Arch. Rational Mech. Anal. 233 (2019) 975–1025
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-019-01373-w

On Navier–Stokes–Korteweg and Euler–Korteweg Systems: Application to Quantum Fluids Models

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Communicated by F. Otto

Abstract

In this paper, the main objective is to generalize to the Navier–Stokes–Korteweg (with density dependent viscosities satisfying the BD relation) and Euler–Korteweg systems a recent relative entropy (proposed by Bresch et al. in C R Math Acad Sci Paris 354(1):45–49, 2016) introduced for the compressible Navier–Stokes equations with a linear density dependent shear viscosity and a zero bulk viscosity. As a concrete application, this helps to justify mathematically the convergence between global weak solutions of the quantum Navier–Stokes system (recently obtained simultaneously by Lacroix-Violet and Vasseur in J Math Pures Appl 114(9):191–210, 2018) and dissipative solutions of the quantum Euler system when the viscosity coefficient tends to zero; this selects a dissipative solution as the limit of a viscous system. We also recover the weak–strong uniqueness for the Quantum-Euler as in Giesselmann et al. (Arch Ration Mech Anal 223:1427–1484, 2017) and extend the result for the Quantum-Navier–Stokes equations. Our results are based on the fact that Euler–Korteweg systems and corresponding Navier–Stokes–Korteweg systems can be reformulated through an augmented system such as the compressible Navier–Stokes system with density dependent viscosities satisfying the BD algebraic relation. This was also observed recently by Bresch et al. (2016) for the Euler–Korteweg system for numerical purposes. As a by-product of our analysis, we show that this augmented formulation helps to define relative entropy estimates for the Euler–Korteweg systems in a simplest way compared to recent works (see Donatelli et al. in Commun Partial Differ Equ 40:1314–1335, 2015; Giesselmann et al. 2017) with less hypothesis required on the capillary coefficient.

1. Introduction

Quantum fluid models have attracted a lot of attention in recent decades due to the variety of their applications. Indeed, such models can be used to describe superfluids [39], quantum semiconductors [25], weakly interacting Bose gases [30] and quantum trajectories of Bohmian mechanics [46]. Recently some dissipative quantum fluid models have been derived. In particular, under some assumptions and using a Chapman-Enskog expansion in Wigner equation, the authors have obtained in [16] the so-called quantum Navier–Stokes model. Roughly speaking, it corresponds to the classical Navier–Stokes equations with a quantum correction term. The main difficulties of such models lie in the highly nonlinear structure of the third order quantum term and the proof of positivity (or non-negativity) of the particle density. Note that formally, the quantum Euler system corresponds to the limit of the quantum Navier–Stokes model when the viscosity coefficient tends to zero. This type of model belongs to more general classes of models: the Navier–Stokes–Korteweg and the Euler–Korteweg systems. Readers interested by Korteweg type systems are referred to the following articles and books: [17,20,32, 35,40,41,43] and references cited therein.

The goal of this paper is to extend to these two Korteweg systems a recent relative entropy proposed in [13] introduced for the compressible Navier–Stokes equations with a linear density dependent shear viscosity and a zero bulk viscosity. This leads, for each system, to the definition of what we call a dissipative solution following the concept introduced by P.-L. Lions in the incompressible setting (see [38]) and later extended to the compressible framework (see [5,23,24,42] for constant viscosities and [13,31] for density dependent viscosities). As a consequence we obtain some weak–strong uniqueness results and as an application, we can use it to show that a global weak solution (proved in [36], which is also a dissipative one) of the quantum Navier–Stokes system converges to a dissipative solution of the quantum Euler system. Our results will be compared to recent results in [21,27] showing that we relax one hypothesis on the capillarity coefficient by introducing entropy-relative solutions of an augmented system. Note also the interesting paper [4] where the authors prove the existence of global weak solutions of the quantum-Navier–Stokes equations with a different method compared to [36]. It is worthy of note we cannot use such global weak solutions because capillarity and viscosity magnitudes are linked together in their study. Let us also mention the interesting new paper [19] where the authors investigate the long-time behavior of solutions to the isothermal Euler–Korteweg system.

Let us now present in more detail the models of interest here. Note that for the convenience of the reader all the operators are defined in Sect. 6.3. Let $\Omega = \mathbb{T}^d$ be the torus in dimension d (in this article $1 \leq d \leq 3$).

Euler–Korteweg system. Following the framework of the paper, we first present the Euler-Kortewg system and then the Navier–Stokes Korteweg one. Note that throughout the paper, the systems are supplemented with the initial conditions

$$\rho|_{t=0} = \rho_0, \quad (\rho u)|_{t=0} = \rho_0 u_0 \quad \text{for } x \in \Omega, \tag{1}$$

with the regularity $\rho_0 \geq 0$, $\rho_0 \in L^\gamma(\Omega)$, $\rho_0 |u_0|^2 \in L^1(\Omega)$, $\sqrt{K(\rho_0)}\nabla\rho_0 \in L^2(\Omega)$. The Euler–Korteweg system describes the time evolution for $t > 0$ of the density $\rho = \rho(t, x)$ and the momentum $J = J(t, x) = \rho(t, x)u(t, x)$ (with u the velocity) for $x \in \Omega$ of an inviscid fluid. The equations can be written in the form ([21]):

$$\partial_t \rho + \operatorname{div} J = 0, \tag{2}$$

$$\partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla(p(\rho)) = \varepsilon^2 \rho \nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right), \tag{3}$$

where $K : (0,\infty) \to (0,\infty)$ is a smooth function and p is the pressure function given by $p(\rho) = \rho^\gamma$ for $\gamma > 1$. Note that it could be interesting to consider non-monotone pressure laws as in [28] and [27]. The coefficient ε stands for the Planck constant. In this paper we will consider a function $K(\rho)$ which behaves as ρ^s with $s \in \mathbb{R}$. As mentioned in [21],

$$\rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right) = \operatorname{div}(\mathbb{K}),$$

with

$$\mathbb{K} = \left(\rho \operatorname{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\right)\mathbb{I}_{\mathbb{R}^d} - K(\rho)\nabla\rho \otimes \nabla\rho.$$

Observing that $\mathbb{K}$ may be written as

$$\mathbb{K} = \left(\operatorname{div}(\rho K(\rho)\nabla\rho) - \frac{1}{2}(K(\rho) + \rho K'(\rho))|\nabla\rho|^2\right)\mathbb{I}_{\mathbb{R}^d} - K(\rho)\nabla\rho \otimes \nabla\rho, \tag{4}$$

and following the ideas of [8] with

$$\mu'(\rho) = \sqrt{\rho K(\rho)}, \tag{5}$$

we can define the drift velocity v by

$$v = \sqrt{\frac{K(\rho)}{\rho}}\nabla\rho = \frac{\nabla(\mu(\rho))}{\rho}$$

and show the following generalization of the Bohm identity:

$$\operatorname{div}(\mathbb{K}) = \operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} v),$$

with

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)).$$

Remark 1. Note that the relation between λ and μ is exactly the BD relation found in [9] in the Navier–Stokes setting: see the Navier–Stokes–Korteweg part below.

We will choose $K(\rho)$ as

$$K(\rho) = \frac{(s+3)^2}{4}\rho^s \text{ with } s \in \mathbb{R} \qquad \text{in order to get} \qquad \mu(\rho) = \rho^{(s+3)/2}.$$

This multiplicative constant in the definition of K does not affect any generality, it suffices to change the definition of ε. Then, we obtain the following augmented formulation for the Euler–Korteweg Equations (2)–(3):

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \tag{6}$$

$$\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) + \nabla p(\rho) = \varepsilon \left[\operatorname{div}(\mu(\rho)\nabla \bar{v}) + \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} \bar{v})\right], \tag{7}$$

$$\partial_t(\rho\, \bar{v}) + \operatorname{div}(\rho\, \bar{v} \otimes u) = \varepsilon \left[-\operatorname{div}(\mu(\rho)\, {}^t\nabla u) - \frac{1}{2}\nabla(\lambda(\rho)\operatorname{div} u)\right], \tag{8}$$

with

$$\lambda(\rho) = 2(\rho\, \mu'(\rho) - \mu(\rho)), \qquad \bar{v} = \varepsilon \nabla \mu(\rho)/\rho. \tag{9}$$

System (6)–(9) is called the Euler–Korteweg augmented system in the sequel. It has been firstly introduced in this conservative form in [8] to propose a useful construction of a numerical scheme with entropy stability property under a hyperbolic CFL condition for such dispersive PDEs. augmented system, the second order operator matrix is skew-symetric.

The Quantum Euler Equations. Note that the choice $K(\rho) = 1/\rho$ (which gives $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$) leads to the Bohm identity

$$\rho\nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right) = \operatorname{div}(\rho\nabla v) = 2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right).$$

In that case the system (6)–(9) becomes

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \tag{10}$$

$$\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) + \nabla(p(\rho)) = \varepsilon\, \operatorname{div}(\rho\, \nabla \bar{v}), \tag{11}$$

$$\partial_t(\rho\, \bar{v}) + \operatorname{div}(\rho\, \bar{v} \otimes u) = -\varepsilon \operatorname{div}(\rho\, {}^t\nabla u), \tag{12}$$

with

$$\bar{v} = \varepsilon \nabla \log \rho, \tag{13}$$

which corresponds to the augmented formulation of the quantum Euler system:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{14}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 2\,\varepsilon^2\, \rho\, \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{15}$$

Then such a choice gives rise the so called quantum fluid system for which the global existence of weak solutions of (14)–(15) has been shown in [2,3] and more recently in [18] assuming the initial velocity irrotational namely $\mathrm{curl}(\rho_0 u_0) = 0$. Note that the quantum term is written as (4) in these papers, namely

$$2\rho\,\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \mathrm{div}\big(\nabla(\rho\nabla\log\rho) - \rho\nabla\log\rho\otimes\nabla\log\rho\big), \tag{16}$$

observing that

$$\sqrt{\rho}\nabla\log\rho = 2\nabla\sqrt{\rho}.$$

The existence of local strong solutions has also been proved (see [6]) and global well-posedness for small irrotational data has been performed recently in [1], assuming a natural stability condition on the pressure. We refer to (10)–(12) as the quantum Euler augmented system in all the paper.

Important remark. Differentiating in space the mass equation in $\mathcal{D}'((0,T)\times\Omega)$ we get

$$\partial_t\nabla\rho + \nabla\,\mathrm{div}(\rho u) = \partial_t\nabla\rho + \mathrm{div}({}^t\nabla(\rho u)) = 0,$$

which may be written

$$\partial_t\nabla\rho + \mathrm{div}(\rho\nabla\log\rho\otimes u) + \mathrm{div}\big({}^t\nabla(\rho u) - \rho\nabla\log\rho\otimes u\big) = 0.$$

This formula will be used to show that global weak solutions of the Quantum-Euler system (14)–(15) with the quantum term written as (16) will be global weak solutions of the Quantum-Euler system in its augmented form.

Note that the quantum correction $(\Delta\sqrt{\rho})/\sqrt{\rho}$ can be interpreted as a quantum potential, the so-called Bohm potential, which is well known in quantum mechanics. This Bohm potential arises from the fluid dynamical formulation of the single-state Schrödinger equation. The non-locality of quantum mechanics is approximated by the fact that the equations of state do not only depend on the particle density but also on its gradient. These equations were employed to model field emissions from metals and steady-state tunneling in metal-insulator-metal structures and to simulate ultra-small semiconductor devices.

Navier–Stokes–Korteweg system. Let us consider the compressible Navier–Stokes–Korteweg system with density dependent viscosities $\mu(\rho)$ and $\lambda(\rho)$ satisfying the BD relation

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)),$$

and with the capillarity coefficient $K(\rho)$ linked to the shear viscosity $\mu(\rho)$ in the following manner:

$$K(\rho) = [\mu'(\rho)]^2/\rho \text{ with } \mu(\rho) = \rho^{(s+3)/2} \text{ with } s\in\mathbb{R}.$$

Remark 2. With this choice of shear viscosity, the relation between the capillarity coefficient and the viscosity gives a capillarity coefficient proportional to ρ^s.

Then using the identity given in the Euler–Korteweg part, the Navier–Stokes–Korteweg system can be written for $x \in \Omega$ and $t > 0$,

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{17}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - 2\nu \operatorname{div}(\mu(\rho) D(u)) - \nu \nabla(\lambda(\rho) \operatorname{div} u)$$
$$= \varepsilon^2 \left[(\operatorname{div}(\mu(\rho)^t \nabla v) + \frac{1}{2} \nabla(\lambda(\rho) \operatorname{div} v) \right], \tag{18}$$

in which the symmetric part of the velocity gradient is $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$. The parameter $\nu > 0$ stands for the viscosity constant. Multiplying (17) by $\mu'(\rho)$ and taking the gradient, we have the following equation on v:

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\mu(\rho)\, {}^t\nabla u) + \frac{1}{2} \nabla(\lambda(\rho) \operatorname{div} u) = 0. \tag{19}$$

Moreover, defining the intermediate velocity, called the effective velocity, $w = u + \nu\, v$, Eqs. (18) and (19) lead to

$$\partial_t(\rho\, w) + \operatorname{div}(\rho\, w \otimes u) + \nabla(p(\rho)) - \nu \operatorname{div}(\mu(\rho)\nabla w) - \frac{\nu}{2} \nabla(\lambda(\rho) \operatorname{div} w)$$
$$= (\varepsilon^2 - \nu^2) \left[\operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2} \nabla(\lambda(\rho) \operatorname{div} v) \right].$$

Then (17)–(18) may be reformulated through the following augmented system:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{20}$$

$$\partial_t(\rho\, w) + \operatorname{div}(\rho\, w \otimes u) + \nabla(p(\rho)) - \nu \operatorname{div}(\mu(\rho)\nabla w) - \frac{\nu}{2} \nabla(\lambda(\rho) \operatorname{div} w)$$
$$= (\varepsilon^2 - \nu^2) \left[\operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2} \nabla(\lambda(\rho) \operatorname{div} v) \right], \tag{21}$$

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\mu(\rho)\, {}^t\nabla u) + \frac{1}{2} \nabla(\lambda(\rho) \operatorname{div} u) = 0, \tag{22}$$

with

$$w = u + \nu \nabla \mu(\rho)/\rho, \qquad v = \nabla \mu(\rho)/\rho, \tag{23}$$

which we call the Navier–Stokes–Korteweg augmented system in the sequel.

The Quantum Navier–Stokes Equations. Note that with the choice $K(\rho) = 1/\rho$, which gives $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$, system (20)–(23) becomes

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{24}$$

$$\partial_t(\rho\, w) + \operatorname{div}(\rho\, w \otimes u) + \nabla(p(\rho)) - \nu \operatorname{div}(\rho \nabla w) = (\varepsilon^2 - \nu^2) \operatorname{div}(\rho \nabla v), \tag{25}$$

$$\partial_t(\rho\, v) + \operatorname{div}(\rho\, v \otimes u) + \operatorname{div}(\rho\, {}^t\nabla u) = 0, \tag{26}$$

with the constraints

$$w = u + \nu \nabla \log \rho, \qquad v = \nabla \log \rho, \tag{27}$$

which is the augmented formulation of the compressible barotropic quantum Navier–Stokes system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{28}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - 2\,\nu\, \operatorname{div}(\rho D(u)) = 2\,\varepsilon^2\,\rho\,\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{29}$$

In [22,33,34], the global existence of weak solutions to (28)–(29) has been shown following the idea introduced in [11] by testing the momentum equation by $\rho\,\phi$ with ϕ a test function. The problem of such formulation is that it requires $\gamma > 3$ for $d = 3$, which is not a suitable assumption for physical cases. In [12] the authors show the existence of solutions for (28)–(29) without quantum term (*i.e.* for $\varepsilon = 0$) by adding a cold pressure term in the momentum equation. The cold pressure is a suitable increasing function p_c satisfying $\lim_{n\to 0} p_c(n) = +\infty$. The key element of the proof is a κ-entropy estimate. In [29], using the same strategy and a κ-entropy with $\kappa = 1/2$, the existence of global weak solutions for (28)–(29) is proven without any extra assumption on γ and the semi-classical limit ε tends to zero is performed. In [44], A. Vasseur and C. Yu consider the compressible barotropic quantum Navier–Stokes equations with damping *i.e.* system (28)–(29) with additional terms in the right hand side of (29): $-r_0 u - r_1\rho|u|^2 u$. They prove the global-in-time existence of weak solutions and their result is still valuable in the case $r_1 = 0$. Their proof is based on a Faedo-Galerkin approximation (following the ideas of [34]) and a Bresch-Desjardins entropy (see [10,11]). In [45], the authors use the result obtained in [44] and pass to the limits ε, r_0, r_1 tend to zero to prove the existence of global-in-time weak solutions to degenerate compressible Navier–Stokes equations. Note that to prove such a result they need uniform (with respect to r_0, r_1) estimates to pass to the limit r_0, r_1 tend to 0. To this end they have to firstly pass to the limit ε tends to 0. The reader interested by the compressible Navier–Stokes equations with density dependent viscosities is also referred to the interesting paper [37] where more general viscosities are consiedered. Recently in [36] and [4], global existence of weak solutions for the quantum Navier–Stokes equations (28)–(29) has been proved without drag terms and without any cold pressure. In the first paper, the method is based on the construction of weak solutions that are renormalized in the velocity variable. Note that the construction being uniform with respect to the Planck constant, the authors also perform the semi-classical limit to the associated compressible Navier–Stokes equations. Note also the recent paper [4] concerning the global existence for the quantum Navier–Stokes system where they use in a very nice way the mathematical structure of the equations. It is important to remark that a global weak solutions of the quantum Navier–Stokes equations in the sense of [36] is also weak solution of the augmented system (due to the regularity which is envolved allowing to write the equation on the drift velocity v). Remark also that there exists no global existence result of weak solutions for the compressible Navier–Stokes–Korteweg system with constant viscosities even in the two-dimensional in space case.

Main objectives of the paper. In this paper, in the authors' point of view, there are several interesting and new results. First, starting with the global weak solutions of the quantum Navier–Stokes equations constructed in [36] (which is a $1/2$-entropy solution in the sense of [12]), we show at the viscous limit the existence of a dissipative solution for the quantum Euler system letting the viscosity goes to zero. This gives the first global existence result of dissipative solution for the quantum Euler system obtained from a quantum Navier–Stokes type system. Note that in [21], it is proved the existence of infinite dissipative solutions of such inviscid quantum system. Here we present a way to select one starting from a Navier–Stokes type system. Secondly, we develop relative entropy estimates for general cases of the Euler–Korteweg and the Navier–Stokes–Korteweg systems extending the augmented formulations introduced recently in [13] and [14]: more general viscosities and third order dispersive terms. This gives a more simple procedure to perform relative entropy than the one developped in [21,27] for the Euler–Korteweg system but asks us to start with an augmented version of the Euler–Korteweg system. This allows us to extend for the Euler–Korteweg the weak–strong uniqueness result already proved for the quantume Euler-system in [27] and extend the result for the Navier–Stokes–Korteweg systems.

