int_0^s\int_0^{s'} g(\mathrm{d}\tau)g(d\tau')\int_{\mathbb{T}^2}\mathrm{d}\ell\mathrm{d}\ell'\\
&\qquad\times e^{-i\omega(\ell)(s-\tau)}e^{i\omega(\ell')(s'-\tau')}\varepsilon\langle\hat\psi(\ell)\hat\psi^*(\ell')\rangle_{\mu_\varepsilon}\\
&= -\frac{\gamma}{4\pi}\varepsilon\lambda\int_{\mathbb{R}}\mathrm{d}\beta\\
&\qquad\int_{\mathbb{T}^2}\mathrm{d}\ell\mathrm{d}\ell'\varepsilon\langle\hat\psi(\ell)\hat\psi^*(\ell')\rangle_{\mu_\varepsilon}\int_0^{+\infty}\int_0^{+\infty} e^{i\beta(t-t')}e^{-\lambda\varepsilon(t+t')/2}\mathrm{d}t\mathrm{d}t'\\
&\qquad\times\int_0^t\int_0^{t'}\mathrm{d}s\mathrm{d}s'\cos(\omega(k)s)\cos(\omega(k)s')\int_0^s\\
&\qquad\int_0^{s'} g(\mathrm{d}\tau)g(d\tau')e^{-i\omega(\ell)(s-\tau)}e^{i\omega(\ell')(s'-\tau')}\\
&= -\frac{\gamma}{4\pi}\varepsilon\lambda\int_{\mathbb{R}}\mathrm{d}\beta\\
&\qquad\int_{\mathbb{T}^2}\varepsilon\langle\hat\psi(\ell)\hat\psi^*(\ell')\rangle_{\mu_\varepsilon}\Omega(\beta,\ell,k,\lambda)\Omega^\star(\beta,\ell',k,\lambda)\mathrm{d}\ell\mathrm{d}\ell',
\end{aligned}
\tag{6.4}
$$

with

$$
\Omega(\beta,\ell,k,\lambda) := \int_0^{+\infty}\cos(\omega(k)s)\mathrm{d}s\left\{\int_0^s e^{-i\omega(\ell)(s-\tau)}g(\mathrm{d}\tau)\int_s^{+\infty} e^{(-\lambda\varepsilon/2+i\beta)t}\mathrm{d}t\right\}.
$$

Integrating out first the t variable, and then the s varable, we obtain

$$
\begin{aligned}
\Omega(\beta,\ell,k,\lambda) &= \frac{1}{\lambda\varepsilon/2-i\beta}\int_0^{+\infty} e^{i\omega(\ell)\tau}g(\mathrm{d}\tau)\\
&\qquad\int_\tau^{+\infty}\cos(\omega(k)s)e^{[-\lambda\varepsilon/2+i(\beta-\omega(\ell))]s}\mathrm{d}s\\
&= \frac{1}{2(\lambda\varepsilon/2-i\beta)}\int_0^{+\infty} g(\mathrm{d}\tau)e^{i\omega(\ell)\tau}\\
&\qquad\left\{\frac{e^{[-\lambda\varepsilon/2+i[\beta+\omega(k)-\omega(\ell)]]\tau}}{\lambda\varepsilon/2-i(\beta+\omega(k)-\omega(\ell))}+\frac{e^{[-\lambda\varepsilon/2+i(\beta-\omega(k)-\omega(\ell))]\tau}}{\lambda\varepsilon/2-i[\beta+\omega(k)+\omega(\ell)]}\right\}
\end{aligned}
$$

$$= \frac{1}{2(\lambda\varepsilon/2 - i\beta)}$$
$$\left\{ \frac{\tilde{g}\,(\lambda\varepsilon/2 - i[\beta + \omega(k)])}{\lambda\varepsilon/2 - i(\beta + \omega(k) - \omega(\ell))} + \frac{\tilde{g}\,(\lambda\varepsilon/2 - i[\beta - \omega(k))])}{\lambda\varepsilon/2 - i[\beta + \omega(k) + \omega(\ell)]} \right\}.$$

Hence, after a change of variables $\beta := \varepsilon\beta'$, we get

$$C_\varepsilon(\lambda, k) = -\frac{\gamma\lambda}{16 \cdot \pi\varepsilon^2} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2 + \beta^2} \int_{\mathbb{T}^2} \mathrm{d}\ell \mathrm{d}\ell' \varepsilon \langle \hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}$$
$$\times \left\{ \frac{\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta + \omega(k)])}{\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell)]\}} + \frac{\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta - \omega(k)])}{\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) + \omega(\ell)]\}} \right\}$$
$$\times \left\{ \frac{\tilde{g}\,(\lambda\varepsilon/2 + i[\varepsilon\beta + \omega(k)])}{\lambda/2 + i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell')]\}} + \frac{\tilde{g}\,(\lambda\varepsilon/2 + i[\varepsilon\beta - \omega(k)]))}{\lambda/2 + i\{\beta + \varepsilon^{-1}[\omega(k) + \omega(\ell')]\}} \right\}. \tag{6.5}$$

A similar calculation leads to

$$S_\varepsilon(\lambda, k) = \frac{\gamma\lambda}{2^4\pi\varepsilon^2} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2 + \beta^2} \int_{\mathbb{T}^2} \mathrm{d}\ell \mathrm{d}\ell' \varepsilon \langle \hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}$$
$$\times \left\{ \frac{\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta + \omega(k)])}{\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell)]\}} - \frac{\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta - \omega(k)])}{\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) + \omega(\ell)]\}} \right\}$$
$$\times \left\{ \frac{\tilde{g}\,(\lambda\varepsilon/2 + i(\varepsilon\beta - \omega(k)]))}{\lambda/2 + i\{\beta - \varepsilon^{-1}[\omega(k) + \omega(\ell')]\}} - \frac{\tilde{g}\,(\lambda\varepsilon/2 + i[\varepsilon\beta - \omega(k)])}{\lambda/2 + i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell')]\}} \right\}. \tag{6.6}$$

Putting (6.1), (6.5) and (6.6) together gives

$$2\mathrm{Re}\, \mathfrak{d}_\varepsilon^2(\lambda, k) = R_\varepsilon(\lambda, k) + \rho_\varepsilon(\lambda, k), \tag{6.7}$$

with

$$R_\varepsilon(\lambda, k) := -\frac{\gamma\lambda}{8\pi\varepsilon^2} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2 + \beta^2} \int_{\mathbb{T}^2} \mathrm{d}\ell \mathrm{d}\ell' \varepsilon \langle \hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}$$
$$\times \frac{|\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta + \omega(k)])\,|^2}{\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell)]\}}$$
$$\times \frac{1}{\lambda/2 + i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell')]\}} \tag{6.8}$$

and

$\rho_\varepsilon(\lambda, k)$

$$
\begin{aligned}
:= & -\frac{\gamma\lambda}{16\cdot\pi\varepsilon^2}\int_{\mathbb{R}}\frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2}\int_{\mathbb{T}^2}\mathrm{d}\ell\mathrm{d}\ell'\varepsilon\langle\hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}\\
& \left\{\frac{\tilde{g}\,(\lambda\varepsilon/2-i[\varepsilon\beta+\omega(k)])}{\lambda/2-i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(\ell)]\}}\right.\\
& \times\left\{\frac{\tilde{g}\,(\lambda\varepsilon/2+i[\varepsilon\beta+\omega(k)])}{\lambda/2+i\{\beta+\varepsilon^{-1}[\omega(k)+\omega(\ell')]\}}+\frac{\tilde{g}\,(\lambda\varepsilon/2+i[\varepsilon\beta-\omega(k)]))}{\lambda/2+i\{\beta-\varepsilon^{-1}[\omega(k)+\omega(\ell')]}\right\}\\
& +\frac{\tilde{g}\,(\lambda\varepsilon/2-i[\varepsilon\beta-\omega(k)])}{\lambda/2-i\{\beta+\varepsilon^{-1}[\omega(k)+\omega(\ell)]\}}\\
& \times\left\{\frac{\tilde{g}\,(\lambda\varepsilon/2+i[\varepsilon\beta+\omega(k)])}{\lambda/2+i\{\beta+\varepsilon^{-1}[\omega(k)+\omega(\ell')]\}}+\frac{\tilde{g}\,(\lambda\varepsilon/2+i[\varepsilon\beta-\omega(k)]))}{\lambda/2+i\{\beta-\varepsilon^{-1}[\omega(k)+\omega(\ell')]}\right\}\\
& -\frac{\tilde{g}\,(\lambda\varepsilon/2-i[\varepsilon\beta+\omega(k)])}{\lambda/2-i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(\ell)]\}}\frac{\tilde{g}\,(\lambda\varepsilon/2+i[\varepsilon\beta-\omega(k)]))}{\lambda/2+i\{\beta-\varepsilon^{-1}[\omega(k)+\omega(\ell')]\}}\\
& \left.-\frac{\tilde{g}\,(\lambda\varepsilon/2-i[\varepsilon\beta-\omega(k)])}{\lambda/2-i\{\beta+\varepsilon^{-1}[\omega(k)+\omega(\ell)]\}}\frac{\tilde{g}\,(\lambda\varepsilon/2+i[\varepsilon\beta-\omega(k)])}{\lambda/2+i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(\ell')]\}}\right\}.
\end{aligned}
\tag{6.9}
$$

The main contribution to $\mathfrak{L}^{\varepsilon}_{scat,21}(\lambda)$ will come from the term $R_\varepsilon(\lambda,k)$ due to the difference $\omega(k)-\omega(\ell)$ in the denominator that can become small. As $\rho_\varepsilon(\lambda,k)$ contains the sum $\omega(k)+\omega(\ell)$, or $\omega(k)+\omega(\ell')$ we expect its contribution to be small in the limit. More precisely, we will show the following result for the limit of $R_\varepsilon(\lambda,k)$:

Lemma 6.1. *Let*

$$
\mathcal{H}_\pm(\lambda,\varepsilon):=\int_{\mathbb{R}\times\mathbb{T}}R_\varepsilon\,(\lambda,k)\,\frac{\hat{G}^*(\eta,k\pm\varepsilon\eta/2)}{\lambda+i\delta^\pm_\varepsilon\omega(k,\eta)}\mathrm{d}\eta\mathrm{d}k \tag{6.10}
$$

and

$$
\mathcal{I}_{tr}(\lambda):=-2\gamma\pi\int_{\mathbb{R}\times\mathbb{T}}\frac{|\nu(k)|^2\widehat{W}_0(\eta',k)\mathrm{d}\eta'\mathrm{d}k}{|\omega'(k)|[\lambda+i\omega'(k)\eta']}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta}, \tag{6.11}
$$

and

$$
\mathcal{I}_{ref}(\lambda):=-2\gamma\pi\int_{\mathbb{R}\times\mathbb{T}}\frac{|\nu(k)|^2\widehat{W}_0(\eta',-k)\mathrm{d}\eta'\mathrm{d}k}{|\omega'(k)|[\lambda-i\omega'(k)\eta']}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)\mathrm{d}\eta}{\lambda+i\omega'(k)\eta}, \tag{6.12}
$$

then

$$
\lim_{\varepsilon\to0+}\mathcal{H}_\pm(\lambda,\varepsilon)=\frac{1}{2}(\mathcal{I}_{tr}(\lambda)+\mathcal{I}_{ref}(\lambda)). \tag{6.13}
$$

On the other hand, $\rho_\varepsilon(\lambda,k)$ vanishes in the limit.

Lemma 6.2. *For each* $\lambda>0$ *we have*

$$
\lim_{\varepsilon\to0+}\int_{\mathbb{T}}|\rho_\varepsilon(\lambda,k)|\mathrm{d}k=0. \tag{6.14}
$$

These two lemmas, together with (5.21)–(5.22) and (6.7), imply (5.24).

Proof of Lemma 6.2

A word on notation: for two functions $f, g : D \to \mathbb{R}$ we say that $f \preceq g$ if there exists $C > 0$ such that $f(x) \le C g(x)$, $x \in D$. We shall use the notation $f \approx g$ if $f \preceq g$ and $g \preceq f$.

