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Global Stability of Large Solutions to the 3D Compressible Navier–Stokes Equations

Lingbing He, Jingchi Huang & Chao Wang

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Abstract

The present paper investigates the global stability of large solutions to the compressible Navier–Stokes equations in the whole space. Our main results and innovations can be stated as follows:

- Under the assumption that the density $\rho(t, x)$ verifies $\rho(0, x) \geq c > 0$ and $\sup_{t\geq 0} \|\rho(t)\|_{C^\alpha} \leq M$ with α arbitrarily small, we establish a new approach for the convergence of the solutions to its associated equilibrium with an explicit decay rate which is the same as that for the heat equation. The main idea of the proof relies on the basic energy identity, techniques from blow-up criterion and a new estimate for the low frequency part of the solutions.
- We prove the global-in-time stability for the equations, i.e, any perturbed solutions will remain close to the reference solutions if initially they are close to one another. This implies that the set of the smooth and bounded solutions is open.
- Inspired by Paicu and Zhang (J Funct Anal 262(8):3556–3584, 2012), we construct global large solutions to the equations with a class of initial data which are in L^p type critical spaces and far away from equilibrium. Here, "large solutions" means that the vertical component of the incompressible part of the velocity could be arbitrarily large.

1. Introduction

In this paper, we are concerned with the global stability of large solutions to 3-D barotropic compressible Navier–Stokes equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\big(2\mu(\rho) D(u) + \lambda(\rho) \operatorname{div} u \,\mathrm{Id}\big) + \nabla P = 0, \\ \lim_{|x|\to\infty} \rho = 1, \end{cases} \tag{CNS}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ stands for the density, $u = (u^1, u^2, u^3)(t,x) = (u^{hor}, u^3) \in \mathbb{R}^3$ is the velocity field and the pressure P is given by a C^2 function $P = P(\rho)$ with $P' > 0$. For simplicity, here we take $P(\rho) = \rho^\gamma$ with $\gamma \geqq 1$. The bulk and shear viscosities are given by $\lambda = \lambda(\rho)$ and $\mu = \mu(\rho)$, which satisfy $\mu > 0$ and $\lambda + 2\mu > 0$. We assume that μ and λ are two constants. Finally, $D(u)$ stands for the deformation tensor; that is, $(D(u))_{ij} := \frac{1}{2}(\partial_i u^j + \partial_j u^i)$.

1.1. Short Review of the System (CNS)

There is a lot of literature on the barotropic compressible Navier–Stokes equations. Here we only review some results which are related to our stability result.

1.1.1. Well-Posedness Results in Sobolev Spaces The local well-posedness for the system (CNS) was proved by NASH [31] for the smooth initial data which is away from vacuum. For the global smooth solutions, it was first proved by MATSUMURA and NISHIDA [29,30] if the initial data is close to equilibrium in $H^3 \times H^3$. For the small energy, ZHANG [40], and HUANG et al. [23] proved the global existence and uniqueness of (CNS). For the general initial data, if $\mu = const, \lambda(\rho) = b\rho^\beta$, the authors in [24,37] established the global existence and uniqueness of classical solutions for large initial data in dimension two.

1.1.2. Well-Posedness Results in Critical Spaces To catch the scaling invariance property of the system (CNS), Danchin first introduced in his series papers [10–13] the "Critical Spaces" which were inspired by the results for the incompressible Navier–Stokes. More precisely, he proved the local well-posedness of (CNS) in the critical Besov spaces $\dot{B}^{\frac{3}{p}}_{p,1} \times \dot{B}^{\frac{3}{p}-1}_{p,1}$ with $2 \leq p < 6$, and global well-posedness of (CNS) for the initial data close to a stable equilibrium in spaces $(\dot{B}^{\frac{1}{2}}_{2,1} \cap \dot{B}^{\frac{3}{2}}_{2,1}) \times \dot{B}^{\frac{1}{2}}_{2,1}$. In [7], Chen, Miao and Zhang introduced a new method (time-weighted Besov space) to establish the local well-posedness for (CNS) in critical spaces. We remark that when $p > 3$, the Besov space $\dot{B}^{\frac{3}{p}-1}_{p,1}$ contains the data which allows to have high oscillation. A typical example is

$$u_0(x) = \phi(x) \sin(\varepsilon^{-1} x \cdot \omega) n,$$

where ω and n stand for any unit vector in $\mathbb{R}^3$ and ϕ for any smooth compactly supported function. It is easy to check that $\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \sim \varepsilon^{1-\frac{3}{p}}$, but $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \sim \varepsilon^{-\frac{1}{2}}$ can be arbitrarily large. The authors of [5,8] constructed global solutions of (CNS) with such kind of the highly oscillating initial data. Later, HASPOT [20] gave an alternative proof to the similar result by using the viscous effective flux. Very recently, the authors of [14] generalized the previous results to allow for the incompressible part of the velocity to belong to $\dot{B}^{\frac{3}{p}-1}_{p,1}$, with $p \in [2,4]$. In this case, by taking the low mach number limit, the solutions for (CNS) will converge to the solutions for the incompressible Navier–Stokes equations constructed by Cannone et al. [3,4].

For the ill-posedness results, the authors of [9] proved (CNS) is ill-posed in the critical Besov spaces with $p > 6$. Finally, we mention some works on global solutions with large initial data to (CNS). For $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$, Haspot [21] constructed solutions which velocity has large rotational part. With assuming that the bulk viscosity is sufficiently large, Danchin and Mucha proved the existence of a global solution with any initial velocity and almost constantly density in [15]. In [19], the authors constructed the large solutions based on the dispersion property of acoustic waves.

1.1.3. Previous Results and the Main Motivation on Global Dynamics and the Stability of (CNS) To the best of our knowledge, all results on the global dynamics and the stability are restricted to the regime that the solutions are close to the equilibrium. These results heavily rely on the analysis of the linearization of the system and the standard perturbation framework. We refer readers to [17,25–30,33] and reference therein for details. More precisely, if we assume that the initial data (ρ_0, u_0) is a small perturbation of equilibrium $(\rho_\infty, 0)$ in $L^p(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ with $p \in [1, 2]$, then by the previous results, we get that

$$\|(\rho - \rho_\infty, u)(t)\|_{L^2} \leqq C(1+t)^{-\frac{3}{4}(\frac{2}{p}-1)}. \tag{1.1}$$

This discloses that under the close-to-equilibrium setting, the rate of the convergence of the solution is the same as that for the heat equations. In this sense, the decay rate in (1.1) is the optimal decay rate for system (CNS). Recently, Danchin and Xu [16] generalized this estimate in the critical L^p framework based on the existence result established in [14].

Our main motivation in the present work is to investigate the global dynamics when the initial data is far away from the equilibrium in the whole space and then apply it to prove the global-in-time stability. There are few results in this direction, but in bounded domains, there are two results describing the longtime behavior of the solution. The first one is due to Villani. By using the hypo-coercivity, in [38], he proved that if the solution remains smooth and bounded in the torus, then it will converge to its associated equilibrium with algebraic rate. The second result is due to Fang, Zhang and Zi. In [18], they showed that the weak solutions constructed by P.-L. Lions and improved by E. Feireisl decayed exponentially to equilibrium in L^2 space, if the density is bounded. The key idea of the proof is in using the Poincare inequality, the energy identity and the Bogovskii operator $\mathfrak{B}$ to get the integrability of ρ.

Unfortunately, the methods used in [18,38] cannot be employed to get the global-in-time stability in the whole space. We first note that to get the longtime behavior of the solution, both methods rely more or less on the fact that domain is bounded. For instance, the Poincare inequality and the L^p bound of Bogovskii operator $\mathfrak{B}$ are used which do not hold in the whole space. Secondly the propagation of the regularity is not considered in both papers which is essential to prove the global-in-time stability. This shows that we need some new idea to prove the desired result.

1.2. Main Idea and Strategy

The present work investigates the global-in-time stability of the large solutions to (CNS). The main difficulties lie in two areas: the propagation of the regularity and the mechanism for the convergence to the equilibrium.

Our strategy is carried out in three steps: getting uniform-in-time bounds for the propagation of the regularity, deriving a dissipation inequality, and using time-frequency splitting method to obtain a new approach for the convergence to the equilibrium with quantitative estimates.

To obtain uniform-in-time bounds for the solution, we borrow some techniques from the blow-up criterion of the system used in [23,24,35,36,39]. There the authors proved that the upper bound of the density will control the propagation of the regularity, but they did not get uniform-in-time bounds. Under the assumption that the density is bounded uniformly in time in C^α with α arbitrarily small, which is a little stronger than the assumption in [23,24,35,36,39], we succeed in proving the uniform-in-time bounds for the propagation of the regularity. We remark that here the key idea is making full use of the basic energy identity and the coupling effect of the system.

When the uniform-in-time bounds for the regularity of the solution are improved, the dissipation inequality can also be improved correspondingly. Thanks to this observation, eventually we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\rho, u) + D(\rho, u) \leq 0,$$

where, roughly speaking, $E(\rho, u) \sim \|\rho - 1\|^2 + \|u\|^2$ and $D(\rho, u) \sim \|\nabla\rho\|^2 + \|\nabla u\|^2$. Now the time-frequency splitting method is evoked if we can get the control for the low frequency part of $(\rho - 1, u)$. However it is difficult to derive the estimate unless the solution is near the equilibrium. To overcome this obstruction, our key idea is to resort to $(\rho - 1, \rho u)$ instead of $(\rho - 1, u)$ to obtain the estimate for the low frequency part thanks to the new observation of cancellation and the coupling effect for (CNS). One may check Lemma 2.3 for details. As a result, we obtain the optimal decay estimate. We comment that the method used here is comparable to the one due to SCHONBEK [34] for the incompressible Navier–Stokes equations. Moreover, the method is robust, considering that we only request that the density is bounded from above uniformly in time.

Once the global dynamics of the equations is clear, we can prove the global-in-time stability for the system (CNS). The strategy falls into three steps:

(1) By the local well-posedness for the system (CNS), we can show that the perturbed solution will remain close to the reference solution for a long time if initially they are close.
(2) The method for the convergence implies that the reference solution is close to the equilibrium after a long time.
(3) Combining these two facts, we can find a large time t_0 such that at this moment the solution is close to the equilibrium. Then it is not difficult to prove the global existence in the perturbation framework.

To show that our result on the global-in-time stability has wide application, we construct large solutions to the system (CNS) with the initial data in some L^p critical spaces. Our main idea is inspired by [32]. The main observation lies in two aspects. The first one is that the equation for $(\mathcal{P}u)^3$, the vertical component of the incompressible part of the velocity u, is actually a linear equation. The second one is that there is no quadratic term for $(\mathcal{P}u)^3$ in the nonlinear terms. Motivated by these two facts, we construct a global solution for compressible Navier–Stokes equations with the initial data such that $(\mathcal{P}u_0)^3$ could be arbitrarily large. Obviously such a type of solution, initially, is far away from the equilibrium.

1.3. Function Spaces and Main Results

Before we state our results, let us introduce the notations and function spaces which are used throughout the paper. We denote the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We use the notation $a \sim b$ whenever $a \le C_1 b$ and $b \le C_2 a$ where C_1 and C_2 are universal constants. We denote $C(\lambda_1, \lambda_2, \cdots, \lambda_n)$ by a constant depending on parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$. If $u = (u^1, u^2, u^3)$, then we set $u^{hor} \stackrel{\text{def}}{=} (u^1, u^2)$.

We recall that a homogeneous Littlewood–Paley decomposition $(\dot\Delta_j)_{j\in\mathbb{Z}}$ is a dyadic decomposition in the Fourier space for $\mathbb{R}^3$. One may, for instance, set $\dot\Delta_j := \varphi(2^{-j}D)$ with $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$, and χ a non-increasing nonnegative smooth function supported in $B(0, 4/3)$, and with value 1 on $B(0, 3/4)$ (see [1, Chap. 2] for more details). We then define, for $1 \leqq p, r \leqq \infty$ and $s \in \mathbb{R}$, the semi-norms

$$\|z\|_{\dot B^s_{p,r}} := \left\|2^{js}\|\dot\Delta_j z\|_{L^p(\mathbb{R}^d)}\right\|_{L^r(\mathbb{Z})}.$$

As in [1], we adopt the following definition of homogeneous Besov spaces, which turns out to be well adapted to the study of nonlinear PDEs:

$$\dot B^s_{p,r} = \Big\{z \in \mathcal{S}'(\mathbb{R}^d) \colon \|z\|_{\dot B^s_{p,r}} < \infty \ \text{ and } \lim_{j\to-\infty} \|\dot S_j z\|_{L^\infty} = 0\Big\} \quad \text{with } \dot S_j := \chi(2^{-j}D).$$

As we shall work with *time-dependent functions* valued in Besov spaces, we introduce the norm

$$\|u\|_{L^q_T(\dot B^s_{p,r})} := \left\| \|u(t,\cdot)\|_{\dot B^s_{p,r}} \right\|_{L^q(0,T)}.$$

As pointed out in [6], when using parabolic estimates in Besov spaces, it is somehow natural to take the time-Lebesgue norm *before* performing the summation for computing the Besov norm. This motivates us to introduce the following quantities:

$$\|u\|_{\widetilde L^q_T(\dot B^s_{p,r})} := \left\|(2^{js}\|\dot\Delta_j u\|_{L^q_T(L^p)})\right\|_{L^r(\mathbb{Z})}.$$

The index T will be omitted if $T = +\infty$ and we shall denote by $\widetilde{\mathcal{C}}_b(\dot B^s_{p,r})$ the subset of functions of $\widetilde L^\infty(\dot B^s_{p,r})$ which are also continuous from $\mathbb{R}_+$ to $\dot B^s_{p,r}$.

Let us emphasize that, owing to Minkowski inequality, we have, if $r \le q$,

$$\|z\|_{L^q_T(\dot{B}^s_{p,r})} \leqq \|z\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}$$

with equality if and only if $q = r$. Of course, the opposite inequality occurs if $r \geqq q$.

An important example where those nonclassical norms are suitable is the heat equation

$$\partial_t z - \mu \Delta z = f, \qquad z|_{t=0} = z_0, \tag{1.2}$$

for which the following family of inequalities holds true:

$$\|z\|_{\widetilde{L}^m_T(\dot{B}^{s+2/m}_{p,r})} \leqq C\big(\|z_0\|_{\dot{B}^s_{p,r}} + \|f\|_{\widetilde{L}^1_T(\dot{B}^s_{p,r})}\big) \tag{1.3}$$

for any $T > 0$, $1 \leqq m, p, r \leqq \infty$ and $s \in \mathbb{R}$.

Now we are in a position to state our main results on the system (CNS). Our first result is concerned with the global dynamics of the equation.

Theorem 1.1. *Let $\mu > \frac{1}{2}\lambda$, and (ρ, u) be a global and smooth solution of* (CNS) *with initial data (ρ_0, u_0) where $\rho_0 \geqq c > 0$. Suppose that the following admissible condition holds:*

$$u_t|_{t=0} = -u_0 \cdot \nabla u_0 + \frac{1}{\rho_0} L u_0 - \frac{1}{\rho_0} \nabla \rho_0^\gamma, \tag{1.4}$$

where operator L is defined by $Lu = -\mathrm{div}\,(\mu \nabla u) - \nabla((\lambda + \mu)\mathrm{div}\, u)$. Assume that $a \stackrel{def}{=} \rho - 1$, and $\sup_{t \ge 0} \|\rho(t)\|_{C^\alpha} \leqq M$ for some $0 < \alpha < 1$. Then, if $a_0, u_0 \in L^{p_0}(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ with $p_0 \in [1, 2]$, we have

(1) **(Lower bound of the density)** *There exists a positive constant $\underline{\rho} = \underline{\rho}(c, M)$ such that for all $t \ge 0$, $\rho(t) \ge \underline{\rho}$.*

(2) **(Uniform-in-time bounds for the regularity of the solution)**

$$\|a\|^2_{L^\infty_t H^2} + \|u\|^2_{L^\infty_t H^2} + \int_0^\infty (\|\nabla a\|^2_{H^1} + \|\nabla u\|^2_{H^2}) \mathrm{d}\tau \le C(\underline{\rho}, M, \|a_0\|_{H^2}, \|u_0\|_{H^2}). \tag{1.5}$$

(3) **(Decay estimate for the solution)**

$$\|u(t)\|_{H^1} + \|a(t)\|_{H^1} \leqq C(\underline{\rho}, M, \|a_0\|_{L^{p_0} \cap H^1}, \|u_0\|_{L^{p_0} \cap H^2})(1+t)^{-\beta(p_0)}, \tag{1.6}$$

where $\beta(p_0) = \frac{3}{4}(\frac{2}{p_0} - 1)$.

(4) **(Decay estimates in Critical spaces and the control of $\|\nabla u\|_{L^1(0,\infty; L^\infty)}$)** *For $p_0 < 2$,*

$$\|u(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|a(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leqq C(\underline{\rho}, M, \|a_0\|_{L^{p_0} \cap H^2}, \|u_0\|_{L^{p_0} \cap H^2})(1+t)^{-\beta(p_0)}, \tag{1.7}$$

$$\|a(t)\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leqq C(\underline{\rho}, M, \|a_0\|_{L^{p_0}\cap H^2}, \|u_0\|_{L^{p_0}\cap H^2})(1+t)^{-\frac{1}{2}\beta(p_0)}, \tag{1.8}$$

$$\int_0^\infty \|\nabla u\|_{L^\infty} \mathrm{d}t \leq C(\beta(p_0), \underline{\rho}, M, \|a_0\|_{L^{p_0}\cap H^2}, \|u_0\|_{L^{p_0}\cap H^2}). \tag{1.9}$$

Several remarks are in order.

Remark 1.1. We fail to quantify the dependence of $\underline{\rho}$ on c and M. The reason results from the fact that $\lim_{t\to\infty} \|\rho^\gamma(t) - 1\|_{L^6} = 0$ is used in the proof, which shows that the lower bound of ρ depends on evolution of the solution. Once the constants $\underline{\rho}$ and M are fixed, our theorem shows that the global dynamics of the solution of (CNS) depends only on the initial data.

Remark 1.2. Our decay estimate (1.6) is optimal compared to the heat equation. The control of $\|\nabla u\|_{L^1((0,\infty);L^\infty)}$ yields the uniform-in-time propagation of the higher regularity(see Corollary 1.1).

Remark 1.3. The condition $\mu > \frac{1}{2}\lambda$ is to used to get the positivity of the left hand side of (2.3) (see Lemma 2.2 below). Of course, we can relax this condition, for instance if we follow the argument in [36,39]. However we do not want to pursue that because our starting point is to describe the global dynamics of the equations when the initial data is far away from the equilibrium.

Remark 1.4. We mention that the C^α assumption for ρ can be relaxed by

$$\sup_{t\geq 0} \|\rho(t)\|_{L^\infty} < \infty \quad \text{and} \quad \lim_{N\to\infty} \sup_{t\geq 0} \sum_{j\geqq N} \|\dot{\Delta}_j(\rho^\gamma - 1)\|_{L^\infty} = 0. \tag{1.10}$$

To see this, the reader may check Remark 2.3 for details. Since the second assumption of (1.10) is not so easy to verify, we prefer to use the C^α assumption instead of (1.10).

As a direct consequence of Theorem 1.1, we prove the uniform-in-time propagation of the regularity under the assumption that the Lipschitz norm of the velocity is integrable in time. That is, we have

Corollary 1.1. *Suppose that the initial data (ρ_0, u_0) satisfies that $0 < c < \rho_0 < M < \infty$ and $\rho_0 - 1, u_0 \in H^m$ with $m \in \mathbb{N}$ and $m \geqq 3$. If the solution (ρ, u) of (CNS) verifies that $\|\nabla u\|_{L^1(0,\infty;L^\infty)} < \infty$, then we have $\inf_{t\geq 0} \rho(t) \geq C(c, \|\nabla u\|_{L^1(0,\infty;L^\infty)})$ and*

$$\sup_{t\geq 0} \big(\|\rho - 1\|_{H^m} + \|u\|_{H^m}\big) \leq C(c, M, \|\rho_0 - 1\|_{H^m}, \|u_0\|_{H^m}, \|\nabla u\|_{L^1(0,\infty;L^\infty)}).$$

Next we want to state our global-in-time stability result for the system (CNS). To prove the result in the largest function space, we solve the problem in critical spaces. From now on, we agree that for $z \in \mathcal{S}'(\mathbb{R}^d)$,

$$z^L := \sum_{2^j \leqq R_0} \dot{\Delta}_j z \quad \text{and} \quad z^H := \sum_{2^j > R_0} \dot{\Delta}_j z \tag{1.11}$$

for some large enough nonnegative integer R_0 depending only on p, d. The corresponding "truncated" semi-norms are defined as follows:

$$\|z\|^L_{\dot{B}^\sigma_{p,r}} := \|z^L\|_{\dot{B}^\sigma_{p,r}} \text{ and } \|z\|^H_{\dot{B}^\sigma_{p,r}} := \|z^H\|_{\dot{B}^\sigma_{p,r}}.$$

To simplify the notation, we introduce the notation

$$\|u\|_{\dot{B}^{s,t}_{r,p}} \overset{\text{def}}{=} \|u\|^L_{\dot{B}^s_{r,1}} + \|u\|^H_{\dot{B}^t_{p,1}}.$$

Our second result can be stated as follows:

Theorem 1.2. *Let* $(\bar\rho, \bar u)$ *be a global and smooth solution for the* (CNS) *with the initial data* $(\bar\rho_0, \bar u_0)$ *verifying that*

$$\|\frac{1}{\bar\rho}, \bar\rho, \nabla\bar\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac12,\frac3p}_{2,p})} + \|\bar u, \bar u_t\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac12,\frac3p-1}_{2,p})} + \|\bar u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac52,\frac3p+1}_{2,p})} \leqq C, \quad (1.12)$$

where $2 \leqq p \leqq 4$. *Assume that* $(\bar\rho_0 - 1, \bar u_0) \in L^{p_0}(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ *with* $p_0 \in [1, 2)$. *There exists a* $\varepsilon_0 = \varepsilon_0(C)$ *depending only on* C *such that for any* $0 < \varepsilon \leqq \varepsilon_0$, *if*

$$\|(\rho_0 - \bar\rho_0)(t)\|_{\dot{B}^{\frac12,\frac3p}_{2,p}} + \|\mathcal{P}(u_0 - \bar u_0)(t)\|_{\dot{B}^{\frac3p-1}_{p,1}} + \|\mathcal{Q}(u_0 - \bar u_0)(t)\|_{\dot{B}^{\frac12,\frac3p-1}_{2,p}} \leqq \varepsilon, \quad (1.13)$$

then (CNS) *admits a unique and global solution* (ρ, u) *with the initial data* (ρ_0, u_0). *Moreover, for any* $t > 0$,

$$\begin{aligned}&\|(\rho - \bar\rho)(t)\|_{\dot{B}^{\frac12,\frac3p}_{2,p}} + \|\mathcal{P}(u - \bar u)(t)\|_{\dot{B}^{\frac3p-1}_{p,1}} + \|\mathcal{Q}(u - \bar u)(t)\|_{\dot{B}^{\frac12,\frac3p-1}_{2,p}}\\ &\quad\lesssim \min\{(1 + \delta|\ln\varepsilon|)^{-\beta(p_0)/2}, (1 + t)^{-\beta(p_0)/2} + \varepsilon\},\end{aligned}$$

where δ *is a constant independent of* ε, $\beta(p_0)$ *is defined in Theorem* 1.1, $\mathcal{P} = I + \nabla(-\Delta)^{-1}\mathrm{div}$ *and* $\mathcal{Q} = -\nabla(-\Delta)^{-1}\mathrm{div}$.

Remark 1.5. Solutions constructed in [5,8,10,14,20] verify that $\|\nabla u\|_{L^1((0,\infty);L^\infty)} < \infty$. Then by Corollary 1.1, (1.12) is satisfied if initially solutions are in H^3.

Finally we want to construct some solutions to the system (CNS) which initially are far away from the equilibrium. Motivated by [22,32], we consider the case that the incompressible part of the initial velocity $\mathcal{P}u_0$ has arbitrarily large vertical component while the other parts of the velocity are sufficiently small. The reason we can deal with this case is that the equation of $(\mathcal{P}u)^3$ becomes linear due to the divergence free condition. So we will deal with the initial data (a_0, u_0) verifying that

$$a_0 \in \dot{B}^{\frac12,\frac3p}_{2,p}, \quad \mathcal{Q}u_0 \in \dot{B}^{\frac12,\frac3p-1}_{2,p}, \quad \mathcal{P}u_0 \in \dot{B}^{\frac3p-1}_{p,1}. \quad (1.14)$$

Theorem 1.3. *Let* $2 \leqq p \leqq 4$. *If* (a_0, u_0) *satisfies that*

$$\Big(\|(a_0, \mathcal{Q}u_0)\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\mathcal{Q}u_0\|^H_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|a_0\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|(\mathcal{P}u_0)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big) \exp\big(C(1 + \|(\mathcal{P}u_0)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}})\big) \leqq \varepsilon, \tag{1.15}$$

where C *is a universal constant, then the system* (CNS) *with initial data* (a_0, u_0) *admits a unique and global solution* (a, u) *satisfying that for all* $t > 0$,

$$\begin{aligned}&\|a(t)\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}} + \|\mathcal{Q}u(t)\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\\ &\quad + \int_0^\infty \Big(\|a\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}}_{2,p}} + \|\mathcal{Q}u\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\Big)\mathrm{d}t\\ &\quad \leqq C\Big(1 + \|(\mathcal{P}u_0)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big)\varepsilon,\end{aligned} \tag{1.16}$$

and

$$\|(\mathcal{P}u)^3(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \int_0^\infty \|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\mathrm{d}t \leqq 2\|(\mathcal{P}u_0)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \varepsilon. \tag{1.17}$$

Remark 1.6. Estimates (1.16) and (1.17) show that the Lipschitz norm of the velocity u is integrable in time. By Corollary 1.1, if additional $\nabla a_0 \in \dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}$ and $u_t|_{t=0} \in \dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}$ then the solution satisfies $\sup_{t\geq 0}\|\rho\|_{C^\alpha} \leqq M$ and the condition (1.12).