This also helps to get rid the concavity assumption on $1/K(\rho)$ which is strongly used in [27]. For the interested readers, we provide a comparison of the quantities appearing in our relative entropy to the ones introduced in [27] and remark that they are equivalent under the assumptions made in [27]. Note that to perform our calculations for the Navier–Stokes–Korteweg system, we need to generalize in a non-trivial way the identity (5) in [13]; see Proposition 30 for the generalized identity.

For reader's convenience, let us explain the simple idea behind all the calculations. The kinetic energy corresponding to the Euler–Korteweg system reads

$$\int_\Omega \left(\frac{1}{2}\rho|u|^2 + H(\rho) + K(\rho)|\nabla\rho|^2\right)$$

with

$$H(\rho) = \rho \int_1^\rho \frac{p(z)}{z^2} \mathrm{d}z.$$

In [27], they consider that it is an energy written in terms of $(\rho, u, \nabla\rho)$ and they write a relative entropy playing with these unknowns. In our calculations, we write the kinetic energy as follows:

$$\int_\Omega \left(\frac{1}{2}\rho|u|^2 + H(\rho) + \rho|v|^2\right),$$

with $v = \sqrt{K(\rho)}\nabla\rho/\sqrt{\rho}$ and we consider three quantities ρ, u and v. This motivates to write an augmented system (ρ, u, v) and to modulate the energy through these three unknowns. This gives the simplest way to define an appropriate relative entropy quantity compared to [27] and [21] and allows to relax the concavity assumption on $1/K(\rho)$ made in the part concerning Euler–Korteweg system in [27]. Our result covers capillarity coefficient under the form

$$K(\rho) \approx \rho^s \text{ with } s+2 \leq \gamma \text{ and } s \geq -1.$$

Finally, our result makes the link between Euler–Korteweg system and Navier–Stokes–Korteweg system. After proving the global existence of $1/2$-entropy solutions of the general Navier–Stokes–Korteweg system (this is the subject of a forthcoming paper [15] still in progress; the case $K(\rho) = 1/\rho$ has been recently proved in [36]), this could give the mathematical justification of a physical dissipative solution of the Euler–Korteweg equations obtained from $1/2$-entropy solutions of the Navier–Stokes–Korteweg equations in the spirit of [12]. Note also the other interesting result in [4] on the Quantum-Navier–Stokes equations but under hypothesis between the magnitude of the viscous and capillarity coefficients. Let us also mention that our relative entropies could be helpful for other singular limits as explained in the book [24] in the case of constant viscosities.

The paper is organized as follows: in Sect. 2, we provide energy estimates and the definition of weak solutions for the augmented Euler–Korteweg and Navier–Stokes–Korteweg systems. In Sect. 3, we give the definition of the relative entropy formula and we established the associated estimate. This one is used to define what we call a dissipative solution for the Euler–Korteweg system and we established a weak/strong uniqueness result. The same results are obtained for the Navier–Stokes–Korteweg system in Sect. 4. In Sect. 5 we use the previous results to show the limit when the viscosity tends to zero in the quantum Navier–Stokes system. Finally we give in Appendix some technical lemmas on modulated quantities and a comparaison between the relative entropy developed here and the one used in [21,27], and we state the definitions used for the operators.

2. Energy Estimates and Definition of Weak Solutions

In this subsection we give the energy equalities for the augmented Euler–Korteweg and Navier–Stokes–Korteweg systems. They will be used in the following to establish the estimates for the relative entropy associated to each one. We also define weak solutions concept for the two augmented systems. First of all, let us recall the definition of the function H called the enthalpy by

$$H(\rho) = \rho e(\rho) = \rho \int_1^\rho \frac{p(z)}{z^2}\,\mathrm{d}z.$$

Namely we have,

$$\rho H'(\rho) - H(\rho) = p(\rho), \qquad H''(\rho) = \frac{p'(\rho)}{\rho}.$$

To be more precise, since $p(\rho) = \rho^\gamma$ with $\gamma > 1$, this yields $H(\rho) = \dfrac{1}{\gamma - 1}p(\rho)$.

Euler–Korteweg system. For the augmented Euler–Korteweg system we can show the following formal proposition:

Proposition 3. *All strong enough solution* (ρ, u, v) *of system (6)–(9) satisfies*

$$\frac{dE_{EuK}(\rho, u, v)}{dt} = 0,$$

where E_{EuK} *is the natural energy density given by*

$$E_{EuK}(t) = E_{EuK}(\rho, u, v) = \int_\Omega \left(\frac{1}{2}\, \rho\, |u|^2 + \frac{1}{2}\, \varepsilon^2\, K(\rho) |\nabla \rho|^2 + H(\rho) \right). \tag{30}$$

Proof. It suffices to take the scalar product of the equation related to u by u and the equation related to v by v and integrate in space using the mass equation, the symmetry of ∇v and the relation $\rho |v|^2 = K(\rho) |\nabla \rho|^2$. □

Global weak solutions of the augmented system. An assumption between $K(\rho)$ and $p(\rho)$ will be required to define global weak solutions of the augmented version of the Euler–Korteweg system namely,

$$K(\rho) = [\mu'(\rho)]^2/\rho \text{ with } \mu(\rho) = \rho^{(s+3)/2} \qquad \text{and} \qquad p(\rho) = \rho^\gamma,$$

with

$$s + 2 \le \gamma, \qquad s \ge -1 \text{ and } \gamma > 1.$$

Assume the initial density ρ_0 positive and in $L^1(\Omega)$, namely

$$\rho_0 \ge 0 \qquad \text{and} \qquad \int_\Omega \rho_0 < +\infty$$

and

$$E_{\text{EuK}}(\rho_0, u_0, \bar{v_0}) < +\infty,$$

where $\bar{v_0}$ and u_0 is zero where ρ_0 vanishes. We can define global weak solutions of the augmented version of the Euler–Korteweg system as solutions satisfying for a.e $t \in [0, T]$:

$$E_{\text{EuK}}(\rho, u, \bar{v})(t) \le E_{\text{EuK}}(\rho, u, \bar{v})|_{t=0} < +\infty$$

with

$$\rho \ge 0 \qquad \text{and} \qquad \int_\Omega \rho = \int_\Omega \rho_0 \qquad \text{and} \qquad \sup_{t\in(0,T)} \int_\Omega \mu(\rho) < +\infty,$$

and satisfying the following augmented system in a distribution sense:

$$\partial_t \rho + \mathrm{div}(\rho\, u) = 0, \tag{31}$$

$$\partial_t(\rho\, u) + \mathrm{div}(\rho\, u \otimes u) + \nabla p(\rho) = \varepsilon\, \mathrm{div}\Big(\mathbb{T}^{EuK}(\bar v) + \frac{\lambda(\rho)}{2\mu(\rho)} \mathrm{Tr}\, (\mathbb{T}^{EuK}(\bar v))\Big) \tag{32}$$

$$\partial_t(\rho\, \bar v) + \mathrm{div}(\rho\, \bar v \otimes u) = -\varepsilon\, \mathrm{div}\Big((\mathbb{T}^{EuK}(u))^t + \frac{\lambda(\rho)}{2\mu(\rho)}\, \mathrm{Tr}\, (\mathbb{T}^{EuK}(u))\Big) \tag{33}$$

with

$$\lambda(\rho) = 2(\rho\, \mu'(\rho) - \mu(\rho)), \qquad \bar v = \varepsilon \nabla \mu(\rho)/\rho, \tag{34}$$

where the tensor valued function $\mathbb{T}^{EuK}(\theta)$ (for $\theta = u$ and $\bar v$) is defined through the relation

$$\mathbb{T}^{EuK}(\theta) = \Big[\nabla(\mu(\rho)\,\theta) - \frac{1}{\varepsilon}\rho\theta \,\otimes \bar v)\Big]$$

with

$$\mathbb{T}^{EuK}(\theta) \in L^\infty(0, T; W^{-1,1}(\Omega)).$$

Important property. Note that the energy estimate provides the bound $L^\infty(0, T; L^\gamma(\Omega))$ on ρ and thus $\mu(\rho)/\sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))$ and, using the mass equation, $\mu(\rho) \in L^\infty(0, T; L^1(\Omega))$.

Important remark. Let us remark that for the global weak solutions of the Euler–Korteweg, the following equation may be checked to be satsified in the distribution sense:

$$\partial_t \mu(\rho) + \mathrm{div}(\mu(\rho)\, u) + \frac{\lambda(\rho)}{2\mu(\rho)} \mathrm{Tr}\, (\mathbb{T}^{EuK}(u))] = 0. \tag{35}$$

Remark that $\lambda(\rho)/\mu(\rho) = 2(\mu'(\rho)\rho - \rho)/\mu(\rho) = \mathrm{Cte}$. Taking the gradient of Eq. (41), we get

$$\partial_t \nabla\mu(\rho) + \mathrm{div}({}^t\nabla(\mu(\rho)u)) + \nabla\left(\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\mathbb{T}^{EuK}(u))\right) = 0,$$

and therefore, by definition of $\mathbb{T}^{EuK}(u)$ and expression of $\overline{v}$, we can write

$$\partial_t(\rho\overline{v}) + \mathrm{div}(\rho\overline{v} \otimes u) + \varepsilon\, \mathrm{div}((\mathbb{T}^{EuK}(u))^t) + \varepsilon\nabla\left(\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\mathbb{T}^{EuK}(u))\right) = 0.$$

This explain why a global weak solution of the Euler–Korteweg system with the extra equality (35) satsified is also global weak solution of the augmented Euler–Korteweg system.

Navier–Stokes–Korteweg system. Concerning the augmented Navier–Stokes–Korteweg system (20)–(23), defining the energy

$$E^{\varepsilon,\nu}_{NSK}(t) = E^{\varepsilon,\nu}_{NSK}(\rho, v, w) = \int_\Omega \left(\frac{\varepsilon^2 - \nu^2}{2}\rho\, |v|^2 + \frac{\rho}{2}|w|^2 + H(\rho)\right),$$

we have the following formal equality:

Proposition 4. *For* (ρ, v, w) *a strong enough solution of* (20)–(23) *we have*

$$\frac{dE^{\varepsilon,\nu}_{NSK}}{dt}(\rho, v, w) + \nu \int_\Omega \Big(\mu(\rho)\Big(|\nabla w|^2 + (\varepsilon^2 - \nu^2)|\nabla v|^2\Big) + \mu'(\rho)H''(\rho)|\nabla\rho|^2\Big)$$
$$+ \nu \int_\Omega \Big(\frac{\lambda(\rho)}{2}\Big((\operatorname{div} w)^2 + (\varepsilon^2 - \nu^2)(\operatorname{div} v)^2\Big)\Big) = 0.$$

It suffices to take the scalar product of (21) with w and to take the scalar product of (22) by $(\varepsilon^2 - \nu^2)v$, using the expressions of w and v, integrate in space and sum to prove the result using the mass equation.

Global weak solutions of the augmented system. Looking at new unknowns $(\rho, \bar v, w)$ with $\bar v = \sqrt{\varepsilon^2 - \nu^2}$, an assumption between $K(\rho)$ and $p(\rho)$ will be required to define global weak solutions of the augmented version of the Navier-Korteweg system, namely

$$K(\rho) = [\mu'(\rho)]^2/\rho \text{ with } \mu(\rho) = \rho^{(s+3)/2} \qquad \text{and} \qquad p(\rho) = \rho^\gamma,$$

with

$$s + 2 \le \gamma, \qquad s \ge -1 \text{ and } \gamma > 1.$$

Note that with this constraint on $\mu(\rho)$, we have

$$\lambda(\rho)/\mu(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))/\mu(\rho) = (s+1) = \text{Cst} \ge 0.$$

Assume that the initial density ρ_0 is positive and in $L^1(\Omega)$ we have

$$\rho_0 \ge 0, \qquad \int_\Omega \rho_0 < +\infty$$

and

$$E_{\text{NSK}}(\rho_0, \bar{v_0}, w_0) < +\infty$$

with

$$E_{\text{NSK}}(\rho_0, \bar{v_0}, w_0) = [E_{\text{NSK}}(\rho, \bar v, w)]_{t=0} = \Big[\int_\Omega \rho|\overline{v}|^2 + \rho|w|^2 + H(\rho)\Big]_{t=0}$$
$$= \int_\Omega \rho_0|\overline{v_0}|^2 + \rho_0|w_0|^2 + H(\rho_0).$$

We can define global weak solutions of the augmented version of the Navier-Korteweg system as solutions satisfying, for $t \in [0, T]$, a.e $\tau \in [0, t]$,

$$E_{NSK}(\rho, \bar v, w)(\tau) + \nu \int_0^t \int_\Omega \Big(\Big(|\mathbb{T}(w)|^2 + |\mathbb{T}(\bar v)|^2\Big) + \frac{1}{\varepsilon^2 - \nu^2}\frac{\rho\, p'(\rho)}{\mu'(\rho)}|\bar v|^2\Big)$$
$$+ \nu \int_0^t \int_\Omega \Big(\frac{\lambda(\rho)}{2\mu(\rho)}\Big(|\text{Tr}\,(\mathbb{T}(w))|^2 + |\text{Tr}\,(\mathbb{T}(\bar v))|^2\Big)\Big) \le E_{NSK}(\rho, \bar v, w)(0), \tag{36}$$

where

$$E_{NSK}(\rho, \bar{v}, w) = \int_\Omega \rho|\bar{v}|^2 + \rho|w|^2 + H(\rho)$$
$$\rho \geq 0, \qquad \int_\Omega \rho = \int_\Omega \rho_0 < +\infty, \qquad \sup_{t\in(0,T)} \int_\Omega \mu(\rho) < +\infty.$$

The augmented system in the distribution is as follows:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{37}$$

$$\begin{aligned} &\partial_t(\rho\, w) + \mathrm{div}(\rho\, w \otimes u) + \nabla(p(\rho)) \\ &\quad -\nu\ \mathrm{div}\big[\sqrt{\mu(\rho)}\mathbb{T}(w) - \frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}(\mathbb{T}(w))\mathrm{Id}\big] \\ &= \sqrt{\varepsilon^2-\nu^2}\ \mathrm{div}\Big[\sqrt{\mu(\rho)}\mathbb{T}(\bar{v}) + \frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}(\mathbb{T}(\bar{v}))\mathrm{Id}\Big], \end{aligned} \tag{38}$$

$$\begin{aligned} &\partial_t(\rho\, \bar{v}) + \mathrm{div}(\rho\, \bar{v} \otimes u) - \nu\, \mathrm{div}\big[\sqrt{\mu(\rho)}\mathbb{T}(\bar{v}) + \frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}\ (\mathbb{T}(\bar{v}))\mathrm{Id}\big] \\ &= -\sqrt{\varepsilon^2-\nu^2}\ \mathrm{div}\Big[\sqrt{\mu(\rho)}(\mathbb{T}(w))^t + \frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}(\mathbb{T}(w))\mathrm{Id}\Big], \end{aligned} \tag{39}$$

with

$$w = u + \nu\nabla\mu(\rho)/\rho, \qquad \bar{v} = \sqrt{\varepsilon^2-\nu^2}\nabla\mu(\rho)/\rho, \tag{40}$$

and where the tensor valued function $\mathbb{T}(\theta)$ (for $\theta = w$ and $\bar{v}$) satisfies $\sqrt{\nu}\,\mathbb{T}(\theta)$ is bounded in $L^2(0, T; L^2(\Omega))$ and satisfies the relation

$$\sqrt{\mu(\rho)}\mathbb{T}(\theta) = \nabla(\mu(\rho)\,\theta) - \frac{1}{\sqrt{\varepsilon^2-\nu^2}}\rho\theta\ \otimes \bar{v},$$

and is chosen equal to zero when ρ vanishes.

1) *Important property.* Note that the energy estimate provides the bound $L^\infty(0, T; L^\gamma(\Omega))$ on ρ and thus $\mu(\rho)/\sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))$ and thus using the mass quation $\mu(\rho) \in L^\infty(0, T; L^1(\Omega))$.

2) *Important Remark.* Let us remark that for the global weak solutions of the Navier–Stokes–Korteweg, the following equation is satisfied in the distribution sense

$$\nu\left[\partial_t\mu(\rho) + \mathrm{div}(\mu(\rho)\,u) + \frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}\ (\mathbb{T}(u))\right] = 0, \tag{41}$$

where $u = w - \nu\,\bar{v}/\sqrt{\varepsilon^2-\nu^2}$. Taking the gradient of Eq. (41), we get

$$\nu\left[\partial_t\nabla\mu(\rho) + \mathrm{div}({}^t\nabla(\mu(\rho)u)) + \nabla\big(\frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}(\mathbb{T}(u))\big)\right] = 0,$$

and therefore by definition of $\sqrt{\mu(\rho)}\mathbb{T}(u)$ and expression of v, we can write

$$\nu\left[\partial_t(\rho v)+\mathrm{div}(\rho v\otimes u)+\mathrm{div}(\sqrt{\mu(\rho)}(\mathbb{T}(u))^t)\right.$$
$$\left.+\nabla\left(\frac{\lambda(\rho)}{2\mu(\rho)}\sqrt{\mu(\rho)}\mathrm{Tr}(\mathbb{T}(u))\right)\right]=0.$$

This explains why a global weak solution of the Navier–Stokes–Korteweg system is also global weak solution of the augmented Navier–Stokes–Korteweg system.

3. The Euler–Korteweg System: Relative Entropy and Dissipative Solution

In this section, we consider the problem (2)–(3) through its augmented formulation (31)–(33). The main goal of this section is to give the definition of what we call a dissipative solution for this problem. To this end we have to establish a relative entropy inequality.

3.1. Relative Entropy Inequality

In [23], Feireisl et al. have introduced relative entropies, suitable weak solutions and weak–strong uniqueness properties for the compressible Navier–Stokes equations with constant viscosities. The goal of this subsection is to establish a relative entropy inequality for the Euler–Korteweg System using the augmented formulation introduced in [8] and extending the ideas in [13] and [14] to such system in order to be able to define what is called a dissipative solution.