Opening the parentheses in (6.9), we can write

$$\rho_\varepsilon(\lambda, k) = \sum_{j=1}^{6} \rho_\varepsilon^j(\lambda, k).$$

We will only show that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{T}} |\rho_\varepsilon^1(\lambda, k)| \mathrm{d}k = 0, \tag{6.15}$$

as the other terms are analyzed in a similar fashion. To verify (6.15), it suffices to show that

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2} \int_{\mathbb{T}^3} \left| \langle \hat\psi(\ell) \hat\psi^*(\ell') \rangle_{\mu_\varepsilon} \right| \\ \times \left| \frac{\tilde g\,(\lambda\varepsilon/2 - i[\varepsilon\beta + \omega(k)])}{\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(\ell)]\}} \right| \\ \times \left| \frac{\tilde g\,(\lambda\varepsilon/2 + i[\varepsilon\beta + \omega(k)])}{\lambda/2 + i\{\beta + \varepsilon^{-1}[\omega(k) + \omega(\ell')]\}} \right| \mathrm{d}k\mathrm{d}\ell\mathrm{d}\ell' = 0. \end{aligned} \tag{6.16}$$

Change variables

$$\ell =: k' + \frac{\varepsilon\eta'}{2}, \quad \ell' =: k' - \frac{\varepsilon\eta'}{2} \tag{6.17}$$

and let

$$T_\varepsilon^2 := \left[(\eta', k') : |\eta'| \le \frac{1}{\varepsilon},\ |k'| \le \frac{1 - \varepsilon|\eta'|}{2} \right] \subset \mathbb{T}_{2/\varepsilon} \times \mathbb{T}, \tag{6.18}$$

be the image of $\mathbb{T}^2$ under the inverse map, as below (5.17). The expression under the limit in (6.16) can then be estimated by

$$\begin{aligned} \frac{\|\tilde g\|_\infty^2}{\varepsilon} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2} \int_{\mathbb{T}^2\times\mathbb{T}_{2/\varepsilon}} \frac{|\hat W_\varepsilon(\eta', k')|}{|\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(k' + \varepsilon\eta'/2)]\}|} \\ \times \frac{\mathrm{d}k\mathrm{d}k'\mathrm{d}\eta'}{|\lambda/2 + i\{\beta + \varepsilon^{-1}[\omega(k) + \omega(k' - \varepsilon\eta'/2)]\}|} \le I_{1,\varepsilon} + I_{2,\varepsilon}, \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} I_{1,\varepsilon} := \frac{\|\tilde g\|_\infty^2}{\varepsilon} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2} \int_{\mathbb{T}^2\times\mathbb{T}_{2/\varepsilon}} \frac{|\hat W_\varepsilon(\eta', k')|}{|\lambda + i\varepsilon^{-1}\left(\omega(k' - \varepsilon\eta'/2) + \omega(k' + \varepsilon\eta'/2)\right)|} \\ \times \frac{\mathrm{d}k\mathrm{d}k'\mathrm{d}\eta'}{|\lambda/2 - i\{\beta + \varepsilon^{-1}[\omega(k) - \omega(k' + \varepsilon\eta'/2)]\}|} \end{aligned}$$

and

$$I_{2,\varepsilon} := \frac{\|\tilde{g}\|_\infty^2}{\varepsilon} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2} \int_{\mathbb{T}^2\times\mathbb{T}_{2/\varepsilon}} \frac{|\hat{W}_\varepsilon(\eta',k')|}{|\lambda+i\varepsilon^{-1}(\omega(k'-\varepsilon\eta'/2)+\omega(k'+\varepsilon\eta'/2))|} \times \frac{\mathrm{d}k\mathrm{d}k'\mathrm{d}\eta'}{|\lambda/2+i\{\beta+\varepsilon^{-1}[\omega(k)+\omega(k'-\varepsilon\eta'/2)]\}|}.$$

We used here the identity

$$\frac{1}{(\lambda/2-ia)(\lambda/2+ib)} = \left(\frac{1}{\lambda/2-ia}+\frac{1}{\lambda/2+ib}\right)\frac{1}{\lambda-i(a-b)}.$$

Now, we can estimate $I_{1,\varepsilon}$ as follows:

$$I_{1,\varepsilon} \le \varepsilon\Gamma_\varepsilon\|\tilde{g}\|_\infty^2 \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2} \int_{\mathbb{T}\times\mathbb{T}_{2/\varepsilon}} \frac{|\hat{W}_\varepsilon(\eta',k')|\mathrm{d}k'\mathrm{d}\eta'}{\omega(k'-\varepsilon\eta'/2)+\omega(k'+\varepsilon\eta'/2)},$$

with

$$\Gamma_\varepsilon := \sup_{A\in\mathbb{R}} \int_{\mathbb{T}} \frac{\mathrm{d}k}{|\varepsilon\lambda/2-i(\omega(k)-A)|} \le \Gamma_\varepsilon^+ + \Gamma_\varepsilon^-,$$

with

$$\Gamma_\varepsilon^\pm := \sup_{A\in\mathbb{R}} \int_0^{\omega_{\max}} \frac{\mathrm{d}u}{|\varepsilon\lambda/2-i(u-A)||\omega'(\omega_\pm(u))|}.$$

Recall that ω_-,ω_+ are the decreasing and increasing branches of the inverse function of the dispersion relation $\omega(\cdot)$. Our assumptions on the dispersion relation imply that

$$\omega'(\omega_\pm(u)) \approx (\omega_{\max}-u)^{1/2}, \text{ for } \omega_{\max}-u \ll 1.$$

The consideration near the minimum of ω is identical unless $\omega_{\min}=0$, in which case $|\omega'(k)|$ stays uniformly positive near the minimum. Therefore, we have

$$\Gamma_\varepsilon^\pm \preceq \sup_{A\in[0,1]} \int_0^1 \frac{\mathrm{d}u}{[\varepsilon+|u-A|]\sqrt{u}} \preceq \varepsilon^{-1/2}\log\varepsilon^{-1}.$$

We therefore obtain

$$I_{1,\varepsilon} \preceq \varepsilon^{1/2}\log\varepsilon^{-1} \int_{\mathbb{T}_\varepsilon\times\mathbb{T}} \frac{|\hat{W}_\varepsilon(\eta',k')|\mathrm{d}\eta'\mathrm{d}k'}{\omega(k'-\varepsilon\eta'/2)+\omega(k'+\varepsilon\eta'/2)}$$
$$\preceq \int_0^{+\infty}\int_0^1 \frac{\varepsilon^{1/2}\log\varepsilon^{-1}\mathrm{d}q\mathrm{d}u}{(u+\varepsilon(q\wedge 1))(1+q^{3+2\kappa})} \to 0,$$

as $\varepsilon\to 0$, due to (5.1) and (2.19), and since if $\omega_{\min}=0$, then $\omega(k)$ behaves as $|k|$ near the minimum $k=0$. One can easily verify that the right hand side vanishes, with $\varepsilon\to 0$. Similarly we obtain that

$$\lim_{\varepsilon\to 0+} I_{2,\varepsilon} = 0,$$

which finishes the proof of Lemma 6.2. □

7. The Proof of Lemma 6.1

7.1. Outline of the Proof

We now turn to the proof of Lemma 6.1, the main ingredient in the computation of the limit of $\mathfrak{L}^{\varepsilon}_{scat,21}(\lambda)$. We will only consider the term $\mathcal{H}_+(\lambda,\varepsilon)$, as the computation of the limit of $\mathcal{H}_-(\lambda,\varepsilon)$ is essentially the same. We will focus on the harder case when the dispersion relation $\omega(k)$ is smooth both at its maximum $k=1/2$ and its minimum $k=0$, so that the inverse function has a square root singularity at each of these points. That is, the two branches of its inverse $\omega_+ : [\omega_{\min},\omega_{\max}] \to [0,1/2]$ and $\omega_- := -\omega_+$ satisfy

$$\omega'_{\pm}(w) = \pm(w-\omega_{\min})^{-1/2}\chi_*(w),\quad w-\omega_{\min} \ll 1,$$

and

$$\omega'_{\pm}(w) = \pm(\omega_{\max}-w)^{-1/2}\chi^*(w),\quad \omega_{\max}-w \ll 1,$$

with $\chi_*, \chi^* \in C^\infty(\mathbb{T})$ that are strictly positive.

Using (6.8) and the change of variables (6.17) we can write

$$\begin{aligned}\mathcal{H}_+(\lambda,\varepsilon) = -\frac{\gamma\lambda}{4\pi\varepsilon}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{\mathbb{T}\times T^2_\varepsilon}\\ \frac{\widehat{W}_\varepsilon(\eta',k')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}k'}{\lambda/2 - i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(k'+\varepsilon\eta'/2)]\}}\\ \times\frac{|\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta+\omega(k)])\,|^2}{\lambda/2 + i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(k'-\varepsilon\eta'/2)]\}}\times\frac{\hat{G}^*(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}.\end{aligned} \tag{7.1}$$

In fact, we may discard the contribution due to large η', thanks to assumption (5.1). More precisely, let $\widetilde{\mathcal{H}}_+(\lambda,\varepsilon)$ be the expression analogous to $\mathcal{H}_+(\lambda,\varepsilon)$ corresponding to integration over η' and k' over

$$T^2_\varepsilon := \left[(\eta',k') : |\eta'| \le \frac{\delta}{2^{100}\varepsilon},\ |k'| \le \frac{1-\varepsilon|\eta'|}{2}\right] \subset \mathbb{T}_{2/\varepsilon}\times\mathbb{T}, \tag{7.2}$$

with δ as in (5.2). Due to (5.1) and (2.19) we have

$$\begin{aligned}&\left|\mathcal{H}_+(\lambda,\varepsilon) - \widetilde{\mathcal{H}}_+(\lambda,\varepsilon)\right|\\ &\preceq \frac{1}{\varepsilon}\int_{|\eta'|\ge\delta/(2^{100}\varepsilon)}\frac{\mathrm{d}\eta'}{(1+(\eta')^2)^{3/2+\kappa}} \approx \varepsilon^{1+2\kappa} \to 0,\quad \text{as } \varepsilon\to 0.\end{aligned} \tag{7.3}$$

In what follows we restrict ourselves therefore to studying the limit of $\widetilde{\mathcal{H}}_+(\lambda,\varepsilon)$.

The main contribution to the limit comes from the regions where $\omega(k)\approx\omega(k')$, that is, where either $k\approx k'$—this generates the transmission term, or $k\approx -k'$—this is responsible for the reflection term in the limit. The smallness of these regions will compensate for the factor $1/\varepsilon$ in front of the integral in (7.1). To distinguish the contributions of these two regions, we decompose

$$\widetilde{\mathcal{H}}_+(\lambda,\varepsilon) = \sum_{\iota\in\{-,+\}} \mathcal{I}_\iota(\lambda,\varepsilon). \tag{7.4}$$

Here $\mathcal{I}_\iota(\lambda,\varepsilon)$ correspond to the integration over the domains $\tilde{T}^3_{\varepsilon,\iota}$, $\iota = \pm$:

$$\tilde{T}^3_{\varepsilon,\iota} := \Big[(k,\eta',k') \in \mathbb{T}\times T^2_\varepsilon : \operatorname{sign} k = \iota \operatorname{sign}(k'+\varepsilon\eta'/2)\Big],$$

so that the integration over $\tilde{T}^3_{\varepsilon,+}$ will generate the transmission term, and over $\tilde{T}^3_{\varepsilon,-}$ the reflection. Changing variables $k' := \iota k + \varepsilon\eta''/2$, and using the fact that $\omega(k)$ is even, gives

$$\begin{aligned}\mathcal{I}_\iota(\lambda,\varepsilon) = -\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{T^3_{\varepsilon,\iota}} &\frac{\widehat{W}_\varepsilon(\eta',\iota k+\varepsilon\eta''/2)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2 - i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(k+\iota(\varepsilon\eta'/2+\varepsilon\eta''/2))]\}}\\ &\times\frac{|\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta+\omega(k)])\,|^2}{\lambda/2 + i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(k+\iota(-\varepsilon\eta'/2+\varepsilon\eta''/2))]\}}\\ &\times\frac{\hat{G}^*(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}.\end{aligned} \tag{7.5}$$

Here, $T^3_{\varepsilon,\iota} \subset \mathbb{T}\times\mathbb{T}_{2/\varepsilon}\times\mathbb{T}_{6/\varepsilon}$ is the pre-image of $\tilde{T}^3_{\varepsilon,\iota}$ under the mapping $(k,\eta',\eta'') \mapsto (k,\eta',\iota k+\varepsilon\eta''/2)$:

$$\begin{aligned}T^3_{\varepsilon,\iota} := \Big[(k,\eta',\eta'') : k\in\mathbb{T},\ |\eta'| \le \frac{\delta}{2^{50}\varepsilon},\ \Big|k+\iota\frac{\varepsilon\eta''}{2}\Big|\\ \le \frac{1-\varepsilon|\eta'|}{2},\ \operatorname{sign} k = \operatorname{sign}\big(k+\iota(\varepsilon\eta'/2+\varepsilon\eta''/2)\big)\Big].\end{aligned} \tag{7.6}$$

