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Large Time Behavior of the Vlasov–Navier–Stokes System on the Torus

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Abstract

We study the large time behavior of Fujita–Kato type solutions to the Vlasov–Navier–Stokes system set on $\mathbb{T}^3 \times \mathbb{R}^3$. Under the assumption that the initial so-called modulated energy is small enough, we prove that the distribution function converges to a Dirac mass in velocity, with exponential rate. The proof is based on the fine structure of the system and on a bootstrap analysis allowing us to get global bounds on moments.

Contents

1. Introduction

We consider the Vlasov–Navier–Stokes system in $\mathbb{T}^3 \times \mathbb{R}^3$:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v)] = 0, \quad (1.1)$$

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = j_f - \rho_f u, \quad (1.2)$$

$$\mathrm{div}\, u = 0, \quad (1.3)$$

where

$$\rho_f(t,x) := \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v,$$
$$j_f(t,x) := \int_{\mathbb{R}^3} vf(t,x,v)\,\mathrm{d}v.$$

This system of nonlinear PDEs describes the transport of particles (described by their density function f) within a fluid (described by its velocity u and its pressure p) and belongs to the broad family of *fluid-kinetic systems* which were introduced in the pioneering works of O'ROURKE [23] and WILLIAMS [25]. Among all possible couplings (we refer to the introduction of [14] for other examples), the Vlasov–Navier–Stokes has been intensively studied because of both its physical relevance (see [5] for instance) and the mathematical challenges that it offers. The Vlasov–Navier–Stokes system is fully coupled: both unknowns f and u depend on each other. This is due to the Brinkman force (the source term in the fluid equation) and the drag acceleration (the inertial term in the kinetic equation). We refer to [5] for the physical justification of these, and to [2,3,11,18,19] for the (partial) mathematical derivation of the former. The physical constants are all normalized in (1.1)–(1.3).

The mathematical analysis of the Vlasov–Navier–Stokes system has been for a long time focused on the existence of (weak or strong) solutions on rather academic domains [4,10,24] like the flat torus that we consider in this paper, or more realistic ones [6,16]. Most of the previous results provide global existence of weak solutions in the following sense: a Leray solution for the fluid equation and a renormalized one (in the sense of DIPERNA AND LIONS [12]) for the kinetic equation (for a more precise definition, see Definition 1.3 below). These global weak solutions are all built by an approximation-compactness argument which is based on the kinetic energy dissipation of the system. More regular solutions can also be constructed. In 2D, thanks to the uniqueness result of [17], they coincide with the weak solutions. In 3D, regular solutions are only known to exist locally (see [10] for instance). This issue is mainly due to the Navier–Stokes part of the system.

Very few articles deal with the long time behavior of this system. At the formal level, one expects a monokinetic behavior in velocity for the distribution function (in other words, concentration to a Dirac mass in velocity), due to the damping of the fluid component and the friction term acting on the kinetic phase. This behavior however has never been completely proven for the Vlasov–Navier–Stokes system. The closest attempt is the paper [10] of Choi and Kwon in which a conditional theorem is provided: the monokinetic behavior is shown to occur under a boundedness assumption that has not been established for any non-trivial global solution up to now. We intend to fill this gap by using the functional introduced by CHOI AND KWON in [10] and proving that this boundedness property (in fact, a stronger one) indeed holds, for appropriate solutions of the Vlasov–Navier–Stokes system, under the assumption that the initial data are (in a sense to be made precise) sufficiently close to equilibrium.

Concerning the long-time behavior of other fluid-kinetic systems, when a Fokker–Planck dissipation is added in the kinetic equation, the situation is less involved because the equilibria are all Maxwellians, which are non-singular and (at least locally) attract all solutions. This has been investigated for instance in [7,15]. Without this dissipation term, apart from [10], we can mention the work of Jabin [21], in which the Navier–Stokes is replaced by a stationary Stokes equation (and a different coupling term) and [14] in which a specific geometry is considered for the Vlasov–Navier–Stokes system, allowing for non-singular stationary solutions.

To the best of our knowledge the results that we present below constitute the first complete and rigorous proof of asymptotic monokinetic behavior for the Vlasov–Navier–Stokes system.

1.1. Weak Solutions of the Vlasov–Navier–Stokes Ssystem

Let us start with a short review of the notion of weak solutions for the Vlasov–Navier–Stokes system, which will give us the opportunity to introduce some notations.

Definition 1.1. The *kinetic energy* of the system (1.1)–(1.3) is given for $t \geqq 0$ by

$$\mathrm{E}(t) := \frac{1}{2}\int_{\mathbb{T}^3} |u(t,x)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)|v|^2\,\mathrm{d}v\,\mathrm{d}x, \tag{1.4}$$

and the *dissipation* is defined as

$$\mathrm{D}(t) := \int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)|u(t,x)-v|^2\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{T}^3} |\nabla u(t,x)|^2\,\mathrm{d}x. \tag{1.5}$$

The kinetic energy and dissipation stem from the seminal papers on the Vlasov–Navier–Stokes sytem [4,16]. One can check that the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) + \mathrm{D}(t) = 0$$

formally holds, which paves the way for a theory of global weak solutions.

Definition 1.2. We shall say that (f_0, u_0) is an admissible initial condition if

$$u_0 \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^3) = \{\mathrm{U} \in \mathrm{L}^2(\mathbb{T}^3),\ \mathrm{div}\,\mathrm{U} = 0\}, \tag{1.6}$$

$$0 \leqq f_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3), \tag{1.7}$$

$$(x,v) \mapsto f_0(x,v)|v|^2 \in \mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3), \tag{1.8}$$

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0\,\mathrm{d}v\,\mathrm{d}x = 1. \tag{1.9}$$

Remark 1.1. The last condition does not play any role for what concerns the properties of existence, uniqueness and long time behavior that we are about to discuss. However, this normalization allows us to simplify the formulas.

We shall also denote $\mathrm{H}^1_{\mathrm{div}}(\mathbb{T}^3) = \mathrm{H}^1(\mathbb{T}^3) \cap \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^3)$.

Definition 1.3. Consider an admissible initial data (u_0, f_0) in the sense of Definition 1.2. A weak solution of the Vlasov–Navier–Stokes system with initial condition (u_0, f_0) is a pair (u, f) with the regularity

$$\begin{aligned} u &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^2(\mathbb{T}^3)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^1_{\mathrm{div}}(\mathbb{T}^3)), \\ f &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)), \\ j_f - \rho_f u &\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^{-1}(\mathbb{T}^3)), \end{aligned}$$

with u being a Leray solution of (1.2)–(1.3) (initiated by u_0) and f a renormalized solution of (1.1) (initiated by f_0), and such that the following energy estimate holds for almost all $t \geqq s \geqq 0$ (including $s = 0$):

$$\mathrm{E}(t) + \int_s^t \mathrm{D}(\sigma)\,\mathrm{d}\sigma \leqq \mathrm{E}(s), \tag{1.10}$$

where the functionals E and D are the energy and dissipation introduced in Definition 1.1.

The existence of weak solutions (u, f) (in the sense of Definition 1.3) to the Vlasov–Navier–Stokes system has been established in [4] (and even on general domains in [6,24]).

Definition 1.4. We say that an initial condition satisfies the *pointwise decay assumption* of order $q > 0$ if

$$(x, v) \longmapsto (1 + |v|^q) f_0(x, v) \in \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3), \tag{1.11}$$

and in that case we denote

$$N_q(f_0) := \sup_{x \in \mathbb{T}^3, v \in \mathbb{R}^3} (1 + |v|^q) f_0(x, v). \tag{1.12}$$

We finally introduce some useful notations for moments in velocity and averages on the torus.

Definition 1.5. For all $\alpha \geqq 0$ and any measurable non-negative function $\mathrm{f} : \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$, we set

$$m_\alpha \mathrm{f}(t, x) := \int_{\mathbb{R}^3} \mathrm{f}|v|^\alpha\,\mathrm{d}v, \tag{1.13}$$

$$M_\alpha \mathrm{f}(t) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} \mathrm{f}|v|^\alpha\,\mathrm{d}v\,\mathrm{d}x. \tag{1.14}$$

For any measurable non-negative function $\mathrm{h} : \mathbb{T}^3 \to \mathbb{R}^d$ (for any $d \in \mathbb{N} \setminus \{0\}$), we denote its average by

$$\langle \mathrm{h} \rangle := \int_{\mathbb{T}^3} \mathrm{h}\,\mathrm{d}x. \tag{1.15}$$

1.2. Heuristics for the Long Time Behavior

In this paper, we focus on the description of the long time behavior of weak solutions to the Vlasov–Navier–Stokes system. To this end, it is enlightning to first have a look at the linear Vlasov equation with friction, around the trivial equilibrium $(0, 0)$. This reads as

$$\partial_t g + v \cdot \nabla_x g - \mathrm{div}_v[gv] = 0. \tag{1.16}$$

Endowed with an initial condition g_0 at $t = 0$, this equation admits the explicit solution

$$g(t, x, v) = e^{3t} g_0(x - (e^t - 1)v, e^t v). \tag{1.17}$$

Definition 1.6. For $\mathrm{U} \in \mathbb{R}^3$, we denote by δ_{U} the Dirac measure in velocity supported at U, defined by

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^3), \qquad \langle \delta_{\mathrm{U}}, \varphi \rangle = \varphi(\mathrm{U}).$$

The long time behavior of the solution to (1.16) is explicit, as we observe from (1.17) that

$$g(t, x, v) \rightharpoonup_{t \to +\infty} \left(\int_{\mathbb{R}^3} g_0(x - v, v)\, \mathrm{d}v \right) \otimes \delta_0.$$

More generally, given $\mathrm{U} \in \mathbb{R}^3$, for the equation

$$\partial_t g + v \cdot \nabla_x g + \mathrm{div}_v[g(\mathrm{U} - v)] = 0, \tag{1.18}$$

the long time behavior of the solution is also explicit and described by

$$g(t, x, v) - \left(\int_{\mathbb{R}^3} g_0(x - v - t\mathrm{U}, v + \mathrm{U})\, \mathrm{d}v \right) \otimes \delta_{\mathrm{U}} \rightharpoonup_{t \to +\infty} 0.$$

The mechanism at stake in (1.16) and (1.18) is a competition between transport and friction. Friction always wins in the end, causing concentration to a Dirac mass in velocity. In view of this behavior, we may expect a similar concentration phenomenon in velocity for the full Vlasov–Navier–Stokes system, at least in a regime close to some equilibrium.

It is actually even possible to push the heuristics a little further. Taking for granted that the kinetic phase concentrates in velocity, with the behavior $f(t, x, v) \sim \rho_f(t, x) \otimes \delta_{u(t,x)}$ as $t \to +\infty$, we observe in particular that the Brinkman force in the Navier–Stokes equations vanishes as $t \to +\infty$. Since it is well known that the solution $u(t)$ of the Navier–Stokes without forcing tends to homogenize to its average in space $\langle u \rangle(t)$, we may expect that $f(t, x, v) \sim \rho_f(t, x) \otimes \delta_{\langle u \rangle(t)}$ as $t \to +\infty$. In particular this entails $\langle j_f \rangle(t) \sim \langle \rho_f \rangle(t) \langle u \rangle(t)$ as $t \to +\infty$, but then, by the conservation laws

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t)\, \mathrm{d}v\, \mathrm{d}x = 1, \quad \langle u + j_f \rangle(t) = \langle u_0 + j_{f_0} \rangle$$

(see (3.2) and (3.3) in Lemma 3.1), we deduce that $\langle u\rangle(t) \sim \frac{\langle u_0+j_{f_0}\rangle}{2}$ as $t \to +\infty$. To summarize, it follows from this heuristic argument that one can expect

$$f(t,x,v) \sim \rho_f(t,x) \otimes \delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}$$

as $t \to +\infty$, that corresponds to concentration to the constant velocity $\frac{\langle u_0+j_{f_0}\rangle}{2}$.

1.3. The Modulated Energy of Choi and Kwon

In [10], Choi and Kwon introduced a *modulated* version of the energy $\mathrm{E}(t)$ of Definition 1.1:

Definition 1.7. We define the *modulated energy* as

$$\begin{aligned}\mathscr{E}(t) := \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)|v-\langle j_f(t,x)\rangle|^2 \,\mathrm{d}v\,\mathrm{d}x \\ +\frac{1}{2}\int_{\mathbb{T}^3} |u(t,x)-\langle u(t)\rangle|^2\,\mathrm{d}x + \frac{1}{4}|\langle j_f(t)\rangle - \langle u(t)\rangle|^2. \quad (1.19)\end{aligned}$$

It is proved in [10] that the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) + \mathrm{D}(t) = 0$$

formally holds. Controlling the modulated energy is interesting in view of the expected long time monokinetic dynamics for the kinetic phase, because of the following statement:

Lemma 1.1. *With the previous notations, we have that for all $t \geqq 0$,*

$$W_1\left(f(t), \rho_f(t)\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\right) + \left\| u(t) - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right\|_{L^2(\mathbb{T}^3)} \lesssim (\mathscr{E}(t))^{1/2}, \tag{1.20}$$

where W_1 is the Wasserstein(-1) distance.

The definition and basic properties of the Wasserstein distance W_1 are given in the Appendix (see Section 9.1). The proof of the previous lemma is postponed to Section 3.2.

2. Main Results

Our main result provides a sharp description of the long time behavior of weak solutions to the Vlasov–Navier–Stokes system.

Theorem 2.1. *There exists $C_\star > 0$ and a nondecreasing onto function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds. Let (u_0, f_0) be an admissible initial condition such that $N_q(f_0) < +\infty$ for some $q > 4$, $M_\alpha f_0 < +\infty$ for some $\alpha > 3$ and $u_0 \in H^{1/2}(\mathbb{T}^3)$. Then, if*

$$\|u_0\|_{\dot{H}^{1/2}(\mathbb{T}^3)} < \frac{1}{C_\star^2},$$

and if the initial modulated energy $\mathscr{E}(0)$ is small enough, in the sense that

$$\begin{aligned}\varphi &\left(N_q(f_0) + M_\alpha f_0 + E(0) + \|u_0\|_{H^{1/2}(\mathbb{T}^3)} + 1\right) \mathscr{E}(0) \\ &< \min\left(1, \frac{1}{C_\star^2} - \|u_0\|^2_{\dot{H}^{1/2}(\mathbb{T}^3)}\right), \end{aligned} \tag{2.1}$$

then for any weak solution (u, f) to the Vlasov–Navier–Stokes system, there exists a profile $\rho^\infty \in L^\infty(\mathbb{T}^3)$ and $\lambda, C_\lambda > 0$ such that, for all $t \geqq 0$,

$$\begin{aligned}&\left\| u(t) - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right\|_{L^2(\mathbb{T}^3)} + W_1\left(f(t), \rho^\infty\left(x - t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \\ &\leqq \sqrt{\mathscr{E}(0)} C_\lambda \exp(-\lambda t), \end{aligned} \tag{2.2}$$

where W_1 is the Wasserstein distance.

We refer to the solutions that we consider as *Fujita–Kato* type, as we require small initial $\dot{\mathrm{H}}^{1/2}$ norm for the fluid velocity.

Remark 2.1. The constant $\mathrm{C}_\star$ is the universal constant given in Proposition 9.10.

We deduce that when $\langle u_0 + j_{f_0}\rangle = 0$, the distribution function $f(t)$ weakly converges to a stationary solution, whereas when $\langle u_0 + j_{f_0}\rangle \neq 0$, the asymptotic behavior is that of a travelling wave.

Remark 2.2. As already said, existence of weak solutions follows from [4] (note by the way that both the pointwise decay assumption and the higher order Sobolev assumption are not relevant for this part).

Remark 2.3. The fact that the asymptotic state for the distribution function is a Dirac mass in velocity, and thus is singular, virtually forbids the use of standard PDE techniques, such as high order Sobolev energy estimates, to prove this result.

Remark 2.4. This result proves that for the Vlasov–Navier–Stokes system, the large time behavior on the torus is very different from that on a domain with partially dissipative boundary conditions (and under adequate geometric control conditions): in [14], it is indeed proved that in the latter case there exist smooth non-trivial equilibria that are locally stable.

Theorem 2.1 will be a consequence of the following result, bearing on the large time behavior of the modulated energy $\mathscr{E}(t)$:

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have that there exists* $\lambda, C'_\lambda > 0$ *such that for all* $t \geqq 0$,

$$\mathscr{E}(t) \leqq \mathscr{E}(0) C'_\lambda e^{-\lambda t}. \tag{2.3}$$

Furthermore, we have the global bounds

$$\sup_{t \geqq 0} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^3)} < +\infty, \tag{2.4}$$

and

$$\int_1^{+\infty} \|\nabla u(\tau)\|_{L^\infty(\mathbb{T}^3)} \, \mathrm{d}\tau \leqq \eta(\mathscr{E}(0)), \tag{2.5}$$

where η *is a continuous nonnegative function such that* $\eta(0) = 0$.

Remark 2.5. The constants C_λ, C'_λ appearing in Theorems 2.1 and 2.2 are uniform with respect to the various (semi-)norms of u_0 and f_0 that appear in the assumptions.

It is actually even possible to describe the structure of the final density ρ^∞.

Proposition 2.3. *For* δ *small enough, under the assumptions of Theorem 2.1, if furthermore* $u \in \mathscr{C}^0(\mathbb{R}_+; W^{1,\infty}(\mathbb{T}^3))$ *and*

$$\int_0^{+\infty} \|\nabla u(\tau)\|_{L^\infty(\mathbb{T}^3)} \, \mathrm{d}\tau \leqq \delta, \tag{2.6}$$

then there exists a vector field

$$\begin{aligned} \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (s, x, v) &\longmapsto Y^s_{\infty,x,v}, \end{aligned}$$

belonging to $\mathscr{C}^0(\mathbb{R}_+; \mathscr{C}^1(\mathbb{T}^3 \times \mathbb{R}^3))$ *and such that we have*

$$\rho^\infty(x) = \int_{\mathbb{R}^3} f_0\left(Y^0_{\infty,x,v}, v\right) |\det \mathscr{A}(\infty, x, v)| \, \mathrm{d}v, \tag{2.7}$$

with

$$\mathscr{A}(\infty, x, v) = I_3 + \int_0^{+\infty} e^s \nabla u\left(\tau, Y^\tau_{\infty,x,v}\right) D_v Y^\tau_{\infty,x,v} \, \mathrm{d}\tau, \tag{2.8}$$

and $s \mapsto Y^s_{\infty,x,v}$ *satisfies*

$$\begin{aligned} Y^s_{\infty,x,v} &= x - e^{-s} v + \frac{\langle u_0 + j_{f_0} \rangle}{2} \left(e^{-s} + s\right) \\ &\quad - \int_0^{+\infty} \left[\mathbf{1}_{[0,s]}(\tau) e^{\tau - s} + \mathbf{1}_{\tau \geqq s}\right] \left(u(\tau, Y^\tau_{\infty,x,v}) - \frac{\langle u_0 + j_{f_0} \rangle}{2}\right) \mathrm{d}\tau. \end{aligned} \tag{2.9}$$

Remark 2.6. The assumption (2.6) on u is restrictive in the sense that the integral goes down to the time $t = 0$. Indeed, the parabolic regularization and the estimates obtained in Section 6 prove that with the assumptions of Theorem 2.1 alone, $u \in \mathscr{C}^0([\varepsilon, +\infty); \mathrm{W}^{1,\infty}(\mathbb{T}^3))$ for all $\varepsilon > 0$. The assumption (2.6) therefore requires higher regularity for the initial fluid velocity u_0. It is also possible to avoid this extra regularity assumption, replacing f_0 by the value of f at time $t = 1$, and all integrals starting from $s = 0$ by the same starting from $s = 1$. The relevant assumption replacing (2.6), namely

$$\int_1^{+\infty} \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \, \mathrm{d}\tau \leqq \delta, \tag{2.10}$$

is then obtained as a consequence of Theorem 2.2, see (2.5), when $\mathscr{E}(0)$ is taken small enough.

