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Vanishing Viscosity Limit of the Navier–Stokes Equations to the Euler Equations for Compressible Fluid Flow with Vacuum

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Abstract

We establish the vanishing viscosity limit of the Navier–Stokes equations to the Euler equations for three-dimensional compressible isentropic flow in the whole space. When the viscosity coefficients are given as constant multiples of the density's power (ρ^δ with $\delta > 1$), it is shown that there exists a unique regular solution of compressible Navier–Stokes equations with arbitrarily large initial data and vacuum, whose life span is uniformly positive in the vanishing viscosity limit. It is worth paying special attention to the fact that, via introducing a "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure to control the behavior of the velocity of the fluid near the vacuum, we can also give some uniform estimates for $\left(\rho^{\frac{\gamma-1}{2}}, u\right)$ in H^3 and $\rho^{\frac{\delta-1}{2}}$ in H^2 with respect to the viscosity coefficients (adiabatic exponent $\gamma > 1$ and $1 < \delta \leqq \min\{3, \gamma\}$), which lead to the strong convergence of the regular solution of the viscous flow to that of the inviscid flow in $L^\infty([0, T]; H^{s'})$ (for any $s' \in [2, 3)$) with the rate of $\varepsilon^{2(1-s'/3)}$. Furthermore, we point out that our framework in this paper is applicable to other physical dimensions, say 1 and 2, with some minor modifications.

1. Introduction

In this paper, we investigate the inviscid limit problem of the 3D isentropic compressible Navier–Stokes equations with degenerate viscosities, when the initial data contain a vacuum and are arbitrarily large. For this purpose, we consider the following isentropic compressible Navier–Stokes equations (ICNS) in $\mathbb{R}^3$:

$$\begin{cases}\rho_t + \operatorname{div}(\rho u) = 0,\\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}\mathbb{T}.\end{cases} \tag{1.1}$$

We look for the above system's local regular solutions with initial data

$$(\rho, u)|_{t=0} = (\rho_0(x), u_0(x)), \quad x \in \mathbb{R}^3, \tag{1.2}$$

and the far field behavior

$$(\rho, u) \to (0,0) \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0. \tag{1.3}$$

Usually, such kinds of far field behavior occurs naturally under some physical assumptions on (1.1)'s solutions, such as finite total mass and total energy.

In system (1.1), $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geqq 0$ are the space and time variables, respectively, ρ is the density, and $u = \left(u^{(1)}, u^{(2)}, u^{(3)}\right)^\top \in \mathbb{R}^3$ is the velocity of the fluid. In considering the polytropic gases, the constitutive relation, which is also called the equations of state, is given by

$$P = A\rho^\gamma, \quad \gamma > 1, \tag{1.4}$$

where $A > 0$ is an entropy constant and γ is the adiabatic exponent. $\mathbb{T}$ denotes the viscous stress tensor with the form

$$\mathbb{T} = \mu(\rho)\left(\nabla u + (\nabla u)^\top\right) + \lambda(\rho)\mathrm{div} u\, \mathbb{I}_3, \tag{1.5}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix,

$$\mu(\rho) = \varepsilon\alpha\rho^\delta, \quad \lambda(\rho) = \varepsilon\beta\rho^\delta, \tag{1.6}$$

$\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho)+\frac{2}{3}\mu(\rho)$ is the bulk viscosity coefficient, $\varepsilon \in (0, 1]$ is a constant, α and β are both constants satisfying

$$\alpha > 0, \quad 2\alpha + 3\beta \geqq 0, \tag{1.7}$$

and in this paper, we assume that the constant δ satisfies

$$1 < \min\{\delta, \gamma\} \leqq 3. \tag{1.8}$$

In addition, when $\varepsilon = 0$, from (1.1), we naturally have the compressible isentropic Euler equations for the inviscid flow:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = 0, \end{cases} \tag{1.9}$$

which is a fundamental example of a system of hyperbolic conservation laws.

Throughout this paper, we adopt the following simplified notations, most of which are for the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{aligned} |f|_p &= \|f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}, \quad |f|_2 = \|f\|_0 = \|f\|_{L^2(\mathbb{R}^3)}, \\ D^{k,r} &= \{f \in L^1_{loc}(\mathbb{R}^3) : |\nabla^k f|_r < +\infty\}, \\ D^k &= D^{k,2}, |f|_{D^{k,r}} = \|f\|_{D^{k,r}(\mathbb{R}^3)} \ (k \geqq 2), \\ D^1 &= \{f \in L^6(\mathbb{R}^3) : |\nabla f|_2 < \infty\}, \quad |f|_{D^1} = \|f\|_{D^1(\mathbb{R}^3)}, \quad \int_{\mathbb{R}^3} f\, \mathrm{d}x = \int f. \end{aligned}$$

A detailed study of homogeneous Sobolev spaces can be found in GALDI [14].

1.1. Existence Theories of Compressible Flow with Vacuum

Before formulating our problem, we first briefly recall a series of frameworks on the well-posedness of multi-dimensional strong solutions with initial vacuum established for the hydrodynamics equations mentioned above in the whole space. For the inviscid flow, in 1987, via writing (1.9) as a symmetric hyperbolic form that allows the density to vanish, MAKINO–UKAI–KAWSHIMA [31] obtained the local-in-time existence of the unique regular solution with $\inf \rho_0 = 0$, which can be shown by

Theorem 1.1. [31] *Let $\gamma > 1$. If the initial data (ρ_0, u_0) satisfy*

$$\rho_0 \geqq 0, \quad \big(\rho_0^{\frac{\gamma-1}{2}}, u_0\big) \in H^3(\mathbb{R}^3), \tag{1.10}$$

then there exist a time $T_0 > 0$ and a unique regular solution (ρ, u) to Cauchy problem (1.9) with (1.2)–(1.3) satisfying

$$\big(\rho^{\frac{\gamma-1}{2}}, u\big) \in C([0, T_0]; H^3), \quad \big((\rho^{\frac{\gamma-1}{2}})_t, u_t\big) \in C([0, T_0]; H^2), \tag{1.11}$$

where the regular solution (ρ, u) to (1.9) with (1.2)–(1.3) is defined by

- (A) $\big(\rho, u\big)$ *satisfies* (1.9) *with* (1.2) − (1.3) *in the sense of distributions*;
- (B) $\rho \geqq 0$, $\big(\rho^{\frac{\gamma-1}{2}}, u\big) \in C^1([0, T_0] \times \mathbb{R}^3)$;
- (C) $u_t + u \cdot \nabla u = 0$ *when* $\rho(t, x) = 0$.

It should be pointed out that the condition (C) ensures the uniqueness of the regular solution and makes the velocity u well defined in vacuum region. Without (C), it is difficult to get enough information on velocity even for considering special cases such as point vacuum or continuous vacuum on some surface. In 1997, by extracting a dispersive effect after some invariant transformation, SERRE [36] obtained the global existence of the regular solution shown in Theorem 1.1 with small density.

For the constant viscous flow (i.e., $\delta = 0$ in (1.6)), the corresponding local-in-time well-posedness of strong solutions with vacuum was firstly solved by CHO–CHOE–KIM [8,9] in 2004–2006; to compensate the lack of a positive lower bound of the initial density, they introduced an initial compatibility condition

$$\mathrm{div}\mathbb{T}(u_0) + \nabla P_0 = \rho_0 f, \text{ for some } f \in D^1 \text{ and } \sqrt{\rho}_0 f \in L^2, \tag{1.12}$$

which plays a key role in getting some uniform a priori estimates with respect to the lower bound of ρ_0; see also DUAN–LUO–ZHENG [12] for the 2D case. Later, based on the uniform estimate on the upper bound of the density, HUANG–LI–XIN [20] extended this solution to be a global one under some initial smallness assumption for the isentropic flow in $\mathbb{R}^3$.

Recently, the degenerate viscous flow (i.e., $\delta = 0$ in (1.6)) described by (1.1) has received extensive attention from the mathematical community (see the review papers [2,28]), based on the following two main considerations: on the one hand, through the second-order Chapman-Enskog expansion from Boltzmann equations to the compressible Navier–Stokes equations, it is known that the viscosity coefficients are not constants but functions of the absolute temperature (cf. CHAPMAN–COWLING [6] and LI–QIN [25]), which can be reduced to the dependence on density for isentropic flow from laws of BOYLE and GAY-LUSSAC (see [25]); on the other hand, we do have some good models in the following 2D shallow water equations for the height h of the free surface and the mean horizontal velocity field U:

$$\begin{cases} h_t + \mathrm{div}(hU) = 0, \\ (hU)_t + \mathrm{div}(hU \otimes U) + \nabla h^2 = \mathcal{V}(h, U). \end{cases} \tag{1.13}$$

It is clear that system (1.13) is a special case or a simple variant of system (1.1) for some appropriately chosen viscous term $\mathcal{V}(h, U)$ (see [5,15,16,32]). Some important progress has been obtained in the development of the global existence of weak solutions with a vacuum for system (1.1) and related models, see BRESH–DESJARDINS [2–4], MELLET–VASSEUR [33] and some other interesting results, c.f. [24,29,40,41].

However, in the presence of a vacuum, compared with the constant viscosity case [8], there appear to be some new mathematical challenges in dealing with such systems for constructing solutions with high regularities. In particular, these systems become highly degenerate, the result of which is that the velocity cannot even be defined in the vacuum domain and hence it is difficult to get uniform estimates for the velocity near the vacuum. Recently LI–PAN–ZHU [26,27]-via carefully analyzing the mathematical structure of these systems-reasonably gave the time evolution mechanism of the fluid velocity in the vacuum domain. Taking $\delta = 1$, for example, via considering the following parabolic equations with a special source term:

$$\begin{cases} u_t + u \cdot \nabla u + \dfrac{2A\gamma}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}\nabla\rho^{\frac{\gamma-1}{2}} + Lu = (\nabla\rho/\rho) \cdot \mathbb{S}(u), \\ Lu = -\alpha\triangle u - (\alpha + \beta)\nabla \mathrm{div} u, \\ \mathbb{S}(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta \mathrm{div} u \mathbb{I}_3, \end{cases} \tag{1.14}$$

we could transfer the degenearcy shown in system (1.1) caused by the far field vacuum to the possible singualrity of the quantity $\nabla\rho/\rho$, which was luckily shown to be well defined in $L^6 \cap D^1$. Based on this, by making full use of the symmetrical structure of the hyperbolic operator and the weak smoothing effect of the elliptic operator, they established a series of a priori estimates independent of the lower bound of ρ_0, and successfully gave the local existence theory of classical solutions with arbitrarily large data and vacuum for the case $1 \leqq \delta \leqq \min\left\{3, \frac{\gamma+1}{2}\right\}$. We refer readers to ZHU [42,43] for more details and progress.

1.2. Vanishing Viscosity Limit from Viscous Flow to Inviscid Flow

Based on the well-posedness theory mentioned above, naturally there is an important question can we regard the regular solution of inviscid flow [31,36] as those of viscous flow [8,12,20,26,27] with vanishing real physical viscosities?

Actually, there is a lot of literature on the uniform bounds and the vanishing viscosity limit in the whole space. The idea of regarding inviscid flow as viscous flow with vanishing real physical viscosity dates back to DAFERMOS [10], HUGONIOT [22], RANKINE [34], RAYLEIGH [35] and STOKES [39]. However, until 1951, GILBARG [17] gave us the first rigorous convergence analysis of vanishing physical viscosities from the Navier–Stokes equations (1.1) to the isentropic Euler equations (1.9), and established the mathematical existence and vanishing viscous limit of the Navier–Stokes shock layers. The framework on the convergence analysis of piecewise smooth solutions has been established by GÙES–MÉTIVIER–WILLIAMS–ZUMBRUN [18], HOFF–LIU [19], and the references cited therein. The convergence of vanishing physical viscosity with general initial data was first studied by SERRE–SHEARER [37] for a 2×2 system in nonlinear elasticity with severe growth conditions on the nonlinear function in the system. In 2009, based on the uniform energy estimates and compactness compensated argument, CHEN–PEREPELITSA [7] established the first convergence result for the vanishing physical viscosity limit of solutions of the Navier–Stokes equations to a finite-energy entropy weak solution of the isentropic Euler equations with finite-energy initial data, which has been extended to the density-dependent viscosity case (experiencing degeneracy near vacuum states) by HUANG–PAN–WANG–WANG–ZHAI [21].

However, even in 1D space, due to the complex mathematical structure of hydrodynamics equations near the vacuum, the existence of strong solutions to the viscous flow and inviscid flow are usually established in totally different frameworks, for example, [8] and [31]. The proofs shown in [8,12,26] essentially depend on the uniform ellipticity of the Lamé operator L, and the a priori estimates on the solutions and their life spans T^v obtained in the above references both strictly depend on the real physical viscosities. For example, when $\delta = 0$ (i.e., $\mu = \varepsilon\alpha$ and $\lambda = \varepsilon\beta$), we have

$$\begin{cases} |u|_{D^{k+2}} \leqq C\Big(\dfrac{1}{\varepsilon\alpha}, \dfrac{1}{\varepsilon\beta}\Big)\big(|u_t + u\cdot\nabla u + \nabla P|_{D^k}\big), \\ \ \ T^v \sim O\big(\varepsilon\alpha\big) + O\big(\varepsilon\beta\big), \end{cases} \tag{1.15}$$

which implies that the current frameworks do not seem to work for verifying the expected limit relation. Thus the vanishing viscosity limit for the multi-dimensional strong solutions in the whole space from Navier–Stokes equations to Euler equations for compressible flow with initial vacuum in some open set or at the far field is still an open problem.

In this paper, motivated by the uniform estimates on $\varepsilon^{\frac{1}{2}}\nabla^3\rho^{\frac{\delta-1}{2}}$ and $\varepsilon^{\frac{1}{2}}\rho^{\frac{\delta-1}{2}}\nabla^4 u$, when the viscous stress tensor has the form (1.6)–(1.8), we aim at giving a positive answer for this question by introducing a "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure to control the behavior of the velocity near the vacuum. We believe that the method developed in this work could give us a good understanding of the mathematical theory of vacuum, and also can be adapted to some other related vacuum problems in a more general framework, such as the inviscid limit problem for multi-dimensional finite-energy weak solutions in the whole space.

1.3. Symmetric Formulation and Main Results

We first need to analyze the mathematical structure of the momentum equations $(1.1)_2$ carefully, which can be divided into hyberbolic, elliptic and source parts as follows:

$$\underbrace{\rho\big(u_t+u\cdot\nabla u\big)+\nabla P}_{\text{Hyberbolic}}=\underbrace{-\varepsilon\rho^\delta Lu}_{\text{Elliptic}}+\underbrace{\varepsilon\nabla\rho^\delta\cdot\mathbb{S}(u)}_{\text{Source}}. \tag{1.16}$$

For smooth solutions (ρ, u) away from the vacuum, these equations could be written as

$$\underbrace{u_t+u\cdot\nabla u+\frac{A\gamma}{\gamma-1}\nabla\rho^{\gamma-1}-\frac{\delta}{\delta-1}\varepsilon\nabla\rho^{\delta-1}\cdot\mathbb{S}(u)}_{\text{Lower order}}=\underbrace{-\varepsilon\rho^{\delta-1}Lu}_{\text{Higher order}}. \tag{1.17}$$

Then if ρ is smooth enough, we could pass to the limit as $\rho\to 0$ on both sides of (1.17) and formally have

$$u_t+u\cdot\nabla u=0 \quad \text{when } \rho=0, \tag{1.18}$$

which, along with (1.17), implies that the velocity u can be governed by a nonlinear degenerate parabolic system if the density function contains vacuum.

In order to establish uniform a priori estimates for u in H^3 that is independent of ε and the lower bound of the initial density, we hope that the first order terms on the left-hand side of (1.17) could be put into a symmetric hyperbolic structure, then we can deal with the estimates on u in H^3 without being affected by the ε-dependent degenerate elliptic operator. However, this is impossible. The problem is that the term $\nabla\rho^{\delta-1}\cdot\mathbb{S}(u)$ is actually a product of two first order derivatives with the form $\sim\rho^{\delta-2}\nabla\rho\cdot\nabla u$, and obviously there is no similar term in the continuity equation $(1.1)_1$. This also tells us that it is not enough if we only have the estimates on ρ. Some more elaborate estimates for the density related quantities are really needed.

By introducing two new quantities,

$$\varphi=\rho^{\frac{\delta-1}{2}} \quad \text{and} \quad \phi=\rho^{\frac{\gamma-1}{2}},$$

equations (1.1) can be rewritten into a new system that consists of a transport equation for φ, and a "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled system with some special lower order source terms for (ϕ, u):

$$\begin{cases}\underbrace{\varphi_t+u\cdot\nabla\varphi+\frac{\delta-1}{2}\varphi\mathrm{div}u=0,}_{\text{Transport equations}}\\ \underbrace{A_0W_t+\sum_{j=1}^{3}A_j(W)\partial_jW=}_{\text{Symmetric hyperbolic}}\ \underbrace{-\varepsilon\varphi^2\mathbb{L}(W)}_{\text{Degenerate elliptic}}+\underbrace{\varepsilon\mathbb{H}(\varphi)\cdot\mathbb{Q}(W)}_{\text{Lower order source}},\end{cases} \tag{1.19}$$

where $W = (\phi, u)^\top$ and

$$\mathbb{L}(W) = \begin{pmatrix} 0 \\ a_1 L u \end{pmatrix}, \quad \mathbb{H}(\varphi) = \begin{pmatrix} 0 \\ \nabla \varphi^2 \end{pmatrix}, \quad \mathbb{Q}(W) = \begin{pmatrix} 0 & 0 \\ 0 & a_1 Q(u) \end{pmatrix}, \tag{1.20}$$

with $a_1 = \frac{(\gamma-1)^2}{4A\gamma} > 0$ and $Q(u) = \frac{\delta}{\delta-1}\mathbb{S}(u)$. Meanwhile, $\partial_j W = \partial x_j W$, and

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(\gamma-1)^2}{4A\gamma}\mathbb{I}_3 \end{pmatrix}, \quad A_j = \begin{pmatrix} u^{(j)} & \frac{\gamma-1}{2}\phi e_j \\ \frac{\gamma-1}{2}\phi e_j^\top & \frac{(\gamma-1)^2}{4A\gamma}u^{(j)}\mathbb{I}_3 \end{pmatrix}, \quad j = 1, 2, 3. \tag{1.21}$$

Here $e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$ $(j = 1, 2, 3)$ is the Kronecker symbol satisfying $\delta_{ij} = 1$, when $i = j$ and $\delta_{ij} = 0$, otherwise. For any $\xi \in \mathbb{R}^4$, we have

$$\xi^\top A_0 \xi \geqq a_2 |\xi|^2 \quad \text{with} \quad a_2 = \min\left\{1, \frac{(\gamma-1)^2}{4A\gamma}\right\} > 0. \tag{1.22}$$

For simplicity, we denote the symmetric hyperbolic structure shown in the right-hand side of $(1.19)_2$ as SH.

Considering the above system (1.19), first SH does not include all the first order terms related on $\left(\rho^{\frac{\gamma-1}{2}}, u\right)$ $\big($ i.e., $\varepsilon \mathbb{H}(\varphi) \cdot \mathbb{Q}(W)\big)$, so we called the equations $(1.19)_2$ a "quasi-symmetric hyperbolic"–"degenerate elliptic" system. Second, the characteristic speeds of SH in the direction $\mathbf{l} \in S^2$ are $u \cdot \mathbf{l}$, with multiplicity two, and $u \cdot \mathbf{l} \pm \sqrt{P_\rho}$, with multiplicity one, which means that this structure fails to be strictly hyperbolic near the vacuum even in one-dimensional or two-dimensional spaces. At last, this formulation implies that if we can give some reasonable analysis on the additional variable $\rho^{\frac{\delta-1}{2}}$, such that the $\rho^{\frac{\delta-1}{2}}$-related terms will vanish as $\varepsilon \to 0$ and also do not influence the estimates on $\left(\rho^{\frac{\gamma-1}{2}}, u\right)$, then it is hopeful for us to get the desired uniform estimates on $\left(\rho^{\frac{\gamma-1}{2}}, u\right)$ in H^3.

Based on the above observations, we first introduce a proper class of solutions to system (1.1) with arbitrarily large initial data and vacuum.

Definition 1.1. (*Regular solution to the Cauchy problem* (1.1)**-**(1.3)) Let $T > 0$ be a finite constant. A solution (ρ, u) to the Cauchy problem (1.1)–(1.3) is called a regular solution in $[0, T] \times \mathbb{R}^3$ if (ρ, u) satisfies this problem in the sense of distributions and:

(A) $\rho \geqq 0, \ \rho^{\frac{\delta-1}{2}} \in C([0, T]; H^3), \quad \rho^{\frac{\gamma-1}{2}} \in C([0, T]; H^3)$;

(B) $u \in C([0, T]; H^{s'}) \cap L^\infty(0, T; H^3), \quad \rho^{\frac{\delta-1}{2}} \nabla^4 u \in L^2(0, T; L^2)$;

(C) $u_t + u \cdot \nabla u = 0 \quad \text{as} \quad \rho(t, x) = 0$,

where $s' \in [2, 3)$ is an arbitrary constant.

In order to establish the vanishing viscosity limit from the viscous flow to the inviscid flow, first we give the following uniform (with respect to ε) local-in-time well-posedness to the Cauchy problem (1.1)–(1.3).