Let us consider the relative entropy functional, denoted $\mathcal{E}_{EuK}(\rho,u,v|r,U,V)$ and defined by

$$\begin{aligned}\mathcal{E}_{EuK}(t)&=\mathcal{E}_{EuK}(\rho,u,v|r,U,V)(t)\\&=\frac{1}{2}\int_\Omega\rho\left(|u-U|^2+\varepsilon^2\left|\sqrt{\frac{K(\rho)}{\rho}}\nabla\rho-\sqrt{\frac{K(r)}{r}}\nabla r\right|^2\right)+\int_\Omega H(\rho|r)\\&=\frac{1}{2}\int_\Omega\rho\left(|u-U|^2+\varepsilon^2|v-V|^2\right)+\int_\Omega H(\rho|r),\end{aligned}\tag{42}$$

with

$$H(\rho|r)=H(\rho)-H(r)-H'(r)(\rho-r),$$

where (ρ,u,v) is a weak solution of System (31)–(34) and (r,U,V) smooth enough target functions. Note that the definition of the relative entropy used here is different from the one used in [27] but we can show that the twice are equivalent in some sense for some range of the capillary coefficient. We refer to appendix 6.2 for more details. Let us just say that such an energy measures the distance between a weak solution (ρ,u,v) of (31)–(34) to any smooth enough test function (r,U,V). The goal here is to prove an inequality of type

$$\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0)\le C\int_0^t\mathcal{E}_{EuK}(\xi)d\xi,$$

with C a positive constant. To this end let us first prove the following proposition:

Proposition 5. *Let us assume that* $\mu(\rho) = \rho^{(s+3)/2}$ *with* $\gamma \geq s+2$ *and* $s \geq -1$. *Let* $(\rho, u, \bar{v})$ *be a global weak solution to the augmented system* (31)–(34). *We have*

$$
\begin{aligned}
\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq & \int_0^t \int_\Omega \rho\,(U-u)\cdot \partial_t U + \int_0^t \int_\Omega \rho(\nabla U\, u)\cdot(U-u) \\
& + \int_0^t \int_\Omega \rho\,(\bar{V}-\bar{v})\cdot \partial_t V \\
& + \int_0^t \int_\Omega \rho(\nabla \bar{V}\, u)\cdot(\bar{V}-\bar{v}) \\
& + \varepsilon \int_0^t \Big\langle \mathbb{T}^{EuK}(\bar{v}) + \frac{\lambda(\rho)}{2\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(\bar{v}))\mathrm{Id};\, \nabla U \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,+\infty}(\Omega)} \\
& - \varepsilon \int_0^t \Big\langle (\mathbb{T}^{EuK}(u))^t + \frac{\lambda(\rho)}{2\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(u))\mathrm{Id};\, \nabla \bar{V} \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,+\infty}(\Omega)} \\
& - \int_0^t \int_\Omega p(\rho)\,\mathrm{div}\, U \\
& - \int_0^t \int_\Omega \big[\partial_t (H'(r))(\rho - r) + \rho \nabla H'(r)\cdot u\big]
\end{aligned}
$$

for all $t \in [0, T]$ *and for all smooth test functions* (r, U, V) *with*

$$
r \in \mathcal{C}^1([0,T]\times\bar{\Omega}), \quad r > 0, \quad U, V \in \mathcal{C}^2([0,T]\times\bar{\Omega}).
$$

Proof. Thanks to the global weak solutions definition given after Proposition 3 we have

$$
\begin{aligned}
\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq & \int_\Omega \bigg(\frac{\rho}{2}|U|^2 - \rho\, u\cdot U + \frac{1}{2}\rho\varepsilon^2 \frac{K(r)}{r}|\nabla r|^2 \\
& -\rho\varepsilon^2 \sqrt{\frac{K(\rho)}{\rho}}\nabla\rho \cdot \sqrt{\frac{K(r)}{r}}\nabla r \bigg)(t) \\
& - \int_\Omega \bigg(\frac{\rho}{2}|U|^2 - \rho\, u\cdot U + \frac{1}{2}\rho\varepsilon^2 \frac{K(r)}{r}|\nabla r|^2 \\
& -\rho\varepsilon^2 \sqrt{\frac{K(\rho)}{\rho}}\nabla\rho \cdot \sqrt{\frac{K(r)}{r}}\nabla r \bigg)(0) \\
& - \int_\Omega \big(H(r) + H'(r)(\rho - r)\big)(t) \\
& + \int_\Omega \big(H(r) + H'(r)(\rho - r)\big)(0),
\end{aligned}
$$

i.e.

$$\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq \int_0^t \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\rho}{2}|U|^2 - \rho\, u \cdot U + \frac{1}{2}\rho\, |\bar{V}|^2 - \rho\; \bar{v} \cdot \bar{V} \right) - \int_0^t \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} \big(H(r) + H'(r)(\rho - r)\big). \tag{43}$$

We multiply (32) by U, (33) by $\bar{V}$ and we integrate with respect to time and space. Writing

$$\partial_t(\rho\, u \cdot U) = \partial_t(\rho\, u) \cdot U + \rho\, u \cdot \partial_t U$$

and

$$\partial_t(\rho\, v \cdot V) = \partial_t(\rho v) \cdot V + \rho\, v \cdot \partial_t V,$$

and thanks to integrations by parts, we obtain

$$\begin{aligned}
&\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \\
&\leq \int_0^t \int_\Omega \partial_t \left(\frac{\rho}{2}|U|^2 \right) - \int_0^t \int_\Omega \rho\, u \cdot \partial_t U - \int_0^t \int_\Omega \rho(\nabla U\, u) \cdot u \\
&\quad + \varepsilon \int_0^t \Big\langle \mathbb{T}^{EuK}(\bar{v}) + \frac{\lambda(\rho)}{2\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(\bar{v}))\mathrm{Id}; \nabla U \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)} \\
&\quad + \int_0^t \int_\Omega \partial_t \left(\frac{\rho}{2}|\bar{V}|^2 \right) - \int_0^t \int_\Omega \rho\, \bar{v} \cdot \partial_t \bar{V} \\
&\quad - \int_0^t \int_\Omega \rho \left(\nabla \bar{V}\, u \right) \cdot \bar{v} \\
&\quad - \varepsilon \int_0^t \Big\langle (\mathbb{T}^{EuK}(u))^t + \frac{\lambda(\rho)}{2\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(u))\mathrm{Id}; \nabla \bar{V} \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)} \\
&\quad - \int_0^t \int_\Omega p(\rho)\, \mathrm{div}\, U - \int_0^t \int_\Omega \partial_t (H(r) + H'(r)(\rho - r)).
\end{aligned}$$

Using (31) and

$$\begin{aligned}
\partial_t \left(\frac{\rho}{2}|U|^2 \right) &= \frac{1}{2} \partial_t \rho\, |U|^2 + \rho\, U \cdot \partial_t U, \\
\partial_t \left(\frac{\rho}{2}|V|^2 \right) &= -\frac{1}{2} \mathrm{div}(\rho u)|V|^2 + \rho\, V \cdot \partial_t V,
\end{aligned}$$

thanks to integrations by parts, we have

$$\begin{aligned}
\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq &\int_0^t \int_\Omega \rho\, (U - u) \cdot \partial_t U + \int_0^t \int_\Omega \rho\, (\bar{V} - \bar{v}) \cdot \partial_t \bar{V} \\
&+ \int_0^t \int_\Omega \rho\, (\nabla U\, u) \cdot (U - u) + \int_0^t \int_\Omega \rho\, (\nabla \bar{V}\, u) \cdot (\bar{V} - \bar{v}) \\
&+ \varepsilon \int_0^t \Big\langle \mathbb{T}^{EuK}(\bar{v}); \nabla U \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)} \\
&- \varepsilon \int_0^t \Big\langle (\mathbb{T}^{EuK}(u))^t; \nabla \bar{V} \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}
\end{aligned}$$

$$+\frac{\varepsilon}{2}\int_0^t\Big\langle\frac{\lambda(\rho)}{\mu(\rho)}\mathrm{Tr}\,(\mathbb{T}^{EuK}(\bar{v}))\mathrm{Id};\,\nabla U\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}$$
$$-\frac{\varepsilon}{2}\int_0^t\Big\langle\frac{\lambda(\rho)}{\mu(\rho)}\mathrm{Tr}\,(\mathbb{T}^{EuK}(u))\mathrm{Id};\,\nabla \bar{V}\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}$$
$$-\int_0^t\int_\Omega p(\rho)\,\operatorname{div} U-\int_0^t\int_\Omega\partial_t(H(r)+H'(r)(\rho-r)).$$

This last inequality gives the result since with Eq. (31) we have

$$\int_\Omega\partial_t(H'(r)(\rho-r))=\int_\Omega\big(\partial_t(H'(r))(\rho-r)+\rho\nabla(H'(r))\cdot u\big).$$

□

Proposition 6. *Let* $(\rho,u,\bar{v})$ *be a global weak solution of the augmented system* (31)–(34) *and* $(r,U,\bar{V})$ *be a strong solution of*

$$\partial_t r+\operatorname{div}(r\,U)=0, \tag{44}$$

$$r\,(\partial_t U+U\cdot\nabla U)+\nabla p(r)-\varepsilon\left[\operatorname{div}(\mu(r)\nabla\bar{V})+\frac{1}{2}\nabla(\lambda(r)\operatorname{div}\bar{V})\right]=0, \tag{45}$$

$$r\left(\partial_t\bar{V}+U\cdot\nabla\bar{V}\right)+\varepsilon\Big[\operatorname{div}(\mu(r)\,{}^t\nabla U)+\frac{1}{2}\nabla(\lambda(r)\operatorname{div}U)\Big]=0 \tag{46}$$

belonging to the class

$$
\begin{aligned}
&0<\inf_{(0,T)\times\Omega}r\le r\le\sup_{(0,T)\times\Omega}r<+\infty\\
&\nabla r\in L^2(0,T;L^\infty(\Omega)\cap L^1(0,T;W^{1,\infty}(\Omega))\\
&U\in L^\infty(0,T;W^{2,\infty}(\Omega))\cap W^{1,\infty}(0,T;L^\infty(\Omega)),\\
&\bar{V}\in L^\infty(0,T;W^{2,\infty}(\Omega))\cap W^{1,\infty}(0,T;L^\infty(\Omega)),\\
&\partial_t H'(r)\in L^1(0,T;L^{\gamma/(\gamma-1)}(\Omega)),\qquad\nabla H'(r)\in L^1(0,T;L^{2\gamma/(\gamma-1)}(\Omega))
\end{aligned}
$$

and $V|_{t=0}=\varepsilon\nabla\mu(r_0)/r_0$. *Then we have*

$$
\begin{aligned}
\mathcal{E}_{EuK}(t)-\mathcal{E}_{EuK}(0)\le&\int_0^t\int_\Omega\rho(u-U)\cdot(\nabla U(U-u))\\
&-\int_0^t\int_\Omega\rho(\bar{V}-\bar{v})\cdot(\nabla\bar{V}(U-u))\\
&-\int_0^t\int_\Omega(p(\rho)-p(r)-(\rho-r)p'(r))\operatorname{div}U\\
&-\int_0^t\int_\Omega\rho\,(\bar{v}-\bar{V})\cdot\nabla U(\bar{v}-\bar{V})\\
&+\int_0^t\int_\Omega\rho(\bar{v}-\bar{V})\cdot\nabla\bar{V}(u-U)\\
&-\varepsilon\int_0^t\int_\Omega\rho\left(\mu''(\rho)\nabla\rho-\mu''(r)\nabla(r)\right)
\end{aligned}
$$

$$\cdot \big((\bar{v} - \bar{V}) \ \mathrm{div}\, U + (U - u) \ \mathrm{div}\, \bar{V}\big)$$
$$- \varepsilon \int_0^t \int_\Omega \rho \, (\mu'(\rho) - \mu'(r)) \big((\bar{v} - \bar{V})$$
$$\cdot \nabla(\mathrm{div}\, U) + (U - u) \ \cdot \nabla(\mathrm{div}\, \bar{V})\big) .$$

Proof. First remark that due to the initial condition hypothesis and the regularity hypothesis on U, we can prove that $\bar{V} = \varepsilon \nabla \mu(r)/r$. Multiplying (45) by $\dfrac{\rho}{r}\,(U - u)$ and (46) by $\dfrac{\rho}{r}\,(\bar{V} - \bar{v})$ and integrating with respect to time and space we have

$$\begin{aligned}
\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \le & - \int_0^t \int_\Omega \rho \, (\nabla U \, (U - u)) \cdot (U - u) \\
& - \int_0^t \int_\Omega \rho \, (\nabla \bar{V} \, (U - u)) \cdot (\bar{V} - \bar{v}) \\
& + \varepsilon \, (I_1^{EuK} + I_2^{EuK}) + I_3^{EuK}
\end{aligned}$$

with

$$\begin{aligned}
I_1^{EuK} = & \int_0^t \int_\Omega \Big(\frac{\rho}{r} \, \mathrm{div}(\mu(r) \nabla \bar{V}) \cdot (U - u) - \frac{\rho}{r} \, \mathrm{div}(\mu(r)^t \nabla U) \cdot (\bar{V} - \bar{v}) \Big) \\
& + \int_0^t \Big\langle \mathbb{T}^{EuK}(\bar{v}); \nabla U \Big\rangle_{W^{-1,1}(\Omega) \times W^{1,\infty}(\Omega)} \\
& - \int_0^t \Big\langle (\mathbb{T}^{EuK}(u))^t; \nabla \bar{V} \Big\rangle_{W^{-1,1}(\Omega) \times W^{1,\infty}(\Omega)} \\
2\, I_2^{EuK} = & \int_0^t \int_\Omega \frac{\rho}{r} (U - u) \cdot \nabla \big(\lambda(r) \ \mathrm{div}\, \bar{V} \big) - \int_0^t \int_\Omega \frac{\rho}{r} (\bar{V} - \bar{v}) \cdot \nabla \, (\lambda(r) \ \mathrm{div}\, U) \\
& + \int_0^t \Big\langle \frac{\lambda(\rho)}{\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(\bar{v})) \mathrm{Id}; \nabla U \Big\rangle_{W^{-1,1}(\Omega) \times W^{1,\infty}(\Omega)} \\
& - \int_0^t \Big\langle \frac{\lambda(\rho)}{\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(u)) \mathrm{Id}; \nabla \bar{V} \Big\rangle_{W^{-1,1}(\Omega) \times W^{1,\infty}(\Omega)} \\
I_3^{EuK} = & \int_0^t \int_\Omega \Big(- p(\rho) \ \mathrm{div}\, U - \frac{\rho}{r} \, \nabla p(r) \cdot (U - u) \\
& - \partial_t (H'(r))(\rho - r) - \rho \, \nabla (H'(r)) \cdot u \Big) .
\end{aligned}$$

Using $r H''(r) = p'(r)$, we have

$$\frac{\rho}{r} \nabla p(r) = \rho \nabla (H'(r)).$$

Mutiplying (44) by $H''(r)$ and using $r H''(r) = p'(r)$, we obtain

$$\partial_t H'(r) + \nabla H'(r) \cdot U + p'(r) \, \mathrm{div}\, U = 0.$$

Using $r H''(r) = p'(r)$ and an integration by parts, we have

$$\int_0^t \int_\Omega r \nabla H'(r) U = - \int_0^t \int_\Omega p(r) \, \mathrm{div}\, U.$$

Then,

$$I_3^{EuK} = \int_0^t \int_\Omega (p(r) - p(\rho) - (r-\rho)p'(r)) \,\mathrm{div}\, U.$$

We have

$$I_1^{EuK} = I_4^{EuK} + I_5^{EuK}, \tag{47}$$

where

$$\begin{aligned}\varepsilon I_4^{EuK} &= \varepsilon \int_0^t \int_\Omega \frac{\rho}{r}\mu(r)\Big[\Delta\bar V\cdot(U-u) - \nabla \mathrm{div} U\cdot(\bar V - \bar v)\Big]\\ &\quad \int_0^t \int_\Omega \rho(\bar V\cdot\nabla\bar V)\cdot(U-u) - \rho((\bar V-\bar v)\cdot\nabla U)\cdot\bar V,\end{aligned}$$

and using the symmetry of $\nabla\bar v$ and $\nabla\bar V$ and the definition the tensor value function $\mathbb{T}^{EuK}(u)$ and $\mathbb{T}^{EuK}(\bar v)$ which may be also written for U and $\bar V$ (recalling that $\mu(\rho)\in L^\infty(0,T;L^1(\Omega))$),

$$\begin{aligned}\varepsilon I_5^{EuK} &= \varepsilon\int_0^t \Big\langle \mathbb{T}^{EuK}(\bar v);\nabla U\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}\\ &\quad -\varepsilon\int_0^t \Big\langle (\mathbb{T}^{EuK}(u))^t;\nabla \bar V\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}\\ &= \varepsilon\int_0^t \Big\langle (\mathbb{T}^{EuK}(\bar v))^t;\nabla U\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}\\ &\quad -\varepsilon\int_0^t \Big\langle \mathbb{T}^{EuK}(u);\nabla \bar V\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}\\ &\quad -\varepsilon\int_0^t \Big\langle ((\mathbb{T}^{EuK}(\bar V))^t;\nabla U\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}\\ &\quad +\varepsilon\int_0^t \Big\langle \mathbb{T}^{EuK}(U);\nabla \bar V\Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}\\ &= \varepsilon\int_0^t\int_\Omega \mu(\rho)\Big[(\bar V-\bar v)\cdot\nabla\mathrm{div} U + (u-U)\Delta\bar V\Big]\\ &\quad + \int_0^t \rho((\bar V-\bar v)\cdot\nabla U)\cdot\bar v - \rho(\bar v\cdot\nabla\bar V)\cdot(U-u).\end{aligned}$$

Then we get

$$\begin{aligned}\varepsilon I_1^{EuK} &= \varepsilon\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho} - \frac{\mu(r)}{r}\right)((\bar v - \bar V)\cdot \mathrm{div}({}^t\nabla U) + (u-U)\cdot\mathrm{div}(\nabla\bar V))\\ &\quad - \int_0^t\int_\Omega \rho(\bar v-\bar V)\cdot\nabla U(\bar v - \bar V) + \int_0^t\int_\Omega \rho(\bar v-\bar V)\cdot\nabla\bar V(u-U).\end{aligned}$$

Let us now look at I_2^{EuK}. We have

$$2I_2^{EuK} = \int_0^t\int_\Omega \frac{\rho}{r}(U-u)\cdot\nabla(\lambda(r)\,\mathrm{div}\,\bar V) - \int_0^t\int_\Omega \frac{\rho}{r}(\bar V-\bar v)\cdot\nabla(\lambda(r)\,\mathrm{div}\,U)$$

$$+\int_0^t \Big\langle \frac{\lambda(\rho)}{\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(\bar{v}))\mathrm{Id}; \nabla U \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)}$$
$$-+\int_0^t \Big\langle \frac{\lambda(\rho)}{\mu(\rho)} \mathrm{Tr}(\mathbb{T}^{EuK}(u))\mathrm{Id}; \nabla \bar{V} \Big\rangle_{W^{-1,1}(\Omega)\times W^{1,\infty}(\Omega)},$$

and therefore, recalling that $\lambda'(\rho) = 2\rho\mu''(\rho)$ and playing as for I_1^{EuK}, we get

$$2I_2^{EuK} = -2\int_0^t \int_\Omega \rho \left(\mu''(\rho)\nabla\rho - \mu''(r)\nabla r\right) \cdot \left((\bar{v} - \bar{V}) \operatorname{div} U + (U - u) \operatorname{div} \bar{V}\right)$$
$$-\int_0^t \int_\Omega \left(\lambda(\rho) - \frac{\rho}{r}\lambda(r)\right) \left((\bar{v} - \bar{V}) \cdot \nabla(\operatorname{div} U) + (U - u) \cdot \nabla(\operatorname{div} \bar{V})\right),$$

and therefore because $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, we get

$$\begin{aligned}
\varepsilon(I_1^{EuK} + I_2^{EuK}) = &-\int_0^t \int_\Omega \rho(\bar{v} - \bar{V}) \cdot \nabla U(\bar{v} - \bar{V}) \\
&+\int_0^t \int_\Omega \rho(\bar{v} - \bar{V}) \cdot \nabla \bar{V}(u - U) \\
&-\varepsilon \int_0^t \int_\Omega \rho \left(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\right) \\
&\cdot \left((\bar{v} - \bar{V}) \operatorname{div} U + (U - u) \operatorname{div} \bar{V}\right) \\
&-\varepsilon \int_0^t \int_\Omega \rho(\mu'(\rho) - \mu'(r)) \left((\bar{v} - \bar{V})\right. \\
&\left.\cdot\nabla(\operatorname{div} U) + (U - u) \cdot \nabla(\operatorname{div} \bar{V})\right).
\end{aligned}$$

This concludes the proof. □

Theorem 7. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s + 2$ and $s \geq -1$. Let (ρ, u, v) be a global weak solution of the augmented system (31)–(34) and (r, U, V) be a strong solution of (44)–(46) in the sense of Proposition 6. We have*

$$\mathcal{E}_{EuK}(t) - \mathcal{E}_{EuK}(0) \leq C(r, U, V) \int_0^t \mathcal{E}_{EuK}(\xi) d\xi,$$

where $C(r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$.