1.4. Organization of the Paper

We first give a rigorous proof to Theorem 1.1, the global dynamics of (CNS) in Section 2. Then we will prove the global-in-time stability for (CNS), in Section 3. In the next section, Section 4, we will construct global large solutions with a class of initial data in L^p critical spaces. Last, we list some basic knowledge on Littlewood-Paley decomposition in the "Appendix".

2. Global Dynamics of the Compressible Navier–Stokes Equations

In this section, we give prove Theorem 1.1. To do this, we separate the proof into two steps: getting uniform-in-time bounds and the dissipation inequality first and then applying the time-frequency splitting method to obtain the convergence to the equilibrium with quantitative estimates. We emphasize that throughout this section $\overline{\rho}$ denotes the upper bound of the density ρ.

2.1. Uniform-in-Time Bounds and the Dissipation Inequality

In what follows, we set $a \overset{\text{def}}{=} \rho - 1$ and $\mathfrak{a} \overset{\text{def}}{=} \rho^\gamma - 1$. Observe that $\mathfrak{a} = \big(\int_0^1 \gamma(\theta\rho + (1-\theta))^{\gamma-1}\mathrm{d}\theta\big)a$. Thus if $\rho \le \overline{\rho}$,

$$|\mathfrak{a}| \sim |a|. \tag{2.1}$$

We begin with two lemmas. We first recall the basic energy identity for (CNS).

Lemma 2.1. *Let* (ρ, u) *be a global and smooth solution of* (CNS). *Then the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int H(\rho|1)\,\mathrm{d}x + \frac{1}{2}\int \rho u^2\,\mathrm{d}x\Big) + \mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\mathrm{div}\,u\|_{L^2}^2 = 0, \tag{2.2}$$

where

$$H(\rho|1) = \begin{cases} \dfrac{1}{\gamma-1}(\rho^\gamma - 1 - \gamma(\rho-1)), & \textit{when} \quad \gamma > 1, \\ \rho\ln\rho - \rho + 1, & \textit{when} \quad \gamma = 1. \end{cases}$$

Remark 2.1. By Taylor expansion, it is not difficult to check that $H(\rho|1) \geqq C(\overline{\rho})(\rho-1)^2$ if $\rho \leqq \overline{\rho}$.

Lemma 2.2. *Let* $\mu > \frac{1}{2}\lambda$, *and* (ρ, u) *be a global solution of* (CNS) *with* $0 \leqq \rho \leqq \bar{\rho}$. *Then the following inequality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho u^4\,\mathrm{d}x + \int |u|^2|\nabla u|^2\,\mathrm{d}x \leqq C\|\nabla u\|_{L^2}^2(\|\mathfrak{a}\|_{L^6}^2 + \|\nabla u\|_{L^2}^2), \tag{2.3}$$

where C *is a positive constant depending on* μ *and* λ.

Proof. Multiplying $4|u|^2u$ to the second equation of (CNS), and then integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int \rho u^4\,\mathrm{d}x + \int \Big[4|u|^2\Big(\mu|\nabla u|^2 \\ &\qquad + (\lambda+\mu)(\mathrm{div}\,u)^2 + 2\mu\big|\nabla|u|\big|^2\Big) + 4(\lambda+\mu)(\nabla|u|^2)\cdot u\,\mathrm{div}\,u\Big]\mathrm{d}x \\ &\quad = 4\int \mathrm{div}\,(|u|^2u)\mathfrak{a}\,\mathrm{d}x \leqq C\int \mathfrak{a}|u|^2|\nabla u|\,\mathrm{d}x \leqq C\|\mathfrak{a}\|_{L^6}\|u^2\|_{L^3}\|\nabla u\|_{L^2} \\ &\quad \leqq C\|\mathfrak{a}\|_{L^6}^2\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4. \end{aligned}$$

Using the inequality $\big|\nabla|u|\big| \leqq |\nabla u|$, we have

$$\begin{aligned} &4|u|^2\Big(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}\,u)^2 + 2\mu\big|\nabla|u|\big|^2\Big) + 4(\lambda+\mu)(\nabla|u|^2)\cdot u\,\mathrm{div}\,u \\ &\quad \geqq 4|u|^2\Big[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}\,u)^2 + 2\mu\big|\nabla|u|\big|^2 - 2(\lambda+\mu)\big|\nabla|u|\big||\mathrm{div}\,u|\Big] \\ &\quad = 4|u|^2\Big[\mu|\nabla u|^2 + (\lambda+\mu)\Big(\mathrm{div}\,u - \big|\nabla|u|\big|\Big)^2\Big] + 4|u|^2(\mu-\lambda)\big|\nabla|u|\big|^2 \\ &\quad \geqq C|u|^2|\nabla u|^2, \end{aligned}$$

where in the last step we use $\mu > \frac{1}{2}\lambda$. Combining these two estimates, we arrive at (2.3). □

2.1.1. The First Attempt for Uniform Bounds and the Dissipation Inequality

First, we prove

Proposition 2.1. *Let $\mu > \frac{1}{2}\lambda$ and (ρ, u) be a global smooth solution of* (CNS) *with $0 \leqq \rho \leqq \bar{\rho}$. Then $u \in L^\infty((0,+\infty); L^4 \cap H^1) \cap L^2((0,+\infty); \dot{H}^1 \cap \dot{H}^2)$, $\mathfrak{a} \in L^\infty((0,+\infty); H^1) \cap L^2((0,+\infty); L^6)$, and $u \cdot \nabla u \in L^2((0,+\infty); L^2)$. Furthermore, the following inequality holds:*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[A_1\|\rho^{\frac{1}{4}}u\|_{L^4}^4 + A_2\Big(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2 - (\mathfrak{a}, \operatorname{div} u)\\
&\quad + \int f(\rho)\mathrm{d}x\Big) + A_3\|\mathfrak{a}\|_{L^6}^2\\
&\quad + A_4\left(\int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2\right)\Big]\\
&\quad + A_5\Big(\||u||\nabla u|\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2\\
&\quad + \|\Delta\mathcal{P}u\|_{L^2}^2 + \|\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\|_{\dot{H}^1}^2\\
&\quad + \|\mathfrak{a}\|_{L^6}^2 + \|\nabla u\|_{L^6}^2\Big) \leqq 0, \qquad (2.4)
\end{aligned}$$

where $A_i (i = 1, \ldots, 5)$ are positive constants depending on μ, λ, and $\bar{\rho}$, and $f(\rho)$ is defined in (2.6) *verifying $|f(\rho)| \lesssim H(\rho|1)$.*

Remark 2.2. Thanks to the energy identity and the fact that $\int f(\rho)\mathrm{d}x \lesssim \int |\rho - 1|^2\,\mathrm{d}x$, choose A_4 large enough and then we can derive that

$$\begin{aligned}
&A_1\|\rho^{\frac{1}{4}}u\|_{L^4}^4 + A_2\Big(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2 - (\mathfrak{a}, \operatorname{div} u)\\
&\quad + \int f(\rho)\mathrm{d}x\Big) + A_3\|\mathfrak{a}\|_{L^6}^2\\
&\quad + A_4\left(\int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2\right) \sim \|\rho^{\frac{1}{4}}u\|_{L^4}^4\\
&\quad + \|\nabla u\|_{L^2}^2 + \int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2 + \|\mathfrak{a}\|_{L^6}^2.
\end{aligned}$$

Proof. To derive the desired results, we split the proof into several steps.

Step 1: Estimate of ∇u. First, multiplying the second equation of (CNS) with u_t and taking the inner product, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mu\|\nabla u\|_{L^2}^2 + \frac{1}{2}(\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2\right) + (\rho u_t, u_t) = -(\nabla\rho^\gamma, u_t) - (\rho u\cdot\nabla u, u_t). \qquad (2.5)$$

Estimate of $-(\nabla\rho^\gamma, u_t)$. Observe that

$$\begin{aligned}-(\nabla\rho^\gamma, u_t) &= -\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\rho^\gamma, u) + (\partial_t(\rho^\gamma), -\operatorname{div} u) = -\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\rho^\gamma, u)\\ &\quad + (\gamma\rho^\gamma \operatorname{div} u + u\cdot\nabla\rho^\gamma, \operatorname{div} u)\\ &\leqq -\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\mathfrak{a}, u) + C\|\rho\|_{L^\infty}^\gamma \|\nabla u\|_{L^2}^2\\ &\quad + (u\cdot\nabla\mathfrak{a}, \operatorname{div} u).\end{aligned}$$

Let us focus on the last term $(u\cdot\nabla\mathfrak{a}, \operatorname{div} u)$. In fact, one can check that

$$\frac{1}{\lambda+2\mu}(u\mathfrak{a}, \nabla\mathfrak{a}) = \partial_t \int f(\rho)\,\mathrm{d}x,$$

where

$$f(\rho) = \begin{cases} \dfrac{1}{\lambda+2\mu}\Big[\dfrac{\gamma^2}{2(2\gamma-1)}(\rho-1)^2 - \big(\dfrac{\gamma-1}{2(2\gamma-1)}\rho^\gamma + \dfrac{\gamma(\gamma-1)}{2(2\gamma-1)}\rho \\ \quad -\frac{\gamma^2+2\gamma-1}{2(2\gamma-1)}\big)H(\rho|1)\Big], \quad \text{when } \gamma > 1, \\ \dfrac{1}{\lambda+2\mu}\Big[\dfrac{1}{2}(\rho-1)^2 - \big(\rho\ln\rho - \rho + 1\big)\Big], \quad \text{when } \gamma = 1. \end{cases} \tag{2.6}$$

Thanks to Remark 2.1 and $\rho \le \bar{\rho}$, we get that $|f(\rho)| \lesssim H(\rho|1)$.

Going back to the estimate of $-(\nabla\rho^\gamma, u_t)$, we have

$$\begin{aligned}(u\cdot\nabla\mathfrak{a}, \operatorname{div} u) &= -(\mathfrak{a}\operatorname{div} u, \operatorname{div} u) - \Big(\mathfrak{a}u, \nabla(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a})\Big) - \frac{1}{\lambda+2\mu}(u\mathfrak{a}, \nabla\mathfrak{a})\\ &\le C\|\mathfrak{a}\|_{L^\infty}\|\operatorname{div} u\|_{L^2}^2 + \|\mathfrak{a}\|_{L^\infty}^{\frac{1}{3}}\|\mathfrak{a}\|_{L^2}^{\frac{2}{3}}\|\nabla u\|_{L^2}\Big\|\nabla\Big(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\Big)\Big\|_{L^2} - \partial_t\int f(\rho)\,\mathrm{d}x,\end{aligned}$$

which yields

$$\begin{aligned}-(\nabla\mathfrak{a}, u_t) &\leqq -\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\mathfrak{a}, u) - \partial_t\int f(\rho)\,\mathrm{d}x + C\|\rho\|_{L^\infty}^\gamma\|\nabla u\|_{L^2}^2 + C\|\mathfrak{a}\|_{L^\infty}\|\operatorname{div} u\|_{L^2}^2\\ &\quad + C\|\nabla u\|_{L^2}\|\nabla(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a})\|_{L^2}.\end{aligned}$$

Estimate of $(\rho u\cdot\nabla u, u_t)$. We have

$$|(\rho u\cdot\nabla u, u_t)| \leqq \|\rho^{\frac{1}{2}}\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}\|\rho^{\frac{1}{2}}u_t\|_{L^2}.$$

Plugging these two estimates into (2.5), we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\mu\|\nabla u\|_{L^2}^2 + \frac{1}{2}(\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2 - (\mathfrak{a}, \operatorname{div} u) - \int f(\rho)\,\mathrm{d}x\Big)\\ &\quad + \|\rho^{\frac{1}{2}}u_t\|_{L^2}^2\\ &\leqq C\|\nabla u\|_{L^2}^2 + C\eta\|\nabla(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a})\|_{L^2}^2 + C\|u\cdot\nabla u\|_{L^2}^2,\end{aligned} \tag{2.7}$$

where η is a small constant, and the constant C depends on the initial data and $\overline{\rho}$.

Step 2: Improving estimate by the elliptic system. The second equation of (CNS) can be rewritten as

$$-\mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u + \nabla \mathfrak{a} = -\rho(u_t + u\cdot\nabla u).$$

Set $b = \mathcal{P}u = (I + \nabla(-\Delta)^{-1}\mathrm{div})u$, $d = \Lambda^{-1}\mathrm{div}\, u$, where Λ is a Fourier multiplier, which satisfies $\Lambda^2 = -\Delta$. Then the above equation turns to be

$$\begin{cases} -\mu\Delta b = \mathcal{P}\big(\rho(u_t + u\cdot\nabla u)\big), \\ -(\lambda+2\mu)\Delta d - \Lambda\mathfrak{a} = \Lambda^{-1}\mathrm{div}\,\big(\rho(u_t + u\cdot\nabla u)\big). \end{cases} \tag{2.8}$$

By the standard elliptic estimate, we have

$$\|\mu\Delta b\|_{L^2}^2 + \|(\lambda+2\mu)\Lambda d - \mathfrak{a}\|_{\dot{H}^1}^2 \leqq (1+\overline{\rho})^2(\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2 + \|u\cdot\nabla u\|_{L^2}^2). \tag{2.9}$$

Combining (2.7) and (2.9), we get that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\mathrm{div}\, u\|_{L^2}^2 \\ &\quad + (\nabla\mathfrak{a}, u) + \int f(\rho)\,\mathrm{d}x\Big) \\ &\quad + \big(\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2 + \|\mu\Delta b\|_{L^2}^2 + \|(\lambda+2\mu)\Lambda d - \mathfrak{a}\|_{\dot{H}^1}^2\big) \\ &\leqq C\|\nabla u\|_{L^2}^2 + C\|u\cdot\nabla u\|_{L^2}^2, \end{aligned} \tag{2.10}$$

where C is a positive constant depending on $\overline{\rho}$ and the initial data.

Step 3: Estimate of $\mathfrak{a}$. The first equation of (CNS) can be rewritten as

$$\frac{1}{\gamma}(\mathfrak{a}_t + u\cdot\nabla\mathfrak{a}) + \frac{1}{\lambda+2\mu}\mathfrak{a} + \mathfrak{a}\mathrm{div}\, u = -\left(\Lambda d - \frac{1}{\lambda+2\mu}\mathfrak{a}\right). \tag{2.11}$$

Then making the inner product to the above equation with $|\mathfrak{a}|^4\mathfrak{a}$, we obtain that

$$\begin{aligned} &\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{a}\|_{L^6}^6 + \frac{\gamma}{\lambda+2\mu}\|\mathfrak{a}\|_{L^6}^6 + \left(\gamma - \frac{1}{6}\right)\int \mathrm{div}\, u|\mathfrak{a}|^6\,\mathrm{d}x \\ &\leqq \gamma\|\left(\Lambda d - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^6}\|\mathfrak{a}^5\|_{L^{\frac{6}{5}}}, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{a}\|_{L^6}^6 + \frac{1}{\lambda+2\mu}\int\left[\gamma + \left(\gamma - \frac{1}{6}\right)\mathfrak{a}\right]\mathfrak{a}^6\,\mathrm{d}x \leqq C\|\left(\Lambda d - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^6}\|\mathfrak{a}^5\|_{L^{\frac{6}{5}}} \\ &\leqq C\|\left(\Lambda d - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^6}\|\mathfrak{a}\|_{L^6}^5. \end{aligned}$$

Dividing the above estimate by $\|\mathfrak{a}\|_{L^6}^4$, and recalling $\gamma + (\gamma - \frac{1}{6})\mathfrak{a} \geqq \frac{1}{6}$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{a}\|_{L^6}^2 + \|\mathfrak{a}\|_{L^6}^2 \leqq C\|\nabla\left(\Lambda d - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^2}^2. \tag{2.12}$$

Step 4: Closing the energy estimates. Combining (2.2), (2.3), (2.10) and (2.12), and choosing η small enough, we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[&A_1\|\rho^{\frac14}u\|_{L^4}^4+A_2\Big(\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\operatorname{div}u\|_{L^2}^2-(\mathfrak{a},\operatorname{div}u)\\&+\int f(\rho)\mathrm{d}x\Big)+A_3\|\mathfrak{a}\|_{L^6}^2\\&+A_4\Big(\int H(\rho|1)\,\mathrm{d}x+\|\sqrt{\rho}u\|_{L^2}^2\Big)\Big]\\&+A_5\Big(\||u||\nabla u|\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2\\&+\|\Delta b\|_{L^2}^2+\|\Lambda d-\frac{1}{\lambda+2\mu}\mathfrak{a}\|_{\dot H^1}^2+\|\mathfrak{a}\|_{L^6}^2\Big)\\&\leqq A_6(\|\mathfrak{a}\|_{L^6}^2+\|\nabla u\|_{L^2}^2)\|\nabla u\|_{L^2}^2,\end{aligned}\tag{2.13}$$

where $A_i(i=1,\dots,6)$ are positive constants depending on λ, μ and $\overline{\rho}$, and which ensure that the term $A_2(\mathfrak{a},\operatorname{div}u)$ can be controlled by $A_2(\lambda+\mu)\|\operatorname{div}u\|_{L^2}^2$ and $A_4\int H(\rho|1)\,\mathrm{d}x$. By Gronwall's inequality, the above estimate implies that $u\in L^\infty((0,+\infty);L^4\cap H^1)\cap L^2((0,+\infty);\dot H^1)$, $u_t\in L^2((0,+\infty);L^2)$, $\mathfrak{a}\in L^\infty((0,+\infty);L^2)\cap L^2((0,+\infty);L^6)$, and $u\cdot\nabla u\in L^2((0,+\infty);L^2)$.

Using these estimates, we can improve the estimate (2.13). Notice that the term in the righthand side of (2.13) can be bounded by $C\|\nabla u\|_{L^2}^2$. Then thanks to the energy identity (2.2), the dissipation inequality in the proposition is followed by the fact that for $i\geq 1$ and $p=2,6$,

$$\begin{aligned}\|\nabla^i u\|_{L^p}&\leq\|\nabla^i\mathcal{P}u\|_{L^p}+\|\nabla^i\mathcal{Q}u\|_{L^p}\leq\|\nabla^i\mathcal{P}u\|_{L^p}+\|\nabla^{i-1}\operatorname{div}u\|_{L^p}\\&\leq\|\nabla^i\mathcal{P}u\|_{L^p}+\|\nabla^{i-1}\Big(\operatorname{div}u\\&\qquad-\frac{1}{\lambda+2\mu}\mathfrak{a}\Big)\|_{L^p}+\frac{1}{\lambda+2\mu}\|\nabla^{i-1}\mathfrak{a}\|_{L^p},\end{aligned}\tag{2.14}$$

$$\begin{aligned}\|\nabla^i u\|_{L^6}&\leq\|\nabla^{i+1}\mathcal{P}u\|_{L^2}+\|\nabla^i\Big(\operatorname{div}u-\frac{1}{\lambda+2\mu}\mathfrak{a}\Big)\|_{L^2}\\&\qquad+\frac{1}{\lambda+2\mu}\|\nabla^{i-1}\mathfrak{a}\|_{L^6}.\end{aligned}\tag{2.15}$$

□

2.1.2. Improving Regularity Estimate for u In order to get the dissipation estimate for a, we first improve the regularity estimate for u in this subsection. We still assume that (ρ,u) is a global and smooth solution of (CNS). We set up some notations first. For a function or a vector field(or even a 3×3 matrix) $f(t,x)$, the material derivative $\dot f$ is defined by

$$\dot f=f_t+u\cdot\nabla f,$$

and $\operatorname{div}(f\otimes u)=\sum_{j=1}^3\partial_j(fu_j)$. For two matrices $A=(a_{ij})_{3\times3}$ and $B=(b_{ij})_{3\times3}$, we use the notation $A:B=\sum_{i,j=1}^3a_{ij}b_{ij}$ and AB is the usual multiplication of matrix.

Proposition 2.2. *Let $\mu > \frac{1}{2}\lambda$ and (ρ, u) be a global and smooth solution of* (CNS) *satisfying* $0 \leqq \rho \leqq \overline{\rho}$ *and the admissible condition* (1.4). *Then there exist constants* $A_i (i = 1, \dots, 6)$ *such that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[A_1\|\rho^{\frac14}u\|_{L^4}^4 + A_2\Big(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\mathrm{div}\, u\|_{L^2}^2 \\
&\quad - (\mathfrak{a}, \mathrm{div}\, u) + \int_{\mathbb{R}^6} f(\rho)\mathrm{d}x\Big) + A_3\|\mathfrak{a}\|_{L^6}^2 \\
&\quad + A_4\left(\int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2\right) \\
&\quad + A_5\|\sqrt{\rho}\dot{u}\|_{L^2}^2\Big] + A_6\Big(\||u||\nabla u|\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\quad\ + \|\Delta\mathcal{P}u\|_{L^2}^2 + \|\mathrm{div}\, u - \frac{1}{\lambda+2\mu}\mathfrak{a}\|_{\dot{H}^1}^2 + \|\mathfrak{a}\|_{L^6}^2 + \|\nabla u\|_{L^6}^2 + \|\nabla\dot{u}\|_{L^2}^2 \\
&\quad\ + \|\mathrm{div}\, u - \frac{1}{2\mu+\lambda}\mathfrak{a}\|_{W^{1,6}}^2 \\
&\quad\ + \|\nabla\mathcal{P}u\|_{W^{1,6}}^2\Big) \leqq 0,
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
&A_1\|\rho^{\frac14}u\|_{L^4}^4 + A_2\Big(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\mathrm{div}\, u\|_{L^2}^2 - (\mathfrak{a}, \mathrm{div}\, u) \\
&\quad + \int_{\mathbb{R}^6} f(\rho)\mathrm{d}x\Big) + A_3\|\mathfrak{a}\|_{L^6}^2 \\
&\quad + A_4\left(\int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2\right) + A_5\|\sqrt{\rho}\dot{u}\|_{L^2}^2 \\
&\sim \|\rho^{\frac14}u\|_{L^4}^4 + \|\nabla u\|_{L^2}^2 + \int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2 + \|\mathfrak{a}\|_{L^6}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2.
\end{aligned}$$

Proof. We rewrite the second equation of (CNS) as

$$\rho\dot{u} + \nabla\mathfrak{a} + Lu = 0.$$

Then it is not difficult to check that

$$\begin{aligned}
&\rho\dot{u}_t + \rho u\cdot\nabla\dot{u} + \nabla\mathfrak{a}_t + \mathrm{div}\,(\nabla\mathfrak{a}\otimes u) \\
&\quad = \mu\big[\Delta u_t + \mathrm{div}\,(\Delta u\otimes u)\big] + (\lambda+\mu)\big[\nabla\mathrm{div}\, u_t + \mathrm{div}\,((\nabla\mathrm{div}\, u)\otimes u)\big].
\end{aligned} \tag{2.17}$$

By the energy estimate, we derive that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \frac12\rho|\dot{u}|^2\,\mathrm{d}x + \underbrace{-\mu\int \dot{u}\cdot\big(\Delta u_t + \mathrm{div}\,(\Delta u\otimes u)\big)\,\mathrm{d}x}_{\stackrel{\mathrm{def}}{=} I} \\
&\quad - (\lambda+\mu)\underbrace{\int \dot{u}\cdot\big((\nabla\mathrm{div}\, u_t) + \mathrm{div}\,((\nabla\mathrm{div}\, u)\otimes u)\big)\,\mathrm{d}x}_{\stackrel{\mathrm{def}}{=} II}
\end{aligned}$$

$$= \underbrace{\int \mathfrak{a}_t \operatorname{div} \dot{u} + (\dot{u} \cdot \nabla u) \cdot \nabla \mathfrak{a} \, \mathrm{d}x}_{\stackrel{\text{def}}{=} III} . \tag{2.18}$$

Estimate of I. It is easy to check that

$$\begin{aligned}
-\int \dot{u} \cdot \big(\Delta u_t + \operatorname{div}(\Delta u \otimes u)\big) \mathrm{d}x &= \int [\nabla \dot{u} \colon \nabla u_t + u \otimes \Delta u \colon \nabla \dot{u}] \, \mathrm{d}x \\
&= \int \Big[|\nabla \dot{u}|^2 - \big((\nabla u \nabla u) + (u \cdot \nabla) \nabla u\big) \colon \nabla \dot{u} - \nabla(u \cdot \nabla \dot{u}) \colon \nabla u \Big] \mathrm{d}x \\
&= \int \Big[|\nabla \dot{u}|^2 - (\nabla u \nabla u) \colon \nabla \dot{u} + \big((u \cdot \nabla) \nabla \dot{u}\big) \colon \nabla u - (\nabla u \nabla \dot{u}) \colon \nabla u - \big((u \cdot \nabla) \nabla \dot{u}\big) \colon \nabla u \Big] \mathrm{d}x \\
&\geqq \int \Big[\frac{3}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \Big] \, \mathrm{d}x.
\end{aligned}$$