Theorem 1.2. (Uniform Regularity) *Let* (1.8) *hold. If initial data* (ρ_0, u_0) *satisfies*

$$\rho_0 \geqq 0, \quad \left(\rho_0^{\frac{\gamma-1}{2}}, \rho_0^{\frac{\delta-1}{2}}, u_0\right) \in H^3, \tag{1.23}$$

then there exists a time $T_* > 0$ *independent of* ε*, and a unique regular solution* (ρ, u) *in* $[0, T_*] \times \mathbb{R}^3$ *to the Cauchy problem* (1.1)–(1.3) *satisfying the following uniform estimates:*

$$\begin{aligned} &\sup_{0 \leqq t \leqq T_*} \left(||\rho^{\frac{\gamma-1}{2}}||_3^2 + ||\rho^{\frac{\delta-1}{2}}||_2^2 + \varepsilon|\rho^{\frac{\delta-1}{2}}|_{D^3}^2 + ||u||_2^2\right)(t) \\ &\qquad + ess \sup_{0 \leqq t \leqq T_*} |u(t)|_{D^3}^2 + \int_0^t \varepsilon|\rho^{\frac{\delta-1}{2}} \nabla^4 u|_2^2 ds \leqq C^0, \end{aligned} \tag{1.24}$$

for arbitrary constant $s' \in [2, 3)$ *and positive constant* $C^0 = C^0(\alpha, \beta, A, \gamma, \delta, \rho_0, u_0)$*. Actually,* (ρ, u) *satisfies the Cauchy problem (1.1)–(1.3) classically in positve time* $(0, T_*]$.

Moreover, if the following condition holds:

$$1 < min\{\delta, \gamma\} \leqq 5/3, \quad or \quad \delta = 2, 3, \quad or \quad \gamma = 2, 3, \tag{1.25}$$

we still have

$$\rho \in C([0, T_*]; H^3), \quad \rho_t \in C([0, T_*]; H^2). \tag{1.26}$$

Remark 1.1. The new varible ϕ is actually the constant multiple of local sound speed c of the hydrodynamics equations:

$$c = \sqrt{\frac{d}{d\rho} P(\rho)} \quad \left(= \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}} \text{ for polytropic flows}\right).$$

Remark 1.2. Compared with [27], we not only extend the viscosity power parameter δ to a broder region $1 < \min\{\delta, \gamma\} \leqq 3$, but also establish a more precise estimate that the life span of the regular solution has a uniformly positive lower bound with respect to ε. Actually, our desired a priori estimates mainly come from the "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure $(1.19)_2$, and the details could be seen in Section 3. Moreover, we point out that the regular solution obtained in the above theorem will break down in finite time, if the initial data contain "isolated mass group" or "hyperbolic singularity set", which could be rigorously proved via the same arguments used in [27].

Remark 1.3. If we relax the initial assumption from $\rho_0^{\frac{\gamma-1}{2}} \in H^3$ to $\rho_0^{\gamma-1} \in H^3$, then the correspongding local-in-time well-posedness for quantities $(\rho^{\gamma-1}, \rho^{\frac{\delta-1}{2}}, u)$ still can be obtained by the similar argument used to prove Theorem 1.2. However, for this case, the uniform positive lower bound of the life span and the uniform a priori estimates with respect to ε are not available, because the change of variable from c to c^2 has directly destroyed the symmetric hyperbolic structure as shown in the left hand side of $(1.19)_2$.

Letting $\varepsilon \to 0$, the solution obtained in Theorem 1.2 will strongly converge to that of the compressible Euler equations (1.9) in $C([0,T]; H^{s'})$ for any $s' \in [1,3)$. Meanwhile, we can also obtain the detail convergence rates, that is

Theorem 1.3. (Inviscid Limit) *Let* (1.8) *hold. Suppose that* $(\rho^\varepsilon, u^\varepsilon)$ *is the regular solution to the Cauchy problem* (1.1)–(1.3) *obtained in Theorem 1.2, and* (ρ, u) *is the regular solution to the Cauchy problem* (1.9) *with* (1.2)–(1.3) *obtained in Theorem 1.1. If*

$$(\rho^\varepsilon, u^\varepsilon)|_{t=0} = (\rho, u)|_{t=0} = (\rho_0, u_0) \tag{1.27}$$

satisfies (1.23)*, then* $\big(\rho^\varepsilon, u^\varepsilon\big)$ *converges to* (ρ, u) *as* $\varepsilon \to 0$ *in the that sense*

$$\lim_{\varepsilon\to 0} \sup_{0\leqq t\leqq T_*} \left(\left\| \left((\rho^\varepsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}} \right)(t) \right\|_{H^{s'}} + \left\| \left(u^\varepsilon - u\right)(t) \right\|_{H^{s'}} \right) = 0 \tag{1.28}$$

for any constant $s' \in [0,3)$*. Moreover, we also have*

$$\begin{aligned} &\sup_{0\leqq t\leqq T_*} \left(\left\| \left((\rho^\varepsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}} \right)(t) \right\|_1 + \left\| \left(u^\varepsilon - u\right)(t) \right\|_1 \right) \leqq C\varepsilon,\\ &\sup_{0\leqq t\leqq T_*} \left(\left| \left((\rho^\varepsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}} \right)(t) \right|_{D^2} + \left| \left(u^\varepsilon - u\right)(t) \right|_{D^2} \right) \leqq C\sqrt{\varepsilon}, \end{aligned} \tag{1.29}$$

where $C > 0$ *is a constant depending only on the fixed constants* $A, \delta, \gamma, \alpha, \beta, T_*$ *and* ρ_0, u_0*.*

Furthermore, if the condition (1.25) *holds, we still have*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \sup_{0\leqq t\leqq T_*} \left(\left\| \left(\rho^\varepsilon - \rho\right)(t) \right\|_{H^{s'}} + \left\| \left(u^\varepsilon - u\right)(t) \right\|_{H^{s'}} \right) = 0,\\ &\sup_{0\leqq t\leqq T_*} \left(\left\| (\rho^\varepsilon - \rho)(t) \right\|_1 + \left\| \left(u^\varepsilon - u\right)(t) \right\|_1 \right) \leqq C\varepsilon,\\ &\sup_{0\leqq t\leqq T_*} \left(\left| \left(\rho^\varepsilon - \rho\right)(t) \right|_{D^2} + \left| \left(u^\varepsilon - u\right)(t) \right|_{D^2} \right) \leqq C\sqrt{\varepsilon}. \end{aligned} \tag{1.30}$$

Remark 1.4. It should be pointed out that conclusions of Theorems 1.2–1.3 still hold when viscosities μ and λ are in the general form

$$\mu(\rho) = \varepsilon\rho^\delta \alpha(\rho), \quad \lambda(\rho) = \varepsilon\rho^\delta \beta(\rho), \tag{1.31}$$

where $\alpha(\rho)$ and $\beta(\rho)$ are functions of ρ, satisfying

$$\big(\alpha(\rho), \beta(\rho)\big) \in C^4(\mathbb{R}^+), \quad \alpha(\rho) \geqq C_0 > 0, \quad \text{and} \quad 2\alpha(\rho) + 3\beta(\rho) \geqq 0. \tag{1.32}$$

For function pairs $(g_1(\rho), g_2(\rho))$ have the form $\sim \rho^p$, it is obviously that they do not belong to $C^4(\mathbb{R}^+)$ when $p < 4$ due to the presence of the vacuum. However, Theorems 1.2–1.3 still hold if the initial data satisfies the additional initial assumptions

$$\begin{cases} \alpha(\rho) = P_{k_1}(\rho) + 1, \quad \beta(\rho) = P_{k_2}(\rho),\\ P_{k_1}(\rho_0) \in H^3, \quad P_{k_2}(\rho_0) \in H^3, \end{cases} \tag{1.33}$$

where $P_{k_i}(\rho)$ $(i = 1, 2)$ is a k_i-th degree polynomial of ρ with vanishing constant term, and the minimum power of density in all the terms of $P_{k_i}(\rho)$ should be greater than or equal to $\frac{\delta-1}{2}$. Furthermore, our framework in this paper is applicable to other physical dimensions, say 1 and 2, after some minor modifications.

The rest of the paper is organized as follows: in Section 2, we list some basic lemmas that will be used in our proof. In Section 3, based on some uniform estimates for

$$\left(\rho^{\frac{\gamma-1}{2}}, u\right) \quad \text{in} \quad H^3, \quad \text{and} \quad \rho^{\frac{\delta-1}{2}} \quad \text{in} \quad H^2,$$

we will give the proof for the uniform (with respect to ε) local-in-time well-posedness of the strong solution to the reformulated Cauchy problem (3.1), which is achieved in the following four steps:

(1) Via introducing a uniform elliptic operator $\varepsilon(\varphi^2+\eta^2)Lu$ with artificial viscosity coefficients $\eta^2 > 0$ in momentum equations, the global well-posedness of the approximation solution to the corresponding linearized problem (3.6) for (φ, ϕ, u) has been established (Section 3.1).
(2) We establish the uniform a priori estimates with respect to (η, ε) for

$$\left(\phi, u\right) \quad \text{in} \quad H^3, \quad \text{and} \quad \varphi \quad \text{in} \quad H^2,$$

to the linearized problem (3.6) in $[0, T_*]$, where the time T_* is also independent of (η, ε) (Section 3.2).
(3) Via passing to the limit as $\eta \to 0$, we obtain the solution of the linearized problem (3.60), which allows that the elliptic operator appearing in the reformulated momentum equations is degenerate (Section 3.3).
(4) Based on the uniform analysis for the linearized problem, we prove the uniform (with respect to ε) local-in-time well-posedness of the non-linear reformulated problem through the Picard iteration approach (Section 3.4).

According to the uniform local-in-time well-posedness and a priori estimates (with respect to ε) to non-linearized problem (3.1) obtained in Section 3, in Section 4, we will give the proof for Theorem 1.2. Finally in Section 5, the convergence rates from the viscous flow to inviscid flow will be obtained, which is the proof of Theorem 1.3.

2. Preliminaries

In this section, we show some basic lemmas that will be frequently used in the proofs to follow. The first one is the well-known Gagliardo-Nirenberg inequality.

Lemma 2.1. [23] *For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for*

$$f \in H^1(\mathbb{R}^3), \quad \textit{and} \quad g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

we have

$$|f|_p^p \leqq C|f|_2^{(6-p)/2}|\nabla f|_2^{(3p-6)/2},$$
$$|g|_\infty \leqq C|g|_q^{q(r-3)/(3r+q(r-3))}|\nabla g|_r^{3r/(3r+q(r-3))}. \tag{2.1}$$

Some special versions of this inequality can be written as

$$|u|_6 \leqq C|u|_{D^1}, \quad |u|_\infty \leqq C\|\nabla u\|_1, \quad |u|_\infty \leqq C\|u\|_{W^{1,r}}, \quad \text{for} \quad r > 3. \tag{2.2}$$

The second one can be found in Majda [30]. Here we omit its proof.

Lemma 2.2. [30] *Let constants r, a and b satisfy the relation*

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad \textit{and} \quad 1 \leqq a,\ b,\ r \leqq \infty.$$

$\forall s \geqq 1$, *if $f, g \in W^{s,a} \cap W^{s,b}(\mathbb{R}^3)$, then we have*

$$|\nabla^s(fg) - f\nabla^s g|_r \leqq C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_b|g|_a\big), \tag{2.3}$$
$$|\nabla^s(fg) - f\nabla^s g|_r \leqq C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_a|g|_b\big), \tag{2.4}$$

where $C_s > 0$ is a constant only depending on s, and $\nabla^s f$ $(s > 1)$ is the set of all $\partial_x^\zeta f$ with $|\zeta| = s$. Here $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$ is a multi-index.

The third one will show some compactness results from the Aubin-Lions Lemma.

Lemma 2.3. [38] *Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Then:*

I) Let G be bounded in $L^p(0, T; X_0)$ where $1 \leqq p < \infty$, and $\frac{\partial G}{\partial t}$ be bounded in $L^1(0, T; X_1)$, then G is relatively compact in $L^p(0, T; X)$.
II) Let F be bounded in $L^\infty(0, T; X_0)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^p(0, T; X_1)$ with $p > 1$, then F is relatively compact in $C(0, T; X)$.

The following lemma will be used to show the time continuity for the higher order terms of our solution:

Lemma 2.4. [1] *If $f(t, x) \in L^2([0, T]; L^2)$, then there exists a sequence s_k such that*

$$s_k \to 0, \quad \textit{and} \quad s_k|f(s_k, x)|_2^2 \to 0, \quad \textit{as} \quad k \to +\infty.$$

Next we give some Sobolev inequalities on the interpolation estimate, product estimate, composite function estimate and so on in the following three lemmas:

Lemma 2.5. [30] *Let $u \in H^s$, then for any $s' \in [0, s]$, there exists a constant C_s only depending on s such that*

$$\|u\|_{s'} \leqq C_s\|u\|_0^{1-\frac{s'}{s}}\|u\|_s^{\frac{s'}{s}}.$$

Lemma 2.6. [30] *Let functions $u,\ v \in H^s$ and $s > \frac{3}{2}$, then $u \cdot v \in H^s$, and there exists a constant C_s only depending on s such that*

$$\|uv\|_s \leqq C_s \|u\|_s \|v\|_s.$$

Lemma 2.7. [30]

(1) For functions $f,\ g \in H^s \cap L^\infty$ and $|\nu| \leqq s$, there exists a constant C_s only depending on s such that

$$\|\nabla^\nu (fg)\|_s \leqq C_s(|f|_\infty |\nabla^s g|_2 + |g|_\infty |\nabla^s f|_2). \tag{2.5}$$

(2) Assume that $g(u)$ is a smooth vector-valued function on G, $u(x)$ is a continuous function with $u \in H^s \cap L^\infty$. Then for $s \geqq 1$, there exists a constant C_s only depending on s such that

$$|\nabla^s g(u)|_2 \leqq C_s \left\| \frac{\partial g}{\partial u} \right\|_{s-1} |u|_\infty^{s-1} |\nabla^s u|_2. \tag{2.6}$$

The last lemma is a useful tool for improving the weak convergence to a strong one.

Lemma 2.8. [30] *If function sequence $\{w_n\}_{n=1}^\infty$ converges weakly in a Hilbert space X to w, then w_n converges strongly to w in X if and only if*

$$\|w\|_X \geqq \lim sup_{n\to\infty} \|w_n\|_X.$$

For simplicity, by introducing four matrices $A_1, A_2, A_3, B = (b_{ij}) = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, a vector $W = (w_1, w_2, w_3)^\top$, and letting $A = (A_1, A_2, A_3)$, we denote

$$\begin{cases} \operatorname{div} A = \sum_{j=1}^3 \partial_j A_j, \quad W^\top B W = \sum_{i,j=1}^3 b_{ij} w_i w_j, \\ |B|_2^2 = B : B = \sum_{i,j=1}^3 b_{ij}^2, \quad W \cdot B = w_1 \mathbf{b}_1 + w_2 \mathbf{b}_2 + w_3 \mathbf{b}_3. \end{cases} \tag{2.7}$$

The above symbols will be used throughtout the rest of the paper.

3. Uniform Regularity

In this section, we will establish the desired uniform regularity shown in Theorem 1.2. As the discussion shown in Subsection 1.3, for this purpose we need to consider the following reformulated problem:

$$\begin{cases} \varphi_t + u \cdot \nabla \varphi + \frac{\delta - 1}{2} \varphi \operatorname{div} u = 0, \\ A_0 W_t + \sum_{j=1}^3 A_j(W) \partial_j W + \varepsilon \varphi^2 \mathbb{L}(W) = \varepsilon \mathbb{H}(\varphi) \cdot \mathbb{Q}(W), \\ (\varphi, W)|_{t=0} = (\varphi_0, W_0), \quad x \in \mathbb{R}^3, \\ (\varphi, W) \to (0, 0), \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0, \end{cases} \tag{3.1}$$

where $W = (\phi, u)^\top$ and

$$(\varphi_0, W_0) = (\varphi, \phi, u)|_{t=0} = (\varphi_0, \phi_0, u_0) \\ = \left(\rho_0^{\frac{\delta-1}{2}}(x), \rho_0^{\frac{\gamma-1}{2}}(x), u_0(x)\right), \quad x \in \mathbb{R}^3. \tag{3.2}$$

The definitions of A_j $(j = 0, 1, ..., 3)$, $\mathbb{L}$, $\mathbb{H}$ and $\mathbb{Q}$ can be found in (1.20)–(1.22).

To prove Theorem 1.2, our first step is to establish the following existence of the unique strong solutions for the reformulated problem (3.1):

Theorem 3.1. *If the initial data* (φ_0, ϕ_0, u_0) *satisfy*

$$\varphi_0 \geqq 0, \quad \phi_0 \geqq 0, \quad (\varphi_0, \phi_0, u_0) \in H^3, \tag{3.3}$$

then there exists a positive time T_* *independent of* ε, *and a unique strong solution* (φ, ϕ, u) *in* $[0, T_*] \times \mathbb{R}^3$ *to the Cauchy problem (3.1) satisfying*

$$\begin{aligned} &\varphi \in C([0, T_*]; H^3), \quad \phi \in C([0, T_*]; H^3), \\ &u \in C([0, T_*]; H^{s'}) \cap L^\infty([0, T_*]; H^3), \\ &\varphi \nabla^4 u \in L^2([0, T_*]; L^2), \quad u_t \in C([0, T_*]; H^1) \cap L^2([0, T_*]; D^2), \end{aligned} \tag{3.4}$$

for any constant $s' \in [2, 3)$. *Moreover, we can also obtain the following uniform estimates:*

$$\begin{aligned} &\sup_{0 \leqq t \leqq T_*} \left(\|\varphi\|_2^2 + \varepsilon|\varphi|_{D^3}^2 + \|\phi\|_3^2 + \|u\|_2^2\right)(t) \\ &+ ess \sup_{0 \leqq t \leqq T_*} |u(t)|_{D^3}^2 + \int_0^{T_*} \varepsilon |\varphi \nabla^4 u|_2^2 \, dt \leqq C^0, \end{aligned} \tag{3.5}$$

where C^0 *is a positive constant depending only on* T_*, (φ_0, ϕ_0, u_0) *and the fixed constants* A, δ, γ, α *and* β, *and is independent of* ε.

We will subsequently prove Theorem 3.1 through the next four Subsections (3.1–3.4), and in the next section, we will show that this theorem indeed implies Theorem 1.2.

3.1. Linearization with an Artificial Strong Elliptic Operator

Let T be any positive time. In order to construct the local strong solutions for the nonlinear problem, we need to consider the following linearized approximation problem:

$$\begin{cases} \varphi_t + v \cdot \nabla \varphi + \dfrac{\delta - 1}{2} \omega \mathrm{div} v = 0, \\ A_0 W_t + \displaystyle\sum_{j=1}^{3} A_j(V) \partial_j W + \varepsilon(\varphi^2 + \eta^2) \mathbb{L}(W) = \varepsilon \mathbb{H}(\varphi) \cdot \mathbb{Q}(V), \\ (\varphi, W)|_{t=0} = (\varphi_0, W_0), \quad x \in \mathbb{R}^3, \\ (\varphi, W) \to (0, 0), \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0, \end{cases} \tag{3.6}$$

where $\eta \in (0, 1]$ is a constant, $W = (\phi, u)^\top$, $V = (\psi, v)^\top$ and $W_0 = (\phi_0, u_0)^\top$. (ω, ψ) are both known functions and $v = (v^{(1)}, v^{(2)}, v^{(3)})^\top \in \mathbb{R}^3$ is a known vector satisfying the initial assumption $(\omega, \psi, v)(t = 0, x) = (\varphi_0, \phi_0, u_0)$ and

$$\begin{aligned} &\omega \in C([0, T]; H^3), \quad \omega_t \in C([0, T]; H^2), \quad \psi \in C([0, T]; H^3),\\ &\psi_t \in C([0, T]; H^2), \quad v \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3),\\ &\omega\nabla^4 v \in L^2([0, T]; L^2), \quad v_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2) \end{aligned} \tag{3.7}$$

for any constant $s' \in [2, 3)$. Moreover, we assume that

$$\varphi_0 \geqq 0, \quad \phi_0 \geqq 0, \quad (\varphi_0, W_0) \in H^3. \tag{3.8}$$

Now we have the following global existence of a strong solution (φ, ϕ, u) to (3.6) by the standard methods at least when $\eta > 0$:

Lemma 3.1. *Assume that the initial data* (φ_0, ϕ_0, u_0) *satisfy* (3.8)*. Then there exists a unique strong solution* (φ, ϕ, u) *in* $[0, T] \times \mathbb{R}^3$ *to* (3.6) *when* $\eta > 0$ *such that*

$$\begin{aligned} &\varphi \in C([0, T]; H^3), \ \phi \in C([0, T]; H^3),\\ &u \in C([0, T]; H^3) \cap L^2([0, T]; D^4), \ u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2). \end{aligned} \tag{3.9}$$

Proof. First, the existence and regularities of a unique solution φ in $(0, T) \times \mathbb{R}^3$ to the equation $(3.6)_1$ can be obtained by the standard theory of transport equation (see [13]).

Second, when $\eta > 0$, based on the regularities of φ, it is not difficult to solve W from the linear symmetric hyperbolic-parabolic coupled system $(3.6)_2$ to complete the proof of this lemma (see [13]). Here we omit its details. □

In the next two subsections, we first establish the uniform estimates for (ϕ, u) in H^3 space with respect to both η and ε, then we pass to the limit for the case: $\eta = 0$.

