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Weak Solutions for Navier–Stokes Equations with Initial Data in Weighted L^2 Spaces

Pedro Gabriel Fernández-Dalgo &
Pierre Gilles Lemarié-Rieusset

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Abstract

We show the existence of global weak solutions to the three dimensional Navier–Stokes equations with initial velocity in the weighted spaces $L^2_{w_\gamma}$, where $w_\gamma(x) = (1 + |x|)^{-\gamma}$ and $0 < \gamma \leqq 2$, using new energy controls. As an application we give a new proof of the existence of global weak discretely self-similar solutions to the three dimensional Navier–Stokes equations for discretely self-similar initial velocities which are locally square integrable.

1. Introduction

Infinite-energy weak Leray solutions to the Navier–Stokes equations were introduced by Lemarié-Rieusset in 1999 [8] (they are presented more completely in [9] and [10]). This has allowed demonstration of the existence of local weak solutions for a uniformly locally square integrable initial data.

Other constructions of infinite-energy solutions for locally uniformly square integrable initial data were given in 2006 by Basson [1] and in 2007 by Kikuchi and Seregin [7]. These solutions allowed Jia and Sverak [6] to construct in 2014 the self-similar solutions for large (homogeneous of degree -1) smooth data. Their result has been extended in 2016 by Lemarié-Rieusset [10] to solutions for rough locally square integrable data. We remark that an homogeneous (of degree -1) and locally square integrable data is automatically uniformly locally L^2.

Recently, Bradshaw and Tsai [2] and Chae and Wolf [3] considered the case of solutions which are self-similar according to a discrete subgroup of dilations. Those solutions are related to an initial data which is self-similar only for a discrete group of dilations; in contrast to the case of self-similar solutions for all dilations, such initial data, when locally L^2, is not necessarily uniformly locally L^2, therefore their results are no consequence of constructions described by Lemarié-Rieusset in [10].

In this paper, we construct an alternative theory to obtain infinite-energy global weak solutions for large initial data, which include the discretely self-similar locally square integrable data. More specifically, we consider the weights

$$w_\gamma(x) = \frac{1}{(1+|x|)^\gamma}$$

with $0 < \gamma$, and the spaces

$$L^2_{w_\gamma} = L^2(w_\gamma \, \mathrm{d}x).$$

Our main theorem is the following one:

Theorem 1. *Let $0 < \gamma \leqq 2$. If $\mathbf{u}_0$ is a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and if $\mathbb{F}$ is a tensor $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ such that $\mathbb{F} \in L^2((0,+\infty), L^2_{w_\gamma})$, then the Navier–Stokes equations with initial value $\mathbf{u}_0$*

$$(NS)\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

have a global weak solution $\mathbf{u}$ such that:

- *for every $0 < T < +\infty$, $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *the pressure p is related to $\mathbf{u}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$p = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (u_i u_j - F_{i,j})$$

  where, for every $0 < T < +\infty$, $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (u_i u_j)$ belongs to $L^4((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j F_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$
- *the map $t \in [0,+\infty) \mapsto \mathbf{u}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t\to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

- *the solution $\mathbf{u}$ is suitable: there exists a non-negative locally finite measure μ on $(0,+\infty)\times\mathbb{R}^3$ such that*

$$\partial_t\left(\frac{|\mathbf{u}|^2}{2}\right) = \Delta\left(\frac{|\mathbf{u}|^2}{2}\right) - |\nabla\mathbf{u}|^2 - \nabla\cdot\left(\left(\frac{|\mathbf{u}|^2}{2} + p\right)\mathbf{u}\right) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

In particular, we have the energy controls

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} + 2\int_0^t \|\nabla \mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s$$
$$\leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} - \int_0^t\int \nabla|\mathbf{u}|^2\cdot\nabla w_\gamma\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int (|\mathbf{u}|^2+2p)\mathbf{u}\cdot\nabla(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s$$
$$-2\sum_{i=1}^3\sum_{j=1}^3\int_0^t\int F_{i,j}(\partial_i u_j)w_\gamma + F_{i,j}u_i\partial_j(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s$$

and

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} \leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma\int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s + C_\gamma\int_0^t \|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}} + \|\mathbf{u}(s,.)\|^6_{L^2_{w_\gamma}}\,\mathrm{d}s$$

Remark. We use the following notations: the vector $\mathbf{u}$ is given by its coordinates $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. The operator $\mathbf{u}\cdot\nabla$ is the differential operator $\mathbf{u}_1\partial_1 + \mathbf{u}_2\partial_2 + \mathbf{u}_3\partial_3$. Thus, $\nabla\cdot(f\mathbf{u}) = f\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla f$.

For $\mathbb{F} = (F_{i,j})$, we write $\nabla\cdot\mathbb{F}$ for the vector $(\sum_{i=1}^3 \partial_i F_{i,1}, \sum_{i=1}^3 \partial_i F_{i,2}, \sum_{i=1}^3 \partial_i F_{i,3})$.

For the vector fields $\mathbf{b}$ and $\mathbf{u}$, we define $\mathbf{b}\otimes\mathbf{u}$ as $(b_i u_j)_{1\leqq i\leqq 3, 1\leqq j\leqq 3}$. Thus, if $\mathbf{b}$ is divergence free (that is if $\nabla\cdot\mathbf{b} = 0$) we have $\nabla\cdot(\mathbf{b}\otimes\mathbf{u}) = (\mathbf{b}\cdot\nabla)\mathbf{u}$.

A key tool for proving Theorem 1 and for applying it to the study of discretely self-similar solutions is given by the following a priori estimates for an advection-diffusion problem:

Theorem 2. *Let* $0 < \gamma \leqq 2$. *Let* $0 < T < +\infty$. *Let* $\mathbf{u}_0$ *be a divergence-free vector field such that* $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ *and* $\mathbb{F}$ *be a tensor* $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ *such that* $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. *Let* $\mathbf{b}$ *be a time-dependent divergence free vector-field (*$\nabla\cdot\mathbf{b} = 0$*) such that* $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.

Let $\mathbf{u}$ *be a solution of the following advection-diffusion problem:*

$$(AD)\begin{cases}\partial_t\mathbf{u} = \Delta\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F}\\ \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0\end{cases}$$

such that

- $\mathbf{u}$ *belongs to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla\mathbf{u}$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$*;*
- *the pressure* p *is related to* $\mathbf{u}$, $\mathbf{b}$ *and* $\mathbb{F}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formula*

$$p = \sum_{i=1}^3\sum_{j=1}^3 R_iR_j(b_iu_j - F_{i,j})$$

where $\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(b_iu_j)$ *belongs to* $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$ *and* $\sum_{i=1}^3\sum_{j=1}^3 R_iR_jF_{i,j}$ *belongs to* $L^2((0,T),L^2_{w_\gamma})$*;*

- *the map* $t\in[0,T)\mapsto \mathbf{u}(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t=0$ *:*

$$\lim_{t\to 0}\|\mathbf{u}(t,.)-\mathbf{u}_0\|_{L^2_{w_\gamma}}=0;$$

- *there exists a non-negative locally finite measure* μ *on* $(0,T)\times\mathbb{R}^3$ *such that*

$$\partial_t\left(\frac{|\mathbf{u}|^2}{2}\right)=\Delta\left(\frac{|\mathbf{u}|^2}{2}\right)-|\nabla\mathbf{u}|^2-\nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right)-\nabla\cdot(p\mathbf{u})+\mathbf{u}\cdot(\nabla\cdot\mathbb{F})-\mu. \quad (1)$$

Then, we have the energy controls

$$\begin{aligned}&\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}}+2\int_0^t\|\nabla\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\\&\quad\leqq\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}-\int_0^t\int\nabla|\mathbf{u}|^2\cdot\nabla w_\gamma\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int|\mathbf{u}|^2\mathbf{b}\cdot\nabla(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s\\&\qquad+2\int_0^t\int p\mathbf{u}\cdot\nabla(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s-2\sum_{i=1}^3\sum_{j=1}^3\int_0^t\int F_{i,j}(\partial_iu_j)w_\gamma\\&\qquad+F_{i,j}u_i\partial_j(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}}+\int_0^t\|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\\&\quad\leqq\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+C_\gamma\int_0^t\|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\\&\qquad+C_\gamma\int_0^t(1+\|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}})\|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s,\end{aligned}$$

where C_γ *depends only on* γ *(and not on* T*, and not on* $\mathbf{b}$*,* $\mathbf{u}$*,* $\mathbf{u}_0$ *nor* $\mathbb{F}$*).*

In particular, we shall prove the following stability result:

Theorem 3. *Let* $0<\gamma\leqq 2$*. Let* $0<T<+\infty$*. Let* $\mathbf{u}_{0,n}$ *be divergence-free vector fields such that* $\mathbf{u}_{0,n}\in L^2_{w_\gamma}(\mathbb{R}^3)$ *and* $\mathbb{F}_n$ *be tensors such that* $\mathbb{F}_n\in L^2((0,T),L^2_{w_\gamma})$*. Let* $\mathbf{b}_n$ *be time-dependent divergence free vector-fields such that* $\mathbf{b}_n\in L^3((0,T),L^3_{w_{3\gamma/2}})$*.*

Let $\mathbf{u}_n$ *be solutions of the advection-diffusion problems*

$$(AD_n)\begin{cases}\partial_t\mathbf{u}_n=\Delta\mathbf{u}_n-(\mathbf{b}_n\cdot\nabla)\mathbf{u}_n-\nabla p_n+\nabla\cdot\mathbb{F}_n\\ \nabla\cdot\mathbf{u}_n=0,\qquad\mathbf{u}_n(0,.)=\mathbf{u}_{0,n}\end{cases}$$

such that

- $\mathbf{u}_n$ *belongs to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla \mathbf{u}_n$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$*;*
- *the pressure* p_n *is related to* $\mathbf{u}_n$*,* $\mathbf{b}_n$ *and* $\mathbb{F}_n$ *by the formula*

$$p_n = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_{n,i} u_{n,j} - F_{n,i,j});$$

- *the map* $t \in [0,T) \mapsto \mathbf{u}_n(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$*:*

$$\lim_{t\to 0} \|\mathbf{u}_n(t,.) - \mathbf{u}_{0,n}\|_{L^2_{w_\gamma}} = 0.$$

- *there exists a non-negative locally finite measure* μ_n *on* $(0,T) \times \mathbb{R}^3$ *such that*

$$\partial_t \left(\frac{|\mathbf{u}_n|^2}{2}\right) = \Delta\left(\frac{|\mathbf{u}_n|^2}{2}\right) - |\nabla \mathbf{u}_n|^2 - \nabla\cdot\left(\frac{|\mathbf{u}_n|^2}{2}\mathbf{b}_n\right)$$
$$-\nabla\cdot(p_n \mathbf{u}_n) + \mathbf{u}_n \cdot (\nabla\cdot\mathbb{F}_n) - \mu_n;$$

If $\mathbf{u}_{0,n}$ *is strongly convergent to* $\mathbf{u}_{0,\infty}$ *in* $L^2_{w_\gamma}$*, if the sequence* $\mathbb{F}_n$ *is strongly convergent to* $\mathbb{F}_\infty$ *in* $L^2((0,T), L^2_{w_\gamma})$*, and if the sequence* $\mathbf{b}_n$ *is bounded in* $L^3((0,T), L^3_{w_{3\gamma/2}})$*, then there exists* p_∞*,* $\mathbf{u}_\infty$*,* $\mathbf{b}_\infty$ *and an increasing sequence* $(n_k)_{k\in\mathbb{N}}$ *with values in* $\mathbb{N}$ *such that*

- $\mathbf{u}_{n_k}$ *converges *-weakly to* $\mathbf{u}_\infty$ *in* $L^\infty((0,T), L^2_{w_\gamma})$*,* $\nabla\mathbf{u}_{n_k}$ *converges weakly to* $\nabla\mathbf{u}_\infty$ *in* $L^2((0,T), L^2_{w_\gamma})$*;*
- $\mathbf{b}_{n_k}$ *converges weakly to* $\mathbf{b}_\infty$ *in* $L^3((0,T), L^3_{w_{3\gamma/2}})$*,* p_{n_k} *converges weakly to* p_∞ *in* $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$*;*
- $\mathbf{u}_{n_k}$ *converges strongly to* $\mathbf{u}_\infty$ *in* $L^2_{\text{loc}}([0,T)\times\mathbb{R}^3)$ *such that for every* $T_0 \in (0,T)$ *and every* $R > 0$*, we have*

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^2 \, ds \, dy = 0.$$

Moreover, $\mathbf{u}_\infty$ *is a solution of the advection-diffusion problem*

$$(AD_\infty) \begin{cases} \partial_t \mathbf{u}_\infty = \Delta\mathbf{u}_\infty - (\mathbf{b}_\infty\cdot\nabla)\mathbf{u}_\infty - \nabla p_\infty + \nabla\cdot\mathbb{F}_\infty \\ \\ \nabla\cdot\mathbf{u}_\infty = 0, \qquad \mathbf{u}_\infty(0,.) = \mathbf{u}_{0,\infty} \end{cases}$$

and is such that

- *the map* $t \in [0,T) \mapsto \mathbf{u}_\infty(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$ *:*

$$\lim_{t\to 0} \|\mathbf{u}_\infty(t,.) - \mathbf{u}_{0,\infty}\|_{L^2_{w_\gamma}} = 0;$$

- *there exists a non-negative locally finite measure* μ_∞ *on* $(0,T)\times\mathbb{R}^3$ *such that*

$$\partial_t \left(\frac{|\mathbf{u}_\infty|^2}{2}\right) = \Delta\left(\frac{|\mathbf{u}_\infty|^2}{2}\right) - |\nabla \mathbf{u}_\infty|^2 - \nabla\cdot\left(\frac{|\mathbf{u}_\infty|^2}{2}\mathbf{b}_\infty\right)$$
$$-\nabla\cdot(p_\infty \mathbf{u}_\infty) + \mathbf{u}_\infty \cdot (\nabla\cdot\mathbb{F}_\infty) - \mu_\infty.$$

Notations

Throughout the text, C_γ is a positive constant whose value may change from line to line but which depends only on γ.