Estimate of II. Observe that

$$\begin{aligned}
&\operatorname{div}\big((\nabla \operatorname{div} u) \otimes u\big) = \nabla(u \cdot \nabla \operatorname{div} u) - \operatorname{div}(\operatorname{div} u \nabla \otimes u) + \nabla (\operatorname{div} u)^2, \\
&\operatorname{div} \dot{u} = \operatorname{div} u_t + \operatorname{div}(u \cdot \nabla u) = \operatorname{div} u_t + u \cdot \nabla \operatorname{div} u + \nabla u \colon (\nabla u)^T,
\end{aligned}$$

where A^T means the transpose of matrix A. Then we get

$$\begin{aligned}
-\int \dot{u} \cdot \Big[\nabla \operatorname{div} u_t + \operatorname{div}\big((\nabla \operatorname{div} u) \otimes u\big) \Big] \mathrm{d}x \\
&= \int \Big[\operatorname{div} \dot{u} \operatorname{div} u_t + \operatorname{div} \dot{u} (u \cdot \nabla \operatorname{div} u) - \operatorname{div} u (\nabla \dot{u})^T \colon \nabla u + \operatorname{div} \dot{u} (\operatorname{div} u)^2 \Big] \mathrm{d}x \\
&= \int \Big[|\operatorname{div} \dot{u}|^2 - \operatorname{div} \dot{u} \nabla u \colon (\nabla u)^T - \operatorname{div} u (\nabla \dot{u})^T \colon \nabla u + \operatorname{div} \dot{u} (\operatorname{div} u)^2 \Big] \mathrm{d}x \\
&\geqq \int \Big[\frac{1}{2} |\operatorname{div} \dot{u}|^2 - \frac{1}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \Big] \mathrm{d}x.
\end{aligned}$$

Estimate of III. We have

$$\begin{aligned}
&\int \mathfrak{a}_t \operatorname{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla \mathfrak{a} \, \mathrm{d}x \\
&= \int -\gamma \rho^\gamma \operatorname{div} u \operatorname{div} \dot{u} - (u \cdot \nabla a) \operatorname{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla \mathfrak{a} \, \mathrm{d}x \\
&= \int -\gamma \rho^\gamma \operatorname{div} u \operatorname{div} \dot{u} + \mathfrak{a} \Big[\operatorname{div}\big((\operatorname{div} \dot{u}) u\big) - \operatorname{div}((u \cdot \nabla \dot{u})) \Big] \mathrm{d}x \\
&= \int -\gamma \rho^\gamma \operatorname{div} u \operatorname{div} \dot{u} + \mathfrak{a} \Big[\operatorname{div} u \operatorname{div} \dot{u} - (\nabla u)^T \colon \nabla \dot{u} \Big] \mathrm{d}x \\
&\leqq C \|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2}.
\end{aligned}$$

Substituting these estimates into (2.18) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \rho |\dot{u}|^2 \mathrm{d}x + \mu \int |\nabla \dot{u}|^2 \mathrm{d}x + (\lambda + \mu) \int |\operatorname{div} \dot{u}|^2 \mathrm{d}x \\
&\quad \leqq C \int |\nabla u|^4 \mathrm{d}x + C \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.19}$$

To conclude the estimate by Gronwall's inequality, we will use $\|\sqrt{\rho}\dot{u}\|_{L^2}$ to control $\|\nabla u\|_{L^4}$. By Proposition 2.1 and (2.8), we have

$$\|\nabla u\|_{L^\infty(0,\infty;L^2)} + \|\mathfrak{a}\|_{L^\infty(0,\infty;L^6)} \leqq C,$$
$$\|\nabla b\|_{L^6} + \|\Lambda d - \frac{\mathfrak{a}}{\lambda+2\mu}\|_{L^6} \leqq \|\rho\dot{u}\|_{L^2} \leqq C\|\sqrt{\rho}\dot{u}\|_{L^2},$$

from which together with (2.15) imply that

$$\begin{aligned}\|\nabla u\|_{L^4}^4 &\leqq \|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3 \leqq C\|\nabla u\|_{L^6}\|\nabla u\|_{L^6}^2\\ &\leqq C\|\nabla u\|_{L^6}^2\big(\|\nabla b\|_{L^6} + \|\nabla d - \frac{\mathfrak{a}}{\lambda+2\mu}\|_{L^6} + \|\mathfrak{a}\|_{L^6}\big)\\ &\leqq C\|\nabla u\|_{L^6}^2\Big(1 + \|\sqrt{\rho}\dot{u}\|_{L^2}\Big) \leq C\|\nabla u\|_{L^6}^2\Big(1 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2\Big).\end{aligned}$$

Substituting this estimate into (2.19) and noting that $\|\nabla u(t)\|_{L^6}^2 \in L^1(0,\infty)$ by Proposition 2.1, we get by Gronwall's inequality that

$$\int \rho|\dot{u}|^2 \mathrm{d}x + \int_0^\infty \int |\nabla\dot{u}|^2 \mathrm{d}x\mathrm{d}t \leqq C, \tag{2.20}$$

with C depending only on $\overline{\rho}$ and ρ_0, u_0. By using (2.20), (2.19) can be improved as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\dot{u}|^2\mathrm{d}x + \mu\int|\nabla\dot{u}|^2\mathrm{d}x + (\lambda+\mu)\int|\mathrm{div}\,\dot{u}|^2\mathrm{d}x \leqq C(\|\nabla u\|_{L^6}^2 + \|\nabla u\|_{L^2}^2),$$

from which together with (2.4), (2.8) and Sobolev embedding theorem will imply (2.16). □

2.1.3. Estimate for the Propagation of $\nabla\mathfrak{a}$ In this subsection, we want to give the proof of the upper bound of $\|\nabla u\|_{L^2((0,+\infty);L^\infty)}$ which is used to estimate the propagation of $\nabla\mathfrak{a}$. More precisely, we want to prove

Proposition 2.3. *Let* $0 < \alpha < 1$, $\mu > \frac{1}{2}\lambda$ *and* (ρ, u) *be a global and smooth solution of* (CNS) *with initial data* (ρ_0, u_0) *verifying that* $\rho_0 \geq c > 0$, *the admissible condition* (1.4) *and*

$$\sup_{t\in\mathbb{R}^+} \|\rho(t,\cdot)\|_{C^\alpha} \leqq M. \tag{2.21}$$

Then

$$\|\mathfrak{a}\|_{L^\infty((0,+\infty);W^{1,6})\cap L^2((0,+\infty);W^{1,6})} + \|\nabla u\|_{L^2((0,+\infty);L^\infty)} \leqq C, \tag{2.22}$$

where C *depends on the initial data* (ρ_0, u_0) *and* M. *As a consequence, there exists a constant* $\underline{\rho} = \underline{\rho}(c, M) > 0$ *such that for all* $t \geq 0$, $\rho(t,x) \geq \underline{\rho}$. *Moreover,*

$$\partial_t\|\nabla\mathfrak{a}\|_{L^2}^2 + \frac{1}{4(\lambda+2\mu)}\|\nabla\mathfrak{a}\|_{L^2}^2 \leqq C(\|\nabla\dot{u}\|_{L^2}^2 + \|u\nabla u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\mathfrak{a}\|_{L^6}^2). \tag{2.23}$$

Proof. Since that x^γ is a convex function when $\gamma > 1$, (2.21) implies that

$$\sup_{t\in\mathbb{R}^+} \|\rho^\gamma(t,\cdot)\|_{C^\alpha} \leqq M.$$

Next, we have the interpolation inequality

$$\begin{aligned}\|\nabla\Lambda^{-1}\mathfrak{a}\|_{L^\infty} &\leqq 2^{\frac{N}{2}}\|\mathfrak{a}\|_{L^6} + \sum_{j\geqq N} 2^{-j\alpha}(2^{j\alpha}\|\dot{\Delta}_j\mathfrak{a}\|_{L^\infty}) \leq 2^{\frac{N}{2}}\|\mathfrak{a}\|_{L^6}\\ &+ 2^{-N\alpha}\|\mathfrak{a}\|_{C^\alpha} \leq C(\eta)\|\mathfrak{a}\|_{L^6} + \eta,\end{aligned} \tag{2.24}$$

where η is a sufficiently small constant depending on N. This implies that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leqq C\left(\|\nabla\mathcal{P}u\|_{L^\infty} + \|\nabla\Lambda^{-1}\left(\operatorname{div} u - \frac{1}{2\mu+\lambda}\mathfrak{a}\right)\|_{L^\infty} + \|\nabla\Lambda^{-1}\mathfrak{a}\|_{L^\infty}\right)\\ &\leqq C\left(\|\nabla\mathcal{P}u\|_{W^{1,6}} + \|\left(\operatorname{div} u - \frac{1}{2\mu+\lambda}\mathfrak{a}\right)\|_{W^{1,6}} + C(\eta)\|\mathfrak{a}\|_{L^6} + \eta\right).\end{aligned} \tag{2.25}$$

On the other hand, it is not difficult to derive that

$$\begin{aligned}&\frac{1}{\gamma}\big(\partial_t\nabla\mathfrak{a} + (u\cdot\nabla)\nabla\mathfrak{a}\big) + \frac{\rho^\gamma}{\lambda+2\mu}\nabla\mathfrak{a}\\ &\quad + \frac{1}{\gamma}\nabla u\nabla\mathfrak{a} + \operatorname{div} u\nabla\mathfrak{a} = -\rho^\gamma\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right).\end{aligned} \tag{2.26}$$

Multiplying (2.26) by $|\nabla\mathfrak{a}|^{p-2}\nabla\mathfrak{a}$ and integrating the resulting equation on $\mathbb{R}^3$, we can derive that for $p \geq 2$,

$$\begin{aligned}&\frac{1}{p}\partial_t\|\nabla\mathfrak{a}\|_{L^p}^p + \frac{\gamma}{\lambda+2\mu}\int \rho^\gamma|\nabla\mathfrak{a}|^p\,\mathrm{d}x\\ &\leqq \frac{1}{p}\int \operatorname{div} u|\nabla\mathfrak{a}|^p\,\mathrm{d}x + C\|\nabla u\|_{L^\infty}\|\nabla\mathfrak{a}\|_{L^p}^p + C\|\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^p}\|\nabla\mathfrak{a}\|_{L^p}^{p-1}\\ &\leqq \frac{1}{p(\lambda+2\mu)}\int \mathfrak{a}|\nabla\mathfrak{a}|^p\,\mathrm{d}x + C\left(\|\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\|_{L^\infty} + \|\nabla u\|_{L^\infty}\right)\|\nabla\mathfrak{a}\|_{L^p}^p\\ &\quad + C\|\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^p}\|\nabla\mathfrak{a}\|_{L^p}^{p-1},\end{aligned}$$

which means that

$$\begin{aligned}&\frac{1}{p}\partial_t\|\nabla\mathfrak{a}\|_{L^p}^p + \frac{1}{(\lambda+2\mu)p}\int[(\gamma p-1)\rho^\gamma+1]|\nabla\mathfrak{a}|^p\,\mathrm{d}x\\ &\leqq C\left(\|\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\|_{L^\infty} + \|\nabla u\|_{L^\infty}\right)\|\nabla\mathfrak{a}\|_{L^p}^p\\ &\quad + C\|\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^p}\|\nabla\mathfrak{a}\|_{L^p}^{p-1}.\end{aligned}$$

Noting that $\gamma \geqq 1$ and $p \geqq 2$, we can obtain that

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\nabla\mathfrak{a}\|_{L^p}^2 + \frac{1}{(\lambda+2\mu)p}\|\nabla\mathfrak{a}\|_{L^p}^2 \\
&\leqq C\left(\|\nabla u\|_{L^\infty}\|\nabla\mathfrak{a}\|_{L^p}^2 + \|\operatorname{div} u - \frac{1}{2\mu+\lambda}\mathfrak{a}\|_{L^\infty}\|\nabla\mathfrak{a}\|_{L^p}^2\right. \\
&\left.+\|\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^p}\|\nabla\mathfrak{a}\|_{L^p}\right),
\end{aligned} \tag{2.27}$$

which implies that

$$\begin{aligned}
&\partial_t\|\nabla\mathfrak{a}\|_{L^p}^2 + \frac{1}{(\lambda+2\mu)p}\|\nabla\mathfrak{a}\|_{L^p}^2 \\
&\leqq C(\|\nabla u\|_{L^\infty}^2\|\nabla\mathfrak{a}\|_{L^p}^2 + \|\operatorname{div} u - \frac{1}{2\mu+\lambda}\mathfrak{a}\|_{W^{1,6}}^2\|\nabla\mathfrak{a}\|_{L^p}^2 \\
&+\|\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^p}^2).
\end{aligned} \tag{2.28}$$

By taking $p = 6$ in (2.28), and using (2.25) and (2.16), we obtain from Gronwall's inequality that $\|\mathfrak{a}\|_{L^\infty((0,+\infty);W^{1,6})\cap L^2((0,+\infty);W^{1,6})} \leqq C$. From which together with (2.25), we obtain that

$$\|\nabla u\|_{L^\infty} \leqq C(\|\nabla\mathcal{P}u\|_{W^{1,6}} + \|\left(\operatorname{div} u - \tfrac{1}{2\mu+\lambda}\mathfrak{a}\right)\|_{W^{1,6}} + \|\mathfrak{a}\|_{W^{1,6}}).$$

This implies that $\|\nabla u\|_{L^2((0,+\infty);L^\infty)} \leqq C$. It completes the proof to (2.22).

Now we go back to (2.28) with $p = 2$. By Gronwall's inequality, we obtain that $\nabla\mathfrak{a} \in L^\infty((0,+\infty);L^2)\cap L^2((0,+\infty);L^2)$. Thanks to uniform-in-time bounds obtained in the above, (2.28) with $p = 2$ will yield that

$$\begin{aligned}
&\partial_t\|\nabla\mathfrak{a}\|_{L^2}^2 + \frac{1}{4(\lambda+2\mu)}\|\nabla\mathfrak{a}\|_{L^2}^2 \\
&\leqq C(\|\nabla\left(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a}\right)\|_{L^2}^2 + \|\operatorname{div} u - \frac{1}{2\mu+\lambda}\mathfrak{a}\|_{W^{1,6}}^2 \\
&+\|\nabla\mathcal{P}u\|_{W^{1,6}}^2) + C_\eta\|\mathfrak{a}\|_{L^6}^2,
\end{aligned}$$

from which together with the elliptic estimate for (2.8), we obtain (2.23).

Now using the equation of density and the estimate (2.22), we have

$$\rho(t,x) \geqq \rho_0(x)\exp^{-\int_0^t\|\operatorname{div} u\|_{L^\infty}\,\mathrm{d}\tau} \geq ce^{-Ct^{\frac{1}{2}}}.$$

On the other hand, thanks to (2.1) and (2.12), we derive that $\lim_{t\to\infty}\|a(t)\|_{L^6} = \lim_{t\to\infty}\|\mathfrak{a}(t)\|_{L^6} = 0$, from which together with the upper bound for ρ in C^α, we derive that $\lim_{t\to\infty}\|a(t)\|_{L^\infty} = 0$. These two facts imply that there exists a constant $\underline{\rho} = \underline{\rho}(c, M) > 0$ such that for all $t \geq 0$, $\rho(t,x) \geq \underline{\rho}$. We complete the proof to the proposition. $\square$

Remark 2.3. Now we show that the result of Proposition 2.3 is still valid when (2.21) is replaced by (1.10). The proof is almost the same as the original one. We only point out the modification. Firstly by using the second assumption in (1.10) and the fact that $\|\nabla\Lambda^{-1}\dot{\Delta}_j f\|_{L^\infty} \lesssim \|\dot{\Delta}_j f\|_{L^\infty}$ for all $j \in \mathbb{Z}$, we can recover the estimate (2.24) by

$$\|\nabla\Lambda^{-1}\mathfrak{a}\|_{L^\infty} \lesssim \sum_{j\in\mathbb{Z}} \|\dot{\Delta}_j\mathfrak{a}\|_{L^\infty} \lesssim 2^{\frac{N}{2}}\|\mathfrak{a}\|_{L^6} + \sum_{j\geq N}\|\dot{\Delta}_j\mathfrak{a}\|_{L^\infty} \lesssim C(\eta)\|\mathfrak{a}\|_{L^6} + \eta,$$

where η is a sufficiently small constant depending on N. Then by the same argument used in the above, we can derive (2.22) and (2.23). To get the lower bound of the density, we notice that

$$\|\mathfrak{a}(t)\|_{L^\infty} \lesssim 2^{\frac{N}{2}}\|\mathfrak{a}\|_{L^6} + \sum_{j\geq N}\|\dot{\Delta}_j\mathfrak{a}\|_{L^\infty},$$

from which together with the fact $\lim_{t\to\infty}\|\mathfrak{a}(t)\|_{L^6} = 0$ and (1.10), we obtain that $\lim_{t\to\infty}\|\mathfrak{a}(t)\|_{L^\infty} = 0$. This implies the lower bound of the density ρ which completes the proof.

2.1.4. Deriving the Dissipation Inequality We want to prove

Proposition 2.4. *Let* $0 < \alpha < 1$, $\mu > \frac{1}{2}\lambda$, *and* (ρ, u) *be a global and smooth solution of* (CNS) *with initial data* (ρ_0, u_0) *verifying that* $\rho_0 \geq c > 0$, *the admissible condition* (1.4) *and* $\sup_{t\in\mathbb{R}^+}\|\rho(t,\cdot)\|_{C^\alpha} \leqq M$. *Then there exist positive constants* $A_i (i = 1, \cdots, 7)$ *which are depending on* μ, λ *and* M *such that*

$$\begin{aligned}X(t) &\stackrel{def}{=} A_1\|\rho^{\frac{1}{4}}u\|_{L^4}^4 + A_2\left(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2 - (\mathfrak{a}, \operatorname{div} u) + \int f(\rho)\mathrm{d}x\right)\\ &+ A_3\|\mathfrak{a}\|_{L^6}^2 + A_4\left(\int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2\right) + A_5\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + A_6\|\nabla\mathfrak{a}\|_{L^2}^2\\ &\sim \|u\|_{H^1}^2 + \|a\|_{H^1}^2 + \|\dot{u}\|_{L^2}^2,\end{aligned}$$

which verifies

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + A_7\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla a\|_{L^2}^2 + \|\nabla\dot{u}\|_{L^2}^2\right) \leqq 0. \quad (2.29)$$

Proof. Thanks to Proposition 2.3, we may assume that $\underline{\rho} \leq \rho \leq M$. From (2.23) and (2.16), we get that there exist positive constants $A_i (i = 1, \cdots, 7)$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[&A_1\|\rho^{\frac{1}{4}}u\|_{L^4}^4 + A_2\big(\mu\|\nabla u\|_{L^2}^2 + (\lambda+\mu)\|\operatorname{div} u\|_{L^2}^2 - (\mathfrak{a}, \operatorname{div} u)\\ &+ \int f(\rho)\mathrm{d}x\big) + A_3\|\mathfrak{a}\|_{L^6}^2\\ &+ A_4\left(\int H(\rho|1)\,\mathrm{d}x + \|\sqrt{\rho}u\|_{L^2}^2\right) + A_5\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + A_6\|\nabla\mathfrak{a}\|_{L^2}^2\bigg]\\ &+ A_7\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\mathfrak{a}\|_{L^2}^2 + \|\nabla\dot{u}\|_{L^2}^2\right) \leqq 0.\end{aligned}$$

Thanks to the energy identity (2.2), the constant A_4 can be chosen large enough to ensure that $X(t) \geq 0$. Due to the condition $\underline{\rho} \leq \rho \leq M$, one has $\|\nabla a\|_{L^2} \sim \|\nabla \mathfrak{a}\|_{L^2}$ and $\int H(\rho|1)\,\mathrm{d}x \sim \|\rho - 1\|_{L^2}^2$, from which together with (2.1) and $\rho\dot{u} + \nabla\mathfrak{a} = \mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}\, u$, we deduce that $X(t) \sim \|u\|_{H^2}^2 + \|a\|_{H^1}^2 + \|\dot{u}\|_{L^2}^2$. It ends the proof of the proposition. □

2.2. Convergence to the Equilibrium

The aim of this subsection is to show the convergence of the solution to the equilibrium. Thanks to Proposition 2.3, now we may assume that $\rho \geqq \underline{\rho}$.

We begin with a crucial lemma about the low frequency part of the solution.

Lemma 2.3. *Let* $0 < \alpha < 1$, $\mu > \frac{1}{2}\lambda$, *and* (ρ, u) *be a global and smooth solution of* (CNS) *with initial data* (ρ_0, u_0) *verifying that* $\rho_0 \geq c > 0$, *the admissible condition* (1.4) *and* $\sup_{t\in\mathbb{R}^+}\|\rho(t,\cdot)\|_{C^\alpha} \leqq M$. *Let* $a_0, \rho_0 u_0 \in L^{p_0}(\mathbb{R}^3)$ *with* $p_0 \in [1,2]$. *Then if* $\rho(t,x) \leq M$, *we have*

$$\int_{S(t)} \big(\gamma|\hat{a}(\xi,t)|^2 + |\widehat{\rho u}(\xi,t)|^2\big)\,\mathrm{d}\xi \leqq C(M)\big(\|a_0\|_{L^{p_0}}^2 + \|\rho_0 u_0\|_{L^{p_0}}^2\big)(1+t)^{-2\beta(p_0)} + C(M)(1+t)^{-\frac{3}{2}}\int_0^t \big(\|u\|_{L^2}^4 + \|a\|_{L^2}^4\big)\,\mathrm{d}s, \tag{2.30}$$

where $S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \leqq C(1+t)^{-\frac{1}{2}}\}$, *and* $\beta(p_0) = \frac{3}{4}(\frac{2}{p_0} - 1)$.