3.2. A Priori Estimates Independent of (η, ε)

Let (φ, ϕ, u) be the unique strong solution to (3.6) in $[0, T] \times \mathbb{R}^3$ obtained in Lemma 3.1. In this subsection, we will get some local (in time) a priori estimates for (ϕ, u) in H^3 space, which are independent of (η, ε) listed in the following Lemmas 3.2–3.5. For this purpose, we fix $T > 0$ and a positive constant c_0 large enough such that

$$2 + \|\varphi_0\|_3 + \|\phi_0\|_3 + \|u_0\|_3 \leqq c_0, \tag{3.10}$$

and

$$\sup_{0\leqq t\leqq T^*}\left(\|\omega(t)\|_1^2+\|\psi(t)\|_1^2+\|v(t)\|_1^2\right)+\int_0^{T^*}\varepsilon|\omega\nabla^2 v|_2^2\,\mathrm{d}t\leqq c_1^2,$$
$$\sup_{0\leqq t\leqq T^*}\left(|\omega(t)|_{D^2}^2+|\psi(t)|_{D^2}^2+|v(t)|_{D^2}^2\right)+\int_0^{T^*}\varepsilon|\omega\nabla^3 v|_2^2\,\mathrm{d}t\leqq c_2^2,$$
$$\operatorname*{ess\,sup}_{0\leqq t\leqq T^*}\left(|\psi(t)|_{D^3}^2+|v(t)|_{D^3}^2+\varepsilon|\omega(t)|_{D^3}^2\right)+\int_0^{T^*}\varepsilon|\omega\nabla^4 v|_2^2\,\mathrm{d}t\leqq c_3^2 \tag{3.11}$$

for some time $T^*\in(0,T)$ and constants c_i $(i=1,2,3)$ such that

$$1<c_0\leqq c_1\leqq c_2\leqq c_3.$$

The constants c_i $(i=1,2,3)$ and T^* will be determined later (see (3.58)) and depend only on c_0 and the fixed constants α, β, γ, A, δ and T.

Hereinafter, we use $C\geqq 1$ to denote a generic positive constant depending only on fixed constants α, β, γ, A, δ and T, but is independent of (η,ε), which may be different from line to line. We start from the estimates for φ.

Lemma 3.2. *Let* (φ, W) *be the unique strong solution to* (3.6) *on* $[0,T]\times\mathbb{R}^3$. *Then*

$$1+|\varphi(t)|_\infty^2+\|\varphi(t)\|_2^2\leqq Cc_0^2,\quad \varepsilon|\varphi(t)|_{D^3}^2\leqq Cc_0^2,$$
$$|\varphi_t(t)|_2^2\leqq Cc_1^4,\quad |\varphi_t(t)|_{D^1}^2\leqq Cc_2^4,\quad \varepsilon|\varphi_t(t)|_{D^2}^2\leqq Cc_3^4,$$

for $0\leqq t\leqq T_1=\min(T^*,(1+c_3)^{-2})$.

Proof. We apply the operator ∂_x^ζ $(0\leqq|\zeta|\leqq 3)$ to $(3.6)_1$, and obtain

$$(\partial_x^\zeta\varphi)_t+v\cdot\nabla\partial_x^\zeta\varphi=-(\partial_x^\zeta(v\cdot\nabla\varphi)-v\cdot\nabla\partial_x^\zeta\varphi)-\frac{\delta-1}{2}\partial_x^\zeta(\omega\mathrm{div}v). \tag{3.12}$$

Then multiplying both sides of (3.12) by $\partial_x^\zeta\varphi$, and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}|\partial_x^\zeta\varphi|_2^2\leqq C|\mathrm{div}v|_\infty|\partial_x^\zeta\varphi|_2^2+C\Lambda_1^\zeta|\partial_x^\zeta\varphi|_2+C\Lambda_2^\zeta|\partial_x^\zeta\varphi|_2, \tag{3.13}$$

where

$$\Lambda_1^\zeta=|\partial_x^\zeta(v\cdot\nabla\varphi)-v\cdot\nabla\partial_x^\zeta\varphi|_2,\quad \Lambda_2^\zeta=|\partial_x^\zeta(\omega\mathrm{div}v)|_2.$$

First, when $|\zeta|\leqq 2$, we consider the term Λ_1^ζ and Λ_2^ζ. It follows from Lemma 2.1 and Hölder's inequality that

$$\begin{aligned}|\Lambda_1^\zeta|_2\leqq&C(|\nabla v\cdot\nabla\varphi|_2+|\nabla v\cdot\nabla^2\varphi|_2+|\nabla^2 v\cdot\nabla\varphi|_2)\\ \leqq&C(|\nabla v|_\infty\|\nabla\varphi\|_1+|\nabla^2 v|_3|\nabla\varphi|_6)\leqq C\|\nabla v\|_2\|\varphi\|_2,\\ |\Lambda_2^\zeta|_2\leqq&C\|w\|_2\|v\|_3,\end{aligned} \tag{3.14}$$

which, along with (3.13)–(3.14), implies that

$$\frac{d}{dt}\|\varphi(t)\|_2 \leqq C\|\nabla v\|_2\|\varphi\|_2 + C\|w\|_2\|v\|_3. \tag{3.15}$$

Then, according to Gronwall's inequality, one has

$$\|\varphi(t)\|_2 \leqq \Big(\|\varphi_0\|_2 + c_3^2 t\Big)\exp(Cc_3 t) \leqq Cc_0^2 \tag{3.16}$$

for $0 \leqq t \leqq T_1 = \min\{T^*, (1+c_3)^{-2}\}$.

Second, when $|\zeta| = 3$, it follows from Lemma 2.1 and Hölder's inequality that

$$\begin{aligned}
|\Lambda_1^\zeta|_2 \leqq& C(|\nabla v\cdot\nabla^3\varphi|_2 + |\nabla^2 v\cdot\nabla^2\varphi|_2 + |\nabla^3 v\cdot\nabla\varphi|_2) \leqq C\|\nabla v\|_2\|\nabla\varphi\|_2,\\
|\Lambda_2^\zeta|_2 \leqq& C(|\omega\nabla^4 v|_2 + |\nabla\omega\cdot\nabla^3 v|_2 + |\nabla^2\omega\cdot\nabla^2 v|_2 + |\nabla^3\omega\cdot\nabla v|_2)\\
\leqq& C|\omega\nabla^4 v|_2 + C\|w\|_3\|v\|_3.
\end{aligned} \tag{3.17}$$

Then, combining (3.13)–(3.17), we arrive at

$$\begin{aligned}
\frac{d}{dt}|\nabla^3\varphi(t)|_2 \leqq& C\big(\|\nabla v\|_2\|\nabla\varphi\|_2 + |\omega\nabla^4 v|_2 + \|w\|_3\|v\|_3\big)\\
\leqq& C(c_3|\nabla^3\varphi|_2 + c_3^2 + c_3^2\varepsilon^{-\frac12} + |\omega\nabla^4 v|_2),
\end{aligned} \tag{3.18}$$

which, along with Gronwall's inequality, implies that

$$|\varphi(t)|_{D^3} \leqq \Big(|\varphi_0|_{D^3} + c_3^2 t + c_3^2\varepsilon^{-\frac12}t + \int_0^t |\omega\nabla^4 v|_2\,\mathrm{d}s\Big)\exp(Cc_3 t). \tag{3.19}$$

Therefore, observing that

$$\int_0^t |\omega\nabla^4 v|_2\,\mathrm{d}s \leqq \varepsilon^{-\frac12}t^{\frac12}\Big(\int_0^t |\varepsilon^{\frac12}\omega\nabla^4 v|_2^2\,\mathrm{d}s\Big)^{\frac12} \leqq Cc_3 t^{\frac12}\varepsilon^{-\frac12},$$

from (3.19), one can obtain that

$$|\varphi(t)|_{D^3} \leqq C(c_0 + \varepsilon^{-\frac12}), \quad \text{for} \quad 0 \leqq t \leqq T_1.$$

At last, the estimates for φ_t follow from the relation:

$$\varphi_t = -v\cdot\nabla\varphi - \frac{\delta-1}{2}\omega\mathrm{div}v.$$

For $0 \leqq t \leqq T_1$, we easily have

$$\begin{aligned}
|\varphi_t(t)|_2 \leqq& C\big(|v(t)|_6|\nabla\varphi(t)|_3 + |\omega(t)|_\infty|\mathrm{div}v(t)|_2\big) \leqq Cc_2^2,\\
|\varphi_t(t)|_{D^1} \leqq& C\big(|v(t)|_\infty|\nabla^2\varphi(t)|_2 + |\nabla v(t)|_6|\nabla\varphi(t)|_3 + |\omega(t)|_\infty|\nabla^2 v(t)|_2\big)\\
&+ C|\nabla v(t)|_6|\nabla\omega(t)|_3 \leqq Cc_2^2,\\
|\varphi_t(t)|_{D^2} \leqq& C\big(|v(t)|_\infty|\nabla^3\varphi(t)|_2 + |\nabla v(t)|_\infty|\nabla^2\varphi(t)|_2 + |\nabla^2 v(t)|_6|\nabla\varphi(t)|_3\big)\\
&+ C\big(|\omega(t)|_\infty|\nabla^3 v(t)|_2 + |\nabla\omega(t)|_6|\nabla^2 v(t)|_3 + |\nabla^2\omega(t)|_2|\nabla v(t)|_\infty\big)\\
\leqq& C(c_3^2 + c_3\varepsilon^{-\frac12}).
\end{aligned} \tag{3.20}$$

Thus, we complete the proof of this lemma. □

Based on Gagliardo-Nirenberg-Sobolev and interpolation inequalities, we firstly present several useful inequalities

$$
\begin{cases}
|\varphi|_\infty \leqq C|\varphi|_6^{\frac{1}{2}}|\nabla\varphi|_6^{\frac{1}{2}} \leqq C|\nabla\varphi|_2^{\frac{1}{2}}|\nabla^2\varphi|_2^{\frac{1}{2}} \leqq Cc_0,\\
|\nabla\varphi|_\infty \leqq C|\nabla\varphi|_6^{\frac{1}{2}}|\nabla^2\varphi|_6^{\frac{1}{2}} \leqq C|\nabla^2\varphi|_2^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}} \leqq Cc_0\varepsilon^{-\frac{1}{4}},\\
|\nabla\varphi|_3 \leqq C|\nabla\varphi|_2^{\frac{1}{2}}|\nabla\varphi|_6^{\frac{1}{2}} \leqq C|\nabla\varphi|_2^{\frac{1}{2}}|\nabla^2\varphi|_2^{\frac{1}{2}} \leqq Cc_0,\\
|\nabla^2\varphi|_3 \leqq C|\nabla^2\varphi|_2^{\frac{1}{2}}|\nabla^2\varphi|_6^{\frac{1}{2}} \leqq C|\nabla^2\varphi|_2^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}} \leqq Cc_0\varepsilon^{-\frac{1}{4}},\\
|\nabla\varphi|_6 \leqq C|\nabla^2\varphi|_2 \leqq Cc_0, \quad |\nabla^2\varphi|_6 \leqq C|\nabla^3\varphi|_2 \leqq Cc_0\varepsilon^{-\frac{1}{2}},
\end{cases}
\tag{3.21}
$$

which will be frequently used in the proofs to follow.

Using the notations in (2.7), now we show the estimate for $\|W\|_1$.

Lemma 3.3. *Let* (φ, W) *be the unique strong solution to (3.6) on* $[0, T]\times\mathbb{R}^3$. *Then*

$$
\|W(t)\|_1^2 + \varepsilon\int_0^t |\sqrt{\varphi^2+\eta^2}\nabla^2 u|_2^2\, ds \leqq Cc_0^2,
$$

for $0 \leqq t \leqq T_2 = \min\{T_1, (1+c_3)^{-4}\}$.

Proof. Applying the operator ∂_x^ζ to $(3.6)_2$, we have

$$
\begin{aligned}
&A_0\partial_x^\zeta W_t + \sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta W + \varepsilon(\varphi^2+\eta^2)\mathbb{L}(\partial_x^\zeta W)\\
&= \mathbb{H}(\varphi)\cdot\partial_x^\zeta\mathbb{Q}(V) - \sum_{j=1}^3\Big(\partial_x^\zeta(A_j(V)\partial_j W) - A_j(V)\partial_j\partial_x^\zeta W\Big)\\
&\quad - \varepsilon\Big(\partial_x^\zeta((\varphi^2+\eta^2)\mathbb{L}(W)) - (\varphi^2+\eta^2)\mathbb{L}(\partial_x^\zeta W)\Big)\\
&\quad + \varepsilon\Big(\partial_x^\zeta(\mathbb{H}(\varphi)\cdot\mathbb{Q}(V)) - \mathbb{H}(\varphi)\cdot\partial_x^\zeta\mathbb{Q}(V)\Big).
\end{aligned}
\tag{3.22}
$$

Then multiplying (3.22) by $\partial_x^\zeta W$ on both sides and integrating over $\mathbb{R}^3$ by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \Big((\partial_x^\zeta W)^\top A_0\partial_x^\zeta W\Big) + a_1\varepsilon\alpha|\sqrt{\varphi^2+\eta^2}\nabla\partial_x^\zeta u|_2^2\\
&\quad + a_1\varepsilon(\alpha+\beta)|\sqrt{\varphi^2+\eta^2}\mathrm{div}\partial_x^\zeta u|_2^2\\
&= \int(\partial_x^\zeta W)^\top \mathrm{div}A(V)\partial_x^\zeta W + a_1\varepsilon\int\Big(\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\\
&\quad - \frac{\delta-1}{\delta}a_1\varepsilon\int\Big(\nabla(\varphi^2+\eta^2)\cdot Q(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
&\quad - \sum_{j=1}^3\int\Big(\partial_x^\zeta(A_j(V)\partial_j W) - A_j(V)\partial_j\partial_x^\zeta W\Big)\cdot\partial_x^\zeta W
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&-a_1\varepsilon\int\Big(\partial_x^\zeta((\varphi^2+\eta^2)Lu)-(\varphi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\
&+a_1\varepsilon\int\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u:=\sum_{i=1}^{6}I_i.
\end{aligned}
$$

Now we consider the terms on the right-hand side of (3.23) when $|\zeta|\leqq 1$. First, it follows from Lemmas 2.1, 3.2, Hölder's inequality and Young's inequality that

$$
\begin{aligned}
I_1=&\int(\partial_x^\zeta W)^\top \mathrm{div}A(V)\partial_x^\zeta W\\
\leqq&C|\nabla V|_\infty|\partial_x^\zeta W|_2^2\leqq C|\nabla V|_6^{\frac12}|\nabla^2 V|_6^{\frac12}|\partial_x^\zeta W|_2^2\\
\leqq&C|\nabla^2 V|_2^{\frac12}|\nabla^3 V|_2^{\frac12}|\partial_x^\zeta W|_2^2\leqq Cc_3|\partial_x^\zeta W|_2^2,\\
I_2=&a_1\varepsilon\int\Big(\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\\
\leqq&C\varepsilon|\varphi|_\infty|\nabla\varphi|_3|\nabla\partial_x^\zeta v|_6|\partial_x^\zeta u|_2\leqq C\varepsilon c_3^3|\partial_x^\zeta u|_2,\\
I_3=&-\frac{\delta-1}{\delta}a_1\varepsilon\int\Big(\nabla(\varphi^2+\eta^2)\cdot Q(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
\leqq&C\varepsilon|\nabla\varphi|_\infty|\varphi\nabla\partial_x^\zeta u|_2|\partial_x^\zeta u|_2\\
\leqq&\frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla\partial_x^\zeta u|_2^2+Cc_0^2\varepsilon^{\frac12}|\partial_x^\zeta u|_2^2,\\
I_4=&-\sum_{j=1}^{3}\int\Big(\partial_x^\zeta(A_j(V)\partial_j W)-A_j(V)\partial_j\partial_x^\zeta W\Big)\partial_x^\zeta W\\
\leqq&C|\partial_x^\zeta(A_j(V)\partial_j W)-A_j(V)\partial_j\partial_x^\zeta W|_2|\partial_x^\zeta W|_2\\
\leqq&C|\nabla V|_\infty|\nabla W|_2^2\leqq Cc_3|\nabla W|_2^2,
\end{aligned}
\tag{3.24}
$$

where we have used the fact (3.21).

Similarly, for the terms I_5-I_6, using (3.21), one can obtain that

$$
\begin{aligned}
I_5=&-a_1\varepsilon\int\Big(\partial_x^\zeta((\varphi^2+\eta^2)Lu)-(\varphi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\
\leqq&C\varepsilon|\nabla\varphi|_\infty|\varphi Lu|_2|\partial_x^\zeta u|_2\\
\leqq&\frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^2 u|_2^2+Cc_0^2\varepsilon^{\frac12}|\partial_x^\zeta u|_2^2,\\
I_6=&a_1\varepsilon\int\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\\
\leqq&C\varepsilon\big(|\varphi|_\infty|\nabla v|_\infty|\nabla^2\varphi|_2+|\nabla\varphi|_6|\nabla\varphi|_3|\nabla v|_\infty\big)|\partial_x^\zeta u|_2\\
\leqq&Cc_3^3\varepsilon|\partial_x^\zeta u|_2\leqq Cc_3^3\varepsilon+Cc_3^3\varepsilon|\partial_x^\zeta u|_2^2.
\end{aligned}
\tag{3.25}
$$

Then from (3.23)–(3.25), we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\big((\partial_x^\zeta W)^\top A_0\partial_x^\zeta W\big)+\frac12 a_1\varepsilon\alpha|\sqrt{\varphi^2+\eta^2}\nabla\partial_x^\zeta u|_2^2\\
&\leqq C\big(c_3^2+c_3^4\varepsilon\big)\|W\|_1^2+Cc_3^2\varepsilon,
\end{aligned}
\tag{3.26}
$$

which, along with Gronwall's inequality, implies that

$$\begin{aligned}&\|W(t)\|_1^2+\varepsilon\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^2 u|_2^2\,\mathrm{d}s\\&\leqq C\big(\|W_0\|_1^2+c_3^2\varepsilon t\big)\exp(C(c_3^2+c_3^4\varepsilon)t)\leqq Cc_0^2,\end{aligned}\tag{3.27}$$

for $0\leqq t\leqq T_2=\min\{T_1,(1+c_3)^{-4}\}$. □

Next we show the estimate for $|W|_{D^2}$.

Lemma 3.4. *Let* (φ, W) *be the unique strong solution to (3.6) on* $[0,T]\times\mathbb{R}^3$. *Then*

$$|W(t)|_{D^2}^2+\varepsilon\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2\,ds\leqq Cc_0^2,,\quad \textit{for}\quad 0\leqq t\leqq T_2;$$

$$|W_t(t)|_2^2+|\phi_t(t)|_{D^1}^2+\int_0^t|\nabla u_t|_2^2\,ds\leqq Cc_2^6,\quad \textit{for}\quad 0\leqq t\leqq T_2.$$

Proof. We divide the proof into two steps.