Remark 2.7. Proposition 2.3 is proved in Section 8. For the sake of clarity the proof focuses on the case $\langle u_0 + j_{f_0} \rangle = 0$. The proof of the general case is similar and adds in only a few lines of computations, see Remark 8.1.

There are mainly two stabilization mechanisms at stake in the large time dynamics of solutions to the Vlasov–Navier–Stokes system. The first one is due to *friction* in the Vlasov equation, that forces the distribution function to concentrate in velocity. The second stabilization mechanism comes from the *dissipation* in the Navier–Stokes equations. There is a competition in the Navier–Stokes equations between this dissipation and the possible growth of the non-linearity and the Brinkman force $F = j_f - \rho_f u$. Loosely speaking, the smallness assumptions we make allow to tame the influence of the forcing.

As already briefly discussed in the introduction, thanks to the fine structure of the system, there happens to be a modulated energy/dissipation identity that follows from the energy identity and the conservation laws of the system, as exhibited by Choi and Kwon [10]. This identity somehow reflects the two stabilization mechanisms we have just discussed.

2.1. The Case of Dimension 2

For the sake of conciseness and physical relevance, we focus in this paper on the case of dimension 3. However, with the same method that we develop, it is possible to study the Vlasov–Navier–Stokes system on $\mathbb{T}^2 \times \mathbb{R}^2$ with weaker regularity assumptions on the initial data. Namely, we can treat admissible data (in all the Definitions, statements or equations discussed in this section, one has to replace $\mathbb{T}^3, \mathbb{R}^3$ by $\mathbb{T}^2, \mathbb{R}^2$ when necessary), without requiring the higher $\dot{\mathrm{H}}^{1/2}$ regularity for u_0 like in dimension 3, see Theorem 2.1 (the fact that more stringent regularity assumptions are required in dimension 3 is due to the well-known difficulties related to the resolution of the Navier–Stokes equations). For the record, we gather in the following statement what we may obtain in dimension 2.

Theorem 2.4. *There exists a nondecreasing onto function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the following holds. Let* (f_0, u_0) *be an admissible initial condition such that* $N_q(f_0) < +\infty$ *for some* $q > 4$. *Assume that the initial modulated energy* $\mathscr{E}(0)$ *is small enough, in the sense that*

$$\varphi\left(N_q(f_0) + E(0) + 1\right)\mathscr{E}(0) < 1. \tag{2.11}$$

Then the weak solution (u, f) *to the Vlasov–Navier–Stokes system satisfies* (2.2).

Note that the statement is also strengthened compared to dimension 3 since there is uniqueness of the weak solution of the Vlasov–Navier–Stokes system: it has been indeed established in [17] that in dimension $d = 2$, under the pointwise decay assumption of order $q > 4$ of Definition 1.4 (and in fact an even less stringent condition is sufficient), uniqueness holds for weak solutions of the Vlasov–Navier–Stokes system.

The proof developed in dimension 3 applies *mutatis mutandis*, with the following significant simplifications:

- the $\dot{\mathrm{H}}^{1/2}$ regularity for the fluid velocity is not required in order to get higher order energy estimates for positive times, and therefore in particular we do not need to propagate $\dot{\mathrm{H}}^{1/2}$ estimates for all times;
- we can rely on various estimates already proved in [17];
- several indices in the Sobolev embeddings are more favorable in dimension 2.

Let us finally mention that Proposition 2.3 holds as well in dimension 2.

2.2. *Outline of the Proof and Organisation of the Paper*

To conclude this section, let us provide a (non-technical) outline of the proof of Theorem 2.2. This also gives the opportunity to describe how this paper is organized.

The purpose of the first Section 3 is to explain how Theorem 2.1 can be deduced from Theorem 2.2, and more strikingly how the proof of the latter boils down to *one* single uniform estimate on the local density of the kinetic phase. In Section 3.1 we gather conservation laws for the Vlasov–Navier–Stokes system. Section 3.2 emphasizes the role of the modulated energy, and we prove therein Lemma 1.1, which explains how the decay of this functional leads to concentration in velocity for the particles. Sections 3.3 and 3.4 detail the following key observation of [10]: up to a control of the $\mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ norm of the local density $\rho_f = \int_{\mathbb{R}^3} f \, \mathrm{d}v$, the modulated energy is essentially controlled by its dissipation, yielding exponential decay. We also explain, following an argument of JABIN [21], how one can recover the existence of the asymptotic profile ρ^∞ appearing in (2.2), once the exponential decay is established.

As a consequence of Section 3, the proof of Theorem 2.2 (and therefore Theorem 2.1) relies only on obtaining the following global bound for ρ_f:

$$\sup_{t \geqq 0} \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} < +\infty, \tag{2.12}$$

and in fact our bootstrap strategy actually will prove the same estimate for $j_f = \int_{\mathbb{R}^3} f v \, \mathrm{d}v$.

In Section 4, we present the main tools we used to obtain such bounds on moments. They are based on the method of characteristics, which allows, considering the characteristics curves (X, V) solving the system

$$\begin{aligned} \dot{\mathrm{X}}(s; t, x, v) &= \mathrm{V}(s; t, x, v), \\ \dot{\mathrm{V}}(s; t, x, v) &= u(s, \mathrm{X}(s; t, x, v)) - \mathrm{V}(s; t, x, v), \end{aligned} \tag{2.13}$$

with $(\mathrm{X}(t; t, x, v), \mathrm{V}(t; t, x, v)) = (x, v)$, to write solutions to the Vlasov equation as

$$f(t, x, v) = e^{3t} f_0(\mathrm{X}(0; t, x, v), \mathrm{V}(0; t, x, v)). \tag{2.14}$$

We deduce that

$$\rho_f(t, x) = e^{3t} \int_{\mathbb{R}^3} f_0(t, \mathrm{X}(0; t, x, v), \mathrm{V}(0; t, x, v)) \, \mathrm{d}v. \tag{2.15}$$

In order to study (2.15), we rely on a change of variables in velocity, referred to as the *straightening* change of variables, namely $v \mapsto \mathrm{V}(0; t, x, v)$. It is not obvious that this map is a diffeomorphism. In, Section 4, we provide a sufficient condition to ensure this: there exists a constant $\delta > 0$ such that, if

$$\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \, \mathrm{d}s < \delta, \tag{2.16}$$

then indeed the straightening change of variable is admissible. Under this smallness condition, the outcome is the estimate

$$\|\rho_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{T}^3))} \lesssim N_q(f_0).$$

Similar bounds for $j_f = \int_{\mathbb{R}^3} v f \, \mathrm{d}v$ can be obtained as well.

This change of variables is inspired by that used by Bardos and Degond [1] for the study of global small solutions to the Vlasov–Poisson system on $\mathbb{R}^3 \times \mathbb{R}^3$.

As a consequence of Section 4, the remaining task is now to prove that for small enough initial modulated energy, the estimate (2.16) holds for all $t \geqq 1$ (small times are handled by local estimates). Sections 5 to 7 are dedicated to this task. To this end, we set up a bootstrap argument. Loosely speaking, we consider

$$t^\star := \sup \left\{ t \geqq 1, \ \int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \, \mathrm{d}s < \delta \right\}, \tag{2.17}$$

and the aim is to show that $t^\star = +\infty$. The general strategy is as follows: assuming $t^\star < +\infty$, we work on the interval of time $[1, t^\star]$, and shall obtain regularity estimates for u using higher order energy estimates for the Navier–Stokes equations and maximal parabolic estimates for the Stokes equations. Such bounds are not relevant in terms of decay in time but on $[1, t^\star]$ we have thanks to the straightening change of variables

$$\sup_{t \in [1, t^\star]} \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim 1.$$

Therefore, by Choi–Kwon's key observation, $\mathscr{E}(t)$ decays exponentially fast on $[1, t^\star)$. The idea is then to interpolate the higher regularity estimate with the pointwise $\mathrm{L}^2(\mathbb{T}^3)$ bound bearing on $u - \langle u \rangle$ which is provided by the exponential decay of the modulated energy. More precisely, we use the Gagliardo–Nirenberg–Sobolev interpolation inequalities to obtain

$$\|u(s) - \langle u(s) \rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \|\mathrm{D}^2 u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}^{\alpha} \|u(s) - \langle u(s) \rangle\|_{\mathrm{L}^2(\mathbb{T}^3)}^{1-\alpha}, \tag{2.18}$$

for $\alpha \in (0, 1)$; we argue similarly for the control of ∇u.

To apply the previous bootstrap strategy, we need enough regularity and integrability on the solutions of the Vlasov–Navier–Stokes system. We prove in Section 5 that any weak solution of the system instantaneously satisfies adequate estimates, which includes:

- a short time control of ρ_f and j_f in $\mathrm{L}^\infty(\mathbb{T}^3)$, using local bounds;
- $\mathrm{L}^\infty_t \mathrm{H}^1_x \cap \mathrm{L}^2_t \mathrm{H}^2_x$ estimates for u, on time intervals *away* from zero, that is to say for $t \geqq 1$

$$\|\nabla u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}^2 + \int_1^t \|\Delta u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}^2 \,\mathrm{d}s \lesssim 1 + \sup_{s \in [1,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}. \tag{2.19}$$

We introduce the convenient notion of *strong existence times* in order to be able to propagate regularity.

In Section 6, we start to implement the interpolation strategy, relying this time on higher order maximal parabolic estimates for the Stokes equation. The outcome is a control of $\mathrm{D}^2 u$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^q(\mathbb{T}^3))$ by $(u \cdot \nabla)u$ and $j_f - \rho_f u$ in the same space.

Then Section 7 is dedicated to the proof of the global bound (2.12) : we explain therein how the previous control of $\mathrm{D}^2 u$ can be iteratively used to produce an estimate of the form

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \mathscr{E}(0)^\gamma.$$

Consequently, if $\mathscr{E}(0)$ is small enough, then we must have $t^\star = +\infty$, which concludes the proof of Theorem 2.2.

Finally Section 8 is devoted to the proof of Proposition 2.3 which provides a sharper description of the asymptotic behavior. The analysis comes down to the study of the limit as $t \to \infty$ of characteristics (more precisely of renormalized versions of them). For the sake of clarity, the proof is written in the particular case $\langle u_0 + j_{f_0} \rangle = 0$ to lighten the computations (see Remark 8.1).

To conclude the paper, Section 9 is an Appendix where we provide some reminders (in particular, we shortly review some well-known basic facts about the Wasserstein distance) and justify H^1 energy estimates for the Navier–Stokes equations with source.

3. Conservation Laws, Energy Dissipation Identities and Consequences

3.1. Conservation Laws

We discuss here some conservations laws for the Vlasov–Navier–Stokes system. We start by describing some basic ones in a first lemma: the first two ones come from the structure of the Vlasov equation alone, while the third one is a consequence of the fine structure of the complete system.

Lemma 3.1. *Any weak solution (in the sense of Definition (1.3)) satisfies the following conservations laws: for almost all* $t \geqq 0$*,*

$$f(t) \geqq 0, \quad \textit{for almost all } (x,v) \in \mathbb{T}^3 \times \mathbb{R}^3, \tag{3.1}$$

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0\,\mathrm{d}v\,\mathrm{d}x = 1, \tag{3.2}$$

$$\langle u + j_f\rangle(t) = \langle u_0 + j_{f_0}\rangle. \tag{3.3}$$

Proof. Considering the results of [4], the only item to prove is (3.3). Let us assume that both u and f are smooth functions. Integrating the Vlasov equation against v, the conservation law satisfied by j_f reads as

$$\partial_t j_f + \mathrm{div}\left(\int_{\mathbb{R}^3} f\, v \otimes v\, dv\right) = \rho_f u - j_f, \tag{3.4}$$

so that $\langle j_f\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle j_f\rangle = \langle \rho_f u - j_f\rangle.$$

On the other hand, from (1.2), $\langle u\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle = \langle j_f - \rho_f u\rangle,$$

from which we deduce $\frac{\mathrm{d}}{\mathrm{d}t}\langle u + j_f\rangle = 0$, and consequently (3.3).

In the general case, for the fluid equation we can directly use $\varphi = 1$ as an admissible test function to recover almost everywhere

$$\langle u(t)\rangle - \langle u_0\rangle = \int_0^t \langle j_f - \rho_f u\rangle(s)\,\mathrm{d}s.$$

For the kinetic equation we use an approximation argument relying on DiPerna and Lions theory [12] for linear transport equations : we consider a sequence of nonnegative distribution functions $(f_n)_n$ solving the Vlasov equation with regularized vector fields $(u_n)_n$ and regularized and truncated initial conditions $(f_{0,n})_n$, and such that for all $n \geqq 1$ and all $t \geqq 0$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n |v|^2\,\mathrm{d}v\,\mathrm{d}x \lesssim 1.$$

By the DiPerna–Lions theory, f is the (strong) limit of $(f_n)_n$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^p(\mathbb{T}^3\times\mathbb{R}^3))$ for all finite values of p ; interpolating with the previous bound we infer that $(j_{f_n})_n \to j_f$ strongly in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3))$ and $(\rho_{f_n}u_n)_n \to \rho_f u$ at least in $\mathrm{L}^1(0,T;\mathrm{L}^1(\mathbb{T}^3))$. This is sufficient to pass to the limit in the following identity (which is justified at the regularized level)

$$\langle j_{f_n}(t)\rangle - \langle j_{f_0}\rangle = \int_0^t \langle \rho_{f_n}u_n - j_{f_n}\rangle(s)\,\mathrm{d}s,$$

and finally, (3.3) follows for almost every t. □

A straightforward consequence of (3.3) in Lemma 3.1 is the following formula:

Lemma 3.2. *For almost all $t \geqq 0$,*

$$\frac{1}{4}|\langle j_f\rangle(t) - \langle u\rangle(t)|^2 = \left|\langle j_f\rangle(t) - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|^2 = \left|\langle u\rangle(t) - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|^2. \quad (3.5)$$

Remark 3.1. We shall use in this paper several times the DiPerna–Lions theory [12], in the same fashion as in the proof of Lemma 3.1. Thanks to the property of strong stability of renormalized solutions, this allows to systematically argue as if both f and u are smooth when looking to establish estimates for the kinetic phase. The argument, as already outlined in the proof of Lemma 3.1, is the following:

- consider an approximating sequence $(u_n)_n$ for u and $(f_n)_n$ the associated solution to the Vlasov equation, with a regularized initial condition;
- prove the desired estimate for the solution f_n (without explicitly using the higher regularity of f_n or u_n);
- pass to the limit using the strong stability property of renormalized solutions (and Fatou's lemma).

In the following, for brevity, we will never write down this argument explicitly but will repeatedly refer to the current remark.

3.2. The Role of the Modulated Energy: Proof of Lemma 1.1

Proof. By the Monge–Kantorovich duality for the W_1 distance (see Proposition 9.2 in the Appendix), we have

$$\begin{aligned}
&\mathrm{W}_1\left(f(t), \rho_f(t)\otimes\delta_{\langle j_f\rangle}\right) \\
&\quad = \sup_{\|\nabla_{x,v}\phi\|_\infty \leqq 1} \left\{\int_{\mathbb{T}^3}\left(\int_{\mathbb{R}^3} f(t,x,v)\phi(x,v)\,\mathrm{d}v - \rho(t,x)\phi(x,\langle j_f\rangle)\right)\mathrm{d}x\right\} \\
&\quad = \sup_{\|\nabla_{x,v}\phi\|_\infty \leqq 1} \left\{\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)(\phi(x,v) - \phi(x,\langle j_f\rangle))\mathrm{d}v\,\mathrm{d}x\right\} \\
&\quad \leqq \int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - \langle j_f\rangle|\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We thus infer, using the Cauchy–Schwarz inequality, the normalization (3.2) and the definition of the modulated energy $\mathscr{E}(t)$

$$\begin{aligned}&\mathrm{W}_1\left(f(t), \rho_f(t)\otimes\delta_{\langle j_f\rangle}\right)\\&\leqq\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v-\langle j_f\rangle|^2\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\leqq\sqrt{2}\,\mathscr{E}(t)^{1/2}.\end{aligned}$$

Likewise,

$$\begin{aligned}&\mathrm{W}_1\Big(\rho_f\otimes\delta_{\langle j_f\rangle}, \rho_f\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\Big)\\&=\sup_{\|\nabla_{x,v}\phi\|_\infty\leqq 1}\int_{\mathbb{T}^3}\rho_f(t,x)\Big(\phi(x,\langle j_f\rangle)-\phi\left(x,\frac{\langle u_0+j_{f_0}\rangle}{2}\right)\Big)\mathrm{d}x\\&\leqq\left|\langle j_f\rangle-\frac{\langle u_0+j_{f_0}\rangle}{2}\right|\int_{\mathbb{T}^3}\rho_f(t,x)\mathrm{d}x.\end{aligned}$$

We therefore deduce, using the normalization (3.2) and the identity (3.5)

$$\mathrm{W}_1\Big(\rho_f\otimes\delta_{\langle j_f\rangle}, \rho_f\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\Big)\leqq\frac{1}{2}\left|\langle j_f\rangle-\langle u\rangle\right|\leqq\mathscr{E}(t)^{1/2},$$

so that by triangular inequality we have established

$$\mathrm{W}_1\left(f(t), \rho_f(t)\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\right)\lesssim\mathscr{E}(t)^{1/2}.$$

On the other hand, using again (3.5), we can also estimate

$$\begin{aligned}\left\|u(t)-\frac{\langle u_0+j_{f_0}\rangle}{2}\right\|_{\mathrm{L}^2(\mathbb{T}^3)}&\leqq\|u(t)-\langle u(t)\rangle\|_{\mathrm{L}^2(\mathbb{T}^3)}+\left\|\langle u(t)\rangle-\frac{\langle u_0+j_{f_0}\rangle}{2}\right\|_{\mathrm{L}^2(\mathbb{T}^3)}\\&\leqq\|u(t)-\langle u(t)\rangle\|_{\mathrm{L}^2(\mathbb{T}^3)}+\frac{1}{4}|\langle j_f\rangle-\langle u\rangle|^2,\end{aligned}$$

and the result follows. □

3.3. Dissipation of the Modulated Energy

As already said in the introduction, Choi and Kwon noticed in [10][1] that the modulated energy (see Definition 1.7) satisfies the following formal identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t)+\mathrm{D}(t)=0. \tag{3.6}$$

At the level of weak solutions, we are only able to obtain the inequality version of (3.6), as stated in the next lemma.