Proof. Note that $\rho_t = a_t$; we take the Fourier transform of (CNS), and then multiply $\gamma\bar{\hat{a}}$ to the first equation, and multiply $\overline{\widehat{\rho u}}$ to the second equation to obtain that

$$\begin{cases} \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\gamma|\hat{a}|^2 + i\gamma\xi\cdot\widehat{\rho u}\bar{\hat{a}} = 0, \\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}|\widehat{\rho u}|^2 + \big(\widehat{\mathrm{div}\,(\rho u\otimes u)} - \mu\widehat{\Delta u} - (\lambda+\mu)\widehat{\nabla\mathrm{div}\, u}\big)\cdot\overline{\widehat{\rho u}} \\ \qquad + i\xi\big((\gamma-1)\widehat{H(\rho|1)} + \gamma\hat{a}\big)\cdot\overline{\widehat{\rho u}} = 0, \end{cases}$$

which implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma|\hat{a}|^2 + |\widehat{\rho u}|^2\big) = Re\Big[-\widehat{\mathrm{div}\,(\rho u\otimes u)} + \mu\widehat{\Delta u} + (\lambda+\mu)\widehat{\nabla\mathrm{div}\, u} + i(\gamma-1)\xi\widehat{H(\rho|1)}\Big]\cdot\overline{\widehat{\rho u}} \stackrel{\text{def}}{=} F(\xi,t).$$

Integrating the above equation with respect to the time t, we get that

$$\gamma|\hat{a}(\xi,t)|^2 + |\widehat{\rho u}(\xi,t)|^2 = \gamma|\hat{a}(\xi,0)|^2 + |\widehat{\rho u}(\xi,0)|^2 + 2\int_0^t F(\xi,s)\,\mathrm{d}s.$$

Let $S(t) \stackrel{\text{def}}{=} \{\xi : |\xi| \leqq C(1+t)^{-\frac{1}{2}}\}$, then we can split the phase space $\mathbb{R}^3$ into two time-dependent regions, $S(t)$ and $S(t)^c$. Integrating the above equation over $S(t)$, and noting that $\widehat{\rho u} = \hat{u} + \widehat{au}$, and

$$\widehat{\Delta u}\bar{\hat{u}} = -|\xi|^2|\hat{u}|^2, \quad \widehat{\nabla\mathrm{div}\, u}\bar{\hat{u}} = -|\xi\cdot\hat{u}|^2,$$

we can obtain that

$$
\begin{aligned}
&\int_{S(t)} \left(\gamma|\hat{a}(\xi,t)|^2 + |\widehat{\rho u}(\xi,t)|^2\right) \mathrm{d}\xi + \int_0^t \int_{S(t)} \left(\mu|\xi|^2|\hat{u}|^2 + (\lambda+\mu)|\xi\cdot\hat{u}|^2\right) \mathrm{d}\xi\,\mathrm{d}s \\
&= \int_{S(t)} \left(\gamma|\hat{a}(\xi,0)|^2 + |\widehat{\rho u}(\xi,0)|^2\right) \mathrm{d}\xi + Re\int_0^t \int_{S(t)} \Big[-\operatorname{div}\widehat{(\rho u\otimes u)}\cdot\overline{\widehat{\rho u}} \\
&\quad + \left(\mu\widehat{\Delta u} + (\lambda+\mu)\widehat{\nabla \operatorname{div} u}\right)\cdot\overline{\widehat{au}} + i(\gamma-1)\xi\widehat{H(\rho|1)}\cdot\overline{\widehat{\rho u}}\Big] \mathrm{d}\xi\,\mathrm{d}s \\
&\overset{\text{def}}{=} \int_{S(t)} \left(\gamma|\hat{a}(\xi,0)|^2 + |\widehat{\rho u}(\xi,0)|^2\right) \mathrm{d}\xi + B_1 + B_2 + B_3.
\end{aligned}
\tag{2.31}
$$

From Lemma 2.1, we have that a, u and ρu all belong to $L^\infty((0,+\infty);L^2)$, which mean that $\widehat{\rho u\otimes u}$ and $\widehat{au}$ belong to $L^\infty((0,+\infty);L^\infty)$. Thanks to these facts, we can give estimates to terms $B_i (i=1,2,3)$. We first have that

$$
\begin{aligned}
|B_1| &\leqq \left|\int_0^t \int_{S(t)} \operatorname{div}\widehat{(\rho u\otimes u)}\cdot(\overline{\hat{u}} + \overline{\widehat{au}})\,\mathrm{d}\xi\,\mathrm{d}s\right| \\
&\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta\int_0^t \int_{S(t)} \left|\widehat{\rho u\otimes u}\right|^2 \mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \int_0^t \int_{S(t)} |\xi|\left|\widehat{\rho u\otimes u}\right||\widehat{au}|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta\int_0^t \|\widehat{\rho u\otimes u}\|_{L^\infty}^2 \int_{S(t)} \mathrm{d}\xi\,\mathrm{d}s \\
&\quad + C(1+t)^{-\frac{1}{2}}\int_0^t \|\widehat{\rho u\otimes u}\|_{L^\infty}\|\widehat{au}\|_{L^\infty}\int_{S(t)} \mathrm{d}\xi\,\mathrm{d}s \\
&\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta(1+t)^{-\frac{3}{2}}\int_0^t \|u\|_{L^2}^4\,\mathrm{d}s \\
&\quad + C(1+t)^{-2}\int_0^t \|u\|_{L^2}^3\|a\|_{L^2}\,\mathrm{d}s.
\end{aligned}
\tag{2.32}
$$

Similarly, one has

$$
\begin{aligned}
|B_2| &\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta\int_0^t \int_{S(t)} |\xi|^2\left|\widehat{au}\right|^2 \mathrm{d}\xi\,\mathrm{d}s \\
&\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta(1+t)^{-\frac{5}{2}}\int_0^t \|u\|_{L^2}^2\|a\|_{L^2}^2\,\mathrm{d}s,
\end{aligned}
\tag{2.33}
$$

and

$$
\begin{aligned}
|B_3| &\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta\int_0^t \int_{S(t)} |\xi|^2\left|\widehat{au}\right|^2 \mathrm{d}\xi\,\mathrm{d}s \\
&\quad + C\int_0^t \int_{S(t)} |\widehat{H(\rho|1)}|^2\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leqq \eta\int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2\,\mathrm{d}\xi\,\mathrm{d}s + C_\eta(1+t)^{-\frac{5}{2}}\int_0^t \|u\|_{L^2}^2\|a\|_{L^2}^2\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ (1+t)^{-\frac{3}{2}} \int_0^t \|H(\rho|1)\|_{L^1}^2 \,\mathrm{d}s \\
&\leqq \eta \int_0^t \int_{S(t)} \mu|\xi|^2|\hat{u}|^2 \,\mathrm{d}\xi\mathrm{d}s + C_\eta(1+t)^{-\frac{5}{2}} \int_0^t \|u\|_{L^2}^2\|a\|_{L^2}^2 \,\mathrm{d}s \\
&+ (1+t)^{-\frac{3}{2}} \int_0^t \|a\|_{L^2}^4 \,\mathrm{d}s.
\end{aligned}
$$

Note that a_0 and $\rho_0 u_0$ belong to $L^{p_0}(\mathbb{R}^3)$ for $1 \leqq p_0 < \frac{3}{2}$. Then for $\frac{1}{p_0}+\frac{1}{p_0'} = 1$, one has

$$
\begin{aligned}
\int_{S(t)} \big(|\hat{a}(\xi,0)|^2 + |\widehat{\rho u}(\xi,0)|^2\big)\,\mathrm{d}\xi &\leqq \big(\|\hat{a}_0\|_{L^{p_0'}}^2 + \|\widehat{\rho_0 u_0}\|_{L^{p_0'}}^2\big) \Big(\int_{S(t)} \mathrm{d}\xi\Big)^{1-\frac{2}{p_0'}} \\
&\leqq C\big(\|a_0\|_{L^{p_0}}^2 + \|\rho_0 u_0\|_{L^{p_0}}^2\big)(1+t)^{-2\beta(p_0)}.
\end{aligned}
\tag{2.34}
$$

Plugging (2.32), (2.33) and (2.34) into (2.31), and choosing η small enough, we arrive at

$$
\begin{aligned}
&\int_{S(t)} \big(|\hat{a}(\xi,t)|^2 + |\widehat{\rho u}(\xi,t)|^2\big)\,\mathrm{d}\xi + C\int_0^t \int_{S(t)} \big(\mu|\xi|^2|\hat{u}|^2 + (\lambda+\mu)|\xi\cdot\hat{u}|^2\big)\,\mathrm{d}\xi\mathrm{d}s \\
&\leqq C\big(\|a_0\|_{L^{p_0}}^2 + \|\rho_0 u_0\|_{L^{p_0}}^2\big)(1+t)^{-2\beta(p_0)} + C(1+t)^{-\frac{3}{2}} \int_0^t \big(\|u\|_{L^2}^4 + \|a\|_{L^2}^4\big)\,\mathrm{d}s.
\end{aligned}
$$

This ends the proof to the lemma. □

Now we are in a position to prove

Proposition 2.5. *Let* $0 < \alpha < 1$, $\mu > \frac{1}{2}\lambda$, *and* (ρ, u) *be a global and smooth solution of* (CNS) *with initial data* (ρ_0, u_0) *verifying that* $\rho_0 \geq c > 0$, *the admissible condition* (1.4) *and* $\sup_{t\in\mathbb{R}^+} \|\rho(t,\cdot)\|_{C^\alpha} \leqq M$. *Suppose that* $a_0 \in L^{p_0}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ *and* $u_0 \in L^{p_0}(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ *with* $p_0 \in [1,2]$. *Then we have*

$$
\|u(t)\|_{H^1} + \|a(t)\|_{H^1} \leqq \bar{C}(1+t)^{-\beta(p_0)}, \tag{2.35}
$$

where $\beta(p_0) = \frac{3}{4}(\frac{2}{p_0} - 1)$, *and the constant* $\bar{C}$ *depends only on* $\underline{\rho}$, μ, λ, M, $\|a_0\|_{L^{p_0}\cap H^1}$, *and* $\|u_0\|_{L^{p_0}\cap H^2}$.

Proof. We separate the proof into several steps.

Step 1: The first sight of the convergence. Thanks to (2.30) and the fact that a and u belong to $L^\infty((0,+\infty); L^2)$, we have

$$
\begin{aligned}
&\int_{S(t)} \big(|\hat{a}(\xi,t)|^2 + |\widehat{\rho u}(\xi,t)|^2\big)\,\mathrm{d}\xi \\
&\leqq C\big(\|a_0\|_{L^{p_0}}^2 + \|\rho_0 u_0\|_{L^{p_0}}^2\big)(1+t)^{-2\beta(p_0)} + C\big(\|u\|_{L^\infty(L^2)}^4 + \|a\|_{L^\infty(L^2)}^4\big)(1+t)^{-\frac{1}{2}} \\
&\leqq C(1+t)^{-r_m},
\end{aligned}
\tag{2.36}
$$

where $r_m = \min\{2\beta(p_0), \frac{1}{2}\}$. Due to the fact $u = \rho u - au$, we have

$$\begin{aligned}\int_{S(t)} |\widehat{u}(\xi,t)|^2 \,\mathrm{d}\xi &\leqq \int_{S(t)} |\widehat{\rho u}(\xi,t)|^2 \,\mathrm{d}\xi + \int_{S(t)} |\widehat{au}(\xi,t)|^2 \,\mathrm{d}\xi \\ &\leqq C(1+t)^{-r_m} + C(1+t)^{-\frac{3}{2}} |\widehat{au}(\xi,t)|^2_{L^\infty} \\ &\leqq C(1+t)^{-r_m}.\end{aligned}$$

Next, because of $\rho\dot{u} = \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u - \nabla \mathfrak{a}$, following the same argument, we can obtain

$$\begin{aligned}&\int_{S(t)} |\widehat{\rho\dot{u}}(\xi,t)|^2 \,\mathrm{d}\xi \leqq \int_{S(t)} |\mu\widehat{\Delta u} + (\lambda+\mu)\widehat{\nabla \operatorname{div} u} \\ &\quad - \big((\gamma-1)\nabla\widehat{H(\rho|1)} + \gamma\widehat{\nabla a}\big)(\xi,t)|^2 \,\mathrm{d}\xi \leqq C(1+t)^{-1-r_m},\end{aligned}$$

which implies that $\int_{S(t)} |\widehat{\dot{u}}(\xi,t)|^2 \,\mathrm{d}\xi \leqq C(1+t)^{-1-r_m}$.

We recall the dissipation inequality (2.29). Then by frequency splitting method, it is not difficult to derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \frac{K}{1+t}X(t) &\leqq \frac{1}{1+t}\int_{S(t)} \big(|\hat{a}(\xi,t)|^2 + |\widehat{u}(\xi,t)|^2 + |\widehat{\dot{u}}(\xi,t)|^2\big) \,\mathrm{d}\xi \\ &\leq C(1+t)^{-1-r_m},\end{aligned}$$

which implies

$$X(t) \leqq C(1+t)^{-r_m}. \tag{2.37}$$

In particular, we have

$$\|u\|_{L^2} + \|a\|_{L^2} \leqq 2C(1+t)^{-r_m/2}. \tag{2.38}$$

Step 2: Improving the decay estimate (I). We want to improve the decay estimate if $\beta(p_0) > \frac{1}{4}$. By definition, $r_m = \frac{1}{2}$. Thanks to (2.30) and (2.38), we improve the estimate for the low frequency part as follows:

$$\int_{S(t)} \big(|\hat{a}(\xi,t)|^2 + |\widehat{\rho u}(\xi,t)|^2\big) \,\mathrm{d}\xi \leqq C(1+t)^{-2\beta(p_0)} + C(1+t)^{-\frac{3}{2}}\log(1+t).$$

Now, following an argument similar to that used in the previous step, we conclude that

$$\begin{aligned}&\int_{S(t)} \big(|\hat{a}(\xi,t)|^2 + |\widehat{u}(\xi,t)|^2 + |\widehat{\dot{u}}(\xi,t)|^2\big) \,\mathrm{d}\xi \\ &\quad \leqq C(1+t)^{-2\beta(p_0)} + C(1+t)^{-\frac{3}{2}}\log(1+t),\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \frac{K}{1+t}X(t) \leqq C(1+t)^{-1}\big((1+t)^{-2\beta(p_0)} + (1+t)^{-\frac{3}{2}}\log(1+t)\big).$$

We obtain that

$$X(t) \leqq C \min\{(1+t)^{-2\beta(p_0)}, (1+t)^{-\frac{3}{2}} \log(1+t)\}. \tag{2.39}$$

In particular, $\|u\|_{L^2} + \|a\|_{L^2} \leqq \min\{(1+t)^{-\beta(p_0)}, (1+t)^{-\frac{3}{4}} \log^{\frac{1}{2}}(1+t)\}$.

Step 3: Improving the decay estimate (II). Finally we deal with the case that $\beta(p_0) > \frac{1}{2}$. By (2.39), we have $\|u\|_{L^2} + \|a\|_{L^2} \leqq C(1+t)^{-\frac{1}{2}}$. Now we may repeat the same process in the above to get that

$$\int_{S(t)} \left(|\hat{a}(\xi,t)|^2 + |\widehat{u}(\xi,t)|^2 + |\widehat{\dot{u}}(\xi,t)|^2\right) \mathrm{d}\xi \leqq C(1+t)^{-2\beta(p_0)} + C(1+t)^{-\frac{3}{2}},$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) + \frac{K}{1+t} X(t) \leqq C(1+t)^{-1}(1+t)^{-2\beta(p_0)}.$$

It is enough to derive (2.35). We ends the proof to the proposition. □

2.3. Proof of Theorem 1.1

Before giving the proof to Theorem 1.1, we first show the propagation of the regularity for $\nabla^2 a$.

Proposition 2.6. *Let* (a, u) *be a solution of* (CNS) *with initial data* (a_0, u_0). *Then under the assumptions of Proposition* 2.5, *there hold* $a \in L^\infty((0,+\infty); H^2)$, $\nabla a \in L^2((0,+\infty); H^1)$ *and* $\nabla u \in L^2((0,\infty); H^2)$.

Proof. Due to Proposition 2.4, we have $\mathfrak{a} \in L^\infty((0,+\infty); H^1) \cap L^2((0,+\infty); \dot{H}^1)$. Thanks to the lower and upper bounds for the density ρ, it is not difficult to check that $\|a\|_{H^2} \sim \|\mathfrak{a}\|_{H^2}$. Then the desired result is reduced to the proof of the propagation of $\nabla^2 \mathfrak{a}$.

We first notice that by Proposition 2.3, $\nabla \mathfrak{a} \in L^\infty((0,\infty); L^p)$ with $p \in [2, 6]$, which will be used frequently in what follows. Recall that

$$\begin{aligned} &\frac{1}{\gamma}(\mathfrak{a}_t + u \cdot \nabla \mathfrak{a}) + \mathfrak{a}\left(\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\right) \\ &\quad + \frac{1}{\mu+2\lambda}\mathfrak{a}^2 + \left(\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\right) + \frac{1}{\mu+2\lambda}\mathfrak{a} = 0. \end{aligned} \tag{2.40}$$

Then it is not difficult to derive that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\mathfrak{a}\|_{L^2}^2 + \frac{1}{\mu+2\lambda}\|\nabla^2\mathfrak{a}\|_{L^2}^2 \leqq \left|\left(\nabla^2(\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}), \nabla^2\mathfrak{a}\right)\right| \\ &\quad + \frac{1}{\mu+2\lambda}\left|\left(\nabla\mathfrak{a}\nabla\mathfrak{a}, \nabla^2\mathfrak{a}\right)\right| \\ &\quad + \left|\left(\nabla^2[\mathfrak{a}(\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a})], \nabla^2\mathfrak{a}\right)\right| + \left|\left(\nabla^2(u\cdot\nabla\mathfrak{a}), \nabla^2\mathfrak{a}\right)\right| \\ &\quad \stackrel{\text{def}}{=} D_1 + D_2 + D_3 + D_4. \end{aligned}$$

By Cauchy–Schwartz inequality and Proposition 2.3, we can estimate $D_i\,(i=1,2)$ easily by

$$D_1 \leqq \|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2}\|\nabla^2\mathfrak{a}\|_{L^2} \leqq C_\eta\|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2}^2 + \eta\|\nabla^2\mathfrak{a}\|_{L^2}^2,$$
$$D_2 \leqq C\|\nabla\mathfrak{a}\|_{L^2}^{\frac12}\|\nabla\mathfrak{a}\|_{L^6}^{\frac32}\|\nabla^2\mathfrak{a}\|_{L^2} \leqq \eta\|\nabla^2\mathfrak{a}\|_{L^2}^2 + C_\eta\|\nabla\mathfrak{a}\|_{L^6}^2.$$

For D_3, we have

$$\begin{aligned} D_3 \leqq\ & \|\mathfrak{a}\|_{L^\infty}\|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2}\|\nabla^2\mathfrak{a}\|_{L^2} \\ &+ \|\nabla\mathfrak{a}\|_{L^3}\|\nabla\left(\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\right)\|_{\dot H^1}\|\nabla^2\mathfrak{a}\|_{L^2} \\ &+ \|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{L^\infty}\|\nabla^2\mathfrak{a}\|_{L^2}^2 \\ \leqq\ & \eta\|\nabla^2\mathfrak{a}\|_{L^2}^2 + C\left(\|\mathfrak{a}\|_{W^{1,6}}^2 + \|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{W^{1,6}}^2\right)\|\nabla^2\mathfrak{a}\|_{L^2}^2 \\ &+ C_\eta\|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2}^2. \end{aligned}$$

For D_4, thanks to integration by parts, we obtain that

$$(u\cdot\nabla\nabla^2\mathfrak{a}, \nabla^2\mathfrak{a}) = -\frac12\big((\operatorname{div} u)\nabla^2\mathfrak{a}, \nabla^2\mathfrak{a}\big),$$

which implies that

$$\begin{aligned} D_4 &\leqq \|\nabla\mathfrak{a}\|_{L^3}\|\nabla^2 u\|_{L^6}\|\nabla^2\mathfrak{a}\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^2\mathfrak{a}\|_{L^2}^2 + \|\operatorname{div} u\|_{L^\infty}\|\nabla^2\mathfrak{a}\|_{L^2}^2 \\ &\leqq \eta\|\nabla^2\mathfrak{a}\|_{L^2}^2 + C_\eta(\|\nabla\mathfrak{a}\|_{L^3}^2\|\nabla^2 u\|_{L^6}^2 + \|\nabla u\|_{L^\infty}^2\|\nabla^2\mathfrak{a}\|_{L^2}^2). \end{aligned}$$

Combining all the estimates in the above, we obtain that

$$\begin{aligned} &\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\mathfrak{a}\|_{L^2}^2 + \frac34\frac{1}{\mu+2\lambda}\|\nabla^2\mathfrak{a}\|_{L^2}^2 \\ &\quad\leqq C\Big(\|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2}^2 + \|\nabla^2 u\|_{L^6}^2 + \|\nabla\mathfrak{a}\|_{L^6}^2\Big) \\ &\qquad + C\big(\|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{W^{1,6}}^2 + \|\mathfrak{a}\|_{W^{1,6}}^2 + \|\nabla u\|_{L^\infty}^2\big)\|\nabla^2\mathfrak{a}\|_{L^2}^2. \end{aligned} \tag{2.41}$$

Thanks to (2.8), we have

$$\begin{aligned} &\|\nabla\mathcal{P}u\|_{\dot H^2} + \|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2} \leqq \|\rho\dot u\|_{\dot H^1} \leqq C(\|\nabla a\dot u\|_{L^2} + \|\nabla\dot u\|_{L^2}) \\ &\leqq C(\|\nabla a\|_{L^3}\|\dot u\|_{L^6} + \|\nabla\dot u\|_{L^2}) \leqq C(\|\nabla\mathfrak{a}\|_{L^3}\|\dot u\|_{L^6} + \|\nabla\dot u\|_{L^2}) \leqq C\|\nabla\dot u\|_{L^2}, \end{aligned} \tag{2.42}$$

which, together with (2.15), implies that

$$\int_0^\infty\Big(\|\operatorname{div} u - \frac{1}{\mu+2\lambda}\mathfrak{a}\|_{\dot H^2}^2 + \|\nabla^2 u\|_{L^6}^2 + \|\nabla\mathfrak{a}\|_{L^6}^2\Big)\mathrm{d}t \leq C.$$

By Gronwall's inequality, we have $\mathfrak{a} \in L^\infty((0,+\infty);\dot{H}^2) \cap L^2((0,+\infty);\dot{H}^2)$, from which together with (2.42), we deduce that $\nabla u \in L^2((0,\infty);H^2)$. This ends the proof of the proposition. □

Finally, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We first note that the first three results of the Theorem 1.1 are proved by Propositions 2.3–2.6.

Finally let us give the proof to the fourth result of the Theorem 1.1. For $q \in [2,4]$, one has

$$\|f\|_{\dot{B}^{-1+\frac{3}{q}}_{q,1}} \leqq \|f\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leqq \|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}, \qquad \|f\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \leqq \|f\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leqq \|\nabla f\|_{L^2}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2}^{\frac{1}{2}},$$

from which, together with (2.35), we can easily get (1.7) and (1.8). On the other hand, for $p_0 < 2$, these two estimates imply that there exists a time t_0 such that

$$\|u(t_0)\|_{\dot{B}^{-1+\frac{3}{q}}_{q,1}} + \|a(t_0)\|_{\dot{B}^{-1+\frac{3}{q}}_{q,1}} + \|a(t_0)\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \leqq \eta,$$

where η is sufficiently small. Then by the global well-posedness for (CNS) in [5,8], we obtain that $\nabla u \in L^1((t_0,\infty);L^\infty)$ which yields (1.9) by recalling that $\nabla u \in L^2((0,\infty);L^\infty)$. □

2.4. Proof of Corollary 1.1

In the end of this section, we give a proof of Corollary 1.1.

Proof. We first claim that if $\|\nabla u\|_{L^1(0,\infty;L^\infty)} < \infty$, then (1.5) holds. To see that, we remark that (1.5) is a consequence of Proposition 2.1, Proposition 2.2, Proposition 2.3 and Proposition 2.6.

Thanks to the assumption $\|\nabla u\|_{L^1((0,\infty);L^\infty)} < \infty$, it is easy to prove that there exist two constants, c_1 and c_2, such that, for $t \geq 0$,

$$c_1 \leq \rho(t) \leq c_2, \tag{2.43}$$

where c_1 and c_2 depend on $\|\nabla u\|_{L^1(0,\infty;L^\infty)} < \infty$ and ρ_0. This implies that Proposition 2.1 and Proposition 2.2 are still valid. To get the result of Proposition 2.3, we only need to modify the original proof. Thanks to (2.27), we get that

$$\begin{aligned}
&\partial_t\|\nabla \mathfrak{a}\|_{L^p}^2 + \frac{1}{(\lambda+2\mu)p}\|\nabla \mathfrak{a}\|_{L^p}^2 \\
&\leqq C(\|\nabla u\|_{L^\infty}\|\nabla \mathfrak{a}\|_{L^p}^2 + \|\operatorname{div} u - \frac{1}{2\mu+\lambda}\mathfrak{a}\|_{W^{1,6}}^2\|\nabla \mathfrak{a}\|_{L^p}^2 \\
&\quad + \|\nabla(\operatorname{div} u - \frac{1}{\lambda+2\mu}\mathfrak{a})\|_{L^p}^2).
\end{aligned}$$

By taking $p = 6$, and using (2.16) and $\|\nabla u\|_{L^1((0,\infty);L^\infty)} < \infty$, we obtain from Gronwall's inequality that

$\|\mathfrak{a}\|_{L^\infty((0,+\infty);W^{1,6})\cap L^2((0,+\infty);W^{1,6})} \leqq C$. From which together with (2.25), we obtain that

$$\|\nabla u\|_{L^\infty} \leqq C(\|\nabla \mathcal{P}u\|_{W^{1,6}} + \|\operatorname{div} u - \tfrac{1}{2\mu+\lambda}\mathfrak{a}\|_{W^{1,6}} + \|\mathfrak{a}\|_{W^{1,6}}).$$

This implies that $\|\nabla u\|_{L^2((0,+\infty);L^\infty)} \leqq C$, and completes the proof of (2.22) and (2.23). In other words, now Proposition 2.3 is available. Recalling that Proposition 2.6 is derived through Proposition 2.1–2.3, we complete the proof of (1.5).

