Arch. Rational Mech. Anal. 232 (2019) 489–529
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-018-1326-1

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Two-Phase Solutions for One-Dimensional Non-convex Elastodynamics

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Communicated by C. De Lellis

Abstract

We explore the local existence and properties of classical weak solutions to the initial-boundary value problem for a class of quasilinear equations of elastodynamics in one space dimension with a non-convex stored-energy function, a model of phase transitions in elastic bars proposed by Ericksen (J Elast 5(3–4):191–201,1975). The instantaneous phase separation and formation of microstructures of such solutions are observed for all smooth initial data with initial strain having its range that overlaps with the phase transition zone of the Piola–Kirchhoff stress. Moreover, we can select those solutions in a way that their phase gauges are close to a certain number inherited from a modified hyperbolic problem and thus give rise to an internal strain–stress hysteresis loop. As a byproduct, we prove the existence of a measure-valued solution to the problem that is generated by a sequence of weak solutions but not a weak solution itself. It is also shown that the problem admits a local weak solution for all smooth initial data and local weak solutions that are smooth for a short period of time and exhibit microstructures thereafter for certain smooth initial data.

1. Introduction

The evolution process of a one-dimensional continuous medium with elastic response can be modeled by quasilinear wave equations of the form

$$u_{tt} = (\sigma(u_x))_x, \tag{1.1}$$

where $u = u(x, t)$ denotes the *displacement* of a reference point x at time t and $\sigma = \sigma(s)$ the *Piola–Kirchhoff stress*, which is the derivative of a *stored-energy function* $W = W(s) \geqq 0$. With $v = u_x$ and $w = u_t$, one may study equation (1.1) as the system of conservation laws

$$\begin{cases} v_t = w_x, \\ w_t = (\sigma(v))_x. \end{cases} \tag{1.2}$$

For the case of a strictly convex stored-energy function, the existence of weak or classical solutions to equation (1.1) and to its vectorial case has been studied extensively. Global weak solutions to system (1.2) and hence to equation (1.1) were established in a classical work by DIPERNA [19] via a vanishing viscosity method in the framework of the compensated compactness of TARTAR [52] for L^∞ data and later by LIN [36] and SHEARER [47] in an L^p setup. This framework was also used to construct global weak solutions to (1.1) via relaxation methods by SERRE [46] and TZAVARAS [54]. An alternative variational scheme was studied by DEMOULINI et al. [16] via time discretization. However the existence of global weak solutions to the vectorial case of (1.1) is still open. In regard to classical solutions to (1.1) and to its vectorial case, one can refer to DAFERMOS AND HRUSA [11] for the local existence of smooth solutions, to KLAINERMAN AND SIDERIS [34] for the global existence of smooth solutions for small initial data in dimension 3, and to DAFERMOS [13] for the uniqueness of a smooth solution in the class of BV weak solutions whose shock intensity is not too strong.

The convexity assumption on the stored-energy function has often been regarded as a severe restriction getting a good viewpoint on the actual behavior of elastic materials (see, example, [24, Section 2] and [7, Section 8]). However there have not been many analytic works dealing with the lack of convexity on the energy function. For the vectorial case of equation (1.1) in dimension 3, measure-valued solutions were constructed for polyconvex energy functions by DEMOULINI *et al.* [17]. Also by the same authors [18], in an identical situation, it was shown that a dissipative measure-valued solution coincides with a strong one provided the latter exists. Assuming convexity on the energy function at infinity but not allowing polyconvexity, measure-valued solutions were obtained by RIEGER [45] for the vectorial case of (1.1) in any dimension. Despite of all these existence results, there has been no known example of a non-convex energy function with which (1.1) admits *classical* weak solutions in general other than the measure-valued ones above.

In this paper, we study the initial-boundary value problem of non-convex elastodynamics in one space dimension:

$$\begin{cases} u_{tt} = (\sigma(u_x))_x & \text{in } \Omega_T := \Omega \times (0, T), \\ u(0, t) = u(1, t) = 0 & \text{for } t \in (0, T), \\ u = g, \ u_t = h & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{1.3}$$

where $\Omega := (0, 1) \subset \mathbb{R}$ is the domain occupied by a reference configuration of an elastic bar, $T > 0$ is a fixed number, $g = g(x)$ is the initial displacement of the bar, $h = h(x)$ is the initial rate of change of the displacement, and the stress $\sigma : (-1, \infty) \to \mathbb{R}$ is given as in Fig. 1. The zero boundary condition here amounts to the physical situation of fixing the end-points of the bar; that is, in the context of elasticity, the bar is held fixed at the left end-point and loaded in a *hard device* with $d(1, t) = 1$, where $d(x, t) = u(x, t) + x$ is the *deformation* of the bar. In