We will pass to the limit $\varepsilon \to 0$ in expression (7.5) in several steps. The first step will be to replace the quotient $\varepsilon^{-1}[\omega(k)-\omega(k+\iota(\varepsilon\eta'/2+\varepsilon\eta''/2))]$ in the first denominator by $-\iota\omega'(k)(\eta'+\eta'')/2$. That is, we will show the following:

Lemma 7.1. *We have*

$$\lim_{\varepsilon\to0}\{\mathcal{I}_\iota(\lambda,\varepsilon) - \mathcal{I}^{(1)}_\iota(\lambda,\varepsilon)\} = 0, \quad \iota\in\{-,+\}, \tag{7.7}$$

where

$$\begin{aligned}\mathcal{I}^{(1)}_\iota(\lambda,\varepsilon) := -\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{T^3_{\varepsilon,\iota}} &\frac{\widehat{W}_\varepsilon\left(\eta',\iota k+\varepsilon\eta''/2\right)}{\lambda/2 - i\{\beta-\iota\omega'(k)(\eta'+\eta'')/2\}}\\ &\times\frac{|\tilde{g}\,(\lambda\varepsilon/2 - i[\varepsilon\beta+\omega(k)])\,|^2}{\lambda/2 + i\{\beta+\varepsilon^{-1}[\omega(k)-\omega(k+\iota(-\varepsilon\eta'/2+\varepsilon\eta''/2))]\}}\\ &\times\frac{\hat{G}^\star(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_\varepsilon\omega(k,\eta)}\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''.\end{aligned} \tag{7.8}$$

Next, we will replace a similar term in the second denominator by $\iota\omega'(k)(\eta'-\eta'')/2$.

Lemma 7.2. *We have*

$$\lim_{\varepsilon\to 0}\{\mathcal{I}_\iota^{(1)}(\lambda,\varepsilon)-\mathcal{I}_\iota^{(2)}(\lambda,\varepsilon)\}=0,\quad \iota\in\{-,+\}, \tag{7.9}$$

where

$$\mathcal{I}_\iota^{(2)}(\lambda,\varepsilon):=-\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{T^3_{\varepsilon,\iota}}\frac{\widehat{W}_\varepsilon\left(\eta',\iota k+\varepsilon\eta''/2\right)}{\lambda/2-i\{\beta-\iota\omega'(k)(\eta'+\eta'')/2\}}$$
$$\times\frac{|\tilde{g}\,(\lambda\varepsilon/2-i[\varepsilon\beta+\omega(k)])\,|^2}{\lambda/2+i\{\beta+\iota\omega'(k)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta_\varepsilon^+\omega(k,\eta)}\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''. \tag{7.10}$$

The third step will be to replace the term $|\tilde{g}\,(\lambda\varepsilon/2-i[\varepsilon\beta+\omega(k)])\,|^2$ in (7.10) by its limit $|\nu(k)|^2$.

Lemma 7.3. *We have*

$$\lim_{\varepsilon\to 0}|\mathcal{I}_\iota^{(2)}(\lambda,\varepsilon)-\mathcal{I}_\iota^{(3)}(\lambda,\varepsilon)|=0,\quad \iota\in\{-,+\}, \tag{7.11}$$

with

$$\mathcal{I}_\iota^{(3)}(\lambda,\varepsilon)=-\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{T^3_{\varepsilon,\iota}}\frac{\widehat{W}_\varepsilon(\eta',\iota k+\varepsilon\eta''/2)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2-i\{\beta-\iota\omega'(k)(\eta'+\eta'')/2\}}$$
$$\times\frac{|\nu(k)|^2}{\lambda/2+i\{\beta+\iota\omega'(k)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta_\varepsilon^+\omega(k,\eta)}. \tag{7.12}$$

Next, we will approximate the Wigner transform $\widehat{W}_\varepsilon(\eta',\iota k+\varepsilon\eta''/2)$ by $\widehat{W}_\varepsilon(\eta',\iota k)$, the test function $G^*(\eta,k+\varepsilon\eta/2)$ by $G^*(\eta,k)$, and $\delta_\varepsilon^+\omega(k,\eta)$ by $\omega'(k)\eta$, respectively.

Lemma 7.4. *We have*

$$\lim_{\varepsilon\to 0}|\mathcal{I}_\iota^{(3)}(\lambda,\varepsilon)-\mathcal{I}_\iota^{(4)}(\lambda,\varepsilon)|=0,\quad \iota\in\{-,+\}, \tag{7.13}$$

with

$$\mathcal{I}_\iota^{(4)}(\lambda,\varepsilon)=-\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{\mathbb{T}\times\mathbb{R}^2}\frac{\widehat{W}_\varepsilon(\eta',\iota k)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2-i\{\beta-\iota\omega'(k)(\eta'+\eta'')/2\}}$$
$$\times\frac{|\nu(k)|^2}{\lambda/2+i\{\beta+\iota\omega'(k)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}. \tag{7.14}$$

The last step will be to pass to the limit in $\widehat{W}_\varepsilon(\eta',\iota k)$ and integrate in β, which is done in Section 7.5.

7.2. The Proof of Lemmas 7.1 and 7.2

We only present the proof of Lemma 7.1 since the proof of Lemma 7.2 is very similar except somewhat simpler. We will also only consider $\iota = +$ (the transmission case) in the proof of Lemma 7.1, as the reflection case can be treated in a similar fashion.

Let us drop the subscript $+$, setting

$$\mathcal{I}(\lambda,\varepsilon) := \mathcal{I}_+(\lambda,\varepsilon),\ \ \mathcal{I}^{(1)}(\lambda,\varepsilon) := \mathcal{I}_+^{(1)}(\lambda,\varepsilon)$$

to reduce the number of subscripts. We split the domain of integration $T^3_{\varepsilon,+}$ into four regions:

$$T^3_{\varepsilon,+,\iota_1,\iota_2} := [(k,\eta',\eta'') \in T^3_{\varepsilon,+} : \iota_1 k > 0,\ \iota_2(k - \varepsilon\eta'/2 + \varepsilon\eta''/2) > 0],\quad \iota_1,\iota_2 \in \{-,+\}, \tag{7.15}$$

and write

$$\mathcal{I}(\lambda,\varepsilon) = \sum_{\iota_1,\iota_2=\pm} \mathcal{I}_{\iota_1,\iota_2}(\lambda,\varepsilon), \quad \mathcal{I}^{(1)}(\lambda,\varepsilon) = \sum_{\iota_1,\iota_2=\pm} \mathcal{I}^{(1)}_{\iota_1,\iota_2}(\lambda,\varepsilon).$$

We will only consider the case $\iota_1 = \iota_2 = +$, as the other cases can be done similarly, and set

$$\tilde{\mathcal{I}}(\lambda,\varepsilon) = \mathcal{I}_{+,+}(\lambda,\varepsilon),\ \ \tilde{\mathcal{I}}^{(1)}(\lambda,\varepsilon) = \mathcal{I}^{(1)}_{+,+}(\lambda,\varepsilon).$$

Our goal is to show that for any $\sigma > 0$ we have

$$\limsup_{\varepsilon\to 0} |\tilde{\mathcal{I}}(\lambda,\varepsilon) - \tilde{\mathcal{I}}^{(1)}(\lambda,\varepsilon)| < \sigma. \tag{7.16}$$

We perform the change of variables

$$w_0 := \omega(k),\quad w_1 := \omega(k - \varepsilon\eta'/2 + \varepsilon\eta''/2),\quad w_2 := \omega(k + \varepsilon\eta'/2 + \varepsilon\eta''/2) \tag{7.17}$$

in the integrals over k, η', η'', to get

$$\begin{aligned}
&\tilde{\mathcal{I}}(\lambda,\varepsilon) - \tilde{\mathcal{I}}^{(1)}(\lambda,\varepsilon)\\
&\quad = \frac{\gamma\lambda i}{4\pi\varepsilon^2}\int_{\mathbb{R}^2} \frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\\
&\qquad \int_{D_\varepsilon} \frac{\widehat{W}_\varepsilon\left(\varepsilon^{-1}[\omega_+(w_2)-\omega_+(w_1)],\,(1/2)[\omega_+(w_2)+\omega_+(w_1)]\right)}{\lambda/2 - i\{\beta+\varepsilon^{-1}(w_0-w_2)\}}\\
&\qquad \times \frac{\Delta_+^{(\varepsilon)}(w_2,w_0,\beta)|\tilde{g}\left(\lambda\varepsilon/2 - i(\varepsilon\beta+w_0)\right)|^2}{\lambda/2 + i\{\beta+\varepsilon^{-1}(w_0-w_1)\}}\\
&\qquad \times \frac{\hat{G}^*(\eta,\omega_+(w_0)+\varepsilon\eta/2)}{\lambda + i\delta_\varepsilon^+\omega(\omega_+(w_0),\eta)}\prod_{j=1}^2 \frac{1}{\omega'(\omega_+(w_j))}\mathrm{d}w_0\mathrm{d}w_1\mathrm{d}w_2,
\end{aligned} \tag{7.18}$$

with $D_\varepsilon \subset [\omega_{\min}, \omega_{\max}]^3$—the image of $T^3_{\varepsilon,+,+,+}$ under the change of variables mapping,

$$\Delta_\pm^{(\varepsilon)}(w', w, \beta) := \frac{\varepsilon^{-1}\delta\omega_+(w', w)}{\lambda/2 \mp i\{\beta + \varepsilon^{-1}(w - w') + \varepsilon^{-1}\omega'(\omega_+(w))\delta\omega_+(w', w)\}}, \tag{7.19}$$

and

$$\delta\omega_+(w', w) := \omega_+(w) - \omega_+(w') - \omega_+'(w)(w - w').$$

Let us explain some difficulties in passing to the limit in (7.18). Formally, we have a factor of ε^{-2} in front of the integral compensated by the terms of the order ε^{-1} in the first two denominators. The factor of ε^{-1} in the first argument in $\widehat{W}_\varepsilon$ seemingly would then bring a collapse of one variable of integration and show that the overall expression is small in the limit. However, there are two obstacles: first, the factors $\omega'(\omega_+(w_j))$ have a square root singularity at $\omega_{\min}$ and $\omega_{\max}$, so that the effect of the ε^{-1} terms in the first two denominators is reduced. Second, the terms of the size $\varepsilon\beta$ are not necessarily small and may influence the limit since the domain of integration in β is all of $\mathbb{R}$.

In order to deal with the first issue, using assumption (5.1), we see that there exists $\delta_0 > 0$ such that for all (w_1, w_2), for which we have

$$\widehat{W}_\varepsilon\left(\frac{1}{\varepsilon}\left[\omega_+(w_2) - \omega_+(w_1)\right], \frac{1}{2}\left[\omega_+(w_2) + \omega_+(w_1)\right]\right) = 0, \tag{7.20}$$

provided that either $w_1,\ w_2 \in [\omega_{\min}, \omega_{\min} + \delta_0)$, or $w_1,\ w_2 \in (\omega_{\max} - \delta_0, \omega_{\max}]$, and $(w_0, w_1, w_2) \in D_\varepsilon$ for some w_0.