Next we turn to the proof of the corollary. Let α be a multi-index with $|\alpha| = m \geq 3$. Then it is easy to check that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha \mathfrak{a}\|_{L^2}^2 + (\partial^\alpha(u\cdot\nabla\mathfrak{a}), \partial^\alpha\mathfrak{a})\\
&\quad + \gamma(\partial^\alpha(\mathfrak{a}\operatorname{div} u), \partial^\alpha\mathfrak{a}) + \gamma(\partial^\alpha \operatorname{div} u, \partial^\alpha\mathfrak{a}) = 0,\\
&(\partial_t(\partial^\alpha(\rho u)), \partial^\alpha u) + (\partial^\alpha \operatorname{div}(\rho u\otimes u), \partial^\alpha u)\\
&\quad + \mu\|\nabla\partial^\alpha u\|_{L^2}^2 + \lambda\|\partial^\alpha \operatorname{div} u\|_{L^2}^2 + (\partial^\alpha\nabla\mathfrak{a}, \partial^\alpha u) = 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha \mathfrak{a}\|_{L^2}^2 + \gamma(\partial_t(\partial^\alpha(\rho u)), \partial^\alpha u) + (\partial^\alpha(u\cdot\nabla\mathfrak{a}), \partial^\alpha\mathfrak{a}) + \gamma(\partial^\alpha(\mathfrak{a}\operatorname{div} u), \partial^\alpha\mathfrak{a})\\
&\quad + \gamma(\partial^\alpha \operatorname{div}(\rho u\otimes u), \partial^\alpha u) + \gamma\mu\|\nabla\partial^\alpha u\|_{L^2}^2 + \gamma\lambda\|\partial^\alpha\operatorname{div} u\|_{L^2}^2 = 0.
\end{aligned}$$

Estimate of $(\partial_t(\partial^\alpha(\rho u)), \partial^\alpha u)$. We first observe that

$$\begin{aligned}
(\partial_t(\partial^\alpha(\rho u)), \partial^\alpha u) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\partial^\alpha u\|_{L^2}^2 + \frac{1}{2}(\rho_t\partial^\alpha u, \partial^\alpha u)\\
&\quad + \sum_{\alpha_1+\alpha_2=\alpha, |\alpha_1|\geq 1} (\partial_t(\partial^{\alpha_1}\rho\partial^{\alpha_2}u), \partial^\alpha u).
\end{aligned}$$

By integration by parts, one has

$$|(\rho_t\partial^\alpha u, \partial^\alpha u)| \leq \|\rho u\|_{L^\infty}\|\partial^\alpha u\|_{L^2}\|\nabla\partial^\alpha u\|_{L^2}.$$

Since $(\partial_t(\partial^{\alpha_1}\rho\partial^{\alpha_2}u), \partial^\alpha u) = (\partial^{\alpha_1}\rho_t\partial^{\alpha_2}u, \partial^\alpha u) + (\partial^{\alpha_1}\rho\partial^{\alpha_2}u_t, \partial^\alpha u)$, thanks to the fact that $|\alpha_1| \geq 1$, we have

$$\begin{aligned}
&|(\partial^{\alpha_1}\rho_t\partial^{\alpha_2}u, \partial^\alpha u)| = |(\partial^{\alpha_1}\operatorname{div}(\rho u)\partial^{\alpha_2}u, \partial^\alpha u)| = |(\partial^{\alpha_1}(\rho u)\nabla\partial^{\alpha_2}u, \partial^\alpha u)\\
&\quad + (\partial^{\alpha_1}(\rho u)\partial^{\alpha_2}u, \nabla\partial^\alpha u)|\\
&\lesssim (\|\rho u\|_{L^\infty}\|\nabla u\|_{H^m} + \|\nabla u\|_{L^\infty}\|\rho u\|_{H^m})\|\partial^\alpha u\|_{L^2} + \|\rho u\|_{H^m}\|\nabla u\|_{H^{\frac{1}{2}}}\|\partial^\alpha u\|_{H^1}\\
&\quad + (\|\rho u\|_{L^\infty}\|u\|_{H^m} + \|u\|_{L^\infty}\|\rho u\|_{H^m} + \|\nabla(\rho u)\|_{H^{\frac{1}{2}}}\|u\|_{H^m})\|\nabla\partial^\alpha u\|_{L^2}
\end{aligned}$$

and

$$|(\partial^{\alpha_1}\rho\partial^{\alpha_2}u_t, \partial^\alpha u)| \lesssim \|\nabla a\|_{L^6}\|u_t\|_{H^{m-1}}\|\partial^\alpha u\|_{H^{\frac{1}{2}}} + \|\nabla a\|_{H^{m-1}}\|u_t\|_{H^{\frac{1}{2}}}\|\partial^\alpha u\|_{H^1}.$$

Finally we obtain that

$$-(\partial_t(\partial^\alpha(\rho u)), \partial^\alpha u) \leq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\partial^\alpha u\|_{L^2}^2 + C\big[\eta^{-1}(\|\nabla u\|_{H^1}^2 + \|\nabla \mathfrak{a}\|_{H^1}^2 + \|\nabla u\|_{L^\infty}) \times(\|u\|_{H^m}^2 + \|\mathfrak{a}\|_{H^m}^2) + \eta\|\nabla\partial^\alpha u\|_{L^2}^2\big] + \|\nabla a\|_{L^6}\|u_t\|_{H^{m-1}}\|\partial^\alpha u\|_{H^{\frac{1}{2}}} + \|\nabla a\|_{H^{m-1}}\|u_t\|_{H^{\frac{1}{2}}}\|\partial^\alpha u\|_{H^1}.$$

Estimate of $(\partial^\alpha(u\cdot\nabla\mathfrak{a}), \partial^\alpha\mathfrak{a})$. Note that

$$(\partial^\alpha(u\cdot\nabla\mathfrak{a}), \partial^\alpha\mathfrak{a}) = (u\cdot\partial^\alpha\nabla\mathfrak{a}, \partial^\alpha\mathfrak{a}) + \sum_{\alpha_1+\alpha_2=\alpha, |\alpha_1|\geq 1}(\partial^{\alpha_1}u\cdot\nabla\partial^{\alpha_2}\mathfrak{a}, \partial^\alpha\mathfrak{a}).$$

Then we infer that

$$|(\partial^\alpha(u\cdot\nabla\mathfrak{a}), \partial^\alpha\mathfrak{a})| \leq \|\nabla u\|_{L^\infty}\|\partial^\alpha\mathfrak{a}\|_{L^2}^2 + (\|\nabla u\|_{L^\infty}\|\nabla\mathfrak{a}\|_{H^{m-1}} + \|\mathfrak{a}\|_{L^\infty}\|\nabla u\|_{H^m} + \|\nabla\mathfrak{a}\|_{H^{\frac{1}{2}}}\|\nabla u\|_{H^{m+1}})\|\partial^\alpha\mathfrak{a}\|_{L^2}.$$

Estimates of $(\partial^\alpha(\mathfrak{a}\mathrm{div}\, u), \partial^\alpha\mathfrak{a})$ and $(\partial^\alpha\mathrm{div}\,(\rho u\otimes u), \partial^\alpha u)$. We have

$$\begin{aligned}|(\partial^\alpha(\mathfrak{a}\mathrm{div}\, u), \partial^\alpha\mathfrak{a})| &\lesssim (\|\mathfrak{a}\|_{L^\infty}\|\nabla u\|_{H^m} + \|\nabla u\|_{L^\infty}\|\mathfrak{a}\|_{H^m})\|\partial^\alpha\mathfrak{a}\|_{L^2},\\ |(\partial^\alpha\mathrm{div}\,(\rho u\otimes u), \partial^\alpha u)| &= |(\partial^\alpha(\rho u\otimes u), \nabla\partial^\alpha u)|\\ &\lesssim (\|\rho u\|_{L^\infty}\|u\|_{H^m} + \|u\|_{L^\infty}\|\rho u\|_{H^m})\|\nabla\partial^\alpha u\|_{L^2}.\end{aligned}$$

Now, summing up all the above estimates, we finally arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathfrak{a}\|_{\dot{H}^m}^2 + \gamma\sum_{|\alpha|=m}\|\sqrt{\rho}\partial^\alpha u\|_{L^2}^2) + \gamma\mu\|\nabla u\|_{\dot{H}^m}^2\\ &\lesssim (\|\nabla u\|_{H^1}^2 + \|\nabla\mathfrak{a}\|_{H^1}^2 + \|\nabla u\|_{L^\infty})(\|u\|_{H^m}^2 + \|\mathfrak{a}\|_{H^m}^2) + \|\nabla a\|_{L^6}\|u_t\|_{H^{m-1}}\|\partial^\alpha u\|_{H^{\frac{1}{2}}}\\ &\quad + \|\nabla a\|_{H^{m-1}}\|u_t\|_{H^{\frac{1}{2}}}\|\partial^\alpha u\|_{H^1}.\end{aligned} \tag{2.44}$$

Observe that $\rho^{-1} = \frac{1}{1+a} = 1 - F(a)$ with $F(a) = \frac{a}{1+a}$. Then, by Lemma 5.6, we have

$$\begin{aligned}\|F(a)\mathrm{div}\, g\|_{H^{m-1}} &\lesssim \|\nabla g\|_{H^{m-1}} + \|g\|_{L^\infty}\|F(a)\|_{H^m}\\ &\lesssim \|\nabla g\|_{H^{m-1}} + C(\|a\|_{L^\infty})\|g\|_{L^\infty}\|a\|_{H^m}.\end{aligned}$$

This yields that

$$\begin{aligned}\|u_t\|_{H^{m-1}} &\leqq \|u\cdot\nabla u\|_{H^{m-1}} + \|\frac{1}{1+a}\left[-\mathrm{div}\left(2\mu D(u) + \lambda\mathrm{div}\, u\,\mathrm{Id}\right) + \nabla\mathfrak{a}\right]\|_{H^{m-1}}\\ &\leqq \|u\|_{W^{1,\infty}}\|u\|_{H^m} + \|\nabla u\|_{H^m} + \|\nabla\mathfrak{a}\|_{H^{m-1}} + (\|\nabla u\|_{L^\infty} + \|\mathfrak{a}\|_{L^\infty})\,\|a\|_{H^m}\end{aligned}$$

from which, together with (2.44), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathfrak{a}\|^2_{\dot H^m}+\gamma\sum_{|\alpha|=m}\|\sqrt{\rho}\partial^\alpha u\|^2_{L^2}\right)+\gamma\mu\|\nabla u\|^2_{\dot H^m}\\&\lesssim(\|\nabla u\|^2_{H^1}+\|\nabla\mathfrak{a}\|^2_{H^1}+\|\nabla u\|_{L^\infty}\\&\quad+\|\nabla a\|^2_{L^6}+\|\nabla u\|^2_{L^\infty}+\|\nabla a\|^4_{L^6})(\|u\|^2_{H^m}+\|a\|^2_{H^m}\\&\quad+\|\mathfrak{a}\|^2_{H^m})+\|\nabla u\|^2_{H^{m-1}}.\end{aligned}$$

From the facts that $\mathfrak{a}=(1+a)^\gamma-1$ and $a=(\mathfrak{a}+1)^{\frac{1}{\gamma}}-1$, we deduce from Lemma 5.6 that $\|a\|_{H^m}\sim\|\mathfrak{a}\|_{H^m}$, which implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathfrak{a}\|^2_{\dot H^m}+\gamma\sum_{|\alpha|=m}\|\sqrt{\rho}\partial^\alpha u\|^2_{L^2})+\gamma\mu\|\nabla u\|^2_{\dot H^m}\\&\lesssim\left[\|\nabla u\|^2_{H^2}+\|\nabla a\|^2_{H^1}(1+\|\nabla a\|^2_{H^1})+\|\nabla u\|_{L^\infty}\right](\|u\|^2_{H^m}+\|\mathfrak{a}\|^2_{H^m})+\|\nabla u\|^2_{H^{m-1}}.\end{aligned}$$

Thanks to (1.5), one has $\|\nabla u\|^2_{H^2}+\|\nabla a\|^2_{H^1}(1+\|\nabla a\|^2_{H^1})+\|\nabla u\|_{L^\infty}\in L^1([0,\infty])$, which enables us to use the inductive method to prove the desired result. Suppose that for $m\geq 3$, $a,u\in L^\infty([0,\infty];H^{m-1})$ and $\nabla u\in L^2([0,\infty];H^{m-1})$. The above inequality immediately implies that $a,u\in L^\infty([0,\infty];H^m)$ and $\nabla u\in L^2([0,\infty];H^m)$. This completes the proof for the corollary. □

3. Global-in-Time Stability for (CNS) System

In this section, we want to prove Theorem 1.2. The proof will fall into two steps:

(1) By the local well-posedness for the system (CNS), we can show that the perturbed solutions will remain close to the reference solutions for a long time if initially they are close.

(2) By the convergence result, we get that the reference solution is close to the equilibrium after a long time. This means that we can find a large time t_0, and at that moment the perturbed solutions are close to the equilibrium. Then it is not difficult to prove the global existence in the perturbation framework.

3.1. Setup of the Problem

Let $(\bar\rho,\bar u)$ be a global smooth solution for the (CNS) with the initial data $(\bar{\rho_0},\bar{u_0})$. And let (ρ,u) be the solution for the (CNS) associated the initial data (ρ_0,u_0), which satisfies (1.13). We denote $h=\rho-\bar\rho$ and $v=u-\bar u$, which satisfy the error equations as follows:

$$\begin{cases}\partial_t h+\mathrm{div}\,((h+\bar\rho)v)+\mathrm{div}\,(h\bar u)=0,\\ \partial_t v+v\cdot\nabla v-\frac{1}{\rho}(\mu\Delta v+(\lambda+\mu)\nabla\mathrm{div}\,v)+\gamma\rho^{\gamma-2}\nabla h=\frac{G}{\rho},\end{cases}$$

where

$$-G = h\bar{u}_t + hv \cdot \nabla \bar{u} + h\bar{u} \cdot \nabla v + h\bar{u} \cdot \nabla \bar{u} + \bar{\rho} v \cdot \nabla \bar{u} + \bar{\rho}\bar{u} \cdot \nabla v + \gamma(\rho^{\gamma-1} - \overline{\rho}^{\gamma-1})\nabla \overline{\rho}.$$

By a slight modification, we rewrite the above system as

$$\begin{cases} \partial_t h + (\bar{u} + v) \cdot \nabla h = -(h + \bar{\rho})\mathrm{div}\, v - h\mathrm{div}\, \bar{u}, \\ \partial_t v + v \cdot \nabla v - \mu \mathrm{div}\, \left(\frac{1}{\rho}\nabla v\right) - (\lambda + \mu)\nabla\left(\frac{1}{\rho}\mathrm{div}\, v\right) \stackrel{\mathrm{def}}{=} H, \end{cases} \tag{ERR}$$

where

$$\begin{aligned} H = &-\frac{1}{\rho}\big[h\bar{u}_t + hv \cdot \nabla \bar{u} + h\bar{u} \cdot \nabla v + h\bar{u} \cdot \nabla \bar{u} + \bar{\rho} v \cdot \nabla \bar{u} \\ &+ \gamma(\rho^{\gamma-1} - \overline{\rho}^{\gamma-1})\nabla \overline{\rho}\big] + \mu \frac{\nabla h}{\rho^2}\nabla v + (\mu + \lambda)\frac{\nabla h}{\rho^2}\mathrm{div}\, v \\ &- \gamma(\rho^{\gamma-2} - \bar{\rho}^{\gamma-2})\nabla h - \gamma\bar{\rho}^{\gamma-2}\nabla h + \left(\frac{\bar{\rho}\bar{u}}{\rho} - \frac{\bar{\rho}\bar{u}}{\bar{\rho}} - \mu\frac{\nabla\rho}{\rho^2} + \mu\frac{\nabla\bar{\rho}}{\bar{\rho}^2}\right) \cdot \nabla v \\ &+ \left(\frac{\bar{\rho}\bar{u}}{\bar{\rho}} - \mu\frac{\nabla\bar{\rho}}{\bar{\rho}^2}\right) \cdot \nabla v \\ &+ (\mu + \lambda)\frac{\nabla\bar{\rho}}{\bar{\rho}^2}\mathrm{div}\, v + (\mu+\lambda)\left(\frac{\nabla\bar{\rho}}{\rho^2} - \frac{\nabla\bar{\rho}}{\bar{\rho}^2}\right)\mathrm{div}\, v. \end{aligned}$$

To catch the dissipation structure of the system, we apply operators $\mathcal{Q}$ and $\mathcal{P}$ to the v-equation of (ERR) individually to obtain that

$$\begin{aligned} &\partial_t \mathcal{Q}v + v \cdot \nabla \mathcal{Q}v - \mu\mathrm{div}\, \left(\frac{1}{\rho}\nabla \mathcal{Q}v\right) - (\lambda + \mu)\nabla\left(\frac{1}{\rho}\mathrm{div}\, \mathcal{Q}v\right) \\ &\quad = \mu\mathrm{div}\, \left(\left[\mathcal{Q}, \frac{1}{\rho}\nabla\right]\right) v + (\lambda + \mu)\nabla\left[\mathcal{Q}, \frac{1}{\rho}\mathrm{div}\right] v + [\mathcal{Q}, v \cdot \nabla]v + \mathcal{Q}H, \\ &\partial_t \mathcal{P}v + v \cdot \nabla \mathcal{P}v - \mu\mathrm{div}\, \left(\frac{1}{\rho}\nabla \mathcal{P}v\right) = \mu\mathrm{div}\, \left(\left[\mathcal{P}, \frac{1}{\rho}\nabla\right]\right) v + [\mathcal{P}, v \cdot \nabla]v + \mathcal{P}H. \end{aligned} \tag{3.1}$$

Before proving the stability, we give the estimate to the term H. We have

Lemma 3.1. *Let* $(\bar{\rho}, \bar{u})$ *be the smooth solution for* (CNS) *satisfying* (1.12). *There exists a* ε_0 *such that, for any* $0 < \varepsilon \leqq \varepsilon_0$, *if*

$$\|h\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p})} \leqq \varepsilon^{\frac{1}{2}},$$

then it holds that

$$\|H\|_{L^1_T(\dot B^{\frac12,\frac3p-1}_{2,p})} \leqq C_1\left(1+\|h\|_{\widetilde L^\infty_T(\dot B^{\frac12,\frac3p}_{2,p})}+\|\mathcal{Q}v\|_{\widetilde L^\infty_T(\dot B^{\frac12,\frac3p-1}_{2,p})}+\|\mathcal{P}v\|_{\widetilde L^\infty_T(\dot B^{\frac3p-1}_{p,1})}\right)$$
$$\times\left(\|h\|_{L^1_T(\dot B^{\frac52,\frac3p}_{2,p})}+\|\mathcal{P}v\|_{L^1_T(\dot B^{\frac3p+1}_{p,1})}+\|\mathcal{Q}v\|_{L^1_T(\dot B^{\frac32,\frac3p}_{2,p})}\right),$$

where C_1 is a positive constant depending only on μ, λ, and C in (1.12).

Proof. We just establish the estimate to the term $\gamma(\rho^{\gamma-2}-\bar\rho^{\gamma-2})\nabla h$; all the other terms can be estimated similarly as to Proposition 5.1.

Note that $\rho^{\gamma-2}-\bar\rho^{\gamma-2}=(\gamma-2)h\int_0^1(\theta\rho+(1-\theta)\bar\rho)^{\gamma-3}\,\mathrm{d}\theta$; then we have

$$\|\gamma(\rho^{\gamma-2}-\bar\rho^{\gamma-2})\nabla h\|_{L^1_T(\dot B^{\frac12,\frac3p-1}_{2,p})}\leqq C\|h\nabla h\|_{L^1_T(\dot B^{\frac12,\frac3p-1}_{2,p})}.$$

Using Proposition 5.1 (b) with $s=\frac32$, $t=\frac12$, $\widetilde s=\frac12$, $\widetilde t=\frac32$, $\theta=0$ yields

$$\sum_{2^j\leqq R_0}2^{\frac12 j}\|\dot\Delta_j(h\nabla h)\|_{L^1_T(L^2)}\leqq C\|h\|_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}\|\nabla h\|_{\widetilde L^2_T(\dot B^{\frac12,\frac3p-1}_{2,p})}$$
$$+\,C\|\nabla h\|_{\widetilde L^2_T(\dot B^{\frac12,\frac3p-1}_{2,p})}\|h\|_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}\leqq C\|h\|^2_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}.$$

On the other hand, from Proposition 5.1 (a) with $\sigma=\frac3p$, $\tau=\frac3p-1$, it follows that

$$\sum_{2^j>R_0}2^{\frac3p-1j}\|\dot\Delta_j(h\nabla h)\|_{L^1_T(L^p)}\leqq C\|h\|_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}\|\nabla h\|_{\widetilde L^2_T(\dot B^{\frac12,\frac3p-1}_{2,p})}\leqq C\|h\|^2_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}.$$

Thus, we deduce that

$$\|h\nabla h\|_{L^1_T(\dot B^{\frac12,\frac3p-1}_{2,p})}\leqq C\|h\|^2_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}.$$

Noting the interpolation inequality

$$\|h\|^2_{\widetilde L^2_T(\dot B^{\frac32,\frac3p}_{2,p})}\leqq\|h\|^{\frac12}_{L^1_T(\dot B^{\frac52,\frac3p}_{2,p})}\|h\|^{\frac12}_{\widetilde L^\infty_T(\dot B^{\frac12,\frac3p}_{2,p})},$$

we have

$$\|h\nabla h\|_{L^1_T(\dot B^{\frac12,\frac3p-1}_{2,p})}\leqq C\|h\|_{L^1_T(\dot B^{\frac52,\frac3p}_{2,p})}\|h\|_{\widetilde L^\infty_T(\dot B^{\frac12,\frac3p}_{2,p})}.$$

□

3.2. Long Time Existence of (ERR)

We want to prove that if the initial data of (ERR) is small, then its associated solution will be still small during a long time interval. More precisely, we have the following proposition:

Proposition 3.1. *Let* $(\bar{\rho}, \bar{u})$ *associated with initial data* $(\bar{\rho}_0, \bar{u}_0)$ *be a global solution of* (CNS) *satisfying* (1.12). *Given an* $\varepsilon > 0$, *if the initial data of (ERR) are determined by the following inequality:*

$$\|(h_0, \mathcal{Q}v_0)\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\mathcal{Q}v_0\|^H_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|h_0\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|\mathcal{P}v_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leqq \varepsilon, \tag{3.2}$$

then there exists a constant δ *independent of* ε, *such that for any* $t \in [0, \delta|\ln\varepsilon|]$, *it holds that*

$$\|(h, \mathcal{Q}v)(t)\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\mathcal{Q}v(t)\|^H_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|h(t)\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|\mathcal{P}v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leqq \varepsilon^{\frac{1}{2}}.$$

Remark 3.1. The assumption (3.2) comes directly from (1.13).

Proof. We use the continuity argument to prove the desired result. Let $\mathcal{T}$ be the maximum time such that for any $t \in [0, \mathcal{T}]$, it holds that

$$\|(h, \mathcal{Q}v)(t)\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\mathcal{Q}v(t)\|^H_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|h(t)\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|\mathcal{P}v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leqq \varepsilon^{\frac{1}{2}}.$$

The existence of $\mathcal{T}$ can be obtained by the local well-posedness for the system. Then the proof of Proposition 3.1 is reduced to prove that $\mathcal{T} \geqq \delta|\ln\varepsilon|$ where $\delta > 0$ is a constant independent of ε.

Step 1: Estimates for the transport equation. Recalling the equation of h, we have

$$\partial_t h + (v + \bar{u}) \cdot \nabla h + v \cdot \nabla\bar{\rho} + h \mathrm{div}\, \bar{u} + (h + \bar{\rho})\mathrm{div}\, v = 0,$$

and with $\dot{\Delta}_j$ acting on both sides, and multiplying by $|\dot{\Delta}_j h|^{p-2}\dot{\Delta}_j h$, we get

$$\begin{aligned}
&\partial_t \|\dot{\Delta}_j h\|^p_{L^p} - \int \mathrm{div}\,(v + \bar{u})|\dot{\Delta}_j h|^p \,\mathrm{d}x \\
&\quad\leqq C \int |[\dot{\Delta}_j, (v + \bar{u}) \cdot \nabla]h \cdot |\dot{\Delta}_j h|^{p-1}| \,\mathrm{d}x \\
&\qquad + C \int |\dot{\Delta}_j((h + \bar{\rho})\mathrm{div}\, v + h\mathrm{div}\, \bar{u})| \cdot |\dot{\Delta}_j h|^{p-1}| \,\mathrm{d}x,
\end{aligned}$$

which implies that

$$\begin{aligned}
\partial_t \|\dot{\Delta}_j h\|_{L^p} \leqq{}& C(\|\mathrm{div}\, \bar{u}\|_{L^\infty} + \|\mathrm{div}\, v\|_{L^\infty})\|\dot{\Delta}_j h\|_{L^p} + C\|[\dot{\Delta}_j, (v + \bar{u}) \cdot \nabla]h\|_{L^p} \\
&+ \|\dot{\Delta}_j((h + \bar{\rho})\mathrm{div}\, v + h\mathrm{div}\, \bar{u})|\|_{L^p}.
\end{aligned}$$

Thus, by the definition the Besov space, for any $t \in [0, \mathcal{T}]$ we have

$$
\begin{aligned}
\|h\|^H_{\widetilde{L}^\infty_t(\dot{B}^{\frac{3}{p}}_{p,1})} &\leqq \|h_0\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} \\
&+ \int_0^t \|\mathrm{div}\, \bar{u}\|_{L^\infty} \|h\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} \, \mathrm{d}\tau + \int_0^t \|h\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} \|v\|_{B^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} \, \mathrm{d}\tau \\
&+ C \int_0^t \sum_{2^j \geqq R_0} 2^{\frac{3j}{p}} \|[\dot{\Delta}_j, \bar{u} \cdot \nabla] h\|_{L^p} \, \mathrm{d}\tau + C \int_0^t \sum_{2^j \geqq R_0} 2^{\frac{3j}{p}} \|[\dot{\Delta}_j, v \cdot \nabla] h\|_{L^p} \, \mathrm{d}\tau \\
&+ C \int_0^t \|(h + \bar{\rho}) \mathrm{div}\, v + h \mathrm{div}\, \bar{u}\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} \, \mathrm{d}\tau, \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
\|h\|^L_{\widetilde{L}^\infty_t(\dot{B}^{\frac{1}{2}}_{2,1})} &\leqq \|h_0\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} \\
&+ \int_0^t \|\mathrm{div}\, \bar{u}\|_{L^\infty} \|h\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} \, \mathrm{d}\tau + \int_0^t \|h\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} \|v\|_{\dot{B}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} \, \mathrm{d}\tau \\
&+ C \int_0^t \sum_{2^j \leqq R_0} 2^{\frac{j}{2}} \|[\dot{\Delta}_j, \bar{u} \cdot \nabla] h\|_{L^2} \, \mathrm{d}\tau + C \int_0^t \sum_{2^j \leqq R_0} 2^{\frac{j}{2}} \|[\dot{\Delta}_j, v \cdot \nabla] h\|_{L^2} \, \mathrm{d}\tau \\
&+ C \int_0^t \|(h + \bar{\rho}) \mathrm{div}\, v + h \mathrm{div}\, \bar{u}\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} \, \mathrm{d}\tau. \qquad (3.4)
\end{aligned}
$$