Using Gronwall's Lemma, we directly obtain

Corollary 8. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s + 2$ and $s \geq -1$. Let (ρ, u, v) be a global weak solution of (31)–(34) and (r, U, V) a strong solution of (44)–(46) in the sense of proposition 6. Then*

$$\mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0) \exp(C\, t),$$

with $C = C(r, U, V)$ a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$. It the initial conditions coincide for (ρ, u, v) and (r, U, V) then $\rho = r$, $u = U$, $v = V$.

Note that theorem 7 is a direct consequence of proposition 6 and the following lemma:

Lemma 9. *We assume that* $\mu(\rho) = \rho^{(s+3)/2}$ *with* $s \geq -1$. *Let* (ρ, u, v) *be a global weak solution of* (31)–(34) *and* (r, U, V) *be a strong solution of* (44)–(46) *in the sense of Proposition* 6. *Then*

$$\varepsilon \left| \int_0^t \int_\Omega \rho \left(\mu''(\rho)\nabla\rho - \mu''(r)\nabla(r)\right) \cdot \left((\bar{v} - \bar{V}) \operatorname{div} U + (U - u) \operatorname{div} \bar{V}\right) \right|$$
$$\leq C \frac{s+1}{2} \int_0^t \int_\Omega \rho(|\bar{v} - \bar{V}|^2 + |u - U|^2),$$

and, if $\gamma \geq 2 + s$, *we have*

$$\left| \int_0^t \int_\Omega \rho(\mu'(\rho) - \mu'(r)) \left((\bar{v} - \bar{V}) \cdot \nabla(\operatorname{div} U) + (U - u) \cdot \nabla(\operatorname{div} \bar{V})\right) \right|$$
$$\leq C \int_0^t \int_\Omega \left(H(\rho|r) + \rho(|\bar{v} - \bar{V}|^2 + |u - U|^2)\right),$$

where $C = C(r, U, V)$ *is a uniformly bounded constant on* $\mathbb{R}^+ \times \Omega$.

Proof. As $(r, U, \bar{V})$ is a strong solution of (44)–(46) then we can prove that $\bar{V} = \varepsilon\nabla(\mu(r))/r$. Since $\mu(\rho) = \rho^{(s+3)/2}$ and $\bar{v} = \varepsilon\nabla(\mu(\rho))/\rho$, we have

$$\varepsilon(\mu''(\rho)\nabla\rho - \mu''(r)\nabla r) = \frac{s+1}{2}(\bar{v} - \bar{V}),$$

which gives the first part of the lemma using Young's inequality. For the second one, using Young's inequality, we have

$$\int_0^t \int_\Omega \left|\rho(\mu'(\rho) - \mu'(r)) \left((\bar{v} - \bar{V}) \cdot \nabla(\operatorname{div} U) + (U - u) \cdot \nabla(\operatorname{div} \bar{V})\right)\right|$$
$$\leq C \left(\frac{1}{2} \int_0^t \int_\Omega \rho|\mu'(\rho) - \mu'(r)|^2 + \int_0^t \int_\Omega \rho|\bar{v} - \bar{V}|^2 + \int_0^t \int_\Omega \rho|u - U|^2 \right),$$

with $C = C(U, V)$ a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$. Using Lemma 35 in the first integral, we obtain the result. □

Let us now give a weak–strong uniqueness result based on solutions that have been already constructed in [3–18] and [6].

Theorem 10. *Let* $(r_0, u_0) \in H^{s+1}(\Omega) \times H^s(\Omega)$ *with* $s > 2 + d/2$ *with* $r_0 > 0$ *such that* $\operatorname{curl}(r_0 u_0) = 0$. *Let* (ρ, u) *be a global weak solution in* $(0, T) \times \Omega$ *of the Quantum-Euler system*

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \tag{48}$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \varepsilon^2 \operatorname{div}(\nabla\nabla\rho - \rho\nabla\log\rho \otimes \nabla\log\rho) \tag{49}$$

corresponding to the initial data $(r_0, r_0 u_0)$ *and let* (r, U) *a local strong solution in* $(0, T^*) \times \Omega$ *of this system for the same initial data with*

$$0 < c \leq r \leq c^{-1} < +\infty,$$

where c is a constant and

$$r \in \mathcal{C}([0,T] \times H^{s+1}(\Omega)) \times \mathcal{C}^1([0,T] \times H^{s-1}(\Omega))$$
$$U \in \mathcal{C}([0,T] \times H^{s}(\Omega)) \times \mathcal{C}^1([0,T] \times H^{s-2}(\Omega)),$$

then $\rho = r$, $u = U$ *and* $w = W$ *on* $(0, \min\{T, T^*\}) \times \Omega$.

Proof. Let us first remark that such existence of local strong solution has been proved for instance in [6] in the whole space without the constraint on $\mathrm{curl}(\rho_0 u_0) = 0$ but may be considered in the periodic case. The global existence of weak solution for the Quantum–Euler System with the constraint $\mathrm{curl}(\rho_0 u_0) = 0$ has been obtained in two papers, namely [3] and [18]. For a strong solution, it is not difficult to prove that it also satisfies the augmented system. Concerning the global weak solution, it suffices to recall the important remark given in the introduction. Differentiating in space the mass equation in $\mathcal{D}'((0,T) \times \Omega)$, we get

$$\partial_t \nabla \rho + \nabla \,\mathrm{div}(\rho u) = \partial_t \nabla \rho + \mathrm{div}({}^t\nabla(\rho u)) = \varepsilon^2 \,\mathrm{div}[\Delta \rho - \nabla\sqrt{\rho} \otimes \sqrt{\rho}],$$

which may be written

$$\partial_t \nabla \rho + \mathrm{div}(\rho \nabla \log \rho \otimes u) + \mathrm{div}\big({}^t\nabla(\rho u) - \rho \nabla \log \rho \otimes u\big) = 0,$$

and therefore

$$\partial_t \nabla \rho + \mathrm{div}(\rho \nabla \log \rho \otimes u) + \mathrm{div}\big(\mathbb{T}(u)^t\big) = 0.$$

Using the definition $\rho \bar{v} = \varepsilon \nabla \rho$, we can rewrite the Quantum-Euler system and the previous relation in its augmented form:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \tag{50}$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = \varepsilon \,\mathrm{div}\ \mathbb{T}(\bar{v}) \tag{51}$$
$$\partial_t \rho \bar{v} + \mathrm{div}(\rho \bar{v} \otimes u) + \varepsilon \,\mathrm{div}\big(\mathbb{T}(u)^t\big) = 0, \tag{52}$$

which is the augmented version of the Quantum-Euler equations. Thus a global weak solution of the Quantum-Euler system is a global weak solution of the augmented Quantum-Euler system and therefore the weak–strong uniqueness corollary 8 may be applied due to the regularity of the strong solution.

3.2. Dissipative Solutions and Weak–Strong Uniqueness Result

In this subsection, we give the definition of what we call a dissipative solution for the Euler–Korteweg System. We recall that $\mathcal{E}_{EuK}(t)$ stands for

$$\mathcal{E}_{EuK}(t) = \mathcal{E}_{EuK}(\rho, u, v | r, U, V)(t),$$

defined in (42). Let U be a smooth function, then we solve the transport equation for r for the initial data r_0 such that $0 < r_0 < +\infty$. We then define the function $\mathscr{E}$ as

$$\mathscr{E}(r, U) = r\,(\partial_t U + U\cdot\nabla U) + \nabla p(r) - \varepsilon^2\,\mathrm{div}(\mu(r)\,{}^t\nabla V) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,V), \tag{53}$$

with $r\,V = \nabla(\mu(r))$. Then we can prove, differentiating (44), that

$$0 = r\,(\partial_t V + U\cdot\nabla V) + \mathrm{div}(\mu(r)\,{}^t\nabla U) + \frac{1}{2}\nabla(\lambda(r)\,\mathrm{div}\,U). \tag{54}$$

Definition 11. Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ (*i.e.* $K(\rho) = \frac{(s+3)^2}{4}\rho^s$) with $\gamma \geq s+2$ and $s \geq -1$. Let ρ_0 and u_0 smooth enough. The triplet (ρ, u, v) is a dissipative solution of the Euler–Korteweg System corresponding to the initial conditions

$$\rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = \rho_0 u_0, \qquad \rho v|_{t=0} = \sqrt{\rho_0 K(\rho_0)}\nabla\rho_0,$$

if the triplet (ρ, u, v) satisfies

$$\mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0)\exp(C\,t) + b_{EuK}(t) + C\int_0^t b_{EuK}(\xi)\exp(C\,(t-\xi))\,d\xi$$

with $C = C(\varepsilon^2, r, U, V)$ a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$, and where

$$b_{EuK}(t) = \int_0^t \int_\Omega \frac{\rho}{r}|\mathscr{E}\cdot(U-u)|$$

for all strong enough U test functions and with $(r, \mathscr{E})$ given respectively through (44) and (53) and the identity (54):

As a direct consequence, we can establish the following weak–strong uniqueness property (see [26]).

Theorem 12. *Let us assume* $\mu(\rho) = \rho^{(s+3)/2}$ *(i.e.* $K(\rho) = \dfrac{(s+3)^2}{4}\rho^s$*) with* $\gamma \geq s+2$ *and* $s \geq -1$*. Let us consider a dissipative solution* (ρ, u, v) *to the Euler–Korteweg system satisfying the initial conditions*

$$\rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = \rho_0 u_0, \qquad \rho v|_{t=0} = \sqrt{\rho_0 K(\rho_0)}\nabla\rho_0.$$

Let us assume that (r, U) *is a strong solution of* (44) *and*

$$r\,(\partial_t U + \nabla U\,U) + \nabla p(r) - \varepsilon^2 r\nabla\left(K(r)\Delta r + \frac{1}{2}K'(r)|\nabla r|^2\right) = 0, \tag{55}$$

with the regularity given in proposition 6 where we denote $\bar{V} = \varepsilon\nabla(\mu(r))/r$ *and with* $(\rho_0, u_0) \in W^{2,\infty}(\Omega)\times W^{1,\infty}(\Omega)$*. If* $r|_{t=0} = \rho_0$*,* $U|_{t=0} = u_0$ *then* $\rho = r$*,* $u = U$ *and* $v = V$*, which means that the problem satisfies a dissipative-strong uniqueness property.*

Proof. If (r, U) is a strong solution of (44), (55) then $\mathscr{E} = 0$ and $b_{EuK}(t) = 0$. We have

$$0 \leq \mathcal{E}_{EuK}(t) \leq \mathcal{E}_{EuK}(0)\exp(C\,t). \tag{56}$$

If $r(t=0) = \rho_0$, $U(t=0) = u_0$ then $v(t=0) = V(t=0)$ and $\mathcal{E}_{EuK}(0) = 0$, then this leads to $\rho = r$, $u = U$, $v = V$ using (56). □

Note that, as already mentioned, all the results and definitions of this section are still valid for the compressible quantum Euler System. Indeed this corresponds to the special case $K(\rho) = 1/\rho$ in the Euler–Korteweg System for which the assumption $2 + s \leq \gamma$ is satisfied since $s = -1$ and $\gamma > 1$. In particular we have the following definition of what we call a dissipative solution of the quantum Euler system this one will be used in Sect. 5:

Definition 13. Let ρ_0 and u_0 smooth enough. The triplet (ρ, u, v) is a dissipative solution of the quantum Euler system (14)–(15) corresponding to the initial conditions

$$\rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = \rho_0 u_0, \qquad \rho v|_{t=0} = \rho_0 \nabla \log \rho_0,$$

if the triplet (ρ, u, v) satisfies

$$\mathcal{E}_{EuQ}(t) \leq \mathcal{E}_{EuQ}(0) \exp(C\,t) + b_{EuQ}(t) + C \int_0^t b_{EuQ}(\xi) \exp(C\,(t-\xi))\, d\xi,$$

where $\mathcal{E}_{EuQ}(0) = \mathcal{E}_{EuQ}|_{t=0}$ and with a constant $C = C(\varepsilon^2, r, U, V)$ uniformly bounded on $\mathbb{R}^+ \times \Omega$, and

$$\begin{aligned} \mathcal{E}_{EuQ}(t) &= \mathcal{E}_{EuK}(t), \text{ for } K(\rho) = 1/\rho, \\ b_{EuQ}(t) &= \int_0^t \int_\Omega \frac{\rho}{r} |\mathscr{E} \cdot (U - u)| \end{aligned}$$

for all smooth U and $(r, V, \mathscr{E})$ defined, respectively, through (44) and

$$V = \nabla \log r, \tag{57}$$

$$\mathscr{E}(r, U) = (\partial_t U + U \cdot \nabla U) + \nabla p(r) - \varepsilon^2 \operatorname{div}(r \nabla V). \tag{58}$$

Remark. Note that, in the definition above, since U is regular and also r, we have V which satisfies

$$r\,(\partial_t V + U \cdot \nabla V) + \operatorname{div}(r\, {}^t\nabla U) = 0. \tag{59}$$

4. The Navier–Stokes–Korteweg System: Relative Entropy and Dissipative Solution

The goal of this section is to define what we call a dissipative solution for the Navier–Stokes–Korteweg System. To this end, we consider the augmented System (20)–(23) and we establish a relative entropy estimate. Here the viscous term adds some difficulties compared to the case of the Euler–Korteweg system.

4.1. Relative Entropy Inequality

In this section, we establish a relative entropy inequality for a weak solution $(\rho, \bar{v}, w)$ of the augmented System (37)–(39). This will then be used to give the definition of what is called a dissipative solution for the Navier–Stokes–Korteweg system. We define the following relative entropy functional:

$$\begin{aligned}
\mathcal{E}_{NSK}(t) &= \mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W) \\
&= \frac{1}{2}\int_\Omega \rho \left(|\bar{v} - \bar{V}|^2 + |w - W|^2\right) + \int_\Omega H(\rho|r) \\
&\quad + \nu \int_0^t \int_\Omega \mu(\rho) \left(|\frac{\mathbb{T}(\bar{v})}{\sqrt{\mu(\rho)}} - \nabla\bar{V}|^2 + |\frac{\mathbb{T}(\bar{w})}{\sqrt{\mu(\rho)}} - \nabla W|^2\right) \\
&\quad + \frac{\nu}{2}\int_0^t \int_\Omega \lambda(\rho) \left((\frac{\mathrm{Tr}\,\mathbb{T}(\bar{v})}{\sqrt{\mu(\rho)}} - \mathrm{div}\,\bar{V})^2 + (\frac{\mathrm{Tr}\,\mathbb{T}(w)}{\sqrt{\mu(\rho)}} - \mathrm{div}\,W)^2\right).
\end{aligned}$$

Proposition 14. *Any global weak solution $(\rho, \bar{v}, w)$ of the augmented system (37)–(40) satisfies the following inequality for all $t \in [0, T]$ and for any test functions*

$$\begin{aligned}
& r \in C^1([0,T] \times \bar{\Omega}), \quad r > 0, \quad \bar{V},\ W \in C^2([0,T] \times \bar{\Omega}), \\
& \mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W)(t) \\
& \quad \le \mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W)(0) \\
& \qquad + \int_0^t \int_\Omega \rho \left(\partial_t \bar{V} \cdot (\bar{V} - \bar{v}) + (\nabla\bar{V}\, u) \cdot (\bar{V} - \bar{v})\right) \\
& \qquad + \int_0^t \int_\Omega \rho\, (\partial_t W \cdot (W - w) + (\nabla W\, u) \cdot (W - w)) \\
& \qquad + \nu \int_0^t \int_\Omega \mu(\rho) \left(|\nabla\bar{V}|^2 + |\nabla W|^2\right) - \sqrt{\mu(\rho)} \left(\mathbb{T}(\bar{v}) : \nabla\bar{V} + \mathbb{T}(w) : \nabla W\right) \\
& \qquad + \sqrt{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \sqrt{\mu(\rho)} \left(\mathbb{T}(\bar{v}) : \nabla W - (\mathbb{T}(\bar{w}))^t : \nabla\bar{V}\right) \\
& \qquad + \sqrt{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \frac{\lambda(\rho)}{2\sqrt{\mu(\rho)}} \left(\mathrm{Tr}(\mathbb{T}(\bar{v})\, \mathrm{div}\, W - \mathrm{Tr}(\mathbb{T}(w))\, \mathrm{div}\, \bar{V}\right) \\
& \qquad + \frac{\nu}{2}\int_0^t \int_\Omega \lambda(\rho) \left((\mathrm{div}\,\bar{V})^2 + (\mathrm{div}\, W)^2\right) \\
& \qquad - \frac{\nu}{2}\int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} \left(\mathrm{Tr}(\mathbb{T}(\bar{v})\, \mathrm{div}\,\bar{V} + \mathrm{Tr}(\mathbb{T}(w))\, \mathrm{div}\, W\right) \\
& \qquad - \int_0^t \int_\Omega \left(\partial_t(H'(r))(\rho - r) + \rho\, \nabla(H'(r)) \cdot u + p(\rho)\, \mathrm{div}\, W\right) \\
& \qquad - \nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla\rho|^2.
\end{aligned} \tag{60}$$

Remark. Note that each of these quantities are defined in the usual sense for weak solution $(\rho, \bar{v}, w)$ and regular test functions $(r, \bar{V}, W)$ as chosen in the proposition above. The main difference compared to the Euler–Korteweg system is that here we control $\sqrt{\nu}\,\mathbb{T}(\bar{v})$ and $\sqrt{\nu}\,\mathbb{T}(w)$ in $L^2(0, T; L^2(\Omega))$ and

$\sqrt{\mu(\rho)} \in L^\infty(0,T;L^2(\Omega))$ to define in the usual way the first order derivative quantities.