2. The Weights w_δ

We consider the weights $w_\delta = \frac{1}{(1+|x|)^\delta}$ where $0 < \delta$ and $x \in \mathbb{R}^3$. A very important feature of those weights is the control of their gradients:

$$|\nabla w_\delta(x)| = \delta \frac{w_\delta(x)}{1+|x|} \tag{2}$$

From this control, we can infer the following Sobolev embedding:

Lemma 1. (Sobolev embeddings) *Let $\delta > 0$. If $f \in L^2_{w_\delta}$ and $\nabla f \in L^2_{w_\delta}$ then $f \in L^6_{w_{3\delta}}$ and*

$$\|f\|_{L^6_{w_{3\delta}}} \leqq C_\delta(\|f\|_{L^2_{w_\delta}} + \|\nabla f\|_{L^2_{w_\delta}}).$$

Proof. Since both f and $w_{\delta/2}$ are locally in H^1, we write

$$\partial_i(f w_{\delta/2}) = w_{\delta/2}\partial_i f + f\partial_i(w_{\delta/2}) = w_{\delta/2}\partial_i f - \frac{\delta}{2}\frac{x_i}{|x|}\frac{1}{1+|x|} w_{\delta/2} f,$$

and thus

$$\|w_{\delta/2} f\|_2^2 + \|\nabla(w_{\delta/2} f)\|_2^2 \leqq \left(1 + \frac{\delta^2}{2}\right) \|w_{\delta/2} f\|_2^2 + 2\|w_{\delta/2}\nabla f\|_2^2.$$

Thus, $w_{\delta/2} f$ belongs to L^6 (since $H^1 \subset L^6$), or equivalently $f \in L^6_{w_{3\delta}}$. □

We shall mainly be interested in the case $\delta \leqq 2$. An important property for $0 < \delta < 3$ is

Lemma 2. (Muckenhoupt weights) *If $0 < \delta < 3$ and $1 < p < +\infty$, then w_δ belongs to the Muckenhoupt class $\mathcal{A}_p$.*

Proof. We recall that a weight w belongs to $\mathcal{A}_p(\mathbb{R}^3)$ for $1 < p < +\infty$ if and only if it satisfies the reverse Hölder inequality

$$\sup_{x\in\mathbb{R}^3, R>0} \left(\frac{1}{|B(x,R)|}\int_{B(x,R)} w(y)\,\mathrm{d}y\right)^{\frac{1}{p}} \left(\frac{1}{|B(x,R)|}\int_{B(x,R)} \frac{\mathrm{d}y}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}} < +\infty. \tag{3}$$

For all $0 < R \leqq 1$ the inequality $|x-y| < R$ implies $\frac{1}{2}(1+|x|) \leqq 1+|y| \leqq 2(1+|x|)$, thus we can control the left side in (3) for w_δ by $4^{\frac{\delta}{p}}$.

For all $R > 1$ and $|x| > 10R$, we have that the inequality $|x - y| < R$ implies $\frac{9}{10}(1+|x|) \leqq 1+|y| \leqq \frac{11}{10}(1+|x|)$, thus we can control the left side in (3) for w_δ by $(\frac{11}{9})^{\frac{\delta}{p}}$.

Finally, for $R > 1$ and $|x| \leqq 10R$, we write

$$\begin{aligned}
&\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} w(y)\,\mathrm{d}y\right)^{\frac{1}{p}} \left(\frac{1}{|B(x,R)|}\int_{B(0,R)} \frac{\mathrm{d}y}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}} \\
&\leqq \left(\frac{1}{|B(0,R)|}\int_{B(x,11\,R)} w(y)\,\mathrm{d}y\right)^{\frac{1}{p}} \left(\frac{1}{|B(0,R)|}\int_{B(0,11\,R)} \frac{\mathrm{d}y}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}} \\
&= \left(\frac{1}{R^3}\int_0^{11\,R} r^2 \frac{dr}{(1+r)^\delta}\right)^{\frac{1}{p}} \left(\frac{1}{R^3}\int_0^{11\,R} r^2 (1+r)^{\frac{\delta}{p-1}}\,dr\right)^{1-\frac{1}{p}} \\
&\leqq c_{\delta,p}\left(\frac{1}{R^3}\int_0^{11\,R} r^2 \frac{dr}{r^\delta}\right)^{\frac{1}{p}} \left(\left(\frac{1}{R^3}\int_0^{11\,R} r^2 dr\right)^{1-\frac{1}{p}}\right. \\
&\quad \left. + \left(\frac{1}{R^3}\int_0^{11\,R} r^{2+\frac{\delta}{p-1}}\,dr\right)^{1-\frac{1}{p}}\right) \\
&= c_{\delta,p}\frac{11^3}{(3-\delta)^{\frac{1}{p}}}\left(\frac{(11R)^{-\frac{\delta}{p}}}{3^{1-\frac{1}{p}}} + \frac{1}{(3+\frac{\delta}{p-1})^{1-\frac{1}{p}}}\right).
\end{aligned}$$

The lemma is proved. □

Lemma 3. *If $0 < \delta < 3$ and $1 < p < +\infty$, then the Riesz transforms R_i and the Hardy–Littlewood maximal function operator are bounded on $L^p_{w_\delta} = L^p(w_\delta(x)\,\mathrm{d}x)$:*

$$\|R_j f\|_{L^p_{w_\delta}} \leqq C_{p,\delta}\|f\|_{L^p_{w_\delta}} \text{ and } \|\mathcal{M}_f\|_{L^p_{w_\delta}} \leqq C_{p,\delta}\|f\|_{L^p_{w_\delta}}.$$

Proof. The boundedness of the Riesz transforms or of the Hardy–Littlewwod maximal function on $L^p(w_\gamma\,\mathrm{d}x)$ are basic properties of the Muckenhoupt class $\mathcal{A}_p$ [5]. □

We will use strategically the next corollary, which is specially useful to obtain discretely self-similar solutions.

Corollary 1. *(Non-increasing kernels) Let $\theta \in L^1(\mathbb{R}^3)$ be a non-negative radial function which is radially non-increasing. Then, if $0 < \delta < 3$ and $1 < p < +\infty$, we have, for $f \in L^p_{w_\delta}$, the inequality*

$$\|\theta * f\|_{L^p_{w_\delta}} \leqq C_{p,\delta}\|f\|_{L^p_{w_\delta}}\|\theta\|_1.$$

Proof. We have the well-known inequality for radial non-increasing kernels [4]

$$|\theta * f(x)| \leqq \|\theta\|_1 \mathcal{M}_f(x)$$

so that we may conclude with Lemma 3. □

We illustrate the utility of Lemma 3 with the following corollaries:

Corollary 2. *Let $0 < \gamma < \frac{5}{2}$ and $0 < T < +\infty$. Let $\mathbb{F}$ be a tensor $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ such that $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let $\mathbf{b}$ be a time-dependent divergence free vector-field ($\nabla\cdot\mathbf{b} = 0$) such that $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.*

Let $\mathbf{u}$ be a solution of the following advection-diffusion problem:

$$\begin{cases} \partial_t \mathbf{u} = \Delta\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla q + \nabla\cdot\mathbb{F} \\ \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{4}$$

such that $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$, and the pressure q belongs to $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

Then, the gradient of the pressure ∇q is necessarily related to $\mathbf{u}$, $\mathbf{b}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$\nabla q = \nabla\left(\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(b_iu_j - F_{i,j})\right)$$

and $\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(b_iu_j)$ belongs to $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3\sum_{j=1}^3 R_iR_jF_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$.

Proof. We define

$$p = \left(\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(b_iu_j - F_{i,j})\right).$$

As $0 < \gamma < \frac{5}{2}$ we can use Lemma 3 to obtain $\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(b_iu_j)$ belongs to $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3\sum_{j=1}^3 R_iR_jF_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$.

Taking the divergence in (4), we obtain $\Delta(q-p) = 0$. We take a test function $\alpha \in \mathcal{D}(\mathbb{R})$ such that $\alpha(t) = 0$ for all $|t| \geqq \varepsilon$, and a test function $\beta \in \mathcal{D}(\mathbb{R}^3)$; then the distribution $\nabla q * (\alpha\otimes\beta)$ is well defined on $(\varepsilon, T-\varepsilon)\times\mathbb{R}^3$.

We fix $t \in (\varepsilon, T-\varepsilon)$ and define

$$A_{\alpha,\beta,t} = (\nabla q * (\alpha\otimes\beta) - \nabla p * (\alpha\otimes\beta))(t,.).$$

We have

$$\begin{aligned} A_{\alpha,\beta,t} =& (\mathbf{u} * (-\partial_t\alpha\otimes\beta + \alpha\otimes\Delta\beta) + (-\mathbf{u}\otimes\mathbf{b} + \mathbb{F})\cdot(\alpha\otimes\nabla\beta))(t,.) \\ & - (p * (\alpha\otimes\nabla\beta))(t,.). \end{aligned} \tag{5}$$

Convolution with a function in $\mathcal{D}(\mathbb{R}^3)$ is a bounded operator on $L^2_{w_\gamma}$ and on $L^{6/5}_{w_{6\gamma/5}}$ (as, for $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have $|f*\varphi| \leqq C_\varphi \mathcal{M}_f$). Thus, we may conclude from (5) that $A_{\alpha,\beta,t} \in L^2_{w_\gamma} + L^{6/5}_{w_{6\gamma/5}}$. If $\max\{\gamma, \frac{\gamma+2}{2}\} < \delta < 5/2$, we have $A_{\alpha,\beta,t} \in L^{6/5}_{w_{6\delta/5}}$.

In particular, $A_{\alpha,\beta,t}$ is a tempered distribution. As we have

$$\Delta A_{\alpha,\beta,t} = (\alpha \otimes \beta) * (\nabla \Delta (q - p))(t, .) = 0,$$

we find that $A_{\alpha,\beta,t}$ is a polynomial. We remark that for all $1 < r < +\infty$ and $0 < \delta < 3$, $L^r_{w_\delta}$ does not contain non-trivial polynomials. Thus, $A_{\alpha,\beta,t} = 0$. We then use an approximation of identity $\frac{1}{\varepsilon^4}\alpha(\frac{t}{\varepsilon})\beta(\frac{x}{\varepsilon})$ and conclude that $\nabla(q - p) = 0$. □

Actually, we can answer a question posed by Bradshaw and Tsai in [2] about the nature of the pressure for self-similar solutions of the Navier–Stokes equations. In effect, we have the next corollary.

Corollary 3. *Let* $1 < \gamma < \frac{5}{2}$ *and* $0 < T < +\infty$. *Let* $\mathbb{F}$ *be a tensor* $\mathbb{F}(t, x) = \big(F_{i,j}(t, x)\big)_{1 \leqq i,j \leqq 3}$ *such that* $\mathbb{F} \in L^2((0, T), L^2_{w_\gamma})$.

Let $\mathbf{u}$ *be a solution of the following problem:*

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla q + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

such that $\mathbf{u}$ *belongs to* $L^\infty([0, +\infty), L^2)_{loc}$ *and* $\nabla \mathbf{u}$ *belongs to* $L^2([0, +\infty), L^2)_{loc}$, *and the pressure* q *is in* $\mathcal{D}'((0, T) \times \mathbb{R}^3)$.

We suppose that there exists $\lambda > 1$ *such that* $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t, x)$ *and* $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t, x)$. *Then, the gradient of the pressure* ∇q *is necessarily related to* $\mathbf{u}$ *and* $\mathbb{F}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formula*

$$\nabla q = \nabla \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (u_i u_j - F_{i,j}) \right)$$

and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (u_i u_j)$ *belongs to* $L^4((0, T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ *and* $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j F_{i,j}$ *belongs to* $L^2((0, T), L^2_{w_\gamma})$.

Proof. We shall use Corollary 2, and thus we need to show that $\mathbf{u}$ belongs to $L^\infty((0, T), L^2_{w_\gamma} \cap L^3((0, T), L^3_{3\gamma/2}))$ and $\nabla \mathbf{u}$ belongs to $L^2((0, T), L^2_{w_\gamma})$. In fact,

$$\|u\|_{L^\infty((0,T),L^2_{w_\gamma})} \leqq \sup_{0 \leqq t \leqq T} \int_{|x|<1} |\mathbf{u}(t, x)|^2 \, \mathrm{d}x + c \sup_{0 \leqq t \leqq T} \sum_{k \in \mathbb{N}} \int_{\lambda^{k-1}<|x|<\lambda^k} \frac{|\mathbf{u}(t, x)|^2}{\lambda^{\gamma k}} \, \mathrm{d}x$$

and

$$\sup_{0 \leqq t \leqq T} \sum_{k \geqq 1} \int_{\lambda^{k-1}<|x|<\lambda^k} \frac{|\mathbf{u}(t, x)|^2}{\lambda^{\gamma k}} \, \mathrm{d}x$$

$$\leqq \sup_{0\leqq t\leqq T} \sum_{k\in\mathbb{N}} \lambda^{(1-\gamma)k} \int_{\lambda^{-1}<|x|<1} |\mathbf{u}(\frac{t}{\lambda^{2k}}, x)|^2 \, \mathrm{d}x$$
$$\leqq c \sup_{0\leqq t\leqq T} \int_{\lambda^{-1}<|x|<1} |\mathbf{u}(t,x)|^2 \, \mathrm{d}x < +\infty.$$

For $\nabla\mathbf{u}$, we compute for $k \in \mathbb{N}$,

$$\int_0^T \int_{\lambda^{k-1}<|x|<\lambda^k} |\nabla\mathbf{u}(t,x)|^2 \, \mathrm{d}t \, \mathrm{d}x = \lambda^k \int_0^{\frac{T}{\lambda^{2k}}} \int_{\frac{1}{\lambda}<|x|<1} |\nabla\mathbf{u}(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

We may conclude that $\nabla\mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$, since for $\gamma > 1$ we have $\sum_{k\in\mathbb{N}} \lambda^{(1-\gamma)k} < +\infty$.

Now, we use the Sobolev embedding described in Lemma 1 to get that $\mathbf{u}$ belongs to $L^2((0,T), L^6_{w_{3\gamma}})$, and thus (by interpolation with $L^\infty((0,T), L^2_{w_\gamma})$) to $L^4((0,T), L^3_{w_{3\gamma/2}})$.