Let us give estimates to the terms on the righthand side of (3.3) and (3.4). By Proposition 5.2, the commutators can be estimated as follows:

$$
\begin{aligned}
&\int_0^t \sum_{2^j \geqq R_0} 2^{\frac{3j}{p}} \|[\dot{\Delta}_j, \bar{u} \cdot \nabla] h\|_{L^p} \, \mathrm{d}\tau + \int_0^t \sum_{2^j \geqq R_0} 2^{\frac{3j}{p}} \|[\dot{\Delta}_j, v \cdot \nabla] h\|_{L^p} \, \mathrm{d}\tau \\
&\leqq C \int_0^t \Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|\mathcal{Q}v\|_{\dot{B}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} + \|\mathcal{P}v\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \Big) \|h\|_{\dot{B}^{\frac{1}{2}, \frac{3}{p}}_{2,p}} \, \mathrm{d}\tau,
\end{aligned} \qquad (3.5)
$$

and

$$
\begin{aligned}
&\int_0^t \sum_{2^j \leqq R_0} 2^{\frac{j}{2}} \|[\dot{\Delta}_j, \bar{u} \cdot \nabla] h\|_{L^2} \, \mathrm{d}\tau + \int_0^t \sum_{2^j \leqq R_0} 2^{\frac{j}{2}} \|[\dot{\Delta}_j, v \cdot \nabla] h\|_{L^2} \, \mathrm{d}\tau \\
&\leqq C \int_0^t \Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|\mathcal{Q}v\|_{\dot{B}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} + \|\mathcal{P}v\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \Big) \|h\|_{\dot{B}^{\frac{1}{2}, \frac{3}{p}}_{2,p}} \, \mathrm{d}\tau.
\end{aligned} \qquad (3.6)
$$

Also, by product estimates (Proposition 5.1), we can get

$$\begin{aligned}&\int_0^t \|(h+\bar\rho)\mathrm{div}\, v + h\mathrm{div}\,\bar u\|^L_{\dot B^{\frac12}_{2,1}}\,\mathrm{d}\tau + \int_0^t \|(h+\bar\rho)\mathrm{div}\, v + h\mathrm{div}\,\bar u\|^H_{\dot B^{\frac3p}_{p,1}}\,\mathrm{d}\tau\\
&\leqq C\int_0^t\Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac3p+1}_{p,1}} + \|\mathcal{Q}v\|_{\dot B^{\frac52,\frac3p+1}_{2,p}} + \|\mathcal{P}v\|_{\dot B^{\frac3p+1}_{p,1}}\Big)\|h\|_{\dot B^{\frac12,\frac3p}_{2,p}}\,\mathrm{d}\tau\\
&\quad + C\int_0^t\Big(\|h\|_{\dot B^{\frac12,\frac3p}_{2,p}} + \|\bar\rho-1\|_{\dot B^{\frac12,\frac3p}_{2,p}}+1\Big)\|\mathrm{div}\, v\|_{\dot B^{\frac32,\frac3p}_{2,p}}\,\mathrm{d}\tau. \qquad (3.7)\end{aligned}$$

Plugging (3.5), (3.6) and (3.7) into (3.3) and (3.4), we obtain that

$$\begin{aligned}\|h\|_{\widetilde L^\infty_t(\dot B^{\frac12,\frac3p}_{2,p})} &\leqq \|h_0\|_{\dot B^{\frac12,\frac3p}_{2,p}} + C\int_0^t\Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}\\
&\quad + \|\mathcal{P}\bar u\|_{\dot B^{\frac3p+1}_{p,1}} + \|\mathcal{Q}v\|_{\dot B^{\frac52,\frac3p+1}_{2,p}} + \|\mathcal{P}v\|_{\dot B^{\frac3p+1}_{p,1}}\Big)\|h\|_{\dot B^{\frac12,\frac3p}_{2,p}}\,\mathrm{d}\tau\\
&\quad + C\int_0^t\Big(\|h\|_{\dot B^{\frac12,\frac3p}_{2,p}} + \|\bar\rho-1\|_{\dot B^{\frac12,\frac3p}_{2,p}}+1\Big)\|\mathcal{Q}v\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}\,\mathrm{d}\tau \qquad (3.8)\end{aligned}$$

for any $t\in[0,\mathcal{T}]$.

Step 2: Estimates of the momentum equation. First, we deal with the compressible part of velocity, $\mathcal{Q}v$. Applying the operator $\dot\Delta_j$ to the equation (3.1) and multiplying $|\dot\Delta_j\mathcal{Q}v|^{p-2}\dot\Delta_j\mathcal{Q}v$, we get that

$$\begin{aligned}&\frac1p\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j\mathcal{Q}v\|^p_{L^p} - \mu\int \mathrm{div}\,\Big(\frac1\rho\nabla\dot\Delta_j\mathcal{Q}v\Big)|\dot\Delta_j\mathcal{Q}v|^{p-2}\dot\Delta_j\mathcal{Q}v\,\mathrm{d}x\\
&-(\lambda+\mu)\int\nabla\Big(\frac1\rho\mathrm{div}\,\dot\Delta_j\mathcal{Q}v\Big)|\dot\Delta_j\mathcal{Q}v|^{p-2}\dot\Delta_j\mathcal{Q}v\,\mathrm{d}x\\
&=\int\dot\Delta_j\mathcal{Q}H|\dot\Delta_j\mathcal{Q}v|^{p-2}\dot\Delta_j\mathcal{Q}v\,\mathrm{d}x - \int\dot\Delta_j\mathcal{Q}(v\cdot\nabla v)\cdot|\dot\Delta_j\mathcal{Q}v|^{p-2}\dot\Delta_j\mathcal{Q}v\,\mathrm{d}x\\
&\quad + C\int\Big\{\mu\mathrm{div}\,\Big([\mathcal{Q}\dot\Delta_j,\frac1\rho\nabla]\Big)v + (\lambda+\mu)\nabla[\mathcal{Q}\dot\Delta_j,\frac1\rho\mathrm{div}\,]v\Big\}|\dot\Delta_j\mathcal{Q}v|^{p-1}\,\mathrm{d}x.\end{aligned}$$

By Lemma A.5 and Lemma A.6 in [11], we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j\mathcal{Q}v\|^p_{L^p} + c_p2^{2j}\|\dot\Delta_j\mathcal{Q}v\|^p_{L^p}\\
&\leqq C\int|\dot\Delta_j\mathcal{Q}H|\cdot|\dot\Delta_j v|^{p-1}\,\mathrm{d}x + C\int|\dot\Delta_j\mathcal{Q}(v\cdot\nabla v)|\cdot|\dot\Delta_j\mathcal{Q}v|^{p-1}\,\mathrm{d}x\\
&\quad + C\int\Big|\mu\mathrm{div}\,\Big(\Big[\mathcal{Q}\dot\Delta_j,\frac1\rho\nabla\Big]\Big)v + (\lambda+\mu)\nabla\Big[\mathcal{Q}\dot\Delta_j,\frac1\rho\mathrm{div}\Big]v\Big|\cdot|\dot\Delta_j\mathcal{Q}v|^{p-1}\,\mathrm{d}x.\end{aligned}$$

Thus, by the definition of Besov space, for any $t \in [0, \mathcal{T}]$, we obtain that

$$\begin{aligned}
&\|\mathcal{Q}v\|_{\widetilde{L}_t^\infty(\dot{B}_{2,p}^{\frac{1}{2},\frac{3}{p}-1})} + \|\mathcal{Q}v\|_{L_t^1(\dot{B}_{2,p}^{\frac{5}{2},\frac{3}{p}+1})} \\
&\quad \leqq C\|\mathcal{Q}v_0\|_{\dot{B}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}} + C\|v\cdot\nabla v\|_{L_t^1(\dot{B}_{2,p}^{\frac{1}{2},\frac{3}{p}-1})} + C\|\mathcal{Q}H\|_{L_t^1(\dot{B}_{2,p}^{\frac{1}{2},\frac{3}{p}-1})} \\
&\qquad + C\int_0^t \sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{1}{\rho}\nabla\right]v\right\|_{L^2} \mathrm{d}\tau + C\int_0^t \sum_{2^j\geqq R_0} 2^{j\frac{3}{p}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{1}{\rho}\nabla\right]v\right\|_{L^p} \mathrm{d}\tau.
\end{aligned} \tag{3.9}$$

For the commutators, applying Proposition 5.2 and Lemma 5.5 yields that

$$\begin{aligned}
&\int_0^t \left(\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{1}{\rho}\nabla\right]v\right\|_{L^2} + \sum_{2^j\geqq R_0} 2^{\frac{3j}{p}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{1}{\rho}\nabla\right]v\right\|_{L^p}\right)\mathrm{d}\tau \\
&\quad \leqq C\int_0^t \left(\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \left(\frac{1}{\bar\rho}-1\right)\nabla\right]\mathcal{P}v\right\|_{L^2}\right. \\
&\qquad \left. + \sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{h}{\rho\bar\rho}\nabla\right]\mathcal{P}v\right\|_{L^2}\right)\mathrm{d}\tau \\
&\qquad + C\int_0^t \left(\sum_{2^j\geqq R_0} 2^{j(\frac{3}{p}-1)}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \left(\frac{1}{\bar\rho}-1\right)\nabla\right]\mathcal{P}v\right\|_{L^p}\right. \\
&\qquad \left. + \sum_{2^j\geqq R_0} 2^{\frac{3j}{p}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{h}{\rho\bar\rho}\nabla\right]\mathcal{P}v\right\|_{L^p}\right)\mathrm{d}\tau \\
&\qquad + C\int_0^t \left(\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \left(\frac{1}{\bar\rho}-1\right)\nabla\right]\mathcal{Q}v\right\|_{L^2}\right. \\
&\qquad \left. + \sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{h}{\rho\bar\rho}\nabla\right]\mathcal{Q}v\right\|_{L^2}\right)\mathrm{d}\tau \\
&\qquad + C\int_0^t \left(\sum_{2^j\geqq R_0} 2^{j(\frac{3}{p}-1)}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \left(\frac{1}{\bar\rho}-1\right)\nabla\right]\mathcal{Q}v\right\|_{L^p}\right. \\
&\qquad \left. + \sum_{2^j\geqq R_0} 2^{\frac{3j}{p}}\left\|\left[\dot{\Delta}_j\mathcal{Q}, \frac{h}{\rho\bar\rho}\nabla\right]\mathcal{Q}v\right\|_{L^p}\right)\mathrm{d}\tau \\
&\quad \leqq C\int_0^t \left(\|\bar\rho-1\|_{\dot{B}_{p,1}^{\frac{3}{p}}\cap\dot{B}_{p,1}^{\frac{3}{p}+1}}\right. \\
&\qquad \left. + \|h\|_{\dot{B}_{2,p}^{\frac{5}{2},\frac{3}{p}}}\right)\left(\|\mathcal{Q}v\|_{\dot{B}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}} + \|\mathcal{P}v\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}\right)\mathrm{d}\tau \\
&\qquad + \varepsilon^{\frac{1}{2}}\left(\|\mathcal{Q}v\|_{L_t^1(\dot{B}_{2,p}^{\frac{5}{2},\frac{3}{p}+1})} + \|\mathcal{P}v\|_{L_t^1(\dot{B}_{p,1}^{\frac{3}{p}+1})}\right).
\end{aligned} \tag{3.10}$$

Using Proposition 5.1, we can get that

$$\|v\cdot\nabla v\|_{L^1_t(\dot B^{\frac12,\frac3p-1}_{2,p})} \leqq C\int_0^t \Big(\|\mathcal{P}v\|_{\dot B^{\frac3p-1}_{p,1}} + \|\mathcal{Q}v\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}\Big)\left(\|\mathcal{P}v\|_{\dot B^{\frac3p+1}_{p,1}} + \|\mathcal{Q}v\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}\right)\mathrm{d}\tau. \tag{3.11}$$

Plugging estimates (3.10), (3.11) and estimates of H (Lemma 3.1) into (3.9), and noting that

$$\|h\|_{\dot B^{\frac52,\frac3p}_{2,p}} \leqq C\|h\|_{\dot B^{\frac12,\frac3p}_{2,p}},$$

we obtain that

$$\begin{aligned}
&\|\mathcal{Q}v\|_{\widetilde L^\infty_t(\dot B^{\frac12,\frac3p-1}_{2,p})} + \|\mathcal{Q}v\|_{L^1_t(\dot B^{\frac52,\frac3p+1}_{2,p})} \leqq C\|\mathcal{Q}v_0\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}\\
&\quad + \varepsilon^{\frac12}\left(\|\mathcal{Q}v\|_{L^1_t(\dot B^{\frac52,\frac3p+1}_{2,p})} + \|\mathcal{P}v\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})}\right)\\
&\quad + C\int_0^t \left(\|\mathcal{P}v\|_{\dot B^{\frac3p-1}_{p,1}} + \|\mathcal{Q}v\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} + \|h\|_{\dot B^{\frac12,\frac3p}_{2,p}}\right)\mathrm{d}\tau
\end{aligned} \tag{3.12}$$

for any $t\in[0,\mathcal{T}]$.

Next, we establish estimates of incompressible part, $\mathcal{P}v$. Applying the operator $\dot\Delta_j$ to both sides of (3.1) and multiplying $|\dot\Delta_j\mathcal{P}v|^{p-2}\dot\Delta_j\mathcal{P}v$, we get that

$$\begin{aligned}
&\frac1p\partial_t\|\dot\Delta_j\mathcal{P}v\|^p_{L^p} - \mu\int \mathrm{div}\left(\frac1\rho\nabla\dot\Delta_j\mathcal{P}v\right)|\dot\Delta_j\mathcal{P}v|^{p-2}\dot\Delta_j\mathcal{P}v\,\mathrm{d}x\\
&\quad= \int \dot\Delta_j\mathcal{P}H|\dot\Delta_j\mathcal{P}v|^{p-2}\dot\Delta_j\mathcal{P}v\,\mathrm{d}x - \int \dot\Delta_j\mathcal{P}(v\cdot\nabla v)\cdot|\dot\Delta_j\mathcal{P}v|^{p-2}\dot\Delta_j\mathcal{P}v\,\mathrm{d}x\\
&\qquad+ \int \mu\mathrm{div}\,([\mathcal{P}\dot\Delta_j,\frac1\rho\nabla])v\cdot|\dot\Delta_j\mathcal{P}v|^{p-2}\dot\Delta_j\mathcal{P}v\,\mathrm{d}x,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\partial_t\|\dot\Delta_j\mathcal{P}v\|^p_{L^p} + c_p2^{2j}\|\dot\Delta_j\mathcal{P}v\|^p_{L^p}\\
&\quad\leqq C\left(\int|\dot\Delta_j\mathcal{P}H||\dot\Delta_j\mathcal{P}v|^{p-1}\,\mathrm{d}x + \int|\dot\Delta_j\mathcal{P}(v\cdot\nabla v)|\cdot|\dot\Delta_j\mathcal{P}v|^{p-1}\,\mathrm{d}x\right)\\
&\qquad+\mu\int|\mathrm{div}\,([\mathcal{P}\dot\Delta_j,\frac1\rho\nabla])v|\cdot|\dot\Delta_j\mathcal{P}v|^{p-1}\,\mathrm{d}x.
\end{aligned}$$

Thus, by the definition of Besov space, for any $t\in[0,\mathcal{T}]$, we derive that

$$\begin{aligned}
&\|\mathcal{P}v\|_{\widetilde L^\infty_t(\dot B^{\frac3p-1}_{p,1})} + \|\mathcal{P}v\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})}\\
&\quad\leqq C\|\mathcal{P}v_0\|_{\dot B^{\frac3p-1}_{p,1}} + C\|v\cdot\nabla v\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})} + C\|\mathcal{P}H\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\\
&\qquad+ C\int_0^t\sum_{j\in\mathbb{N}}2^{j\frac3p}\|[\dot\Delta_j\mathcal{P},\frac1\rho\nabla]v\|_{L^2}\,\mathrm{d}\tau.
\end{aligned} \tag{3.13}$$

By the same argument as in the proof of inequality (3.10), we have

$$
\begin{aligned}
\int_0^t \Big(\sum_{j\in\mathbb{N}} 2^{\frac{3j}{p}} \|[\dot\Delta_j \mathcal{P}, \frac{1}{\rho}\nabla]v\|_{L^p}\Big)\, \mathrm{d}x &\leqq C\Big(\|\mathcal{Q}v\|_{L^1_t(\dot B^{\frac12,\frac3p-1}_{2,p})} \\
&+ \|\mathcal{P}v\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\Big) \\
&+ \varepsilon^{\frac12}\Big(\|\mathcal{Q}v\|_{L^1_t(\dot B^{\frac52,\frac3p+1}_{2,p})} + \|\mathcal{P}v\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})}\Big).
\end{aligned} \tag{3.14}
$$

Plugging estimates (3.14) and (3.11) into (3.13) will imply that, for any $t \in [0, \mathcal{T}]$, it holds that

$$
\begin{aligned}
&\|\mathcal{P}v\|_{\widetilde L^\infty_t(\dot B^{\frac3p-1}_{p,1})} + \|\mathcal{P}v\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})} \\
&\quad\leqq C\|\mathcal{P}v_0\|_{\dot B^{\frac3p-1}_{p,1}} + C\int_0^t \Big(\|\mathcal{P}v\|_{\dot B^{\frac3p-1}_{p,1}} + \|\mathcal{Q}v\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} + \|h\|_{\dot B^{\frac12,\frac3p}_{2,p}}\Big)\mathrm{d}\tau \\
&\qquad + \varepsilon^{\frac12}\Big(\|\mathcal{Q}v\|_{L^1_t(\dot B^{\frac52,\frac3p+1}_{2,p})} + \|\mathcal{P}v\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})}\Big).
\end{aligned} \tag{3.15}
$$

Step 3: Closing the energy estimate. Combining estimates (3.8), (3.12) and (3.15), and choosing a suitable δ_1 such that $\delta_1 C \leqq \frac12$, we obtain that

$$
\begin{aligned}
&\delta_1\|h\|_{\widetilde L^\infty_t(\dot B^{\frac12,\frac3p}_{2,p})} + \|\mathcal{Q}v\|_{\widetilde L^\infty_t(\dot B^{\frac12,\frac3p-1}_{2,p})} \\
&\quad + \|\mathcal{P}v\|_{\widetilde L^\infty_t(\dot B^{\frac3p-1}_{p,1})} \leqq C\Big(\|h_0\|_{\dot B^{\frac12,\frac3p}_{p,1}} + \|\mathcal{Q}v_0\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} + \|\mathcal{P}v_0\|_{\dot B^{\frac3p-1}_{p,1}}\Big) \\
&\quad + C\int_0^t \Big(\delta_1\|h\|_{\dot B^{\frac12,\frac3p}_{2,p}} + \|\mathcal{Q}v\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} + \|\mathcal{P}v\|_{\dot B^{\frac3p-1}_{p,1}}\Big)\mathrm{d}\tau
\end{aligned}
$$

for any $t \in [0, \mathcal{T}]$. By the Gronwall's inequality, we get that for any $t \in [0, \mathcal{T}]$, it holds that

$$
\begin{aligned}
&\delta_1\|h\|_{\widetilde L^\infty_t(\dot B^{\frac12,\frac3p}_{2,p})} + \|\mathcal{Q}v\|_{\widetilde L^\infty_t(\dot B^{\frac12,\frac3p-1}_{2,p})} + \|\mathcal{P}v\|_{\widetilde L^\infty_t(\dot B^{\frac3p-1}_{p,1})} \\
&\quad\leqq C\Big(\|h_0\|_{\dot B^{\frac12,\frac3p}_{p,1}} + \|\mathcal{Q}v_0\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} + \|\mathcal{P}v_0\|_{\dot B^{\frac3p-1}_{p,1}}\Big)e^t.
\end{aligned}
$$

According to the definition of $\mathcal{T}$, this implies that $\mathcal{T} \geqq \delta|\ln\varepsilon|$ for a suitable δ independent of ε. Then the proof is completed. □

3.3. Proof of Theorem 1.2

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. First, thanks to Theorem 1.1, we can choose $t_0 = \frac{1}{2}(1 + |\delta \ln \varepsilon|)$ such that

$$\|(\bar{\rho} - 1)(t_0)\|_{\dot{B}^{\frac{1}{2},\frac{3}{2}}_{2,2}} + \|\bar{u}(t_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \lesssim (1 + |\delta \ln \varepsilon|)^{-\beta(p_0)/2}.$$

Recall that $\rho - 1 = h + (\bar{\rho} - 1)$, $u = v + \bar{u}$, then from Proposition 3.1, we derive that

$$\begin{aligned}\|(\rho - 1, \mathcal{Q}u)(t_0)\|^L_{\dot{B}^{\frac{1}{2}}_{2,1}} &+ \|\mathcal{Q}u(t_0)\|^H_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|(\rho - 1)(t_0)\|^H_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|(\mathcal{P}u)(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \\ &\lesssim \varepsilon^{\frac{1}{2}} + (1 + |\delta \ln \varepsilon|)^{-\beta(p_0)/2} \lesssim (1 + |\delta \ln \varepsilon|)^{-\beta(p_0)/2}.\end{aligned}$$

This means that at time t_0, the system (CNS) is in the close-to-equilibrium regime. Then thanks to the results in [8,29,30], we obtain the global existence for $(\rho - 1, u)$. Moreover, due to the definition of $\mathcal{T}$ and (1.7–1.8), we conclude that

$$\begin{aligned}&\delta_1 \|h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p})} + \|\mathcal{Q}v\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p})} \\ &\quad + \|\mathcal{P}v\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim \min\{(1 + |\delta \ln \varepsilon|)^{-\beta(p_0)/2}, (1 + t)^{-\beta(p_0)/2} + \varepsilon\}.\end{aligned}$$

This completes the proof to Theorem 1.2. □

4. Construction of a Global Solutions with a Class of Large Initial Data

iNSPIRED by [22,32], in this section, we construct a global solution for compressible Navier–Stokes equations with the vertical component of the initial data $(\mathcal{P}u_0)^3$ could be arbitrarily large.

4.1. Reduction of the Problem

Given $a_0 \in \dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}$, $\mathcal{Q}u_0 \in \dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}$ and $\mathcal{P}u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$ with $\|a_0\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}}$ being sufficiently small, it follows by a similar argument as that in [7,8] that there exists a positive time T so that (CNS) has a unique solution (a, u) with

$$\begin{aligned}&a \in \mathcal{C}((0,T]; \dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}), \qquad \mathcal{Q}u \in L^\infty((0,T]; \dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}) \cap L^1((0,T); \dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}), \\ &\mathcal{P}u \in L^\infty((0,T]; \dot{B}^{\frac{3}{p}-1}_{p,1}) \cap L^1((0,T); \dot{B}^{\frac{3}{p}+1}_{p,1}). \qquad (4.1)\end{aligned}$$

Next we only need to give an *a priori* estimate to the solution. Observe that the system (CNS) can be recast as

$$\begin{cases} \partial_t a + u\cdot\nabla a + \operatorname{div}\mathcal{Q}u = -a\operatorname{div} u, \\ \partial_t \mathcal{Q}u + u\cdot\nabla\mathcal{Q}u - \mu\Delta\mathcal{Q}u - (\mu+\lambda)\nabla\operatorname{div}\mathcal{Q}u + \gamma\nabla a = \mathcal{Q}W_{\mathcal{Q}}, \\ \partial_t(\mathcal{P}u)^{hor} + u\cdot\nabla(\mathcal{P}u)^{hor} - \Delta(\mathcal{P}u)^{hor} = [\mathcal{P}, u\cdot\nabla]u^{hor} + (\mathcal{P}W_{\mathcal{P}})^{hor}, \\ \partial_t(\mathcal{P}u)^3 + \mathcal{P}(u\cdot\nabla u^3) - \Delta(\mathcal{P}u)^3 = (\mathcal{P}W_{\mathcal{P}})^3, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} W_{\mathcal{Q}} = &-\frac{\nabla a}{(1+a)^2}\nabla u - \frac{a\nabla a}{1+a} + \mu\operatorname{div}\left((\frac{1}{1+a}-1)\nabla u\right) \\ &+ (\mu+\lambda)\nabla\left((\frac{1}{1+a}-1)\operatorname{div} u\right) + [\mathcal{Q}, u\cdot\nabla]u + \gamma(\rho^{\gamma-1}-1)\nabla a, \end{aligned}$$

and

$$W_{\mathcal{P}} = -\frac{\nabla a}{(1+a)^2}\nabla u + \mu\operatorname{div}\left((\frac{1}{1+a}-1)\nabla u\right).$$

Now, to establish uniform estimates for the solution, we first give estimates to $\mathcal{Q}W_{\mathcal{Q}}$ and $\mathcal{P}W_{\mathcal{P}}$.

Lemma 4.1. *There exists a universal constant C such that*

$$\begin{aligned} \|\mathcal{Q}W_{\mathcal{Q}}\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} \leqq{} & C(\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}} + \|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}} \\ &+ \|a\|_{\dot B^{\frac52,\frac3p}_{2,p}})\left(\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}} + \|\mathcal{Q}u\|_{\dot B^{\frac12,\frac3p-1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\right), \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{P}W_{\mathcal{P}}\|_{\dot B^{\frac3p-1}_{p,1}} \leqq{} & C\|a\|_{B^{\frac12,\frac3p}_{2,p}}(\|\mathcal{Q}u\|_{B^{\frac52,\frac3p+1}_{2,p}} + \|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}}) \\ &+ C\|a\|_{B^{\frac52,\frac3p}_{2,p}}\left(\|\mathcal{Q}u\|_{B^{\frac12,\frac3p-1}_{2,p}} + \|\mathcal{P}u\|_{\dot B^{\frac3p-1}_{p,1}}\right). \end{aligned}$$

Proof. For $\mathcal{Q}W_{\mathcal{Q}}$ and $\mathcal{P}W_{\mathcal{P}}$, the most difficult term is $[\mathcal{Q}, u\cdot\nabla]u$; the others can be estimated by the Propositions 5.1–5.4 directly, so we just focus on the term $[\mathcal{Q}, u\cdot\nabla]u$.