Proof. Thanks to (36) , we have

$$
\begin{aligned}
\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq & \int_\Omega \Big(\frac{\rho}{2}|\bar V|^2 - \rho \bar v \cdot \bar V + \frac{\rho}{2}|W|^2 - \rho w \cdot W\Big)(t) \\
& - \int_\Omega \Big(\frac{\rho}{2}|\bar V|^2 - \rho \bar v \cdot \bar V + \frac{\rho}{2}|W|^2 - \rho w \cdot W\Big)(0) \\
& - \int_\Omega \big(H(r) + H'(r)(\rho - r)\big)(t) \\
& + \int_\Omega \big(H(r) + H'(r)(\rho - r)\big)(0) \\
& -\nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla \rho|^2 + \nu \Big(I_1^{NS} + I_2^{NS}\Big),
\end{aligned}
\tag{61}
$$

where

$$
\begin{aligned}
2I_1^{NS} = & \int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} \Big(|\operatorname{div} \bar V|^2 + |\operatorname{div} W|^2\Big), \\
& -2\int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} \big(\mathrm{Tr}(\mathbb{T}(\bar v)) \operatorname{div} \bar V + \mathrm{Tr}(\mathbb{T}(w)) \operatorname{div} W\big),
\end{aligned}
\tag{62}
$$

$$
I_2^{NS} = \int_0^t \int_\Omega \mu(\rho) \Big(|\nabla \bar V|^2 + |\nabla W|^2\Big) \tag{63}
$$

$$
- 2\int_0^t \int_\Omega \sqrt{\mu(\rho)} \big(\mathbb{T}(\bar v) : \nabla \bar V + \mathbb{T}(w) : \nabla W\big). \tag{64}
$$

Using (38),

$$
\begin{aligned}
\partial_t(\rho\, w \cdot W) &= \partial_t(\rho\, w) \cdot W + \rho\, w \cdot \partial_t W \\
&= \Big\langle -\operatorname{div}(\rho\, w \otimes u) - \nabla p(\rho) + \nu \operatorname{div}(\sqrt{\mu(\rho)}\mathbb{T}(w)) \\
&\quad + \sqrt{\varepsilon^2 - \nu^2} \operatorname{div}(\sqrt{\mu(\rho)}\mathbb{T}(\bar v));\, W\Big\rangle_{W^{-2,1}(\Omega)\times W^{2,\infty}(\Omega)} \\
&\quad + \, < A_1; W >_{W^{-2,1}(\Omega)\times W^{2,\infty}(\Omega)} + \rho\, w \cdot \partial_t W,
\end{aligned}
$$

where

$$
A_1 = \frac{\nu}{2}\nabla\left(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}(w))\right) + \frac{\sqrt{\varepsilon^2-\nu^2}}{2}\nabla\left(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}(\bar v))\right).
$$

Using (39),

$$
\begin{aligned}
\partial_t(\rho\, \bar v \cdot \bar V) &= \partial_t(\rho\, \bar v) \cdot \bar V + \rho\, \bar v \cdot \partial_t \bar V \\
&= \Big\langle -\operatorname{div}(\rho\, \bar v \otimes u) + \nu \operatorname{div}(\sqrt{\mu(\rho)}\mathbb{T}(\bar v))
\end{aligned}
$$

$$-\sqrt{\varepsilon^2-\nu^2}\,\mathrm{div}(\sqrt{\mu(\rho)}(\mathbb{T}(w))^t);\,\bar V\Big\rangle_{W^{-2,1}(\Omega)\times W^{2,\infty}(\Omega)}$$
$$+<A_2;\bar V>_{W^{-2,1}(\Omega)\times W^{2,\infty}(\Omega)}+\rho\,\bar v\cdot\partial_t\bar V, \tag{65}$$

where

$$A_2=\frac{\nu}{2}\nabla\left(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}(\bar v))\right)-\frac{1}{2}\sqrt{\varepsilon^2-\nu^2}\,\nabla\left(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}(w))\right).$$

Then, Using (37),

$$\begin{aligned}\int_0^t\int_\Omega\partial_t(H(r)+H'(r)(\rho-r))&=\int_0^t\int_\Omega\big(H'(r)\,\partial_t r+\partial_t(H'(r))(\rho-r)\\&\quad+H'(r)\,\partial_t\rho-H'(r)\,\partial_t r\big)\\&=\int_0^t\int_\Omega\big(\partial_t(H'(r))(\rho-r)-H'(r)\mathrm{div}\,(\rho\,u)\big)\\&=\int_0^t\int_\Omega\big(\partial_t(H'(r))(\rho-r)+\rho\nabla(H'(r))\cdot u\big).\end{aligned}$$

Since

$$\partial_t\left(\frac{\rho}{2}|\bar V|^2\right)=\frac{1}{2}\partial_t\rho|\bar V|^2+\rho\bar V\cdot\partial_t\bar V,\quad\partial_t\left(\frac{\rho}{2}|W|^2\right)=\frac{1}{2}\partial_t\rho|W|^2+\rho W\cdot\partial_t W,$$

and since $\nabla\bar v,\nabla\bar V$ are symmetric matrices (recall that v and V are gradient of functions), thanks to (37) and integrations by parts we obtain

$$\begin{aligned}\mathcal{E}_{NSK}(t)-\mathcal{E}_{NSK}(0)\le&\int_0^t\int_\Omega\rho\,\partial_t\bar V\cdot(\bar V-\bar v)+\int_0^t\int_\Omega\rho\,(\nabla\bar V\,u)\cdot(\bar V-\bar v)\\&+\int_0^t\int_\Omega\rho\,\partial_t W\cdot(W-w)+\int_0^t\int_\Omega\rho\,(\nabla W\,u)\cdot(W-w)\\&+\nu\int_0^t\int_\Omega\sqrt{\mu(\rho)}\,\big(\mathbb{T}(\bar v):\nabla\bar V+\mathbb{T}(w):\nabla W\big)\\&+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega\sqrt{\mu(\rho)}\,\big(\mathbb{T}(\bar v):\nabla W-(\mathbb{T}(w))^t:\nabla\bar V\big)\\&-\int_0^t\int_\Omega p(\rho)\,\mathrm{div}W-\nu\int_0^t\int_\Omega\mu'(\rho)H''(\rho)|\nabla\rho|^2\\&-\int_0^t\int_\Omega\partial_t(H'(r))(\rho-r)-\int_0^t\int_\Omega\rho\nabla(H'(r))\cdot u\\&+\int_0^t\int_\Omega(\nu\,A_3-\sqrt{\varepsilon^2-\nu^2}\,A_4)+\nu\left(I_1^{NS}+I_2^{NS}\right),\end{aligned}$$

where

$$2\,A_3=\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\,\big(\mathrm{Tr}(\mathbb{T}(\bar v))\,\mathrm{div}\,\bar V+\mathrm{Tr}(\mathbb{T}(w))\,\mathrm{div}\,W\big),$$
$$2\,A_4=\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\,\big(\mathrm{Tr}(\mathbb{T}(w))\,\mathrm{div}\,\bar V-\mathrm{Tr}(\mathbb{T}(\bar v))\,\mathrm{div}\,W\big)$$

and I_1^{NS} and I_2^{NS} are given through (62)–(63), which gives the proposition. □

Let us introduce that there exists a strong solution of

$$\partial_t r + \operatorname{div}(r\, U) = 0, \tag{66}$$

$$r\,(\partial_t W + \nabla W\, U) + \nabla p(r) - \nu\, \operatorname{div}(\mu(r)\,\nabla W) - \frac{\nu}{2}\nabla(\lambda(r)\operatorname{div} W) \\ = \sqrt{\varepsilon^2 - \nu^2}\,\left(\operatorname{div}(\mu(r)\,\nabla \bar V) + \frac{1}{2}\nabla(\lambda(r)\operatorname{div}\bar V)\right), \tag{67}$$

$$r\,\left(\partial_t \bar V + \nabla \bar V\, U\right) - \nu\, \operatorname{div}(\mu(r)\,\nabla \bar V) - \frac{\nu}{2}\nabla(\lambda(r)\operatorname{div}\bar V) \\ + \sqrt{\varepsilon^2 - \nu^2}\,\left(\operatorname{div}(\mu(r)\,{}^t\nabla W) + \frac{1}{2}\nabla(\lambda(r)\operatorname{div} W)\right) = 0, \tag{68}$$

with

$$U = W - \nu\, V, \quad \bar V = \sqrt{\varepsilon^2 - \nu^2}\, V$$

belonging to the class

$$\begin{aligned} & 0 < \inf_{(0,T)\times\Omega} r \le r \le \sup_{(0,T)\times\Omega} r < +\infty \\ & \nabla r \in L^2(0,T; L^\infty(\Omega) \cap L^1(0,T; W^{1,\infty}(\Omega)) \\ & W \in L^\infty(0,T; W^{2,\infty}(\Omega)) \cap W^{1,\infty}(0,T; L^\infty(\Omega)), \\ & \bar V \in L^\infty(0,T; W^{2,\infty}(\Omega)) \cap W^{1,\infty}(0,T; L^\infty(\Omega)), \\ & \partial_t H'(r) \in L^1(0,T; L^{\gamma/(\gamma-1)}(\Omega)), \qquad \nabla H'(r) \in L^1(0,T; L^{2\gamma/(\gamma-1)}(\Omega)), \end{aligned} \tag{69}$$

and where $\bar V|_{t=0} = \sqrt{\varepsilon^2 - \nu^2}\nabla\mu(r_0)/r_0$. Defining

$$\begin{aligned} I_3^{NS} = & \int_0^t \int_\Omega \lambda(\rho)\left((\operatorname{div}\bar V)^2 + (\operatorname{div} W)^2\right) \\ & - \int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\left(\operatorname{Tr}(\mathbb{T}(\bar v))\operatorname{div}\bar V + \operatorname{Tr}(\mathbb{T}(w))\operatorname{div} W\right) \\ & + \int_0^t \int_\Omega \frac{\rho}{r}\Big(\nabla(\lambda(r)\operatorname{div}\bar V)\cdot(\bar V - \bar v) + \nabla(\lambda(r)\operatorname{div} W)\cdot(W - w)\Big), \end{aligned} \tag{70}$$

$$\begin{aligned} I_4^{NS} = & \int_0^t \int_\Omega \mu(\rho)\left(|\nabla \bar V|^2 + |\nabla W|^2\right) \\ & - \int_0^t \int_\Omega \sqrt{\mu(\rho)}\left(\mathbb{T}(\bar v) : \nabla \bar V + \nabla W : \mathbb{T}(w)\right) \\ & + \int_0^t \int_\Omega \frac{\rho}{r}\Big(\operatorname{div}\left(\mu(r)\nabla \bar V\right)\cdot(\bar V - \bar v) + \operatorname{div}\left(\mu(r)\nabla W\right)\cdot(W - w)\Big), \end{aligned} \tag{71}$$

$$\begin{aligned} I_5^{NS} = & \int_0^t \int_\Omega \frac{\rho}{r}\Big(\operatorname{div}\left(\mu(r)\nabla \bar V\right)\cdot(W - w) - \operatorname{div}\left(\mu(r)^t\nabla W\right)\cdot(\bar V - \bar v)\Big) \\ & + \int_0^t \int_\Omega \sqrt{\mu(\rho)}\left(\mathbb{T}(\bar v) : \nabla W - (\mathbb{T}(w))^t : \nabla \bar V\right), \end{aligned} \tag{72}$$

$$2\, I_6^{NS} = \int_0^t \int_\Omega \frac{\rho}{r} \left(\nabla(\lambda(r) \operatorname{div} \bar{V}) \cdot (W - w) - \nabla(\lambda(r) \operatorname{div} W) \cdot (\bar{V} - \bar{v})\right) \quad (73)$$
$$- \int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} \left(\operatorname{Tr}(\mathbb{T}(w)) \operatorname{div} \bar{V} - \operatorname{Tr}(\mathbb{T}(\bar{v})) \operatorname{div} W\right),$$

we have

Proposition 15. *Let* $(r, \bar{V}, W)$ *be a strong solution of* (66)–(68) *belonging to the class* (69)*. Let us assume that* $\bar{V}_0 = \sqrt{\varepsilon^2 - \nu^2}\nabla\mu(r_0)/r_0$*. Any weak solution* $(\rho, \bar{v}, w)$ *of the augmented system* (37)–(40) *satisfies the inequality*

$$\begin{aligned}
\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq & \int_0^t \int_\Omega \rho \left[(\nabla \bar{V}\,(u - U)) \cdot (\bar{V} - \bar{v})\right.\\
&\left. + (\nabla W\,(u - U)) \cdot (W - w)\right]\\
& - \int_0^t \int_\Omega \left(p(\rho) - p(r) - p'(r)\,(\rho - r)\right) \operatorname{div} U\\
& + \frac{\nu}{\sqrt{\varepsilon^2 - \nu^2}} \int_0^t \int_\Omega \left(\rho\, \nabla(H'(r)) \cdot (\bar{v} - \bar{V}) - p(\rho) \operatorname{div} \bar{V}\right)\\
& - \nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla\rho|^2\\
& + \frac{\nu}{2} I_3^{NS} + \nu I_4^{NS} + \sqrt{\varepsilon^2 - \nu^2} \left(I_5^{NS} + I_6^{NS}\right),
\end{aligned}$$

where I_i^{NS} *for* $i = 3, 4, 5, 6$ *are given by* (70)–(73).

Proof. Multiplying (67) by $\frac{\rho}{r}(W - w)$ and (68) by $\frac{\rho}{r}(\bar{V} - \bar{v})$, integrating with respect to time and space, and using (66), we obtain

$$\begin{aligned}
\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq\ & \nu \int_0^t \int_\Omega \frac{\rho}{r} \left[\operatorname{div}(\mu(r)\nabla\bar{V}) \cdot (\bar{V} - \bar{v})\right.\\
& \left. + \frac{1}{2}\nabla(\lambda(r) \operatorname{div} \bar{V}) \cdot (\bar{V} - \bar{v})\right]\\
& - \sqrt{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \frac{\rho}{r} \left[\operatorname{div}(\mu(r)^t\nabla W) \cdot (\bar{V} - \bar{v})\right.\\
& \left. + \frac{1}{2}\nabla(\lambda(r) \operatorname{div} W) \cdot (\bar{V} - \bar{v})\right]\\
& + \int_0^t \int_\Omega \rho\, (\nabla \bar{V}\,(u - U)) \cdot (\bar{V} - \bar{v})\\
& + \int_0^t \int_\Omega \rho\, (\nabla W\,(u - U)) \cdot (W - w)\\
& + \nu \int_0^t \int_\Omega \frac{\rho}{r} \left[\operatorname{div}(\mu(r)\nabla W) \cdot (W - w)\right.\\
& \left. + \frac{1}{2}\, \nabla(\lambda(r) \operatorname{div} W) \cdot (W - w)\right]
\end{aligned}$$

$$+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \frac{\rho}{r}\Big[\operatorname{div}(\mu(r)\nabla\bar{V})\cdot(W-w)$$
$$+\frac{1}{2}\nabla(\lambda(r)\operatorname{div}\bar{V})\cdot(W-w)\Big]$$
$$+\nu\int_0^t\int_\Omega \mu(\rho)\,\Big(|\nabla\bar{V}|^2+|\nabla W|^2\Big)$$
$$-\nu\int_0^t\int_\Omega \sqrt{\mu(\rho)}\,\big(\mathbb{T}(\bar{v}):\nabla\bar{V}+\mathbb{T}(w):\nabla W\big)$$
$$+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \sqrt{\mu(\rho)}\,\big(\mathbb{T}(\bar{v}):\nabla W-(\mathbb{T}(w))^t:\nabla\bar{V}\big)$$
$$+\sqrt{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \frac{\lambda(\rho)}{2\sqrt{\mu(\rho)}}\,(\operatorname{Tr}(\mathbb{T}(\bar{v}))\operatorname{div} W$$
$$-\operatorname{Tr}(\mathbb{T}(w))\operatorname{div}\bar{V}\big)$$
$$-\frac{\nu}{2}\int_0^t\int_\Omega \lambda(\rho)\,\Big((\operatorname{div}\bar{V})^2+(\operatorname{div} W)^2\Big)$$
$$+\frac{\nu}{2}\int_0^t\int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}}\,\big(\operatorname{Tr}(\mathbb{T}(\bar{v}))\operatorname{div}\bar{V}+\operatorname{Tr}(\mathbb{T}(w)\operatorname{div} W\big)$$
$$+\int_0^t\int_\Omega (p(r)\operatorname{div} U-p(\rho)\operatorname{div} W)$$
$$-\nu\int_0^t\int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2+I_7^{NS},$$

where

$$I_7^{NS}=-\int_0^t\int_\Omega \frac{\rho}{r}\nabla(p(r))\cdot(W-w)-\int_0^t\int_\Omega \partial_t(H'(r))(\rho-r) \tag{74}$$
$$-\int_0^t\int_\Omega \rho\nabla(H'(r))\cdot u+\int_0^t\int_\Omega H'(r)\,\partial_t r.$$

Using (66) $H'(r)\partial_t r+H'(r)\operatorname{div}(r\,U)=0$ which leads, with an integration by parts, to

$$\int_0^t\int_\Omega (H'(r)\partial_t r-r\nabla(H'(r))\cdot U)=0.$$

Then

$$\int_0^t\int_\Omega (H'(r)\partial_t r-\nabla p(r)\cdot U)=0,$$

or

$$\int_0^t\int_\Omega (H'(r)\partial_t r+p(r)\operatorname{div}(U))=0. \tag{75}$$

Moreover,

$$\partial_t(H'(r))=-p'(r)\operatorname{div}U-H''(r)\nabla r\cdot U=-p'(r)\operatorname{div}U-\nabla(H'(r))\cdot U.$$