In particular, $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (u_i u_j)$ belongs to $L^4((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$, since we have

$$\|(\mathbf{u}\otimes\mathbf{u})w_\gamma\|_{L^{6/5}} \leqq \|\sqrt{w_\gamma}\mathbf{u}\|_{L^2} \|\sqrt{w_\gamma}\mathbf{u}\|_{L^3} \leqq \|\sqrt{w_\gamma}\mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\sqrt{w_\gamma}\mathbf{u}\|_{L^6}^{\frac{1}{2}}.$$

□

3. A Priori Estimates for the Advection-Diffusion Problem

3.1. Proof of Theorem 2

Let $0 < t_0 < t_1 < T$. We take a function $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ which is non-decreasing, with $\alpha(t)$ equal to 0 for $t < 1/2$ and equal to 1 for $t > 1$. For $0 < \eta < \min(\frac{t_0}{2}, T - t_1)$, we define

$$\alpha_{\eta,t_0,t_1}(t) = \alpha\Big(\frac{t-t_0}{\eta}\Big) - \alpha\Big(\frac{t-t_1}{\eta}\Big).$$

We take as well a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equal to 1 for $|x| \leqq 1$ and to 0 for $|x| \geqq 2$. For $R > 0$, we define $\phi_R(x) = \phi(\frac{x}{R})$. Finally, we define, for $\varepsilon > 0$, $w_{\gamma,\varepsilon} = \Big(1 + \sqrt{\varepsilon^2 + |x|^2}\Big)^{-\gamma}$. We have $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\varepsilon}(x) \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ and $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\varepsilon}(x) \geqq 0$. Thus, using the local energy balance (1) and the fact that $\mu \geqq 0$, we find

$$-\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\varepsilon} \, \mathrm{d}x \, \mathrm{d}s$$
$$\leqq -\sum_{i=1}^3 \iint \partial_i \mathbf{u}\cdot\mathbf{u} \, \alpha_{\eta,t_0,t_1} (w_{\gamma,\varepsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \, \mathrm{d}x \, \mathrm{d}s$$

$$
\begin{aligned}
&- \iint |\nabla \mathbf{u}|^2 \ \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\varepsilon} \mathrm{d}x \ \mathrm{d}s \\
&+ \sum_{i=1}^{3} \iint \frac{|\mathbf{u}|^2}{2} b_i \alpha_{\eta,t_0,t_1} (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&+ \sum_{i=1}^{3} \iint \alpha_{\eta,t_0,t_1} p u_i (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&- \sum_{i=1}^{3} \sum_{j=1}^{3} \iint F_{i,j} u_j \alpha_{\eta,t_0,t_1} (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&- \sum_{i=1}^{3} \sum_{j=1}^{3} \iint F_{i,j} \partial_i u_j \ \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\varepsilon} \ \mathrm{d}x \ \mathrm{d}s.
\end{aligned}
$$

We remark that, independently of $R > 1$ and $\varepsilon > 0$, we have (for $0 < \gamma \leqq 2$)

$$
|w_{\gamma,\varepsilon} \partial_i \phi_R| + |\phi_R \partial_i w_{\gamma,\varepsilon}| \leqq C_\gamma \frac{w_\gamma(x)}{1+|x|} \leqq C_\gamma w_{3\gamma/2}(x).
$$

Moreover, we know that $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma}) \cap L^2((0,T), L^6_{w_{3\gamma}})$ hence to $L^4((0,T), L^3_{w_{3\gamma/2}})$. Since $T < +\infty$, we have as well $\mathbf{u} \in L^3((0,T), L^3_{w_{3\gamma/2}})$. (This is the same type of integrability as required for $\mathbf{b}$). Moreover, we have $p u_i \in L^1_{w_{3\gamma/2}}$ since $w_\gamma p \in L^2((0,T), L^{6/5} + L^2)$ and $w_{\gamma/2} \mathbf{u} \in L^2((0,T), L^2 \cap L^6)$. All those remarks will allow us to use dominated convergence.

We first let η go to 0. We find that

$$
\begin{aligned}
&- \lim_{\eta \to 0} \iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\varepsilon} \ \mathrm{d}x \ \mathrm{d}s \\
&\leqq - \sum_{i=1}^{3} \int_{t_0}^{t_1} \int \partial_i \mathbf{u} \cdot \mathbf{u} \, (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&\quad - \int_{t_0}^{t_1} \int |\nabla \mathbf{u}|^2 \ \phi_R w_{\gamma,\varepsilon} \mathrm{d}x \ \mathrm{d}s \\
&\quad + \sum_{i=1}^{3} \int_{t_0}^{t_1} \int \frac{|\mathbf{u}|^2}{2} b_i (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&\quad + \sum_{i=1}^{3} \int_{t_0}^{t_1} \int p u_i (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&\quad - \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{t_0}^{t_1} \int F_{i,j} u_j (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \ \mathrm{d}x \ \mathrm{d}s \\
&\quad - \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{t_0}^{t_1} \int F_{i,j} \partial_i u_j \ \phi_R w_{\gamma,\varepsilon} \ \mathrm{d}x \ \mathrm{d}s.
\end{aligned}
$$

Let us define

$$A_{R,\varepsilon}(t) = \int |\mathbf{u}(t,x)|^2 \phi_R(x) w_{\gamma,\varepsilon}(x)\, \mathrm{d}x.$$

As we have

$$-\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\varepsilon}\, \mathrm{d}x\, \mathrm{d}s = -\frac{1}{2}\int \partial_t \alpha_{\eta,t_0,t_1} A_{R,\varepsilon}(s)\, \mathrm{d}s$$

we find that, when t_0 and t_1 are Lebesgue points of the measurable function $A_{R,\varepsilon}$

$$\lim_{\eta\to 0} -\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\varepsilon}\, \mathrm{d}x\, \mathrm{d}s = \frac{1}{2}(A_{R,\varepsilon}(t_1) - A_{R,\varepsilon}(t_0)).$$

Then, by continuity, we can let t_0 go to 0 and thus replace t_0 by 0 in the inequality. Moreover, if we let t_1 go to t, then by weak continuity, we find that $A_{R,\varepsilon}(t) \leqq \lim_{t_1\to t} A_{R,\varepsilon}(t_1)$, so that we may as well replace t_1 by $t \in (0,T)$. Thus we find that for every $t \in (0,T)$, we have

$$\begin{aligned}
&\int \frac{|\mathbf{u}(t,x)|^2}{2} \phi_R w_{\gamma,\varepsilon}\, \mathrm{d}x \\
&\quad\leqq \int \frac{|\mathbf{u}_0(x)|^2}{2} \phi_R w_{\gamma,\varepsilon}\, \mathrm{d}x \\
&\qquad - \sum_{i=1}^3 \int_0^t \int \partial_i \mathbf{u} \cdot \mathbf{u}\, (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon})\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad - \int_0^t \int |\nabla \mathbf{u}|^2\ \phi_R w_{\gamma,\varepsilon} \mathrm{d}x\, \mathrm{d}s \\
&\qquad + \sum_{i=1}^3 \int_0^t \int \frac{|\mathbf{u}|^2}{2} b_i (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon})\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad + \sum_{i=1}^3 \int_0^t \int p u_i (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon})\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad - \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{i,j} u_j (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon})\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad - \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{i,j} \partial_i u_j\ \phi_R w_{\gamma,\varepsilon}\, \mathrm{d}x\, \mathrm{d}s. \qquad (6)
\end{aligned}$$

Thus, letting R go to $+\infty$ and then ε go to 0, we find by dominated convergence that, for every $t \in (0,T)$, we have

$$\begin{aligned}
&\|\mathbf{u}(t,.)\|_{L^2_{w_\gamma}}^2 + 2\int_0^t \|\nabla \mathbf{u}(s,.)\|_{L^2_{w_\gamma}}^2 \, \mathrm{d}s \\
&\quad \leqq \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 - \int_0^t \int \nabla |\mathbf{u}|^2 \cdot \nabla w_\gamma \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad + \int_0^t \int (|\mathbf{u}|^2 \mathbf{b} + 2p\mathbf{u}) \cdot \nabla (w_\gamma) \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad - 2\sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{i,j} (\partial_i u_j) w_\gamma + F_{i,j} u_i \partial_j (w_\gamma) \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}$$

Now we write

$$\begin{aligned}
\left| \int_0^t \int \nabla |\mathbf{u}|^2 \cdot \nabla w_\gamma \, \mathrm{d}s \, \mathrm{d}s \right| &\leqq 2\gamma \int_0^t \int |\mathbf{u}| |\nabla \mathbf{u}| \, w_\gamma \, \mathrm{d}x \, \mathrm{d}s \\
&\leqq \frac{1}{4} \int_0^t \|\nabla \mathbf{u}\|_{L^2_{w_\gamma}}^2 \, \mathrm{d}s + 4\gamma^2 \int_0^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2 \, \mathrm{d}s.
\end{aligned}$$

Writing

$$p_1 = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_i u_j) \text{ and } p_2 = -\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (F_{i,j}),$$

and using the fact that $w_{6\gamma/5} \in \mathcal{A}_{6/5}$ and $w_\gamma \in \mathcal{A}_2$, we get

$$\begin{aligned}
&\left| \int_0^t \int (|\mathbf{u}|^2 \mathbf{b} + 2p_1 \mathbf{u}) \cdot \nabla (w_\gamma) \, \mathrm{d}x \, \mathrm{d}s \right| \leqq \gamma \int_0^t \int (|\mathbf{u}|^2 |\mathbf{b}| + 2|p_1| \, |\mathbf{u}|) \, w_\gamma^{3/2} \, \mathrm{d}x \, \mathrm{d}s \\
&\quad \leqq \gamma \int_0^t \|w_\gamma^{1/2} \mathbf{u}\|_6 (\|w_\gamma |\mathbf{b}| |\mathbf{u}|\|_{6/5} + \|w_\gamma p_1\|_{6/5}) \mathrm{d}s \\
&\quad \leqq C_\gamma \int_0^t \|w_\gamma^{1/2} \mathbf{u}\|_6 \|w_\gamma |\mathbf{b}| |\mathbf{u}|\|_{6/5} \, \mathrm{d}s \\
&\quad \leqq C_\gamma \int_0^t \|w_\gamma^{1/2} \mathbf{u}\|_6 \|w_\gamma^{1/2} \mathbf{b}\|_3 \|w_\gamma^{1/2} \mathbf{u}\|_2 \, \mathrm{d}s \\
&\quad \leqq C'_\gamma \int_0^t (\|\nabla \mathbf{u}\|_{L^2_{w_\gamma}} + \|\mathbf{u}\|_{L^2_{w_\gamma}}) \, \|\mathbf{b}\|_{L^3_{w_{3\gamma/2}}} \|\mathbf{u}\|_{L^2_{w_\gamma}} \, \mathrm{d}s \\
&\quad \leqq \frac{1}{4} \int_0^t \|\nabla \mathbf{u}\|_{L^2_{w_\gamma}}^2 \, \mathrm{d}s + C''_\gamma \int_0^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2 (\|\mathbf{b}\|_{L^3_{w_{3\gamma/2}}} + \|\mathbf{b}\|_{L^3_{w_{3\gamma/2}}}^2) \, \mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_0^t \int 2p_2 \mathbf{u} \cdot \nabla (w_\gamma) \, \mathrm{d}x \, \mathrm{d}s \right| \\
&\quad \leqq 2\gamma \int_0^t \int |p_2| \, |\mathbf{u}| \, w_\gamma \, \mathrm{d}x \, \mathrm{d}s \\
&\quad \leqq \gamma \int_0^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2 + \|p_2\|_{L^2_{w_\gamma}}^2 \, \mathrm{d}s \\
&\quad \leqq C_\gamma \int_0^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2 + \|\mathbb{F}\|_{L^2_{w_\gamma}}^2 \, \mathrm{d}s.
\end{aligned}$$

Finally, we have

$$\begin{aligned}\left|2\sum_{i=1}^{3}\sum_{j=1}^{3}\int_0^t\int F_{i,j}(\partial_i u_j)w_\gamma + F_{i,j}u_i\partial_j(w_\gamma)\,dx\,ds\right| &\\ \leqq 2\int_0^t\int |F|\,(|\nabla\mathbf{u}|+\gamma|\mathbf{u}|)\,w_\gamma\,dx\,ds &\\ \leqq \frac{1}{4}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds + C_\gamma\int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds.\end{aligned}$$

We have obtained

$$\begin{aligned}&\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} + \int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds\\ &\quad\leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma\int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\,ds\\ &\qquad + C_\gamma\int_0^t\left(1+\|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}}\right)\|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,ds\end{aligned} \tag{7}$$

and Theorem 2 is proven. □

3.2. Passive Transportation

From inequality (7), we have the following direct consequence:

Corollary 4. *Under the assumptions of Theorem 2, we have*

$$\sup_{0<t<T}\|\mathbf{u}\|_{L^2_{w_\gamma}} \leqq (\|\mathbf{u}_0\|_{L^2_{w_\gamma}} + C_\gamma\|\mathbb{F}\|_{L^2((0,T),L^2_{w_\gamma})})\,e^{C_\gamma(T+T^{1/3}\|\mathbf{b}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

and

$$\|\nabla\mathbf{u}\|_{L^2((0,T),L^2_{w_\gamma})} \leqq (\|\mathbf{u}_0\|_{L^2_{w_\gamma}} + C_\gamma\|\mathbb{F}\|_{L^2((0,T),L^2_{w_\gamma})})\,e^{C_\gamma(T+T^{1/3}\|\mathbf{b}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})},$$

where the constant C_γ depends only on γ.

Another direct consequence is the following uniqueness result for the advection-diffusion problem with a (locally in time), bounded $\mathbf{b}$:

Corollary 5. *Let $0<\gamma<2$. Let $0<T<+\infty$. Let $\mathbf{u}_0$ be a divergence-free vector field such that $\mathbf{u}_0\in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}$ be a tensor $\mathbb{F}(t,x)=\big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ such that $\mathbb{F}\in L^2((0,T),L^2_{w_\gamma})$. Let $\mathbf{b}$ be a time-dependent divergence free vector-field ($\nabla\cdot\mathbf{b}=0$) such that $\mathbf{b}\in L^3((0,T),L^3_{w_{3\gamma/2}})$. Assume moreover that $\mathbf{b}$ belongs to $L^2_tL^\infty_x(K)$ for every compact subset K of $(0,T)\times\mathbb{R}^3$.*

Let $(\mathbf{u}_1, p_1)$ *and* $(\mathbf{u}_2, p_2)$ *be two solutions of the following advection-diffusion problem:*

$$(AD)\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \nabla\cdot\mathbf{u} = 0, \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

such that, for $k = 1$ *and* $k = 2$*,*

- $\mathbf{u}_k$ *belongs to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla\mathbf{u}_k$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$*;*
- *the pressure* p_k *is related to* $\mathbf{u}_k$*,* $\mathbf{b}$ *and* $\mathbb{F}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formula*

$$p_k = \sum_{i=1}^3\sum_{j=1}^3 R_i R_j (b_i u_{k,j} - F_{i,j});$$

- *the map* $t\in[0,T)\mapsto \mathbf{u}_k(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$ *:*

$$\lim_{t\to 0}\|\mathbf{u}_k(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

Then $\mathbf{u}_1 = \mathbf{u}_2$.