We can further write

$$\mathcal{I}(\lambda, \varepsilon) - \tilde{\mathcal{I}}^{(1)}(\lambda, \varepsilon) = \sum_{j=1}^{2} \mathcal{J}_{j,\varepsilon},$$

where the integration is split into the regions $|w_1 - w_2| \ge \delta_0/4$ and otherwise. From (5.1) and (7.18) we conclude that $\left|\mathcal{J}_{1,\varepsilon}\right| \preceq \varepsilon$. If, on the other hand $|w_1 - w_2| < \delta_0/4$ we only need to be concerned with the integration over $w_2 \in I(\delta_0/2)$, where $I(\delta) := [\omega_{\min} + \delta, \omega_{\max} - \delta]$, as otherwise the integrand vanishes because of (7.20). The above implies that $w_1 \in I(\delta_0/4)$. Since $\omega_+ \in C^\infty(I(\delta_0/4))$ we can can find $C > 0$ such that, after integration in η, we have, with a constant depending on λ:

$$|\mathcal{J}_{2,\varepsilon}| \preceq \mathcal{R}_\varepsilon := \frac{1}{\varepsilon^2}\int_{\mathbb{R}} \frac{d\beta}{1+\beta^2} \int_{\omega_{\min}}^{\omega_{\max}} dw_0 \int_{I(\delta_0/4)} \int_{I(\delta_0/2)} \frac{\varphi\left(C\varepsilon^{-1}(w_2 - w_1)\right)}{1 + |\beta + \varepsilon^{-1}(w_0 - w_2)|}$$
$$\times \frac{|\Delta_+^{(\varepsilon)}(w_2, w_0, \beta)| dw_1 dw_2}{1 + |\beta + \varepsilon^{-1}(w_0 - w_1)|} = \mathcal{R}_\varepsilon^1 + \mathcal{R}_\varepsilon^2.$$

The two terms above correspond to the integration in w_0 over the regions $I'(\rho) := [\omega_{\min}, \omega_{\max}] \setminus I(\rho)$ and $I(\rho)$, with $\rho < \delta_0/8$. We have

$$\begin{aligned}\mathcal{R}^1_\varepsilon \preceq \frac{1}{\varepsilon^3} \int_{I'(\rho)} \left(1 + \frac{1}{\omega'(\omega_+(w_0))}\right) \mathrm{d}w_0 \\ \int_{I(\delta_0/4)} \int_{I(\delta_0/2)} \varphi\left(C\varepsilon^{-1}(w_2 - w_1)\right) \mathrm{d}w_1 \mathrm{d}w_2 \\ \times \int_{\mathbb{R}} \frac{1}{1 + |\beta + \varepsilon^{-1}(w_0 - w_2)|} \times \frac{1}{1 + |\beta + \varepsilon^{-1}(w_0 - w_1)|} \times \frac{\mathrm{d}\beta}{1 + \beta^2}\end{aligned} \tag{7.21}$$

$$\begin{aligned}\preceq \frac{1}{\varepsilon^3} \sum_{j=1}^{2} \int_{I'(\rho)} \left(1 + \frac{1}{\omega'(\omega_+(w_0))}\right) \mathrm{d}w_0 \int_{I(\delta_0/4)} \\ \int_{I(\delta_0/2)} \varphi\left(C\varepsilon^{-1}(w_2 - w_1)\right) \mathrm{d}w_1 \mathrm{d}w_2 \\ \times \int_{\mathbb{R}} \frac{1}{1 + |\beta + \varepsilon^{-1}(w_0 - w_j)|^2} \times \frac{\mathrm{d}\beta}{1 + \beta^2}.\end{aligned} \tag{7.22}$$

An elementary estimate

$$\int_{\mathbb{R}} \frac{1}{1 + (\beta + a)^2} \times \frac{\mathrm{d}\beta}{1 + \beta^2} \preceq \frac{1}{1 + a^2}, \quad a \in \mathbb{R} \tag{7.23}$$

implies that

$$\begin{aligned}\mathcal{R}^1_\varepsilon \preceq \frac{1}{\varepsilon^3} \sum_{j=1}^{2} \int_{I'(\rho)} \left(1 + \frac{1}{\omega'(\omega_+(w_0))}\right) \mathrm{d}w_0 \int_{I(\delta_0/4)} \\ \int_{I(\delta_0/2)} \varphi\left(C\varepsilon^{-1}(w_2 - w_1)\right) \frac{\mathrm{d}w_1 \mathrm{d}w_2}{1 + \varepsilon^{-2}(w_0 - w_j)^2}.\end{aligned}$$

If $w_0 \in I'(\rho)$ we have $|w_0 - w_j| \geq \delta_0/8$, thus

$$\mathcal{R}^1_\varepsilon \preceq \frac{1}{\delta_0^2} \int_{I'(\rho)} \left(1 + \frac{1}{\omega'(\omega_+(w_0))}\right) \mathrm{d}w_0 \int_0^1 \int_0^1 \frac{1}{\varepsilon} \varphi\left(C\varepsilon^{-1}(w_2 - w_1)\right) \mathrm{d}w_1 \mathrm{d}w_2.$$

We conclude that

$$\limsup_{\varepsilon \to 0} \mathcal{R}^1_\varepsilon \preceq \frac{1}{\delta_0^2} \int_{I'(\rho)} \left(1 + \frac{1}{\omega'(\omega_+(w_0))}\right) \mathrm{d}w_0, \quad \rho \in (0, 1). \tag{7.24}$$

Selecting $\rho > 0$ sufficiently small, we deduce

$$\limsup_{\varepsilon \to 0} \mathcal{R}^1_\varepsilon < \sigma. \tag{7.25}$$

Next, we fix $\rho > 0$ sufficiently small, so that (7.25) holds and look at the term $\mathcal{R}^2_\varepsilon$, that involves integration in w_0 over the region $I(\rho)$. Note that $\omega_+ \in C^\infty(I(\rho))$ and

$$\inf_{w \in I(\rho)} \omega'(\omega_+(w)) > 0,$$

hence

$$\mathcal{R}_\varepsilon^2 \preceq \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\frac{d\beta}{1+\beta^2}\int_{I(\rho)}\int_{I(\delta_0/4)}\int_{I(\delta_0/2)}\frac{\varphi\left(C\varepsilon^{-1}(w_2-w_1)\right)}{1+|\beta+\varepsilon^{-1}(w_0-w_2)|}\times\frac{|\Delta_+^{(\varepsilon)}(w_2,w_0)|dw_0dw_1dw_2}{1+|\beta+\varepsilon^{-1}(w_0-w_1)|}.$$

After the change of variables $w_1' := \varepsilon^{-1}(w_1-w_0)$, $w_2' := \varepsilon^{-1}(w_2-w_0)$, $\beta' := \beta - w_2'$, the expression in the right side can be estimated by

$$\mathcal{I}_\varepsilon := \int_{I(\rho)} dw_0\int_{I_\varepsilon(\delta_0)} dw_1dw_2\int_{\mathbb{R}}\frac{\varphi\left(C(w_2-w_1)\right)}{1+|\beta+w_2-w_1|}\times\frac{|\tilde{\Delta}_+^{(\varepsilon)}(w_2,w_0,\beta)|}{1+(\beta+w_2)^2}\times\frac{d\beta}{1+|\beta|} = \mathcal{I}_\varepsilon^{(1)}+\mathcal{I}_\varepsilon^{(2)}, \tag{7.26}$$

with

$$\tilde{\Delta}_\pm^{(\varepsilon)}(w',w,\beta) := \frac{\varepsilon^{-1}\tilde{\delta}_\varepsilon\omega_+(w',w)}{\lambda/2\mp i\{\beta+\varepsilon^{-1}\omega'(\omega_+(w))\tilde{\delta}_\varepsilon\omega_+(w',w)\}}$$

and

$$\tilde{\delta}_\varepsilon\omega_+(w',w) = -\int_w^{w+\varepsilon w'}(\omega_+'(v)-\omega_+'(w))dv = \omega_+(w)+\omega_+'(w)\varepsilon w' -\omega_+(w+\varepsilon w'). \tag{7.27}$$

The two terms in the right side of (7.26) correspond to splitting the region $I_\varepsilon(\delta_0) \subset [-C_1\varepsilon^{-1}, C_1\varepsilon^{-1}]^2$ of integration in w_1, w_2 (the image of $I(\delta_0/4)\times I(\delta_0/2)$ under the above map) into two sub-regions, corresponding to the integration over

$$\mathcal{B}_\varepsilon(\rho') := [w_2 : |w_2| \le \rho'/\varepsilon]$$

and its complement, with $\rho' > 0$ is to be determined later. Note that in both regions we have the estimates

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\tilde{\delta}_\varepsilon\omega_+(w',w) = 0 \quad \text{for each } w, w', \tag{7.28}$$

and

$$\int_{\mathbb{R}}\frac{\varphi\left(Cw\right)dw}{1+|\beta+w|} \preceq \frac{1}{1+|\beta|}, \quad d\beta \in \mathbb{R}. \tag{7.29}$$

As the domain of integration in (7.26) depends on ε, even with (7.28) in hand, we still can not apply the Lebesgue dominated convergence theorem directly. In addition, we have the estimate

$$|\tilde{\Delta}_+^{(\varepsilon)}(w,w_0,\beta)| \preceq |\varepsilon^{-1}\tilde{\delta}_\varepsilon\omega_+(w,w_0)| \preceq \varepsilon w^2, \tag{7.30}$$

for all w_0 and w in the domain of integration in (7.26). Integrating out the w_1-variable using (7.29) and (7.30), we obtain

$$\mathcal{I}_\varepsilon^{(1)} \preceq \int_{I(\rho)} \mathrm{d}w_0 \int_{-\rho'/\varepsilon}^{\rho'/\varepsilon} \mathrm{d}w \int_{\mathbb{R}} \frac{\varepsilon w^2}{1+(\beta+w)^2} \times \frac{\mathrm{d}\beta}{1+\beta^2} \preceq \int_{I(\rho)} \mathrm{d}w_0 \int_{-\rho'/\varepsilon}^{\rho'/\varepsilon} \frac{\varepsilon w^2 \mathrm{d}w}{1+w^2} \preceq \rho'. \tag{7.31}$$

It follows that

$$\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon^{(1)} \le \sigma, \tag{7.32}$$

for a sufficiently small $\rho' \in (0,1)$.

For the second term in the right side of (7.26), we use (7.29) to integrate out the w_1-variable once again, and write

$$\mathcal{I}_\varepsilon^{(2)} = \int_{I(\rho)} \mathrm{d}w_0 \int_{I'_\varepsilon} \mathrm{d}w \int_{\mathbb{R}} \frac{|\tilde{\Delta}_+^{(\varepsilon)}(w,w_0)|}{1+(\beta+w)^2} \times \frac{\mathrm{d}\beta}{1+\beta^2} = \mathcal{I}_\varepsilon^{(2,1)} + \mathcal{I}_\varepsilon^{(2,2)}. \tag{7.33}$$

Here, $I'_\varepsilon \subset [\rho'/\varepsilon \le |w| \le C_1/\varepsilon]$ is the projection of $I_\varepsilon \cap \mathcal{B}_\varepsilon^c(\rho')$ onto the w_2-axis. The first integral in the right side of (7.33) corresponds to integration over the set

$$[(\beta,w)\in\mathbb{R}^2 : |\beta+w| \le |w|^{3/4}]$$

and the second over its complement. We split again to get that

$$\mathcal{I}_\varepsilon^{(2,1)} = \mathcal{I}_{\varepsilon,+}^{(2,1)} + \mathcal{I}_{\varepsilon,-}^{(2,1)},$$

according to the integration in w over $I_\varepsilon^{\pm} = I'_\varepsilon \cap [w>0]$ and its complement, so that

$$\mathcal{I}_{\varepsilon,\pm}^{(2,1)} \preceq \int_{I(\rho)} \mathrm{d}w_0 \int_{I_\varepsilon^+} \frac{\varepsilon^{-1}|\tilde{\delta}_\varepsilon \omega_+(w,w_0)|\mathrm{d}w}{1+w^2} \int_{[|\beta-w|\le|w|^{3/4}]} \frac{\mathrm{d}\beta}{1+|\beta-\varepsilon^{-1}\omega'(\omega_+(w_0))\tilde{\delta}_\varepsilon\omega_+(w,w_0)|}. \tag{7.34}$$

Let us set

$$z_\varepsilon(w,w_0) := \varepsilon^{-1}\omega'(\omega_+(w_0))\tilde{\delta}_\varepsilon\omega_+(w,w_0) = w - \frac{\omega_+(w_0+\varepsilon w)-\omega_+(w_0)}{\varepsilon\omega'_+(w_0)},$$

so that

$$w - z_\varepsilon(w,w_0) > w^{4/5}, \text{ for all } w_0 \in I(\rho) \text{ and } w \in I_\varepsilon^+ \tag{7.35}$$

for all $\varepsilon > 0$ sufficiently small. Then, we have

$$Z_\varepsilon(w,w_0) := \int_{w-w^{3/4}}^{w+w^{3/4}} \frac{\mathrm{d}\beta}{1+|\beta-z_\varepsilon|} = \int_{w-w^{3/4}-z_\varepsilon}^{w+w^{3/4}-z_\varepsilon} \frac{\mathrm{d}\beta}{1+|\beta|} = \log\Big(\frac{1+w+w^{3/4}-z_\varepsilon}{1+w-w^{3/4}-z_\varepsilon}\Big).$$