[1] As a matter of fact, they consider the more general Vlasov-*inhomogeneous* Navier–Stokes system but we recover the system (1.1)–(1.3) as soon as we stick to the case of constant fluid density.

Lemma 3.3. *For any weak solution (u, f) in the sense of Definition 1.3, for almost all $t \geqq 0$,*

$$\mathscr{E}(t) - E(t) = -\frac{1}{4}|\langle u_0 + j_{f_0}\rangle|^2.$$

In particular, we have the following modulated energy/dissipation inequality for almost all $0 \leqq s \leqq t < +\infty$ (including $s = 0$),

$$\mathscr{E}(t) + \int_s^t D(\sigma)\, \mathrm{d}\sigma \leqq \mathscr{E}(s). \tag{3.7}$$

Proof. Let us first write

$$\mathscr{E}(t) = \mathrm{E}(t) + \frac{1}{2}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f\, \mathrm{d}v\, \mathrm{d}x\right)\langle j_f\rangle^2 - \langle j_f\rangle^2 - \frac{1}{2}\langle u\rangle^2 + \frac{1}{4}|\langle j_f\rangle - \langle u\rangle|^2,$$

that we can simplify in the following way thanks to (3.2)

$$\begin{aligned}\mathscr{E}(t) &= \mathrm{E}(t) - \frac{1}{2}\langle j_f\rangle^2 - \frac{1}{2}\langle u\rangle^2 + \frac{1}{4}|\langle j_f\rangle - \langle u\rangle|^2 \\ &= \mathrm{E}(t) - \frac{1}{4}|\langle j_f\rangle + \langle u\rangle|^2,\end{aligned}$$

so that $\mathscr{E}(t) - \mathrm{E}(t)$ does not depend on t thanks to (3.3). Estimate (3.7) follows then from the energy estimate (1.10). □

3.4. Conditional Long Time Behavior

Definition 3.1. Let c_P be the Poincaré constant, that is the best constant such that the Poincaré–Wirtinger inequality holds:

$$\|g - \langle g\rangle\|_{\mathrm{L}^2(\mathbb{T}^3)} \leqq c_P\|\nabla g\|_{\mathrm{L}^2(\mathbb{T}^3)}, \quad \forall g \in \mathrm{H}^1(\mathbb{T}^3). \tag{3.8}$$

The following result relating the dissipation and the modulated energy is a variant of [10, Theorem 1.2]:

Lemma 3.4. *There exists a continuous nonincreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds, for any weak solution of the VNS system (in the sense of Definition 1.3) for which $\rho_f \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{T}^3))$. Fix $T > 0$ and define*

$$\lambda := \psi\left(\sup_{[0,T]} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^3)}\right). \tag{3.9}$$

Then

$$\forall t \in [0, T], \quad D(t) \geqq \lambda\mathscr{E}(t), \tag{3.10}$$

and we have the exponential estimate

$$\forall t \in [0, T], \quad \mathscr{E}(t) \lesssim e^{-\lambda t}\mathscr{E}(0), \tag{3.11}$$

where $\lesssim$ depends only on λ.

Proof. First we note that (3.10) $\Rightarrow$ (3.11). Indeed, combining with estimate (3.7) of Lemma 3.3, we get for almost all $0 \leqq s \leqq t \leqq T$,

$$\mathscr{E}(t) + \lambda \int_s^t \mathscr{E}(\sigma)\,\mathrm{d}\sigma \leqq \mathscr{E}(s),$$

so that from Lemma 9.3 of the Appendix we get $\mathscr{E}(t) \lesssim \mathscr{E}(0)e^{-\lambda t}$, where $\lesssim$ depends only on λ. We therefore focus on (3.10) and try to find $\lambda > 0$ of the form (3.9).

Define

$$\widetilde{\mathscr{E}}(t) := \mathscr{E}(t) - \frac{1}{2}\|u(t) - \langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^3)}.$$

The Poincaré-Wirtinger inequality gives us a constant $c_P > 0$ such that

$$\mathrm{D}(t) \geqq \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)|v - u(t)|^2\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}c_P\|u(t) - \langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^3)}.$$

Therefore to get (3.10) for some $\lambda > 0$, it is sufficient to prove that for some $\gamma, \beta > 0$ we have

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)|v - u(t)|^2\,\mathrm{d}v\,\mathrm{d}x \geqq \gamma\,\widetilde{\mathscr{E}}(t) - \beta\|u(t) - \langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^3)}, \qquad (3.12)$$

with β small enough (namely $\beta < c_P$) : in that case we have $\mathrm{D}(t) \geqq \lambda\mathscr{E}(t)$ with $\lambda := \min(\gamma, c_P - \beta)$.

For the sake of clarity, we omit the time variable for a few lines. We also denote $\|\rho\|_{\infty,T} := \sup_{[0,T]}\|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$. We start with the identity

$$|v - u|^2 = |v - \langle u\rangle|^2 + 2(v - \langle u\rangle)\cdot(\langle u\rangle - u) + |\langle u\rangle - u|^2,$$

from which we infer

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - u|^2\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - \langle u\rangle|^2\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{T}^3}\rho_f|\langle u\rangle - u|^2\mathrm{d}x$$
$$+2\int_{\mathbb{T}^3\times\mathbb{R}^3} f(v - \langle u\rangle)\cdot(\langle u\rangle - u)\,\mathrm{d}v\,\mathrm{d}x. \qquad (3.13)$$

Now for any $\alpha \in (0, 1)$, Young's inequality entails that

$$2\int_{\mathbb{T}^3\times\mathbb{R}^3} f(\langle u\rangle - v)\cdot(u - \langle u\rangle)\,\mathrm{d}v\,\mathrm{d}x$$
$$\geqq -\alpha\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - \langle u\rangle|^2\,\mathrm{d}v\,\mathrm{d}x - \alpha^{-1}\int_{\mathbb{T}^3}\rho_f|u - \langle u\rangle|^2\,\mathrm{d}x.$$

Combining with (3.13) we have therefore

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - u|^2\,\mathrm{d}v\,\mathrm{d}x \geqq (1-\alpha)\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - \langle u\rangle|^2\,\mathrm{d}v\,\mathrm{d}x$$
$$-(\alpha^{-1} - 1)\int_{\mathbb{T}^3}\rho_f|\langle u\rangle - u|^2\mathrm{d}x. \qquad (3.14)$$

On the other hand, we have

$$|v - \langle u\rangle|^2 = |\langle j_f\rangle - \langle u\rangle|^2 + 2(v - \langle j_f\rangle)\cdot(\langle j_f\rangle - \langle u\rangle) + |v - \langle j_f\rangle|^2,$$

from which we deduce

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - \langle u\rangle|^2 \,\mathrm{d}v\,\mathrm{d}x = |\langle j_f\rangle - \langle u\rangle|^2 + \int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - \langle j_f\rangle|^2 \,\mathrm{d}v\,\mathrm{d}x,$$

where we used the normalization property (3.2) and

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(v - \langle j_f\rangle)\cdot(\langle j_f\rangle - \langle u\rangle)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

In particular, we have

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)|v - \langle u(t)\rangle|^2 \,\mathrm{d}v\,\mathrm{d}x \geqq \widetilde{\mathscr{E}}(t).$$

Since $\alpha \in (0, 1)$ we deduce from (3.14)

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3} f|v - u|^2 \,\mathrm{d}v\,\mathrm{d}x &\geqq (1-\alpha)\widetilde{\mathscr{E}}(t) - (\alpha^{-1} - 1)\int_{\mathbb{T}^3} \rho_f|\langle u\rangle - u|^2 \mathrm{d}x \\ &\geqq (1-\alpha)\widetilde{\mathscr{E}}(t) - (\alpha^{-1} - 1)\|\rho_f\|_{\infty,T}\int_{\mathbb{T}^3} |\langle u\rangle - u|^2 \mathrm{d}x,\end{aligned}$$

which is exactly (3.12) with $\gamma := 1 - \alpha$ and $\beta = (\alpha^{-1} - 1)\|\rho_f\|_{\infty,T}$. Picking α close enough to 1 (to ensure $\beta < c_P$), $\lambda := \min(\gamma, c_P - \beta)$ satisfies (3.10). To check that λ can indeed be chosen of the form (3.9) we have to make more explicit the choice of α by imposing for instance the condition $\beta = c_P/2$ above, that is $\alpha^{-1} = c_P/(2\|\rho\|_{\infty,T}) + 1$ which is a continuous, nonincreasing, nonvanishing function of $\|\rho\|_{\infty,T}$: α is then continuous and increasing and $\lambda := \min(1-\alpha, c_P/2)$ is of the form (3.9). □

Once exponential decay of the modulated energy is ensured, one can prove the existence of an asymptotic profile ρ^∞ for which we have the following convergence statement:

Proposition 3.5. *For any weak solution (u, f) to the Vlasov–Navier–Stokes system for which $\sup_{t\geq 0}\|\rho_f(t)\|_{L^\infty(\mathbb{T}^3)} < +\infty$ and $\mathscr{E}(t) \to_{t\to+\infty} 0$ with exponential decay, there exists a profile $\rho^\infty \in L^\infty(\mathbb{T}^3)$ such that*

$$W_1\left(f(t), \rho^\infty\left(x - t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right)\otimes\delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \longrightarrow_{t\to+\infty} 0, \tag{3.15}$$

exponentially fast.

Proof. We rely on an argument of Jabin [21] used in the context of the large time behavior of the Vlasov–Stokes system. The proof heavily relies on the exponential decay of the modulated energy. Recall the conservation of the mass

$$\partial_t \rho_f = -\nabla \cdot j_f.$$

For any smooth function $\psi \in \mathscr{C}^\infty(\mathbb{T}^3)$ we have therefore for $0 \leqq s \leqq t$

$$\int_{\mathbb{T}^3} \psi \rho_f(t) - \int_{\mathbb{T}^3} \psi \rho(s) = \int_s^t \int_{\mathbb{T}^3} \nabla \psi \cdot j_f(\tau)\, \mathrm{d}\tau.$$

Keeping in mind the definition of the Wasserstein distance (see Section 9.1), one sees that the large time convergence of ρ_f (which would imply that the Cauchy criterion is verified for this metric) is in a way or another linked with the decay of $j_f(\tau)$ as $\tau \to +\infty$. In the general case, this property is not expected, as j_f is "supposed" to converge to $\rho_f \langle u_0 + j_{f_0} \rangle /2$. This justifies to consider the following renormalized density:

$$\overline{\rho}_f(t,x) := \rho_f\Big(t, x + t\frac{\langle u_0 + j_{f_0}\rangle}{2}\Big),$$

for which we have, denoting as well $\overline{j}_f := j_f\left(t, x + t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right)$,

$$\partial_t \overline{\rho}_f = \nabla \cdot \left(\overline{\rho}_f \frac{\langle u_0 + j_{f_0}\rangle}{2} - \overline{j}_f\right).$$

The previous computation implies

$$\int_{\mathbb{T}^3} \psi \overline{\rho}_f(t) - \int_{\mathbb{T}^3} \psi \overline{\rho}_f(s) = \int_s^t \int_{\mathbb{T}^3} \nabla \psi \cdot \left(\overline{j}_f - \overline{\rho}_f \frac{\langle u_0 + j_{f_0}\rangle}{2}\right)(\tau)\, \mathrm{d}\tau,$$

and the integrand is now expected to decay for large time. More precisely if $\|\nabla \psi\|_\infty \leqq 1$ we have, by translation invariance of the integration over $\mathbb{T}^3$

$$\begin{aligned}\left|\int_{\mathbb{T}^3} \psi \overline{\rho}_f(t) - \int_{\mathbb{T}^3} \psi \overline{\rho}_f(s)\right| &\leqq \int_s^t \int_{\mathbb{T}^3} \left|\overline{j}_f - \overline{\rho}_f \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|(\tau)\, \mathrm{d}\tau \\ &= \int_s^t \int_{\mathbb{T}^3} \left|j_f - \rho_f \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|(\tau)\, \mathrm{d}\tau,\end{aligned}$$

and we thus deduce, by Cauchy–Schwarz inequality, that

$$\begin{aligned}&\left|\int_{\mathbb{T}^3} \psi \overline{\rho}_f(t) - \int_{\mathbb{T}^3} \psi \overline{\rho}_f(s)\right| \\ &\quad \leqq \int_s^t \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} f\right)^{1/2} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} f \left|v - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|^2\right)^{1/2} (\tau)\, \mathrm{d}\tau.\end{aligned}$$

On the one hand, thanks to Lemma 3.1, the integral of f over $\mathbb{T}^3 \times \mathbb{R}^3$ equals 1. On the other hand, thanks to Lemma 3.2 we have

$$\begin{aligned}\left|v-\frac{\langle u_0+j_{f0}\rangle}{2}\right|^2 &\lesssim \left|v-\langle j_f\rangle\right|^2+\left|\langle j_f\rangle-\frac{\langle u_0+j_{f0}\rangle}{2}\right|^2\\ &=\left|v-\langle j_f\rangle\right|^2+\frac{1}{4}\left|\langle j_f\rangle-\langle u\rangle\right|^2.\end{aligned}$$

All in all, using the the Definition 1.7 of the modulated energy we have established for any $\psi\in\mathscr{C}^\infty(\mathbb{R}^3)$ such that $\|\nabla\psi\|_\infty\leqq 1$ that

$$\begin{aligned}\left|\int_{\mathbb{T}^3}\psi\overline{\rho}_f(t)-\int_{\mathbb{T}^3}\psi\overline{\rho}_f(s)\right| &\lesssim \int_s^t\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f\left|v-\langle j_f\rangle\right|^2\right)^{1/2}(\tau)\,\mathrm{d}\tau\\ &\quad+\int_s^t|\langle j_f\rangle-\langle u\rangle|(\tau)\,\mathrm{d}\tau\\ &\lesssim\int_s^t\mathscr{E}(\tau)^{1/2}\,\mathrm{d}\tau.\end{aligned}$$

This estimate extends to Lipschitz functions ψ satisfying $\|\nabla\psi\|_\infty\leqq 1$ by a standard approximation argument and the Monge–Kantorovich duality formula allows us to write

$$\mathrm{W}_1(\overline{\rho}_f(t),\overline{\rho}_f(s))\lesssim\int_s^t\mathscr{E}(\tau)^{1/2}\,\mathrm{d}\tau. \tag{3.16}$$

The exponential decay of the modulated energy leads to integrability of $\mathscr{E}^{1/2}$ and therefore the Cauchy criterion for $\overline{\rho}_f(t)$ is verified for $t\to+\infty$: we recover in this way the convergence of $\rho_f^\infty(t)\to\rho^\infty$ for some measure ρ^∞ as $t\to+\infty$. Since $t\mapsto\rho^\infty(t)$ is uniformly bounded in $\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^3))$, we must have $\rho^\infty\in\mathrm{L}^\infty(\mathbb{T}^3)$. Note that the convergence is indeed exponential, thanks to the exponential decay of $\mathscr{E}^{1/2}$: this can be seen when letting $t\to+\infty$ in (3.16). Now by a change of variable we have

$$\mathrm{W}_1(\overline{\rho}_f(s),\rho^\infty)=\mathrm{W}_1\Big(\rho_f(s),\rho^\infty\Big(x-s\frac{\langle u_0+j_{f0}\rangle}{2}\Big)\Big), \tag{3.17}$$

which concludes the proof. □

4. Changes of Variables and L^∞ Bounds on Moments

In this section we aim at establishing tools for obtaining bounds on the moments ρ_f and j_f. We first obtain rough unconditional integrability results for ρ_f and j_f thanks to some interpolation estimates. Next, using some adequate change of variables in velocity, we get refined estimates on ρ_f and j_f, which can be controlled along the flow in the following way. Assuming a suitable control on the quantity $\|\nabla u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}$, it is possible to prove that (cf. Lemma 4.5)

$$\begin{aligned}&\|\rho_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}\lesssim 1,\\ &\|j_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}\lesssim\left(\int_0^t\|u(s)-\langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s+e^{-t}\left(1+\int_0^t e^s|\langle u(s)\rangle|\mathrm{d}s\right)\right),\end{aligned}$$

which can be exploited in long time : the core of the bootstrap argument presented in Section 7 is to prove that the control on ∇u holds as long as $\mathscr{E}(0)$ is small.

Many proofs in this section rely on the representation of the solution to the Vlasov equation using characteristics, which holds at least when u is a smooth vector field.

Definition 4.1. Assume u is smooth (say $\mathscr{C}^1$). We define the characteristic curves $\mathrm{X}(s; t, x, v)$ and $\mathrm{V}(s; t, x, v)$ associated with u as the solution to the system of ODEs

$$\begin{aligned} \dot{\mathrm{X}}(s; t, x, v) &= \mathrm{V}(s; t, x, v), \\ \dot{\mathrm{V}}(s; t, x, v) &= u(s, \mathrm{X}(s; t, x, v)) - \mathrm{V}(s; t, x, v), \end{aligned} \tag{4.1}$$

with the initial condition $(\mathrm{X}(t; t, x, v), \mathrm{V}(t; t, x, v)) = (x, v)$.

By the method of characteristics, for a smooth vector field u, we can write the solution f to the Vlasov equation as

$$f(t, x, v) = e^{3t} f_0(\mathrm{X}(0; t, x, v), \mathrm{V}(0; t, x, v)). \tag{4.2}$$

As explained in Remark 3.1, we then rely on DiPerna–Lions theory to ensure that the estimates we are able to prove with this representation formula still hold even if u is not smooth enough. For instance, a rough bound on the L^∞ norm of f can be directly deduced from (4.2).