Proof. Let $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ and $q = p_1 - p_2$. Then we have

$$\begin{cases} \partial_t \mathbf{v} = \Delta\mathbf{v} - (\mathbf{b}\cdot\nabla)\mathbf{v} - \nabla q \\ \nabla\cdot\mathbf{v} = 0, \qquad \mathbf{v}(0,.) = 0. \end{cases}$$

Moreover on every compact subset K of $(0,T)\times\mathbb{R}^3$, $\mathbf{b}\otimes\mathbf{v}$ is in $L^2_t L^2_x$, while it belongs globally to $L^3_t L^{6/5}_{w_{6\gamma/5}}$. Writing, for $\varphi, \psi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ such that $\psi = 1$ on the neigborhood of the support of φ,

$$\varphi q = q_1 + q_2 = \varphi\sum_{i=1}^3\sum_{j=1}^3 R_i R_j(\psi b_i v_j) + \varphi\sum_{i=1}^3\sum_{j=1}^3 R_i R_j((1-\psi)b_i v_j),$$

we find that $\|q_1\|_{L^2L^2} \leqq C_{\varphi,\psi}\|\psi\mathbf{b}\otimes\mathbf{v}\|_{L^2L^2}$ and

$$\|q_2\|_{L^3L^\infty} \leqq C_{\varphi,\psi}\|\mathbf{b}\otimes\mathbf{v}\|_{L^3L^{6/5}_{w_{6\gamma/5}}}$$

with

$$C_{\varphi,\psi} \leqq C\|\varphi\|_\infty\|1-\psi\|_\infty \sup_{x\in\operatorname{Supp}\varphi}\left(\int_{y\in\operatorname{Supp}(1-\psi)}\left(\frac{(1+|y|)^\gamma}{|x-y|^3}\right)^6\right)^{1/6} < +\infty.$$

Thus, we may take the scalar product of $\partial_t\mathbf{v}$ with $\mathbf{v}$ and find that

$$\partial_t(\frac{|\mathbf{v}|^2}{2}) = \Delta\Big(\frac{|\mathbf{v}|^2}{2}\Big) - |\nabla\mathbf{v}|^2 - \nabla\cdot\left(\frac{|\mathbf{v}|^2}{2}\mathbf{b}\right) - \nabla\cdot(q\mathbf{v}).$$

Thus we are under the assumptions of Theorem 2 and we may use Corollary 4 to find that $\mathbf{v} = 0$. □

3.3. Active Transportation

We begin with the following lemma:

Lemma 4. *Let α be a non-negative bounded measurable function on $[0, T)$ such that, for two constants $A, B \geqq 0$, we have*

$$\alpha(t) \leqq A + B \int_0^t \alpha(s) + \alpha(s)^3 \, \mathrm{d}s.$$

If $T_0 > 0$ and $T_1 = \min(T, T_0, \frac{1}{8B(A+2BT_0)^2})$, we have, for every $t \in [0, T_1]$, $\alpha(t) \leqq \sqrt{2}(A + 2BT_0)$.

Proof. We write $\alpha \leqq 1 + \alpha^3$. We define

$$\Phi(t) = A + 2BT_0 + 2B \int_0^t \alpha^3 \, \mathrm{d}s \text{ and } \Psi(t) = A + 2BT_0 + 2B \int_0^t \Phi^3(s) \, \mathrm{d}s.$$

We have, for $t \in [0, T_1]$, $\alpha \leqq \Phi \leqq \Psi$. Since Ψ is $\mathcal{C}^1$, we may write

$$\Psi'(t) = 2B\Phi(t)^3 \leqq 2B\Psi(t)^3$$

and thus

$$\frac{1}{\Psi(0)^2} - \frac{1}{\Psi(t)^2} \leqq 4Bt.$$

We thus find

$$\Psi(t)^2 \leqq \frac{\Psi(0)^2}{1 - 4B\Psi(0)^2 t} \leqq 2\Psi(0)^2.$$

The lemma is proven. □

Corollary 6. *Assume that $\mathbf{u}_0$, $\mathbf{u}$, p, $\mathbb{F}$ and $\mathbf{b}$ satisfy assumptions of Theorem 2. Assume moreover that $\mathbf{b}$ is the inequality in the next line expresses in which way $\mathbf{b}$ is controlled by $\mathbf{u}$: for every $t \in (0, T)$,*

$$\|\mathbf{b}(t, .)\|_{L^3_{w_{3\gamma/2}}} \leqq C_0 \|\mathbf{u}(t, .)\|_{L^3_{w_{3\gamma/2}}}.$$

Then there exists a constant $C_\gamma \geqq 1$ such that if $T_0 < T$ is such that

$$C_\gamma(1 + C_0^4) \left(1 + C_0^4 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, \mathrm{d}s\right)^2 T_0 \leqq 1$$

then

$$\sup_{0 \leqq t \leqq T_0} \| \mathbf{u}(t, .)\|^2_{L^2_{w_\gamma}} \leqq C_\gamma \left(1 + C_0^4 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, \mathrm{d}s\right)$$

and

$$\int_0^{T_0} \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}} \, \mathrm{d}s \leqq C_\gamma \left(1 + C_0^4 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, \mathrm{d}s\right).$$

Proof. We start from inequality (7):

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} + \int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$\leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$+ C_\gamma \int_0^t \left(1 + \|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}}\right)\|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$

We write

$$\|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}} \leqq C_0^2\|\mathbf{u}(s,.)\|^2_{L^3_{w_{3\gamma/2}}} \leqq C_0^2 C_\gamma \|\mathbf{u}\|_{L^2_{w_\gamma}}(\|u\|_{L^2_{w_\gamma}} + \|\nabla\mathbf{u}\|_{L^2_{w_\gamma}}).$$

This gives

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} + \frac{1}{2}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$\leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$+ C_\gamma \int_0^t \|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}} + C_0^2\|\mathbf{u}(s,.)\|^4_{L^2_{w_\gamma}} + C_0^4\|\mathbf{u}(s,.)\|^6_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$\leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$+ 2C_\gamma \int_0^t \|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}} + C_0^4\|\mathbf{u}(s,.)\|^6_{L^2_{w_\gamma}}\, \mathrm{d}s.$$

For $t \leqq T_0$, we get

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} + \frac{1}{2}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$\leqq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, \mathrm{d}s$$
$$+ C_\gamma(1 + C_0^4)\int_0^t \|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}} + \|\mathbf{u}(s,.)\|^6_{L^2_{w_\gamma}}\, \mathrm{d}s$$

and we may conclude with Lemma 4. □

4. Stability of Solutions for the Advection-Diffusion Problem

4.1. The Rellich Lemma

We recall the Rellich lemma:

Lemma 5. (Rellich) *If $s > 0$ and (f_n) is a sequence of functions on $\mathbb{R}^d$ such that*

- *the family (f_n) is bounded in $H^s(\mathbb{R}^d)$,*

- *there is a compact subset of $\mathbb{R}^d$ such that the support of each f_n is included in K,*

then there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2(\mathbb{R}^d)$.

We shall use a variant of this lemma (see [9]):

Lemma 6. (space-time Rellich) *If $s > 0$, $\sigma \in \mathbb{R}$ and (f_n) is a sequence of functions on $(0, T) \times \mathbb{R}^d$ such that, for all $T_0 \in (0, T)$ and all $\varphi \in \mathcal{D}(\mathbb{R}^3)$,*

- *φf_n is bounded in $L^2((0, T_0), H^s)$,*
- *$\varphi \partial_t f_n$ is bounded in $L^2((0, T_0), H^\sigma)$,*

then there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2_{\mathrm{loc}}([0, T) \times \mathbb{R}^3)$: if f_∞ is the limit, we have for all $T_0 \in (0, T)$ and all $R_0 > 0$

$$\lim_{n_k \to +\infty} \int_0^{T_0} \int_{|x| \leqq R} |f_{n_k} - f_\infty|^2 \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Proof. With no loss of generality, we may assume that $\sigma < \min(1, s)$. Define g by $g_n(t, x) = \alpha(t)\varphi(x) f_n(t, x)$ if $t > 0$ and $g_n(t, x) = \alpha(t)\varphi(x) f_n(-t, x)$ if $t < 0$, where $\alpha \in \mathcal{C}^\infty$ on $(0, T)$, is equal to 1 on $[0, T_0]$ and equal to 0 for $t > \frac{T+T_0}{2}$, and $\varphi(x) = 1$ on $B(0, R_0)$. Then the support of g_n is contained in $[-\frac{T+T_0}{2}, \frac{T+T_0}{2}] \times \operatorname{Supp}\varphi$. Moreover, g_n is bounded in $L^2_t H^s$ and $\partial_t g_n$ is bounded in $L^2 H^\sigma$ so that g_n is bounded in $H^\rho(\mathbb{R} \times \mathbb{R}^3)$ with $\rho = \frac{s}{s+1-\sigma}$ (just write $(1+\tau^2+\xi^2)^{\frac{s}{s+1-\sigma}} \leqq \left((1+\tau^2)(1+\xi^2)^\sigma\right)^{\frac{s}{s+1-\sigma}} \left((1+\xi^2)^s\right)^{\frac{1-\sigma}{s+1-\sigma}}$). By the Rellich lemma, we know that there is a subsequence g_{n_k} which is strongly convergent in $L^2(\mathbb{R} \times \mathbb{R}^3)$, thus a subsequence f_{n_k} which is strongly convergent in $L^2((0, T_0) \times B(0, R_0))$.

We then iterate this argument for an increasing sequence of times $T_0 < T_1 < \cdots < T_N \to T$ and an increasing sequence of radii $R_0 < R_1 < \cdots < R_N \to +\infty$ and finish the proof by the classical diagonal process of Cantor. □

4.2. Proof of Theorem 3

Assume that $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_{0,\infty}$ in $L^2_{w_\gamma}$ and that the sequence $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}_\infty$ in $L^2((0, T), L^2_{w_\gamma})$, and assume that the sequence $\mathbf{b}_n$ is bounded in $L^3((0, T), L^3_{w_{3\gamma/2}})$. Then, by Theorem 2 and Corollary 4, we know that $\mathbf{u}_n$ is bounded in $L^\infty((0, T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0, T), L^2_{w_\gamma})$. In particular, writing $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_{n,i} u_{n,j}) \text{ and } p_{n,2} = -\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (F_{n,i,j}),$$

we get that $p_{n,1}$ is bounded in $L^3((0, T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $p_{n,2}$ is bounded in $L^2((0, T), L^2_{w_\gamma})$.

If $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we find that $\varphi\mathbf{u}_n$ is bounded in $L^2((0,T), H^1)$ and, writing

$$\partial_t \mathbf{u}_n = \Delta\mathbf{u}_n - \left(\sum_{i=1}^{3} \partial_i(b_{n,i}\mathbf{u}_n) + \nabla p_{n,1}\right) + \left(\nabla\cdot\mathbb{F}_n - \nabla p_{n,2}\right),$$

$\varphi\partial_t\mathbf{u}_n$ is bounded in $L^2L^2 + L^2W^{-1,6/5} + L^2H^{-1} \subset L^2((0,T), H^{-2})$. Thus, by Lemma 6, there exist $\mathbf{u}_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}_\infty$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$, and for every $T_0 \in (0,T)$ and every $R > 0$, we have

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^2 \,\mathrm{d}y\,\mathrm{d}s = 0.$$

As $\mathbf{u}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$, the convergence of $\mathbf{u}_{n_k}$ to $\mathbf{u}_\infty$ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ implies that $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}_\infty$ in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$.

By Banach–Alaoglu's theorem, we may assume that there exists $\mathbf{b}_\infty$ such that $\mathbf{b}_{n_k}$ converges weakly to $\mathbf{b}_\infty$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$. In particular $b_{n_k,i}u_{n_k,j}$ is weakly convergent in $(L^{6/5}L^{6/5})_{\mathrm{loc}}$ and thus in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$; as it is bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$, it is weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $b_{\infty,i}u_{\infty,j}$. Let

$$p_{\infty,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(b_{\infty,i}u_{\infty,j}) \text{ and } p_{\infty,2} = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(F_{\infty,i,j}).$$

As the Riesz transforms are bounded on $L^{6/5}_{w_{\frac{6\gamma}{5}}}$ and on $L^2_{w_\gamma}$, we find that $p_{n_k,1}$ is weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $p_{\infty,1}$ and that $p_{n_k,2}$ is strongly convergent in $L^2((0,T), L^2_{w_\gamma})$ to $p_{\infty,2}$.

In particular, we find that in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$,

$$\partial_t\mathbf{u}_\infty = \Delta\mathbf{u}_\infty - \sum_{i=1}^{3}\partial_i(b_{\infty,i}\mathbf{u}_\infty) - \nabla(p_{\infty,1} + p_{\infty,2}) + \nabla\cdot\mathbb{F}_\infty.$$

In particular, $\partial_t\mathbf{u}_\infty$ is locally in L^2H^{-2}, and thus $\mathbf{u}_\infty$ has representative such that $t \mapsto \mathbf{u}_\infty(t,.)$ is continuous from $[0,T)$ to $\mathcal{D}'(\mathbb{R}^3)$ and coincides with $\mathbf{u}_\infty(0,.) + \int_0^t \partial_t\mathbf{u}_\infty\,\mathrm{d}s$. In $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, we have that

$$\begin{aligned}\mathbf{u}_\infty(0,.) + \int_0^t \partial_t\mathbf{u}_\infty\,\mathrm{d}s = \mathbf{u}_\infty = \lim_{n_k\to+\infty}\mathbf{u}_{n_k}\\ = \lim_{n_k\to+\infty}\mathbf{u}_{0,n_k} + \int_0^t \partial_t\mathbf{u}_{n_k}\,\mathrm{d}s = \mathbf{u}_{0,\infty} + \int_0^t \partial_t\mathbf{u}_\infty\,\mathrm{d}s\end{aligned}$$

Thus, $\mathbf{u}_\infty(0,.) = \mathbf{u}_{0,\infty}$, and $\mathbf{u}_\infty$ is a solution of (AD_∞).