Step 1: the estimate of $|W|_{D^2}$. First we need to consider the terms on the right-hand side of (3.23) when $|\zeta|=2$. It follows from Lemmas 2.1 and (3.23), Hölder's inequlaity and Young's inequality that

$$\begin{aligned}I_1=&\frac{1}{2}\int \mathrm{div}A(V)|\partial_x^\zeta W|^2\leqq C|\mathrm{div}A(V)|_\infty|\partial_x^\zeta W|_2^2\leqq Cc_3|\partial_x^\zeta W|_2^2,\\I_2=&a_1\varepsilon\int\Big(\nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\leqq C\varepsilon|\nabla\varphi|_3|\nabla\partial_x^\zeta v|_2|\varphi\partial_x^\zeta u|_6\\\leqq&C\varepsilon c_3^2\big(c_0\varepsilon^{-\frac14}|\partial_x^\zeta u|_2+|\varphi\nabla\partial_x^\zeta u|_2\big)\\\leqq&\frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2+Cc_3^2\varepsilon^{\frac12}|\partial_x^\zeta u|_2^2+Cc_3^4\varepsilon,\end{aligned}\tag{3.28}$$

where we have used (3.21) and

$$\begin{aligned}|\varphi\partial_x^\zeta u|_6\leqq&C|\varphi\partial_x^\zeta u|_{D^1}\leqq C\big(|\varphi\nabla\partial_x^\zeta u|_2+|\nabla\varphi|_\infty|\partial_x^\zeta u|_2\big)\\\leqq&C\big(|\varphi\nabla\partial_x^\zeta u|_2+c_0\varepsilon^{-\frac14}|\partial_x^\zeta u|_2\big).\end{aligned}\tag{3.29}$$

For the term I_3, via integration by parts and (3.21), one has

$$\begin{aligned}I_3=&-\frac{\delta-1}{\delta}a_1\varepsilon\int\Big(\nabla(\varphi^2+\eta^2)\cdot Q(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\\leqq&C\varepsilon|\nabla\varphi|_\infty|\varphi\nabla\partial_x^\zeta u|_2|\partial_x^\zeta u|_2\\\leqq&\frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla\partial_x^\zeta u|_2^2+Cc_0^2\varepsilon^{\frac12}|\partial_x^\zeta u|_2^2.\end{aligned}\tag{3.30}$$

For the term I_4, one gets

$$\begin{aligned}I_4=&-\sum_{j=1}^3\int\Big(\partial_x^\zeta(A_j(V)\partial_j W)-A_j(V)\partial_j\partial_x^\zeta W\Big)\partial_x^\zeta W\\\leqq&C|\partial_x^\zeta(A_j(V)\partial_j W)-A_j(V)\partial_j\partial_x^\zeta W|_2|\partial_x^\zeta W|_2\\\leqq&C|\nabla V|_\infty|\nabla^2 W|_2^2+C|\nabla^2 V|_3|\nabla W|_6|\partial_x^\zeta W|_2\leqq Cc_3|\nabla^2 W|_2^2.\end{aligned}\tag{3.31}$$

For the terms I_5-I_6, using (3.21)and (3.29), one has

$$
\begin{aligned}
I_5 =& - a_1\varepsilon \int \Big(\partial_x^\zeta((\varphi^2+\eta^2)Lu) - (\varphi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot \partial_x^\zeta u\\
\leqq& C\varepsilon\big(|\varphi\nabla Lu|_2 + |\nabla\varphi|_\infty |Lu|_2\big)|\nabla\varphi|_\infty|\partial_x^\zeta u|_2 + C\varepsilon|\nabla^2\varphi|_3|Lu|_2|\varphi\partial_x^\zeta u|_6\\
\leqq& Cc_0\varepsilon^{\frac{3}{4}}\Big(\big(|\varphi\nabla Lu|_2 + c_0\varepsilon^{-\frac{1}{4}}|Lu|_2\big)|\partial_x^\zeta u|_2\\
&+\big(|\varphi\nabla\partial_x^\zeta u|_2 + c_0\varepsilon^{-\frac{1}{2}}|\partial_x^\zeta u|_2\big)|Lu|_2\Big)\\
\leqq& \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2 + Cc_0^2\varepsilon^{\frac{1}{2}}|\nabla^2 u|_2^2,
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
I_6 =& a_1\varepsilon\int\Big(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v)) - \nabla\varphi^2\cdot Q(\partial_x^\zeta v)\Big)\cdot\partial_x^\zeta u\\
\leqq& C\varepsilon\big(|\varphi|_\infty|\nabla v|_\infty|\nabla^3\varphi|_2 + |\nabla\varphi|_\infty|\nabla^2\varphi|_2|\nabla v|_\infty\big)|\partial_x^\zeta u|_2\\
&+ C\varepsilon|\nabla\varphi|_\infty|\nabla\varphi|_3|\nabla^2 v|_6|\partial_x^\zeta u|_2 + C\varepsilon|\nabla^2\varphi|_2|\nabla^2 v|_3|\varphi\partial_x^\zeta u|_6\\
\leqq& \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2^2 + Cc_3^2\varepsilon^{\frac{1}{2}}|\nabla^2 u|_2^2 + Cc_3^4\varepsilon.
\end{aligned}
$$

Then, from (3.23) and (3.28)–(3.32), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\big((\partial_x^\zeta W)^\top A_0\partial_x^\zeta W\big) + \frac{1}{2}a_1\varepsilon\alpha|\sqrt{\varphi^2+\eta^2}\nabla\partial_x^\zeta u|_2^2\\
&\leqq Cc_3^2(1+\varepsilon)|W|_{D^2}^2 + Cc_3^4\varepsilon,
\end{aligned} \tag{3.33}
$$

which, along with Gronwall's inequaltiy, implies that

$$
\begin{aligned}
&|W(t)|_{D^2}^2 + \varepsilon\int_0^t |\sqrt{\varphi^2+\eta^2}\nabla\partial_x^\zeta u|_2^2\,\mathrm{d}s\\
&\leqq C\big(|W_0|_{D^2}^2 + c_3^4\varepsilon t\big)\exp(Cc_3^2(1+\varepsilon)t) \leqq Cc_0^2, \quad \text{for} \quad 0\leqq t\leqq T_2.
\end{aligned} \tag{3.34}
$$

Step 2: the estimate for W_t. First, the estimate for ϕ_t follows from

$$
\phi_t = -v\cdot\nabla\phi - \frac{\gamma-1}{2}\psi\,\mathrm{div}u. \tag{3.35}
$$

For $0\leqq t\leqq T_2$, we easily have that

$$
\begin{aligned}
|\phi_t(t)|_2 \leqq& C\big(|v(t)|_6|\nabla\phi(t)|_3 + |\psi(t)|_6|\mathrm{div}u(t)|_3\big) \leqq Cc_1^2,\\
|\phi_t(t)|_{D^1} \leqq& C\big(|v(t)|_\infty|\nabla^2\phi(t)|_2 + |\nabla v(t)|_6|\nabla\phi(t)|_3 + |\psi(t)|_\infty|\nabla^2 u(t)|_2\big)\\
&+ C|\nabla u(t)|_6|\nabla\psi(t)|_3 \leqq Cc_2^2.
\end{aligned} \tag{3.36}
$$

Second, we consider the estimate for $|\partial_x^\zeta u_t|_2$ when $|\zeta|\leqq 1$. From the relation

$$
u_t + v\cdot\nabla u + \frac{2A\gamma}{\gamma-1}\psi\nabla\phi + \varepsilon(\varphi^2+\eta^2)Lu = \varepsilon\nabla\varphi^2\cdot Q(v),
$$

one can obtain

$$
\begin{aligned}
|u_t|_2 =&\Big|v\cdot\nabla u+\frac{2A\gamma}{\gamma-1}\psi\nabla\phi+\varepsilon(\varphi^2+\eta^2)Lu-\varepsilon\nabla\varphi^2\cdot Q(v)\Big|_2\\
\leqq & C\Big(|v|_6|\nabla u|_3+|\psi|_6|\nabla\phi|_3+\varepsilon|\varphi^2+\eta^2|_\infty|u|_{D^2}+\varepsilon|\varphi|_\infty|\nabla\varphi|_\infty|\nabla v|_2\Big)\\
\leqq & Cc_1^3.
\end{aligned}
\tag{3.37}
$$

Similarly, for $|u_t|_{D^1}$, using (3.29), we have

$$
\begin{aligned}
|u_t|_{D^1} =&\Big|v\cdot\nabla u+\frac{2A\gamma}{\gamma-1}\psi\nabla\phi+\varepsilon(\varphi^2+\eta^2)Lu-\varepsilon\nabla\varphi^2\cdot Q(v)\Big|_{D^1}\\
\leqq & C\|v\|_2\|\nabla u\|_1+C|\psi|_\infty|\nabla^2\phi|_2+C|\nabla\psi|_3|\nabla\phi|_6\\
&+C\varepsilon|\sqrt{\phi^2+\eta^2}|_\infty|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2\\
&+C\varepsilon|\varphi\nabla^2 u|_6|\nabla\varphi|_3+C\varepsilon\|\varphi\|_2^2\|v\|_2\\
\leqq & Cc_2^3+Cc_0\varepsilon|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2,
\end{aligned}
\tag{3.38}
$$

which implies that

$$
\int_0^t |u_t|_{D^1}^2\,\mathrm{d}s \leqq C\int_0^t \big(c_2^3+c_0^2\varepsilon|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2^2\big)\,\mathrm{d}s \leqq Cc_2^4, \quad \text{for} \quad 0\leqq t\leqq T_2.
$$

□

Finally, we give the estimates on the highest order terms: $\nabla^3 W$ and $\varphi\nabla^4 u$.

Lemma 3.5. *Let* (φ, W) *be the unique strong solution to (3.6) on* $[0,T]\times\mathbb{R}^3$. *Then*

$$
\begin{aligned}
&|W(t)|_{D^3}^2+\varepsilon\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2\,ds \leqq Cc_0^4, \quad \textit{for} \quad 0\leqq t\leqq T_2;\\
&|u_t(t)|_{D^1}^2+|\phi_t(t)|_{D^2}^2+\int_0^t|\nabla^2 u_t|_2^2\,ds \leqq Cc_3^6, \quad \textit{for} \quad 0\leqq t\leqq T_2.
\end{aligned}
$$

Proof. We divide the proof into two steps.

Step 1: the estimate of $|W|_{D^3}$. First we need to consider the terms on the right-hand side of (3.23) when $|\zeta|=3$. For the term I_1, it is easy to see that

$$
\begin{aligned}
I_1 =&\int(\partial_x^\zeta W)^\top \mathrm{div}A(V)\partial_x^\zeta W\\
\leqq & C|\mathrm{div}A(V)|_\infty|\partial_x^\zeta W|_2^2 \leqq Cc_3|\partial_x^\zeta W|_2^2.
\end{aligned}
\tag{3.39}
$$

For the terms I_2-I_3, via integration by parts, (3.21) and (3.29), one has

$$\begin{aligned}
I_2 =& a_1\varepsilon \int \Big(\nabla\varphi^2 \cdot Q(\partial_x^\zeta v)\Big)\cdot \partial_x^\zeta u\\
\leqq& C\varepsilon \int \Big(|\nabla^2\varphi^2||\nabla^3 v||\nabla^3 u| + |\nabla\varphi^2||\nabla^3 v||\nabla^4 u|\Big)\\
\leqq& C\varepsilon|\nabla^3 v|_2\big(|\nabla\varphi|_\infty^2|\nabla^3 u|_2 + |\nabla\varphi|_\infty|\varphi\nabla^4 u|_2 + |\nabla^2\varphi|_3|\varphi\nabla^3 u|_6\big)\\
\leqq& \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_3^4\varepsilon^{\frac12}|\nabla^3 u|_2^2 + Cc_3^4\varepsilon^{\frac12}, \qquad (3.40)\\
I_3 =& -\frac{\delta-1}{\delta}a_1\varepsilon \int \Big(\nabla(\varphi^2+\eta^2)\cdot Q(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
\leqq& C\varepsilon|\nabla\varphi|_\infty|\varphi\nabla\partial_x^\zeta u|_2|\partial_x^\zeta u|_2\\
\leqq& \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla_x^4 u|_2^2 + Cc_0^2\varepsilon^{\frac12}|\nabla^3 u|_2^2.
\end{aligned}$$

For the term I_4, letting $r=b=2$, $a=\infty$, $f=A_j$, $g=\partial_j W$ in (2.3) of Lemma 2.2, one can obtain

$$\begin{aligned}
I_4 =& -\sum_{j=1}^{3}\int \Big(\partial_x^\zeta(A_j(V)\partial_j W) - A_j(V)\partial_j\partial_x^\zeta W\Big)\partial_x^\zeta W\\
\leqq& C\sum_{j=1}^{3}|\partial_x^\zeta(A_j\partial_j W) - A_j(V)\partial_j\partial_x^\zeta W|_2|\partial_x^\zeta W|_2 \qquad (3.41)\\
\leqq& C\big(|\nabla V|_\infty|\nabla^3 W|_2 + |\nabla^3 V|_2|\nabla W|_\infty\big)|\nabla^3 W|_2\\
\leqq& C\big(|\nabla V|_\infty|\nabla^3 W|_2 + |\nabla^3 V|_2\|\nabla W\|_2\big)|\nabla^3 W|_2 \leqq Cc_3|\nabla^3 W|_2^2 + Cc_3^3.
\end{aligned}$$

For the term I_5, using (3.21) and (3.29), one gets

$$\begin{aligned}
I_5 =& -a_1\varepsilon\int\Big(\partial_x^\zeta((\varphi^2+\eta^2)Lu) - (\varphi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\
\leqq& C\varepsilon\int\Big(|\nabla^3\varphi||Lu||\varphi\partial_x^\zeta u| + |\nabla\varphi||\nabla^2\varphi||Lu||\partial_x^\zeta u| + |\nabla\varphi|^2|\nabla Lu||\partial_x^\zeta u|\Big)\\
&+ C\varepsilon\int\Big(|\nabla^2\varphi||\varphi\nabla Lu||\partial_x^\zeta u| + |\varphi\nabla^2 Lu||\nabla\varphi||\partial_x^\zeta u|\Big)\\
\leqq& C\varepsilon|\nabla^3\varphi|_2|\nabla^2 u|_3|\varphi\nabla^3 u|_6 + C\varepsilon|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^2 u|_6|\nabla^3 u|_2 \qquad (3.42)\\
&+ C\varepsilon|\nabla\varphi|_\infty^2|\nabla^3 u|_2^2\\
&+ C\varepsilon|\nabla^2\varphi|_3|\nabla^3 u|_2|\varphi\nabla^3 u|_6 + C\varepsilon|\nabla\varphi|_\infty|\varphi\nabla^4 u|_2|\nabla^3 u|_2\\
\leqq& \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_0^2(1+\varepsilon^{\frac12})|u|_{D^3}^2 + Cc_0^4.
\end{aligned}$$

For the term I_6, we notice that

$$
\begin{aligned}
&\partial_x^{\zeta}(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot\partial_x^{\zeta}Q(v)\\
&\quad=\sum_{i,j,k} l_{ijk}\Big(C_{1ijk}\nabla\partial_x^{\zeta^i}\varphi^2\cdot\partial_x^{\zeta^j+\zeta^k}Q(v)+C_{2ijk}\nabla\partial_x^{\zeta^j+\zeta^k}\varphi^2\cdot\partial_x^{\zeta^i}Q(v)\Big)\\
&\qquad+\nabla\partial_x^{\zeta}\varphi^2\cdot Q(v),
\end{aligned}
$$

where $\zeta=\zeta^1+\zeta^2+\zeta^3$ are three multi-indexes $\zeta^i\in\mathbb{R}^3$ $(i=1,2,3)$ satisfying $|\zeta^i|=1$; C_{1ijk} and C_{2ijk} are all constants; $l_{ijk}=1$ if i, j and k are different from each other, otherwise $l_{ijk}=0$. Then, one has

$$
\begin{aligned}
I_6=&a_1\varepsilon\int\big(\partial_x^{\zeta}(\nabla\varphi^2\cdot Q(v))-\nabla\varphi^2\cdot\partial_x^{\zeta}Q(v)\big)\cdot\partial_x^{\zeta}u\\
=&a_1\varepsilon\int\Big(\sum_{i,j,k} l_{ijk}C_{1ijk}\nabla\partial_x^{\zeta^i}\varphi^2\cdot\partial_x^{\zeta^j+\zeta^k}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
&+a_1\varepsilon\int\Big(\sum_{i,j,k} l_{ijk}C_{2ijk}\nabla\partial_x^{\zeta^j+\zeta^k}\varphi^2\cdot\partial_x^{\zeta^i}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
&+a_1\varepsilon\int\nabla\partial_x^{\zeta}\varphi^2\cdot Q(v)\cdot\partial_x^{\zeta}u\equiv: I_{61}+I_{62}+I_{63}.
\end{aligned}
\tag{3.43}
$$

Using (3.21) and (3.29), we first consider the first two terms on the right-hand side of (3.43):

$$
\begin{aligned}
I_{61}=&a_1\varepsilon\int\Big(\sum_{i,j,k} l_{ijk}C_{1ijk}\nabla\partial_x^{\zeta^i}\varphi^2\partial_x^{\zeta^j+\zeta^k}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
\leqq&C\varepsilon|\nabla\varphi|_\infty^2|\nabla^3u|_2|\nabla^3v|_2+C\varepsilon|\nabla^3v|_2|\varphi\nabla^3u|_6|\nabla^2\varphi|_3\\
\leqq&\frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4u|_2^2+Cc_3^2\varepsilon^{\frac12}|u|_{D^3}^2+Cc_3^4\varepsilon^{\frac12},\\
I_{62}=&a_1\varepsilon\int\Big(\sum_{i,j,k} l_{ijk}C_{2ijk}\nabla\partial_x^{\zeta^j+\zeta^k}\varphi^2\cdot\partial_x^{\zeta^i}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
\leqq&C\varepsilon|\nabla^3\varphi|_2|\nabla^2v|_3|\varphi\nabla^3u|_6+C\varepsilon|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^2v|_6|\nabla^3u|_2\\
\leqq&\frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4u|_2^2+Cc_3^2\varepsilon^{\frac12}|u|_{D^3}^2+Cc_3^4.
\end{aligned}
\tag{3.44}
$$

For the term I_{63}, it follows from the integration by parts that

$$
\begin{aligned}
I_{63}=a_1\varepsilon\int\Big(\nabla\partial_x^{\zeta}\varphi^2Q(v)\Big)\cdot\partial_x^{\zeta}u=&-a_1\varepsilon\int\sum_{i=1}^3\Big(\partial_x^{\zeta-\zeta^i}\nabla\varphi^2\cdot\partial_x^{\zeta^i}Q(v)\cdot\partial_x^{\zeta}u\\
&+\partial_x^{\zeta-\zeta^i}\nabla\varphi^2\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^i}u\Big)=\sum_{i=1}^3\big((I_{631}^{(i)}+I_{632}^{(i)}\big).
\end{aligned}
\tag{3.45}
$$

For simplicity, we only consider the case that $i=1$, the rest terms can be estimated similarly. When $i=1$, similarly to the estimates on I_{61} in (3.44), we first have

$$I_{631}^{(1)} \leqq \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_3^2\varepsilon^{\frac{1}{2}}|u|_{D^3}^2 + Cc_3^4\varepsilon^{\frac{1}{2}}. \tag{3.46}$$

Next, for the term I_{632}, one has

$$\begin{aligned} I_{632}^{(1)} =& -a_1\varepsilon\int \partial_x^{\zeta^2+\zeta^3}\nabla\varphi^2\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u \\ =& -2a_1\varepsilon\int\Big(\varphi\partial_x^{\zeta^2+\zeta^3}\nabla\varphi+\partial_x^{\zeta^2}\nabla\varphi\partial_x^{\zeta^3}\varphi\Big)\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u \\ &-2a_1\varepsilon\int\Big(\partial_x^{\zeta^3}\nabla\varphi\partial_x^{\zeta^2}\varphi+\nabla\varphi\partial_x^{\zeta^2+\zeta^3}\varphi\Big)\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u \\ =& I_A+I_B+I_C+I_D. \end{aligned} \tag{3.47}$$

For the term I_A, it is not hard to show that

$$\begin{aligned} I_A =& -2a_1\varepsilon\int\Big(\varphi\partial_x^{\zeta^2+\zeta^3}\nabla\varphi\Big)\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u \\ \leqq& C\varepsilon|\nabla^3\varphi|_2|\nabla v|_\infty|\varphi\nabla^4 u|_2 \leqq \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_3^4. \end{aligned} \tag{3.48}$$

For the term I_B, it follows from integration by parts that

$$\begin{aligned} I_B =& -2a_1\varepsilon\int\Big(\partial_x^{\zeta^2}\nabla\varphi\partial_x^{\zeta^3}\varphi\Big)\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^1}u \\ \leqq& C\varepsilon\int\Big(\big(|\nabla\varphi||\nabla^3\varphi|+|\nabla^2\varphi|^2\big)|\nabla^3 u||\nabla v|+|\nabla^2\varphi||\nabla\varphi||\nabla^3 u||\nabla^2 v|\Big) \\ \leqq& C\varepsilon|\nabla^3 u|_2|\nabla v|_\infty\big(|\nabla^3\varphi|_2|\nabla\varphi|_\infty+|\nabla^2\varphi|_3|\nabla^2\varphi|_6\big) \\ &+C\varepsilon|\nabla^3 u|_2|\nabla^2 v|_3|\nabla^2\varphi|_6|\nabla\varphi|_\infty \leqq Cc_3^3\varepsilon^{\frac{1}{4}}|\nabla^3 u|_2. \end{aligned} \tag{3.49}$$

Via an argument similar to that used in (3.49), we also have

$$I_C+I_D \leqq Cc_3^3\varepsilon^{\frac{1}{4}}|\nabla^3 u|_2, \tag{3.50}$$

which, along with (3.43)–(3.48), implies that

$$I_6 \leqq \frac{a_1\varepsilon\alpha}{20}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_3^4|u|_{D^3}^2 + Cc_3^4. \tag{3.51}$$

Then from (3.39)–(3.42) and (3.51), one has

$$\frac{1}{2}\frac{d}{dt}\int\big((\partial_x^\zeta W)^\top A_0\partial_x^\zeta W\big)+\frac{1}{2}a_1\varepsilon\alpha|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 \leqq Cc_3^4|W|_{D^3}^2+Cc_3^4, \tag{3.52}$$

which, along with Gronwall's inequaltiy, implies that

$$|W(t)|^2_{D^3}+\varepsilon\int_0^t|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2\,\mathrm{d}s\leqq C\big(|W_0|^2_{D^3}+c_3^4 t\big)\exp(Cc_3^4 t)\leqq Cc_0^2 \tag{3.53}$$

for $0\leqq t\leqq T_2$.