Lemma 4.1. *For almost all $t \geqq 0$,*

$$\|f(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leqq \|f_0\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} e^{3t}. \tag{4.3}$$

In the remaining paragraphs of this section we will systematically use the approximation procedure described in Remark 3.1, without refering to it explicitely. This is in particular the case for each of the proofs which rely on the characteristic curves.

4.1. Rough Local Bounds on Moments

We recall the notations M_α and m_α introduced in Definition 1.5.

Lemma 4.2. *Consider $\alpha \geqq 1$ such that $u \in L^1_{loc}(\mathbb{R}_+; L^{\alpha+3} \cap W^{1,1}(\mathbb{T}^3))$ and $M_\alpha f_0 < \infty$. Then $M_\alpha f(t) < \infty$ and for all $t > 0$ and*

$$M_\alpha f(t) \lesssim_\alpha \left(M_\alpha f_0 + e^{\frac{3t}{\alpha+3}} \int_0^t \|u(s)\|_{L^{\alpha+3}(\mathbb{T}^3)}\, \mathrm{d}s \right)^{\alpha+3}. \tag{4.4}$$

Proof. Multiplying the Vlasov equation by $|v|^\alpha$ and integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} M_\alpha f(t) + \alpha M_\alpha f(t) = \alpha \int_{\mathbb{T}^3} u(t, x) \cdot m_{\alpha-1}(t, x)\, \mathrm{d}x. \tag{4.5}$$

Recall that, for $0 \leqq \ell \leqq k$, the following interpolation estimate

$$\|m_\ell g\|_{\mathrm{L}^{\frac{k+3}{\ell+3}}(\mathbb{T}^3)} \lesssim (M_k g)^{\frac{\ell+3}{k+3}} \|g\|_{\mathrm{L}^\infty(\mathbb{T}^3)}^{\frac{k-\ell}{k+3}}, \tag{4.6}$$

holds for any non-negative $g \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$. In particular for $(\ell, k) = (\alpha - 1, \alpha)$ we get

$$\|m_{\alpha-1} g\|_{\mathrm{L}^{\frac{\alpha+3}{\alpha+2}}(\mathbb{T}^3)} \lesssim (M_\alpha g)^{\frac{\alpha+2}{\alpha+3}} \|g\|_{\mathrm{L}^\infty(\mathbb{T}^3)}^{\frac{1}{\alpha+3}}.$$

We can control $\|g\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$ by Lemma 4.1, so that using Hölder's inequality in (4.5), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t} M_\alpha f(t)^{\frac{1}{\alpha+3}} + \frac{\alpha}{\alpha+3} M_\alpha f(t)^{\frac{1}{\alpha+3}} \lesssim e^{\frac{3t}{\alpha+3}} \|u(t)\|_{\mathrm{L}^{\alpha+3}(\mathbb{T}^3)},$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ e^{\frac{\alpha t}{\alpha+3}} M_\alpha f(t)^{\frac{1}{\alpha+3}} \right\} \lesssim e^t \|u(t)\|_{\mathrm{L}^{\alpha+3}(\mathbb{T}^3)},$$

from which (4.4) follows. □

Lemma 4.3. *Assuming $M_3 f_0 < +\infty$, we have the following:*

(i) $M_3 f \in L^\infty_{loc}(\mathbb{R}_+)$;
(ii) $\rho_f \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{T}^3))$;
(iii) $j_f \in L^\infty_{loc}(\mathbb{R}_+; L^{3/2}(\mathbb{T}^3))$.

Proof. By Lemma 4.2, we have

$$M_3 f(t) \lesssim \left(M_3 f_0 + e^{\frac{t}{2}} \int_0^t \|u(s)\|_{\mathrm{L}^6(\mathbb{T}^3)} \,\mathrm{d}s \right)^6.$$

However, using the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$ and the Poincaré-Wirtinger inequality and the energy estimate (1.10), we infer

$$\begin{aligned} \int_0^t \|u(s)\|_{\mathrm{L}^6(\mathbb{T}^3)} \,\mathrm{d}s &\leqq \int_0^t \|u(s) - \langle u(s) \rangle\|_{\mathrm{L}^6(\mathbb{T}^3)} \,\mathrm{d}s + \sqrt{t}\mathrm{E}(0)^{1/2} \\ &\lesssim \sqrt{t} \left(\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \,\mathrm{d}s \right)^{1/2} + \sqrt{t}\mathrm{E}(0)^{1/2} \\ &\lesssim \sqrt{t}\mathrm{E}(0)^{1/2}. \end{aligned}$$

This concludes the proof of (i). By the interpolation estimate (4.6) for $(\ell, k) = (0, 3)$ and $(\ell, k) = (1, 3)$ we have

$$\begin{aligned} \|\rho_f(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} &= \|m_0 f(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim M_3 f(t)^{1/2} \|f(t)\|_{\mathrm{L}^\infty}^{1/2}, \\ \|j_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} &\leqq \|m_1 f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} \lesssim M_3 f(t)^{2/3} \|f(t)\|_{\mathrm{L}^\infty}^{1/3}. \end{aligned}$$

We therefore obtain (ii) and (iii) thanks to (i) and Lemma 4.1. □

4.2. The Straightening Change of Variables

We discuss in this section the change of variables in velocity that will allow us, as explained at the beginning of this section, to prove long time estimates. The idea is to come down to the "free" case (that is to say to the characteristics associated with the vector field $(x, v) \to (v, -v)$ here), by using an appropriate diffeomorphism in velocity. In doing this, a smallness condition bearing on $\|\nabla u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}$ will naturally appear in our calculations.

This change of variables is close in spirit to that employed in [1] by Bardos and Degond in the study of small data solutions to the Vlasov–Poisson system on $\mathbb{R}^3 \times \mathbb{R}^3$. We note however that the stabilization mechanism for Vlasov–Poisson on $\mathbb{R}^3 \times \mathbb{R}^3$ is based on the dispersion properties of the free transport operator, which is significantly different from that used in our work.

We also mention that similar ideas were recently used in the context of the inertialess limit of the Vlasov–Stokes system in [20].

Lemma 4.4. *Fix $\delta > 0$ such that $\delta e^\delta < 1/9$. Then, for any $t \in \mathbb{R}_+$ satisfying*

$$\int_0^t \|\nabla u(s)\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}s \leqq \delta, \tag{4.7}$$

and any $x \in \mathbb{R}^3$, the map

$$\Gamma_{t,x} : v \mapsto V(0; t, x, v),$$

is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself satisfying furthermore

$$\forall v \in \mathbb{R}^3, \quad |\det D_v \Gamma_{t,x}(v)| \geqq \frac{e^{3t}}{2}. \tag{4.8}$$

Proof. The proof is directly inspired from the arguments outlined in [1, Proposition 1 and Corollary 1].
(i) Consider a generic vector-valued flow $\mathrm{Y}^s_{t,z} := \mathrm{Y}(s; t, z)$ associated with a smooth vector field $w(t, z)$ defined on $\mathbb{R}_+ \times X$ and assume that $\|\mathrm{D}_z w(t)\|_{\mathrm{L}^\infty(X)} \leqq 1 + \psi(t)$, for some function $\psi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$. We have $\partial_s \mathrm{Y}^s_{t,z} = w(s, \mathrm{Y}^s_{t,z})$ which after differentiation with respect to z (introducing $\Theta^s_{t,z} := \mathrm{D}_z \mathrm{Y}^s_{t,z}$) leads to

$$\partial_s \Theta^s_{t,z} = \mathrm{D}_z w(s, \mathrm{Y}^s_{t,z}) \cdot \Theta^s_{t,z},$$

from which we get by Gronwall's inequality, for $s \leqq t$, that

$$\begin{aligned} \|\Theta^s_{t,z}\|_{\mathrm{L}^\infty(X)} &\leqq \|\Theta^t_{t,z}\|_{\mathrm{L}^\infty(X)} \exp\left(\int_s^t \|\mathrm{D}_z w(\sigma)\|_{\mathrm{L}^\infty(X)} \mathrm{d}\sigma\right) \\ &\leqq e^{t-s} \exp\left(\int_s^t |\psi(\sigma)|\,\mathrm{d}\sigma\right), \end{aligned} \tag{4.9}$$

where we used $\Theta^t_{t,z} = \mathrm{Id}$.

Now, let us get back to our system. Introducing the state variable $z := (x, v)$ which belongs to $X = \mathbb{T}^3 \times \mathbb{R}^3$, the vector field $w(t, z) := (v, u(t, x) - v)$

satisfies the assumption for the above abstract result, since $\|\mathrm{D}_z w(t)\|_{\mathrm{L}^\infty(X)} \leqq 1 + \|\nabla u\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$. If we denote by $(\mathrm{X}(s;t,z), \mathrm{V}(s;t,z))$ the characteristics associated with u, integrating the equation defining $s \mapsto \mathrm{V}(s;t,z)$ we have

$$\mathrm{V}(0;t,z) = e^t v - \int_0^t e^s u(s, \mathrm{X}(s;t,z))\,\mathrm{d}s, \tag{4.10}$$

which leads to

$$\mathrm{D}_v\mathrm{V}(0;t,z) - e^t\mathrm{Id} = -\int_0^t e^s \nabla u(s, \mathrm{X}(s;t,z))\mathrm{D}_v\mathrm{X}(s,t;z)\,\mathrm{d}s.$$

We thus infer from (4.9) with $\psi = \|\nabla u\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$ that

$$\|\mathrm{D}_v\mathrm{X}(s;t,z)\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leqq e^{t-s}\exp\left(\int_s^t \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}\tau\right),$$

and thus that

$$\begin{aligned}&\|e^{-t}\mathrm{D}_v\mathrm{V}(0;t,z) - \mathrm{Id}\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}\\ &\quad\leqq \exp\left(\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathrm{d}s\right)\int_0^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathrm{d}s.\end{aligned}$$

In particular, if (4.7) holds with $\delta > 0$ such that $\delta e^\delta \leqq \frac{1}{9}$, then Lemma 9.4 applies and we can conclude. □

Thanks to the change of variables of Lemma 4.4, we deduce the following control on moments:

Lemma 4.5. *If assumption* (4.7) *of Lemma* 4.4 *is satisfied, we have for almost all* $t \geqq 0$,

$$\|\rho_f(t)\|_{L^\infty(\mathbb{T}^3)} \leqq 2I_q N_q(f_0), \tag{4.11}$$

$$\|j_f(t)\|_{L^\infty(\mathbb{T}^3)} \leqq 2I_q e^{-t}\left(\int_0^t e^s\|u(s)\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}s + 1\right)N_q(f_0), \tag{4.12}$$

where $N_q(f_0)$ *is given by* (1.12) *and*

$$I_q := \int_{\mathbb{R}^3} \frac{1+|v|}{1+|v|^q}\,\mathrm{d}v.$$

Proof. Let $(\mathrm{X}(s,t;x,v), \mathrm{V}(s,t;x,v))$ be the characteristics (4.1) associated with u. We start again from the representation formula

$$\rho_f(t,x) = e^{3t}\int_{\mathbb{R}^3} f_0(\mathrm{X}(0;t,x,v), \mathrm{V}(0;t,x,v))\,\mathrm{d}v.$$

By Lemma 4.4, the mapping $v \mapsto \Gamma_{t,x}(v) = \mathrm{V}(0;t,x,v)$ defines an admissible change of variable of which we deduce

$$\rho_f(t,x) = e^{3t}\int_{\mathbb{R}^3} f_0(\mathrm{X}(0;t,x,\Gamma_{t,x}(w)), w)\left|\mathrm{D}_v(\Gamma_{t,x})(\Gamma_{t,x}(w))\right|\,\mathrm{d}w,$$

which implies (the control of the jacobian is given by Lemma 4.4)

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \leqq 2N_q(f_0)I_q. \tag{4.13}$$

For j_f we proceed similarly and write the representation formula (valid for the same reasons)

$$j_f(t,x) = e^{3t}\int_{\mathbb{R}^3} \Gamma_{t,x}(w) f_0(\mathrm{X}(0;t,x,\Gamma_{t,x}(w)),w)\left|\mathrm{D}_v(\Gamma_{t,x})(\Gamma_{t,x}w))\right| \mathrm{d}w.$$

By definition of $\Gamma_{t,x}(w)$, we have the identity

$$w = e^t\Gamma_{t,x}(w) - \int_0^t e^s u(s,\mathrm{X}(s;t,x,\Gamma_{t,x}(w)))\,\mathrm{d}s, \tag{4.14}$$

from which we deduce

$$|\Gamma_{t,x}(w)| \leqq e^{-t}\left[|w| + \int_0^t e^s \|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s\right],$$

hence the claimed result. □

In the next lemma, we study how the pointwise decay condition of Definition 1.4 can be locally propagated.

Lemma 4.6. *Let $t_0 > 0$. If f_0 satisfies (1.11) and $u \in L^1_{loc}(\mathbb{R}_+; H^1 \cap L^\infty(\mathbb{T}^3))$, then $f_{t_0} := f(t_0)$ satisfies also (1.11) and*

$$N_q(f_{t_0}) \lesssim (1 + \|u\|^q_{L^1(0,t_0;L^\infty(\mathbb{T}^3))})N_q(f_0).$$

Proof. We write

$$f(t_0,x,v) = e^{3t_0} f_0(\mathrm{X}(0;t_0,x,v),\mathrm{V}(0;t_0,x,v)).$$

Thanks to the differential equation satisfied by $s \mapsto \mathrm{V}(s;t,x,v)$ we have

$$\begin{aligned}\mathrm{V}(0;t_0,x,v) &= e^{t_0}v - \int_0^{t_0} e^s u(s,\mathrm{X}(0;s,x,v))\,\mathrm{d}s \\ &= e^{t_0}\left(v - \int_0^{t_0} e^{s-t_0}\langle u(s)\rangle\,\mathrm{d}s\right) \\ &\quad - \int_0^{t_0} e^s\Big(u(s,\mathrm{X}(0;s,x,v)) - \langle u(s)\rangle\,\mathrm{d}s\Big).\end{aligned} \tag{4.15}$$

We deduce

$$|v| \leqq |V(0;t_0,x,v)| + \int_0^{t_0} \|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s,$$

and therefore

$$(1+|v|^q)f(t_0,x,v) \lesssim e^{3t_0}\left(1 + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^3))}\right)N_q(f_0).$$

□

This allows us to obtain another version of Lemma 4.5 with a control like (4.7) starting only from some time $t_0 > 0$.

Lemma 4.7. *Let $t_0 > 0$. With the same assumptions and notations as in Lemma 4.4, except that we replace (4.7) by*

$$\int_{t_0}^{t} \|\nabla u(s)\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}s \leqq \delta, \tag{4.16}$$

we have, for all $t \geqq t_0$,

$$\|\rho_f(t)\|_{L^\infty(\mathbb{T}^3)} \lesssim N_q(f_0)(1 + \|u\|^q_{L^1(0,t_0;L^\infty(\mathbb{T}^3))}), \tag{4.17}$$

$$\|j_f(t)\|_{L^\infty(\mathbb{T}^3)} \lesssim e^{-t}\left(\int_0^t e^s\|u(s)\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}s+1\right) N_q(f_0)(1+\|u\|^q_{L^1(0,t_0;L^\infty(\mathbb{T}^3))}). \tag{4.18}$$

Proof. We can reproduce Lemma 4.4 and Lemma 4.5 replacing the initial time $t = 0$ by $t = t_0$ and thus f_0 by $f(t_0)$. Using Lemma 4.6, we obtain the claimed estimates. □

5. Regularity Estimates for Solutions of the Vlasov–Navier–Stokes System

This section is devoted to the following two tasks:

- obtaining a precise short time control for the L^∞ norm of ρ_f and j_f (relying on local estimates and Lemma 4.6);
- obtaining $\mathrm{L}^\infty_t\mathrm{H}^1_x \cap \mathrm{L}^2_t\mathrm{H}^2_x$ estimates for u, on time intervals *away* from zero, as developed in Proposition 9.10.

Such estimates will be crucial to prove Theorem 2.2, combined with the higher order estimates proved in Section 6.

We shall also introduce in this section the notion of *strong existence times* (see Definition 5.2). Loosely speaking, this corresponds to times t for which the solution u of the Navier–Stokes equation is *strong* on the interval of time $[0, t]$, which means in this context that it enjoys $\mathrm{H}^{1/2}(\mathbb{T}^3)$ regularity. A smallness criterion bearing both on u and on the Brinkman force $j_f - \rho_f u$ (see (5.4)) will be used.

Notation 5.1. *From now on, $A \lesssim_0 B$ will mean*

$$A \leqq \varphi\left(\|u_0\|_{H^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + N_q(f_0) + E(0) + 1\right) B,$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is onto, continuous and nondecreasing, and $q > 4$ and $\alpha > 3$ are the exponents given in the statements of Theorems 2.1 and 2.2. Note that $\lesssim_0$ may depend on the integration exponents appearing in the inequality, but this will always be harmless.

Notation 5.2. *We will use the following notations:*

$$F := j_f - \rho_f u, \qquad S := F - (u \cdot \nabla)u.$$

5.1. Local Estimates

In this paragraph we establish local estimates on both the fluid and the particle densities. Namely, we prove $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ and deduce from this estimate that $\rho_f, j_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ and then $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^2(\mathbb{T}^3))$.

Proposition 5.1. *We have $u \in L^1_{loc}(\mathbb{R}_+; L^\infty(\mathbb{T}^3))$ and $\rho_f, j_f \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{T}^3))$. Moreover there exists a continuous nondecreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|u\|_{L^1(0,t;L^\infty(\mathbb{T}^3))} \lesssim_0 \eta(t), \tag{5.1}$$

$$\|\rho_f\|_{L^\infty(0,t;L^\infty(\mathbb{T}^3))} + \|j_f\|_{L^\infty(0,t;L^\infty(\mathbb{T}^3))} \lesssim_0 \eta(t). \tag{5.2}$$

Proof. In the proof we denote by η a generic continuous function (as in the statement of the proposition), which may vary from line to line.