Next, we define

$$\begin{aligned} A_n &= = |\nabla \mathbf{u}_n|^2 + \mu_n \\ &= -\partial_t\Big(\frac{|\mathbf{u}_n|^2}{2}\Big) + \Delta\Big(\frac{|\mathbf{u}_n|^2}{2}\Big) - \nabla\cdot\left(\frac{|\mathbf{u}_n|^2}{2}\mathbf{b}_n\right) - \nabla\cdot(p_n\mathbf{u}_n) + \mathbf{u}_n\cdot(\nabla\cdot\mathbb{F}_n). \end{aligned}$$

As $\mathbf{u}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$, it is bounded in $L^2((0,T), L^6_{w_{3\gamma/2}})$ and by interpolation with $L^\infty((0,T), L^2_{w_\gamma})$ it is bounded in $L^{10/3}((0,T), L^{10/3}_{w_{5\gamma/3}})$. Thus, u_{n_k} is locally bounded in $L^{10/3}L^{10/3}$ and locally strongly convergent in L^2L^2; it is then strongly convergent in L^3L^3. Thus, A_{n_k} is convergent in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ to

$$A_\infty = -\partial_t\left(\frac{|\mathbf{u}_\infty|^2}{2}\right) + \Delta\left(\frac{|\mathbf{u}_\infty|^2}{2}\right) - \nabla\cdot\left(\frac{|\mathbf{u}_\infty|^2}{2}\mathbf{b}_\infty\right) - \nabla\cdot(p_\infty\mathbf{u}_\infty) + \mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty).$$

In particular, $A_\infty = \lim_{n_k\to+\infty}|\nabla\mathbf{u}_{n_k}|^2 + \mu_{n_k}$. If $\Phi\in\mathcal{D}((0,T)\times\mathbb{R}^3)$ is non-negative, we have

$$\begin{aligned} \iint A_\infty\Phi\,\mathrm{d}x\,\mathrm{d}s &= \lim_{n_k\to+\infty}\iint A_{n_k}\Phi\,\mathrm{d}x\,\mathrm{d}s \\ &\geqq \limsup_{n_k\to+\infty}\iint|\nabla\mathbf{u}_{n_k}|^2\Phi\,\mathrm{d}x\,\mathrm{d}s \geqq \iint|\nabla\mathbf{u}_\infty|^2\Phi\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

(since $\sqrt{\Phi}\nabla\mathbf{u}_{n_k}$ is weakly convergent to $\sqrt{\Phi}\nabla\mathbf{u}_\infty$ in L^2L^2). Thus, there exists a non-negative locally finite measure μ_∞ on $(0,T)\times\mathbb{R}^3$ such that $A_\infty = |\nabla\mathbf{u}_\infty|^2 + \mu_\infty$, that is such that

$$\begin{aligned} \partial_t\Big(\frac{|\mathbf{u}_\infty|^2}{2}\Big) = \Delta\Big(\frac{|\mathbf{u}_\infty|^2}{2}\Big) - |\nabla\mathbf{u}_\infty|^2 - \nabla\cdot\left(\frac{|\mathbf{u}_\infty|^2}{2}\mathbf{b}_\infty\right) \\ -\nabla\cdot(p_\infty\mathbf{u}_\infty) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}_\infty) - \mu_\infty. \end{aligned}$$

Finally, we start from inequality (6):

$$\begin{aligned} \int\frac{|\mathbf{u}_n(t,x)|^2}{2}\phi_R w_{\gamma,\varepsilon}\,\mathrm{d}x \leqq& \int\frac{|\mathbf{u}_{0,n}(x)|^2}{2}\phi_R w_{\gamma,\varepsilon}\,\mathrm{d}x \\ &- \sum_{i=1}^3\int_0^t\int\partial_i\mathbf{u}_n\cdot\mathbf{u}_n\,(w_{\gamma,\varepsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\varepsilon})\,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t\int|\nabla\mathbf{u}_n|^2\;\phi_R w_{\gamma,\varepsilon}\mathrm{d}x\,\mathrm{d}s \\ &+ \sum_{i=1}^3\int_0^t\int\frac{|\mathbf{u}_n|^2}{2}b_{n,i}(w_{\gamma,\varepsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\varepsilon})\,\mathrm{d}x\,\mathrm{d}s \\ &+ \sum_{i=1}^3\int_0^t\int p_n u_{n,i}(w_{\gamma,\varepsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\varepsilon})\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

$$- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int F_{n,i,j} u_{n,j} (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \, dx \, ds$$
$$- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int F_{n,i,j} \partial_i u_{n,j} \ \phi_R w_{\gamma,\varepsilon} \, dx \, ds.$$

This gives

$$\begin{aligned}
\limsup_{n_k \to +\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2} \phi_R w_{\gamma,\varepsilon} \, dx + \int_0^t \int |\nabla \mathbf{u}_{n_k}|^2 \ \phi_R w_{\gamma,\varepsilon} dx \, ds \\
\leqq \int \frac{|\mathbf{u}_{0,\infty}(x)|^2}{2} \phi_R w_{\gamma,\varepsilon} \, dx \\
- \sum_{i=1}^{3} \int_0^t \int \partial_i \mathbf{u}_\infty \cdot \mathbf{u}_\infty \, (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \, dx \, ds \\
+ \sum_{i=1}^{3} \int_0^t \int \frac{|\mathbf{u}_\infty|^2}{2} b_{\infty,i} (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \, dx \, ds \\
+ \sum_{i=1}^{3} \int_0^t \int p_\infty u_{\infty,i} (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \, dx \, ds \\
- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int F_{\infty,i,j} u_{\infty,j} (w_{\gamma,\varepsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\varepsilon}) \, dx \, ds \\
- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int F_{\infty,i,j} \partial_i u_{\infty,j} \ \phi_R w_{\gamma,\varepsilon} \, dx \, ds.
\end{aligned}$$

As we have

$$\mathbf{u}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t \mathbf{u}_{n_k} \, ds,$$

we see that $\mathbf{u}_{n_k}(t,.)$ is convergent to $\mathbf{u}_\infty(t,.)$ in $\mathcal{D}'(\mathbb{R}^3)$, hence is weakly convergent in L^2_{loc} (as it is bounded in $L^2_{w_\gamma}$), so that:

$$\int \frac{|\mathbf{u}_\infty(t,x)|^2}{2} \phi_R w_{\gamma,\varepsilon} \, dx \leqq \limsup_{n_k \to +\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2} \phi_R w_{\gamma,\varepsilon} \, dx.$$

Similarly, as $\nabla \mathbf{u}_{n_k}$ is weakly convergent in $L^2 L^2_{w_\gamma}$, we have

$$\int_0^t \int \frac{|\nabla \mathbf{u}_\infty(s,x)|^2}{2} \phi_R w_{\gamma,\varepsilon} \, dx \, ds \leqq \limsup_{n_k \to +\infty} \int_0^t \int \frac{|\nabla \mathbf{u}_{n_k}(s,x)|^2}{2} \phi_R w_{\gamma,\varepsilon} \, dx \, ds.$$

Thus, letting R go to $+\infty$ and then ε go to 0, we find by dominated convergence that, for every $t \in (0, T)$, we have

$$\begin{aligned}
&\|\mathbf{u}_\infty(t,.)\|_{L^2_{w_\gamma}}^2 + 2\int_0^t \|\nabla \mathbf{u}_\infty(s,.)\|_{L^2_{w_\gamma}}^2 \,\mathrm{d}s \\
&\quad \leqq \|\mathbf{u}_{0,\infty}\|_{L^2_{w_\gamma}}^2 - \int_0^t \int \nabla |\mathbf{u}_\infty|^2 \cdot \nabla w_\gamma \,\mathrm{d}x\,\mathrm{d}s \\
&\quad\quad + \int_0^t \int (|\mathbf{u}_\infty|^2 \mathbf{b}_\infty + 2p_\infty \mathbf{u}_\infty) \cdot \nabla(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\quad - 2\sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{\infty,i,j}(\partial_i u_{\infty,j}) w_\gamma + F_{\infty,i,j} u_{\infty,i} \partial_j(w_\gamma)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Letting t go to 0, we find

$$\limsup_{t\to 0} \|\mathbf{u}_\infty(t,.)\|_{L^2_{w_\gamma}}^2 \leqq \|\mathbf{u}_{0,\infty}\|_{L^2_{w_\gamma}}^2 .$$

On the other hand, we know that $\mathbf{u}_\infty$ is weakly continuous in $L^2_{w_\gamma}$ and thus we have

$$\|\mathbf{u}_{0,\infty}\|_{L^2_{w_\gamma}}^2 \leqq \liminf_{t\to 0} \|\mathbf{u}_\infty(t,.)\|_{L^2_{w_\gamma}}^2 .$$

This gives $\|\mathbf{u}_{0,\infty}\|_{L^2_{w_\gamma}}^2 = \lim_{t\to 0} \|\mathbf{u}_\infty(t,.)\|_{L^2_{w_\gamma}}^2$, which allows to turn the weak convergence into a strong convergence. Theorem 3 is proven. □

5. Solutions of the Navier–Stokes Problem with Initial Data in $L^2_{w_\gamma}$

We now prove Theorem 1. The idea is to approximate the problem by a Navier–Stokes problem in L^2, then use the a priori estimates (Theorem 2) and the stability theorem (Theorem 3) to find a solution to the Navier–Stokes problem with data in $L^2_{w_\gamma}$).

5.1. Approximation by Square Integrable Data

Lemma 7. (Leray's projection operator) *Let $0 < \delta < 3$ and $1 < r < +\infty$. If $\mathbf{v}$ is a vector field on $\mathbb{R}^3$ such that $\mathbf{v} \in L^r_{w_\delta}$, then there exists a unique decompostion*

$$\mathbf{v} = \mathbf{v}_\sigma + \mathbf{v}_\nabla$$

such that

- $\mathbf{v}_\sigma \in L^r_{w_\delta}$ *and* $\nabla \cdot \mathbf{v}_\sigma = 0$,
- $\mathbf{v}_\nabla \in L^r_{w_\delta}$ *and* $\nabla \wedge \mathbf{v}_\nabla = 0$.

We shall write $\mathbf{v}_\sigma = \mathbb{P}\mathbf{v}$*, where* $\mathbb{P}$ *is Leray's projection operator.*

Similarly, if $\mathbf{v}$ *is a distribution vector field of the type* $\mathbf{v} = \nabla \cdot \mathbb{G}$ *with* $\mathbb{G} \in L^r_{w_\delta}$ *then there exists a unique decompostion*

$$\mathbf{v} = \mathbf{v}_\sigma + \mathbf{v}_\nabla$$

such that

- *there exists* $\mathbb{H} \in L^r_{w_\delta}$ *such that* $\mathbf{v}_\sigma = \nabla \cdot \mathbb{H}$ *and* $\nabla \cdot \mathbf{v}_\sigma = 0$,
- *there exists* $q \in L^r_{w_\delta}$ *such that* $\mathbf{v}_\nabla = \nabla q$ *(and thus* $\nabla \wedge \mathbf{v}_\nabla = 0$*).*

We shall still write $\mathbf{v}_\sigma = \mathbb{P}\mathbf{v}$. *Moreover, the function* q *is given by*

$$q = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (G_{i,j}).$$

Proof. As $w_\delta \in \mathcal{A}_r$ the Riesz transforms are bounded on $L^r_{w_\delta}$. Using the identity

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \wedge (\nabla \wedge \mathbf{v})$$

we find (if the decomposition exists) that

$$\Delta \mathbf{v}_\sigma = -\nabla \wedge (\nabla \wedge \mathbf{v}_\sigma) = -\nabla \wedge (\nabla \wedge \mathbf{v}) \text{ and } \Delta \mathbf{v}_\nabla = \nabla(\nabla \cdot \mathbf{v}_\nabla) = \nabla(\nabla \cdot \mathbf{v}).$$

This proves the uniqueness. By linearity, we just have to prove that $\mathbf{v} = 0 \implies \mathbf{v}_\nabla = 0$. We have $\Delta \mathbf{v}_\nabla = 0$, and thus $\mathbf{v}_\nabla$ is harmonic; as it belongs to $\mathcal{S}'$, we find that it is a polynomial. But a polynomial which belongs to $L^r_{w_\delta}$ must be equal to 0. Similarly, if $\mathbf{v}_\nabla = \nabla q$, then $\Delta q = \nabla \cdot \mathbf{v}_\nabla = \nabla \cdot \mathbf{v} = 0$; thus q is harmonic and belongs to $L^r_{w_\delta}$, hence $q = 0$.

For the existence, it is enough to check that $v_{\nabla,i} = -\sum_{j=1}^3 R_i R_j v_j$ in the first case and $\mathbf{v}_\nabla = \nabla q$ with $q = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (G_{i,j})$ in the second case fulfill the conclusions of the lemma. □

Lemma 8. *Let* $0 < \gamma < 2$. *Let* $\mathbf{u}_0$ *be a divergence-free vector field such that* $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ *and* $\mathbb{F}$ *be a tensor* $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1 \leqq i,j \leqq 3}$ *such that* $\mathbb{F} \in L^2((0,+\infty), L^2_{w_\gamma})$. *Let* $\phi \in \mathcal{D}(\mathbb{R}^3)$ *be a non-negative function which is equal to* 1 *for* $|x| \leqq 1$ *and to* 0 *for* $|x| \geqq 2$. *For* $R > 0$*, we define* $\phi_R(x) = \phi(\frac{x}{R})$, $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$ *and* $\mathbb{F}_R = \phi_R \mathbb{F}$. *Then* $\mathbf{u}_{0,R}$ *is a divergence-free square integrable vector field and* $\lim_{R \to +\infty} \|\mathbf{u}_{0,R} - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0$. *Similarly,* $\mathbb{F}_R$ *belongs to* $L^2 L^2$ *and* $\lim_{R \to +\infty} \|\mathbb{F}_R - \mathbb{F}\|_{L^2((0,+\infty), L^2_{w_\gamma})} = 0$.