Step 2: the estimates for $|\partial_x^\zeta W_t|_2$ when $1\leqq|\zeta|\leqq 2$. First from (3.35), we have

$$\begin{aligned}|\phi_t(t)|_{D^2}\leqq&C\big(|v(t)|_\infty|\nabla^3\phi(t)|_2+|\nabla v(t)|_6|\nabla^2\phi(t)|_3+|\nabla^2 v|_6|\nabla\phi|_3\big)\\&+C\big(|\psi(t)|_\infty|\nabla^3 u(t)|_2+|\nabla\psi(t)|_\infty|\nabla^2 u(t)|_2+|\nabla^2\psi|_3|\mathrm{div}u|_6\big)\\\leqq&Cc_3^2.\end{aligned} \tag{3.54}$$

Second, it follows from (3.38) that

$$|u_t|_{D^1}\leqq Cc_2^3+Cc_0|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2\leqq Cc_2^3. \tag{3.55}$$

Finally, for $|W_t|_{D^2}$, from Lemma 2.6, one gets

$$\begin{aligned}|u_t|_{D^2}=&\Big|v\cdot\nabla u+\frac{2A\gamma}{\gamma-1}\psi\nabla\phi+\varepsilon(\varphi^2+\eta^2)Lu-\varepsilon\nabla\varphi^2\cdot Q(v)\Big|_{D^2}\\\leqq&C\big(\|v\|_2\|\nabla u\|_2+\|\psi\|_2\|\nabla\phi\|_2\big)+C\varepsilon|\sqrt{\varphi^2+\eta^2}|_\infty|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2\\&+C\varepsilon|\varphi|_\infty|\nabla\varphi|_\infty|u|_{D^3}+C\varepsilon\big(|\varphi|_\infty|\nabla^2\varphi|_3|Lu|_6+|\nabla\varphi|_6|\nabla\varphi|_6|u|_6\big)\\&+C\varepsilon\|\nabla\varphi^2\|_2\|\nabla v\|_2\leqq Cc_3^4+Cc_0\varepsilon|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2,\end{aligned} \tag{3.56}$$

which implies that

$$\int_0^t|u_t|^2_{D^2}\,\mathrm{d}s\leqq C\int_0^t\big(c_3^8+\varepsilon^2c_0^2|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2^2\big)\,\mathrm{d}s\leqq Cc_3^4,\quad\text{for}\quad 0\leqq t\leqq T_2.$$

□

Combining the estimates obtained in Lemmas 3.2-3.5, we have

$$\begin{aligned}1+|\varphi(t)|^2_\infty+\|\varphi(t)\|_2^2\leqq Cc_0^2,\quad\varepsilon|\varphi|^2_{D^3}\leqq&Cc_0^2,\\|\varphi_t(t)|_2^2\leqq Cc_1^4,\quad|\varphi_t(t)|^2_{D^1}\leqq Cc_2^4,\quad\varepsilon|\varphi_t(t)|^2_{D^2}\leqq&Cc_3^4,\\\|W(t)\|_1^2+\varepsilon\int_0^t|\varphi\nabla^2 u|_2^2\,\mathrm{d}s\leqq&Cc_0^2,\\|W(t)|^2_{D^2}+\varepsilon\int_0^t|\varphi\nabla^3 u|_2^2\,\mathrm{d}s\leqq&Cc_0^2,\\|W_t(t)|_2^2+|\phi_t(t)|^2_{D^1}+\int_0^t|\nabla u_t|_2^2\,\mathrm{d}s\leqq&Cc_2^6,\\|W(t)|^2_{D^3}+\varepsilon\int_0^t|\varphi\nabla^4 u|_2^2\,\mathrm{d}s\leqq&Cc_0^2,\\|u_t(t)|^2_{D^1}+|\phi_t(t)|^2_{D^2}+\int_0^t|\nabla^2 u_t|_2^2\,\mathrm{d}s\leqq&Cc_3^6\end{aligned} \tag{3.57}$$

for $0 \leqq t \leqq \min\{T, (1+c_3)^{-4}\}$. Therefore, if we define the constants c_i $(i = 1, 2, 3)$ and T^* by

$$c_1 = c_2 = c_3 = C^{\frac{1}{2}} c_0, \quad T^* = \min\{T, (1 + c_3)^{-4}\}, \tag{3.58}$$

then we deduce that

$$\begin{aligned}
&\sup_{0\leqq t\leqq T^*} \left(\|\varphi(t)\|_1^2 + \|\phi(t)\|_1^2 + \|u(t)\|_1^2\right) + \int_0^{T^*} \varepsilon|\varphi\nabla^2 u|_2^2 \leqq c_1^2,\\
&\sup_{0\leqq t\leqq T^*} \left(|\varphi(t)|_{D^2}^2 + |\phi(t)|_{D^2}^2 + |u(t)|_{D^2}^2\right) + \int_0^{T^*} \varepsilon|\varphi\nabla^3 u|_2^2 \,\mathrm{d}t \leqq c_2^2,\\
&\operatorname{ess}\sup_{0\leqq t\leqq T^*} \left(|\phi(t)|_{D^3}^2 + |u(t)|_{D^3}^2 + \varepsilon|\varphi(t)|_{D^3}^2\right) + \int_0^{T^*} \varepsilon|\varphi\nabla^4 u|_2^2 \,\mathrm{d}t \leqq c_3^2,\\
&\operatorname{ess}\sup_{0\leqq t\leqq T^*} \left(\|W_t(t)\|_1^2 + |\phi_t(t)|_{D^2}^2 + \varepsilon|\varphi_t(t)|_{D^2}^2\right) + \int_0^{T^*} |u_t|_{D^2}^2 \,\mathrm{d}t \leqq c_3^6.
\end{aligned} \tag{3.59}$$

In other words, given fixed c_0 and T, there exist positive constants T^* and c_i $(i = 1, 2, 3)$, depending solely on c_0, T and the generic constant C, independent of (η, ε), such that if (3.11) holds for ω and V, then (3.59) holds for the strong solution of (3.6) in $[0, T^*] \times \mathbb{R}^3$.

3.3. Passing to the Limit as $\eta \to 0$

Based on the local (in time) a priori estimates (3.59), we have the following existence result under the assumption that $\varphi_0 \geqq 0$ to the following Cauchy problem:

$$\begin{cases}
\varphi_t + v \cdot \nabla\varphi + \dfrac{\delta - 1}{2}\omega \operatorname{div} v = 0,\\
A_0 W_t + \displaystyle\sum_{j=1}^{3} A_j(V)\partial_j W + \varepsilon\varphi^2 \mathbb{L}(W) = \varepsilon\mathbb{H}(\varphi)\cdot\mathbb{Q}(V),\\
(\varphi, W)|_{t=0} = (\varphi_0, W_0), \quad x \in \mathbb{R}^3,\\
(\varphi, W) \to (0, 0), \quad \text{as } |x| \to +\infty, \quad t \geqq 0.
\end{cases} \tag{3.60}$$

Lemma 3.6. *Assume (φ_0, W_0) satisfy (3.8). Then there exists a time $T^* > 0$ that is independent of ε, and a unique strong solution (φ, W) in $[0, T^*] \times \mathbb{R}^3$ to (3.60) such that*

$$\begin{aligned}
&\varphi \in C([0, T^*]; H^3), \quad \phi \in C([0, T^*]; H^3),\\
&u \in C([0, T^*]; H^{s'}) \cap L^\infty([0, T^*]; H^3),\\
&\varphi\nabla^4 u \in L^2([0, T^*]; L^2), \quad u_t \in C([0, T^*]; H^1) \cap L^2([0, T^*]; D^2)
\end{aligned} \tag{3.61}$$

for any constant $s' \in [2, 3)$. Moreover, (φ, W) also satisfies the a priori estimates (3.59).

Proof. We prove the existence, uniqueness and time continuity in three steps.

Step 1 existence. Due to Lemma 3.1 and the uniform estimates (3.59), for every $\eta > 0$, there exists a unique strong solution (φ^η, W^η) in $[0, T^*] \times \mathbb{R}^3$ to the linearized problem (3.6) satisfying estimates (3.59), where the time $T^* > 0$ is also independent of (η, ε).

By virtue of the uniform estimates (3.59) independent of (η, ε) and the compactness in Lemma 2.3 (see [38]), we know that for any $R > 0$, there exists a subsequence of solutions (still denoted by) (φ^η, W^η), which converges to a limit $(\varphi, W) = (\varphi, \phi, u)$ in the following strong sense:

$$(\varphi^\eta, W^\eta) \to (\varphi, W) \quad \text{in } C([0, T^*]; H^2(B_R)), \quad \text{as } \eta \to 0. \tag{3.62}$$

Again by virtue of the uniform estimates (3.59) independent of (η, ε), we also know that there exists a subsequence of solutions (still denoted by) (φ^η, W^η), which converges to (φ, W) in the following weak or weak* sense:

$$\begin{aligned} (\varphi^\eta, W^\eta) &\rightharpoonup (\varphi, W) \quad \text{weakly* in } L^\infty([0, T^*]; H^3(\mathbb{R}^3)),\\ (\phi^\eta_t, \varphi^\eta_t) &\rightharpoonup (\phi_t, \varphi_t) \quad \text{weakly* in } L^\infty([0, T^*]; H^2(\mathbb{R}^3)),\\ u^\eta_t &\rightharpoonup u_t \quad \text{weakly* in } L^\infty([0, T^*]; H^1(\mathbb{R}^3)),\\ u^\eta_t &\rightharpoonup u_t \quad \text{weakly in } L^2([0, T^*]; D^2(\mathbb{R}^3)), \end{aligned} \tag{3.63}$$

which, along with the lower semi-continuity of weak convergence, implies that (φ, W) also satisfies the corresponding estimates (3.59) except those of $\varphi\nabla^4 u$.

Combining the strong convergence in (3.62) and the weak convergence in (3.63), we easily obtain that (φ, W) also satisfies the local estimates (3.59) and

$$\varphi^\eta \nabla^4 u^\eta \rightharpoonup \varphi \nabla^4 u \quad \text{weakly in } L^2([0, T^*] \times \mathbb{R}^3). \tag{3.64}$$

Now we want to show that (φ, W) is a weak solution in the sense of distributions to the linearized problem (3.60). Multiplying $(3.6)_2$ by test function $f(t, x) = (f^1, f^2, f^3) \in C^\infty_c([0, T^*) \times \mathbb{R}^3)$ on both sides, and integrating over $[0, T^*] \times \mathbb{R}^3$, we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^3} u^\eta \cdot f_t \,\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla) u^\eta \cdot f \,\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^3} \frac{2A\gamma}{\gamma - 1} \psi \nabla \phi^\eta f \,\mathrm{d}x\mathrm{d}s\\ &= -\int u_0 \cdot f(0, x) + \int_0^t \int_{\mathbb{R}^3} \varepsilon \Big(((\varphi^\eta)^2 + \eta^2) L u^\eta - \nabla (\varphi^\eta)^2 \cdot Q(v) \Big) \cdot f \,\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.65}$$

Combining the strong convergence in (3.62) and the weak convergences in (3.63)–(3.64), and letting $\eta \to 0$ in (3.65), we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^3} u \cdot f_t \,\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla) u \cdot f \,\mathrm{d}x\mathrm{d}s - \frac{2A\gamma}{\gamma - 1} \int_0^t \int_{\mathbb{R}^3} \psi \nabla \phi f \,\mathrm{d}x\mathrm{d}s\\ &= -\int u_0 \cdot f(0, x) + \int_0^t \int_{\mathbb{R}^3} \varepsilon \Big(\varphi^2 L u - \nabla \varphi^2 \cdot Q(v) \Big) \cdot f \,\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.66}$$

Thus it is obvious that (φ, W) is a weak solution in the sense of distributions to the linearized problem (3.60), satisfying the regularities

$$\begin{aligned}&\varphi \in L^\infty([0,T^*];H^3),\ \varphi_t \in L^\infty([0,T^*];H^2),\ \phi \in L^\infty([0,T^*];H^3),\\&\phi_t \in L^\infty([0,T^*];H^2),\ u \in L^\infty([0,T^*];H^3),\\&\varphi\nabla^4 u \in L^2([0,T^*];L^2),\quad u_t \in L^\infty([0,T^*];H^1)\cap L^2([0,T^*];D^2).\end{aligned} \tag{3.67}$$

Step 2 uniqueness. Let (φ_1, W_1) and (φ_2, W_2) be two solutions obtained in the above step. We denote

$$\overline{\varphi} = \varphi_1 - \varphi_2, \quad \overline{W} = W_1 - W_2.$$

Then from $(3.60)_1$, we have

$$\overline{\varphi}_t + v\cdot\nabla\overline{\varphi} = 0,$$

which quickly implies that $\overline{\varphi} = 0$.

Let $\overline{W} = (\overline{\phi}, \overline{u})^\top$, from $(3.60)_2$ and $\varphi_1 = \varphi_2$, we have

$$A_0\overline{W}_t + \sum_{j=1}^{3} A_j(V)\partial_j\overline{W} = -\varepsilon\varphi_1^2 L(\overline{W}). \tag{3.68}$$

Then multiplying (3.68) by $\overline{W}$ on both sides, and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \overline{W}^\top A_0\overline{W} + a_1\varepsilon\alpha|\varphi_1\nabla\overline{u}|_2^2\\&\leqq C|\nabla V|_\infty|\overline{W}|_2^2 + \varepsilon|\overline{u}|_2|\nabla\varphi_1|_\infty|\varphi_1\nabla\overline{u}|_2\\&\leqq \frac{a_1\varepsilon\alpha}{10}|\varphi_1\nabla\overline{u}|_2^2 + C(|\nabla V|_\infty + \varepsilon|\nabla\varphi_1|_\infty^2)|\overline{W}|_2^2.\end{aligned} \tag{3.69}$$

From Gronwall's inequality, we obtain that $\overline{W} = 0$ in $\mathbb{R}^3$, which gives the uniqueness.

Step 3 time continuity. First for φ, via the regularities shown in (3.67) and the classical Sobolev imbedding theorem, we have

$$\varphi \in C([0,T^*];H^2)\cap C([0,T^*];\text{weak}-H^3). \tag{3.70}$$

Using the same arguments as in Lemma 3.2, we have

$$\begin{aligned}\|\varphi(t)\|_3^2 \leqq &\Big(\|\varphi_0\|_3^2 + C\int_0^t \big(\|\nabla\omega\|_2^2\|v\|_3^2 + |w\nabla^4 v|_2^2\big)\,\mathrm{d}s\Big)\\&\exp\Big(C\int_0^t \|v(s)\|_3\,\mathrm{d}s\Big),\end{aligned} \tag{3.71}$$

which implies that

$$\limsup_{t\to 0}\|\varphi(t)\|_3 \leqq \|\varphi_0\|_3. \tag{3.72}$$

Then according to Lemma 2.8 and (3.70), we know that φ is right continuous at $t=0$ in H^3 space. From the reversibility on the time to equation $(3.60)_1$, we know

$$\varphi \in C([0,T^*];H^3). \tag{3.73}$$

For φ_t, from

$$\varphi_t = -v\cdot\nabla\varphi - \frac{\delta-1}{2}\omega\mathrm{div}v, \tag{3.74}$$

we only need to consider the term $\omega\mathrm{div}v$. Due to

$$\omega\mathrm{div}v \in L^2([0,T^*];H^3), \quad (\omega\mathrm{div}v)_t \in L^2([0,T^*];H^1) \tag{3.75}$$

and the Sobolev imbedding theorem, we have

$$\omega\mathrm{div}v \in C([0,T^*];H^2), \tag{3.76}$$

which implies that

$$\varphi_t \in C([0,T^*];H^2).$$

The similar arguments can be used to deal with ϕ.

For velocity u, from the regularity shown in (3.67) and Sobolev's imbedding theorem, we obtain that

$$u \in C([0,T^*];H^2)\cap C([0,T^*];\text{weak}-H^3). \tag{3.77}$$

Then from Lemma 2.5, for any $s'\in[2,3)$, we have

$$\|u\|_{s'} \leqq C_3\|u\|_0^{1-\frac{s'}{3}}\|u\|_3^{\frac{s'}{3}}.$$

Together with the upper bound shown in (3.59) and the time continuity (3.77), we have

$$u \in C([0,T^*];H^{s'}). \tag{3.78}$$

Finally, we consider u_t. From equations $(3.60)_2$ we have

$$u_t = -v\cdot\nabla u - \frac{2A\gamma}{\gamma-1}\psi\nabla\phi + \varepsilon\varphi^2 Lu + \varepsilon\nabla\varphi^2\cdot Q(v), \tag{3.79}$$

where

$$Q(v) = \frac{\delta}{\delta-1}\left(\alpha(\nabla v+(\nabla v)^\top)+\beta\mathrm{div}v\mathbb{I}_3\right) \in L^2([0,T^*];H^2).$$

From (3.67), we have

$$\varepsilon\varphi^2 Lu \in L^2([0,T^*];H^2), \quad \varepsilon(\varphi^2 Lu)_t \in L^2([0,T^*];L^2), \tag{3.80}$$

which means that

$$\varphi^2 Lu \in C([0,T^*];H^1). \tag{3.81}$$

Then combining (3.7), (3.73), (3.78) and (3.81), we deduce that

$$u_t \in C([0,T^*];H^1).$$

□

3.4. Proof of Theorem 3.1

Our proof is based on the classical iteration scheme and the existence results for the linearized problem obtained in Section 3.4. Like in Section 3.3, we define constants c_0 and c_i $(i = 1, 2, 3)$, and assume that

$$2 + \|\varphi_0\|_3 + \|W_0\|_3 \leqq c_0.$$

Let $(\varphi^0, W^0 = (\phi^0, u^0))$, with the regularities

$$\begin{aligned}
&\varphi^0 \in C([0, T^*]; H^3), \quad \phi^0 \in C([0, T^*]; H^3), \quad \varphi^0\nabla^4 u^0 \in L^2([0, T^*]; L^2),\\
&u^0 \in C([0, T^*]; H^{s'}) \cap L^\infty([0, T^*]; H^3) \quad \text{for any } \ s' \in [2, 3),
\end{aligned} \tag{3.82}$$

be the solution to the problem

$$\begin{cases}
X_t + u_0 \cdot \nabla X = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3,\\
Y_t + u_0 \cdot \nabla Y = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3,\\
Z_t - X^2 \triangle Z = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3,\\
(X, Y, Z)|_{t=0} = (\varphi_0, \phi_0, u_0) \quad \text{in} \quad \mathbb{R}^3,\\
(X, Y, Z) \to (0, 0, 0) \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0.
\end{cases} \tag{3.83}$$

We take a time $T^{**} \in (0, T^*]$ small enough such that

$$\begin{aligned}
&\sup_{0\leqq t\leqq T^{**}} \big(||\varphi^0(t)||_1^2 + ||\phi^0(t)||_1^2 + ||u^0(t)||_1^2\big) + \int_0^{T^{**}} \varepsilon|\varphi^0\nabla^2 u^0| \,\mathrm{d}t \leqq c_1^2,\\
&\sup_{0\leqq t\leqq T^{**}} \big(|\varphi^0(t)|_{D^2}^2 + |\phi^0(t)|_{D^2}^2 + |u^0(t)|_{D^2}^2\big) + \int_0^{T^{**}} \varepsilon|\varphi^0\nabla^3 u^0| \,\mathrm{d}t \leqq c_2^2,\\
&\operatorname{ess}\sup_{0\leqq t\leqq T^{**}} \big(\varepsilon|\varphi^0(t)|_{D^3}^2 + |\phi^0(t)|_{D^3}^2 + |u^0(t)|_{D^3}^2\big) + \int_0^{T^{**}} \varepsilon|\varphi^0\nabla^4 u^0| \,\mathrm{d}t \leqq c_3^2.
\end{aligned} \tag{3.84}$$

Proof. We prove the existence, uniqueness and time continuity in three steps.

Step 1 existence. Let $(\omega, \psi, v) = (\varphi^0, \phi^0, u^0)$, we define (φ^1, W^1) as a strong solution to problem (3.60). Then we construct approximate solutions

$$(\varphi^{k+1}, W^{k+1}) = (\varphi^{k+1}, \phi^{k+1}, u^{k+1})$$

inductively, by assuming that (φ^k, W^k) was defined for $k \geqq 1$, let (φ^{k+1}, W^{k+1}) be the unique solution to problem (3.60) with (ω, ψ, v) replaced by (φ^k, W^k) as follows:

$$\begin{cases} \varphi_t^{k+1} + u^k \cdot \nabla \varphi^{k+1} + \frac{\delta - 1}{2} \varphi^k \mathrm{div} u^k = 0, \\ A_0 W_t^{k+1} + \sum_{j=1}^{3} A_j(W^k) \partial_j W^{k+1} + \varepsilon (\varphi^{k+1})^2 \mathbb{L}(u^{k+1}) = \varepsilon \mathbb{H}(\varphi^{k+1}) \cdot \mathbb{Q}(u^k), \\ (\varphi^{k+1}, W^{k+1})|_{t=0} = (\varphi_0, W_0), \quad x \in \mathbb{R}^3, \\ (\varphi^{k+1}, W^{k+1}) \to (0,0) \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0. \end{cases} \tag{3.85}$$

It follows from Lemma 3.6 that the sequence (φ^k, W^k) satisfies the uniform a priori estimates (3.59) for $0 \leqq t \leqq T^{**}$.