Then

$$I_7^{NS} = \int_0^t \int_\Omega \rho\, \nabla(H'(r)) \cdot (-W + w + U - u) + \int_0^t \int_\Omega p'(r) \mathrm{div} U(\rho - r)$$
$$= \nu \int_0^t \int_\Omega \rho \nabla(H'(r)) \cdot (v - V) + \int_0^t \int_\Omega p'(r) \mathrm{div} U(\rho - r).$$

Therefore

$$\begin{aligned}\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq & \int_0^t \int_\Omega \rho \left[(\nabla \bar{V}\,(u - U)) \cdot (\bar{V} - \bar{v})\right.\\ & \left.+(\nabla W\,(u - U)) \cdot (W - w)\right]\\ & + \int_0^t \int_\Omega [p(r)\,\mathrm{div} U - p(\rho)\,\mathrm{div} U - \nu\, p(\rho)\,\mathrm{div} V]\\ & -\nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla \rho|^2\\ & +\frac{\nu}{2} \int_0^t \int_\Omega \lambda(\rho) \left((\mathrm{div}\,\bar{V})^2 + (\mathrm{div}\,W)^2\right)\\ & -\frac{\nu}{2} \int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} \left(\mathrm{Tr}(\mathbb{T}(\bar{v}))\,\mathrm{div}\,\bar{V} + \mathrm{Tr}(\mathbb{T}(w))\,\mathrm{div}\,W\right)\\ & +I_7^{NS} + \nu\, I_4^{NS} + \nu I_8^{NS} + \sqrt{\varepsilon^2 - \nu^2}\,(I_5^{NS} + I_6^{NS}),\end{aligned}$$

where

$$I_8^{NS} = \frac{1}{2} \int_0^t \int_\Omega \frac{\rho}{r} \left(\nabla(\lambda(r)\,\mathrm{div}\,\bar{V}) \cdot (\bar{V} - \bar{v}) + \nabla(\lambda(r)\,\mathrm{div}\,W) \cdot (W - w)\right) \tag{76}$$

and I_i^{NS} for $i = 4, 5, 6, 7$ are given by (71)–(74). Finally,

$$\begin{aligned}\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq & \int_0^t \int_\Omega \left[\rho\,(\nabla \bar{V}\,(u - U)) \cdot (\bar{V} - \bar{v})\right.\\ & \left.+\rho\,(\nabla W\,(u - U)) \cdot (W - w)\right]\\ & + \int_0^t \int_\Omega \left[(p'(r)(\rho - r) - p(\rho) + p(r))\,\mathrm{div}\,U\right.\\ & \left.-\nu\, p(\rho)\,\mathrm{div}\,V\right]\\ & -\frac{\nu}{\sqrt{\varepsilon^2 - \nu^2}} \int_0^t \int_\Omega \rho \nabla(H'(r)) \cdot (\bar{V} - \bar{v})\\ & -\nu \int_0^t \int_\Omega \mu'(\rho) H''(\rho) |\nabla \rho|^2\\ & +\frac{\nu}{2} \int_0^t \int_\Omega \lambda(\rho) \left((\mathrm{div}\,\bar{V})^2 + (\mathrm{div}\,W)^2\right)\\ & -\frac{\nu}{2} \int_0^t \int_\Omega \frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} \left(\mathrm{Tr}(\mathbb{T}(\bar{v}))\,\mathrm{div}\,\bar{V} + \mathrm{Tr}(\mathbb{T}(w))\,\mathrm{div}\,W\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu}{2}\int_0^t\int_\Omega \frac{\rho}{r}\left(\nabla(\lambda(r)\operatorname{div}\bar V)\cdot(\bar V-\bar v)\right.\\
&+\nabla(\lambda(r)\operatorname{div}W)\cdot(W-w))\\
&+\nu\int_0^t\int_\Omega \mu(\rho)\left(|\nabla\bar V|^2+|\nabla W|^2\right)\\
&-\nu\int_0^t\int_\Omega \sqrt{\mu(\rho)}\left(\mathbb{T}(\bar v):\nabla\bar V+\nabla W:\mathbb{T}(w)\right)\\
&+\nu\int_0^t\int_\Omega \frac{\rho}{r}\Big(\operatorname{div}\left(\mu(r)\nabla\bar V\right)\cdot(\bar V-\bar v)\\
&+\operatorname{div}\left(\mu(r)\nabla W\right)\cdot(W-w)\Big)\\
&+\sqrt{\varepsilon^2-\nu^2}\left(I_5^{NS}+I_6^{NS}\right),
\end{aligned}
$$

with I_5^{NS} and I_6^{NS} given by (72) and (73). This gives the proposition. □

Lemma 16. *Let I_5^{NS} given by (72) and I_6^{NS} given by (73). Under the assumptions of Proposition 15, we have*

$$
\begin{aligned}
I_5^{NS} = &-\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)\left(\operatorname{div}(\nabla\bar V)\cdot(W-w)+\operatorname{div}({}^t\nabla W)\cdot(\bar v-\bar V)\right)\\
&-\int_0^t\int_\Omega \rho\left(\nabla\bar V(W-w)+\nabla W(\bar v-\bar V)\right)\cdot(v-V),
\end{aligned}
$$

and

$$
\begin{aligned}
2\,I_6^{NS} = -2&\int_0^t\int_\Omega \rho\left(\mu''(\rho)\nabla\rho-\mu''(r)\nabla r\right)\cdot\left((W-w)\operatorname{div}\bar V+(\bar v-\bar V)\operatorname{div}W\right)\\
&-\int_0^t\int_\Omega \left(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\right)\left((W-w)\cdot\nabla(\operatorname{div}\bar V)+(\bar v-\bar V)\cdot\nabla(\operatorname{div}W)\right).
\end{aligned}
$$

Proof. The proof follows the same lines that the ones for (47) in the Euler–Korteweg section. □

Lemma 17. *Let I_5^{NS} given by (72) and I_6^{NS} given by (73). Let us assume $\mu(\rho)=\rho^{(s+3)/2}$ with $\gamma\geq s+2$ and $s\geq -1$. Under the assumptions of Proposition 15 we have*

$$
\left|I_5^{NS}+I_6^{NS}\right|\leq C\int_0^t\int_\Omega\left(\rho|w-W|^2+\rho|v-V|^2+\rho|\bar v-\bar V|^2+H(\rho|r)\right),
$$

where $C=C(r,\bar V,W)$ is a uniformly bounded constant on $\mathbb{R}^+\times\Omega$.

Proof. By definition of $\lambda(\rho)$, lemma 16 directly leads to

$$
I_5^{NS}+I_6^{NS}=-\int_0^t\int_\Omega \rho\,(\nabla\bar V(W-w))\cdot(v-V)
$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega \rho\,(\nabla W(\bar V-\bar v))\cdot(v-V)\\
&-\int_0^t\int_\Omega \rho\left(\mu''(\rho)\nabla\rho-\mu''(r)\nabla(r)\right)\\
&\cdot\left((W-w)\ \mathrm{div}\,\bar V+(\bar v-\bar V)\ \mathrm{div}\,W\right)\\
&-\int_0^t\int_\Omega \rho(\mu'(\rho)-\mu'(r))\,((W-w)\\
&\cdot\nabla(\mathrm{div}\,\bar V)+(\bar v-\bar V)\cdot\nabla(\mathrm{div}\,W)\big).
\end{aligned}
$$

Moreover, more of in an analogous way than for Lemma 9, we can show that

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega \rho\left(\mu''(\rho)\nabla\rho-\mu''(r)\nabla(r)\right)\cdot\left((W-w)\ \mathrm{div}\,\bar V+(\bar v-\bar V)\ \mathrm{div}\,W\right)\right|\\
&\quad\le C\int_0^t\int_\Omega \rho\left(|W-w|^2+|\bar v-\bar V|^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega \rho(\mu'(\rho)-\mu'(r))\left((W-w)\cdot\nabla(\mathrm{div}\,\bar V)+(\bar v-\bar V)\cdot\nabla(\mathrm{div}\,W)\right)\right|\\
&\quad\le C\int_0^t\int_\Omega \left(H(\rho|r)+\rho(|w-W|^2+|\bar v-\bar V|^2)\right).
\end{aligned}
$$

□

Lemma 18. *Let I_3^{NS} given by (70) and I_4^{NS} given by (71). Under the assumptions of Proposition 15, we have*

$$
\begin{aligned}
I_3^{NS}=&-2\int_0^t\int_\Omega \rho(\mu''(\rho)\nabla\rho-\mu''(r)\nabla r)\cdot((\bar V-\bar v)\,\mathrm{div}\,\bar V+(W-w)\,\mathrm{div}\,W)\\
&-\int_0^t\int_\Omega\left(\lambda(\rho)-\frac{\rho}{r}\lambda(r)\right)(\nabla(\mathrm{div}\,\bar V)\cdot(\bar V-\bar v)+\nabla(\mathrm{div}\,W)\cdot(W-w)),
\end{aligned}
$$

and

$$
\begin{aligned}
I_4^{NS}=&-\int_0^t\int_\Omega \rho(v-V)\cdot\left(\nabla\bar V(\bar V-\bar v)+{}^t\nabla W(W-w)\right)\\
&-\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\mathrm{div}(\nabla\bar V)\cdot(\bar V-\bar v)+\mathrm{div}(\nabla W)\cdot(W-w)).
\end{aligned}
$$

Proof. The proof follows the same lines that the ones for (47) in the Euler–Korteweg section. □

Using the previous lemma and the symmetry of $\nabla\bar V$, we obtain the following lemma:

Lemma 19. *Let* I_3^{NS} *given by* (70) *and* I_4^{NS} *given by* (71). *We assume* $\mu(\rho) = \rho^{(s+3)/2}$ *with* $\gamma \geq s+2$ *and* $s \geq -1$. *Under the assumptions of Proposition* 15, *we have*

$$\left|\frac{1}{2}I_3^{NS} + I_4^{NS}\right| \leq C\int_0^t\int_\Omega \Big(\rho|v-V|^2 + \rho|\bar{v}-\bar{V}|^2 + \rho|w-W|^2 + H(\rho|r)\Big),$$

where $C = C(r, \bar{V}, W)$ *is a uniformly bounded constant on* $\mathbb{R}^+ \times \Omega$.

Proof. We have:

$$\begin{aligned}
\frac{1}{2}I_3^{NS} + I_4^{NS} = &-\int_0^t\int_\Omega \rho\,(v-V)\cdot\big(\nabla\bar{V}(\bar{V}-\bar{v}) + {}^t\nabla W(W-w)\big)\\
&-\int_0^t\int_\Omega \rho\,(\mu''(\rho)\nabla\rho - \mu''(r)\nabla r)\cdot((\bar{V}-\bar{v})\operatorname{div}\bar{V}\\
&+(W-w)\operatorname{div}W)\\
&-\int_0^t\int_\Omega \rho\,(\mu'(\rho)-\mu'(r))\big(\nabla(\operatorname{div}\bar{V})\cdot(\bar{V}-\bar{v})\\
&+\nabla(\operatorname{div}W)\cdot(W-w))\\
&-\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\operatorname{div}(\nabla W)-\nabla(\operatorname{div}W))\cdot(W-w).
\end{aligned}$$

In more of an analogous way than for the Lemma 9, we can show

$$\begin{aligned}
&\int_0^t\int_\Omega \rho\left(\frac{\mu(\rho)}{\rho}-\frac{\mu(r)}{r}\right)(\operatorname{div}(\nabla W)-\nabla(\operatorname{div}W))\cdot(W-w)\\
&\quad\leq C\int_0^t\int_\Omega\Big(H(\rho|r)+\rho\,|W-w|^2\Big).
\end{aligned}$$

Then using more of an analogous result than the one used in the proof of Lemma 17, we obtain the result. □

Let us now define

$$\begin{aligned}
I_{11}^{NS} = &-\frac{\nu}{\sqrt{\varepsilon^2-\nu^2}}\int_0^t\int_\Omega\Big(\rho\nabla(H'(r))\cdot(\bar{V}-\bar{v}) + p(\rho)\operatorname{div}V\Big)\\
&-\nu\int_0^t\int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2.
\end{aligned}$$

Using the definition of H and an integration by parts, we obtain

$$I_{11}^{NS} = -\frac{\nu}{\varepsilon^2-\nu^2}\int_0^t\int_\Omega \rho\left(\frac{p'(r)}{\mu'(r)}\bar{V} - \frac{p'(\rho)}{\mu'(\rho)}\bar{v}\right)\cdot(\bar{V}-\bar{v}), \tag{77}$$

with $v = \nabla(\mu(\rho))/\rho,\ \bar{v} = \sqrt{\varepsilon^2-\nu^2}\,v,\ V = \nabla(\mu(r))/r,\ \bar{V} = \sqrt{\varepsilon^2-\nu^2}\,V$. We can now show the following proposition:

Proposition 20. *Let I_{11}^{NS} given by (77). Assuming $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$, $s \geq -1$ and the hypothesis of Proposition 15, there exists a contant $C = C(r, U, \bar{V}, W)$ uniformly bounded on $\mathbb{R}^+ \times \Omega$ such that*

$$I_{11}^{NS} \leq C \frac{\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega H(\rho|r).$$

Proof. Using Lemma 36, we can write

$$I_{11}^{NS} = -\frac{\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \rho \frac{p'(\rho)}{\mu'(\rho)} |\bar{V} - \bar{v}|^2 - I_{12}^{NS}, \tag{78}$$

where

$$I_{12}^{NS} = \frac{\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \left(\sqrt{\varepsilon^2 - \nu^2} \nabla \phi_1(\rho|r) + \phi_2(\rho|r) \bar{V} \right) \cdot \bar{V}. \tag{79}$$

Using an integration by parts,

$$I_{12}^{NS} = -\frac{\nu}{\sqrt{\varepsilon^2 - \nu^2}} \int_0^t \int_\Omega \phi_1(\rho|r) \operatorname{div}(\bar{V}) + \frac{\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega \phi_2(\rho|r) \bar{V} \cdot \bar{V}.$$

Now, using lemma 34, we obtain

$$\begin{aligned} I_{12}^{NS} &\leq \frac{C\nu}{\sqrt{\varepsilon^2 - \nu^2}} \int_0^t \int_\Omega H(\rho|r) + \frac{C\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega H(\rho|r) \\ &\leq \frac{C\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega H(\rho|r), \end{aligned}$$

which gives the result due to Expression (78) and the sign of the first quantity in the right-hand side. □

Theorem 21. *Assuming $\mu(\rho) = \rho^{(s+3)/2}$, $\gamma \geq s+2$ and $s \geq -1$, any weak solution $(\rho, \bar{v}, w)$ of System (37)–(40) satisfies the following inequality:*

$$\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq C \left(1 + \frac{\nu}{\varepsilon^2 - \nu^2} \right) \int_0^t \mathcal{E}_{NSK}(\xi) d\xi, \tag{80}$$

where $(r, \bar{V}, W)$ is a strong solution of (66)–(68) belonging to the class (69) and where $C = C(r, U, \bar{V}, W)$ is a constant uniformly bounded on $\mathbb{R}^+ \times \Omega$.

Proof. Thanks to Proposition 15, we have

$$\begin{aligned} \mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq & \int_0^t \int_\Omega \rho \left[(\nabla \bar{V} (u - U)) \cdot (\bar{V} - \bar{v}) \right. \\ & \left. + (\nabla W (u - U)) \cdot (W - w) \right] \\ & - \int_0^t \int_\Omega \left(p(\rho) - p(r) - p'(r)(\rho - r) \right) \operatorname{div} U \\ & + I_{11}^{NS} + \frac{\nu}{2} I_3^{NS} + \nu I_4^{NS} + \sqrt{\varepsilon^2 - \nu^2} \left(I_5^{NS} + I_6^{NS} \right), \end{aligned}$$

with I_i^{NS} for $i = 3, 4, 5, 6$ given by (70)–(73) and I_{11}^{NS} given by (77). This gives, with the regularity of U, $\bar{V}$ and W and the previous lemmas,

$$
\begin{aligned}
\mathcal{E}_{NSK}(t) - \mathcal{E}_{NSK}(0) \leq C \int_0^t \int_\Omega \rho \, \left(|u-U|^2 + |\bar{v} - \bar{V}|^2 + |w - W|^2\right) \\
- \int_0^t \int_\Omega \left(p(\rho) - p(r) - p'(r)\,(\rho - r)\right) \operatorname{div} U \\
+ C \frac{\nu}{\varepsilon^2 - \nu^2} \int_0^t \int_\Omega H(\rho | r) \\
\leq C \left(1 + \frac{\nu}{\varepsilon^2 - \nu^2}\right) \int_0^t \mathcal{E}_{NSK}(\xi) d\xi.
\end{aligned}
$$

□

Corollary 22. *Let* $(r, \bar{V}, W)$ *be a strong solution of* (66)–(68) *in the class belonging to the class* (69). *Assuming* $\mu(\rho) = \rho^{(s+3)/2}$, $\gamma \geq s + 2$ *and* $s \geq -1$ *any weak solution* $(\rho, w, \bar{v})$ *of* (37)–(40) *satisfies the following inequality:*

$$
\begin{aligned}
&\mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W)(t) \\
&\leq \mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W)(0) \exp\left(C\left(1 + \frac{\nu}{\varepsilon^2 - \nu^2}\right) t\right),
\end{aligned}
$$

where $C = C(r, U, \bar{V}, W)$ *is a constant uniformly bounded on* $\mathbb{R}^+ \times \Omega$.

Proof. Thanks to the previous proposition and the Gronwall's Lemma, we have the inequality. □

Let U be a given and smooth function. We define r as the strong solution of (44), and we introduce the functions $\mathscr{E}_1^\nu$ and $\mathscr{E}_2^\nu$ such that

$$
\begin{aligned}
\mathscr{E}_1^\nu(r, \bar{V}, W), = r\, \left(\partial_t W + U \cdot \nabla W\right) + \nabla p(r) \\
-\nu \operatorname{div}(\mu(r) \nabla W) - \frac{\nu}{2} \nabla(\lambda(r) \operatorname{div} W) \\
-\sqrt{\varepsilon^2 - \nu^2} \left(\operatorname{div}(\mu(r) \nabla \bar{V}) + \frac{1}{2} \nabla(\lambda(r) \operatorname{div} \bar{V})\right)
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
0 = r\, \left(\partial_t \bar{V} + U \cdot \nabla \bar{V}\right) - \nu \operatorname{div}(\mu(r) \nabla \bar{V}) - \frac{\nu}{2} \nabla(\lambda(r) \operatorname{div} \bar{V}) \\
+ \sqrt{\varepsilon^2 - \nu^2} \left(\operatorname{div}(\mu(r)\, {}^t\nabla W) + \frac{1}{2} \nabla(\lambda(r) \operatorname{div} W)\right),
\end{aligned}
\tag{82}
$$

where $\bar{V} = \sqrt{\varepsilon^2 - \nu^2} \nabla \mu(r)/r$. In a same way than for the proof of Theorem 21, we have the following result:

Proposition 23. *Let us assume* $\mu(\rho) = \rho^{(s+3)/2}$ *(i.e.* $K(\rho) = \dfrac{(s+3)^2}{4} \rho^s$*),* $\gamma \geq s + 2$ *and* $s \geq -1$. *Let* $(\rho, \bar{v}, w)$ *be a global weak solution of System* (37)–(40) *and* $(r, \bar{V}, W)$ *a strong solution of* (44), (82)–(82) *in the class* (69). *Then*

$$\mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W)(t) - \mathcal{E}_{NSK}(\rho, \bar{v}, w | r, \bar{V}, W)(0)$$
$$\leq C\left(1 + \frac{\nu}{\varepsilon^2 - \nu^2}\right) \int_0^t \mathcal{E}_{NSK} + b^\nu(t),$$

with

$$b^\nu(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r}\mathscr{E}_1^\nu \cdot (W - w)\right],$$

and where $C = C(r, U, \bar{V}, W)$ is a constant uniformly bounded on $\mathbb{R}^+ \times \Omega$.