It follows from (7.35) that by taking ε sufficiently small we may ensure that

$$\limsup_{\varepsilon\to 0}\ \sup_{w_0\in I(\rho),w\in I_\varepsilon^+}|Z_\varepsilon(w,w_0)|=0. \tag{7.36}$$

Then, using (7.30), we get

$$\mathcal{I}_{\varepsilon,+}^{(2,1)}\preceq\int_{I(\rho)}\mathrm{d}w_0\int_{|w-z|\ge w^{4/5}}|Z_\varepsilon(w,w_0)|\frac{\varepsilon w^2\mathrm{d}w}{1+w^2}\le\frac{\sigma}{2}, \tag{7.37}$$

for $\varepsilon>0$ sufficiently small. A similar calculation yields the same estimate for $\mathcal{I}_{\varepsilon,-}^{(2,1)}$, thus

$$\limsup_{\varepsilon\to 0}\mathcal{I}_\varepsilon^{(2,1)}<\sigma. \tag{7.38}$$

As for $\mathcal{I}_\varepsilon^{(2,2)}$ we can write

$$\mathcal{I}_\varepsilon^{(2,2)}\preceq\int_{I(\rho)}\mathrm{d}w_0\int_{I'_\varepsilon}\frac{\varepsilon^{-1}|\tilde{\delta}_\varepsilon\omega_+(w,w_0)|\mathrm{d}w}{1+w^{3/2}}$$
$$\int_{\mathbb{R}}\frac{\mathrm{d}\beta}{(1+|\beta+\varepsilon^{-1}\omega'(\omega_+(w_0))\tilde{\delta}_\varepsilon\omega_+(w,w_0)|)(1+\beta^2)}.$$

Using an elementary estimate

$$\int_{\mathbb{R}}\frac{1}{1+|\beta+a|}\times\frac{\mathrm{d}\beta}{1+\beta^2}\preceq\frac{1}{1+|a|},\quad a\in\mathbb{R}, \tag{7.39}$$

we obtain

$$\mathcal{I}_\varepsilon^{(2,2)}\preceq\int_{I(\rho)}\mathrm{d}w_0\int_{I'_\varepsilon}\frac{\varepsilon^{-1}|\tilde{\delta}_\varepsilon\omega_+(w,w_0)|\mathrm{d}w}{[1+|\varepsilon^{-1}\omega'(\omega_+(w_0))\tilde{\delta}_\varepsilon\omega_+(w,w_0)|](1+w^{3/2})}.$$

Here, we can use the Lebesgue dominated convergence theorem and (7.28) to conclude that

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon^{(2,2)}=0.$$

This finishes the proof of (7.16).

7.3. The Proof of Lemma 7.3

Let us note that the integration in η'' both in expression (7.10) for $\mathcal{I}_t^{(2)}$ and (7.12) for $\mathcal{I}_t^{(3)}$ would bring out the factor of $[\omega'(k)]^{-1}$ that is not integrable. This singularity should be compensated by the $\tilde{g}$-term in (7.10) and by its limit $|\nu(k)|^2$ in (7.12), as can be seen from (2.27), (2.28) and (2.31). The following auxiliary result will allow us to use this argument:

Lemma 7.5. *For each k_* such that $\omega'(k_*)=0$, we have*

$$\lim_{\delta'\to 0}\limsup_{\varepsilon\to 0}\sup_{\beta\in(-\delta',\delta')}|\tilde{g}\left(\varepsilon-i[\beta+\omega(k_*)]\right)|=0. \tag{7.40}$$

Proof. As follows from (2.25), it suffices to show that

$$\lim_{\delta'\to 0}\liminf_{\varepsilon\to 0}\inf_{\beta\in(-\delta',\delta')}|\tilde{J}(\varepsilon-i[\beta+\omega(k_*)])|=+\infty, \tag{7.41}$$

with $\tilde{J}(\cdot)$ as in (2.24). Consider the point $k_*=1/2$ where ω attains its maximum $\omega_{\max}=\omega(k_*)>0$, and write

$$\begin{aligned}&\tilde{J}(\varepsilon-i[\beta+\omega(k_*)])\\&\quad=\frac{i}{2}\Big\{\int_{\mathbb{T}}\frac{d\ell}{i\varepsilon+\beta+\omega(k_*)+\omega(\ell)}+\int_{\mathbb{T}}\frac{d\ell}{i\varepsilon+\beta+\omega(k_*)-\omega(\ell)}\Big\}.\end{aligned}$$

Hence, (7.41) would follow if we can show that for each $M>0$ there exist $\varepsilon_0,\delta_0\in(0,1)$ such that

$$\Big|\int_{\mathbb{T}}\frac{d\ell}{i\varepsilon+\beta+\omega(k_*)-\omega(\ell)}\Big|>M,\quad \beta\in(-\delta_0,\delta_0),\ \varepsilon\in(0,\varepsilon_0). \tag{7.42}$$

The real and imaginary parts of the expression under the absolute value in (7.42) are

$$r_\varepsilon(\beta):=\int_{\mathbb{T}}\frac{[\beta+\omega(k_*)-\omega(\ell)]d\ell}{\varepsilon^2+[\beta+\omega(k_*)-\omega(\ell)]^2},\quad j_\varepsilon(\beta):=-\int_{\mathbb{T}}\frac{\varepsilon d\ell}{\varepsilon^2+[\beta+\omega(k_*)-\omega(\ell)]^2}.$$

Changing variables $u:=\omega(\ell)-\beta$, we obtain

$$|j_\varepsilon(\beta)|\ge\int_{\omega_{\min}-\beta}^{\omega_{\max}-\beta}\frac{\varepsilon}{\varepsilon^2+[\omega_{\max}-u]^2}\times\frac{du}{|\omega'(\omega_+(u+\beta))|}.$$

Choosing a sufficiently small $\delta_0>0$, we see that

$$|\omega'(\omega_+(u+\beta))|\le\pi/(2M)\text{ for }|\beta|<\delta_0\text{ and }u\in(\omega_{\max}-\delta_0,\omega_{\max}+\delta_0),$$

hence

$$\inf_{\beta\in(-\delta_0,\delta_0)}|j_\varepsilon(\beta)|\ge\frac{2M}{\pi}\int_{\omega_{\max}-\delta_0}^{\omega_{\max}+\delta_0}\frac{\varepsilon du}{\varepsilon^2+[\omega_{\max}-u]^2}.$$

It follows that for a sufficiently small ε_0 we have

$$\inf_{\beta\in(-\delta_0,\delta_0)}|j_\varepsilon(\beta)|\ge M,\quad \varepsilon\in(0,\varepsilon_0),$$

and (7.42) follows. □

We now turn to the proof of Lemma 7.3. Once again, we will only consider $\iota=+$ and drop the subscript $+$ in the notation. Let $\sigma>0$ be arbitrary. For $\rho\in(0,\delta/4)$, with $\delta>0$ as in (5.2), we let

$$L(\rho):=[k:\operatorname{dist}(k,\Omega_*)<\rho],\quad \Omega_*:=[k\in\mathbb{T}:\ \omega'(k)=0]\subset\{0,1/2\}, \tag{7.43}$$

with ρ to be specified further later. We can write

$$\mathcal{I}^{(2)}(\lambda,\varepsilon)-\mathcal{I}^{(3)}(\lambda,\varepsilon)=\tilde{\mathcal{I}}^{1}(\lambda,\varepsilon)+\tilde{\mathcal{I}}^{2}(\lambda,\varepsilon),$$

with the two terms corresponding to the integration in (7.10) and (7.12) in the k-variable over $L^c(\rho)$, the complement of $L(\rho)$, and $L(\rho)$ itself, respectively. Since $|\omega'(k)|$ is bounded away from 0 on $L^c(\rho)$, an elementary application of the Lebesgue dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0}\tilde{\mathcal{I}}^{1}(\lambda,\varepsilon)=0. \tag{7.44}$$

Assumption (5.1), (5.2) on the support of $\widehat{W}_\varepsilon(\eta,k)$ in k allows us to write

$$|\tilde{\mathcal{I}}^{2}(\lambda,\varepsilon)|\preceq\tilde{\mathcal{I}}^{2,1}(\lambda,\varepsilon)+\tilde{\mathcal{I}}^{2,2}(\lambda,\varepsilon), \tag{7.45}$$

where

$$\begin{aligned}\tilde{\mathcal{I}}^{2,j}(\lambda,\varepsilon):=&\int_{\mathbb{R}}\frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2}\int_{L(\rho)\times B(\delta,\varepsilon)\times A(\delta,\varepsilon)}\frac{|\delta_{j,\varepsilon}(\beta,k)|}{1+|\beta+\omega'(k)(\eta'-\eta'')/2|}\\&\times\frac{\varphi(\eta')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+|\beta-\omega'(k)(\eta'+\eta'')/2|},\end{aligned}$$

with

$$A(\delta,\varepsilon):=[\eta'':\ \delta/(2\varepsilon)\le|\eta''|\le 6/\varepsilon],\quad B(\delta,\varepsilon):=[\eta':\ |\eta'|\le\delta/(2^{100}\varepsilon)] \tag{7.46}$$

and

$$\delta_{1,\varepsilon}(\beta,k):=|\tilde{g}\left(\lambda\varepsilon/2-i[\varepsilon\beta+\omega(k)]\right)|^2,\qquad\delta_{2,\varepsilon}(\beta,k):=|\nu(k)|^2\preceq|\omega'(k)|. \tag{7.47}$$

The last inequality above follows from (2.29). It follows that

$$\begin{aligned}\tilde{\mathcal{I}}^{2,2}(\lambda,\varepsilon)\preceq&\int_{\mathbb{R}}\frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2}\int_{L(\rho)\times B(\delta,\varepsilon)\times A(\delta,\varepsilon)}\frac{|\omega'(k)|}{1+|\beta+\omega'(k)(\eta'-\eta'')/2|}\\&\frac{\varphi(\eta')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+|\beta-\omega'(k)(\eta'+\eta'')/2|}\\\le&\ C(J_++J_-),\end{aligned} \tag{7.48}$$

with

$$J_\pm:=\int_{L(\rho)\times B(\delta,\varepsilon)\times A(\delta,\varepsilon)}\frac{\varphi(\eta')|\omega'(k)|\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+[\omega'(k)(\eta'\pm\eta'')]^2}, \tag{7.49}$$

and a constant $C>0$ independent of ε, ρ. We used the Cauchy-Schwarz inequality and (7.23) in the last inequality in (7.48). Note that

$$J_\pm:=\int_{L(\rho)\times B(\delta,\varepsilon)\times A(\delta,\varepsilon)}\frac{\varphi(\eta')|\omega'(k)|\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+[\omega'(k)(\eta'\pm\eta'')]^2}. \tag{7.50}$$

Changing variables $\eta'' := \omega'(k)(\eta' \pm \eta'')$ we conclude that

$$J_\pm \le \int_{L(\rho)\times\mathbb{R}^2} \frac{\varphi(\eta')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+|\eta''|^2} \le \sigma, \tag{7.51}$$

provided that $\rho > 0$ is sufficiently small.