Now we prove the convergence of the whole sequence (φ^k, W^k) of approximate solutions to a limit (φ, W) in some strong sense. Let

$$\overline{\varphi}^{k+1} = \varphi^{k+1} - \varphi^k, \quad \overline{W}^{k+1} = (\overline{\phi}^{k+1}, \overline{u}^{k+1})^\top,$$

with

$$\overline{\phi}^{k+1} = \phi^{k+1} - \phi^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k.$$

Then, from (3.85), one has

$$\begin{cases} \overline{\varphi}_t^{k+1} + u^k \cdot \nabla \overline{\varphi}^{k+1} + \overline{u}^k \cdot \nabla \varphi^k + \frac{\delta - 1}{2} (\overline{\varphi}^k \mathrm{div} u^{k-1} + \varphi^k \mathrm{div} \overline{u}^k) = 0, \\ A_0 \overline{W}_t^{k+1} + \sum_{j=1}^{3} A_j(W^k) \partial_j \overline{W}^{k+1} + \varepsilon (\varphi^{k+1})^2 \mathbb{L}(\overline{W}^{k+1}) \\ = \sum_{j=1}^{3} A_j(\overline{W}^k) \partial_j W^k - \varepsilon \overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k) \mathbb{L}(W^k) \\ + \varepsilon \big(\mathbb{H}(\varphi^{k+1}) - \mathbb{H}(\varphi^k) \big) \cdot \mathbb{Q}(W^k) + \mathbb{H}(\varphi^{k+1}) \cdot \mathbb{Q}(\overline{W}^k). \end{cases} \tag{3.86}$$

First, we consider $|\overline{\varphi}^{k+1}|_2$. Multiplying $(3.86)_1$ by $2\overline{\varphi}^{k+1}$ and integrating over $\mathbb{R}^3$, one has

$$\begin{aligned} \frac{d}{dt} |\overline{\varphi}^{k+1}|_2^2 = & -2 \int \Big(u^k \cdot \nabla \overline{\varphi}^{k+1} + \overline{u}^k \cdot \nabla \varphi^k \\ & + \frac{\delta - 1}{2} (\overline{\varphi}^k \mathrm{div} u^{k-1} + \varphi^k \mathrm{div} \overline{u}^k) \Big) \overline{\varphi}^{k+1} \\ \leqq & C |\nabla u^k|_\infty |\overline{\varphi}^{k+1}|_2^2 + C |\overline{\varphi}^{k+1}|_2 \big(|\overline{u}^k|_2 |\nabla \varphi^k|_\infty \\ & + |\overline{\varphi}^k|_2 |\nabla u^{k-1}|_\infty + |\varphi^k \mathrm{div} \overline{u}^k|_2 \big), \end{aligned}$$

which means that ($0 < \nu \leqq \frac{1}{10}$ is a constant)

$$\frac{d}{dt} |\overline{\varphi}^{k+1}(t)|_2^2 \leqq A_\nu^k(t) |\overline{\varphi}^{k+1}(t)|_2^2 + \nu \Big(\varepsilon^{-\frac{1}{2}} |\overline{u}^k(t)|_2^2 + |\overline{\varphi}^k(t)|_2^2 + |\varphi^k \mathrm{div} \overline{u}^k(t)|_2^2 \Big) \tag{3.87}$$

with $A_\nu^k(t) = C\big(1 + \nu^{-1}\big)$.

Second, we consider $|\overline{W}^{k+1}|_2$. Multiplying (3.86)$_2$ by $\overline{W}^{k+1}$ and integrating over $\mathbb{R}^3$, one gets

$$\begin{aligned}
&\frac{d}{dt}\int(\overline{W}^{k+1})^\top A_0\overline{W}^{k+1}+2a_1\varepsilon\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+2(\alpha+\beta)a_1\varepsilon|\varphi^{k+1}\mathrm{div}\overline{u}^{k+1}|_2^2\\
&\leqq\int(\overline{W}^{k+1})^\top\mathrm{div}A(W^k)\overline{W}^{k+1}+\int\sum_{j=1}^3(\overline{W}^{k+1})^\top A_j(\overline{W}^k)\partial_jW^k\\
&\quad-2a_1\varepsilon\frac{\delta-1}{\delta}\int\nabla(\varphi^{k+1})^2\cdot Q(\overline{u}^{k+1})\cdot\overline{u}^{k+1}\\
&\quad-2a_1\varepsilon\int\Big(\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)\cdot L(u^k)\\
&\quad-\nabla(\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k))\cdot Q(u^k)\Big)\cdot\overline{u}^{k+1}\\
&\quad+2a_1\varepsilon\int\nabla(\varphi^k)^2\cdot(Q(u^k)-Q(u^{k-1}))\cdot\overline{u}^{k+1}:=\sum_{i=1}^6J_i.
\end{aligned}\tag{3.88}$$

For the terms J_1-J_4, it follows from (3.21) and (3.29) that

$$\begin{aligned}
J_1=&\int(\overline{W}^{k+1})^\top\mathrm{div}A(W^k)\overline{W}^{k+1}\leqq C|\nabla W^k|_\infty|\overline{W}^{k+1}|_2^2\leqq C|\overline{W}^{k+1}|_2^2,\\
J_2=&\int\sum_{j=1}^3A_j(\overline{W}^k)\partial_jW^k\cdot\overline{W}^{k+1}\\
\leqq&C|\nabla W^k|_\infty|\overline{W}^k|_2|\overline{W}^{k+1}|_2\leqq C\nu^{-1}|\overline{W}^{k+1}|_2^2+\nu|\overline{W}^k|_2^2,\\
J_3=&-2a_1\frac{\delta-1}{\delta}\varepsilon\int\nabla(\varphi^{k+1})^2\cdot Q(\overline{u}^{k+1})\cdot\overline{u}^{k+1}\\
\leqq&C\varepsilon|\nabla\varphi^{k+1}|_\infty|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2|\overline{u}^{k+1}|_2\\
\leqq&C\varepsilon^{\frac12}|\overline{W}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2,\\
J_4=&-2a_1\varepsilon\int\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)Lu^k\cdot\overline{u}^{k+1}\\
\leqq&C\varepsilon|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}\overline{u}^{k+1}|_6|Lu^k|_3+C\varepsilon|\overline{\varphi}^{k+1}|_2|\varphi^kLu^k|_\infty|\overline{u}^{k+1}|_2\\
\leqq&C\varepsilon|Lu^k|_3^2|\overline{\varphi}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+C\varepsilon|\overline{\varphi}^{k+1}|_2^2\\
&+C\varepsilon|\nabla\varphi^{k+1}|_\infty^2|Lu^k|_3^2|\overline{u}^{k+1}|_2^2+C\varepsilon(\varepsilon^{-1}+|\varphi^k\nabla^4u^k|_2^2)|\overline{u}^{k+1}|_2^2\\
\leqq&C\varepsilon|\overline{\varphi}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+C(1+\varepsilon|\varphi^k\nabla^4u^k|_2^2)|\overline{W}^{k+1}|_2^2,
\end{aligned}\tag{3.89}$$

where we have used the fact (see Lemma 2.1) that

$$\begin{aligned}|\varphi^k\nabla^2u^k|_\infty \leqq& C|\varphi^k\nabla^2u^k|_6^{\frac{1}{2}}|\nabla(\varphi^k\nabla^2u^k)|_6^{\frac{1}{2}}\\ \leqq& C|\varphi^k\nabla^2u^k|_{D^1}^{\frac{1}{2}}|\nabla(\varphi^k\nabla^2u^k)|_{D^1}^{\frac{1}{2}} \leqq C\|\nabla(\varphi^k\nabla^2u^k)\|_1\\ \leqq& C(|\nabla\varphi^k|_\infty\|\nabla^2u^k\|_1+|\varphi^k|_\infty|\nabla^3u|_2+|\varphi^k\nabla^4u|_2+|\nabla^2\varphi^k|_6|\nabla^2u^k|_3)\\ \leqq& C(\|\nabla\varphi^k\|_2\|\nabla^2u^k\|_1+|\varphi^k\nabla^4u^k|_2)\leqq C(\varepsilon^{-\frac{1}{2}}+|\varphi^k\nabla^4u^k|_2).\end{aligned}\tag{3.90}$$

Next, we begin to consider the term J_5. First we have

$$\begin{aligned}J_5=&2a_1\varepsilon\int\nabla(\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k))\cdot Q(u^k)\cdot\overline{u}^{k+1}\\ =&-2a_1\varepsilon\int\sum_{ij}\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)\partial_i(a_k^{ij}\overline{u}^{k+1,j})\\ =&J_{51}+J_{52}+J_{53}+J_{54},\end{aligned}\tag{3.91}$$

where $u^{k,j}$ represents the j-th component of u^k ($k\geqq 1$),

$$\overline{u}^{k,j}=u^{k,j}-u^{k-1,j},\quad \text{for } k\geqq 1,\quad j=1,2,3,$$

and the quantity a_k^{ij} is given by

$$a_k^{ij}=\frac{\delta}{\delta-1}\Big(\alpha(\partial_iu^{k,j}+\partial_ju^{k,i})+\mathrm{div}u^k\delta_{ij}\Big)\quad\text{for } i,\ j=1,2,3,$$

and δ_{ij} is the Kronecker symbol satisfying $\delta_{ij}=1,\ i=j$ and $\delta_{ij}=0$, otherwise.

For terms J_{51}-J_{53}, using (3.21), (3.29) and (3.90), one can obtain

$$\begin{aligned}J_{51}=&-2a_1\varepsilon\int\sum_{ij}\overline{\varphi}^{k+1}\varphi^{k+1}\partial_ia_k^{ij}\overline{u}^{k+1,j}\\ \leqq& C\varepsilon|\nabla^2u^k|_6|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}\overline{u}^{k+1}|_3\\ \leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}\overline{u}^{k+1}|_2^{\frac{1}{2}}|\varphi^{k+1}\overline{u}^{k+1}|_6^{\frac{1}{2}}\\ \leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2^2+C\varepsilon|\varphi^{k+1}\overline{u}^{k+1}|_2|\varphi^{k+1}\overline{u}^{k+1}|_6\\ \leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2^2+C\varepsilon^{\frac{1}{2}}|\overline{W}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2,\\ J_{52}=&-2a_1\varepsilon\int\sum_{i,j}\overline{\varphi}^{k+1}\varphi^k\partial_ia_k^{ij}\overline{u}^{k+1,j}\\ \leqq& C\varepsilon|\varphi^k\nabla^2u^k|_\infty|\overline{\varphi}^{k+1}|_2|\overline{u}^{k+1}|_2\\ \leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2^2+C\varepsilon(\varepsilon^{-1}+|\varphi^k\nabla^4u^k|_2^2)|\overline{u}^{k+1}|_2^2\\ \leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2^2+C(1+\varepsilon|\varphi^k\nabla^4u^k|_2^2)|\overline{W}^{k+1}|_2^2,\\ J_{53}=&-2a_1\varepsilon\int\sum_{i,j}\overline{\varphi}^{k+1}\varphi^{k+1}a_k^{ij}\partial_i\overline{u}^{k+1,j}\\ \leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2|\nabla u^k|_\infty\end{aligned}\tag{3.92}$$

$$\leqq C\varepsilon|\nabla u^k|_\infty^2|\overline{\varphi}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2$$
$$\leqq C\varepsilon|\overline{\varphi}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2.$$

For the last term on the right-hand side of (3.91), one has

$$\begin{aligned}J_{54}=&-2a_1\varepsilon\int\sum_{i,j}\overline{\varphi}^{k+1}\varphi^k a_k^{ij}\partial_i\overline{u}^{k+1,j}\\=&-2a_1\varepsilon\int\sum_{i,j}\overline{\varphi}^{k+1}(\varphi^k-\varphi^{k+1}+\varphi^{k+1})a_k^{ij}\partial_i\overline{u}^{k+1,j}\\\leqq& C\varepsilon|\nabla u^k|_\infty|\nabla\overline{u}^{k+1}|_\infty|\overline{\varphi}^{k+1}|_2^2+C\varepsilon|\nabla u^k|_\infty|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2|\overline{\varphi}^{k+1}|_2\\\leqq& C\varepsilon|\nabla u^k|_\infty^2|\overline{\varphi}^{k+1}|_2^2+C\varepsilon|\nabla u^k|_\infty|\nabla\overline{u}^{k+1}|_\infty|\overline{\varphi}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\\\leqq& C\varepsilon|\overline{\varphi}^{k+1}|_2^2+\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2,\end{aligned}\tag{3.93}$$

which, along with (3.91)–(3.93), implies that

$$J_5\leqq\frac{a_1\varepsilon\alpha}{20}|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2+C(+\varepsilon|\varphi^k\nabla^4u^k|_2^2)|\overline{W}^{k+1}|_2^2+C\varepsilon|\overline{\varphi}^{k+1}|_2^2.\tag{3.94}$$

For the term J_6, we have

$$\begin{aligned}J_6=&2a_1\varepsilon\int\nabla(\varphi^k)^2\cdot(Q(u^k)-Q(u^{k-1}))\cdot\overline{u}^{k+1}\\\leqq& C\varepsilon|\nabla\varphi^k|_\infty|\varphi^k\nabla\overline{u}^k|_2|\overline{u}^{k+1}|_2\leqq C\nu^{-1}|\overline{W}^{k+1}|_2^2+\nu\varepsilon|\varphi^k\nabla\overline{u}^k|_2^2,\end{aligned}\tag{3.95}$$

which, together with (3.88)-(3.89) and (3.94)–(3.95), immediately implies that

$$\begin{aligned}&\frac{d}{dt}\int(\overline{W}^{k+1})^\top A_0\overline{W}^{k+1}+a_1\varepsilon\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\\&\leqq C(\nu^{-1}+\varepsilon|\varphi^k\nabla^4u^k|_2^2)|\overline{W}^{k+1}|_2^2+C\varepsilon|\overline{\varphi}^{k+1}|_2^2\\&\quad+\nu(\varepsilon|\varphi^k\nabla\overline{u}^k|_2^2+\varepsilon|\varphi^k|_2^2+|\overline{W}^k|_2^2).\end{aligned}\tag{3.96}$$

Finally, we denote

$$\Gamma^{k+1}(t)=\sup_{s\in[0,t]}|\overline{W}^{k+1}(s)|_2^2+\sup_{s\in[0,t]}\varepsilon|\overline{\varphi}^{k+1}(s)|_2^2.$$

From (3.87) and (3.96), one has

$$\begin{aligned}&\frac{d}{dt}\int\Big((\overline{W}^{k+1})^\top A_0\overline{W}^{k+1}+\varepsilon|\overline{\varphi}^{k+1}(t)|_2^2\Big)+a_1\varepsilon\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\\&\leqq E_\nu^k(|\overline{W}^{k+1}|_2^2+\varepsilon|\overline{\varphi}^{k+1}|_2^2)+\nu(\varepsilon|\varphi^k\nabla\overline{u}^k|_2^2+\varepsilon|\overline{\varphi}^k|_2^2+|\overline{W}^k|_2^2),\end{aligned}$$

for some E_ν^k such that $\int_0^t E_\nu^k(s)\,\mathrm{d}s \leqq C + C_\nu t$. From Gronwall's inequality, one gets

$$\begin{aligned}
&\Gamma^{k+1} + \int_0^t a_1\varepsilon\alpha|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\,\mathrm{d}s\\
&\leqq C\nu\int_0^t \Big(\varepsilon|\varphi^k\nabla\overline{u}^k|_2^2 + \varepsilon|\overline{\varphi}^k|_2^2 + |\overline{W}^k|_2^2\Big)\,\mathrm{d}s\exp\left(C + C_\nu t\right)\\
&\leqq \Big(C\nu\int_0^t \varepsilon|\varphi^k\nabla\overline{u}^k|_2^2\,\mathrm{d}s + Ct\nu\sup_{s\in[0,t]}[|\overline{W}^k|_2^2 + \varepsilon|\overline{\varphi}^k|_2^2]\Big)\exp\left(C + C_\nu t\right).
\end{aligned}$$

We can choose $\nu_0 > 0$ and $T_* \in (0, \min(1, T^{**}))$ small enough such that

$$C\nu_0\exp C = \frac{1}{8}\min\left\{1, a_1\alpha\right\}, \quad \exp(C_\nu T_*) \leqq 2,$$

which implies that

$$\sum_{k=1}^{\infty}\Big(\Gamma^{k+1}(T_*) + \int_0^{T_*}\alpha\varepsilon|\varphi^{k+1}\nabla\overline{u}^{k+1}|_2^2\,\mathrm{d}t\Big) \leqq C < +\infty.$$

Thus, from the above estimate for $\Gamma^{k+1}(T_*)$ and (3.59), we know that the whole sequence (φ^k, W^k) converges to a limit (φ, W) in the following strong sense:

$$(\varphi^k, W^k) \to (\varphi, W) \text{ in } L^\infty([0, T_*]; H^2(\mathbb{R}^3)). \tag{3.97}$$

Due to the local estimates (3.59) and the lower-continuity of norm for weak or weak* convergence, we know that (φ, W) satisfies the estimates (3.59). According to the strong convergence in (3.97), it is easy to show that (φ, W) is a weak solution of (3.1) in the sense of distribution with the regularities:

$$\begin{aligned}
&\varphi \in L^\infty([0, T_*]; H^3), \quad \varphi_t \in L^\infty([0, T_*]; H^2), \quad \phi \in L^\infty([0, T_*]; H^3),\\
&\phi_t \in L^\infty([0, T_*]; H^2), \quad u \in L^\infty([0, T_*]; H^3),\\
&\varphi\nabla^4 u \in L^2([0, T_*]; L^2), \quad u_t \in L^\infty([0, T_*]; H^1) \cap L^2([0, T_*]; D^2).
\end{aligned}\tag{3.98}$$

Thus the existence of strong solutions is proved.

Step 2 uniqueness. Let (W_1, φ_1) and (W_2, φ_2) be two strong solutions to Cauchy problem (3.1) satisfying the uniform a priori estimates (3.59). We denote that

$$\overline{\varphi} = \varphi_1 - \varphi_2, \quad \overline{W} = (\overline{\phi}, \overline{u}) = (\phi_1 - \phi_2, u_1 - u_2),$$

then according to (3.86), $(\overline{\varphi}, \overline{\phi}, \overline{u})$ satisfies the system

$$\begin{cases}
\overline{\varphi}_t + u_1\cdot\nabla\overline{\varphi} + \overline{u}\cdot\nabla\varphi_2 + \dfrac{\delta-1}{2}(\overline{\varphi}\mathrm{div}u_2 + \varphi_1\mathrm{div}\overline{u}) = 0,\\
A_0\overline{W}_t + \sum\limits_{j=1}^{3}A_j(W_1)\partial_j\overline{W} + \varepsilon\varphi_1^2\mathbb{L}(\overline{u})\\
= -\sum\limits_{j=1}^{3}A_j(\overline{W})\partial_j W_2 - \overline{\varphi}(\varphi_1 + \varphi_2)\mathbb{L}(u_2)\\
+\varepsilon\big(\mathbb{H}(\varphi_1) - \mathbb{H}(\varphi_2)\big)\cdot\mathbb{Q}(W_2) + \mathbb{H}(\varphi_1)\cdot\mathbb{Q}(\overline{W}).
\end{cases}\tag{3.99}$$

Using the same arguments as in the derivation of (3.87)–(3.96), and letting

$$\Lambda(t) = |\overline{W}(t)|_2^2 + \varepsilon|\overline{\varphi}(t)|_2^2,$$

we similarly have

$$\begin{cases} \dfrac{d}{dt}\Lambda(t) + C\varepsilon|\varphi_1\nabla\overline{u}(t)|_2^2 \leqq F(t)\Lambda(t), \\ \displaystyle\int_0^t F(s)ds \leqq C \quad \text{for} \quad 0 \leqq t \leqq T_*. \end{cases} \tag{3.100}$$

From Gronwall's inequality, we have $\overline{\varphi} = \overline{\phi} = \overline{u} = 0$. Then the uniqueness is obtained.

Step 3 the time-continuity. It can be obtained via the same arguments used in in the proof of Lemma 3.6. □

4. Proof of Theorem 1.2

Based on Theorem 3.1, now we are ready to prove the uniform local-in-time well-posedenss (with respect to ε) of the regular solution to the original Cauchy problem (1.1)–(1.3), i.e., the proof of Theorem 1.2. Moreover, we will show that the regular solutions that we obtained satisfy system (1.1) classically in positive time $(0, T_*]$.

Proof. We divide the proof into three steps.

Step 1 existence of regular solutions. First, for the initial assumption (1.23), it follows from Theorem 3.1 that there exists a positive time T_* independent of ε such that the problem (3.1) has a unique strong solution (φ, ϕ, u) in $[0, T_*] \times \mathbb{R}^3$ satisfying the regularities in (3.4), which means that

$$(\rho^{\frac{\delta-1}{2}}, \rho^{\frac{\gamma-1}{2}}) = (\varphi, \phi) \in C^1((0, T_*) \times \mathbb{R}^3), \quad \text{and} \quad (u, \nabla u) \in C((0, T_*) \times \mathbb{R}^3). \tag{4.1}$$

Noticing that $\rho = \varphi^{\frac{2}{\delta-1}}$ and $\frac{2}{\delta-1} \geqq 1$, it is easy to show that

$$\rho \in C^1((0, T_*) \times \mathbb{R}^3).$$

Second, the system $(3.1)_2$ for $W = (\phi, u)$ could be written as

$$\begin{cases} \phi_t + u\cdot\nabla\phi + \dfrac{\gamma-1}{2}\phi\mathrm{div}u = 0, \\ u_t + u\cdot\nabla u + \dfrac{A\gamma}{\gamma-1}\nabla\phi^2 + \varepsilon\varphi^2 Lu = \varepsilon\nabla\varphi^2\cdot Q(u). \end{cases} \tag{4.2}$$

Multiplying $(4.2)_1$ by $\frac{\partial\rho}{\partial\phi}(t,x) = \frac{2}{\gamma-1}\phi^{\frac{3-\gamma}{\gamma-1}}(t,x) \in C((0, T_*) \times \mathbb{R}^3)$ on both sides, we get the continuity equation in $(1.1)_1$:

$$\rho_t + u\cdot\nabla\rho + \rho\mathrm{div}u = 0. \tag{4.3}$$

Multiplying (4.2)$_2$ by $\phi^{\frac{2}{\gamma-1}} = \rho(t,x) \in C^1((0,T_*) \times \mathbb{R}^3)$ on both sides, we get the momentum equations in (1.1)$_2$:

$$\rho u_t + \rho u \cdot \nabla u + \nabla P = \mathrm{div}\Big(\mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\mathrm{div} u I_3\Big). \tag{4.4}$$

Finally, recalling that ρ can be represented by the formula

$$\rho(t,x) = \rho_0(U(0,t,x)) \exp\Big(\int_0^t \mathrm{div} u(s, U(s,t,x))\,\mathrm{d}s\Big),$$

where $U \in C^1([0,T_*] \times [0,T_*] \times \mathbb{R}^3)$ is the solution to the initial value problem

$$\begin{cases} \frac{d}{ds} U(t,s,x) = u(s, U(s,t,x)), & 0 \leqq s \leqq T_*, \\ U(t,t,x) = x, & 0 \leqq t \leqq T_*, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.5}$$

it is obvious that

$$\rho(t,x) \geqq 0, \ \forall (t,x) \in (0,T_*) \times \mathbb{R}^3.$$

That is to say, (ρ, u) satisfies the problem (1.1)–(1.3) in the sense of distributions, and has the regularities shown in Definition 1.1, which means that the Cauchy problem (1.1)–(1.3) has a unique regular solution (ρ, u). □

Step 2 the smoothness of regular solutions. Now we will show that the regular solution that we obtained in the above step is indeed a classcial one in positive time $(0, T_*]$.