Since $M_2 f_0 < +\infty$ (see the Definition 1.2 of admissible initial data), we have also $M_3 f_0 \lesssim M_2 f_0 + M_\alpha f_0 < +\infty$. We infer from the proof of Lemma 4.3 that

$$\|\rho_f(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} \lesssim_0 \eta(t).$$

In particular, recalling the notation $S = j_f - \rho_f u - (u \cdot \nabla)u$, we infer, using Hölder's inequality and the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$ and the energy estimate (1.10),

$$\int_0^t \|S(s)\|^2_{\mathrm{L}^{3/2}(\mathbb{T}^3)}\, \mathrm{d}s \lesssim_0 \eta(t).$$

Now, if $\mathbb{P}$ stands for the Leray projector (that is the projection on divergence free vector fields), let w be the unique solution of

$$\begin{aligned} \partial_t w - \Delta w &= \mathbb{P}S, \\ \operatorname{div} w &= 0, \\ w(0) &= 0, \end{aligned}$$

so that $u - w = e^{t\Delta}u_0$. Since $u_0 \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T}^3)$, we infer from [13, Lemma 3.3] that $u - w \in \mathrm{L}^2(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ with the estimate

$$\int_0^\infty \|(u-w)(s)\|^2_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s \lesssim \|u_0\|^2_{\mathrm{H}^{\frac{1}{2}}(\mathbb{T}^3)}.$$

Thanks to the $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{3/2}(\mathbb{T}^3))$ estimate on S that we obtained above, we infer from the continuity of $\mathbb{P}$ on $\mathrm{L}^{3/2}(\mathbb{T}^3)$ and the maximal regularity of the heat operator on the torus (see Corollary 9.8) that $\Delta w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{3/2}(\mathbb{T}^3))$. Therefore, from a standard elliptic estimate, we deduce $\mathrm{D}^2 w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{3/2}(\mathbb{T}^3))$ and thus $w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^p(\mathbb{T}^3))$ for all $p < \infty$, by Sobolev's embedding. We have even more precisely (keeping track of the different constants)

$$\int_0^t \|w(s)\|^2_{\mathrm{L}^p(\mathbb{T}^3)}\, \mathrm{d}s \lesssim_0 \eta(t).$$

Up to now we have thus established (for any $p < \infty$) that $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^p(\mathbb{T}^3))$ with

$$\int_0^t \|u(s)\|^2_{\mathrm{L}^p(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_0 \eta(t). \tag{5.3}$$

In particular, we get $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{\alpha+3}(\mathbb{T}^3))$. Using estimate (4.4) of Lemma 4.2 we first have

$$M_\alpha f(t) \lesssim \left(M_\alpha f_0 + e^{\frac{3t}{\alpha+3}} \int_0^t \|u(s)\|_{\alpha+3}\,\mathrm{d}s \right)^{\alpha+3} \lesssim_0 \eta(t).$$

We use the interpolation estimate (4.6) with $k = \alpha$ and $\ell \in \{0, 1\}$ to obtain this time

$$\|\rho_f(t)\|_{\mathrm{L}^{\frac{\alpha+3}{3}}(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^{\frac{\alpha+3}{4}}(\mathbb{T}^3)} \lesssim_0 \eta(t),$$

where the integration exponents are strictly larger than $3/2$. Using (5.3) we can estimate $(u \cdot \nabla)u$ in some $\mathrm{L}^\gamma_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^r(\mathbb{T}^3))$ for $\gamma > 1$ and $r > 3/2$ leading to the following estimate on the source S:

$$\int_0^t \|S(s)\|^\gamma_{\mathrm{L}^r(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_0 \eta(t).$$

Since $r > 3/2$, using like before the maximal regularity of the heat operator we eventually infer by the Sobolev embedding $\mathrm{W}^{2,r}(\mathbb{T}^3) \hookrightarrow \mathrm{L}^\infty(\mathbb{T}^3)$

$$\int_0^t \|w(s)\|^\gamma_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_0 \eta(t).$$

All in all, we have obtained that $u = (u - w) + w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$. Finally using Lemma 4.6 and the straightforward bound

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim N_q(f(t)),$$

we infer that both ρ_f, j_f belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ with the estimate

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim_0 \eta(t).$$

□

Lemma 5.2. *Recalling Notation 5.2, we have $F \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{T}^3))$ and moreover*

$$\int_0^t \|F(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s \leqq \min(E(0), \mathscr{E}(0)) \sup_{s\in[0,t]} \|\rho_f(s)\|_{L^\infty(\mathbb{T}^3)}.$$

Proof. By Cauchy–Schwarz's inequality, we have, almost everywhere,

$$|F| = \left| \int_{\mathbb{R}^3} f(v-u)\,\mathrm{d}v \right| \leqq \rho_f^{1/2} \left(\int_{\mathbb{R}^3} f|v-u|^2\,\mathrm{d}v \right)^{1/2},$$

from which we infer, for almost all $s \geqq 0$,

$$\|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \leqq \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \mathrm{D}(s),$$

where D is the dissipation introduced in (1.5). The estimate follows thus from the energy (1.10) and modulated energy (3.7) estimates. □

5.2. Parabolic Regularization for the Fluid

We state here a consequence of the parabolic regularization result of Proposition 9.10 of the appendix. This roughly establishes the instantaneous gain of two derivatives for the Navier–Stokes equation, if the right-hand side is square-integrable. However, such an estimate can only be obtained if a suitable smallness condition is satisfied.

Proposition 5.3. *Assume that for some $T > 0$ there holds*

$$\|u_0\|^2_{H^{1/2}(\mathbb{T}^3)} + C_\star \int_0^T \|F(s)\|^2_{H^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s < \frac{1}{C_\star^2}, \tag{5.4}$$

where $C_\star$ is the universal constant given by Proposition 9.10. Then one has for all $1/2 \leqq t \leqq T$ the estimate

$$\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \int_{1/2}^t \|\Delta u(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s \lesssim E(0)\Big(1 + \sup_{[0,t]} \|\rho_f(s)\|_{L^\infty(\mathbb{T}^3)}\Big), \tag{5.5}$$

where $\lesssim$ depends only on $C_\star$.

Proof. If (5.4) is indeed satisfied, we can directly use the well-posedness framework given by Proposition 9.10. Thanks to Lemma 5.2 we have also (9.5) which here reduces to (5.5) because the decay of the energy (1.10) ensures $\mathrm{A}(t) \leqq \mathrm{E}(0)$. □

5.3. Strong Existence Times

Thanks to Proposition 5.1, we know that ρ_f and j_f both belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$. We can therefore focus on the boundedness over $[1, +\infty)$. For this purpose, the following notations will be convenient:

Definition 5.1. We set for $t \geqq 1$

$$\mathrm{M}_{\rho_f}(t) := \sup_{[1,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}, \quad \mathrm{M}_{j_f}(t) := \sup_{[1,t]} \|j_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}, \tag{5.6}$$

$$\mathrm{M}_{\rho_f, j_f}(t) := \mathrm{M}_{\rho_f}(t) + \mathrm{M}_{j_f}(t). \tag{5.7}$$

In order to use the regularization offered by Proposition 5.3, we need to ensure that the smallness condition (5.4) remains satisfied. For this reason, we introduce the following definition:

Definition 5.2. *(Strong existence times)* A real number $T \geqq 0$ will be said to be a *strong existence time* whenever (5.4) holds.

The following lemma asserts that within our set of assumptions, we have a lower bound for strong existence times:

Lemma 5.4. *The smallness condition (2.1) of Theorem 2.1 suffices to ensure that $T = 1$ is a strong existence time in the sense of Definition 5.2.*

Proof. Using Lemma 5.2 and estimate and (5.2), we straightforwardly have

$$\begin{aligned}\int_0^1 \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s &\leqq \int_0^1 \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\ &\leqq \min(\mathrm{E}(0), \mathscr{E}(0)) \sup_{s\in[0,1]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \\ &\lesssim_0 \mathscr{E}(0),\end{aligned}$$

and recalling the meaning of $\lesssim_0$ (see Notation 5.1), one sees that the smallness condition (2.1) is indeed sufficient. □

6. Estimates on the Convection and the Brinkman Force

Our ultimate bootstrap argument requires high order estimates bearing on u, for which, as in the proof of Proposition 5.1, we will see the Navier–Stokes equation as

$$\partial_t u - \Delta u = \mathbb{P}F - \mathbb{P}(u\cdot\nabla)u,$$

where $\mathbb{P}$ is the Leray projector. We use the maximal regularity of the heat operator on the previous identity to get estimates on $\mathrm{D}^2 u$ in terms of the Brinkman force F and the the convection term $(u\cdot\nabla)u$. In this short section we explain in Proposition 6.1 the maximal regularity argument and give $\mathrm{L}^p_t\mathrm{L}^q_x$ estimates for the source terms in Lemma 6.2 and Lemma 6.3. As explained in Corollary 6.4, these estimates are already sufficient to justify the $\mathrm{L}^1_t\mathrm{W}^{1,\infty}_x$ regularity needed to express the condition (4.16) (and a quantitative version with the required smallness will be provided afterwards).

Proposition 6.1. *Fix $a, b, r \in (1,\infty)$ and $\lambda > 0$. For any $t \geqq 1$, and any exponent $1 \leqq q \leqq a, b$ there holds (with a possible infinite right-hand side)*

$$\begin{aligned}&\int_1^t e^{-\lambda s}\|D^2u(s)\|^q_{L^r(\mathbb{T}^3)}\,\mathrm{d}s \\ &\quad\lesssim_0 \Phi(\lambda)\Big(1 + \|(u\cdot\nabla)u\|^q_{L^a(1/2,t;L^r(\mathbb{T}^3))} + \|F\|^q_{L^b(1/2,t;L^r(\mathbb{T}^3))}\Big),\end{aligned} \tag{6.1}$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing.

Proof. Similarly to what we have done in the proof of Proposition 5.1, we introduce w_1 and w_2 as the unique divergence-free solutions on $[1/2, +\infty)$ of

$$\begin{aligned}\partial_t w_1 - \Delta w_1 &= \mathbb{P}(u\cdot\nabla)u, \\ \partial_t w_2 - \Delta w_2 &= \mathbb{P}F,\end{aligned}$$

with initial conditions $w_1(1/2) = w_2(1/2) = 0$ so that, denoting $u_h := u - (w_1 + w_2)$, we have $u_h(t+1/2) = e^{t\Delta}u(1/2)$. Now, thanks to the maximal regularity of the heat operator (see Corollary 9.8) and the continuity of $\mathbb{P}$ on $\mathrm{L}^r(\mathbb{T}^3)$, we infer for $t \geqq 1/2$

$$\left(\int_{1/2}^t \|\mathrm{D}^2 w_1(s)\|^a_{\mathrm{L}^r(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/a} \lesssim \left(\int_{1/2}^t \|(u\cdot\nabla u)(s)\|^a_{\mathrm{L}^r(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/a}, \tag{6.2}$$

$$\left(\int_{1/2}^t \|\mathrm{D}^2 w_2(s)\|^b_{\mathrm{L}^r(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/b} \lesssim \left(\int_{1/2}^t \|F(s)\|^b_{\mathrm{L}^r(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/b}. \tag{6.3}$$

On the other hand, since $u_h(t+1/2) = e^{t\Delta}u(1/2)$, where we write

$$u(1/2,x) =: \sum_{k\in\mathbb{Z}^3} c_k e^{2i\pi k\cdot x} \in \mathrm{L}^2(\mathbb{T}^3),$$

we have, for $t \geqq 1/2$,

$$u_h(t,x) = \sum_{k\in\mathbb{Z}^3} c_k e^{-(2\pi|k|)^2(t-1/2)} e^{2i\pi k\cdot x},$$

and in particular, for $t \geqq 1$ and $\ell \geqq 1$,

$$\begin{aligned}
\|u_h(t)\|^2_{\dot{\mathrm{H}}^\ell(\mathbb{T}^3)} &= \sum_{k\in\mathbb{Z}^3} |c_k|^2 |k|^{2\ell} e^{-(2\pi|k|)^2(t-1/2)} \\
&\lesssim \sum_{k\in\mathbb{Z}^3} |c_k|^2 e^{-|k|^2(t-1/2)} \\
&\lesssim \|u(1/2)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} e^{-(t-1/2)},
\end{aligned}$$

so that for any $\ell \geqq 1$ we obtain

$$\int_1^{+\infty} \|u_h(s)\|^q_{\dot{\mathrm{H}}^\ell(\mathbb{T}^3)}\,\mathrm{d}s \lesssim \|u(1/2)\|^q_{\mathrm{L}^2(\mathbb{T}^3)} \int_1^{+\infty} e^{-q(s-1/2)/2}\,\mathrm{d}s \lesssim \|u(1/2)\|^q_{\mathrm{L}^2(\mathbb{T}^3)}. \tag{6.4}$$

By the energy estimate (1.10), we have $\|u(1/2)\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim_0 1$, so using (6.4) for ℓ large enough, we infer

$$\left(\int_1^t \|\mathrm{D}^2 u_h(s)\|^q_{\mathrm{L}^r(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/q} \lesssim_0 1. \tag{6.5}$$

Using the decomposition $u = w_1 + w_2 + u_h$ and combining (6.2), (6.3) and (6.5), we infer by Hölder's inequality the estimate (6.1). □

Lemma 6.2. *There exists $a \in (2,4)$ and $r_a > 2$ such that the following interpolation estimate holds for $t \geqq 1$:*

$$\|(u\cdot\nabla)u\|_{L^a(1/2,t;L^{r_a}(\mathbb{T}^3))} \lesssim_0 1 + M_{\rho_f,j_f}(t). \tag{6.6}$$

Proof. The proof boils down to the interpolation inequality

$$\|(u\cdot\nabla)u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^{r_a}(\mathbb{T}^3))} \leqq \|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\|\nabla u\|^{\frac{2}{a}}_{\mathrm{L}^2(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\|\nabla u\|^{1-\frac{2}{a}}_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^2(\mathbb{T}^3))}.$$

Indeed, if the later is satisfied, since t is a strong existence time, we have thanks to the regularization estimate (5.5) and the energy estimate (1.10), together with the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3)\hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$,

$$\|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}+\|\nabla u\|_{\mathrm{L}^2(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}+\|\nabla u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^2(\mathbb{T}^3))} \lesssim_0 1+\mathrm{M}_{\rho_f,j_f}(t).$$

To justify the interpolation above, notice that for any $a>2$, we have by Hölder inequality and interpolation $[(2,6),(\infty,2)]_\theta$,

$$\|u\cdot\nabla u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^{r_a}(\mathbb{T}^3))}\leqq\|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\|\nabla u\|^{\theta}_{\mathrm{L}^2(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\|\nabla u\|^{1-\theta}_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^2(\mathbb{T}^3))},$$

with the following equality:

$$\left(\frac{1}{a},\frac{1}{r_a}\right)=\left(0,\frac{1}{6}\right)+\theta\left(\frac{1}{2},\frac{1}{6}\right)+(1-\theta)\left(0,\frac{1}{2}\right).$$

We deduce $\theta=2/a$. From the previous identity we also deduce the value of r_a, because $\frac{1}{r_a}=\frac{1}{6}(1+\frac{2}{a})+\frac{1}{2}(1-\frac{2}{a})$. In the limit case $a=2$ we get $r_a=3$, so that taking $|a-2|$ small enough we have indeed $r_a>2$ and $a\in(2,4)$. □

Lemma 6.3. *For any finite $b>4$, the following estimate holds for some $r_b>3$ and all strong existence times $t\geqq 1$:*

$$\|F\|_{L^b(1/2,t;L^{r_b}(\mathbb{T}^3))}\lesssim_0 1+M_{\rho_f,j_f}(t)^{\frac{3}{2}-\frac{2}{b}}. \tag{6.7}$$

Proof. Thanks to Lemma 5.2 and (5.2) we have

$$\|F\|_{\mathrm{L}^2(1/2,t;\mathrm{L}^2(\mathbb{T}^3))}\lesssim 1+\mathrm{M}_{\rho_f}(t)^{1/2}\leqq 1+\mathrm{M}_{\rho_f,j_f}(t)^{1/2}. \tag{6.8}$$

By interpolation $[(2,2);(\infty,6)]_\theta$, we have

$$\|F\|_{\mathrm{L}^b(1/2,t;\mathrm{L}^{r_b}(\mathbb{T}^3))}\leqq\|F\|^{\theta}_{\mathrm{L}^2(1/2,t;\mathrm{L}^2(\mathbb{T}^3))}\|F\|^{1-\theta}_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}, \tag{6.9}$$

where θ and r_b are defined by the equality $(\frac{1}{b},\frac{1}{r_b})=\theta(\frac{1}{2},\frac{1}{2})+(1-\theta)(0,\frac{1}{6})$ from which we get $\theta=2/b$ and $\frac{1}{r_b}=\frac{2}{3b}+\frac{1}{6}$; we notice that $b>4$ implies $r_b>3$.

By the triangle inequality, we get

$$\begin{aligned}\|F\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}&=\|j_f-\rho_f u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\\&\lesssim_0(1+\mathrm{M}_{\rho_f,j_f}(t))(1+\|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}).\end{aligned}$$

Using the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3)\hookrightarrow\mathrm{L}^6(\mathbb{T}^3)$ together with (5.5) and the energy estimate (1.10) we have $\|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\lesssim_0\mathrm{M}_{\rho_f,j_f}(t)^{1/2}$ which implies

$$\|F\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\lesssim_0 1+\mathrm{M}_{\rho_f,j_f}(t)^{3/2}.$$

Combining the previous estimate with (6.8) in (6.9) we therefore get

$$\|F\|_{\mathrm{L}^b(1/2,t;\mathrm{L}^{r_b}(\mathbb{T}^3))}\lesssim 1+\mathrm{M}_{\rho_f,j_f}(t)^{3/2-\theta},$$

which is exactly (6.7) because $b=2/\theta$. □

Corollary 6.4. *For any strong existence time $t \geqq 1$, one has $\nabla u \in L^1(1, t; L^\infty(\mathbb{T}^3))$.*

Proof. Thanks to Proposition 5.1, the right-hand sides of estimates (6.6) and 6.8 are finite. By Lemmas 6.2 and 6.3, we can therefore take $r > 3/2$ in (6.1) and thus, by Sobolev's embedding and Hölder's inequality, we finally obtain the claimed regularity. □

7. Exponential Decay of the Modulated Energy

In this section, we finish the proof of Theorem 2.2 by setting up a bootstrap procedure. Define

$$t^\star := \sup \left\{ \text{strong existence times } t \text{ such that } \int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s < \delta \right\}. \tag{7.1}$$

where δ is given in Lemma 4.4. Thanks to the change of variables of Section 4, we have that $\mathrm{M}_{\rho_f, j_f}(t) \lesssim_0 1$ on for $t < t^\star$ (see Proposition 7.1). The main goal will be to prove that $t^\star = +\infty$. In order to do so, we shall combine the higher order estimates of Section 6 with the exponential decay estimates provided by Lemma 3.4.