Because $u = \mathcal{Q}u + \mathcal{P}u$, we have,

$$\begin{aligned} [\mathcal{Q}, u\cdot\nabla]u &= [\mathcal{Q}, (\mathcal{Q}u+\mathcal{P}u)\cdot\nabla](\mathcal{Q}u+\mathcal{P}u) \\ &= [\mathcal{Q}, (\mathcal{Q}u)\cdot\nabla]\mathcal{Q}u + [\mathcal{Q}, (\mathcal{Q}u)\cdot\nabla]\mathcal{P}u + [\mathcal{Q}, (\mathcal{P}u)\cdot\nabla]\mathcal{Q}u + [\mathcal{Q}, (\mathcal{P}u)\cdot\nabla]\mathcal{P}u. \end{aligned}$$

Here, by Propositions 5.1–5.4, it is not difficult to get that

$$
\begin{aligned}
&\|[\mathcal{Q}, (\mathcal{Q}u)\cdot\nabla]\mathcal{Q}u + [\mathcal{Q}, (\mathcal{Q}u)\cdot\nabla]\mathcal{P}u + [\mathcal{Q}, (\mathcal{P}u)\cdot\nabla]\mathcal{Q}u \\
&\quad + [\mathcal{Q}, (\mathcal{P}u)\cdot\nabla](\mathcal{P}u)^{hor}\|_{L^{\frac{1}{2}}_{\dot{B}_{2,1}}} \\
&\leqq C\left(\|\mathcal{Q}u\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} + \|\mathcal{P}u\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\right)\left(\|\mathcal{Q}u\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\right).
\end{aligned}
$$

For the remainder term $[\mathcal{Q}, (\mathcal{P}u)\cdot\nabla](\mathcal{P}u)^3$, by using $\operatorname{div}\mathcal{P}u = 0$, we can obtain

$$
\begin{aligned}
[\mathcal{Q}, (\mathcal{P}u)\cdot\nabla](\mathcal{P}u)^3 &= \mathcal{Q}((\mathcal{P}u)\cdot\nabla(\mathcal{P}u)^3) \\
&= \mathcal{Q}\big((\mathcal{P}u)^{hor}\cdot\nabla_h(\mathcal{P}u)^3 - (\mathcal{P}u)^3\operatorname{div}_h(\mathcal{P}u)^{hor}\big) \\
&= \mathcal{Q}\big((\mathcal{P}u)^{hor}\cdot\nabla_h(\mathcal{P}u)^3\big) - [\mathcal{Q}, (\mathcal{P}u)^3]\operatorname{div}_h(\mathcal{P}u)^{hor}.
\end{aligned}
$$

Thus, by Proposition 5.1–5.4, we complete the proof of this lemma. □

4.2. Proof of Theorem 1.3

We finally give the proof to Theorem 1.3.

Proof of Theorem 1.3. The proof of the theorem falls into four steps.

Step 1: Estimates for the low frequency part of the solution. Applying $\dot{\Delta}_j$ on the both side of the first equation of (4.2) and multiplying $\dot{\Delta}_j a$, we have

$$
\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j a\|_{L^2}^2 - \int \operatorname{div}\mathcal{Q}u|\dot{\Delta}_j a|^2\,\mathrm{d}x + \int \dot{\Delta}_j\operatorname{div}\mathcal{Q}u\cdot\dot{\Delta}_j a\,\mathrm{d}x \\
&\quad\leqq C\int |[\dot{\Delta}_j, u\cdot\nabla]a\cdot\dot{\Delta}_j a|\,\mathrm{d}x + C\int |\dot{\Delta}_j(a\operatorname{div}\mathcal{Q}u)\cdot\dot{\Delta}_j a|\,\mathrm{d}x,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j a\|_{L^2}^2 + \int \dot{\Delta}_j\operatorname{div}\mathcal{Q}u\cdot\dot{\Delta}_j a\,\mathrm{d}x \\
&\quad\leqq C\|\operatorname{div}\mathcal{Q}u\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}^2 + C\big(\|[\dot{\Delta}_j, u\cdot\nabla]a\|_{L^2} + \|\dot{\Delta}_j(a\operatorname{div}\mathcal{Q}u)\|_{L^2}\big)\|\dot{\Delta}_j a\|_{L^2}.
\end{aligned}
\tag{4.3}
$$

Next, taking $\Lambda^{-1}\operatorname{div}$ on the both sides of the second equation of (4.2) yields that

$$
\partial_t d + u\cdot\nabla d - (2\mu+\lambda)\Delta d - \gamma\Lambda a = \Lambda^{-1}\operatorname{div}\mathcal{Q}W_{\mathcal{Q}} - [\Lambda^{-1}\operatorname{div}, u\cdot\nabla]\mathcal{Q}u,
\tag{4.4}
$$

where $d = \Lambda^{-1}\operatorname{div}\mathcal{Q}u$.

Acting as the operator $\dot{\Delta}_j$ on the both sides of (4.4) and multiplying by $\dot{\Delta}_j d$, we get that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j d\|_{L^2}^2 + (2\mu+\lambda)\|\nabla\dot{\Delta}_j d\|_{L^2}^2 - \gamma\int \dot{\Delta}_j a\cdot\dot{\Delta}_j\Lambda d\,\mathrm{d}x \\ \leqq C\|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot{\Delta}_j d\|_{L^2}^2 + \|\dot{\Delta}_j d\|_{L^2}\|[\dot{\Delta}_j, u\cdot\nabla]d\|_{L^2} \\ + \|\dot{\Delta}_j d\|_{L^2}\big(\|\dot{\Delta}_j\mathcal{Q}W_{\mathcal{Q}}\|_{L^2} + \|\dot{\Delta}_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^2}\big).\end{aligned} \tag{4.5}$$

Now, we introduce a new auxiliary function $w \overset{def}{=} (2\mu+\lambda)\Lambda a - d$, which satisfies that

$$\begin{aligned}&\partial_t w + u\cdot\nabla w + \gamma\Lambda a \\ &\quad = -(2\mu+\lambda)[\Lambda, u\cdot\nabla]a - (2\mu+\lambda)\Lambda(a\mathrm{div}\,u) \\ &\qquad - \Lambda^{-1}\mathrm{div}\,\mathcal{Q}W_{\mathcal{Q}} + [\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u.\end{aligned} \tag{4.6}$$

Applying $\dot{\Delta}_j$ on the both sides of (4.6) and multiplying by $\dot{\Delta}_j w$, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j w\|_{L^2}^2 + (2\mu+\lambda)\gamma\|\dot{\Delta}_j\Lambda a\|_{L^2}^2 - \gamma\int \dot{\Delta}_j a\cdot\dot{\Delta}_j\Lambda d\,\mathrm{d}x \\ &\leqq C\Big(\|\mathrm{div}\,u\|_{L^\infty}\|\dot{\Delta}_j w\|_{L^2}^2 + \|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|\dot{\Delta}_j w\|_{L^2}\Big) \\ &\quad + C\Big(\|\dot{\Delta}_j[\Lambda, u\cdot\nabla]a\|_{L^2} + \|\dot{\Delta}_j\Lambda(a\mathrm{div}\,u)\|_{L^2} + \|\dot{\Delta}_j\mathcal{Q}W_{\mathcal{Q}}\|_{L^2}\Big)\|\dot{\Delta}_j w\|_{L^2} \\ &\quad + C\|\dot{\Delta}_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2}.\end{aligned} \tag{4.7}$$

Putting together estimates (4.3), (4.5) and (4.7), we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\gamma\|\dot{\Delta}_j a\|_{L^2}^2 + (1-\delta)\|\dot{\Delta}_j d\|_{L^2}^2 + \delta\|\dot{\Delta}_j w\|_{L^2}^2\Big) + \|\nabla\dot{\Delta}_j d\|_{L^2}^2 + \delta\|\Lambda\dot{\Delta}_j a\|_{L^2}^2 \\ &\leqq C\|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}^2 + C\Big(\|[\dot{\Delta}_j, u\cdot\nabla]a\|_{L^2} + \|\dot{\Delta}_j(a\mathrm{div}\,\mathcal{Q}u)|\|_{L^2}\Big)\|\dot{\Delta}_j a\|_{L^2} \\ &\quad + C\|\dot{\Delta}_j d\|_{L^2}\Big(\|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot{\Delta}_j d\|_{L^2} + \|[\dot{\Delta}_j, u\cdot\nabla]d\|_{L^2}\Big) \\ &\quad + C\Big(\|\dot{\Delta}_j\mathcal{Q}W_{\mathcal{Q}}\|_{L^2} + \|\dot{\Delta}_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^2}\Big)\|\dot{\Delta}_j d\|_{L^2} \\ &\quad + C\delta\Big(\|\mathrm{div}\,u\|_{L^\infty}\|\dot{\Delta}_j w\|_{L^2}^2 + \|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|\dot{\Delta}_j w\|_{L^2}\Big) \\ &\quad + C\delta\Big(\|\dot{\Delta}_j[\Lambda, u\cdot\nabla]a\|_{L^2} + \|\dot{\Delta}_j\Lambda(a\mathrm{div}\,u)\|_{L^2} + \|\dot{\Delta}_j\mathcal{Q}W_{\mathcal{Q}}\|_{L^2}\Big)\|\dot{\Delta}_j w\|_{L^2} \\ &\quad + C\delta\|\dot{\Delta}_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2}.\end{aligned} \tag{4.8}$$

When $2^j \leqq R_0$, it holds that $\|\Lambda\dot{\Delta}_j a\|_{L^2} \leqq R_0\|\dot{\Delta}_j a\|_{L^2}$. Thus, we could find a $\delta > 0$ (small enough) such that

$$\|\dot{\Delta}_j a\|_{L^2}^2 + (1-\delta)\|\dot{\Delta}_j d\|_{L^2}^2 + \delta\|\dot{\Delta}_j w\|_{L^2}^2 \geqq \frac{1}{C}(\|\dot{\Delta}_j a\|_{L^2}^2 + \|\dot{\Delta}_j d\|_{L^2}^2).$$

Integrating (4.8) over $[0, T]$, we get that

$$
\begin{aligned}
&\|\dot\Delta_j a\|_{L^2} + \|\dot\Delta_j d\|_{L^2} + 2^{2j}\int_0^T (\|\dot\Delta_j d\|_{L^2} + \delta\|\dot\Delta_j a\|_{L^2})\,\mathrm{d}t \\
&\leqq C\int_0^T \|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot\Delta_j a\|_{L^2}\,\mathrm{d}t \\
&\quad + C\int_0^T \Big(\|[\dot\Delta_j, u\cdot\nabla]a\|_{L^2} + \|\dot\Delta_j(a\mathrm{div}\,\mathcal{Q}u)\|_{L^2}\Big)\,\mathrm{d}t \\
&\quad + C\int_0^T \|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot\Delta_j d\|_{L^2}\,\mathrm{d}t \\
&\quad + C\int_0^T \Big(\|[\dot\Delta_j, u\cdot\nabla]d\|_{L^2} + \|\dot\Delta_j \mathcal{Q}W_{\mathcal{Q}}\|_{L^2}\Big)\,\mathrm{d}t \\
&\quad + C\int_0^T \Big(\|\dot\Delta_j[\Lambda, u\cdot\nabla]a\|_{L^2} + \|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla] \\
&\qquad \times\mathcal{Q}u\|_{L^2} + \|[\dot\Delta_j, u\cdot\nabla]\Lambda a\|_{L^2}\Big)\,\mathrm{d}t,
\end{aligned}
$$

where C depends on the μ, λ and R_0. By the definition of Besov space, we obtain

$$
\begin{aligned}
&\|a\|^L_{\dot B^{\frac12}_{2,1}} + \|d\|^L_{\dot B^{\frac12}_{2,1}} + \int_0^T (\|d\|^L_{\dot B^{\frac52}_{2,1}} + \delta\|a\|^L_{\dot B^{\frac52}_{2,1}})\,\mathrm{d}t \\
&\leqq C\int_0^T \|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|a\|^L_{\dot B^{\frac12}_{2,1}}\,\mathrm{d}t \\
&\quad + C\int_0^T \Big(\sum_{2^j\leqq R_0} 2^{\frac j2}\|[\dot\Delta_j, u\cdot\nabla]a\|_{L^2} + \|a\mathrm{div}\,\mathcal{Q}u\|^L_{\dot B^{\frac12}_{2,1}}\Big)\,\mathrm{d}t \\
&\quad + C\int_0^T \|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|d\|^L_{\dot B^{\frac12}_{2,1}}\,\mathrm{d}t \\
&\quad + C\int_0^T \Big(\sum_{2^j\leqq R_0} 2^{\frac j2}\|[\dot\Delta_j, u\cdot\nabla]d\|_{L^2} + \|\mathcal{Q}W_{\mathcal{Q}}\|^L_{\dot B^{\frac12}_{2,1}}\Big)\,\mathrm{d}t \\
&\quad + C\int_0^T \sum_{2^j\leqq R_0} 2^{\frac j2}\|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^2}\,\mathrm{d}t \\
&\quad + C\int_0^T \sum_{2^j\leqq R_0} 2^{\frac j2}\|\dot\Delta_j[\Lambda, u\cdot\nabla]a\|_{L^2}\,\mathrm{d}t \\
&\quad + C\int_0^T \sum_{2^j\leqq R_0} 2^{\frac j2}\|[\dot\Delta_j, u\cdot\nabla]\Lambda a\|_{L^2}\,\mathrm{d}t.
\end{aligned}
$$

Let us give estimates to terms in the righthand side one by one. Due to Proposition 5.2 and Lemma 5.5, we deduce that

$$\begin{aligned}\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\|[\dot\Delta_j, u\cdot\nabla]a\|_{L^2} &\leqq \sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\|[\dot\Delta_j, \mathcal{P}u\cdot\nabla]a\|_{L^2}\\ &\quad+\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\|[\dot\Delta_j, \mathcal{Q}u\cdot\nabla]a\|_{L^2}\\ &\leqq C(\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}})\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}. \end{aligned}\tag{4.9}$$

A similar argument yields that

$$\begin{aligned}&\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\|[\dot\Delta_j, u\cdot\nabla]d\|_{L^2}\leqq C(\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}})\|d\|_{\dot B^{\frac12,\frac3p-1}_{2,p}},\\ &\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^2}\leqq C(\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}})\|d\|_{\dot B^{\frac12,\frac3p-1}_{2,p}},\\ &\sum_{2^j\leqq R_0} 2^{\frac{j}{2}}\|\dot\Delta_j[\Lambda, u\cdot\nabla]a\|_{L^2}\leqq C(\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}})\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}.\end{aligned}$$

Finally, by Proposition 5.1, we have

$$\|a\mathrm{div}\,\mathcal{Q}u\|^L_{\dot B^{\frac12}_{2,1}}\leqq C\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}.$$

Now, putting all estimates together and applying Lemma 4.1, we have

$$\begin{aligned}&\|a\|^L_{\dot B^{\frac12}_{2,1}}+\|d\|^L_{\dot B^{\frac12}_{2,1}}+\int_0^T\left(\|d\|^L_{\dot B^{\frac52}_{2,1}}+\|a\|^L_{\dot B^{\frac52}_{2,1}}\right)\mathrm{d}t\leqq\|a_0\|^L_{\dot B^{\frac12}_{2,1}}+\|\mathcal{Q}u_0\|^L_{\dot B^{\frac12}_{2,1}}\\ &\quad+C\int_0^T\Bigg(\|\mathcal{Q}u\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}}\\ &\quad+\|a\|_{\dot B^{\frac52,\frac3p}_{2,p}}\Bigg)\Bigg(\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|\mathcal{Q}u\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\Bigg)\mathrm{d}t.\end{aligned}\tag{4.10}$$

Step 2: Estimates for the high frequency part of the solution. Applying $\dot\Delta_j$ on the both side of equation of d, (4.4), and taking the inner product with $|\dot\Delta_j d|^{p-2}\dot\Delta_j d$, we derive that

$$\begin{aligned}&\frac1p\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j d\|^p_{L^p}-(2\mu+\lambda)\int\Delta\dot\Delta_j d\cdot|\dot\Delta_j d|^{p-2}\dot\Delta_j d\,\mathrm{d}x\\ &\quad-\gamma\int\dot\Delta_j\Lambda a\cdot|\dot\Delta_j d|^{p-2}\dot\Delta_j d\,\mathrm{d}x\\ &\leqq C\|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot\Delta_j d\|^p_{L^p}+\|\dot\Delta_j d\|^{p-1}_{L^p}\|[\dot\Delta_j, u\cdot\nabla]d\|_{L^p}\\ &\quad+\|\dot\Delta_j d\|^{p-1}_{L^p}(\|\dot\Delta_j\mathcal{Q}W_{\mathcal{Q}}\|_{L^2}+\|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p}).\end{aligned}$$

By using Lemma A.5 and A.6 in [11], we have that

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j d\|_{L^p}^p + (c_p 2^{2j}-1)\|\dot\Delta_j d\|_{L^p}^p \\
&\quad \leqq C\|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot\Delta_j d\|_{L^p}^p + \|\dot\Delta_j d\|_{L^p}^{p-1}\|[\dot\Delta_j, u\cdot\nabla]d\|_{L^p} \\
&\qquad + \|\dot\Delta_j d\|_{L^p}^{p-1}(\|\dot\Delta_j \mathcal{Q}W_{\mathcal{Q}}\|_{L^p} + \|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p} + \|\dot\Delta_j w\|_{L^p}),
\end{aligned} \tag{4.11}$$

where c_p is a positive constant depending only on p.

By the same argument applied to (4.6), we get that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j w\|_{L^p}^p + \|\dot\Delta_j w\|_{L^p}^p \\
&\quad \leqq C\|\dot\Delta_j d\|_{L^p}\|\dot\Delta_j w\|_{L^p}^{p-1} \\
&\qquad + C\|\mathrm{div}\,u\|_{L^\infty}\|\dot\Delta_j w\|_{L^p}^p + C\|[\dot\Delta_j, u\cdot\nabla]w\|_{L^p}\|\dot\Delta_j w\|_{L^p}^{p-1} \\
&\qquad + C\Big(\|\dot\Delta_j[\Lambda, u\cdot\nabla]a\|_{L^p} + \|\dot\Delta_j\Lambda(a\mathrm{div}\,u)\|_{L^p} \\
&\qquad + \|\dot\Delta_j\Lambda^{-1}\mathrm{div}\,\mathcal{Q}W_{\mathcal{Q}}\|_{L^p}\Big)\|\dot\Delta_j w\|_{L^p}^{p-1} \\
&\qquad + C\|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p}\|\dot\Delta_j w\|_{L^p}^{p-1}.
\end{aligned} \tag{4.12}$$

Thanks to (4.11) and (4.12), we have

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\dot\Delta_j d\|_{L^p}^p + \delta\|\dot\Delta_j w\|_{L^p}^p\Big) + (c_p 2^{2j}-2)\|\dot\Delta_j d\|_{L^p}^p + \delta\|\dot\Delta_j w\|_{L^p}^p \\
&\quad \leqq 2\|\dot\Delta_j d\|_{L^p}\|\dot\Delta_j w\|_{L^p}^{p-1} + C\|\mathrm{div}\,\mathcal{Q}u\|_{L^\infty}\|\dot\Delta_j d\|_{L^p}^p \\
&\qquad + \|\dot\Delta_j d\|_{L^p}^{p-1}\|[\dot\Delta_j, u\cdot\nabla]d\|_{L^p} \\
&\qquad + \|\dot\Delta_j d\|_{L^p}^{p-1}\Big(\|\dot\Delta_j\mathcal{Q}W_{\mathcal{Q}}\|_{L^2} + \|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p}\Big) \\
&\qquad + C\|\mathrm{div}\,u\|_{L^\infty}\|\dot\Delta_j w\|_{L^p}^p \\
&\qquad + C\Big(\|[\dot\Delta_j, u\cdot\nabla]w\|_{L^p} + \|\dot\Delta_j[\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p}\Big)\|\dot\Delta_j w\|_{L^p}^{p-1} \\
&\qquad + C\Big(\|\dot\Delta_j[\Lambda, u\cdot\nabla]a\|_{L^p} + \|\dot\Delta_j\Lambda(a\mathrm{div}\,u)\|_{L^p} \\
&\qquad + \|\dot\Delta_j\Lambda^{-1}\mathrm{div}\,\mathcal{Q}W_{\mathcal{Q}}\|_{L^p}\Big)\|\dot\Delta_j w\|_{L^p}^{p-1}.
\end{aligned} \tag{4.13}$$

Observe that

$$\|\dot\Delta_j d\|_{L^p}\|\dot\Delta_j w\|_{L^p}^{p-1} \leqq \delta/2\|\dot\Delta_j w\|_{L^p}^p + C_\delta\|\dot\Delta_j d\|_{L^p}^p,$$

where the constant δ is chosen to satisfy the following inequality:

$$\|\dot\Delta_j d\|_{L^p} + \delta\|\dot\Delta_j w\|_{L^p} \geqq \frac{1}{2}(\|\dot\Delta_j d\|_{L^p} + \delta\|\dot\Delta_j\Lambda a\|_{L^p}).$$

Meanwhile, choosing a suitable R_0 such that for any $2^j \geqq R_0$ and $2 \leqq p \leqq 4$, we have $c_p 2^{2j} - 2 - C_\delta \geqq \frac{c_p}{2} 2^{2j}$. Thus, when $2^j \geqq R_0$, we get that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\dot\Delta_j d\|_{L^p} + \delta \|\dot\Delta_j \Lambda a\|_{L^p} \Big) + 2^{2j} \|\dot\Delta_j d\|_{L^p} + \|\dot\Delta_j \Lambda a\|_{L^p} \\
&\leqq C \|\mathrm{div}\, \mathcal{Q}u\|_{L^\infty} \|\dot\Delta_j d\|_{L^p} + \|[\dot\Delta_j, u\cdot\nabla] d\|_{L^p} \\
&\quad + \|\dot\Delta_j \mathcal{Q} W_{\mathcal{Q}}\|_{L^p} + C\|\dot\Delta_j \Lambda^{-1} \mathrm{div}\, \mathcal{Q} W_{\mathcal{Q}}\|_{L^p} \\
&\quad + \|\dot\Delta_j [\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p} + C\|\mathrm{div}\, u\|_{L^\infty} \|\dot\Delta_j \Lambda a\|_{L^p} \\
&\quad + C\|[\dot\Delta_j, u\cdot\nabla]\Lambda a\|_{L^p} \\
&\quad + C\|\dot\Delta_j [\Lambda, u\cdot\nabla] a\|_{L^p} + C\|\dot\Delta_j \Lambda(a \mathrm{div}\, u)\|_{L^p}. \qquad (4.14)
\end{aligned}$$

By the definition of Besov space, we deduce that

$$\begin{aligned}
&\|d, \Lambda a\|^H_{\dot B^{\frac{3}{p}-1}_{p,1}} + \int_0^T \|d\|^H_{\dot B^{\frac{3}{p}+1}_{p,1}} + \|\Lambda a\|^H_{\dot B^{\frac{3}{p}-1}_{p,1}} \,\mathrm{d}t \\
&\leqq \int_0^T \|\mathrm{div}\, \mathcal{Q}u\|_{L^\infty} \|d, \Lambda a\|^H_{\dot B^{\frac{3}{p}-1}_{p,1}} \,\mathrm{d}t + \|\mathcal{Q}W_{\mathcal{Q}}\|^H_{L^1_T(\dot B^{\frac{3}{p}-1}_{p,1})} \\
&\quad + \|a \mathrm{div}\, u\|^H_{L^1_T(\dot B^{\frac{3}{p}}_{p,1})} + \int_0^T \sum_{2^j \geqq R_0} 2^{j(3/p-1)} (\|[\dot\Delta_j, u\cdot\nabla] d\|_{L^p} \\
&\quad + \|\dot\Delta_j [\Lambda, u\cdot\nabla] a\|_{L^p}) \,\mathrm{d}t \\
&\quad + \int_0^T \sum_{2^j \geqq R_0} 2^{j(3/p-1)} (\|\dot\Delta_j [\Lambda^{-1}\mathrm{div}\,, u\cdot\nabla]\mathcal{Q}u\|_{L^p} \\
&\quad + \|[\dot\Delta_j, u\cdot\nabla]\Lambda a\|_{L^p}) \,\mathrm{d}t.
\end{aligned}$$