Due to the definition of regular solution and the classical Sobolev imbedding theorem, we immediately know that

$$(\rho, \nabla\rho, \rho_t, u, \nabla u) \in C([0,T_*] \times \mathbb{R}^3).$$

Now we only need to prove that

$$(u_t, \mathrm{div}\mathbb{T}) \in C((0,T_*] \times \mathbb{R}^3).$$

Proof. According to Theorem 1.2 in Section 3, the solution (φ, ϕ, u) of problem (3.1) satisfies the regularities (3.4) and $(\phi, \varphi) \in C^1([0,T_*] \times \mathbb{R}^3)$.

Next, we first give the continuity of u_t. We differentiate (4.2)$_2$ with respect to t:

$$u_{tt} + \varepsilon\varphi^2 L u_t = -(\varphi^2)_t L u - (u \cdot \nabla u)_t - \frac{A\gamma}{\gamma-1}\nabla(\phi^2)_t + \varepsilon(\nabla\varphi^2 \cdot Q(u))_t, \tag{4.6}$$

which, along with (3.4), easily implies that

$$u_{tt} \in L^2([0,T_*]; L^2). \tag{4.7}$$

Applying the operator ∂_x^ζ ($|\zeta|=2$) to (4.6), multiplying the resulting equations by $\partial_x^\zeta u_t$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\partial_x^\zeta u_t|_2^2+\alpha\varepsilon|\varphi\nabla\partial_x^\zeta u_t|_2^2+(\alpha+\beta)\varepsilon|\varphi\mathrm{div}\partial_x^\zeta u_t|_2^2\\
&\quad=\int\Big(-\varepsilon\nabla\varphi^2\cdot Q(\partial_x^\zeta u_t)-\varepsilon\Big(\partial_x^\zeta(\varphi^2Lu_t)-\varphi^2L\partial_x^\zeta u_t\Big)\Big)\cdot\partial_x^\zeta u_t\\
&\qquad+\int\Big(-\varepsilon\partial_x^\zeta\big((\varphi^2)_tLu\big)-\partial_x^\zeta(u\cdot\nabla u)_t-\frac{A\gamma}{\gamma-1}\partial_x^\zeta\nabla(\phi^2)_t\Big)\cdot\partial_x^\zeta u_t\\
&\qquad+\int\partial_x^\zeta(\varepsilon\nabla\varphi^2\cdot Q(u))_t\cdot\partial_x^\zeta u_t\equiv:\sum_{i=7}^{12}J_i.
\end{aligned}\tag{4.8}$$

Now we consider the terms on the right-hand side of (4.8). It follows from the Hölder's inequality, Lemma 2.1 and Young's inequality that

$$\begin{aligned}
J_7=&\varepsilon\int\Big(-\nabla\varphi^2\cdot Q(\partial_x^\zeta u_t)\Big)\cdot\partial_x^\zeta u_t\\
\leqq&C\varepsilon|\varphi\nabla^3u_t|_2|\nabla^2u_t|_2|\nabla\varphi|_\infty\leqq\frac{\alpha\varepsilon}{20}|\varphi\nabla^3u_t|_2^2+C\varepsilon^{\frac{1}{2}}|u_t|_{D^2}^2,\\
J_8=&\int-\varepsilon\Big(\partial_x^\zeta(\varphi^2Lu_t)-\varphi^2L\partial_x^\zeta u_t\Big)\cdot\partial_x^\zeta u_t\\
\leqq&C\varepsilon\Big(|\varphi\nabla^3u_t|_2|\nabla\varphi|_\infty+|\nabla\varphi|_\infty^2|u_t|_{D^2}+|\nabla^2\varphi|_3|\varphi\nabla^2u_t|_6\Big)|u_t|_{D^2}\\
\leqq&\frac{\alpha\varepsilon}{20}|\varphi\nabla^3u_t|_2^2+C\varepsilon^{\frac{1}{2}}|u_t|_{D^2}^2,
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
J_9=&\int-\varepsilon\partial_x^\zeta\big((\varphi^2)_tLu\big)\cdot\partial_x^\zeta u_t\\
\leqq&C\varepsilon\Big(|\nabla^2\varphi|_3|Lu|_6|\varphi_t|_\infty|u_t|_{D^2}+|\varphi\nabla^2u_t|_6|\varphi_t|_{D^2}|Lu|_3\\
&+|\nabla\varphi|_\infty|\nabla\varphi_t|_6|Lu|_3|u_t|_{D^2}+|\varphi\nabla^3u|_6|\nabla\varphi_t|_3|u_t|_{D^2}\\
&+|\nabla\varphi|_\infty|\varphi_t|_\infty|\nabla^3u|_2|u_t|_{D^2}+|\varphi_t|_\infty|u_t|_{D^2}|\varphi\nabla^4u|_2\Big)\\
\leqq&\frac{\alpha\varepsilon}{20}|\varphi\nabla^3u_t|_2^2+C\varepsilon^{\frac{1}{2}}|u_t|_{D^2}^2+C\varepsilon|\varphi\nabla^4u|_2^2+C\varepsilon^{\frac{1}{2}},\\
J_{10}=&\int-\partial_x^\zeta(u\cdot\nabla u)_t\cdot\partial_x^\zeta u_t\\
\leqq&C(\|u_t\|_1+|u_t|_{D^2})\|u\|_3-\int\big(u\cdot\nabla\big)\partial_x^\zeta u_t\cdot\partial_x^\zeta u_t\\
\leqq&C+C|u_t|_{D^2}^2+C|\nabla u|_\infty|\partial_x^\zeta u_t|_2^2\leqq C+C|u_t|_{D^2}^2,\\
J_{11}=&\int-\frac{A\gamma}{\gamma-1}\partial_x^\zeta\nabla(\phi^2)_t\cdot\partial_x^\zeta u_t\\
\leqq&C\Big(|\nabla^2\phi_t|_2|\phi\nabla^3u_t|_2+|\phi_t|_\infty|\nabla^3\phi|_2|\nabla^2u_t|_2\\
&+C|\nabla^2\phi|_6|\nabla\phi_t|_3|\nabla^2u_t|_2+|\nabla^2\phi_t|_2|\nabla\phi|_\infty|\nabla u_t|_2\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqq \frac{\alpha}{20}|\phi\nabla^3 u_t|_2^2 + C(1+|u_t|_{D^2}),\\
J_{12} =& \int \varepsilon\partial_x^\zeta(\nabla\varphi^2\cdot Q(u))_t\cdot\partial_x^\zeta u_t\\
&\leqq C\varepsilon\Big(\|\nabla\varphi\|_2^2|u_t|_{D^2}^2 + \big(\|\nabla\varphi\|_2|\nabla u_t|_3 + \|u\|_3\|\varphi_t\|_2\big)|\varphi\nabla^2 u_t|_6\\
&+\big(\|\nabla\varphi\|_2|\varphi\nabla^3 u_t|_2 + \|\nabla\varphi\|_2|\varphi\nabla^2 u_t|_6\big)|u_t|_{D^2}\\
&+\big(\|\nabla\varphi\|_2\|\varphi_t\|_2\|u\|_3 + \|\varphi_t\|_2|\varphi\nabla^3 u|_6\big)|u_t|_{D^2}\Big)\\
&+\varepsilon\int\partial_x^\zeta(\nabla\varphi^2)_t\cdot Q(u)\cdot\partial_x^\zeta u_t (= J_{121})\\
&\leqq \frac{\alpha\varepsilon}{20}|\varphi\nabla^3 u_t|_2^2 + C\varepsilon|u_t|_{D^2}^2 + C\varepsilon|\varphi\nabla^4 u|_2^2 + J_{121},
\end{aligned}
$$

where the term J_{121} can be estimated as follows:

$$
J_{121} \leqq \varepsilon\|\nabla\varphi\|_2\|\varphi_t\|_2\|u\|_3|u_t|_{D^2} + \varepsilon\int\varphi\partial_x^\zeta\nabla\varphi_t\cdot Q(u)\cdot\partial_x^\zeta u_t (= J_{1211)}. \tag{4.10}
$$

Via intergration by parts, for the last term J_{1211} on the right-hand side of (4.10), one has

$$
\begin{aligned}
J_{1211} \leqq& C\varepsilon\Big(|\nabla\varphi|_\infty|\varphi_t|_{D^2}|\nabla u|_\infty|u_t|_{D^2} + |\varphi_t|_{D^2}|\nabla^2 u|_3|\varphi\nabla^2 u_t|_6\\
&+ |\varphi\nabla^3 u_t|_2|\nabla u|_\infty\|\varphi_t\|_2\Big) \leqq \frac{\alpha\varepsilon}{20}|\varphi\nabla^3 u_t|_2^2 + C\varepsilon^{\frac{1}{2}}|u_t|_{D^2}^2 + C\varepsilon.
\end{aligned} \tag{4.11}
$$

It follows from (4.8)–(4.11) that

$$
\frac{1}{2}\frac{d}{dt}|u_t|_{D^2}^2 + \frac{\alpha}{2}|\varphi\nabla^3 u_t|_2^2 \leqq C\varepsilon^{\frac{1}{2}}|u_t|_{D^2}^2 + C\varepsilon|\varphi\nabla^4 u|_2^2 + C\varepsilon. \tag{4.12}
$$

Then multiplying both sides of (4.12) with t and integrating over $[\tau, t]$ for any $\tau \in (0, t)$, one gets

$$
t|u_t|_{D^2}^2 + \int_\tau^t s|\varphi\nabla^3 u_t|_2^2\,\mathrm{d}s \leqq C\tau|u_t(\tau)|_{D^2}^2 + C(1+t). \tag{4.13}
$$

According to the definition of the regular solution, we know that

$$
\nabla^2 u_t \in L^2([0, T_*]; L^2),
$$

which, along with Lemma 2.4, implies that there exists a sequence s_k such that

$$
s_k \to 0, \quad \text{and} \quad s_k|\nabla^2 u_t(s_k,\cdot)|_2^2 \to 0, \quad \text{as} \quad k \to +\infty.
$$

Then, letting $\tau = s_k \to 0$ in (4.13), we have

$$
t|u_t|_{D^2}^2 + \int_0^t s|\varphi\nabla^3 u_t|_2^2\,\mathrm{d}s \leqq C(1+t) \leqq C, \tag{4.14}
$$

so we have

$$t^{\frac{1}{2}} u_t \in L^\infty([0, T_*]; H^2). \tag{4.15}$$

Based on the classical Sobolev imbedding theorem

$$L^\infty([0, T]; H^1) \cap W^{1,2}([0, T]; H^{-1}) \hookrightarrow C([0, T]; L^q), \tag{4.16}$$

for any $q \in (3, 6)$, from (4.7) and (4.15), we have

$$t u_t \in C([0, T_*]; W^{1,4}),$$

which implies that $u_t \in C((0, T_*] \times \mathbb{R}^3)$.

Finally, we consider the continuity of $\mathrm{div}\mathbb{T}$. Denote $\mathbb{N} = \varepsilon\varphi^2 Lu - \varepsilon\nabla\varphi^2 \cdot Q(u)$. Based on (3.4) and (4.15), we have

$$t\mathbb{N} \in L^\infty(0, T_*; H^2).$$

From $\mathbb{N}_t \in L^2(0, T_*; L^2)$ and (4.16), we obtain $t\mathbb{N} \in C([0, T_*]; W^{1,4})$, which implies that $\mathbb{N} \in C((0, T_*] \times \mathbb{R}^3)$. Since $\rho \in C([0, T_*] \times \mathbb{R}^3)$ and $\mathrm{div}\mathbb{T} = \rho\mathbb{N}$, then we obtain the desired conclusion. □

Step 3 the proof of (1.26). If $1 < \delta \leqq \frac{5}{3}$, that is $\frac{2}{\delta - 1} \geqq 3$. Due to

$$\phi \in C([0, T_*]; H^3) \cap C^1([0, T_*]; H^2), \quad \text{and} \quad \rho(t, x) = \phi^{\frac{2}{\delta-1}}(t, x),$$

then we have

$$\rho(t, x) \in C([0, T_*]; H^3).$$

Noticing

$$\begin{aligned} & u \in C([0, T_*]; H^{s'}) \cap L^\infty(0, T_*; H^{s'}) \quad \text{for} \quad s' \in [2, 3), \\ & \rho^{\frac{\delta-1}{2}} \nabla^4 u \in C(0, T_*; L^2), \quad u_t \in C([0, T_*]; H^1) \cap L^2(0, T_*; D^2), \end{aligned} \tag{4.17}$$

it is not difficult to show that

$$\rho \mathrm{div} u \in L^2(0, T_*; H^3) \cap C([0, T^*]; H^2). \tag{4.18}$$

From the continuity equation $(1.1)_1$, and (4.17)-(4.18), we have

$$\rho \in C([0, T_*]; H^3) \cap C^1([0, T_*]; H^2).$$

Similarly, we can deal with cases: $\delta = 2$ or 3. Then the proof of Theorem 1.2 is finished.

5. Vanishing Viscosity Limit

In this section, we will establish the vanishing viscosity limit stated in Theorem 1.3. First we denote by

$$(\varphi^\varepsilon, W^\varepsilon) = (\varphi^\varepsilon, \phi^\varepsilon, u^\varepsilon)^\top = ((\rho^\varepsilon)^{\frac{\delta-1}{2}}, (\rho^\varepsilon)^{\frac{\gamma-1}{2}}, u^\varepsilon)^\top$$

the solution of problem (3.1), that is

$$\begin{cases} \varphi^\varepsilon_t + u \cdot \nabla \varphi^\varepsilon + \dfrac{\delta-1}{2}\varphi^\varepsilon \mathrm{div} u^\varepsilon = 0, \\ A_0 W^\varepsilon_t + \displaystyle\sum_{j=1}^3 A_j(W^\varepsilon)\partial_j W^\varepsilon = -\varepsilon\Big((\varphi^\varepsilon)^2 \mathbb{L}(W^\varepsilon) - \mathbb{H}(\varphi^\varepsilon)\cdot \mathbb{Q}(W^\varepsilon)\Big), \\ (\varphi^\varepsilon, W^\varepsilon)|_{t=0} = (\varphi_0, W_0), \\ (\varphi, W) \to (0,0), \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0. \end{cases} \tag{5.1}$$

The definitions of A_j $(j = 0, 1, ..., 3)$, $\mathbb{L}$, $\mathbb{H}$ and $\mathbb{Q}$ could be find in (1.20)–(1.22).

Second, based on [31], we denote by

$$W = (\phi, u) = (\rho^{\frac{\gamma-1}{2}}, u)$$

the regular solution of compressible Euler equations (1.9), which can be written as the following symmetric system:

$$\begin{cases} A_0 W_t + \displaystyle\sum_{j=1}^3 A_j(W)\partial_j W = 0, \\ W(x,0) = W_0 = (\phi_0, u_0), \\ W \to 0, \quad \text{as} \quad |x| \to +\infty, \quad t \geqq 0. \end{cases} \tag{5.2}$$

From Theorem 3.1, there exists a time $T^1_* > 0$ that is independent of ε such that the solution $(\varphi^\varepsilon, W^\varepsilon)$ of Cauchy problem (5.1) satisfies

$$\begin{aligned} &\sup_{0\leqq t\leqq T^1_*} \left(\|\varphi^\varepsilon(t)\|_2^2 + \|\phi^\varepsilon(t)\|_3^2 + \|u^\varepsilon(t)\|_2^2 + \varepsilon|\varphi^\varepsilon(t)|^2_{D^3}\right) \\ &\quad + \mathrm{ess}\sup_{0\leqq t\leqq T^1_*} |u^\varepsilon(t)|^2_{D^3} + \int_0^{T^1_*} \varepsilon|\varphi^\varepsilon \nabla^4 u^\varepsilon(t)|_2^2\, \mathrm{d}t \leqq C^0, \end{aligned} \tag{5.3}$$

where C^0 is a positive constant depending only on T^1_*, (φ_0, W_0) and the fixed constants A, δ, γ, α and β, and is independent of ε.

From [31] (see Theorem 1.1), we know that there exits a time T^2_* such that there is a unique regular solution W of Cauchy problem (5.2) satisfies

$$\sup_{0\leqq t\leqq T^2_*} \|W(t)\|_3^2 \leqq C^0. \tag{5.4}$$

Denote $T^* = \min\{T_*^1, T_*^2\}$. Then, letting $\overline{W}^\varepsilon = W^\varepsilon - W$, $(5.1)_2$ and $(5.2)_1$ lead to

$$\begin{aligned} &A_0\overline{W}_t^\varepsilon + \sum_{j=1}^{3} A_j(W^\varepsilon)\partial_j\overline{W}^\varepsilon \\ &= -\sum_{j=1}^{3} A_j(\overline{W}^\varepsilon)\partial_j W - \varepsilon\Big((\varphi^\varepsilon)^2\mathbb{L}(W^\varepsilon) - \mathbb{H}(\varphi^\varepsilon)\cdot\mathbb{Q}(W^\varepsilon)\Big). \end{aligned} \tag{5.5}$$

If the initial data

$$(\rho^\varepsilon, u^\varepsilon)|_{t=0} = (\rho, u)|_{t=0} = (\rho_0, u_0) \tag{5.6}$$

satisfies (1.23), it is obviously to see that $\overline{W}^\varepsilon(x,0) \equiv 0$.

Next we give some lemmas for establishing the vanishing viscosity limit.

Lemma 5.1. *If $(\varphi^\varepsilon, W^\varepsilon)$ and W are the regular solutions of Navier–Stokes (5.1) and Euler equations (5.2) respectively, then we have*

$$|\overline{W}^\varepsilon(t)|_2 \leqq C\varepsilon, \quad \textit{for} \quad 0 \leqq t \leqq T_*, \tag{5.7}$$

where the constant $C > 0$ depends on A, α, β, δ, (ρ_0, u_0) and T_*.

Proof. Multiplying (5.5) by $2\overline{W}^\varepsilon$ on both sides and integrating over $\mathbb{R}^3$, then one has

$$\begin{aligned} &\frac{d}{dt}\int (\overline{W}^\varepsilon)^\top A_0\overline{W}^\varepsilon + 2\sum_{j=1}^{3}\int (\overline{W}^\varepsilon)^\top A_j(W^\varepsilon)\partial_j\overline{W}^\varepsilon \\ &= -2\sum_{j=1}^{3}\int (\overline{W}^\varepsilon)^\top A_j(\overline{W}^\varepsilon)\partial_j W \\ &\quad - 2\varepsilon\int\Big((\varphi^\varepsilon)^2\mathbb{L}(W^\varepsilon) - \mathbb{H}(\varphi^\varepsilon)\cdot\mathbb{Q}(W^\varepsilon)\Big)\cdot\overline{W}^\varepsilon. \end{aligned} \tag{5.8}$$

It follows from the integrating by parts and Hölder's inequality that

$$\begin{aligned} \frac{d}{dt}\int (\overline{W}^\varepsilon)^\top A_0\overline{W}^\varepsilon \leqq &\int (\overline{W}^\varepsilon)^\top \mathrm{div}A(W^\varepsilon)\overline{W}^\varepsilon + C|\nabla W|_\infty|\overline{W}^\varepsilon|_2^2 \\ &+ 2\varepsilon\Big|(\varphi^\varepsilon)^2\mathbb{L}(W^\varepsilon) - \mathbb{H}(\varphi^\varepsilon)\cdot\mathbb{Q}(W^\varepsilon)\Big|_2|\overline{W}^\varepsilon|_2 \\ \leqq &C\big(|\nabla W^\varepsilon|_\infty + |\nabla W|_\infty\big)|\overline{W}^\varepsilon|_2^2 \\ &+ C\varepsilon\big(|\varphi^\varepsilon|_\infty^2|\nabla^2 u^\varepsilon|_2 + |\varphi^\varepsilon|_\infty|\nabla\varphi^\varepsilon|_3|\nabla u^\varepsilon|_6\big)|\overline{W}^\varepsilon|_2 \\ \leqq &C|\overline{W}^\varepsilon|_2^2 + C\varepsilon^2, \end{aligned}$$

where we used (3.21). According to the Gronwall's inequality, (5.7) follows immediately. □

Now we consider the estimate of $|\partial_x^\zeta\overline{W}^\varepsilon|_2$, as $|\zeta| = 1$.

Lemma 5.2. *If* $(\varphi^\varepsilon, W^\varepsilon)$ *and* W *are the regular solutions of Navier–Stokes* (5.1) *and Euler equations* (5.2) *respectively, then we have*

$$|\overline{W}^\varepsilon|_{D^1} \leqq C\varepsilon, \quad \text{for} \quad 0 \leqq t \leqq T_*, \tag{5.9}$$

where the constant $C > 0$ *depends on* A, α, β, δ, (ρ_0, u_0) *and* T_*.