As for the term $\tilde{\mathcal{I}}^{2,1}(\lambda,\varepsilon)$, using Cauchy-Schwarz inequality we obtain

$$\tilde{\mathcal{I}}^{2,1}(\lambda,\varepsilon) \preceq K_{\varepsilon,+} + K_{\varepsilon,-},$$

with

$$\begin{aligned} K_{\varepsilon,\pm}(\rho) \;&:= \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{1+\beta^2} \int_{L(\rho)\times B(\delta,\varepsilon)\times A(\delta,\varepsilon)} \frac{\delta_{1,\varepsilon}(\beta,k)\varphi(\eta')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+|\beta-\omega'(k)(\eta''\pm\eta')/2|^2} \\ &= K^1_{\varepsilon,+}(\rho,\rho') + K^2_{\varepsilon,+}(\rho,\rho'). \end{aligned} \tag{7.52}$$

The terms in the right side correspond to integration over the regions

$$\begin{aligned} \mathcal{K}^1_{\varepsilon,+}(\rho,\rho') &:= [(\beta,k,\eta',\eta'') \in \mathbb{R}\times L(\rho)\times\mathbb{R}\times A(\delta,\varepsilon)\;:|\eta'| \\ &\quad < \delta/(2^{100}\varepsilon),\; |\beta| \ge \rho'\varepsilon^{-1}], \\ \mathcal{K}^2_{\varepsilon,+}(\rho,\rho') &:= [(\beta,k,\eta',\eta'') \in \mathbb{R}\times L(\rho)\times\mathbb{R}\times A(\delta,\varepsilon)\;:|\eta'| \\ &\quad < \delta/(2^{100}\varepsilon),\; |\beta| < \rho'\varepsilon^{-1}], \end{aligned}$$

with $\rho' > 0$ to be chosen later. Since $\omega'(k_*) = 0$, for each $\rho' > 0$ we can find ρ sufficiently small so that

$$|\beta - \omega'(k)(\eta''+\eta')/2| \ge |\beta|/2, \quad \text{on } \mathcal{K}^1_{\varepsilon,+}(\rho,\rho').$$

Therefore, for each $\rho' > 0$ we can find $\rho > 0$ sufficiently small so that

$$K^1_{\varepsilon,+}(\rho,\rho') \preceq \frac{1}{\varepsilon}\int_{[|\beta|\ge\rho'\varepsilon^{-1}]} \frac{\mathrm{d}\beta}{(1+\beta^2)^2} \to 0, \quad \text{as } \varepsilon\to 0, \tag{7.53}$$

with the pre-factor ε^{-1} coming again from the integration over η'' in (7.52) . Finally, we can write

$$\begin{aligned} K^2_{\varepsilon,+}(\rho,\rho') \le m_\varepsilon(\rho,\rho') \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{1+\beta^2} \int_{L(\rho)\times[|\eta'|<\delta/(2^{100}\varepsilon)]\times A(\delta,\varepsilon)} \\ \frac{\varphi(\eta')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+|\beta-\omega'(k)(\eta''+\eta')/2|^2}, \end{aligned}$$

where

$$\delta' := \rho' + \sup_{k\in L(\rho)} |\omega(k)-\omega(k_*)|, \quad m_\varepsilon(\rho,\rho') := \sup_{\beta'\in(-\delta',\delta')} |\tilde{g}(\varepsilon\lambda - i[\beta'+\omega(k_*)])|^2.$$

Using (7.23) again gives

$$K^2_{\varepsilon,+}(\rho,\rho') \preceq m_\varepsilon(\rho,\rho') \int_{L(\rho)\times[|\eta'|<\delta/(2^{100}\varepsilon)]\times A(\delta,\varepsilon)} \frac{\varphi(\eta')\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{1+|\omega'(k)(\eta''+\eta')|^2}$$
$$\preceq m_\varepsilon(\rho,\rho') \int_{L(\rho)\times A(\delta/10,\varepsilon/2)} \frac{\mathrm{d}k\mathrm{d}\eta''}{1+|\omega'(k)\eta''|^2}$$
$$\preceq m_\varepsilon(\rho,\rho') \int_{L(\rho)} \frac{\mathrm{d}k}{|\omega'(k)|}\left[\arctan\left(\frac{12|\omega'(k)|}{\varepsilon}\right) - \arctan\left(\frac{\delta|\omega'(k)|}{5\varepsilon}\right)\right].$$

Using a well known trigonometric identity we write

$$\arctan\left(\frac{12|\omega'(k)|}{\varepsilon}\right) - \arctan\left(\frac{\delta|\omega'(k)|}{5\varepsilon}\right)$$
$$= \arctan\left(\frac{(12-\delta/5)|\omega'(k)|}{\varepsilon}\left\{1+\frac{(12/5)\delta|\omega'(k)|^2}{\varepsilon^2}\right\}^{-1}\right),$$

therefore

$$K^2_{\varepsilon,+}(\rho,\rho') \preceq m_\varepsilon(\rho,\rho') \int_0^\rho \frac{\mathrm{d}k}{k} \arctan\left(\frac{(k/\varepsilon)}{1+(k/\varepsilon)^2}\right)$$
$$\preceq m_\varepsilon(\rho,\rho') \int_0^\infty \frac{\mathrm{d}k}{k} \arctan\left(\frac{k}{1+k^2}\right) \preceq m_\varepsilon(\rho,\rho').$$

Lemma 7.5 implies now that we can choose ρ, ρ' so small that

$$\limsup_{\varepsilon\to 0} K^2_{\varepsilon,+}(\rho,\rho') \preceq \sigma. \tag{7.54}$$

Combining (7.53) and (7.54) we conclude that for each $\sigma > 0$ there exists $\rho \in (0,1)$ such that

$$\limsup_{\varepsilon\to 0} K_{\varepsilon,+}(\rho) \preceq \sigma. \tag{7.55}$$

The analysis for $K_{\varepsilon,-}(\rho)$ is very similar, finishing the proof of Lemma 7.3.

7.4. The Proof of Lemma 7.4

As usual, we only consider $\iota = +$ and drop the corresponding subscript $+$. A straightforward computation using (5.1), the regularity of the test function $\hat{G}(\eta,k)$, and (7.23) shows that we can replace $\hat{G}^\star(\eta, k+\varepsilon\eta/2)$ in (7.12) by $\hat{G}^\star(\eta,k)$, and $\delta^+_\varepsilon\omega(k,\eta)$ by $\omega'(k)\eta$, so that

$$|\mathcal{I}^{(3)}(\lambda,\varepsilon) - \tilde{\mathcal{I}}^{(3)}(\lambda,\varepsilon)| \to 0, \text{ as } \varepsilon \to 0, \tag{7.56}$$

where

$$\tilde{\mathcal{I}}^{(3)}(\lambda,\varepsilon) = -\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2} \frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2} \int_{T^3_{\varepsilon,\iota}} \frac{\widehat{W}_\varepsilon(\eta', k+\varepsilon\eta''/2)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2 - i\{\beta - \iota\omega'(k)(\eta'+\eta'')/2\}}$$
$$\times \frac{|\nu(k)|^2}{\lambda/2 + i\{\beta+\iota\omega'(k)(\eta'-\eta'')/2\}} \times \frac{\hat{G}^\star(\eta,k)}{\lambda + i\omega'(k)\eta}. \tag{7.57}$$

We change variables $k' := k + \varepsilon\eta''/2$ in the right side to obtain

$$\begin{aligned}\tilde{\mathcal{I}}^{(3)}(\lambda,\varepsilon) &= -\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{U_\varepsilon}\frac{\widehat{W}_\varepsilon(\eta',k)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2-i\{\beta-\omega'(k-\varepsilon\eta''/2)(\eta'+\eta'')/2\}}\\ &\quad\times\frac{|\nu(k-\varepsilon\eta''/2)|^2}{\lambda/2+i\{\beta+\omega'(k-\varepsilon\eta''/2)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}+o(1)\\ &= -\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{U_\varepsilon}\frac{\widehat{W}_\varepsilon(\eta',k)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2-i\{\beta-\omega'(k)(\eta'+\eta'')/2\}}\\ &\quad\times\frac{|\nu(k)|^2}{\lambda/2+i\{\beta+\omega'(k)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}+\Delta_\varepsilon+o(1),\end{aligned} \tag{7.58}$$

with, cf (7.6),

$$\begin{aligned}U_\varepsilon := \Big[(k,\eta',\eta'') : k\in\mathbb{T},\ |\eta'|\le\frac{\delta}{2^{100}\varepsilon},\ |k|\le\frac{1-\varepsilon|\eta'|}{2},\ \operatorname{sign}(k-\varepsilon\eta''/2)\\ =\operatorname{sign}\big(k+\varepsilon\eta'/2\big)\Big].\end{aligned} \tag{7.59}$$

The term $o(1)$ in the right side of (7.58) appears because we have, once again, approximated the arguments in $\hat{G}^\star$ and ω' by k in the very last factor, despite the latest change of variables. The error Δ_ε, that we now need to estimate, appears in (7.58) because we have replaced the arguments of ω' by k in the first two factors.

Thanks to assumption (5.1), the integration over k in (7.58) is only over the complement of the set $L(\delta)$, see (5.3). We can then write (cf (7.46)) that

$$\begin{aligned}|\Delta_\varepsilon| &\preceq \int_{L^c(\delta)\times\mathbb{R}\times B(\delta,\varepsilon)\times\mathbb{R}} d_\varepsilon(k,\eta',\eta'')\varphi(\eta')\frac{|\hat{G}^\star(\eta,k)|}{|\lambda+i\omega'(k)\eta|}\mathrm{d}k\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}\eta''\\ &= \Delta'_\varepsilon+\Delta''_\varepsilon.\end{aligned} \tag{7.60}$$

The terms Δ'_ε and Δ''_ε correspond to the integration in η'' over the domains $A'(\delta,\varepsilon) := [|\eta''|\le\delta/(2\varepsilon)]$, and $A''(\delta,\varepsilon) = [|\eta''|\ge\delta/(2\varepsilon)]$, and

$$\begin{aligned}d_\varepsilon(k,\eta',\eta'') := \int_{\mathbb{R}}\frac{\mathrm{d}\beta}{(\lambda/2)^2+\beta^2}\Bigg|&\frac{1}{\lambda/2-i\{\beta-\omega'(k-\varepsilon\eta''/2)(\eta'+\eta'')/2\}}\\ &\times\frac{|\nu(k-\varepsilon\eta''/2)|^2}{\lambda/2+i\{\beta+\omega'(k-\varepsilon\eta''/2)(\eta'-\eta'')/2\}}\\ &-\frac{1}{\lambda/2-i\{\beta-\omega'(k)(\eta'+\eta'')/2\}}\\ &\times\frac{|\nu(k)|^2}{\lambda/2+i\{\beta+\omega'(k)(\eta'-\eta'')/2\}}\Bigg|.\end{aligned} \tag{7.61}$$

Using (7.23) we can estimate, for $(k,\eta',\eta'')\in L^c(\delta)\times B(\delta,\varepsilon)\times A'(\delta,\varepsilon)$, that

$$d_\varepsilon(k,\eta',\eta'')\preceq\frac{1}{1+(\eta'-\eta'')^2}+\frac{1}{1+(\eta'+\eta'')^2}. \tag{7.62}$$

As $d_\varepsilon(k,\eta',\eta'') \to 0$ pointwise, we can apply the dominated convergence theorem in (7.60), to get

$$\lim_{\varepsilon\to 0} \Delta'_\varepsilon = 0. \tag{7.63}$$

To estimate Δ''_ε, observe that (7.23) implies

$$d_\varepsilon(k,\eta',\eta'') \preceq \sum_{\iota=\pm} \left(d_\varepsilon^{1,\iota}(k,\eta',\eta'') + d_\varepsilon^{2,\iota}(k,\eta',\eta'') \right),$$

with

$$d_\varepsilon^{1,\iota}(k,\eta',\eta'') := \frac{|\nu(k-\varepsilon\eta''/2)|^2}{1+[\omega'(k-\varepsilon\eta''/2)(\eta'+\iota\eta'')]^2}, \quad d_\varepsilon^{2,\iota}(k,\eta',\eta'') := \frac{|\nu(k)|^2}{1+[\omega'(k)(\eta'+\iota\eta'')]^2}, \quad \iota\in\{-,+\}. \tag{7.64}$$

As $|\eta''|$ is larger than δ/ε on $A''(\delta,\varepsilon)$ and $|\omega'(k)|$ is bounded away from 0 on $L^c(\delta)$, the decay of $\varphi(\eta')$ allows us to apply the dominated convergence theorem, to obtain

$$\lim_{\varepsilon\to 0} \int_{L^c(\delta)\times\mathbb{R}\times B(\delta,\varepsilon)\times A''(\delta,\varepsilon)} d_\varepsilon^{2,\iota}(k,\eta',\eta'')\varphi(\eta') \frac{|\hat{G}^\star(\eta,k)|}{|\lambda+i\omega'(k)\eta|} \mathrm{d}k\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}\eta'' = 0, \quad \iota\in\{-,+\}. \tag{7.65}$$

For the terms $d_\varepsilon^{1,\iota}$, we consider only the case $\iota=+$, as the other case can be done similarly. Note that

$$B(\delta,\varepsilon)\times A''(\delta,\varepsilon) \subset A_1(\delta,\varepsilon) := [(\eta',\eta'')\in\mathbb{R}\times A''(\delta,\varepsilon) :\ |\eta'+\eta''| \ge |\eta''|/2].$$