Thanks to Proposition 5.1-Proposition 5.4 as well as Lemma 4.1, the above inequality can be written as

$$\begin{aligned}
&\|d, \Lambda a\|^H_{\dot B^{\frac{3}{p}-1}_{p,1}} + \int_0^T \|d\|^H_{\dot B^{\frac{3}{p}+1}_{p,1}} + \|\Lambda a\|^H_{\dot B^{\frac{3}{p}-1}_{p,1}} \,\mathrm{d}t \leqq \|d_0, \Lambda a_0\|^H_{\dot B^{\frac{3}{p}-1}_{p,1}} \\
&\quad + C\int_0^T \Big(\|\mathcal{Q}u\|_{\dot B^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} + \|\mathcal{P}u\|_{\dot B^{\frac{3}{p}+1}_{p,1}} \\
&\quad + \|a\|_{\dot B^{\frac{5}{2}, \frac{3}{p}}_{2,p}} \Big)\Big(\|a\|_{\dot B^{\frac{1}{2}, \frac{3}{p}}_{2,p}} + \|\mathcal{Q}u\|_{\dot B^{\frac{1}{2}, \frac{3}{p}-1}_{2,p}} \\
&\quad + \|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac{3}{p}-1}_{p,1}} \Big) \,\mathrm{d}t. \qquad (4.15)
\end{aligned}$$

Step 3: Estimates for the incompressible part of the velocity. To close the estimates, we need to estimate $\mathcal{P}u$. For $(\mathcal{P}u)^{hor}$. Applying Proposition 5.5 to the third equation of (4.2), we obtain that

$$
\begin{aligned}
&\|(\mathcal{P}u)^{hor}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|(\mathcal{P}u)^{hor}\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})} \\
&\leqq \|(\mathcal{P}u_0)^{hor}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + C\int_0^T \|\mathrm{div}\, u\|_{L^\infty}\|\mathcal{P}u^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\,\mathrm{d}t \\
&\quad + \int_0^T \|(\mathcal{P}W_{\mathcal{P}})^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\,\mathrm{d}t \\
&\quad + C\int_0^T \sum_{j\in\mathbb{N}} 2^{j(\frac{3}{p}-1)}(\|[\dot{\Delta}_j, u\cdot\nabla](\mathcal{P}u)^{hor}\|_{L^p} + \|[\mathcal{P}, u\cdot\nabla]u^{hor}\|_{L^p})\,\mathrm{d}t \qquad (4.16)\\
&\leqq \|(\mathcal{P}u_0)^{hor}\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + C\int_0^T \|a\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}}(\|(\mathcal{Q}u)^{hor}\|_{\dot{B}^{5/2,3/p+1}_{2,p}} \\
&\quad + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}})\,\mathrm{d}t \\
&\quad + C\int_0^T (\|(\mathcal{Q}u)^{hor}\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{2,p}})(\|\mathcal{Q}u\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} \\
&\quad + \|\mathcal{P}u\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}})\,\mathrm{d}t,
\end{aligned}
$$

where we used Lemma 4.1 and

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}} 2^{j(\frac{3}{p}-1)}(\|[\dot{\Delta}_j, u\cdot\nabla](\mathcal{P}u)^{hor}\|_{L^p} + \|[\mathcal{P}, u\cdot\nabla]u^{hor}\|_{L^p}) \\
&\quad \leqq C(\|(\mathcal{Q}u)^{hor}\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}})(\|\mathcal{Q}u\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} + \|\mathcal{P}u\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}).
\end{aligned}
$$

For $(\mathcal{P}u)^3$, deducing from the divergence free condition $\mathrm{div}\,\mathcal{P}u = 0$, we have

$$
\mathcal{P}(u\cdot\nabla u^3) = \mathcal{P}\big(((\mathcal{Q}u)^{hor} + (\mathcal{P}u)^{hor})\partial_h u^3\big) - \mathcal{P}(u^3\mathrm{div}_h(\mathcal{P}u)^{hor}) + \mathcal{P}(u^3\partial_3(\mathcal{Q}u)^3).
$$

Thus, applying Proposition 5.5 again to the last equation of (4.2), we obtain that

$$
\begin{aligned}
&\|(\mathcal{P}u)^3\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|(\mathcal{P}u)^3\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})} \\
&\leqq \|(\mathcal{P}u_0)^3\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \int_0^T \|((\mathcal{Q}u)^{hor} + (\mathcal{P}u)^{hor})\partial_h u^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\,\mathrm{d}t \qquad (4.17)\\
&\quad + \int_0^T \left(\|u^3\mathrm{div}_h(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|u^3\partial_3(\mathcal{Q}u)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|(\mathcal{P}W_{\mathcal{P}})^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\right)\mathrm{d}t.
\end{aligned}
$$

Thanks to Proposition 5.1 and Lemma 5.4, we have

$$\|((\mathcal{Q}u)^{hor}+(\mathcal{P}u)^{hor})\partial_h u^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leqq C\Big(\|(\mathcal{Q}u)^{hor}\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big)\|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}},$$

and

$$\begin{aligned}&\|u^3\operatorname{div}_h(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|u^3\partial_3(\mathcal{Q}u)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|(\mathcal{P}W_{\mathcal{P}})^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\\ &\leqq C\Big(\|(\mathcal{Q}u)^3\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big)\Big(\|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|(\mathcal{Q}u)^3\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}}\Big)\\ &\quad + C\|a\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}}\Big(\|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\Big),\end{aligned}$$

from which, together with Lemma 4.1 and (4.17), we derive that

$$\begin{aligned}&\|(\mathcal{P}u)^3\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|(\mathcal{P}u)^3\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})}\\ &\leqq \|(\mathcal{P}u_0)^3\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\quad + C\int_0^T (\|(\mathcal{Q}u)^{hor}\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}})\|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\,\mathrm{d}t\\ &\quad + C\int_0^T \Big(\|(\mathcal{Q}u)^3\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big)\\ &\quad \times\Big(\|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|(\mathcal{Q}u)^3\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}}\Big)\,\mathrm{d}t\\ &\quad + C\int_0^T \|a\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}}\Big(\|(\mathcal{P}u)^{hor}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\Big)\,\mathrm{d}t.\end{aligned} \tag{4.18}$$

Step 4: Continuity argument. We first deduce from (4.10), (4.15) and (4.16) that

$$\begin{aligned}&\|a(t)\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}} + \|d(t)\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u)^{hor}(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\\ &\quad + \int_0^t\Big(\|a\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}}_{2,p}} + \|d\|_{\dot{B}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} + \|\mathcal{P}u\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\Big)\mathrm{d}\tau\\ &\leqq C\Big(\|a_0\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}}_{2,p}} + \|d_0\|_{\dot{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + \|(\mathcal{P}u_0)^{hor}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\Big)\end{aligned}$$

$$+\int_0^t\left(\|a\|_{\dot B^{\frac52,\frac3p}_{2,p}}+\|d\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|\mathcal{P}u\|_{\dot B^{\frac3p+1}_{p,1}}\right)\left(\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\right)\mathrm{d}\tau.$$

Let $\varepsilon^{\frac12}\le c_1\ll 1$. We define $\bar T$ by

$$\bar T\overset{\text{def}}{=}\sup\left\{T>0\colon \|a\|_{\widetilde L^\infty_T(\dot B^{\frac12,\frac3p}_{2,p})}+\|d\|_{\widetilde L^\infty_T(\dot B^{\frac12,\frac3p-1}_{2,p})}+\|(\mathcal{P}u)^{hor}\|_{\widetilde L^\infty_T(\dot B^{\frac3p-1}_{p,1})}\leqq c_1\right\}. \tag{4.19}$$

According to the local existence and blow up criterion for the system, it is obvious that $\bar T>0$. We shall prove $\bar T=\infty$ under the assumption (1.15). For any $t\in[0,\bar T]$, the above inequality can be recast by

$$\begin{aligned}
&\|a(t)\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d(t)\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u)^{hor}(t)\|_{\dot B^{\frac3p-1}_{p,1}}\\
&\quad+(1-c_1)\int_0^t\left(\|a\|_{\dot B^{\frac52,\frac3p}_{2,p}}+\|d\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p+1}_{p,1}}\right)\mathrm{d}\tau\\
&\leqq\|a_0\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d_0\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u_0)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\\
&\quad+C\int_0^t\|(\mathcal{P}u)^3\|_{\dot B^{\frac3p+1}_{p,1}}\left(\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\right)\mathrm{d}\tau,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|a(t)\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d(t)\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u)^{hor}(t)\|_{\dot B^{\frac3p-1}_{p,1}}\\
&\leqq C\left(\|a_0\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d_0\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u_0)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\right)\\
&\quad\times\exp\left(\int_0^t\|(\mathcal{P}u)^3\|_{\dot B^{\frac3p+1}_{p,1}}\mathrm{d}\tau\right)
\end{aligned}\tag{4.20}$$

for any $t\in[0,\bar T]$.

On the other hand, from (4.18) and (4.19) , we get that for any $t\in[0,\bar T]$,

$$\begin{aligned}
&\|(\mathcal{P}u)^3\|_{\widetilde L^\infty_t(\dot B^{\frac3p-1}_{p,1})}+\frac23\|(\mathcal{P}u)^3\|_{L^1_t(\dot B^{\frac3p+1}_{p,1})}\\
&\leqq\|(\mathcal{P}u_0)^3\|_{\dot B^{\frac3p-1}_{p,1}}+\int_0^t\left(\|a\|_{\dot B^{\frac52,\frac3p}_{2,p}}+\|d\|_{\dot B^{\frac52,\frac3p+1}_{2,p}}+\|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p+1}_{p,1}}\right)\\
&\quad\times\left(\|a\|_{\dot B^{\frac12,\frac3p}_{2,p}}+\|d\|_{\dot B^{\frac12,\frac3p-1}_{2,p}}+\|(\mathcal{P}u)^{hor}\|_{\dot B^{\frac3p-1}_{p,1}}\right)\mathrm{d}\tau,
\end{aligned}$$

from which, together with (4.19), we obtain that for any $t \in [0, \bar{T}]$,

$$\int_0^t \|(\mathcal{P}u)^3\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \mathrm{d}\tau \le C\Big(\|a_0\|_{\dot{B}^{\frac12,\frac3p}_{2,p}} + \|d_0\|_{\dot{B}^{\frac12,\frac3p-1}_{2,p}} + \|(\mathcal{P}u_0)^{hor}\|_{\dot{B}^{\frac3p-1}_{p,1}} + \|(\mathcal{P}u_0)^3\|_{\dot{B}^{\frac3p-1}_{p,1}}\Big) \tag{4.21}$$

Plugging this estimate into (4.20) and using the condition (1.15), we obtain that

$$\begin{aligned}
&\|a(t)\|_{\dot{B}^{\frac12,\frac3p}_{2,p}} + \|d(t)\|_{\dot{B}^{\frac12,\frac3p-1}_{2,p}} + \|(\mathcal{P}u)^{hor}(t)\|_{\dot{B}^{\frac3p-1}_{p,1}} \\
&\leqq C\Big(\|a_0\|_{\dot{B}^{\frac12,\frac3p}_{2,p}} + \|d_0\|_{\dot{B}^{\frac12,\frac3p-1}_{2,p}} \\
&\quad + \|(\mathcal{P}u_0)^{hor}\|_{\dot{B}^{\frac3p-1}_{p,1}}\Big) \exp\big(C(\varepsilon + \|(\mathcal{P}u_0)^3\|_{\dot{B}^{\frac3p-1}_{p,1}})\big) \\
&\leqq C\varepsilon \leqq \frac{c_1}{2},
\end{aligned} \tag{4.22}$$

for any $t \in [0, \bar{T}]$, which is in contradiction with the definition of $\bar{T}$. We conclude that $\bar{T} = \infty$ and (4.22) holds for all time, from which together with (4.21) will imply (1.16) and (1.17). This ends the proof of Theorem 1.3. □

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5. Appendix

For the convenience of readers, in this appendix, we list some basic facts about the Littlewood-Paley theory.

5.1. Littlewood-Paley Decomposition

Let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^3,\ \frac34 \le |\xi| \le \frac83\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0.$$

The frequency localization operators Δ_j and S_j are defined by

$$\Delta_j f = \varphi(2^{-j}D)f, \quad S_j f = \sum_{k\leq j-1} \Delta_k f \quad \text{for} \quad j \in \mathbb{Z}.$$

With our choice of φ, one can easily verify that

$$\Delta_j\Delta_k f = 0 \quad \text{if} \quad |j-k| \geq 2 \quad \text{and} \quad \Delta_j(S_{k-1}f\Delta_k f) = 0 \quad \text{if} \quad |j-k| \geq 5. \tag{5.1}$$

Next we recall Bony's decomposition from [2]:

$$uv = T_u v + T_v u + R(u, v), \tag{5.2}$$

with

$$T_u v = \sum_{j\in\mathbb{Z}} S_{j-1}u\Delta_j v, \quad R(u, v) = \sum_{j\in\mathbb{Z}} \Delta_j u\widetilde{\Delta}_j v, \quad \widetilde{\Delta}_j v = \sum_{|j'-j|\leq 1} \Delta_{j'} v.$$

5.2. Product Estimates in Besov Spaces

We first recall the Bernstein lemma which will be frequently used (see [1]).

Lemma 5.1. *Let* $1 \leq p \leq q \leq +\infty$. *Assume that* $f \in L^p(\mathbb{R}^3)$, *then for any* $\gamma \in (\mathbb{N}\cup\{0\})^3$, *there exist constants* C_1, C_2 *independent of* f, j *such that*

$$\begin{aligned}
&\mathrm{supp}\hat{f} \subseteq \{|\xi| \leq A_0 2^j\} \Rightarrow \|\partial^\gamma f\|_q \leq C_1 2^{j|\gamma|+3j(\frac{1}{p}-\frac{1}{q})}\|f\|_p,\\
&\mathrm{supp}\hat{f} \subseteq \{A_1 2^j \leq |\xi| \leq A_2 2^j\} \Rightarrow \|f\|_p \leq C_2 2^{-j|\gamma|} \sup_{|\beta|=|\gamma|} \|\partial^\beta f\|_p.
\end{aligned}$$

As a consequence, the estimates for the paraproduct and remainder operators can be given by

Lemma 5.2. *Let* $1 \leq p, q, q_1, q_2 \leq \infty$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. *Then we have*

(a) *if* $s_2 \leq \frac{3}{p}$, *we have*

$$\|T_g f\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,1})} \leq C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})};$$

(b) *if* $s_1 \leq \frac{3}{p} - 1$, *we have*

$$\|T_f g\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,1})} \leq C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})};$$

(c) *if* $s_1 + s_2 > 3\max(0, \frac{2}{p} - 1)$, *we have*

$$\|R(f, g)\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,1})} \leq C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}.$$

We refer readers to [1] for detailed proof. Then we arrive at the following product estimates:

Lemma 5.3. *Let* $s_1 \le \frac{3}{p} - 1, s_2 \le \frac{3}{p}, s_1 + s_2 > 3\max(0, \frac{2}{p} - 1)$, *and* $1 \le p, q, q_1, q_2 \le \infty$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. *Then it holds that*

$$\|fg\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,1})} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}.$$

Lemma 5.4. *Let* $s_1, s_2 \le \frac{3}{p},\ s_1+s_2 > 3\max(0, \frac{2}{p}-1)$, *and* $1 \le p, q, q_1, q_2 \le \infty$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. *Then it holds that*

$$\|fg\|_{\widetilde{L}^q_T(\dot{B}^{s_1+s_2-\frac{3}{p}}_{p,1})} \le C\|f\|_{\widetilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\widetilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}.$$

Next, the commutator between the frequency localization operator and the function can be estimated by

Lemma 5.5. *Let* $p \in [1, \infty)$ *and* $s \in (-3\min(\frac{1}{p}, \frac{1}{p'}), \frac{3}{p}]$. *Then it holds that*

$$\big\|2^{js}\|[\Delta_j, f]\nabla g\|_{L^1_T(L^p)}\big\|_{L^1} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|g\|_{L^1_T(\dot{B}^{s+1}_{p,1})},$$

$$\big\|2^{js}\|[\Delta_j, f]\nabla g\|_{L^1_T(L^p)}\big\|_{L^1} \le C\|f\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{3}{p}+1}_{p,1})}\|g\|_{L^1_T(\dot{B}^{s}_{p,1})}.$$

We recall the following composite result:

Lemma 5.6. *Let* $s > 0$ *and* $1 \le p, q, r \le \infty$. *Assume that* $F \in W^{[s]+3,\infty}_{loc}(\mathbb{R})$ *with* $F(0) = 0$. *Then it holds that*

$$\|F(f)\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} \le C(1 + \|f\|_{L^\infty_T(L^\infty)})^{[s]+2}\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}.$$

In the anisotropic spaces, the above results still hold true. We refer readers to [5,8] for details. We begin with the product estimates.

Proposition 5.1. *Let* $s, t, \widetilde{s}, \widetilde{t}, \sigma, \tau \in \mathbb{R}$, $2 \leqq p \leqq 4$, *and* $1 \leqq r, r_1, r_2 \leqq \infty$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. *Then we have the following estimates:*

(a) *If* $\sigma, \tau \leqq \frac{n}{p}$ *and* $\sigma + \tau > 0$, *then*

$$\sum_{2^j > R_0} 2^{j(\sigma+\tau-\frac{n}{p})}\|\dot{\Delta}_j(fg)\|_{L^r_T(L^p)}$$
$$\leqq C(\|f\|^L_{\widetilde{L}^{r_1}_T(\dot{B}^{n/2-n/p+\sigma}_{2,1})} + \|f\|^H_{\widetilde{L}^{r_1}_T(\dot{B}^{\sigma}_{p,1})})(\|g\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^{n/2-n/p+\tau}_{2,1})} + \|g\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^{\tau}_{p,1})}).$$

(b) *If* $s, \widetilde{s} \leqq \frac{n}{p}$ *and* $s + t > n - \frac{2n}{p}$ *with* $s + t = \widetilde{s} + \widetilde{t}$ *and* $\theta \in \mathbb{R}$, *then*

$$\sum_{2^j \leqq R_0} 2^{j(s+t-\frac{n}{2})}\|\dot{\Delta}_j(fg)\|_{L^r_T(L^2)}$$
$$\leqq C(\|f\|^L_{\widetilde{L}^{r_1}_T(\dot{B}^{s}_{2,1})} + \|f\|^H_{\widetilde{L}^{r_1}_T(\dot{B}^{s-n/2+n/p}_{p,1})})(\|g\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^{t}_{2,1})} + \|g\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^{t-n/2+n/p+\theta}_{p,1})})$$
$$+ C(\|g\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^{\widetilde{s}}_{2,1})} + \|g\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^{\widetilde{s}-n/2+n/p}_{p,1})})(\|f\|^L_{\widetilde{L}^{r_1}_T(\dot{B}^{\widetilde{t}}_{2,1})} + \|f\|^H_{\widetilde{L}^{r_1}_T(\dot{B}^{\widetilde{t}-n/2+n/p}_{p,1})}).$$

(c) *If* $s, \widetilde{s} \leqq \frac{n}{2}$ *and* $s + t > \frac{n}{2} - \frac{n}{p}$ *with* $s + t = \widetilde{s} + \widetilde{t}$, *then*

$$\sum_{j \in \mathbb{Z}} 2^{j(s+t-\frac{n}{2})} \|\dot{\Delta}_j(fg)\|_{L^r_T(L^2)} \leqq C(\|f\|^L_{\widetilde{L}^{r_1}_T(\dot{B}^s_{2,1})} + \|f\|^H_{\widetilde{L}^{r_1}_T(\dot{B}^{s-n/2+n/p}_{p,1})})\|g\|_{\widetilde{L}^{r_2}_T(\dot{B}^t_{2,1})}$$
$$+ C(\|g\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^{\widetilde{s}}_{2,1})} + \|g\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^{\widetilde{s}-n/2+n/p}_{p,1})})\|f\|_{\widetilde{L}^{r_1}_T(\dot{B}^{\widetilde{t}}_{2,1})}.$$

Let $\{c(j)\}$ be a sequence in l^1 with the norm $\|\{c(j)\}\|_{l^1} \leqq 1$. Then the estimates for the commutators can be stated as follows:

Proposition 5.2. *Let* $2 \leqq p \leqq 4, -\frac{n}{p} < s \leqq \frac{n}{2} + 1, -\frac{n}{p} < \sigma \leqq \frac{n}{p} + 1$, *and* $1 \leqq r, r_1, r_2 \leqq \infty$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. *Then, for* $2^j > R_0$, *it holds that*

$$\|[v, \dot{\Delta}_j] \cdot \nabla f\|_{L^r_T(L^p)} \leqq Cc(j)(2^{-j\sigma} + 2^{j(\frac{n}{2}-\frac{n}{p}-s)})(\|f\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^s_{2,1})} + \|f\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^\sigma_{p,1})})$$
$$\times (\|v\|^L_{\widetilde{L}^{r_1}_T(\dot{B}^{n/2+1}_{2,1})} + \|v\|^H_{\widetilde{L}^{r_1}_T(\dot{B}^{1+n/p}_{p,1})}),$$
$$\|[v, \dot{\Delta}_j] \cdot \nabla f\|_{L^r_T(L^p)} \leqq Cc(j)(2^{-j\sigma} + 2^{j(\frac{n}{2}-\frac{n}{p}-s)})(\|f\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^s_{2,1})}$$
$$+ \|f\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^\sigma_{p,1})})\|v\|_{\widetilde{L}^{r_1}_T(\dot{B}^{1+n/p}_{p,1})}.$$

Moreover, if $-\frac{n}{p} < s \leqq \frac{n}{p} + 1$, *then*

$$\|[v, \dot{\Delta}_j] \cdot \nabla f\|_{L^r_T(L^2)} \leqq Cc(j)2^{-js}\|f\|_{\widetilde{L}^{r_2}_T(\dot{B}^s_{2,1})}(\|v\|^L_{\widetilde{L}^{r_1}_T(\dot{B}^{n/2+1}_{2,1})} + \|v\|^H_{\widetilde{L}^{r_1}_T(\dot{B}^{1+n/p}_{p,1})}).$$

Proposition 5.3. *Under the assumption of Proposition 5.2, if* $S \in S^m_{1,0}$, *then, for* $2^j > R_0$,

$$\|[S, \dot{\Delta}_j] \cdot \nabla f\|_{L^r_T(L^p)} \leqq Cc(j)(2^{-j\sigma} + 2^{j(\frac{n}{2}-\frac{n}{p}-s)})(\|f\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^{s+m}_{2,1})} + \|f\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^{\sigma+m}_{p,1})}),$$

and, for $2^j \leqq R_0$, *if* $-\frac{n}{p} < s \leqq \frac{n}{p} + 1$, *then*

$$\|[S, \dot{\Delta}_j] \cdot \nabla f\|_{L^r_T(L^2)} \leqq Cc(j)2^{-js}(\|f\|^L_{\widetilde{L}^{r_2}_T(\dot{B}^{s+m}_{2,1})} + \|f\|^H_{\widetilde{L}^{r_2}_T(\dot{B}^{\sigma+m}_{p,1})}).$$

Finally the composite result can be proven.

Proposition 5.4. *Let* $2 \leqq p \leqq 4, s, \sigma > 0$, *and* $s \geqq \sigma - \frac{n}{2} + \frac{n}{p}, r \geqq 1$. *Assume that* $F \in W^{[s]+2}_{loc} \cap W^{[\sigma]+2}_{loc}$ *with* $F(0) = 0$. *Then it holds that*

$$\|F(f)\|^L_{\widetilde{L}^r_T(\dot{B}^s_{2,1})} + \|F(f)\|^H_{\widetilde{L}^r_T(\dot{B}^\sigma_{p,1})}$$
$$\leqq C(1 + \|f\|^L_{\widetilde{L}^\infty_T(\dot{B}^{n/p}_{2,1})} + \|f\|^H_{\widetilde{L}^\infty_T(\dot{B}^{n/p}_{p,1})})^{\max([s],[\sigma])+1}(\|f\|^L_{\widetilde{L}^r_T(\dot{B}^s_{2,1})} + \|f\|^H_{\widetilde{L}^r_T(\dot{B}^\sigma_{p,1})}).$$

For any $s > 0$ *and* $p \geqq 1$, *it holds that*

$$\|F(f)\|_{\widetilde{L}^r_T(\dot{B}^s_{p,1})} \leqq C(1 + \|f\|_{L^\infty_T(L^\infty)})^{[s]+1}\|f\|_{\widetilde{L}^r_T(\dot{B}^s_{p,1})}.$$

For Stokes equations, the maximum regularity estimate can be concluded as

Proposition 5.5. *Let $p \in (1, \infty)$, and $s \in \mathbb{R}$. Let $u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^3)$ be a divergence-free field and $g \in \widetilde{L}^1_T(\dot{B}^s_{p,1})$. If u solves*

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\Pi = g, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{5.3}$$

then (5.3) has a unique solution u so that

$$\|u\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})} + \mu\|u\|_{L^1_T(\dot{B}^{s+2}_{p,1})} + \|\nabla\Pi\|_{L^1_T(\dot{B}^s_{p,1})} \leqq \|u_0\|_{\dot{B}^s_{p,1}} + C\|g\|_{L^1_T(\dot{B}^s_{p,1})}.$$

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LINGBING HE
Department of Mathematical Sciences,
Tsinghua University,
Beijing
100084,
People's Republic of China.
e-mail: hlb@mail.tsinghua.edu.cn

and

JINGCHI HUANG
School of Mathematics,
Sun Yat-sen University,
Guangzhou
510275, Guangdong,
People's Republic of China.
e-mail: huangjch25@mail.sysu.edu.cn

and

CHAO WANG
School of Mathematical Sciences,
Peking University,
Beijing
100871,
People's Republic of China.
e-mail: wangchao@math.pku.edu.cn

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