Proof. Applying the operator ∂_x^ζ on (5.5), one gets

$$\begin{aligned}
&A_0(\partial_x^\zeta \overline{W}^\varepsilon)_t + \sum_{j=1}^{3} A_j(W^\varepsilon)\partial_j(\partial_x^\zeta \overline{W}^\varepsilon)\\
&\quad= \sum_{j=1}^{3} \Big(\partial_x^\zeta (A_j(W^\varepsilon)\partial_j \overline{W}^\varepsilon) - A_j(W^\varepsilon)\partial_j(\partial_x^\zeta \overline{W}^\varepsilon)\Big) - \sum_{j=1}^{3} \partial_x^\zeta (A_j(\overline{W}^\varepsilon))\partial_j W\\
&\qquad - \sum_{j=1}^{3} \Big(\partial_x^\zeta \big(A_j(\overline{W}^\varepsilon)\partial_j W\big) - \partial_x^\zeta \big(A_j(\overline{W}^\varepsilon)\big)\partial_j W\Big)\\
&\qquad - \varepsilon(\varphi^\varepsilon)^2 \mathbb{L}(\partial_x^\zeta W^\varepsilon) + \varepsilon \mathbb{H}(\varphi^\varepsilon)^2 \cdot \partial_x^\zeta \mathbb{Q}(W^\varepsilon)\\
&\qquad - \varepsilon \Big(\partial_x^\zeta \big((\varphi^\varepsilon)^2 \mathbb{L}(W^\varepsilon)\big) - (\varphi^\varepsilon)^2 \mathbb{L}(\partial_x^\zeta W^\varepsilon)\Big)\\
&\qquad + \varepsilon \Big(\partial_x^\zeta \big(\mathbb{H}(\varphi^\varepsilon)^2 \cdot \mathbb{Q}(W^\varepsilon)\big) - \mathbb{H}(\varphi^\varepsilon)^2 \cdot \partial_x^\zeta \mathbb{Q}(W^\varepsilon)\Big).
\end{aligned} \tag{5.10}$$

Then multiplying (5.10) by $2\partial_x^\zeta \overline{W}^\varepsilon$ and integrating over $\mathbb{R}^3$ by parts, one can obtain that

$$\begin{aligned}
&\frac{d}{dt}\int (\partial_x^\zeta \overline{W}^\varepsilon)^\top A_0 \partial_x^\zeta \overline{W}^\varepsilon\\
&\quad= \int (\partial_x^\zeta \overline{W}^\varepsilon)^\top \mathrm{div} A(W^\varepsilon) \partial_x^\zeta \overline{W}^\varepsilon - 2\sum_{j=1}^{3} \int (\partial_x^\zeta \overline{W}^\varepsilon)^\top \partial_x^\zeta (A_j(\overline{W}^\varepsilon))\partial_j W\\
&\qquad + 2\sum_{j=1}^{3} \int \Big(\partial_x^\zeta \Big(A_j(W^\varepsilon)\partial_j \overline{W}^\varepsilon\Big) - A_j(W^\varepsilon)\partial_j(\partial_x^\zeta \overline{W}^\varepsilon)\Big) \cdot \partial_x^\zeta \overline{W}^\varepsilon\\
&\qquad - 2\sum_{j=1}^{3} \int \Big(\partial_x^\zeta \big(A_j(\overline{W}^\varepsilon)\partial_j W\big) - \partial_x^\zeta A_j(\overline{W}^\varepsilon)\partial_j W\Big) \cdot \partial_x^\zeta \overline{W}^\varepsilon\\
&\qquad - 2a_1\varepsilon \int \Big((\varphi^\varepsilon)^2 L(\partial_x^\zeta u^\varepsilon)\Big) \cdot \partial_x^\zeta \overline{u}^\varepsilon + 2a_1\varepsilon \int \Big(\nabla(\varphi^\varepsilon)^2 \cdot Q(\partial_x^\zeta u^\varepsilon)\Big) \cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\qquad - 2a_1\varepsilon \int \Big(\partial_x^\zeta ((\varphi^\varepsilon)^2 L u^\varepsilon) - (\varphi^\varepsilon)^2 L \partial_x^\zeta u^\varepsilon\Big) \cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\qquad + 2a_1\varepsilon \int \Big(\partial_x^\zeta (\nabla(\varphi^\varepsilon)^2 \cdot Q(u^\varepsilon)) - \nabla(\varphi^\varepsilon)^2 \cdot Q(\partial_x^\zeta u^\varepsilon)\Big) \cdot \partial_x^\zeta \overline{u}^\varepsilon := \sum_{i=1}^{8} I_i,
\end{aligned} \tag{5.11}$$

where $\overline{u}^\varepsilon = u^\varepsilon - u$. For $|\zeta| = 1$, one gets

$$
\begin{aligned}
I_1 &= \int (\partial_x^\zeta \overline{W}^\varepsilon)^\top \mathrm{div} A(W^\varepsilon)\partial_x^\zeta \overline{W}^\varepsilon \leqq C|\nabla W^\varepsilon|_\infty |\nabla \overline{W}^\varepsilon|_2^2 \leqq C|\overline{W}^\varepsilon|_{D^1}^2,\\
I_2 &= -2\sum_{j=1}^{3}\int (\partial_x^\zeta \overline{W}^\varepsilon)^\top \partial_x^\zeta (A_j(\overline{W}^\varepsilon))\partial_j W \leqq C|\nabla W|_\infty |\overline{W}^\varepsilon|_{D^1}^2 \leqq C|\overline{W}^\varepsilon|_{D^1}^2,\\
I_3 &= 2\sum_{j=1}^{3}\int \Big(\partial_x^\zeta (A_j(W^\varepsilon)\partial_j \overline{W}^\varepsilon) - A_j(W^\varepsilon)\partial_j(\partial_x^\zeta \overline{W}^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{W}^\varepsilon\\
&\leqq C|\nabla W^\varepsilon|_\infty |\nabla \overline{W}^\varepsilon|_2^2 \leqq C|\overline{W}^\varepsilon|_{D^1}^2, \qquad (5.12)\\
I_4 &= -2\sum_{j=1}^{3}\int \Big(\partial_x^\zeta\big(A_j(\overline{W}^\varepsilon)\partial_j W\big) - \partial_x^\zeta (A_j(\overline{W}^\varepsilon))\partial_j W\Big)\cdot \partial_x^\zeta \overline{W}^\varepsilon\\
&\leqq C|\nabla^2 W|_3 |\nabla \overline{W}^\varepsilon|_2 |\overline{W}^\varepsilon|_6 \leqq C|\overline{W}^\varepsilon|_{D^1}^2,\\
I_5 &= -2a_1\varepsilon \int \Big((\varphi^\varepsilon)^2 \cdot L(\partial_x^\zeta u^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty^2 |\nabla^3 u^\varepsilon|_2 |\nabla \overline{u}^\varepsilon|_2 \leqq C\varepsilon|\nabla \overline{u}^\varepsilon|_2 \leqq C\varepsilon^2 + C|\overline{W}^\varepsilon|_{D^1}^2,\\
I_6 &= 2a_1\varepsilon \int \Big(\nabla(\varphi^\varepsilon)^2 \cdot Q(\partial_x^\zeta u^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty |\nabla\varphi^\varepsilon|_3 |\nabla^2 u^\varepsilon|_6 |\nabla \overline{u}^\varepsilon|_2 \leqq C\varepsilon|\nabla \overline{u}^\varepsilon|_2 \leqq C\varepsilon^2 + C|\overline{W}^\varepsilon|_{D^1}^2,\\
I_7 &= -2a_1\varepsilon \int \Big(\partial_x^\zeta((\varphi^\varepsilon)^2 L u^\varepsilon) - (\varphi^\varepsilon)^2 L\partial_x^\zeta u^\varepsilon\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty |\nabla\varphi^\varepsilon|_3 |\nabla^2 u^\varepsilon|_6 |\nabla \overline{u}^\varepsilon|_2 \leqq C\varepsilon^2 + C|\overline{W}^\varepsilon|_{D^1}^2,\\
I_8 &= 2a_1\varepsilon \int \Big(\partial_x^\zeta(\nabla(\varphi^\varepsilon)^2 \cdot Q(u^\varepsilon)) - \nabla(\varphi^\varepsilon)^2 \cdot Q(\partial_x^\zeta u^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon(|\varphi^\varepsilon|_\infty |\nabla u^\varepsilon|_\infty |\nabla^2\varphi^\varepsilon|_2 |\nabla \overline{u}^\varepsilon|_2 + |\nabla u^\varepsilon|_\infty |\nabla\varphi^\varepsilon|_3 |\nabla\varphi^\varepsilon|_6 |\nabla \overline{u}^\varepsilon|_2)\\
&\leqq C\varepsilon|\nabla \overline{u}^\varepsilon|_2 \leqq C\varepsilon^2 + C|\overline{W}^\varepsilon|_{D^1}^2.
\end{aligned}
$$

Substituting (5.12) into (5.11), one has

$$
\frac{d}{dt}\int (\partial_x^\zeta \overline{W}^\varepsilon)^\top A_0 \partial_x^\zeta \overline{W}^\varepsilon \leqq C\varepsilon^2 + C|\overline{W}^\varepsilon|_{D^1}^2.
$$

Then, according to the Gronwall's inequality, (5.9) follows immediately. □

For $|\zeta| = 2$, we have

Lemma 5.3. *If* $(\varphi^\varepsilon, W^\varepsilon)$ *and* W *are the regular solutions of Navier–Stokes* (5.1) *and Euler equations* (5.2) *respectively, then we have*

$$
|\overline{W}^\varepsilon|_{D^2} \leqq C\varepsilon^{\frac{1}{2}}, \quad \text{for} \quad 0 \leqq t \leqq T_*, \qquad (5.13)
$$

where the constant $C > 0$ *depends on* A, α, β, δ, (ρ_0, u_0) *and* T_*.

Proof. For $|\zeta|=2$, it follows from (3.21) and (5.11) that

$$
\begin{aligned}
I_1 &= \int (\partial_x^\zeta \overline{W}^\varepsilon)^\top \mathrm{div} A(W^\varepsilon)\partial_x^\zeta \overline{W}^\varepsilon \leqq C|\nabla W^\varepsilon|_\infty |\nabla^2 \overline{W}^\varepsilon|_2^2 \leqq C|\overline{W}^\varepsilon|_{D^2}^2,\\
I_2 &= 2\sum_{j=1}^3 \int \Big(\partial_x^\zeta (A_j(W^\varepsilon)\partial_j \overline{W}^\varepsilon) - A_j(W^\varepsilon)\partial_j(\partial_x^\zeta \overline{W}^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{W}^\varepsilon\\
&\leqq C(|\nabla W^\varepsilon|_\infty |\nabla^2 \overline{W}^\varepsilon|_2 + |\nabla^2 W^\varepsilon|_3 |\nabla \overline{W}^\varepsilon|_6)|\nabla^2 \overline{W}^\varepsilon|_2 \leqq C|\overline{W}^\varepsilon|_{D^2}^2,
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
I_3 &= -2\sum_{j=1}^3 \int (\partial_x^\zeta \overline{W}^\varepsilon)^\top \partial_x^\zeta (A_j(\overline{W}^\varepsilon))\partial_j W\\
&\leqq C|\nabla W|_\infty |\overline{W}^\varepsilon|_{D^2}^2 \leqq C|\overline{W}^\varepsilon|_{D^2}^2,\\
I_4 &= -2\sum_{j=1}^3 \int \Big(\partial_x^\zeta \big(A_j(\overline{W}^\varepsilon)\partial_j W\big) - \partial_x^\zeta (A_j(\overline{W}^\varepsilon))\partial_j W\Big)\cdot \partial_x^\zeta \overline{W}^\varepsilon\\
&\leqq C\int \Big(|\nabla \overline{W}^\varepsilon||\nabla^2 W| + |\overline{W}^\varepsilon||\nabla^3 W|\Big)|\nabla^2 \overline{W}^\varepsilon|\\
&\leqq C\Big(|\nabla \overline{W}^\varepsilon|_6 |\nabla^2 W|_3 |\nabla^2 \overline{W}^\varepsilon|_2 + |\nabla^3 W|_2 |\overline{W}^\varepsilon|_\infty |\nabla^2 \overline{W}^\varepsilon|_2\Big)\\
&\leqq C\left(|\nabla^2 \overline{W}^\varepsilon|_2^2 + \varepsilon^{\frac12}|\nabla^2 \overline{W}^\varepsilon|_2^{\frac32}\right) \leqq C|\overline{W}^\varepsilon|_{D^2}^2 + C\varepsilon^2,\\
I_5 &= -2a_1\varepsilon \int \Big((\varphi^\varepsilon)^2 \cdot L(\partial_x^\zeta u^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty |\varphi^\varepsilon \nabla^4 u^\varepsilon|_2 |\nabla^2 \overline{u}^\varepsilon|_2 \leqq C\varepsilon^2 |\varphi^\varepsilon \nabla^4 u^\varepsilon|_2^2 + C|\overline{W}^\varepsilon|_{D^2}^2,\\
I_6 &= 2a_1\varepsilon \int \Big(\nabla(\varphi^\varepsilon)^2 \cdot Q(\partial_x^\zeta u^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty |\nabla\varphi^\varepsilon|_\infty |\nabla^3 u^\varepsilon|_2 |\nabla^2 \overline{u}^\varepsilon|_2 \leqq C\varepsilon^{\frac34}|\nabla^2 \overline{u}^\varepsilon|_2 \leqq C\varepsilon^{\frac32} + C|\overline{W}^\varepsilon|_{D^2}^2,\\
I_7 &= -2a_1\varepsilon \int \Big(\partial_x^\zeta((\varphi^\varepsilon)^2 L u^\varepsilon) - (\varphi^\varepsilon)^2 L\partial_x^\zeta u^\varepsilon\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon \int \Big(\big(|\varphi^\varepsilon||\nabla^2\varphi^\varepsilon| + |\nabla\varphi^\varepsilon|^2\big)|\nabla^2 u^\varepsilon| + |\varphi^\varepsilon||\nabla\varphi^\varepsilon||\nabla^3 u^\varepsilon|\Big)|\nabla^2 \overline{u}^\varepsilon|\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty |\nabla^2\varphi^\varepsilon|_3 |\nabla^2 u^\varepsilon|_6 |\nabla^2 \overline{u}^\varepsilon|_2 + C\varepsilon|\nabla\varphi^\varepsilon|_6^2 |\nabla^2 u^\varepsilon|_6 |\nabla^2 \overline{u}^\varepsilon|_2\\
&\quad + C\varepsilon|\varphi^\varepsilon|_\infty |\nabla\varphi^\varepsilon|_\infty |\nabla^3 u^\varepsilon|_2 |\nabla^2 \overline{u}^\varepsilon|_2 \leqq C\varepsilon^{\frac32} + C|\overline{u}^\varepsilon|_{D^2}^2,\\
I_8 &= \varepsilon \int \Big(\partial_x^\zeta\big(\nabla(\varphi^\varepsilon)^2 \cdot Q(u^\varepsilon)\big) - \nabla(\varphi^\varepsilon)^2 \cdot Q(\partial_x^\zeta u^\varepsilon)\Big)\cdot \partial_x^\zeta \overline{u}^\varepsilon\\
&\leqq C\varepsilon \int \big(|\varphi^\varepsilon||\nabla^3\varphi^\varepsilon| + |\nabla\varphi^\varepsilon||\nabla^2\varphi^\varepsilon|\big)|\nabla u^\varepsilon||\nabla^2 \overline{u}^\varepsilon|\\
&\quad + C\varepsilon \int \big(|\varphi^\varepsilon||\nabla^2\varphi^\varepsilon| + |\nabla\varphi^\varepsilon|^2\big)|\nabla^2 u^\varepsilon||\nabla^2 \overline{u}^\varepsilon|\\
&\leqq C\varepsilon|\varphi^\varepsilon|_\infty |\nabla u^\varepsilon|_\infty |\nabla^3\varphi^\varepsilon|_2 |\nabla^2 \overline{u}^\varepsilon|_2 + C\varepsilon|\nabla\varphi^\varepsilon|_3 |\nabla^2\varphi^\varepsilon|_6 |\nabla u^\varepsilon|_\infty |\nabla^2 \overline{u}^\varepsilon|_2\\
&\quad + C\varepsilon|\varphi^\varepsilon|_\infty |\nabla^2\varphi^\varepsilon|_3 |\nabla^2 u^\varepsilon|_6 |\nabla^2 \overline{u}^\varepsilon|_2 + C\varepsilon|\nabla\varphi^\varepsilon|_6 |\nabla\varphi^\varepsilon|_6 |\nabla^2 u^\varepsilon|_6 |\nabla^2 \overline{u}^\varepsilon|_2
\end{aligned}
$$

$$\leqq C\varepsilon^{\frac{1}{2}}|\nabla^2\overline{u}^\varepsilon|_2 \leqq C\varepsilon + C|\nabla^2\overline{u}^\varepsilon|^2_{D^2}.$$

Substituting (5.14) into (5.11), we have

$$\frac{d}{dt}\int(\partial_x^\zeta \overline{W}^\varepsilon)^\top A_0\partial_x^\zeta \overline{W}^\varepsilon \leqq C\varepsilon + C|\overline{W}^\varepsilon|^2_{D^2} + C\varepsilon^2|\varphi^\varepsilon\nabla^4 u^\varepsilon|^2_2.$$

Then, according to Gronwall's inequality, (5.13) follows immediately. □

Finally, we have

Lemma 5.4. *If* $(\varphi^\varepsilon, W^\varepsilon)$ *and* W *are the regular solutions of Navier–Stokes* (5.1) *and Euler equations* (5.2) *respectively, then we have*

$$\sup_{0\leqq t\leqq T_*}\|\overline{W}^\varepsilon(t)\|_{H^{s'}} \leqq C\varepsilon^{1-\frac{s'}{3}}, \tag{5.15}$$

where $s' \in (2, 3)$ *and the constant* $C > 0$ *depends on* A, α, β, δ, (ρ_0, u_0) *and* T_*.

Proof. Based on Lemma 2.5, (3.3), (5.3) and (5.4), we have

$$\|\overline{W}^\varepsilon(t)\|_{s'} \leqq C\|\overline{W}_\varepsilon(t)\|_0^{1-\frac{s'}{3}}\|\overline{W}_\varepsilon(t)\|_3^{\frac{s'}{3}} \leqq C\varepsilon^{1-\frac{s'}{3}}.$$

□

Finally, we give the proof for Theorem 1.3.

Proof. First, according to Lemmas 5.1-5.4, when $\varepsilon \to 0$, the solutions $(\rho^\varepsilon, u^\varepsilon)$ of compressible Navier–Stokes solutions converges to the solution (ρ, u) of compressible Euler equations in the following sense:

$$\lim_{\varepsilon\to 0}\sup_{0\leqq t\leqq T_*}\left(\left\|\left((\rho^\varepsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}}\right)(t)\right\|_{H^{s'}} + \left\|\left(u^\varepsilon - u\right)(t)\right\|_{H^{s'}}\right) = 0, \tag{5.16}$$

for any constant $s' \in [0, 3)$. Moreover, one can also obtain that

$$\begin{aligned}
&\sup_{0\leqq t\leqq T_*}\left(\left\|\left((\rho^\varepsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}}\right)(t)\right\|_1 + \|\left(u^\varepsilon - u\right)(t)\|_1\right) \leqq C\varepsilon,\\
&\sup_{0\leqq t\leqq T_*}\left(\left|\left((\rho^\varepsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}}\right)(t)\right|_{D^2} + |\left(u^\varepsilon - u\right)(t)|_{D^2}\right) \leqq C\sqrt{\varepsilon},
\end{aligned} \tag{5.17}$$

where $C > 0$ is a constant depending only on the fixed constants $A, \delta, \gamma, \alpha, \beta, T_*$ and ρ_0, u_0.

Further more, if the condition (1.25) holds, one has $\frac{2}{\delta-1} \geqq 3$. Then, from (5.16)–(5.17), it is not hard to see that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\sup_{0\leqq t\leqq T_*}\left(\left\|\left(\rho^\varepsilon - \rho\right)(t)\right\|_{H^{s'}} + \left\|\left(u^\varepsilon - u\right)(t)\right\|_{H^{s'}}\right) &= 0,\\
\sup_{0\leqq t\leqq T_*}\left(\left\|\rho^\varepsilon - \rho)(t)\right\|_1 + \|\left(u^\varepsilon - u\right)(t)\|_1\right) &\leqq C\varepsilon,\\
\sup_{0\leqq t\leqq T_*}\left(\left|\left(\rho^\varepsilon - \rho\right)(t)\right|_{D^2} + |\left(u^\varepsilon - u\right)(t)|_{D^2}\right) &\leqq C\sqrt{\varepsilon}.
\end{aligned} \tag{5.18}$$

Thus the proof of Theorem 1.3 is finished. □

Remark 5.1. For the case $\delta = 1$ with vacuum at the far field, via introducing two different symmetric structures in Ding–Zhu [11], some uniform estimates with respect to the viscosity coefficients for $\left(\rho^{\frac{\gamma-1}{2}}, u\right)$ in $H^3(\mathbb{R}^2)$ and $\nabla\rho/\rho$ in $L^6 \cap D^1(\mathbb{R}^2)$ have been obtained, which lead the convergence of the regular solution of the viscous flow to that of the inviscid flow still in $L^\infty([0,T]; H^{s'}(\mathbb{R}^2))$ (for any $s' \in [2,3)$) with the rate of $\varepsilon^{2(1-\frac{s'}{3})}$; their conclusion also applies to the 2-D shallow water equations (1.13).

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