Hence,

$$\int_{L^c(\delta)\times\mathbb{R}\times B(\delta,\varepsilon)\times A''(\delta,\varepsilon)} d_\varepsilon^{1}(k,\eta',\eta'')\varphi(\eta') \frac{|\hat{G}^\star(\eta,k)|}{|\lambda+i\omega'(k)\eta|} \mathrm{d}k\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}\eta'' \le \mathcal{D}_\varepsilon := \int_{L^c(\delta)\times\mathbb{R}\times A_1(\delta,\varepsilon)} d_\varepsilon^{1}(k,\eta',\eta'')\varphi(\eta') \frac{|\hat{G}^\star(\eta,k)|}{|\lambda+i\omega'(k)\eta|} \mathrm{d}k\mathrm{d}\eta\mathrm{d}\eta'\mathrm{d}\eta''. \tag{7.66}$$

For any $\kappa'\in(0,1)$ we can write

$$\begin{aligned}
\mathcal{D}_\varepsilon &\preceq \int_{L^c(\delta)\times A''(\delta,\varepsilon)} \frac{|\nu(k-\varepsilon\eta''/2)|^2}{1+[\omega'(k-\varepsilon\eta''/2)\eta'']^2} \mathrm{d}k\mathrm{d}\eta'' \\
&\preceq \int_{L^c(\delta)\times A''(\delta,\varepsilon)} \frac{|\nu(k-\varepsilon\eta''/2)|^2}{|\omega'(k-\varepsilon\eta''/2)|} \times \frac{\mathrm{d}k\mathrm{d}\eta''}{|\omega'(k-\varepsilon\eta''/2)|^{\kappa'}|\eta''|^{1+\kappa'}} \\
&\preceq \int_{L^c(\delta)\times A''(\delta,\varepsilon)} \frac{\mathrm{d}k\mathrm{d}\eta''}{|\omega'(k-\varepsilon\eta''/2)|^{\kappa'}|\eta''|^{1+\kappa'}} \\
&\preceq \int_{A''(\delta,\varepsilon)} \frac{\mathrm{d}\eta''}{|\eta''|^{1+\kappa'}} \preceq \varepsilon^{\kappa'} \to 0, \text{ as } \varepsilon\to 0.
\end{aligned} \tag{7.67}$$

We obtain from (7.65) and (7.67) and its analog for $\iota = -$ that

$$\lim_{\varepsilon\to 0} \Delta''_{\varepsilon} = 0, \tag{7.68}$$

which, together with (7.63) gives

$$\lim_{\varepsilon\to 0} \Delta_{\varepsilon} = 0. \tag{7.69}$$

We have shown that

$$\tilde{\mathcal{I}}^{(3)}(\lambda,\varepsilon) = -\frac{\gamma\lambda}{8\pi}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{U_\varepsilon}\frac{\widehat{W}_\varepsilon(\eta',k)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2-i\{\beta-\omega'(k)(\eta'+\eta'')/2\}} \times\frac{|\nu(k)|^2}{\lambda/2+i\{\beta+\omega'(k)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}+o(1). \tag{7.70}$$

Now, the dominated convergence theorem allows us to pass to the limit in the domains of integration in (7.58), leading to (7.14).

7.5. The End of the Proof of Lemma 6.1

As a result of Lemmas 7.1–7.4, together with (7.4), we know that

$$\mathcal{H}_+(\lambda,\varepsilon) = -\frac{\gamma\lambda}{8\pi}\sum_{\iota=\pm}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{\mathbb{T}\times\mathbb{R}^2}\frac{\widehat{W}_\varepsilon(\eta',\iota k)\mathrm{d}k\mathrm{d}\eta'\mathrm{d}\eta''}{\lambda/2-i\{\beta-\iota\omega'(k)(\eta'+\eta'')/2\}} \times\frac{|\nu(k)|^2}{\lambda/2+i\{\beta+\iota\omega'(k)(\eta'-\eta'')/2\}}\times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}+o(1), \tag{7.71}$$

as $\varepsilon \ll 1$. Recall the elementary formula: for $q_\pm \in \mathbb{C}$ such that $\operatorname{Im} q_+ > 0 > \operatorname{Im} q_-$ we have

$$\int_{\mathbb{R}}\frac{\mathrm{d}q}{(q-q_+)(q-q_-)} = \frac{2\pi i}{q_+-q_-}. \tag{7.72}$$

Performing the integral in the η'' variable in (7.71) we obtain

$$\mathcal{H}_+(\lambda,\varepsilon) = -\frac{\gamma\lambda}{2}\sum_{\iota=\pm}\int_{\mathbb{R}^2}\frac{\mathrm{d}\beta\mathrm{d}\eta}{(\lambda/2)^2+\beta^2}\int_{\mathbb{T}\times\mathbb{R}}\frac{|\nu(k)|^2\widehat{W}_\varepsilon(\eta',\iota k)\mathrm{d}k\mathrm{d}\eta'}{|\omega'(k)|[\lambda+\iota i\omega'(k)\eta']} \times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}+o(1). \tag{7.73}$$

Integrating out the β-variable we get (recall that $\bar{\omega}'(k) = \omega'(k)/(2\pi)$)

$$\mathcal{H}_+(\lambda,\varepsilon) = -\frac{\gamma}{2}\sum_{\iota=\pm}\int_{\mathbb{T}\times\mathbb{R}^2}\frac{|\nu(k)|^2\widehat{W}_\varepsilon(\eta',\iota k)}{|\bar{\omega}'(k)|[\lambda+\iota i\omega'(k)\eta']} \times\frac{\hat{G}^\star(\eta,k)}{\lambda+i\omega'(k)\eta}\mathrm{d}k\mathrm{d}\eta\mathrm{d}\eta'+o(1). \tag{7.74}$$

An analogous formula holds for $\mathcal{H}_-(\lambda,\varepsilon)$. Letting $\varepsilon \to 0$ we obtain (6.13), finishing the proof of Lemma 6.1.

8. Proof of Lemma 5.2: The Limit of $\mathfrak{L}^{\varepsilon}_{scat,22}(\lambda)$

We now turn to the computation that leads to (5.25) the second and final ingredient in Lemma 5.2:

$$\lim_{\varepsilon\to 0+} \mathfrak{L}^{\varepsilon}_{scat,22}(\lambda) = 0. \tag{8.1}$$

Observe that, as follows from (5.21) and (5.22), we have

$$\mathfrak{L}^{\varepsilon}_{scat,22}(\lambda) = -\frac{i\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}} [\mathrm{Im}\mathfrak{d}^2_{\varepsilon}(\lambda,k)]\Big[\frac{\hat{G}^*(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_{\varepsilon}\omega(k,\eta)} - \frac{\hat{G}^*(\eta,k-\varepsilon\eta/2)}{\lambda+i\delta^-_{\varepsilon}\omega(k,\eta)}\Big]\mathrm{d}\eta\mathrm{d}k \tag{8.2}$$

with

$$\mathfrak{d}^2_{\varepsilon}(\lambda,k) = -\gamma\varepsilon\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathrm{d}t\Big\{\int_0^t e^{i\omega(k)(t-s)}\big\langle g\star\mathfrak{p}^0_0(s) g\star\mathfrak{p}^0_0(t)\big\rangle_{\mu_\varepsilon}\Big\}\mathrm{d}s. \tag{8.3}$$

A lengthy calculation, similar to that at the beginning of Section 6, leads to an expression

$$\begin{aligned} i\mathrm{Im}\,\mathfrak{d}^2_{\varepsilon}(\lambda,k) = &-\frac{i\varepsilon\lambda\gamma\omega(k)}{4\pi}\int_{\mathbb{R}} \frac{\beta\,|\tilde{g}(\varepsilon\lambda/2-i\beta)|^2\mathrm{d}\beta}{\{(\varepsilon\lambda/2)^2+[\beta+\omega(k)]^2\}\{(\varepsilon\lambda/2)^2+[\beta-\omega(k)]^2\}} \\ &\times\int_{\mathbb{T}^2} \frac{\varepsilon\langle\hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}\mathrm{d}\ell\mathrm{d}\ell'}{\{\varepsilon\lambda/2-i[\beta-\omega(\ell)]\}\{\varepsilon\lambda/2+i[\beta-\omega(\ell')]\}}, \end{aligned} \tag{8.4}$$

hence

$$\begin{aligned} \mathfrak{L}^{\varepsilon}_{scat,22}(\lambda) = &\frac{i\varepsilon\lambda\gamma^2}{8\pi}\int_{\mathbb{R}^2\times\mathbb{T}^3}\left[\frac{\hat{G}^\star(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_{\varepsilon}\omega(k,\eta)} - \frac{\hat{G}^\star(\eta,k-\varepsilon\eta/2)}{\lambda+i\delta^-_{\varepsilon}\omega(k,\eta)}\right] \\ &\times\frac{\omega(k)\beta\,|\tilde{g}(\varepsilon\lambda/2-i\beta)|^2}{\{(\varepsilon\lambda/2)^2+[\beta+\omega(k)]^2\}\{(\varepsilon\lambda/2)^2+[\beta-\omega(k)]^2\}} \\ &\frac{\varepsilon\langle\hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}\mathrm{d}\beta\mathrm{d}\eta\mathrm{d}k\mathrm{d}\ell\mathrm{d}\ell'}{\{\varepsilon\lambda/2-i[\beta-\omega(\ell)]\}\{\varepsilon\lambda/2+i[\beta-\omega(\ell')]\}}. \end{aligned} \tag{8.5}$$

After the change of variables $\beta' := \varepsilon^{-1}\beta$, we get

$$\begin{aligned} \mathfrak{L}^{\varepsilon}_{scat,22}(\lambda) = &-\frac{\lambda\gamma^2}{8\pi\varepsilon}\int_{\mathbb{T}}\omega(k)\mathcal{G}_{\varepsilon}(k)\mathrm{d}k \\ &\int_{\mathbb{R}}\frac{\beta\,|\tilde{g}(\varepsilon\lambda/2-i\varepsilon\beta)|^2\mathrm{d}\beta}{\{(\lambda/2)^2+[\beta+\varepsilon^{-1}\omega(k)]^2\}\{(\lambda/2)^2+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \\ &\times\int_{T^2_{\varepsilon}}\frac{\varepsilon\langle\hat{\psi}(\ell)\hat{\psi}^*(\ell')\rangle_{\mu_\varepsilon}\mathrm{d}\ell\mathrm{d}\ell'}{\{\lambda/2-i[\beta-\varepsilon^{-1}\omega(\ell)]\}\{\lambda/2+i[\beta-\varepsilon^{-1}\omega(\ell')]\}}, \end{aligned} \tag{8.6}$$

with

$$\mathcal{G}_{\varepsilon}(k) := -i\int_{\mathbb{R}}\left[\frac{\hat{G}^\star(\eta,k+\varepsilon\eta/2)}{\lambda+i\delta^+_{\varepsilon}\omega(k,\eta)} - \frac{\hat{G}^\star(\eta,k-\varepsilon\eta/2)}{\lambda+i\delta^-_{\varepsilon}\omega(k,\eta)}\right]\mathrm{d}\eta$$

$$= \int_{\mathbb{R}} \frac{\hat{G}^\star(\eta, k+\varepsilon\eta/2)[2\omega(k) - \omega(k+\varepsilon\eta) - \omega(k-\varepsilon\eta)]}{\varepsilon\{\lambda^2 + [\delta_\varepsilon^+\omega(k,\eta)]^2\}} \mathrm{d}\eta. \quad (8.7)$$

Let us first assume that $\omega \in C^\infty(\mathbb{T})$. Then we can estimate

$$|\mathcal{G}_\varepsilon(k)| \preceq \varepsilon \|\omega''\|_\infty \int_{\mathbb{R}} \eta^2 \|\hat{G}^\star(\eta,\cdot)\|_\infty \mathrm{d}\eta \preceq \varepsilon, \quad (8.8)$$

while the last integral in the right side of (8.6) is bounded by

$$\frac{4\varepsilon}{\lambda^2}\left\langle \left[\int_{\mathbb{T}^2} |\hat{\psi}(\ell)|\mathrm{d}\ell\right]^2 \right\rangle_{\mu_\varepsilon} \le \frac{4\varepsilon}{\lambda^2}\left\langle \|\hat{\psi}\|^2_{L^2(\mathbb{T})}\right\rangle_{\mu_\varepsilon} \preceq 1. \quad (8.9)$$