Proof. By dominated convergence, we have $\lim_{R \to +\infty} \|\phi_R \mathbf{u}_0 - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0$. We conclude by writing $\mathbf{u}_{0,R} - \mathbf{u}_0 = \mathbb{P}(\phi_R \mathbf{u}_0 - \mathbf{u}_0)$. □

5.2. Leray's Mollification

We want to solve the Navier–Stokes equations with initial value $\mathbf{u}_0$:

$$(NS)\begin{cases}\partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0\end{cases}$$

We begin with Leray's method [11] for solving the problem in L^2:

$$(NS_R)\begin{cases}\partial_t \mathbf{u}_R = \Delta \mathbf{u}_R - (\mathbf{u}_R\cdot\nabla)\mathbf{u}_R - \nabla p_R + \nabla\cdot\mathbb{F}_{\mathbb{R}} \\ \nabla\cdot\mathbf{u}_R = 0, \qquad \mathbf{u}_R(0,.) = \mathbf{u}_{0,R}\end{cases}$$

The idea of Leray is to mollify the non-linearity by replacing $\mathbf{u}_R\cdot\nabla$ by $(\mathbf{u}_R * \theta_\varepsilon)\cdot\nabla$, where $\theta(x) = \frac{1}{\varepsilon^3}\theta(\frac{x}{\varepsilon})$, $\theta\in\mathcal{D}(\mathbb{R}^3)$, θ is non-negative and radially decreasing and $\int\theta\,\mathrm{d}x = 1$. We thus solve the problem

$$(NS_{R,\varepsilon})\begin{cases}\partial_t \mathbf{u}_{R,\varepsilon} = \Delta \mathbf{u}_{R,\varepsilon} - ((\mathbf{u}_{R,\varepsilon} * \theta_\varepsilon)\cdot\nabla)\mathbf{u}_{R,\varepsilon} - \nabla p_{R,\varepsilon} + \nabla\cdot\mathbb{F}_R \\ \nabla\cdot\mathbf{u}_{R,\varepsilon} = 0, \qquad \mathbf{u}_{R,\varepsilon}(0,.) = \mathbf{u}_{0,R}\end{cases}$$

The classical result of Leray states that the problem $(NS_{R,\varepsilon})$ is well-posed:

Lemma 9. *Let $\mathbf{v}_0\in L^2$ be a divergence-free vector field. Let $\mathbb{G}\in L^2((0,+\infty),L^2)$. Then the problem*

$$(NS_\varepsilon)\begin{cases}\partial_t \mathbf{v}_\varepsilon = \Delta \mathbf{v}_\varepsilon - ((\mathbf{v}_\varepsilon * \theta_\varepsilon)\cdot\nabla)\mathbf{v}_\varepsilon - \nabla q_\varepsilon + \nabla\cdot\mathbb{G} \\ \nabla\cdot\mathbf{v}_\varepsilon = 0, \qquad \mathbf{v}_\varepsilon(0,.) = \mathbf{v}_0\end{cases}$$

has a unique solution $\mathbf{v}_\varepsilon$ in $L^\infty((0,+\infty),L^2)\cap L^2((0,+\infty),\dot H^1)$. Moreover, this solution belongs to $\mathcal{C}([0,+\infty),L^2)$.

5.3. Proof of Theorem 1 (Local Existence)

We use Lemma 9 and find a solution $\mathbf{u}_{R,\varepsilon}$ to the problem $(NS_{R,\varepsilon})$. Then we check that $\mathbf{u}_{R,\varepsilon}$ fulfills the assumptions of Theorem 2 and of Corollary 6:

- $\mathbf{u}_{R,\varepsilon}$ belongs to $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla\mathbf{u}_{R,\varepsilon}$ belongs to $L^2((0,T),L^2_{w_\gamma})$;
- the map $t\in[0,+\infty)\mapsto\mathbf{u}_{R,\varepsilon}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$:

$$\lim_{t\to 0}\|\mathbf{u}_{R,\varepsilon}(t,.)-\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} = 0,$$

- on $(0, T) \times \mathbb{R}^3$, $\mathbf{u}_{R,\varepsilon}$ fulfills the energy equality

$$\partial_t\Big(\frac{|\mathbf{u}_{R,\varepsilon}|^2}{2}\Big) = \Delta\Big(\frac{|\mathbf{u}_{R,\varepsilon}|^2}{2}\Big) - |\nabla \mathbf{u}_{R,\varepsilon}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}_{R,\varepsilon}\right) - \nabla \cdot (p_{R,\varepsilon}\mathbf{u}_{R,\varepsilon}) + \mathbf{u}_{R,\varepsilon} \cdot (\nabla \cdot \mathbb{F}_R).$$

  with $\mathbf{b}_{R,\varepsilon} = \mathbf{u}_{R,\varepsilon} * \theta_\varepsilon$;
- $\mathbf{b}_{R,\varepsilon}$ is controlled by $\mathbf{u}_{R,\varepsilon}$: for every $t \in (0, T)$,

$$\|\mathbf{b}_{R,\varepsilon}(t,.)\|_{L^3_{w_{3\gamma/2}}} \leqq \|\mathcal{M}_{\mathbf{u}_{R,\varepsilon}(t,.)}\|_{L^3_{w_{3\gamma/2}}} \leqq C_0\|\mathbf{u}_{R,\varepsilon}(t,.)\|_{L^3_{w_{3\gamma/2}}}.$$

Thus, we know that, for every time T_0 such that

$$C_\gamma(1 + C_0^4)\left(1 + C_0^4 + \|\mathbf{u}_{0,R}\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}_R\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \leqq 1,$$

we have

$$\sup_{0\leqq t\leqq T_0} \|\mathbf{u}_{R,\varepsilon}(t,.)\|^2_{L^2_{w_\gamma}} \leqq C_\gamma(1 + C_0^4 + \|\mathbf{u}_{0,R}\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}_R\|^2_{L^2_{w_\gamma}}\, ds)$$

and

$$\int_0^{T_0} \|\nabla\mathbf{u}_{R,\varepsilon}\|^2_{L^2_{w_\gamma}}\, ds \leqq C_\gamma(1 + C_0^4 + \|\mathbf{u}_{0,R}\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}_R\|^2_{L^2_{w_\gamma}}\, ds).$$

Moreover, we have that

$$\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} \leqq C_\gamma\|\mathbf{u}_0\|_{L^2_{w_\gamma}} \text{ and } \|\mathbb{F}_R\|_{L^2_{w_\gamma}} \leqq \|\mathbb{F}\|_{L^2_{w_\gamma}},$$

so that

$$\begin{aligned}\|\mathbf{b}_{R,\varepsilon}\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})} &\leqq C_\gamma\|\mathbf{u}_{R,\varepsilon}\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})} \\ &\leqq C'_\gamma T_0^{\frac{1}{12}}\Big((1+\sqrt{T_0})\|\mathbf{u}_{R,\varepsilon}\|_{L^\infty((0,T_0),L^2_{w_\gamma})} \\ &\quad +\|\nabla\mathbf{u}_{R,\varepsilon}\|_{L^2((0,T_0),L^2_{w_\gamma})}\Big) \\ &\leqq C''_\gamma\sqrt{1 + C_0^4 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds}.\end{aligned}$$

Let $R_n \to +\infty$ and $\varepsilon_n \to 0$. Let $\mathbf{u}_{0,n} = \mathbf{u}_{0,R_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{b}_n = \mathbf{b}_{R_n,\varepsilon_n}$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\varepsilon_n}$. We may then apply Theorem 3, since $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_0$ in $L^2_{w_\gamma}$, $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}$ in $L^2((0, T_0), L^2_{w_\gamma})$, and the sequence $\mathbf{b}_n$ is bounded in $L^3((0, T_0), L^3_{w_{3\gamma/2}})$. Thus there exists p, $\mathbf{u}$, $\mathbf{b}$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T_0),L^2_{w_\gamma})$, $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}$ in $L^2((0,T_0),L^2_{w_\gamma})$;
- $\mathbf{b}_{n_k}$ converges weakly to $\mathbf{b}$ in $L^3((0,T_0),L^3_{w_{3\gamma/2}})$, p_{n_k} converges weakly to p in $L^3((0,T_0),L^{6/5}_{w_{\frac{6\gamma}{5}}})+L^2((0,T_0),L^2_{w_\gamma})$;
- $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\mathrm{loc}}([0,T_0)\times\mathbb{R}^3)$.

Moreover, $\mathbf{u}$ is a solution of the advection-diffusion problem

$$\begin{cases}\partial_t\mathbf{u}=\Delta\mathbf{u}-(\mathbf{b}\cdot\nabla)\mathbf{u}-\nabla p+\nabla\cdot\mathbb{F}\\ \\ \nabla\cdot\mathbf{u}=0,\qquad \mathbf{u}(0,.)=\mathbf{u}_0,\end{cases}$$

and is such that

- the map $t\in[0,T_0)\mapsto\mathbf{u}(t,.)$ is weakly continuous from $[0,T_0)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$:

$$\lim_{t\to0}\|\mathbf{u}(t,.)-\mathbf{u}_0\|_{L^2_{w_\gamma}}=0;$$

- there exists a non-negative locally finite measure μ on $(0,T_0)\times\mathbb{R}^3$ such that

$$\partial_t\Big(\frac{|\mathbf{u}|^2}{2}\Big)=\Delta\Big(\frac{|\mathbf{u}|^2}{2}\Big)-|\nabla\mathbf{u}|^2-\nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right)-\nabla\cdot(p\mathbf{u})+\mathbf{u}\cdot(\nabla\cdot\mathbb{F})-\mu,$$

Finally, as $\mathbf{b}_n=\theta_{\varepsilon_n}*(\mathbf{u}_n-\mathbf{u})+\theta_{\varepsilon_n}*\mathbf{u}$, we see that $\mathbf{b}_{n_k}$ is strongly convergent to $\mathbf{u}$ in $L^3_{\mathrm{loc}}([0,T_0)\times\mathbb{R}^3)$, so that $\mathbf{b}=\mathbf{u}$: thus, $\mathbf{u}$ is a solution of the Navier–Stokes problem on $(0,T_0)$. (It is easy to check that

$$p=\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(u_iu_j-F_{i,j})$$

as $u_{i,n_k}u_{j,n_k}$ is weakly convergent to u_iu_j in $L^4((0,T_0),L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $w_{\frac{6\gamma}{5}}\in\mathcal{A}_{6/5}$.)

5.4. Proof of Theorem 1 (Global Existence)

In order to finish the proof, we shall use the scaling properties of the Navier–Stokes equations : if $\lambda>0$, then $\mathbf{u}$ is a solution of the Cauchy initial value problem for the Navier–Stokes equations on $(0,T)$ with initial value $\mathbf{u}_0$ and forcing tensor $\mathbb{F}$ if and only if $\mathbf{u}_\lambda(t,x)=\lambda\mathbf{u}(\lambda^2t,\lambda x)$ is a solution of the Navier–Stokes equations on $(0,T/\lambda^2)$ with initial value $\mathbf{u}_{0,\lambda}(x)=\lambda\mathbf{u}_0(\lambda x)$ and forcing tensor $\mathbb{F}_\lambda(t,x)=\lambda^2\mathbb{F}(\lambda^2t,\lambda x)$.

We take $\lambda>1$ and for $n\in\mathbb{N}$ we consider the Navier–Stokes problem with initial value $\mathbf{v}_{0,n}=\lambda^n\mathbf{u}_0(\lambda^n\cdot)$ and forcing tensor $\mathbb{F}_n=\lambda^{2n}\mathbb{F}(\lambda^{2n}\cdot,\lambda^n\cdot)$. Then we have seen that we can find a solution $\mathbf{v}_n$ on $(0,T_n)$, with

$$C_\gamma\left(1+\|\mathbf{v}_{0,n}\|^2_{L^2_{w_\gamma}}+\int_0^{+\infty}\|\mathbb{F}_n\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right)^2T_n=1.$$

Of course, we have $\mathbf{v}_n(t,x)=\lambda^n\mathbf{u}_n(\lambda^{2n}t,\lambda^nx)$ where $\mathbf{u}_n$ is a solution of the Navier–Stokes equations on $(0,\lambda^{2n}T_n)$ with initial value $\mathbf{u}_0$ and forcing tensor $\mathbb{F}$.

Lemma 10.

$$\lim_{n\to+\infty} \frac{\lambda^n}{1+\|\mathbf{v}_{0,n}\|^2_{L^2_{w_\gamma}}+\int_0^{+\infty}\|\mathbb{F}_n\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s}=+\infty.$$

Proof. We have

$$\|\mathbf{v}_{0,n}\|^2_{L^2_{w_\gamma}}=\int |\mathbf{u}_0(x)|^2\lambda^{n(\gamma-1)}\frac{(1+|x|)^\gamma}{(\lambda^n+|x|)^\gamma}w_\gamma(x)\,\mathrm{d}x.$$

We have

$$\lambda^{n(\gamma-1)}\leqq\lambda^n$$

as $\gamma\leqq 2$ and we have, by dominated convergence,

$$\lim_{n\to+\infty}\int |\mathbf{u}_0(x)|^2\frac{(1+|x|)^\gamma}{(\lambda^n+|x|)^\gamma}w_\gamma(x)\,\mathrm{d}x=0.$$

Similarly, we have

$$\int_0^{+\infty}\|\mathbb{F}_n\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s=\int_0^{+\infty}\int |\mathbb{F}(s,x)|^2\lambda^{n(\gamma-1)}\frac{(1+|x|)^\gamma}{(\lambda^n+|x|)^\gamma}w_\gamma(x)\,\mathrm{d}x\,\mathrm{d}s=o(\lambda^n).$$

Thus, $\lim_{n\to+\infty}\lambda^{2n}T_n=+\infty$.