Proposition 7.1. *We have $t^\star > 1$. Moreover, for any $t < t^\star$, one has $M_{\rho_f, j_f}(t) \lesssim_0 1$.*

Proof. By a view of the proof of Lemma 5.4 (reducing $\mathscr{E}(0)$ if necessary), we remark that for $\varepsilon > 0$ small enough, $t = 1+\varepsilon$ is a strong existence time, and Corollary 6.4 ensures that for t close enough to 1, the inequality $\int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s < \delta$ is satisfied, ensuring $t^\star > 1$.

For $t \in [1, t^\star)$ we can invoke Lemma 4.7 with $t_0 = 1$ and (5.1), to obtain that $\mathrm{M}_{\rho_f}(t) \lesssim_0 1$ and

$$\|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim_0 e^{-t} \int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s.$$

Thanks to Sobolev's embedding $\mathrm{H}^2(\mathbb{T}^3) \hookrightarrow \mathrm{L}^\infty(\mathbb{T}^3)$ we infer

$$\int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s \lesssim \int_1^t e^s \|u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\, \mathrm{d}s + \int_1^t e^s \|\mathrm{D}^2 u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\, \mathrm{d}s,$$

and therefore (using Cauchy–Schwarz's inequality)

$$\begin{aligned} \int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s &\lesssim (e^t - 1) \sup_{[1,t]} \|u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)} \\ &\quad + \left(\int_1^t e^{2s}\, \mathrm{d}s \right)^{1/2} \left(\int_1^t \|\mathrm{D}^2 u(s)\|_2^2\, \mathrm{d}s \right)^{1/2}. \end{aligned}$$

Thanks to (5.5) and the energy estimate (1.10) we eventually infer

$$e^{-t} \int_1^t e^s \|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s \lesssim_0 1 + \mathrm{M}_{\rho_f}(t),$$

and we have already proved that $\mathrm{M}_{\rho_f}(t) \lesssim_0 1$. We deduce that $\mathrm{M}_{j_f}(t) \lesssim_0 1$ and this concludes the proof. □

We now combine Proposition 6.1 with Lemma 3.4.

Lemma 7.2. *Assume that* $t_\star < \infty$. *For any* $\alpha \in [1/2, 1)$, $c \in [1, \infty)$ *and any finite* $a, b \geqq \max(1, c\alpha)$, *the following estimate holds (with a possible infinite right-hand side):*

$$\left(\int_1^{t^\star} \|\nabla u(s)\|^c_{L^p(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/c} \lesssim_0 \mathscr{E}(0)^{\frac{1-\alpha}{2}}\left(1 + \|(u\cdot\nabla)u\|^\alpha_{L^a(1/2,t^\star;L^r(\mathbb{T}^3))} + \|F\|^\alpha_{L^b(1/2,t^\star;L^r(\mathbb{T}^3))}\right) \tag{7.2}$$

for $p \in [1, \infty]$ *and* $r \in (1, \infty)$ *satisfying*

$$\frac{1}{p} = \frac{1}{3} + \alpha\left(\frac{1}{r} - \frac{2}{3}\right) + \frac{1-\alpha}{2}. \tag{7.3}$$

Proof. Owing to Lemma 3.4, if $t^\star < +\infty$, there is, on $[0, t^\star]$, an exponential decay of the modulated energy with decay rate $\lambda^\star$. The Gagliardo–Nirenberg–Sobolev estimate of Theorem 9.9 for $(j, m, q) = (1, 2, 2)$ allows us to write for any $\alpha \in [1/2, 1)$ and $s \geqq 1$

$$\|\nabla u(s)\|_{\mathrm{L}^p(\mathbb{T}^3)} \lesssim \|\mathrm{D}^2 u(s)\|^\alpha_{\mathrm{L}^r(\mathbb{T}^3)}\|u(s) - \langle u(s)\rangle\|^{1-\alpha}_{\mathrm{L}^2(\mathbb{T}^3)},$$

for p, r satisfying (7.3). By definition of the modulated energy and using its exponential decay on $[1, t^\star]$, we have, therefore,

$$\|\nabla u(s)\|_{\mathrm{L}^p(\mathbb{T}^3)} \lesssim \mathscr{E}(0)^{\frac{1-\alpha}{2}} e^{-\lambda s}\|\mathrm{D}^2 u(s)\|^\alpha_{\mathrm{L}^r(\mathbb{T}^3)},$$

for $\lambda = \lambda^\star(1-\alpha)$. We apply Proposition 6.1 to infer that for any exponent c such that $c\alpha \leqq a, b$

$$\int_1^{t^\star} \|\nabla u(s)\|^c_{\mathrm{L}^p(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_0 \Phi(\lambda^\star)\mathscr{E}(0)^{c\frac{1-\alpha}{2}}\left(1 + \|(u\cdot\nabla)u\|^{c\alpha}_{\mathrm{L}^a(1/2,t^\star;\mathrm{L}^r(\mathbb{T}^3))} + \|F\|^{c\alpha}_{\mathrm{L}^b(1/2,t^\star;\mathrm{L}^r(\mathbb{T}^3))}\right),$$

where Φ is nonincreasing. But by Lemma 3.4, $\lambda^\star$ itself is a nonincreasing function of $\mathrm{M}_{\rho_f}(t^\star) \lesssim_0 1$, which yields (7.2). □

Lemma 7.3. *There exists* $\gamma > 0$ *such that, if* $t^\star < +\infty$, *then the following estimate holds:*

$$\int_1^{t^\star} \|\nabla u(s)\|_{L^\infty(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_0 \mathscr{E}(0)^\gamma. \tag{7.4}$$

Proof. We start by combining Lemma 7.2 with Lemma 6.2 and Lemma 6.3. Since (by Proposition 7.1) $\mathrm{M}_{\rho_f, j_f} \lesssim_0 1$ on $[1, t^\star]$, these results give us for some $b > 4 > a > 2$ and $r = \min(r_a, r_b) > 2$, the following estimate:

$$\left(\int_1^{t^\star} \|\nabla u(s)\|^c_{\mathrm{L}^p(\mathbb{T}^3)} \,\mathrm{d}s\right)^{1/c} \lesssim_0 \mathscr{E}(0)^{\frac{1-\alpha}{2}}, \tag{7.5}$$

which holds for any $\alpha \in [1/2, 1)$ and p defined by (7.3), provided that that $\alpha c \leqq \min(a, b)$.

It is important to note that $p = \infty$ is not yet reachable at this stage, due to the constraint $\alpha \in [1/2, 1]$. However, we can first use Lemma 7.2 with $c = a < b$ in (7.5). In that case, going back to (7.3), we see that the limit case $\alpha = 1$ leads to the equality

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{3},$$

which, since $r > 2$, implies $\frac{1}{p} < \frac{1}{6}$, that is $p > 6$. Taking $\alpha \in [1/2, 1)$ close enough to 1, we therefore infer the existence of $p > 6$ such that

$$\|\nabla u\|_{\mathrm{L}^a(1,t^\star;\mathrm{L}^p(\mathbb{T}^3))} \lesssim_0 \mathscr{E}(0)^{(1-\alpha)/2}.$$

Since $p > 6$, we infer from Hölder's inequality, for some $\widetilde{r}_a > 3$, that

$$\begin{aligned}\left(\int_{1/2}^{t^\star} \|(u\cdot\nabla)u(s)\|^a_{\mathrm{L}^{\tilde{r}_a}(\mathbb{T}^3)} \,\mathrm{d}s\right)^{1/a} &\leqq \|u\|_{\mathrm{L}^\infty(1/2,t^\star;\mathrm{L}^6(\mathbb{T}^3))} \|\nabla u\|_{\mathrm{L}^a(1/2,t^\star;\mathrm{L}^p(\mathbb{T}^3))} \\ &\lesssim_0 \mathscr{E}(0)^{(1-\alpha)/2} \|u\|_{\mathrm{L}^\infty(1/2,t^\star;\mathrm{L}^6(\mathbb{T}^3))}, \\ &\lesssim_0 \mathscr{E}(0)^{(1-\alpha)/2}.\end{aligned}$$

The point is that this last inequality can now replace Lemma 6.2 that we used earlier: we can perform the same analysis as before with the advantage that, now $\widetilde{r}_a > 3$. This yields that $\widetilde{r} := \min(r_b, \widetilde{r}_a) > 3$ and hence, taking

$$\widetilde{\alpha} = 5\left(7 - \frac{6}{\widetilde{r}}\right)^{-1} < 1,$$

we can check that $\widetilde{\alpha} \in [1/2, 1)$ and satisfies

$$0 = \frac{1}{3} + \widetilde{\alpha}\left(\frac{1}{\widetilde{r}} - \frac{2}{3}\right) + \frac{1-\widetilde{\alpha}}{2}.$$

Thus we invoke Lemma 7.2 another time with $r = \widetilde{r} > 3$, $c = 1$ and $\widetilde{\alpha}$ as above to infer

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \,\mathrm{d}s \lesssim_0 \mathscr{E}(0)^{(1-\tilde{\alpha})/2}\left(1 + \mathscr{E}(0)^{(1-\alpha)/2}\right),$$

which is an estimate of the form (7.4). □

We are finally in position to conclude the proof of Theorem 2.2.

Proof. Applying Proposition 7.1, the question thus reduces to ensure $t^\star = +\infty$. Assuming $t^\star < +\infty$, we will reach a contradiction by proving (for a small enough $\mathscr{E}(0)$) the existence of $t > t^\star$ which is still a strong existence time and for which the inequality (7.1) is satisfied.

The first task is to exhibit strong existence times larger than $t^\star$. Thanks to Proposition 7.1 and Proposition 5.1, recalling the meaning of the symbol $\lesssim_0$ (see Notation 5.1), we have the existence of nondecreasing function φ such for any $t \in [1, t^\star]$,

$$\sup_{s\in[0,1]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \mathrm{M}_{\rho_f, j_f}(t) \leqq \varphi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + \mathrm{E}(0) + N_q(f_0) + 1\right). \tag{7.6}$$

Recall that by assumption, we have $\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} < \frac{1}{\mathrm{C}_\star^2}$. Using Lemma 5.2, we thus infer that for all strong existence times $t \leqq t^\star$,

$$\begin{aligned}
&\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathrm{C}_\star \int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \\
&\quad\leqq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathrm{C}_\star \int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\
&\quad\leqq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathscr{E}(0)\mathrm{C}_\star\left(\mathrm{M}_{\rho_f, j_f}(t) + \sup_{s\in[0,1]} \|\rho(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\right),
\end{aligned}$$

where we used the embedding $\mathrm{L}^2(\mathbb{T}^3) \hookrightarrow \mathrm{H}^{-\frac{1}{2}}(\mathbb{T}^3)$, with constant 1. Combining this with (7.6), we get, for some nondecreasing function still denoted φ,

$$\begin{aligned}
&\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathrm{C}_\star \int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \\
&\quad\leqq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathscr{E}(0)\varphi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + \mathrm{E}(0) + N_q(f_0) + 1\right),
\end{aligned}$$

Therefore, choosing $\mathscr{E}(0)$ small enough so that

$$\begin{aligned}
&\varphi\left(N_q(f_0) + M_\alpha f_0 + \mathrm{E}(0) + \|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + 1\right)\mathscr{E}(0) \\
&\quad< \min\left(1, \frac{1}{\mathrm{C}_\star^2} - \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}\right),
\end{aligned}$$

we deduce that

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathrm{C}_\star \int_0^{t^\star} \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s < \frac{1}{\mathrm{C}_\star^2},$$

hence proving by continuity the existence of strong existence times larger than $t^\star$.

To check that (7.1) is satisfied after $t^\star$ we use Lemma 7.3 to infer the existence of an universal onto nondecreasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s \leqq \varphi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + N_q(f_0) + \mathrm{E}(0) + 1\right) \mathscr{E}(0)^\gamma,$$

and we observe that a smallness condition as (2.1) ensures

$$\int_1^{t^\star} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s < \delta.$$

Therefore we can find a strong existence time $t > t^\star$ such that

$$\int_1^{t} \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, \mathrm{d}s < \delta.$$

This is a contradiction with the definition of $t^\star$ and finally concludes the proof. □

8. Further Description of the Asymptotic State

Once the exponential decay of the modulated energy is established, Proposition 3.5 leads to the existence of a profile $\rho^\infty \in \mathrm{L}^\infty(\mathbb{T}^3)$ which allows to describe the asymptotic behavior of f in the space variable. The content of Proposition 3.5 is quite implicit as the profile is obtained by an abstract argument. It is in fact possible to describe ρ^∞ in a finer way (but still, *via* implicit equations); this is the purpose of Proposition 2.3 that we aim at proving in this last section.

Before doing this, it is interesting to compare the statement of Proposition 2.3 with the explicit asymptotic behavior of solutions to the linearized equation when $\langle u_0 + j_{f_0}\rangle = 0$, that is the Vlasov equation with friction

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(-vf) = 0,$$

for which we recall that we have

$$\mathrm{W}_1\left(f(t,x,v), \widetilde{\rho}_0 \otimes \delta_0\right) \longrightarrow_{t\to\infty} 0,$$

with

$$\widetilde{\rho}_0(x) := \int_{\mathbb{R}^3} f_0\left(x - v, v\right)\, \mathrm{d}v.$$

From 2.9 , we therefore see that the deviation from the linearized behavior is small, as

$$\mathrm{Y}^0_{\infty,x,v} - (x - v) = -\int_0^{+\infty} u(\tau, \mathrm{Y}^\tau_{\infty,x,v})\, \mathrm{d}\tau,$$

is small in $\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, as it is controlled by the initial modulated energy $\mathscr{E}(0)$ and $|\det \mathscr{A}(\infty, x, v)| - 1$ is also small in $\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, as we will see in the upcoming proof.

We will detail the proof of Proposition 2.3 only in the particular case $\langle u_0 + j_{f_0}\rangle = 0$ for which the computations are a bit less tedious. The general case is a straightforward generalization (see Remark 8.1).

Proof of Proposition 2.3 in the case $\langle u_0 + j_{f_0}\rangle = 0$

Proof. Recall the map $\Gamma_{t,x} : v \mapsto \mathrm{V}(0; t, x, v)$ that we already used in Lemma 4.4 of Section 4.2 : this very lemma ensures that, for δ small enough ($\delta e^{\delta} < 1/9$ is sufficient), if (2.6) is satisfied, $\Gamma_{t,x}$ is a $\mathscr{C}^1(\mathbb{R}^3)$-diffeomorphism. In order to capture the asymptotic profile of $\rho_f(t)$ we look at its action on a continuous function ψ:

$$\int_{\mathbb{T}^3} \rho_f(t,x)\,\psi(x)\,\mathrm{d}x.$$

Since ρ_f does not solve a transport equation we cannot link it to the initial density $\rho_f(0)$, however we can write

$$\begin{aligned}\int_{\mathbb{T}^3} \rho_f(t,x)\,\psi(x)\,\mathrm{d}x &= \int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)\,\psi(x)\,\mathrm{d}v\,\mathrm{d}x \\ &= \int_{\mathbb{T}^3\times\mathbb{R}^3} e^{3t} f_0(\mathrm{X}(0;t,x,v),\mathrm{V}(0;t,x,v))\,\psi(x)\,\mathrm{d}v\,\mathrm{d}x \\ &= e^{3t}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(\mathrm{Y}(0;t,x,v),v)\,\psi(x)|\det \mathrm{D}_v\Gamma_{t,x}|^{-1}\,\mathrm{d}v\,\mathrm{d}x,\end{aligned}$$

where $\mathrm{Y}(0;t,x,v) := \mathrm{X}(0;t,x,\Gamma_{t,x}^{-1}(v))$. Recall that

$$\Gamma_{t,x}(v) = e^t v - \int_0^t e^{\tau} u(\tau,\mathrm{X}(\tau;t,x,v))\,\mathrm{d}\tau,$$

hence (with the notation $\mathrm{Y}(\tau;t,x,v) := \mathrm{X}(\tau;t,x,\Gamma_{t,x}^{-1}(v)))$

$$\Gamma_{t,x}^{-1}(v) = e^{-t}v + \int_0^t e^{\tau-t}u(\tau,\mathrm{Y}(\tau;t,x,v))\,\mathrm{d}\tau, \tag{8.1}$$

from which we infer

$$e^t\mathrm{D}_v\Gamma_{t,x}^{-1}(v) = \mathrm{I}_3 + \int_0^t e^{\tau}\nabla u(\tau,\mathrm{Y}(\tau;t,x,v))\,\mathrm{D}_v\mathrm{Y}(\tau;t,x,v)\mathrm{d}\tau.$$

All in all, introducing the variable $z := (x,v)$ and denoting $\mathrm{Y}^s_{t,z} := \mathrm{Y}(s;t,z)$, we have established

$$\int_{\mathbb{T}^3} \rho_f(t,x)\,\psi(x)\,\mathrm{d}x = \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(\mathrm{Y}^0_{t,z},v)\psi(x)\,|\det\mathscr{A}(t,z)|\,\mathrm{d}z, \tag{8.2}$$

where

$$\mathscr{A}(t,z) := \mathrm{I}_3 + \int_0^t e^{\tau}\nabla u(\tau,\mathrm{Y}^{\tau}_{t,z})\,\mathrm{D}_v\mathrm{Y}^{\tau}_{t,z}\,\mathrm{d}\tau. \tag{8.3}$$

In order to understand the behavior of $\rho_f(t)$ as $t \to +\infty$ it is therefore natural to follow the curves $t \mapsto \mathrm{Y}^s_{t,z}$ as $t \to +\infty$, and this is the purpose of □

Lemma 8.1. *For $\delta > 0$ small enough, the following holds: for all $0 \leqq s \leqq t$ and $z := (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$ we have*

$$|D_x Y^s_{t,z}| \leqq 2, \tag{8.4}$$

$$|e^s D_v Y^s_{t,z}| \leqq 4. \tag{8.5}$$

Furthermore, the family of maps $(s, z) \mapsto Y^s_{t,z}$ converges in $\mathscr{C}^0(\mathbb{R}_+; \mathscr{C}^1(\mathbb{T}^3 \times \mathbb{R}^3))$, as $t \to +\infty$, to a map $(s, z) \mapsto Y^s_{\infty,z}$ that satisfies

$$Y^s_{\infty,z} = x - e^{-s}v - \int_0^{+\infty} \Big[\mathbf{1}_{[0,s]}(\tau)e^{\tau-s} + \mathbf{1}_{\tau \geqq s}\Big] u(\tau, Y^\tau_{\infty,z})\, d\tau.$$