Using Gronwall's Lemma, we immediately obtain the following corollary:

Corollary 24. *Let us assume $\mu(\rho) = \rho^{(s+3)/2}$ (i.e. $K(\rho) = \dfrac{(s+3)^2}{4}\rho^s$) with $\gamma \geq s + 2$ and $s \geq -1$. Let $(\rho, \bar{v}, w)$ be a weak solution of System (37)–(40) and $(r, \bar{V}, W)$ a strong solution of (44), (82)–(82) in the class (69). Then*

$$\mathcal{E}_{NSK}(t) \leq \mathcal{E}_{NSK}(0)\, \exp(F^\nu\, t) + F^\nu \int_0^t b^\nu(\xi)\, \exp(F^\nu\,(t - \xi))\, d\xi + b^\nu(t),$$

where b^ν is defined in Proposition 23 and

$$F^\nu = C\left(1 + \frac{\nu}{\varepsilon^2 - \nu^2}\right),$$

with $C = C(r, U, \bar{V}, W)$ a constant uniformly bounded on $\mathbb{R}^+ \times \Omega$.

4.2. Dissipative Solution and Weak–Strong Uniqueness Result

Let us now give the definition of what is called a dissipative solution of the compressible Navier–Stokes–Korteweg System. To this end, let U be a smooth function, then $(r, \mathcal{E}^\nu(r, U))$ defined through Eq. (44) and

$$\mathscr{E}^\nu(r, U) = r\,(\partial_t U + U \cdot \nabla U) + \nabla p(r) - 2\,\nu\, \mathrm{div}(\mu(r) D(U)) - \nu \nabla(\lambda(\rho)\, \mathrm{div}\, U)$$
$$+ \varepsilon^2 \left[(\mathrm{div}(\mu(r)^t \nabla V) + \frac{1}{2}\nabla(\lambda(r)\, \mathrm{div}\, V)\right], \tag{83}$$

where $V = \nabla\mu(r)/r$. Denoting

$$\bar{V} = \sqrt{\varepsilon^2 - \nu^2}\, V,\;\; W = U + \nu\, V,$$

we then have the following:

$$\mathscr{E}^\nu(r, U) = r\,\,(\partial_t W + U \cdot \nabla W) + \nabla p(r) - \nu\,\, \mathrm{div}(\mu(r)\, \nabla W) \tag{84}$$
$$-\frac{\nu}{2}\nabla(\lambda(r)\, \mathrm{div}\, W) - \sqrt{\varepsilon^2 - \nu^2}\left(\mathrm{div}(\mu(r)\, \nabla \bar{V}) + \frac{1}{2}\nabla(\lambda(r)\, \mathrm{div}\, \bar{V})\right),$$

$$0 = r\,\,\left(\partial_t \bar{V} + U \cdot \nabla \bar{V}\right) + \sqrt{\varepsilon^2 - \nu^2}\,\,\left(\mathrm{div}(\mu(r)\,^t \nabla U) + \frac{1}{2}\nabla(\lambda(r)\, \mathrm{div}\, U\right). \tag{85}$$

Before giving the definition, let us recall that $\mathcal{E}_{NSK}(t)$ stands for

$$\mathcal{E}_{NSK}(t) = \mathcal{E}_{NSK}(\rho, \bar{v}, w|r, \bar{V}, W)(t).$$

Definition 25. Let us assume $\mu(\rho) = \rho^{(s+3)/2}$, $\gamma \geq s+2$ and $s \geq -1$. Let ρ_0 and u_0 smooth enough. The pair (ρ, u) is a dissipative solution of (17)–(18), (1) if the triplet $(\rho, \bar{v}, w)$ (with $\rho\, v = \nabla\mu(\rho)$, $\bar{v} = \sqrt{\varepsilon^2 - \nu^2}\, v$, $w = u + \nu\, v$) satisfies

$$\mathcal{E}_{NSK}(t) \leq \mathcal{E}_{NSK}(0) \exp(F^\nu t) + F^\nu \int_0^t b_{NSK}(\xi) \exp(F^\nu (t-\xi))\, d\xi + b_{NSK}(t),$$

with F^ν given in Corollary 24 and

$$b_{NSK}(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r} \mathscr{E}^\nu \cdot (W - w)\right]$$

with $(r, \bar{V}, W)$ and $\mathscr{E}^\nu$ are defined as mentioned above from all given smooth function U.

Noticing that each global weak solutions of the Navier–Stokes–Korteweg is global weak solutions of the augmented Navier–Stokes–Korteweg system, a direct consequence of the method is the following weak–strong uniqueness result:

Theorem 26. *Let us assume* $\mu(\rho) = \rho^{(s+3)/2}$, $\gamma \geq s+2$ *and* $s \geq -1$. *Let us consider* (ρ, u) *a global weak solution to the compressible Navier–Stokes–Korteweg system and define* $w = u + \nu\, \nabla\mu(\rho)/\rho$ *and* $\bar{v} = \sqrt{\varepsilon^2 - \nu^2}\, \nabla\mu(\rho)/\rho$. *Let us assume that there exists* (r, U) *a strong solution of the compressible Navier–Stokes–Korteweg System and let us define* $W = U + \nu\nabla\mu(r)/r$ *and* $\bar{V} = \sqrt{\varepsilon^2 - \nu^2}\, \nabla\mu(r)/r$. *Assume that* $(r, W, \bar{V})$ *satisfies hypothesis* (69). *If* $(\rho_0, u_0) = (r, U)(t = 0)$ *then* $(\rho, \bar{v}, w) = (r, \bar{V}, W)$ *or* $(\rho, u) = (r, U)$, *which corresponds to a weak–strong uniqueness property.*

Finally, let us give Definition 25 in the particular case of $K(\rho) = 1/\rho$ which corresponds to the quantum Navier–Stokes system. This one will be used in Sect. 5. To this end we introduce the function $\mathscr{E}^\nu_{NSQ}$ given by

$$\mathscr{E}^\nu_{NSQ}(r, U) = r\, (\partial_t U + U \cdot \nabla U) + \nabla p(r) - 2\,\nu\, \mathrm{div}(r\, D(U)) + \varepsilon^2\, \mathrm{div}(r^t \nabla V) \tag{86}$$

with U a given smooth enough function, r a strong solution of the mass equation (44) and $r V = \nabla\mu(r)$. Defining

$$\bar{V} = \sqrt{\varepsilon^2 - \nu^2}\, V, \qquad W = U + \nu\, V,$$

and using Eq. (44), we obtain

$$\mathscr{E}^\nu(r, U) = r\ (\partial_t W + U \cdot \nabla W) + \nabla p(r) - \nu\ \mathrm{div}(r\, \nabla W) - \sqrt{\varepsilon^2 - \nu^2}\, \mathrm{div}(r\, \nabla \bar{V}), \tag{87}$$

$$0 = r\ \left(\partial_t \bar{V} + U \cdot \nabla \bar{V}\right) + \sqrt{\varepsilon^2 - \nu^2}\ \mathrm{div}(r^{\,t} \nabla U). \tag{88}$$

We define $\mathcal{E}_{NSQ}(t)$ by

$$\mathcal{E}_{NSQ}(t) = \mathcal{E}_{NSK}(t) \text{ with } K(\rho) = 1/\rho.$$

Definition 27. Let ρ_0 and u_0 smooth enough. The pair (ρ, u) is a dissipative solution of (28), (29), (1) if the triplet $(\rho, \bar{v}, w)$ (with $\rho\, v = \nabla\mu(\rho)$, $\bar{v} = \sqrt{\varepsilon^2 - \nu^2}\, v$, $w = u + \nu\, v$) satisfies

$$\mathcal{E}_{NSQ}(t) \leq \mathcal{E}_{NSQ}(0)\, \exp(F^\nu t) + F^\nu \int_0^t b_{NSQ}(\xi)\, \exp(F^\nu\,(t-\xi))\, d\xi + b_{NSQ}(t),$$

with F^ν given in Corollary 24 and

$$b_{NSQ}(t) = \int_0^t \int_\Omega \Big[\frac{\rho}{r}\mathscr{E}^\nu_{NSQ} \cdot (W - w)\Big],$$

with $(r, \bar{V}, W)$ and $\mathscr{E}^\nu_{NSQ}$ defined as mentioned above from all given smooth functions U.

Remark 28. Note that by definition, using Corollary 24, all weak solution of (28)–(29), (1) is also a dissipative solution in the sense of Definition 27.

5. From the Quantum Navier–Stokes System to the Quantum Euler System: The Viscous Limit

We can now perform the limit of a dissipative solution of the quantum Navier–Stokes system to one of the quantum Euler system when the viscosity constant ν tends to zero. Thanks to the entropies, we have the following regularities on the global weak solution of the quantum Navier–Stokes equations:

$$\sqrt{\rho^\nu}\,\bar{v}^\nu \in L^\infty(0,T;L^2(\Omega)),\ \sqrt{\rho^\nu}\,w^\nu \in L^\infty(0,T;L^2(\Omega)),\ H(\rho^\nu) \in L^\infty(0,T,L^1(\Omega)),$$

where

$$\bar{v}^\nu = \sqrt{\varepsilon^2 - \nu^2}\,\nabla\log\rho^\nu, \quad w^\nu = u^\nu + \nu\,\nabla\log\rho^\nu.$$

The goal of this section is then to prove the following result:

Theorem 29. *Let ρ_0 and u_0 smooth enough. Let (ρ^ν, u^ν) be a global weak solution to the quantum Navier–Stokes system (28)-(29) with initial conditions (1). Let (ρ, u) be the weak limit of (ρ^ν, u^ν) when ν tends to 0 in the sense that*

$$\begin{aligned} &\rho^\nu \rightharpoonup \rho \text{ weakly} \star \text{ in } L^\infty(0,T;L^\gamma(\Omega)),\\ &\sqrt{\rho^\nu}\,w^\nu \rightharpoonup \sqrt{\rho}u \text{ weakly} \star \text{ in } L^\infty(0,T;L^2(\Omega)),\\ &\sqrt{\rho^\nu}\,\bar{v}^\nu \rightharpoonup \varepsilon\sqrt{\rho}v \text{ weakly} \star \text{ in } L^\infty(0,T;L^2(\Omega)),\end{aligned}$$

with $\rho\, v = \nabla\rho$. Then (ρ, u) is a dissipative solution of the quantum Euler system (14)–(15) with initial conditions (1).

Proof. According to Remark 28, the pair (ρ^ν, u^ν) being an entropic weak solution, it is also a dissipative one. We want to prove that (ρ, u), which is the limit of (ρ^ν, u^ν) when ν tends to zero, is a dissipative solution of (14)–(15) satisfying the initial conditions (1). The goal is then to prove that (ρ, u) satisfies Definition 13. Let us define $v = \nabla \log \rho$ (because in this case $\mu(\rho) = \rho$). Let U be smooth function and let $(r, \mathscr{E}(r, U), V)$ be defined with $V = \nabla \log r$, (44) and (58). We define

$$\bar{V}^\nu = \sqrt{\varepsilon^2 - \nu^2}\, V, \qquad W^\nu = U + \nu V.$$

Then it is easy to see that $(r, \bar{V}^\nu, W^\nu)$ is a candidate for (44), (87)–(88) with

$$\mathscr{E}^\nu_{NSQ}(r, U) = \mathscr{E}(r, U) - 2\,\nu\, \mathrm{div}(r D(U)).$$

Then, using Definition 27, and with (ρ^ν, u^ν) being a dissipative solution, we have

$$\begin{aligned} \mathcal{E}_{NSQ}(t) \le \mathcal{E}_{NSQ}(0)\, \exp(F^\nu t) \\ + F^\nu \int_0^t b^\nu_{NSQ}(\xi)\, \exp(F^\nu\,(t-\xi))\, d\xi + b^\nu_{NSQ}(t), \end{aligned} \tag{89}$$

with

$$F^\nu = C\left(1 + \frac{\nu}{\varepsilon^2 - \nu^2}\right),$$

and

$$b^\nu_{NSQ}(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r}\,(\mathscr{E} - 2\,\nu\, \mathrm{div}(r D(U))) \cdot (W - w)\right].$$

Since, by definition, we have

$$\begin{aligned} \mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(t) &= \frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right) \\ &+ \int_\Omega \left(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\right) \\ &+ \nu \int_0^t \int_\Omega \rho^\nu\,\left(|\nabla \bar{v}^\nu - \nabla \bar{V}|^2 + |\nabla w^\nu - \nabla W|^2\right), \end{aligned}$$

we easily obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right) + \int_\Omega \left(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\right) \\ &\le \mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(t) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(0) &= \frac{1}{2}\int_\Omega \rho^\nu\,\left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right)(0) \\ &+ \int_\Omega \left(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\right)(0). \end{aligned}$$

Then (89) gives

$$\frac{1}{2}\int_\Omega \rho^\nu \left(|\bar{v}^\nu - \bar{V}^\nu|^2 + |w^\nu - W^\nu|^2\right)(t) + \int_\Omega \left(H(\rho^\nu) - H(r) - H'(r)(\rho^\nu - r)\right)(t) \le \mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(0)\exp(F^\nu t) + F^\nu \int_0^t b^\nu_{NSQ}(\xi)\, \exp(F^\nu\,(t-\xi))\, d\xi + b^\nu_{NSQ}(t).$$

It remains now to pass to the limit ν tends to zero in this inequality. Clearly, using the lower semi-continuity of the term $\mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)$, the left-hand side is greater than

$$\frac{1}{2}\int_\Omega \rho \left(\varepsilon^2 |\, v - V|^2 + |u - U|^2\right)(t) + \int_\Omega H(\rho|r)(t),$$

which is $\mathcal{E}_{EuQ}(\rho, u, v | r, U, V)(t)$ (*i.e.* $\mathcal{E}_{EuK}(\rho, u, v | r, U, V)(t)$ given by (42) with $K(\rho) = 1/\rho$). For the right hand side, we use the direct limit of the term $\mathcal{E}_{NSQ}(\rho^\nu, \bar{v}^\nu, w^\nu | r, \bar{V}^\nu, W^\nu)(0)$ (through the expression of the initial data) and b^ν_{NSQ} tends to

$$b_{EuQ}(t) = \int_0^t \int_\Omega \left[\frac{\rho}{r}\mathscr{E} \cdot (U - u)\right],$$

to conclude that

$$\mathcal{E}_{EuQ}(t) \le \mathcal{E}_{EuQ}(0)\exp(C\,t) + b_{EuQ}(t) + C\int_0^t \exp(C\,(t-\xi)) b_{EuQ}(\xi)\, d\xi,$$

where $C = C(\varepsilon^2, r, U, V)$ is a uniformly bounded constant on $\mathbb{R}^+ \times \Omega$. Therefore we finally obtain that (ρ, u) satisfies the Definition 13 and then is a dissipative solution of (14)–(15), (1). $\square$

Acknowledgements. The third author acknowledges support from the team INRIA/RAPSODI and the Labex CEMPI (ANR-11-LABX-0007-01). The first author acknowledges the project TELLUS INSU-INSMI “Approche croisée pour fluides visco-élasto-plastiques: vers une meilleure compréhension des zones solides/fluides”. The first and the second authors acknowledge the ANR Project FRAISE managed by C. Ruyer-Quil.

Conflict of interest The authors have no conflicts of interest to declare.

6. Appendix

6.1. Technical Lemmas on Modulated Quantities

In this section we give some technical lemmas which are used in the paper.