Now, for a given $T>0$, if $\lambda^{2n}T_n>T$ for $n\geqq n_T$, then $\mathbf{u}_n$ is a solution of the Navier-Stokes problem on $(0,T)$. Let $\mathbf{w}_n(t,x)=\lambda^{n_T}\mathbf{u}_n(\lambda^{2n_T}t,\lambda^{n_T}x)$. For $n\geqq n_T$, $\mathbf{w}_n$ is a solution of the Navier-Stokes problem on $(0,\lambda^{-2n_T}T)$ with initial value $\mathbf{v}_{0,n_T}$ and forcing tensor $\mathbb{F}_{n_T}$. As $\lambda^{-2n_T}T\leqq T_{n_T}$, we have

$$C_\gamma\left(1+\|\mathbf{v}_{0,n_T}\|^2_{L^2_{w_\gamma}}+\int_0^{+\infty}\|\mathbb{F}_{n_T}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right)^2\lambda^{-2n_T}T\leqq 1.$$

By Corollary 6, we have

$$\sup_{0\leqq t\leqq\lambda^{-2n_T}T}\|\mathbf{w}_n(t,.)\|^2_{L^2_{w_\gamma}}\leqq C_\gamma\left(1+\|\mathbf{v}_{0,n_T}\|^2_{L^2_{w_\gamma}}+\int_0^{\lambda^{-2n_T}T}\|\mathbb{F}_{n_T}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right)$$

and

$$\int_0^{\lambda^{-2n_T}T}\|\nabla\mathbf{w}_n\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\leqq C_\gamma\left(1+\|\mathbf{v}_{0,n_T}\|^2_{L^2_{w_\gamma}}+\int_0^{\lambda^{-2n_T}T}\|\mathbb{F}_{n_T}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right).$$

We have

$$\begin{aligned}\|\mathbf{w}_n\|^2_{L^2_{w_\gamma}}&=\int |\mathbf{u}_n(\lambda^{2n_T}t,x)|^2\lambda^{n_T(\gamma-1)}\frac{(1+|x|)^\gamma}{(\lambda^{n_T}+|x|)^\gamma}w_\gamma(x)\,\mathrm{d}x\\&\geqq\lambda^{-n_T\gamma}\|\mathbf{u}_n(\lambda^{2n_T}t,.)\|^2_{L^2_{w_\gamma}}\end{aligned}$$

and

$$\int_0^{\lambda^{-2n_T}T} \|\nabla \mathbf{w}_n\|_{L^2_{w_\gamma}}^2 \, ds = \int_0^T \int |\nabla \mathbf{u}_n(s,x)|^2 \lambda^{n_T(\gamma-1)} \frac{(1+|x|)^\gamma}{(\lambda^{n_T}+|x|)^\gamma} w_\gamma(x) \, dx \, ds$$
$$\geqq \lambda^{-n_T} \int_0^T \|\nabla \mathbf{u}_n\|_{L^2_{w_\gamma}}^2 \, ds.$$

Thus, we have a uniform control of $\mathbf{u}_n$ and of $\nabla \mathbf{u}_n$ on $(0, T)$ for $n \geqq n_T$. We may then apply the Rellich lemma (Lemma 6) and Theorem 3 to find a subsequence $\mathbf{u}_{n_k}$ that converges to a global solution of the Navier–Stokes equations. Theorem 1 is proven. □

6. Solutions of the Advection-Diffusion Problem with Initial Data in $L^2_{w_\gamma}$

The proof of Theorem 1 on the Navier–Stokes problem can be easily adapted to the case of the advection-diffusion problem:

Theorem 4. *Let* $0 < \gamma \leqq 2$. *Let* $0 < T < +\infty$. *Let* $\mathbf{u}_0$ *be a divergence-free vector field such that* $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ *and* $\mathbb{F}$ *be a tensor* $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ *such that* $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. *Let* $\mathbf{b}$ *be a time-dependent divergence free vector-field* ($\nabla \cdot \mathbf{b} = 0$) *such that* $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.

Then the advection-diffusion problem

$$(AD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

has a solution $\mathbf{u}$ *such that:*

- $\mathbf{u}$ *belongs to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla \mathbf{u}$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$;
- *the pressure* p *is related to* $\mathbf{u}$, $\mathbf{b}$ *and* $\mathbb{F}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formula*

$$p = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_i u_j - F_{i,j});$$

- *the map* $t \in [0,T) \mapsto \mathbf{u}(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$, *and is strongly continuous at* $t = 0$*:*

$$\lim_{t\to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0;$$

- *there exists a non-negative locally finite measure* μ *on* $(0,T)\times\mathbb{R}^3$ *such that*

$$\partial_t\Big(\frac{|\mathbf{u}|^2}{2}\Big) = \Delta\Big(\frac{|\mathbf{u}|^2}{2}\Big) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right) - \nabla \cdot (p\mathbf{u}) + \mathbf{u}\cdot(\nabla \cdot \mathbb{F}) - \mu.$$

Proof. Again, we define $\phi_R(x) = \phi(\frac{x}{R})$, $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$ and $\mathbb{F}_R = \phi_R \mathbb{F}$. Moreover, we define $\mathbf{b}_R = \mathbb{P}(\phi_R \mathbf{b})$. We then solve the mollified problem

$$(AD_{R,\varepsilon}) \begin{cases} \partial_t \mathbf{u}_{R,\varepsilon} = \Delta \mathbf{u}_{R,\varepsilon} - ((\mathbf{b}_R * \theta_\varepsilon) \cdot \nabla) \mathbf{u}_{R,\varepsilon} - \nabla p_{R,\varepsilon} + \nabla \cdot \mathbb{F}_{R,\varepsilon} \\ \nabla \cdot \mathbf{u}_{R,\varepsilon} = 0, \qquad \mathbf{u}_{R,\varepsilon}(0,.) = \mathbf{u}_{0,R}, \end{cases}$$

for which we easily find a unique solution $\mathbf{u}_{R,\varepsilon}$ in $L^\infty((0,T), L^2) \cap L^2((0,T), \dot{H}^1)$. Moreover, this solution belongs to $\mathcal{C}([0,T), L^2)$.

Again, $\mathbf{u}_{R,\varepsilon}$ fulfills the assumptions of Theorem 2:

- $\mathbf{u}_{R,\varepsilon}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_{R,\varepsilon}$ belongs to $L^2((0,T), L^2_{w_\gamma})$
- the map $t \in [0,T) \mapsto \mathbf{u}_{R,\varepsilon}(t,.)$ is weakly continuous from $[0,T)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:

$$\lim_{t \to 0} \|\mathbf{u}_{R,\varepsilon}(t,.) - \mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} = 0.$$

- on $(0,T) \times \mathbb{R}^3$, $\mathbf{u}_{R,\varepsilon}$ fulfills the energy equality:

$$\partial_t \Big(\frac{|\mathbf{u}_{R,\varepsilon}|^2}{2} \Big) = \Delta \Big(\frac{|\mathbf{u}_{R,\varepsilon}|^2}{2} \Big) - |\nabla \mathbf{u}_{R,\varepsilon}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2} \mathbf{b}_{R,\varepsilon} \right) - \nabla \cdot (p_{R,\varepsilon} \mathbf{u}_{R,\varepsilon}) + \mathbf{u}_{R,\varepsilon} \cdot (\nabla \cdot \mathbb{F}_R).$$

with $\mathbf{b}_{R,\varepsilon} = \mathbf{b}_R * \theta_\varepsilon$.

Thus, by Corollary 4 we know that,

$$\sup_{0<t<T} \|\mathbf{u}_{R,\varepsilon}\|_{L^2_{w_\gamma}} \leqq (\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} + C_\gamma \|\mathbb{F}_R\|_{L^2((0,T),L^2_{w_\gamma})})\, e^{C_\gamma (T + T^{1/3} \|\mathbf{b}_{R,\varepsilon}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

and

$$\|\nabla \mathbf{u}_{R,\varepsilon}\|_{L^2((0,T),L^2_{w_\gamma})} \leqq (\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} + C_\gamma \|\mathbb{F}_R\|_{L^2((0,T),L^2_{w_\gamma})})\, e^{C_\gamma (T + T^{1/3} \|\mathbf{b}_{R,\varepsilon}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})},$$

where the constant C_γ depends only on γ.

Moreover, we have that

$$\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} \leqq C_\gamma \|\mathbf{u}_0\|_{L^2_{w_\gamma}}, \ \|\mathbb{F}_R\|_{L^2_{w_\gamma}} \leqq \|\mathbb{F}\|_{L^2_{w_\gamma}}$$

and

$$\|\mathbf{b}_{R,\varepsilon}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leqq \|\mathcal{M}_{\mathbf{b}_R}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leqq C'_\gamma \|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}.$$

Let $R_n \to +\infty$ and $\varepsilon_n \to 0$. Let $\mathbf{u}_{0,n} = \mathbf{u}_{0,R_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{b}_n = \mathbf{b}_{R_n,\varepsilon_n}$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\varepsilon_n}$. We may then apply Theorem 3, since $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_0$ in $L^2_{w_\gamma}$, $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}$ in $L^2((0,T), L^2_{w_\gamma})$, and the sequence $\mathbf{b}_n$ is strongly convergent to $\mathbf{b}$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Thus there exists p, $\mathbf{u}$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with values in $\mathbb{N}$ such that

- $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T),L^2_{w_\gamma})$, $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}$ in $L^2((0,T),L^2_{w_\gamma})$;
- p_{n_k} converges weakly to p in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})+L^2((0,T),L^2_{w_\gamma})$;
- $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\text{loc}}([0,T)\times\mathbb{R}^3)$.

We then easily finish the proof. $\square$

7. Application to the Study of λ-Discretely Self-similar Solutions

We may now apply our results to the study of λ-discretely self-similar solutions for the Navier–Stokes equations.

Definition 1. Let $\mathbf{u}_0\in L^2_{\text{loc}}(\mathbb{R}^3)$. We say that $\mathbf{u}_0$ is a λ-discretely self-similar function (λ-DSS) if there exists $\lambda>1$ such that $\lambda\mathbf{u}_0(\lambda x)=\mathbf{u}_0$.

A vector field $\mathbf{u}\in L^2_{\text{loc}}([0,+\infty)\times\mathbb{R}^3)$ is λ-DSS if there exists $\lambda>1$ such that $\lambda\mathbf{u}(\lambda^2 t,\lambda x)=\mathbf{u}(t,x)$.

A forcing tensor $\mathbb{F}\in L^2_{\text{loc}}([0,+\infty)\times\mathbb{R}^3)$ is λ-DSS if there exists $\lambda>1$ such that $\lambda^2\mathbb{F}(\lambda^2 t,\lambda x)=\mathbb{F}(t,x)$.

We shall speak of self-similarity if $\mathbf{u}_0$, $\mathbf{u}$ or $\mathbb{F}$ are λ-DSS for every $\lambda>1$.

Examples. • Let $\gamma>1$ and $\lambda>1$. Then, for two positive constants $A_{\gamma,\lambda}$ and $B_{\gamma,\lambda}$, we have : if $\mathbf{u}_0\in L^2_{\text{loc}}(\mathbb{R}^3)$ is λ-DSS, then $\mathbf{u}_0\in L^2_{w_\gamma}$ and

$$A_{\gamma,\lambda}\int_{1<|x|\leqq\lambda}|\mathbf{u}_0(x)|^2\,\mathrm{d}x\leqq\int|\mathbf{u}_0(x)|^2w_\gamma(x)\,\mathrm{d}x\leqq B_{\gamma,\lambda}\int_{1<|x|\leqq\lambda}|\mathbf{u}_0(x)|^2\,\mathrm{d}x.$$

- $\mathbf{u}_0\in L^2_{\text{loc}}$ is self-similar if and only if it is of the form $\mathbf{u}_0=\frac{\mathbf{w}_0(\frac{x}{|x|})}{|x|}$ with $\mathbf{w}_0\in L^2(S^2)$.
- $\mathbb{F}$ belongs to $L^2((0,+\infty),L^2_{w_\gamma})$ with $\gamma>1$ and is self-similar if and only if it is of the form $\mathbb{F}(t,x)=\frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$ with $\int|\mathbb{F}_0(x)|^2\frac{1}{|x|}\,\mathrm{d}x<+\infty$.

Proof. • If $\mathbf{u}_0$ is λ-DSS and if $k\in\mathbb{Z}$ we have

$$\int_{\lambda^k<|x|<\lambda^{k+1}}|\mathbf{u}_0(x)|^2w_\gamma(x)\,\mathrm{d}x\leqq\frac{\lambda^k}{(1+\lambda^k)^\gamma}\int_{1<|x|<\lambda}|\mathbf{u}_0(x)|^2\,\mathrm{d}x$$

with $\sum_{k\in\mathbb{Z}}\frac{\lambda^k}{(1+\lambda^k)^\gamma}<+\infty$ for $\gamma>1$.

- If $\mathbf{u}_0$ is self-similar, we have $\mathbf{u}_0(x)=\frac{1}{|x|}\mathbf{u}_0(\frac{x}{|x|})$. From this equality, we find that, for $\lambda>1$

$$\int_{1<|x|<\lambda}|\mathbf{u}_0(x)|^2\,\mathrm{d}x=(\lambda-1)\int_{S^2}|\mathbf{u}_0(\sigma)|^2\,\mathrm{d}\sigma.$$

- If $\mathbb{F}$ is self-similar, then it is of the form $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$. Moreover, we have

$$\int_0^{+\infty}\int |\mathbb{F}(t,x)|^2\, w_\gamma(x)\,\mathrm{d}x\,\mathrm{d}s = \int_0^{+\infty}\int |\mathbb{F}_0(x)|^2 w_\gamma(\sqrt{t}\,x)\,\mathrm{d}x\,\frac{\mathrm{d}t}{\sqrt{t}}$$
$$= C_\gamma \int |\mathbb{F}_0(x)|^2\,\frac{\mathrm{d}x}{|x|}.$$

with $C_\gamma = \int_0^{+\infty} \frac{1}{(1+\sqrt{\theta})^\gamma}\frac{\mathrm{d}\theta}{\sqrt{\theta}} < +\infty$. □

In this section, we are going to give a new proof of the results of Chae and Wolf [3] and Bradshaw and Tsai [2] on the existence of λ-DSS solutions of the Navier–Stokes problem (and of Jia and Šverák [6] for self-similar solutions) :

Theorem 5. *Let* $4/3 < \gamma < 2$ *and* $\lambda > 1$*. If* $\mathbf{u}_0$ *is a λ-DSS divergence-free vector field (such that* $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$*) and if* $\mathbb{F}$ *is a λ-DSS tensor* $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ *such that* $\mathbb{F} \in L^2_{\mathrm{loc}}([0,+\infty)\times\mathbb{R}^3)$*, then the Navier–Stokes equations with initial value* $\mathbf{u}_0$

$$(NS)\begin{cases}\partial_t\mathbf{u} = \Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F}\\ \\ \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0\end{cases}$$

have a global weak solution $\mathbf{u}$ *such that*

- $\mathbf{u}$ *is a λ-DSS vector field;*
- *for every* $0 < T < +\infty$*,* $\mathbf{u}$ *belongs to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla\mathbf{u}$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$*;*
- *the map* $t \in [0,+\infty) \mapsto \mathbf{u}(t,.)$ *is weakly continuous from* $[0,+\infty)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$*:*

$$\lim_{t\to 0}\|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0;$$

- *the solution* $\mathbf{u}$ *is suitable, and there exists a non-negative locally finite measure* μ *on* $(0,+\infty)\times\mathbb{R}^3$ *such that*

$$\partial_t\Big(\frac{|\mathbf{u}|^2}{2}\Big) = \Delta\Big(\frac{|\mathbf{u}|^2}{2}\Big) - |\nabla\mathbf{u}|^2 - \nabla\cdot\Big(\big(\frac{|\mathbf{u}|^2}{2} + p\big)\mathbf{u}\Big) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