Proof. We start by recalling

$$\mathrm{X}(s; t, x, v) = x + (1 - e^{t-s})v + \int_s^t \left(e^{\tau-s} - 1\right) u(\tau, \mathrm{X}(\tau; t, x, v))\, d\tau,$$

from which, together with (8.1), we deduce the following formula for $s \leqq t$:

$$\mathrm{Y}^s_{t,z} = x + (e^{-t} - e^{-s})v + \int_0^{+\infty} \Big[e^{\tau-t}\mathbf{1}_{\tau \leqq t} - e^{\tau-s}\mathbf{1}_{\tau \leqq s} - \mathbf{1}_{s \leqq \tau \leqq t}\Big] u(\tau, \mathrm{Y}^\tau_{t,z})\, d\tau. \tag{8.6}$$

From the previous expression, we infer for $s \leqq t$ that

$$|\mathrm{D}_x \mathrm{Y}^s_{t,z}| \leqq 1 + 2\int_0^{+\infty} \mathbf{1}_{\tau \leqq t} |\nabla u(\tau, \mathrm{Y}^\tau_{t,z}) \mathrm{D}_x \mathrm{Y}^\tau_{t,z}|\, d\tau.$$

In particular, this implies

$$\sup_{0 \leqq \tau \leqq t} |\mathrm{D}_x \mathrm{Y}^\tau_{t,z}| \leqq 1 + 2 \sup_{0 \leqq \tau \leqq t} |\mathrm{D}_x \mathrm{Y}^\tau_{t,z}| \int_0^{+\infty} \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\, d\tau,$$

which together with the assumption (2.6) implies, for $s \leqq t$, that

$$|\mathrm{D}_x \mathrm{Y}^s_{t,z}| \leqq \sup_{0 \leqq \tau \leqq t} |\mathrm{D}_x \mathrm{Y}^\tau_{t,z}| \leqq \frac{1}{1 - 2\delta},$$

which implies (8.4) for $\delta \leqq 1/4$. Similarly, and returning to (8.6), we have for $s \leqq t$ that

$$|e^s \mathrm{D}_v \mathrm{Y}^s_{t,z}| = 2 + 2\int_0^t e^\tau \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} |\mathrm{D}_v \mathrm{Y}^\tau_{t,z}|\, d\tau,$$

and we can proceed in the same way to obtain (8.5). To establish the existence of $(s, z) \mapsto \mathrm{Y}^s_{\infty,z}$, we shall prove that $(s, z) \mapsto \mathrm{Y}^s_{t,z}$ satisfies Cauchy's criterion as $t \to +\infty$, with respect to the local uniform metric. Since $\langle u_0 + j_{f_0}\rangle = 0$, we have by Lemma 3.2 and by definition of the modulated energy that $t \mapsto \langle u(t)\rangle$ is integrable over $\mathbb{R}_+$ (due to its exponential decay). In particular, we infer the integrability over $\mathbb{R}_+$ of $t \mapsto \|u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \leqq |\langle u(t)\rangle| + \|\nabla u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$, thanks to the assumption (2.6). In particular, by dominated convergence we infer that

$$\mathrm{Y}^s_{t,z} = x - e^{-s}v - \int_0^{+\infty} \Big[e^{\tau - s}\mathbf{1}_{\tau \leqq s} + \mathbf{1}_{s \leqq \tau \leqq t}\Big] u(\tau, \mathrm{Y}^\tau_{t,z})\,\mathrm{d}\tau + \mathrm{o}(1), \tag{8.7}$$

where the notation o(1) refers a term going to 0 in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3)$ in the limit $t \to +\infty$. □

Remark 8.1. In the general case $\langle u_0 + j_{f_0}\rangle \neq 0$, $t \mapsto \langle u(t)\rangle$ is not integrable, as it converges to $\langle u_0 + j_{f_0}\rangle/2$. One needs to replace $u(\tau, \mathrm{Y}^\tau_{s,t})$ by $u(\tau, \mathrm{Y}^\tau_{s,t}) - \langle u_0 + j_{f_0}\rangle/2$ in the integrand of (8.7) and, by doing so, adds a diverging drift term to the equation. In a similar fashion as the proof of Proposition 3.5, this can be counterbalanced by considering the renormalized characteristics $\mathrm{Y}(\tau; t, x + \langle u_0 + j_{f_0}\rangle/2, v)$ instead of $\mathrm{Y}(\tau, t, x, v)$. The equations for these shifted trajectories are a bit different, but the convergence properties are proved in the same way, resulting in the implicit equation (2.9).

In particular, taking the difference of this identity (8.7) at times $t_1 < t_2$,

$$\begin{aligned}
|\mathrm{Y}^s_{t_2,z} - \mathrm{Y}^s_{t_1,z}| &\leqq 2\int_0^{+\infty} \mathbf{1}_{\tau \leqq t_2} |u(\tau, \mathrm{Y}^\tau_{t_2,z}) \\
&\qquad - u(\tau, \mathrm{Y}^\tau_{t_1,z})|\,\mathrm{d}\tau + \int_{t_1}^{t_2} |u(\tau, \mathrm{Y}^\tau_{t_1,z}))|\,\mathrm{d}\tau + \mathrm{o}(1) \\
&\leqq 2\int_0^{+\infty} \mathbf{1}_{\tau \leqq t_2} \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} |\mathrm{Y}^\tau_{t_2,z} - \mathrm{Y}^\tau_{t_1,z}|\,\mathrm{d}\tau \\
&\qquad + \int_{t_1}^{t_2} \|u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}\tau + \mathrm{o}(1),
\end{aligned}$$

where o(1) refers here to the asymptotic $t_1 \wedge t_2 \to +\infty$, with the same uniformity as before. Using once more the integrability of $t \mapsto \|u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$, for any compact $K \subset \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$, if

$$\varphi(t_1, t_2) := \sup_{(\tau,z)\in K} |\mathrm{Y}^\tau_{t_2,z} - \mathrm{Y}^\tau_{t_1,z}|,$$

we have established

$$\varphi(t_1, t_2) \leqq 2\varphi(t_1, t_2) \int_0^{+\infty} \|\nabla u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}\tau + \mathrm{o}(1),$$

with a similar (uniform) asymptotic term o(1). From assumption (2.6), this proves

$$\sup_{(s,z)\in K} |\mathrm{Y}^s_{t_2,z} - \mathrm{Y}^s_{t_1,z}| = \mathrm{o}(1),$$

which yields Cauchy's criterion. We deduce the existence of $(s, z) \mapsto \mathrm{Y}^s_{\infty,z}$, as the (local uniform) limit of $(s, z) \mapsto \mathrm{Y}^s_{t,z}$ as $t \to +\infty$. By the dominated convergence and continuity of u for positive times, $\mathrm{Y}^s_{\infty,z}$ must satisfy the equation

$$\mathrm{Y}^s_{\infty,z} = x - e^{-s}v - \int_0^{+\infty} \Big[\mathbf{1}_{[0,s]}(\tau)e^{\tau - s} + \mathbf{1}_{\tau \geqq s}\Big] u(\tau, \mathrm{Y}^\tau_{\infty,z})\,\mathrm{d}\tau. \tag{8.8}$$

For now $Y^{\tau}_{\infty,z}$ is merely continuous (as a uniform limit) in all its variables, but it turns out that the derivatives $(s;t,z) \mapsto D_z Y^s_{t,z}$ enjoys the same Cauchy criterion as $Y^s_{t,z}$. Indeed, going back to (8.6), we infer, using integrability of $\tau \mapsto \|\nabla u(\tau)\|_{L^\infty(\mathbb{T}^3)}$ over $\mathbb{R}_+$ and dominated convergence

$$D_z Y^s_{t,z} = -\int_0^{+\infty} \Big[e^{\tau-s}\mathbf{1}_{\tau\leqq s} + \mathbf{1}_{s\leqq\tau\leqq t}\Big]\nabla u(\tau, Y^{\tau}_{t,z}) D_z Y^{\tau}_{t,z}\, d\tau + o(1) + r_{s,z},$$

where o(1) refers to the asymptotic $t \to +\infty$ and is locally uniform in s, z, while $r_{s,z}$ is some irrelevant function which does not depend on t. For any $t_1 < t_2$ we thus have that

$$|D_z Y^s_{t_2,z} - D_z Y^s_{t_1,z}| \leqq 2\int_0^{+\infty} \mathbf{1}_{\tau\leqq t_2}|\nabla u(\tau, Y^{\tau}_{t_2,z}) D_z Y^{\tau}_{t_2,z} - \nabla u(\tau, Y^{\tau}_{t_1,z}) D_z Y^{\tau}_{t_1,z}|\, d\tau$$
$$+ \int_{t_1}^{t_2} |\nabla u(\tau, Y^{\tau}_{t_1,z}) D_z Y^{\tau}_{t_1,z}|\, d\tau + o(1),$$

where o(1) refers to $t_1 \wedge t_2 \to +\infty$ and is locallly uniform in s, z. Owing to the integrability of $\tau \mapsto \|\nabla u(\tau)\|_{L^\infty(\mathbb{T}^3)}$ over $\mathbb{R}_+$ and the uniform bound on $(s,t,z) \mapsto \mathbf{1}_{s\leqq t} D_z Y^s_{t,z}$ due to estimates (8.4) – (8.5), we infer

$$|D_z Y^s_{t_2,z} - D_z Y^s_{t_1,z}| \leqq 2\int_0^{+\infty} \mathbf{1}_{\tau\leqq t_2}|\nabla u(\tau, Y^{\tau}_{t_2,z})\big[D_z Y^{\tau}_{t_2,z} - D_z Y^{\tau}_{t_1,z}\big]|\, d\tau$$
$$+ \int_0^{+\infty} \mathbf{1}_{\tau\leqq t_2}|\big[\nabla u(\tau, Y^{\tau}_{t_2,z}) - \nabla u(\tau, Y^{\tau}_{t_1,z})\big] D_z Y^{\tau}_{t_1,z}|\, d\tau + o(1).$$

Since $(Y^s_{t,z})_t \to Y^s_{\infty,z}$ pointwisely, the continuity of ∇u for positive times, its belonging to $L^1(\mathbb{R}_+; L^\infty(\mathbb{T}^3))$ and the aforementioned uniform boundedness of $(s,t,z) \mapsto \mathbf{1}_{s\leqq t} D_z Y^s_{t,z}$ entail, by dominated convergence,

$$|D_z Y^s_{t_2,z} - D_z Y^s_{t_1,z}| \leqq 2\int_0^{+\infty} \mathbf{1}_{\tau\leqq t_2}\|\nabla u(\tau)\|_{L^\infty(\mathbb{T}^3)}|D_z Y^{\tau}_{t_2,z} - D_z Y^{\tau}_{t_1,z}|\, d\tau + o(1),$$

and we can then proceed as we have done for $Y^s_{t,z}$ to establish the local uniform Cauchy criterion. □

If f_0 was assumed to be continuous in the space variable, we would now able to pass to the limit into formula (8.2) ; indeed we would have then by dominated convergence, using the bounds that we have established on $(s,t,z) \mapsto e^s D_v Y^s_{t,z}$ and the integrability of $v \mapsto \sup_{\mathbb{T}^3} f_0(\cdot, v)$,

$$\int_{\mathbb{T}^3} \rho_f(t,x)\psi(x)\, dx \longrightarrow_{t\to+\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(Y^0_{\infty,z}, v)\psi(x)|\det \mathscr{A}(\infty, z)|\, dz,$$

with

$$\mathscr{A}(\infty, z) = I_3 + \int_0^{+\infty} e^{\tau}\nabla u(\tau, Y^{\tau}_{\infty,z})\, D_v Y^{\tau}_{\infty,z}\, d\tau.$$

Notice that here the convergence $z \mapsto \mathscr{A}(t,z)$ towards $z \mapsto \mathscr{A}(\infty, z)$ is also locally uniform in z. However, we are not in position to replace f_0 by a regularized version; to do so we would need a uniqueness result for the whole coupling, and such a result is only known in dimension 2 (see [17]). It turns out that the above convergence holds, but to establish it we have to use another change of variable. More precisely, in (8.2) we consider the change of variable $x \mapsto \Lambda_{t,v}(x) := \mathrm{Y}^0_{t,x,v}$. This is admissible thanks to Lemma 9.4 and the estimate

$$\|\mathrm{D}_x \mathrm{Y}^0_{t,z} - \mathrm{I}_3\|_\infty \leqq \frac{1}{9}, \tag{8.9}$$

which itself is a consequence of (8.6), (8.4) and assumption (2.6), if δ is small enough. We have therefore that

$$\begin{aligned}\int_{\mathbb{T}^3} &\rho_f(t,x)\,\psi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)\,\psi(\Lambda_{t,v}^{-1}(x))|\det\mathscr{A}(t,\Lambda_{t,v}^{-1}(x),v)\det \mathrm{D}_x\Lambda_{t,v}^{-1}(x)|\,\mathrm{d}z.\end{aligned} \tag{8.10}$$

The long-time behavior of $\Lambda_{t,v}^{-1}(x)$ is given by

Lemma 8.2. *For all $v \in \mathbb{R}^3$ the map $\Lambda_{\infty,v} : x \mapsto Y^0_{\infty,x,v}$ is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{T}^3$ onto itself and we have $\Lambda_{t,v}^{-1}(x) \to_t \Lambda_{\infty,v}^{-1}(x)$ in $\mathscr{C}^1(\mathbb{T}^3\times\mathbb{R}^3)$, as $t \to +\infty$ and also $|\det D_x\Lambda_{t,v}^{-1}(x)| \leqq 2$ for all x, v, t.*

Proof. First, we infer from (8.9) the same estimate (by uniform convergence) for $\Lambda_{\infty,v}$, which is therefore also (thanks to Lemma 9.4) a $\mathscr{C}^1$-diffeomorphism. The same lemma gives also $\det\Lambda_{t,v} \geqq 1/2$ for all $t \in [1,\infty]$. Again, thanks to Lemma 9.4, we infer uniformly in t, x, v, $|\mathrm{D}_x\Lambda_{\infty,v}^{-1}(x)| \leqq 9/8$ and $\det \mathrm{D}_x\Lambda_{t,v}(x) \geqq 1/2$. For the convergence, we write

$$\begin{aligned}|\Lambda_{t,v}^{-1}(x) - \Lambda_{\infty,v}^{-1}(x)| &= |\Lambda_{\infty,v}^{-1}\circ\Lambda_{\infty,v}\circ\Lambda_{t,v}^{-1}(x) - \Lambda_{\infty,v}^{-1}(x)| \\ &\leqq \frac{9}{8}|\Lambda_{\infty,v}\circ\Lambda_{t,v}^{-1}(x) - x| \\ &= \frac{9}{8}|\Lambda_{\infty,v}\circ\Lambda_{t,v}^{-1}(x) - \Lambda_{t,v}\circ\Lambda_{t,v}^{-1}(x)|,\end{aligned}$$

which goes to 0 locally uniformly in x, v thanks to Lemma 8.1. Since the inversion map is $\mathscr{C}^1$ on $\mathrm{GL}_3(\mathbb{R})$, using the previous lower bound on the determinants, we infer from the equatity $\mathrm{D}_x\Lambda_{t,v}^{-1} = (\mathrm{D}_x\Lambda_{t,v})^{-1}\circ\Lambda_{t,v}^{-1}$ and the previous convergence the announced convergence in $\mathscr{C}^1(\mathbb{T}^3\times\mathbb{R}^3)$. □

Since $\mathscr{A}(t,z)$ is uniformly bounded and continuous and converges (locally uniformly) towards $\mathscr{A}(\infty,z)$, we infer from Lemma 8.2 and the dominated convergence theorem (using $f_0 \in \mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)$) that

$$\begin{aligned}\int_{\mathbb{T}^3} &\rho_f(t,x)\,\psi(x)\,\mathrm{d}x \\ &\longrightarrow_{t\to+\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)\,\psi(\Lambda_{\infty,v}^{-1}(x))|\det\mathscr{A}(\infty,\Lambda_{\infty,v}^{-1}(x),v)\det\mathrm{D}_x\Lambda_{\infty,v}^{-1}(x)|\,\mathrm{d}z,\end{aligned}$$

and, using the change of variable $x \hookleftarrow \Lambda_{\infty,v}(x)$ (which is admissible thanks to Lemma 8.2), we have eventually proved that

$$(\rho_f(t))_t \rightharpoonup_{t\to+\infty} \rho^\infty,$$

where

$$\rho^\infty(x) := \int_{\mathbb{R}^3} f_0(\mathrm{Y}^0_{\infty,x,v}, v)|\det \mathscr{A}(\infty, x, v)|\,\mathrm{d}v,$$

which concludes the proof. □

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9. Appendix

9.1. Wasserstein Distance

To simplify the presentation, X here will denote either $\mathbb{T}^3$ or $\mathbb{T}^3 \times \mathbb{R}^3$.

Definition 9.1. For $m > 0$ we denote by $\mathcal{M}_{1,m}(X)$ the set of all measures μ such that

$$\int_X |z|\,\mathrm{d}\mu(z) < +\infty, \qquad \mu(X) = m.$$

Definition 9.2. Fix $m > 0$ and consider μ and ν in $\mathcal{M}_{1,m}(X)$. The *Wassertein distance* between μ and ν is

$$\mathrm{W}_1(\mu, \nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{X^2} |z - z'|\,\mathrm{d}\gamma(z, z'),$$

where $\Gamma(\mu, \nu)$ denotes the collection of all measures on $X \times X$ with first and second marginal respectively equal to μ and ν.

Proposition 9.1. (W_1 metrizes the weak-$\star$ convergence) *Fix $m > 0$. Given $(\mu_n)_n \in \mathcal{M}_{1,m}(X)^{\mathbb{N}}$ and $\mu \in \mathcal{M}_{1,m}(X)$, the two following facts are equivalent:*

(i) For all $f \in \mathscr{C}^0_b(X)$,

$$\int_X (f(z) + |z|)\,\mathrm{d}\mu_n(z) \longrightarrow_{n\to+\infty} \int_X (f(z) + |z|)\,\mathrm{d}\mu(z).$$

(ii) $(W_1(\mu_n, \mu))_n \to_n 0$.