Hence, we have

$$\begin{aligned}
|\mathfrak{L}^\varepsilon_{scat,22}(\lambda)| &\preceq \varepsilon \int_{\mathbb{T}} \mathrm{d}k \int_0^{+\infty} \frac{\varepsilon^{-1}\omega(k)\beta\,\mathrm{d}\beta}{\{1+[\beta+\varepsilon^{-1}\omega(k)]^2\}\{1+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \\
&= \varepsilon \int_{\mathbb{T}} \mathrm{d}k \int_0^{\omega(k)/\varepsilon} \frac{\varepsilon^{-1}\omega(k)\beta\,\mathrm{d}\beta}{\{1+[\beta+\varepsilon^{-1}\omega(k)]^2\}\{1+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \\
&\quad + \varepsilon \int_{\mathbb{T}} \mathrm{d}k \int_{\omega(k)/\varepsilon}^{+\infty} \frac{\varepsilon^{-1}\omega(k)\beta\,\mathrm{d}\beta}{\{1+[\beta+\varepsilon^{-1}\omega(k)]^2\}\{1+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \\
&= \varepsilon \int_{\mathbb{T}} \mathrm{d}k \int_0^{\omega(k)/\varepsilon} \frac{\varepsilon^{-1}\omega(k)\left(\varepsilon^{-1}\omega(k)-\beta\right)}{1+[2\varepsilon^{-1}\omega(k)-\beta]^2}\frac{\mathrm{d}\beta}{1+\beta^2} \\
&\quad + \varepsilon \int_{\mathbb{T}} \mathrm{d}k \int_0^{+\infty} \frac{\varepsilon^{-1}\omega(k)\left(\varepsilon^{-1}\omega(k)+\beta\right)}{1+[\beta+2\varepsilon^{-1}\omega(k)]^2}\frac{\mathrm{d}\beta}{1+\beta^2}. \quad (8.10)
\end{aligned}$$

Using the dominated convergence theorem, we conclude that

$$\lim_{\varepsilon\to 0} \mathfrak{L}^\varepsilon_{scat,22}(\lambda) = 0. \quad (8.11)$$

Finally, consider (8.6)–(8.7) when $\omega \in C^\infty(\mathbb{T}\setminus\{0\})$. Let $\sigma > 0$ be arbitrary, and take $A > 0$, to be chosen later. We can write

$$\mathfrak{L}^\varepsilon_{scat,22}(\lambda) = \mathfrak{L}^{\varepsilon,1}_{scat,22}(\lambda) + \mathfrak{L}^{\varepsilon,2}_{scat,22}(\lambda),$$

where the terms in the right hand side correspond to the integration over $[k : |k| \le A\varepsilon]$ and its complement. As ω is Lipschitz, we have

$$|\mathcal{G}_\varepsilon(k)| \preceq \int_{\mathbb{R}} |\eta| \|\hat{G}^\star(\eta,\cdot)\|_\infty \mathrm{d}\eta \preceq 1$$

Using (8.9) we write

$$\begin{aligned}
|\mathfrak{L}^{\varepsilon,1}_{scat,22}(\lambda)| &\preceq \int_{[|k|\le A\varepsilon]} \mathrm{d}k \int_0^{+\infty} \frac{\varepsilon^{-1}\omega(k)\beta\mathrm{d}\beta}{\{1+[\beta+\varepsilon^{-1}\omega(k)]^2\}\{1+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \\
&\le \int_{[|k|\le A\varepsilon]} \mathrm{d}k \int_0^{+\infty} \frac{\varepsilon^{-1}\omega(k)\beta\mathrm{d}\beta}{\{1+\varepsilon^{-1}\omega(k)\beta\}\{1+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \preceq A\varepsilon. \quad (8.12)
\end{aligned}$$

Finally, we write

$$\mathfrak{L}^{\varepsilon,2}_{scat,22}(\lambda) = \mathfrak{L}^{\varepsilon,21}_{scat,22}(\lambda) + \mathfrak{L}^{\varepsilon,22}_{scat,22}(\lambda),$$

corresponding to the partition of the integration domain in η into $[\eta : |\eta| < A/4]$ and its complement. In the first case, as $|k| > A\varepsilon$ and $|\eta| < A/4$, we can still use estimate (8.8), hence

$$\lim_{\varepsilon\to 0} \mathfrak{L}^{\varepsilon,21}_{scat,22}(\lambda) = 0.$$

In the other case, we can estimate

$$\begin{aligned}|\mathfrak{L}^{\varepsilon,22}_{scat,22}(\lambda)| &\preceq \int_{[|\eta|>A/4]} |\eta| \|\hat{G}^\star(\eta,\cdot)\|_\infty \mathrm{d}\eta \int_{[|k|>A\varepsilon]} \mathrm{d}k \\ &\quad \int_0^{+\infty} \frac{\varepsilon^{-1}\omega(k)\beta \mathrm{d}\beta}{\{1+\varepsilon^{-1}\omega(k)\beta\}\{1+[\beta-\varepsilon^{-1}\omega(k)]^2\}} \\ &\preceq \int_{[|\eta|>A/4]} |\eta| \|\hat{G}^\star(\eta,\cdot)\|_\infty \mathrm{d}\eta \le \sigma, \end{aligned} \tag{8.13}$$

provided that A is sufficiently large. This finishes the proof of (5.25), and that of Lemma 5.2 as well.

9. End of Proof of Theorem 2.1

In the present section we show Theorem 2.1 assuming that the Fourier-Wigner transform of the initial data satisfies (2.18) rather than the stronger assumption (5.1). Suppose that $\sigma > 0$ and $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ are arbitrary. Let us decompose the solution of (2.6) as

$$\hat{\psi}(t,k) = \hat{\psi}^1(t,k) + \hat{\psi}^2(t,k),$$

where

$$\begin{aligned} \mathrm{d}\hat{\psi}^1(t,k) &= \left\{ -i\omega(k)\hat{\psi}^1(t,k) - \frac{\gamma}{2i}\int_{\mathbb{T}} [\hat{\psi}^1(t,k') - (\hat{\psi}^1(t,k'))^\star] \mathrm{d}k' \right\} \mathrm{d}t \\ &\quad + i\sqrt{2\gamma T} \mathrm{d}w(t), \\ \hat{\psi}^1(0,k) &= \hat{\psi}(k)\chi_\delta(k) \end{aligned} \tag{9.1}$$

and

$$\begin{aligned} \frac{\mathrm{d}\hat{\psi}^2(t,k)}{\mathrm{d}t} &= -i\omega(k)\hat{\psi}^2(t,k) - \frac{\gamma}{2i}\int_{\mathbb{T}} \left[\hat{\psi}^2(t,k') - (\hat{\psi}^2(t,k'))^\star\right] \mathrm{d}k', \\ \hat{\psi}^2(0,k) &= \hat{\psi}(k)[1-\chi_\delta(k)], \end{aligned} \tag{9.2}$$

with $\chi_\delta \in C(\mathbb{T})$ such that $0 \le \chi \le 1$, $\chi_\delta \equiv 0$ on $L(\delta)$ (see (5.3)), $\chi_\delta \equiv 1$ on $L^c(2\delta)$ and δ chosen so small that

$$\limsup_{\varepsilon\to 0+} \varepsilon \mathbb{E}_\varepsilon \|\hat{\psi}(1-\chi_\delta)\|^2_{L^2(\mathbb{T})} < \sigma. \tag{9.3}$$

Let $\widehat{w}_\varepsilon(\lambda,\eta,k)$ and $\widehat{w}^1_\varepsilon(\lambda,\eta,k)$ be the Laplace transforms of the Fourier-Wigner functions corresponding to $\hat{\psi}(t,k)$ and $\hat{\psi}^1(t,k)$ via (2.11). Using estimates (2.14) and (9.3) we see that

$$\limsup_{\varepsilon\to 0+}\sup_{\eta\in\mathbb{T}_{2/\varepsilon}}\int_{\mathbb{T}}\left|\widehat{w}_\varepsilon(\lambda,\eta,k)-\widehat{w}^1_\varepsilon(\lambda,\eta,k)\right|\mathrm{d}k\preceq\sigma,\ \text{ for each }\lambda>0.$$

It follows, in particular, that

$$\limsup_{\varepsilon\to 0+}\left|\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}_\varepsilon(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k-\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}^1_\varepsilon(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k\right|\preceq\sigma. \tag{9.4}$$

In addition, the initial condition for $\hat{\psi}^1(t,k)$ satisfies assumption (I3') in (5.1). As we have already proved Theorem 2.1 under this hypothesis, we conclude that

$$\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}^1_\varepsilon(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k=\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}^1(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k, \tag{9.5}$$

with $\widehat{w}^1(\lambda,\eta,k)$ given by (3.18), but with $\widehat{W}_0(\eta,k)$ replaced by $\chi^2_\delta(k)\widehat{W}_0(\eta,k)$. Thus, for a sufficiently small $\delta>0$ we have

$$\left|\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}^1(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k-\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k\right|<\sigma. \tag{9.6}$$

We have thus shown that

$$\limsup_{\varepsilon\to 0+}\left|\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}_\varepsilon(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k-\int_{\mathbb{R}\times\mathbb{T}}\hat{G}^*(\eta,k)\widehat{w}(\lambda,\eta,k)\mathrm{d}\eta\mathrm{d}k\right|\preceq\sigma, \tag{9.7}$$

which ends the proof of Theorem 2.1.

10. The Properties of $\nu(k)$

In this section, we prove relation (2.29). The function

$$\nu(k):=\lim_{\varepsilon\to 0}\tilde{g}(\varepsilon-i\omega(k))$$

can be determined from the identity

$$\nu(k)\left(1+\gamma\lim_{\varepsilon\to 0}\tilde{J}(\varepsilon-i\omega(k))\right)=1.$$

Recalling (2.24), we write

$$\lim_{\varepsilon\to 0}\tilde{J}(\varepsilon - i\omega(k)) = \lim_{\varepsilon\to 0}\int_{\mathbb{T}}\frac{(\varepsilon - i\omega(k))\mathrm{d}\ell}{(\varepsilon - i\omega(k))^2+\omega^2(\ell)} = \frac{1}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\frac{\mathrm{d}\ell}{\varepsilon - i\omega(k)+i\omega(\ell)}$$
$$+\frac{1}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\frac{\mathrm{d}\ell}{\varepsilon - i\omega(k)-i\omega(\ell)} = \frac{i}{2}\int_{\mathbb{T}}\frac{\mathrm{d}\ell}{\omega(k)+\omega(\ell)}$$
$$+\frac{i}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\frac{\mathrm{d}\ell}{i\varepsilon+\omega(k)-\omega(\ell)}.$$

Let us set

$$G(u) := \frac{1}{2}\int_{\mathbb{T}}\frac{\mathrm{d}\ell}{u+\omega(\ell)} = \int_0^{1/2}\frac{\mathrm{d}\ell}{u+\omega(\ell)} = \int_{\omega_{min}}^{\omega_{max}}\frac{\mathrm{d}v}{|\omega'(\omega_+^{-1}(v))|(u+v)},$$

and

$$H(u) := \frac{1}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\frac{\mathrm{d}\ell}{i\varepsilon+u-\omega(\ell)} = \lim_{\varepsilon\to 0}\int_{\omega_{min}}^{\omega_{max}}\frac{\mathrm{d}v}{|\omega'(\omega_+^{-1}(v))|(i\varepsilon+u-v)}, \quad \text{a.e.}$$

so that

$$\nu(k) = \frac{1}{1+i\gamma[G(\omega(k))+H(\omega(k))]}. \tag{10.1}$$

In our situation, with $u=\omega(k)\in(\omega_{min},\omega_{max})$, we have

$$H(\omega(k)) = H^r(\omega(k)) + iH^i(\omega(k)), \tag{10.2}$$

with $H^r(u)$, $H^i(u)$ real valued functions equal

$$H^r(u) := \lim_{\varepsilon\to 0}\int_{\omega_{min}}^{\omega_{max}}\frac{(u-v)\mathrm{d}v}{|\omega'(\omega_+^{-1}(v))|[\varepsilon^2+(u-v)^2]} \tag{10.3}$$

and

$$H^i(u) := -\lim_{\varepsilon\to 0}\int_{\omega_{min}}^{\omega_{max}}\frac{\varepsilon\mathrm{d}v}{|\omega'(\omega_+^{-1}(v))|[\varepsilon^2+(u-v)^2]}. \tag{10.4}$$

Both limits exist for all $v\in\mathbb{R}\setminus\{\omega_{min},\omega_{max}\}$. After an elementary calculation we obtain

$$H^i(u) = -\frac{\pi}{|\omega'(\omega_+^{-1}(u))|}. \tag{10.5}$$

Substituting into (10.1) immediately gives

$$\mathrm{Re}\,\nu(k) = \Big(1+\frac{\pi\gamma}{|\omega'(k)|}\Big)|\nu(k)|^2, \tag{10.6}$$

which is (2.29).

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