7.1. The Linear Problem

Following Chae and Wolf, we consider an approximation of the problem that is consistent with the scaling properties of the equations: let θ be a non-negative and radially decreasing function in $\mathcal{D}(\mathbb{R}^3)$ with $\int\theta\,\mathrm{d}x = 1$. We define $\theta_{\varepsilon,t}(x) = \frac{1}{(\varepsilon\sqrt{t})^3}\,\theta(\frac{x}{\varepsilon\sqrt{t}})$. We then will study the "mollified" problem

$$(NS_\varepsilon)\begin{cases}\partial_t\mathbf{u}_\varepsilon = \Delta\mathbf{u}_\varepsilon - ((\mathbf{u}_\varepsilon * \theta_{\varepsilon,t})\cdot\nabla)\mathbf{u}_\varepsilon - \nabla p_\varepsilon + \nabla\cdot\mathbb{F}\\ \\ \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0\end{cases}$$

and begin with the linearized problem

$$(LNS_\varepsilon)\begin{cases}\partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{b} * \theta_{\varepsilon,t})\cdot\nabla)\mathbf{v} - \nabla q + \nabla\cdot\mathbb{F}\\ \nabla\cdot\mathbf{v} = 0, \qquad \mathbf{v}(0,.) = \mathbf{u}_0.\end{cases}$$

Lemma 11. *Let $1 < \gamma < 2$. Let $\lambda > 1$ Let $\mathbf{u}_0$ be a λ-DSS divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}$ be a λ-DSS tensor $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leqq i,j\leqq 3}$ such that, for every $T > 0$, $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let $\mathbf{b}$ be a λ-DSS time-dependent divergence free vector-field ($\nabla\cdot\mathbf{b} = 0$) such that, for every $T > 0$, $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.*

Then the advection-diffusion problem

$$(LNS_\varepsilon)\begin{cases}\partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{b} * \theta_{\varepsilon,t})\cdot\nabla)\mathbf{v} - \nabla q + \nabla\cdot\mathbb{F}\\ \nabla\cdot\mathbf{v} = 0, \qquad \mathbf{v}(0,.) = \mathbf{u}_0\end{cases}$$

has a unique solution $\mathbf{v}$ such that:

- *for every positive T, $\mathbf{v}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{v}$ belongs to $L^2((0,T), L^2_{w_\gamma})$;*
- *the pressure p is related to $\mathbf{v}$, $\mathbf{b}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$p = \sum_{i=1}^3\sum_{j=1}^3 R_iR_j((b_i * \theta_{\varepsilon,t})v_j - F_{i,j});$$

- *the map $t \in [0,+\infty) \mapsto \mathbf{v}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t\to 0}\|\mathbf{v}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

This solution $\mathbf{v}$ is a λ-DSS vector field.

Proof. As we have $|\mathbf{b}(t,.) * \theta_{\varepsilon,t}| \leqq \mathcal{M}_{\mathbf{b}(t,.)}$ and thus

$$\|\mathbf{b}(t,.) * \theta_{\varepsilon,t}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leqq C_\gamma \|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})},$$

we see that we can use Theorem 4 to get a solution $\mathbf{v}$ on $(0,T)$.

As clearly $\mathbf{b} * \theta_{\varepsilon,t}$ belongs to $L^2_tL^\infty_x(K)$ for every compact subset K of $(0,T)\times\mathbb{R}^3$, we can use Corollary 5 to see that $\mathbf{v}$ is unique.

Let $\mathbf{w}(t,x) = \frac{1}{\lambda}\mathbf{v}(\frac{t}{\lambda^2},\frac{x}{\lambda})$. As $b * \theta_{\varepsilon,t}$ is still λ-DSS, we see that $\mathbf{w}$ is solution of (LNS_ε) on $(0,T)$, so that $\mathbf{w} = \mathbf{v}$. This means that $\mathbf{v}$ is λ-DSS. □

7.2. The Mollified Navier–Stokes Equations

The solution $\mathbf{v}$ provided by Lemma 11 belongs to $L^3((0,T), L^3_{w_{3\gamma/2}})$ (as $\mathbf{v}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{v}$ belongs to $L^2((0,T), L^2_{w_\gamma})$). Thus we have a mapping $L_\varepsilon : \mathbf{b} \mapsto \mathbf{v}$ which is defined from

$$X_{T,\gamma} = \{\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}}) \ / \ \mathbf{b} \text{ is } \lambda - \text{DSS}\}$$

to $X_{T,\gamma}$ by $L_\varepsilon(\mathbf{b}) = \mathbf{v}$.

Lemma 12. *For $4/3 < \gamma$, $X_{T,\gamma}$ is a Banach space for the equivalent norms $\|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$ and $\|\mathbf{b}\|_{L^3((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))}$.*

Proof. We have

$$\int_0^T \int_{B(0,1)} |\mathbf{b}(t,x)|^3 \, \mathrm{d}x \, \mathrm{d}t = \lambda^2 \int_0^{\frac{T}{\lambda^2}} \int_{B(0,\frac{1}{\lambda})} |\mathbf{b}(t,x)|^3 \, \mathrm{d}x \, \mathrm{d}t$$

and , for $k \in \mathbb{N}$,

$$\int_0^T \int_{\lambda^{k-1}<|x|<\lambda^k} |\mathbf{b}(t,x)|^3 \, \mathrm{d}x \, \mathrm{d}t = \lambda^{2k} \int_0^{\frac{T}{\lambda^{2k}}} \int_{\frac{1}{\lambda}<|x|<1} |\mathbf{b}(t,x)|^3 \, \mathrm{d}x \, \mathrm{d}t.$$

We may conclude, since for $\gamma > 4/3$ we have $\sum_{k\in\mathbb{N}} \lambda^{k(2-\frac{3\gamma}{2})} < +\infty$. □

Lemma 13. *For $4/3 < \gamma < 2$, the mapping L_ε is continuous and compact on $X_{T,\gamma}$.*

Proof. Let $\mathbf{b}_n$ be a bounded sequence in $X_{T,\gamma}$ and let $\mathbf{v}_n = L_\varepsilon(\mathbf{b}_n)$. We remark that the sequence $\mathbf{b}_n(t,.) * \theta_{\varepsilon,t}$ is bounded in $X_{T,\gamma}$. Thus, by Theorem 2 and Corollary 4, the sequence $\mathbf{v}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{v}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

We now use Theorem 3 and get that then there exists q_∞, $\mathbf{v}_\infty$, $\mathbf{B}_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- $\mathbf{v}_{n_k}$ converges *-weakly to $\mathbf{v}_\infty$ in $L^\infty((0,T), L^2_{w_\gamma})$, $\nabla\mathbf{v}_{n_k}$ converges weakly to $\nabla\mathbf{v}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$;
- $\mathbf{b}_{n_k} * \theta_{\varepsilon,t}$ converges weakly to $\mathbf{B}_\infty$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$;
- the associated pressures q_{n_k} converge weakly to q_∞ in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$;
- $\mathbf{v}_{n_k}$ converges strongly to $\mathbf{v}_\infty$ in $L^2_{\text{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\mathbf{v}_{n_k}(s,y) - \mathbf{v}_\infty(s,y)|^2 \, \mathrm{d}s \, \mathrm{d}y = 0.$$

As $\sqrt{w_\gamma}\mathbf{v}_n$ is bounded in $L^\infty((0,T),L^2)$ and in $L^2((0,T),L^6)$, it is bounded in $L^{10/3}((0,T)\times\mathbb{R}^3)$. The strong convergence of $\mathbf{v}_{n_k}$ in $L^2_{\rm loc}([0,T)\times\mathbb{R}^3)$ then implies the strong convergence of $\mathbf{v}_{n_k}$ in $L^3_{\rm loc}((0,T)\times\mathbb{R}^3)$.

Moreover, $\mathbf{v}_\infty$ is still λ-DSS (a property that is stable under weak limits).We find that $\mathbf{v}_\infty\in X_{T,\gamma}$ and that

$$\lim_{n_k\to+\infty}\int_0^{\frac{T}{\lambda^2}}\int_{B(0,\frac{1}{\lambda})}|\mathbf{v}_{n_k}(s,y)-\mathbf{v}_\infty(s,y)|^3\,\mathrm{d}s\,\mathrm{d}y=0.$$

This proves that L_ε is compact.

If we assume moreover that $\mathbf{b}_n$ is convergent to $\mathbf{b}_\infty$ in $X_{T,\gamma}$, then necessarily we have $\mathbf{B}_\infty=\mathbf{b}_\infty*\theta_{\varepsilon,t}$, and $\mathbf{v}_\infty=L_\varepsilon(\mathbf{b}_\infty)$. Thus, the relatively compact sequence $\mathbf{v}_n$ can have only one limit point; thus it must be convergent. This proves that L_ε is continuous. □

Lemma 14. *Let* $4/3<\gamma<2$*. If, for some* $\mu\in[0,1]$*,* $\mathbf{v}$ *is a solution of* $\mathbf{v}=\mu L_\varepsilon(\mathbf{v})$ *then*

$$\|\mathbf{v}\|_{X_{T,\gamma}}\leqq C_{\mathbf{u}_0,\mathbb{F},\gamma,T},$$

where the constant $C_{\mathbf{u}_0,\mathbb{F},\gamma,T}$ *depends only on* $\mathbf{u}_0$*,* $\mathbb{F}$*,* γ *and* T *(but not on* μ *nor on* ε*).*

Proof. We have $\mathbf{v}=\mu\mathbf{w}$; with

$$\begin{cases}\partial_t\mathbf{w}=\Delta\mathbf{w}-((\mathbf{v}*\theta_{\varepsilon,t})\cdot\nabla)\mathbf{w}-\nabla q+\nabla\cdot\mathbb{F}\\[2ex] \nabla\cdot\mathbf{w}=0,\qquad \mathbf{w}(0,.)=\mathbf{u}_0.\end{cases}$$

Multiplying by μ, we find that

$$\begin{cases}\partial_t\mathbf{v}=\Delta\mathbf{v}-((\mathbf{v}*\theta_{\varepsilon,t})\cdot\nabla)\mathbf{v}-\nabla(\mu q)+\nabla\cdot\mu\mathbb{F}\\[2ex] \nabla\cdot\mathbf{v}=0,\qquad \mathbf{v}(0,.)=\mu\mathbf{u}_0.\end{cases}$$

We then use Corollary 6. We choose $T_0\in(0,T)$ such that

$$C_\gamma\left(1+\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right)^2T_0\leqq 1.$$

Then, as

$$C_\gamma\left(1+\|\mu\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mu\mathbb{F}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right)^2T_0\leqq 1.$$

we know that

$$\sup_{0\leqq t\leqq T_0}\|\mathbf{v}(t,.)\|^2_{L^2_{w_\gamma}}\leqq C_\gamma\left(1+\mu^2\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\mu^2\int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,\mathrm{d}s\right)$$

and

$$\int_0^{T_0} \|\nabla \mathbf{v}\|_{L^2_{w_\gamma}}^2 \, ds \leqq C_\gamma \left(1 + \mu^2 \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + \mu^2 \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w_\gamma}}^2 \, ds\right).$$

In particular, we have

$$\int_0^{T_0} \|\mathbf{v}\|_{L^3_{w_{3\gamma/2}}}^3 \, ds \leqq C_\gamma T_0^{1/4} \left(1 + \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w_\gamma}}^2 \, ds\right)^{\frac{3}{2}}.$$

As $\mathbf{v}$ is λ-DSS, we can go back from T_0 to T. □

Lemma 15. *Let $4/3 < \gamma \leqq 2$. There is at least one solution $\mathbf{u}_\varepsilon$ of the equation $\mathbf{u}_\varepsilon = L_\varepsilon(\mathbf{u}_\varepsilon)$.*

Proof. Obvious due to the Leray–Schauder principle (and the Schaefer theorem), since L_ε is continuous and compact and since we have uniform a priori estimates for the fixed points of μL_ε for $0 \leqq \mu \leqq 1$. □

7.3. Proof of Theorem 5

We may now finish the proof of Theorem 5. We consider the solutions $\mathbf{u}_\varepsilon$ of $\mathbf{u}_\varepsilon = L_\varepsilon(\mathbf{u}_\varepsilon)$.

By Lemma 14, $\mathbf{u}_\varepsilon$ is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$, and so is $\mathbf{u}_\varepsilon * \theta_{\varepsilon,t}$. We then know, by Theorem 2 and Corollary 4, that the familly $\mathbf{u}_\varepsilon$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_\varepsilon$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

We now use Theorem 3 and get that then there exists p, $\mathbf{u}$, $\mathbf{B}$ and a decreasing sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ (converging to 0) with values in $(0,+\infty)$ such that

- $\mathbf{u}_{\varepsilon_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T), L^2_{w_\gamma})$, $\nabla \mathbf{u}_{\varepsilon_k}$ converges weakly to $\nabla \mathbf{u}$ in $L^2((0,T), L^2_{w_\gamma})$;
- $\mathbf{u}_{\varepsilon_k} * \theta_{\varepsilon_k,t}$ converges weakly to $\mathbf{B}$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$;
- the associated pressures p_{ε_k} converge weakly to p in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$;
- $\mathbf{u}_{\varepsilon_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$.

Moreover we easily see that $\mathbf{B} = \mathbf{u}$. Indeed, we have that $\mathbf{u} * \theta_{\varepsilon,t}$ converges strongly in $L^2_{\mathrm{loc}}((0,T) \times \mathbb{R}^3)$ as ε goes to 0 (since it is bounded by $\mathcal{M}_{\mathbf{u}}$ and converges, for each fixed t, strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$); moreover, we have $|(\mathbf{u} - \mathbf{u}_\varepsilon) * \theta_{\varepsilon,t}| \leqq \mathcal{M}_{\mathbf{u}-\mathbf{u}_\varepsilon}$, so that the strong convergence of $\mathbf{u}_{\varepsilon_k}$ to $\mathbf{u}$ is kept by convolution with $\theta_{\varepsilon,t}$ as far as we work on compact subsets of $(0,T) \times \mathbb{R}^3$ (and thus don't allow t to go to 0).

Thus, Theorem 5 is proven. □

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Pedro Gabriel Fernández-Dalgo & Pierre Gilles Lemarié-Rieusset
Université Paris-Saclay, CNRS, Univ Evry,
Laboratoire de Mathématiques et Modélisation d'Evry,
91037 Evry,
France.
e-mail: pedro.fernandez@univ-evry.fr
e-mail: pedrodalgo16@gmail.com

and

Pierre Gilles Lemarié-Rieusset
e-mail: pierregilles.lemarierieusset@univ-evry.fr

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