Proposition 9.2. (Monge-Kantorovitch duality) *Fix* $m > 0$ *and consider* μ *and* ν *in* $\mathcal{M}_{1,m}(X)$. *Then*

$$W_1(\mu, \nu) = \sup\left\{\int_X \phi(z)\mathrm{d}\mu(z) - \int_X \phi(z)\mathrm{d}\nu(z) \; : \; \phi \in Lip(X), \|\nabla\phi\|_\infty \leqq 1\right\}.$$

9.2. Exponential Decay

Lemma 9.3. *Consider* $u : \mathbb{R}_+ \to \mathbb{R}_+$ *a non-increasing integrable function satisfying, for some* $\lambda > 0$ *and almost all* $t \geqq 0$,

$$\lambda \int_t^\infty u(s)\,\mathrm{d}s \leqq u(t).$$

Then, for $t \geqq 0$, *it holds that*

$$u(t) \lesssim_{u(0),\lambda} e^{-\lambda t}.$$

Proof. If $v(t)$ denotes the integral in the estimate, $v \in \mathrm{W}^{1,\infty}(\mathbb{R}_+)$ satisfies $v' \leqq -\lambda v$, so the standard version of the Gronwall Lemma implies $v(t) \leqq v(0)e^{-\lambda t}$. Since $u \leqq u(0)$ w.l.o.g. we can assume $t \geqq 1$ and since u is non-increasing, we have

$$u(t) \leqq \int_{t-1}^t u(s)\,\mathrm{d}s \leqq v(t-1) \leqq v(0)e^{-\lambda t} \leqq \frac{1}{\lambda}u(0)e^{-\lambda t}.$$

□

9.3. Perturbation of the Identity Map

We use in this work the following version of the inverse function theorem:

Lemma 9.4. *For* $\Omega = \mathbb{T}^3$ *or* $\Omega = \mathbb{R}^3$, *if* $\phi : \Omega \to \Omega$ *is* $\mathscr{C}^1$ *and satisfies* $\|\nabla\phi\|_\infty < 1$, *then* $f := Id + \phi$ *is a* $\mathscr{C}^1$*-diffeomorphism of* Ω *onto itself satisfying* $\|\nabla f\|_\infty \leqq (1 - \|\nabla\phi\|_\infty)^{-1}$. *If furthermore* $\|\nabla\phi\|_\infty \leqq 1/9$, *then* $\det \nabla f \geqq 1/2$.

9.4. Maximal Regularity

The maximal regularity estimate for the heat equation, on the whole space, can be stated in the following way:

Theorem 9.5. *For* $p, q \in (1, \infty)$, *and* $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ *such that* $\varphi(0, \cdot) = 0$, *it holds that*

$$\|\Delta\varphi\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}^3))} \lesssim_{p,q} \|\partial_t\varphi - \Delta\varphi\|_{L^p(\mathbb{R}_+;L^q(\mathbb{R}^3))}.$$

This estimate is for instance a consequence of [22]. Naturally, one expects an analogous estimate on the torus, but we did not manage to exhibit a precise reference in the literature. For the sake of completeness we give therefore a proof of the following corollary:

Corollary 9.6. *For $p, q \in (1,\infty)$ and $\psi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^3)$ which is $\mathbb{Z}^3$-periodic in the space variable and such that $\psi(0,\cdot) = 0$, we have*

$$\|\Delta\psi\|_{L^p(\mathbb{R}_+;L^q(\mathbb{T}^3))} \lesssim_{p,q} \|\partial_t\psi - \Delta\psi\|_{L^p(\mathbb{R}_+;L^q(\mathbb{T}^3))}.$$

Proof. Let's use the Dirac comb $Ш := \sum_{n\in\mathbb{Z}^3}\delta_n$ as a getway between functions defined on $\mathbb{R}^3$ and $\mathbb{Z}^3$-periodic functions (identified as functions defined on the torus $\mathbb{T}^3$). In the sequel C denotes the open unit cube $(0,1)^3$. □

Lemma 9.7. *For any $g \in \mathcal{S}(\mathbb{R}^3)$ which is $\mathbb{Z}^3$-periodic there exists $h \in \mathscr{D}(C)$ such that $g = Ш \star h$. Furthermore, for any such function h, and for any $\in [1,\infty]$ there holds $\|g\|_{L^q(\mathbb{R}^3)} = \|h\|_{L^q(\mathbb{T}^3)}$.*

Proof. Fix a non-zero $\theta \in \mathscr{D}(C)$, then $h := g\theta/(Ш\star\theta)$ is a well-defined element of $\mathscr{D}(C)$ satisfying $g = Ш \star h = \sum_{n\in\mathbb{Z}^3}\tau_n h$. Since $h \in \mathscr{D}(C)$, the functions $\tau_n h$ have disjoint supports which justifies the equality of the L^q-norms. □

Obviously the previous lemma holds also when adding a time variable. In particular we have the existence of $\varphi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^3)$, such that $\varphi(0,\cdot) = 0$, $\varphi(t,\cdot) \in \mathscr{D}(C)$ for all t and $\psi = Ш \star \varphi$ (here the convolution is to be understood in the space variable only). From this representation formula we also deduce $\Delta\psi = Ш \star \Delta\varphi$ and $\partial_t\psi - \Delta\psi = Ш \star (\partial_t\varphi - \Delta\varphi)$, where the spatial support of $\Delta\varphi$ and $\partial_t\varphi - \Delta\varphi$ are still included in C : the previous lemma applies therefore to write

$$\begin{aligned}\|\Delta\psi\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{L}^q(\mathbb{T}^3))} &= \|\Delta\varphi\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{L}^q(\mathbb{R}^3))}\\ &\lesssim_{p,q} \|\partial_t\varphi - \Delta\varphi\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{L}^q(\mathbb{R}^3))} = \|\partial_t\psi - \Delta\psi\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{L}^q(\mathbb{T}^3))},\end{aligned}$$

where the inequality is obtained from (9.5). □

In the current article we will use the following consequence of Corollary 9.6, which is obtained by a standard approximation argument:

Corollary 9.8. *For $p, q \in (1,\infty)$ and $T > 0$ if $S \in L^p(0,T; L^q(\mathbb{T}^3))$, the unique tempered solution u of*

$$\partial_t u - \Delta u = S, \qquad u|_{t=0} = 0, \tag{9.1}$$

satisfies

$$\|\Delta u\|_{L^p(0,T;L^q(\mathbb{T}^3))} \lesssim_{p,q} \|S\|_{L^p(0,T;L^q(\mathbb{T}^3))}. \tag{9.2}$$

9.5. Interpolation

The following classical interpolation estimate can be for instance found in [9, Thm 1.5.2].

Theorem 9.9. (Gagliardo-Nirenberg-Sobolev) *Consider $1 \leqq p, q, r \leqq \infty$ and $m \in \mathbb{N}$. Assume that $j \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy*

$$\frac{1}{p} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1-\alpha}{q},$$
$$\frac{j}{m} \leqq \alpha \leqq 1,$$

with the exception $\alpha < 1$ *if* $m - j - 3/r \in \mathbb{N}$. *Then, the following holds. For any* $g \in L^q(\mathbb{T}^3)$, *if* $D^m g \in L^r(\mathbb{T}^3)$, *then* $D^j g \in L^p(\mathbb{T}^3)$ *and we have the following estimate for* g

$$\|D^j g\|_{L^p(\mathbb{T}^3)} \lesssim \|D^m g\|^{\alpha}_{L^r(\mathbb{T}^3)} \|g\|^{1-\alpha}_{L^q(\mathbb{T}^3)} + \|g\|_{L^q(\mathbb{T}^3)},$$

where the constant behind $\lesssim$ *does not depend on* g. *If* $\langle D^j g\rangle = 0$, *then the term* $\|g\|_{L^q(\mathbb{T}^3)}$ *in the right-hand side can be dispensed with.*

9.6. Parabolic Regularization for the Navier–Stokes Equations with a Source Term

The main result of this section is Proposition 9.10, which gives higher order energy estimates for the Navier-Stokes system together with a form of regularization along time. These estimates seem to be folklore but we give here the proof for the sake of completeness.

Proposition 9.10. *There exists a universal constant* $C_\star > 0$ *such that the following holds: consider* $u_0 \in H^{1/2}_{div}(\mathbb{T}^3)$, $F \in L^2_{loc}(\mathbb{R}_+; H^{-1/2}(\mathbb{T}^3))$ *and* $T > 0$ *such that*

$$\|u_0\|^2_{H^{1/2}(\mathbb{T}^3)} + C_\star \int_0^T \|F(s)\|^2_{H^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \leqq \frac{1}{C_\star^2}. \tag{9.3}$$

Then, there exists on $[0,T]$ *a unique Leray solution of the Navier-Stokes system with source* F *and with initial data* u_0. *This solution* u *belongs to* $L^\infty([0,T]; H^{1/2}(\mathbb{T}^3)) \cap L^2(0,T; H^{3/2}(\mathbb{T}^3))$ *and satisfies for almost everywhere* $0 \leqq t \leqq T$

$$\begin{aligned}\|u(t)\|^2_{H^{1/2}(\mathbb{T}^3)} &+ \int_0^t \|\nabla u(s)\|^2_{\dot H^{1/2}(\mathbb{T}^3)}\,\mathrm{d}s\\ &\leqq \|u_0\|^2_{H^{1/2}(\mathbb{T}^3)} + C_\star \int_0^t \|F(s)\|^2_{H^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s. \end{aligned} \tag{9.4}$$

Furthermore, if $F \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{T}^3))$, *we have, for almost everywhere* $1/2 \leqq t \leqq T$,

$$\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \int_{1/2}^t \|\Delta u(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s \lesssim A(t) + \int_0^t \|F(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s, \tag{9.5}$$

where $\lesssim$ *depends only on* $C_\star$, *and* A *is defined by*

$$A(t) := \frac{1}{2}\sup_{[0,t]} \|u(s)\|^2_{L^2(\mathbb{T}^3)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s. \tag{9.6}$$

Proof. The proof proceeds in two different steps. First, if such a Leray solution of the Navier-Stokes exists, because of the interpolation estimate

$$\|\cdot\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \leqq \|\cdot\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \|\cdot\|^2_{\dot{\mathrm{H}}^{3/2}(\mathbb{T}^3)},$$

we have in particular $u \in \mathrm{L}^4(0, T; \mathrm{H}^1(\mathbb{T}^3))$. This is a known case of weak-strong uniqueness, see for instance the stability result [8, Theorem 3.3]. The second step is to prove that such a solution indeed exists. This follows by a simple compactness argument, using Proposition 9.11 and Proposition 9.12 below, choosing for γ an appropriate regularization of $t \mapsto 2t\mathbf{1}_{0\leqq t\leqq 1/2} + \mathbf{1}_{t>1/2}$. □

In order to prove the existence of a solution as in Proposition 9.10, we rely on the following standard approximation procedure: we consider, for $\chi \in \mathscr{C}^\infty(\mathbb{T}^3)$, the regularized system

$$\partial_t u + (\widetilde{u}_\chi \cdot \nabla)u - \Delta u + \nabla p = F, \tag{9.7}$$

$$\operatorname{div} u = 0, \tag{9.8}$$

$$u(0, \cdot) = u_0, \tag{9.9}$$

where $\widetilde{u}_\chi := u \star \chi$. When u_0 and F are smooth, the existence of a unique smooth solution to system (9.7)–(9.9) is standard.

Proposition 9.11. *Consider a nondecreasing function $\gamma \in \mathscr{C}^1_b(\mathbb{R})$ vanishing at 0 and such that $\|\gamma\|_{W^{1,\infty}(\mathbb{R})} \leqq 1$. There exists $C > 0$ and an onto nondecreasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that for any $u_0 \in \mathscr{C}^\infty_{div}(\mathbb{T}^3)$, $F \in \mathscr{C}^\infty(\mathbb{R}_+ \times \mathbb{T}^3)$ and any $\chi \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $\|\chi\|_1 = 1$, the unique solution u of (9.7)–(9.9) satisfies for $t \geqq 0$,*

$$\gamma(t)\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \int_0^t \gamma(s)\|\Delta u(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s$$
$$\lesssim \left(A(t) + \int_0^t \gamma(s)\|F(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s\right)\Phi(h(t)), \tag{9.10}$$

where the constant behind $\lesssim$ is universal, $\Phi(z) := (1+z)e^z$, A is given by (9.6) and

$$h(t) := C\int_0^t \|\nabla u(s)\|^2_{L^3(\mathbb{T}^3)}\,\mathrm{d}s. \tag{9.11}$$

Proof. We multiply the equation by $-\gamma(t)\Delta u$, and use adequate integrations by parts together with Young's and Hölder's inequality, to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\right\} + \frac{\gamma(t)}{2}\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}$$
$$\leqq \frac{1}{2}\gamma'(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}$$
$$+\frac{\gamma(t)}{2}\|F(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \gamma(t)\|\Delta u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}\|u(t)\|_{\mathrm{L}^6(\mathbb{T}^3)}\|\nabla u(t)\|_{\mathrm{L}^3(\mathbb{T}^3)}. \tag{9.12}$$

We use then another time Young's inequality and the Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$ to write

$$\frac{d}{dt}\left\{\gamma(t)\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)}\right\} + \gamma(t)\|\Delta u(t)\|^2_{L^2(\mathbb{T}^3)} \\ \lesssim \gamma'(t)\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \gamma(t)\|F(t)\|^2_{L^2(\mathbb{T}^3)} + \gamma(t)\|u(t)\|^2_{H^1(\mathbb{T}^3)}\|\nabla u(t)\|^2_{L^3(\mathbb{T}^3)}.$$

Using the definition (9.6) of $A(t)$ and the fact that $\|\gamma\|_{W^{1,\infty}(\mathbb{R})} \leqq 1$, we infer, introducing $\ell(t) := \gamma(t)\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)}$

$$\ell'(t) + \gamma(t)\|\Delta u(t)\|^2_{L^2(\mathbb{T}^3)} \\ \lesssim \|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + A(t)\|\nabla u(t)\|^2_{L^3(\mathbb{T}^3)} + \gamma(t)\|F(t)\|^2_{L^2(\mathbb{T}^3)} + \ell(t)\|\nabla u(t)\|^2_{L^3(\mathbb{T}^3)},$$

which implies by Gronwall's inequality (since $\ell(0) = 0$), using once again the definition of $A(t)$,

$$\ell(t) + \int_0^t \gamma(s)\|\Delta u(s)\|^2_{L^2(\mathbb{T}^3)}\,ds \\ \lesssim \left(A(t)(1 + h(t)) + \int_0^t \gamma(s)\|F(s)\|^2_{L^2(\mathbb{T}^3)}\,ds\right)\exp(h(t)),$$

where h is given by (9.11), for some univeral constant $C > 0$; this last estimate can be recasted into (9.10). □

Recall the notation $\|\cdot\|_{\dot{H}^s(\mathbb{T}^3)}$ for the L^2 norm associated with the multiplier $|\xi|^s$.

Proposition 9.12. *There exists a universal constant $C_\star$ such that the following holds. For any $u_0 \in \mathscr{C}^\infty_{div}(\mathbb{T}^3)$, $F \in \mathscr{C}^\infty(\mathbb{R}_+ \times \mathbb{T}^3)$ and any $\chi \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $\|\chi\|_1 = 1$, if, for some $T > 0$, one has*

$$\|u_0\|^2_{H^{1/2}(\mathbb{T}^3)} + C_\star \int_0^T \|F(s)\|^2_{H^{-1/2}(\mathbb{T}^3)}\,ds \leqq \frac{1}{C_\star^2}, \tag{9.13}$$

then the unique solution u of (9.7)–(9.9) satisfies for $t \in [0, T]$,

$$\|u(t)\|^2_{H^{1/2}(\mathbb{T}^3)} + \int_0^t \|\nabla u(s)\|^2_{\dot{H}^{1/2}(\mathbb{T}^3)}\,ds \leqq \|u_0\|^2_{H^{1/2}(\mathbb{T}^3)} + C_\star \int_0^t \|F(s)\|^2_{H^{-1/2}(\mathbb{T}^3)}\,ds.$$

In particular, recalling the definition (9.11), on $[0, T]$ we have $h \leqq C/C_\star$ where C is the universal constant given in Proposition 9.11.

Proof. Let us first recall the fundamental energy estimate

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{T}^3)} + \|u\|^2_{\dot{H}^1(\mathbb{T}^3)} \lesssim \|F\|^2_{H^{-1}(\mathbb{T}^3)}. \tag{9.14}$$

Consider Λ the Fourier multiplier associated with $|\xi|$. After taking the scalar product with Λu, thanks to Plancherel's formula, Hölder's inequality, to the continuity of the Leray projector $\mathbb{P}$ on $L^{3/2}(\mathbb{T}^3)$, we can also obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} + \|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\Lambda u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|\nabla u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|u\|_{\mathrm{L}^3(\mathbb{T}^3)} + \|\Lambda u\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\|F\|_{\dot{\mathrm{H}}^{-1/2}(\mathbb{T}^3)}.$$

Using Young's inequality and combining with (9.14) we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}+\|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\Lambda u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|\nabla u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|u\|_{\mathrm{L}^3(\mathbb{T}^3)}+\|F\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}.$$

We therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}+\|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\Lambda u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|\nabla u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|u\|_{\mathrm{L}^3(\mathbb{T}^3)}+\|F\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}.$$

We have, by Sobolev embedding,

$$\|g - \langle g\rangle\|_{\mathrm{L}^3(\mathbb{T}^3)} \lesssim \|g\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)},$$
$$\|g\|_{\mathrm{L}^3(\mathbb{T}^3)} \lesssim \|g\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)}.$$

Since Λu and ∇u have a vanishing mean, we therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\|u\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \|F\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}.$$

This is a differential inequality of the form

$$x'(t) + y(t) \leqq \mathrm{C}\Big(x(t)^{1/2}y(t) + z(t)\Big),$$

where C is some universal constant and

$$x(t) = \|u(t)\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}, \quad y(t) = \|\nabla u(t)\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}, \quad z(t) = \|F(t)\|^2_{\dot{\mathrm{H}}^{-1/2}(\mathbb{T}^3)}. \tag{9.15}$$

After integration, we thus have

$$x(t) + \int_0^t y(s)\,\mathrm{d}s \leqq x(0) + \int_0^t y(s)\left(\mathrm{C}x(s)^{1/2} - 1\right)\mathrm{d}s + \mathrm{C}\int_0^t z(s)\,\mathrm{d}s.$$

In particular, if, for some $T > 0$, one has (this precisely corresponds to the assumption (9.3))

$$x(0) + \mathrm{C}\int_0^T z(s)\,\mathrm{d}s \leqq \frac{1}{\mathrm{C}^2}, \tag{9.16}$$

then by a standard continuity argument we can show that the inequality $x(t)^{1/2} \leqq 1/C$ which is true for $t = 0$ remains valid up to $t = T$, entailing on $[0, T]$,

$$x(t) + \int_0^t y(s)\,\mathrm{d}s \leqq x(0) + \mathrm{C}\int_0^t z(s)\,\mathrm{d}s, \tag{9.17}$$

which corresponds to the desired inequality, recalling (9.15). $\square$

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