We introduce the function ϕ defined by

$$\phi(\tau) = \int_0^\tau \frac{p'(\mu^{-1}(s))}{\mu'(\mu^{-1}(s))} \mathrm{d}s, \tag{90}$$

and the two functions

$$\phi_1(\rho|r) = \phi(\mu(\rho)) - \phi(\mu(r)) - \phi'(\mu(r))(\mu(\rho) - \mu(r)), \tag{91}$$
$$\phi_2(\rho|r) = \phi''(\mu(r))(\mu(\rho) - \mu(r))\, r - \rho\, (\phi'(\mu(\rho)) - \phi'(\mu(r))). \tag{92}$$

Remark 30. Note that in the case $K(\rho) = 1/\rho$, which gives (using (5)) $\mu(\rho) = \rho$, these two functions are directly linked to $H(\rho|r)$. Indeed, in this case we have

$$\phi_1(\rho|r) = p(\rho) - p(r) - p'(r)(\rho - r) = (\gamma - 1)H(\rho|r),$$
$$\phi_2(\rho|r) = \rho p'(r) - \rho p'(\rho) + r p''(r)(\rho - r) = -\gamma(\gamma - 1)H(\rho|r). \square$$

As is usual in compressible flows (see [24]) let us define the set $\mathcal{F}$ by

$$\mathcal{F} = \left\{\rho \le \frac{r}{2} \text{ or } \rho \ge 2r\right\}.$$

Let us now give some technical lemmas which will be used in what follows. First of all, following [23] we have

Lemma 31. *Assuming p smooth, $p(0) = 0$, $p'(\rho) > 0 \quad \forall \rho > 0$, $\lim_{\rho\to\infty} \dfrac{p'(\rho)}{\rho^{\alpha-1}} = a > 0$ for $\alpha > 1$, we have*

$$H(\rho|r) \ge C(r)(\rho - r)^2 \text{ if } \rho \in \mathcal{F}^c \text{ and } H(\rho|r) \ge C(r)(1+\rho)^\gamma \text{ otherwise},$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Concerning the functions ϕ_1 and ϕ_2, we have

Lemma 32. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \ge s + 2$ and $s \ge -1$. Assume ϕ_i with $i = 1, 2$ defined by (90)–(92). Then*

$$|\phi_i(\rho|r)| \le C(r)|\rho - r|^2 \text{ if } \rho \in \mathcal{F}^c \text{ and } |\phi_i(\rho|r)| \le C(r)(1+\rho)^\gamma \text{ otherwise},$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Remark 33. Let us remark that the choice $\mu(\rho) = \rho^{(s+3)/2}$ with $s \in \mathbb{R}$ and the assumption $\gamma \ge 2 + s$ correspond to the case considered in [27] because $K(\rho)$ is of order ρ^s. Moreover, for the particular case of interest in this paper $K(\rho) = 1/\rho$ (*i.e.* $s = -1$)*, the assumption $2 + s \le \gamma$ is trivially satisfied since we have $\gamma > 1$.*

Proof of the lemma for ϕ_1. Using Taylor expansions and the fact that $\phi''(\mu(c))$, $\mu'(c)$ are bounded with c in a compact we easily obtain

$$|\phi_1(\rho|r)| \le C(r)|\mu(\rho)-\mu(r)|^2 \le C(r)|\rho - r|^2 \text{ on } \mathcal{F}^c.$$

Moreover, since

$$p(\rho)=\rho^\gamma,\ K(\rho)=\frac{(s+3)^2}{4}\rho^s,\ \mu'(\rho)=\sqrt{\rho K(\rho)},$$

we have $\phi(\tau)=\tau^{2\gamma/(s+3)}$, and then by definition,

$$|\phi_1(\rho|r)| = \left|\rho^\gamma - r^\gamma - \frac{2\gamma}{s+3} r^{\frac{2\gamma-(s+3)}{2}}\left(\rho^{\frac{s+3}{2}} - r^{\frac{s+3}{2}}\right)\right|,$$

which gives

$$|\phi_1(\rho|r)| \le C(r)(1+\rho)^\gamma \text{ on } \mathcal{F},$$

since, by assumption, $2\gamma \ge 2(s+2) \ge s+3$ with $s \ge -1$.
Proof of the lemma for ϕ_2. Let us write $\theta = \dfrac{s+3}{2}$ then $\mu(\rho)=\rho^\theta$ and $\phi(\rho)=\rho^{\gamma/\theta}$. Then

$$\begin{aligned}
\phi_2(\rho|r) &= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-3}(\rho^\theta - r^\theta)r - \rho(\rho^{\gamma-\theta} - r^{\gamma-\theta})\right]\\
&= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - \rho^{1+\gamma-\theta} + \rho\, r^{\gamma-\theta}\right]\\
&= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\mu(\rho)) + \rho\, r^{\gamma-\theta}\right]
\end{aligned}$$

with $f(\rho)=\rho^{\frac{\gamma+1}{\theta}-1}$. Note that we have

$$\begin{aligned}
f(\rho|r) &= f(\mu(\rho)) - f(\mu(r)) - f'(\mu(r))(\mu(\rho)-\mu(r))\\
&= f(\mu(\rho)) - (r^\theta)^{\frac{\gamma+1}{\theta-1}} - \left(\frac{\gamma+1}{\theta}-1\right)(r^\theta)^{\frac{\gamma+1}{\theta}-2}(\rho^\theta - r^\theta).
\end{aligned}$$

Then

$$\begin{aligned}
\phi_2(\rho|r) &= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\rho|r) - (r^\theta)^{\frac{\gamma+1}{\theta}-1}\right]\\
&\quad - \frac{2\gamma}{s+3}\left[\left(\frac{2(\gamma+1)}{s+3}-1\right)(r^\theta)^{\frac{\gamma+1}{\theta}-2}(\rho^\theta - r^\theta) - \rho\, r^{\gamma-\theta}\right]\\
&= \frac{2\gamma}{s+3}\left[\left(\frac{2\gamma}{s+3}-1\right) r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\rho|r) - r^{1+\gamma-\theta}\right]\\
&\quad - \frac{2\gamma}{s+3}\left[\left(\frac{2(\gamma+1)}{s+3}-1\right)(r^{\gamma-s-2}(\rho^\theta - r^\theta) - \rho\, r^{\gamma-\theta}\right]
\end{aligned}$$

$$= \frac{2\gamma}{s+3}\left[-\frac{1}{\theta}r^{\gamma-s-2}(\rho^\theta - r^\theta) - f(\rho|r) + \rho\, r^{\gamma-\theta} - r^{1+\gamma-\theta}\right].$$

This can be written $\phi_2(\rho|r) = \dfrac{2\gamma}{s+3}(-f(\rho|r) + g(\rho|r))$ with

$$\begin{aligned} g(\rho|r) &= r^{\gamma-\theta}\left[\rho - r - \frac{1}{\theta}r^{\tau-s-2}(\rho^\theta - r^\theta)\right] \\ &= r^{\gamma-\theta}\left[(\rho^\theta)^{1/\theta} - (r^\theta)^{1/\theta} - \frac{1}{\theta}(r^\theta)^{1/\theta-1}(\rho^\theta - r^\theta)\right]. \end{aligned}$$

In the case $\rho \in \mathcal{F}^c$, using Taylor expansions, this leads to

$$\begin{aligned} |f(\rho|r)| &\le C(r)|\mu(\rho) - \mu(r)]^2 \le C(r)|\rho - r|^2, \\ |g(\rho|r)| &\le C(r)|\rho^\theta - r^\theta|^2 \le C(r)|\rho - r|^2, \end{aligned}$$

and then

$$|\phi_2(\rho|r)| \le C(r)|\rho - r|^2.$$

When $\rho \in \mathcal{F}$, since $2\gamma \ge 2s+4 \ge s+3$ and $s+3 \ge 2$,

$$|\phi_2(\rho|r)| \le C(r)|r^{\gamma-(s+3)}(\rho^{\frac{s+3}{2}} - r^{\frac{s+3}{2}}) - \rho(\rho^{\gamma-\frac{s+3}{2}} - r^{\gamma-\frac{s+3}{2}})| \le C(r)(1+\rho)^\gamma.$$

This completes the proof of Lemma 32. □

Using Lemmas 31 and 32, we directly obtain

Lemma 34. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \ge s+2$ and $s \ge -1$. We have*

$$|\phi_1(\rho|r)| \le C(r)H(\rho|r) \quad \text{and} \quad |\phi_2(\rho|r)| \le C(r)H(\rho|r),$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Let us now prove

Lemma 35. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \ge s+2$ and $s \ge -1$. We have*

$$\rho|\mu'(\rho) - \mu'(r)|^2 \le C(r)H(\rho|r),$$

with $C(r)$ uniformly bounded for r belonging to compact sets in $\mathbb{R}^+ \times \Omega$.

Proof.

$$\rho|\mu'(\rho) - \mu'(r)|^2 = \rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}} + \rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}^c}.$$

We have

$$\begin{aligned} \rho|\mu'(\rho) - \mu'(r)|^2 1_{\mathcal{F}} &\le 2\rho(|\mu'(\rho)|^2 + |\mu'(r)|^2)1_{\mathcal{F}} \\ &\le \frac{(s+3)^2}{2}\rho^{s+2}1_{\mathcal{F}} + 2C(r)\rho\, 1_{\mathcal{F}}. \end{aligned}$$

Using $\rho^{s+2} \leq (1+\rho)^{s+2}$ and the assumption $\gamma \geq s+2$ in the first term, and, the assumption $\gamma > 1$ in the second one, we obtain

$$\begin{aligned}\rho|\mu'(\rho)-\mu'(r)|^2 1_{\mathcal{F}} &\leq \frac{(s+3)^2}{2}(1+\rho)^\gamma 1_{\mathcal{F}} + 2C(r)(1+\rho)^\gamma 1_{\mathcal{F}} \\ &\leq C(r)(1+\rho)^\gamma 1_{\mathcal{F}}.\end{aligned}$$

Moreover,

$$\rho|\mu'(\rho)-\mu'(r)|^2 1_{\mathcal{F}^c} = \frac{s+3}{2}\rho\left|\rho^{\frac{s+1}{2}} - r^{\frac{s+1}{2}}\right|^2 1_{\mathcal{F}^c} = \frac{s+3}{2}\rho\frac{|\rho^{s+1}-r^{s+1}|^2}{|\rho^{\frac{s+1}{2}}+r^{\frac{s+1}{2}}|^2} 1_{\mathcal{F}^c},$$

and then

$$\begin{aligned}\rho|\mu'(\rho)-\mu'(r)|^2 1_{\mathcal{F}^c} &\leq \frac{s+3}{2}\rho\frac{|\rho^{s+1}-r^{s+1}|^2}{|r^{\frac{s+1}{2}}|^2} 1_{\mathcal{F}^c} \\ &\leq C(r)|\rho^{s+1}-r^{s+1}|^2 1_{\mathcal{F}^c} \leq C(r)|\rho-r|^2 1_{\mathcal{F}^c}.\end{aligned}$$

Using lemma 31, we finally obtain the result. □

An important relation. The last technical and important lemma is

Lemma 36. *Let us assume that $\mu(\rho) = \rho^{(s+3)/2}$ with $\gamma \geq s+2$ and $s \geq -1$. We have*

$$\rho\left(\frac{p'(\rho)}{\mu'(\rho)}v - \frac{p'(r)}{\mu'(r)}V\right)\cdot(v-V) = [\nabla\phi_1(\rho|r) + \phi_2(\rho|r)V] \cdot V + \rho\frac{p'(\rho)}{\mu'(\rho)}|V-v|^2$$

with ϕ_1 and ϕ_2 defined by (90)–(92).

Remark 37. This lemma generalizes to general $\mu(\rho)$ the relation (5) established in [13] when $\mu(\rho) = \rho$. This is an important lemma which helps to control the terms coming from the pressure in the relative entropy at the Navier–Stokes level.

Proof. Remark first that

$$\begin{aligned}\rho\left(\frac{p'(\rho)}{\mu'(\rho)}v - \frac{p'(r)}{\mu'(r)}V\right)\cdot(v-V) &= \rho\frac{p'(\rho)}{\mu'(\rho)}|V-v|^2 \\ &+\rho\left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)(v-V)\cdot V.\end{aligned}$$

We have

$$\begin{aligned}&\rho\left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)(v-V) \\ &= \left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)(\rho v - \rho V)\end{aligned}$$

$$= \left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)\nabla(\mu(\rho)) - \frac{\rho}{r}\left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)\nabla(\mu(r)).$$

Moreover, it is easy to see that, by definition,

$$\begin{aligned}\nabla(\phi_1(\rho|r)) &= \phi'(\mu(\rho))\nabla\mu(\rho) - \phi''(\mu(r))(\mu(\rho)-\mu(r))\nabla\mu(r)\\ &\quad -\phi'(\mu(r))\nabla\mu(\rho)\\ &= \left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)\nabla\mu(\rho) - \phi''(\mu(r))(\mu(\rho)-\mu(r))\nabla\mu(r),\end{aligned}$$

and then using the definition of $\phi_2(\rho|r)$,

$$\rho\left(\frac{p'(\rho)}{\mu'(\rho)} - \frac{p'(r)}{\mu'(r)}\right)(v-V) = \nabla\phi_1(\rho|r) + \phi_2(\rho|r)V.$$

□

6.2. Equivalence of $\mathcal{E}_{EuK}$ and the Relative Entropy in [27]

Let us consider the relative entropy functional, denoted $\mathcal{E}_{EuK}(\rho, u, v|r, U, V)$ and defined by (42). The goal of this section is to prove that this relative entropy is equivalent to the relative entropy defined by (2.23) in [27] under the concavity assumption on K with $K(\rho) = \rho^s$. Let us first recall the relative entropy $\mathcal{E}_{EuK}^{GLT}$ defined in [27], which reads

$$\mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho|r, U, \nabla r) = \frac{1}{2}\int_\Omega \rho|u-U|^2 + \frac{1}{2}\varepsilon^2\int_\Omega I_T + \int_\Omega H(\rho|r), \tag{93}$$

where

$$I_T = K(\rho)|\nabla\rho|^2 - K(r)|\nabla r|^2 - K'(r)|\nabla r|^2(\rho - r) - 2K(r)\nabla r(\nabla\rho - \nabla r).$$

Note that I_T corresponds to the term $K(\rho)|\nabla\rho|^2$ linearized in the variables (ρ, q) where $q = \nabla\rho$. Let us now introduce the quantity

$$I_{EuK}^2 = \rho\left|\sqrt{\frac{K(\rho)}{\rho}}\nabla\rho - \sqrt{\frac{K(r)}{r}}\nabla r\right|^2 = \left|\sqrt{K(\rho)}\nabla\rho - \sqrt{\frac{\rho}{r}}\sqrt{K(r)}\nabla r\right|^2.$$

Then our Euler–Korteweg modulated energy reads

$$\mathcal{E}_{EuK}(\rho, u, v|r, U, V) = \frac{1}{2}\int \rho|u-U|^2 + \frac{1}{2}\varepsilon^2\int I_{EuK}^2 + \int H(\rho|r),$$

where $v = \sqrt{K(\rho)}\nabla\rho/\sqrt{\rho}$ and $V = \sqrt{K(r)}\nabla r/\sqrt{r}$. Let us prove that under the hypothesis on K introduced in [27]

$$\mathcal{E}_{EuK}(\rho, u, v|r, U, V) = 0 \qquad \Leftrightarrow \qquad \mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho|r, U, \nabla r) = 0.$$

If this is so, we prove has that our relative entropy and the one in [27] are equivalent under the hypothesis in [27]. Our convergence result will therefore be more general that the one in [27] because it has not asked for a concavity hypothesis on $K(\rho)$. First let us prove the following lemma:

Lemma 38. *We have the equality*

$$
\begin{aligned}
&I_{\mathrm{EuK}}^2 + K(r)\,|\nabla r|^2 \left|\sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}\right|^2 \\
&\quad - K(r)^2\,|\nabla r|^2 \left(\frac{1}{K(\rho)} - \frac{1}{K(r)} + \frac{K'(r)}{K(r)^2}(\rho - r)\right) \\
&= I_T + 2\sqrt{K(r)}\nabla r\, I_{\mathrm{EuK}} \left(\sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}\right).
\end{aligned}
$$

Proof. After computations, we check that

$$
\begin{aligned}
I_{EuK}^2 &= I_T + K'(r)|\nabla r|^2(\rho - r) + \frac{\rho}{r}K(r)|\nabla r|^2 \\
&\quad -2\sqrt{\frac{\rho}{r}}\sqrt{K(\rho)}\sqrt{K(r)}\nabla\rho\cdot\nabla r + 2K(r)\nabla r\cdot\nabla\rho - K(r)|\nabla r|^2, \\
&= I_T + I_1
\end{aligned}
$$

where

$$
I_1 = K(r)|\nabla r|^2\left(\frac{\rho}{r} - 1 + \frac{K'(r)}{K(r)}(\rho - r) + 2\frac{\sqrt{K(r)}}{\sqrt{K(\rho)}}\sqrt{\frac{\rho}{r}} - 2\frac{\rho}{r}\right) + I_2,
$$

with

$$
I_2 = 2\sqrt{K(r)}\,\nabla r\; I_3\; I_{EuK} \qquad \text{and} \qquad I_3 = \sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}.
$$

Corollary 39. *Let $K(\rho) = \rho^s$ with $-1 \le s \le 0$, then*

$$
\mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho | r, U, \nabla r) = 0 \qquad \Leftrightarrow \qquad \mathcal{E}_{EuK}(\rho, u, v | r, U, V) = 0.
$$

Proof. Under the assumption on K, we check that

$$
\begin{aligned}
I_3^2 &= \left(\sqrt{\frac{K(r)}{K(\rho)}} - \sqrt{\frac{\rho}{r}}\right)^2 \\
&= \left(\sqrt{\left(\frac{\rho}{r}\right)^{-s}} - \sqrt{\frac{\rho}{r}}\right)^2 \\
&\le 2\left(\sqrt{\left(\frac{\rho}{r}\right)^{-s}} - 1\right)^2 + 2\left(1 - \sqrt{\frac{\rho}{r}}\right)^2 \\
&\le 2\frac{1}{r^{-s}}(\sqrt{\rho^{-s}} - \sqrt{r^{-s}})^2 + \frac{2}{r}(\sqrt{r} - \sqrt{\rho})^2 \\
&\le \frac{2}{r^{-s}}|\rho^{-s} - r^{-s}| + \frac{2}{r}|r - \rho|,
\end{aligned}
$$

with $0 \le -s \le 1$. Assume $\mathcal{E}_{EuK}^{GLT}(\rho, u, \nabla\rho | r, U, \nabla r) = 0$, then $I_3 = 0$ and

$$\left(\frac{1}{K(\rho)} - \frac{1}{K(r)} + \frac{K'(r)}{K(r)^2}(\rho - r)\right) = 0.$$

Therefore, using Lemma 38, we conclude that $\mathcal{E}_{EuK}(\rho, u, v | r, U, V) = 0$ (the inverse follows along the same lines). This ends the proof. □

6.3. Definition of the Operators

For the convenience of the reader we recall in this section all the definitions of the operators used in this article. The definitions used here are the ones presented in [7] in Appendix A.

Let f be a scalar, u, v two vectors, and $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ a tensor field defined on $\Omega \subset \mathbb{R}^d$ smooth enough.

- Denoting by $v_1, \cdots, v_d$ the coordinates of v, we call *divergence* of v the scalar given by:

$$\operatorname{div}(v) = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}.$$

- We call *laplacian* of f the scalar given by:

$$\Delta f = \operatorname{div}(\nabla f) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$

- We call *gradient* of v the tensor given by:

$$\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{1 \le i,j \le d}.$$

- We call *divergence* of σ the vector given by:

$$\operatorname{div}(\sigma) = \left(\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1 \le i \le d}.$$

- We call *laplacian* of v the vector given by:

$$\Delta v = \operatorname{div}(\nabla v).$$

- We call *tensor product* of u and v the tensor given by:

$$u \otimes v = \left(u_i v_j\right)_{1 \le i,j \le d}.$$

Proposition 40. *Let u, v, w three smooth enough vectors on Ω and r a scalar smooth enough on Ω. We have the following properties:*

- $(u \otimes v)w = (v \cdot w)u,$
- $\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + (v \cdot \nabla)u,$
- $\operatorname{div}(r\, u) = \nabla r \cdot u + r \ \operatorname{div} u,$
- $\operatorname{div}(r\, u \otimes v) = (\nabla r \cdot v)u + r(v \cdot \nabla)u + r \ \operatorname{div}(v)u.$

Definition 41. Let τ and σ be two tensors of order 2. We call scalar product of the two tensors the real defined by

$$\sigma : \tau = \sum_{1 \leq i, j \leq d} \sigma_{ij} \tau_{ij}.$$

The norm associated to this scalar product is simply denoted by $|\cdot|$ in such a way that

$$|\sigma|^2 = \sigma : \sigma.$$

Remark 42. By definition, we have

$$\sigma : \tau = {}^t\sigma : {}^t\tau.$$

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(*Received June 18, 2018 / Accepted March 15, 2019*)
Published online April 9, 2019