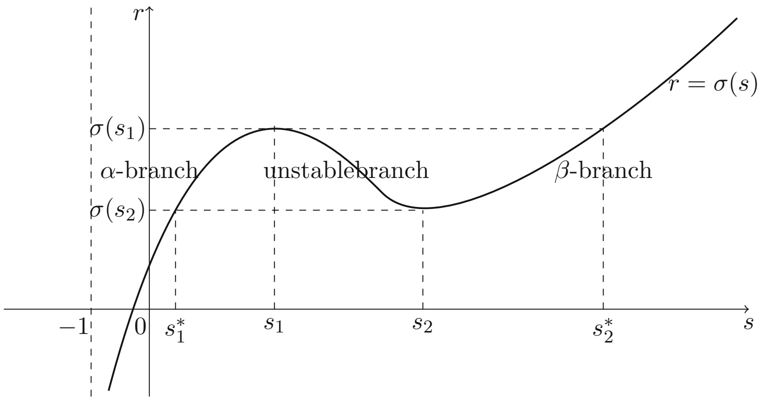

Fig. 1. Non-monotone Piola–Kirchhoff stress σ

this case, the energy function $W : (-1, \infty) \to [0, \infty)$ may satisfy $W(s) \to \infty$ as $s \to -1^+$, but this is not required in this work.

Problem (1.3) with a non-monotone stress σ as in Fig. 1 was proposed by Ericksen [21] as a model of the phenomena of phase transitions in elastic bars, and the stability analysis for the corresponding elastostatics was thoroughly carried out by James [25] (see also [51]). Beyond these works, there have been many studies on this problem that usually fall into two types. One direction of study is to consider the Riemann problem of the system of conservation laws of mixed type (1.2), initiated by James [26] and followed by numerous works (see, example, Shearer [48], Pego and Serre [42] and Hattori [23]). Another path is to study the viscoelastic version of equation (1.1). In this regard, Dafermos [12] considered the equation $u_{tt} = \sigma(u_x, u_{xt})_x + f(x, t)$ under certain parabolicity and growth conditions and established the global existence and uniqueness of a smooth solution with its asymptotic behavior as $t \to \infty$. Following the work of Andrews [2], Andrews and Ball [3] proved the global existence of weak solutions to the equation $u_{tt} = u_{xxt} + \sigma(u_x)_x$ for non-smooth initial data and studied their large-time behaviors. For the same equation, Pego [41] characterized the large-time convergence of weak solutions in a strong sense to several different types of stationary state. Nonetheless, to our best knowledge, the main theorem below may be the first general existence result on weak solutions to (1.3), not in the stream of the Riemann problem nor that of non-convex viscoelastodynamics. Moreover, we go beyond the simple existence result to explore some interesting properties that obtained solutions can satisfy (see below and Section 2).

Let σ be given as in Fig. 1 (see Subsection 2.1 for precise assumptions). We adopt a natural definition of weak solutions to problem (1.3) as follows:

Definition 1.1. For an initial datum $(g, h) \in W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)$, a function $u \in W^{1,\infty}(\Omega_T)$ is called a *weak solution* to (1.3) provided that the *strain* $u_x \geqq -1 + \theta$ a.e. in Ω_T, for some constant $\theta > 0$, that for all $\varphi \in C_c^\infty(\Omega \times [0, T))$ one has

$$\int_{\Omega_T} (u_t\varphi_t - \sigma(u_x)\varphi_x)\,\mathrm{d}x\mathrm{d}t = -\int_0^1 h(x)\varphi(x,0)\,\mathrm{d}x, \tag{1.4}$$

and that

$$\begin{cases} u(0,t) = u(1,t) = 0 \text{ for } t \in (0,T), \\ u(x,0) = g(x) \qquad\quad \text{for } x \in \Omega. \end{cases} \tag{1.5}$$

We may informally call such a weak solution a *classical weak solution* to mean that it is not merely a generalized *measure-valued solution* as in the next definition, but also truly a weak solution in the sense of distributions (see [15,37]).

Definition 1.2. Let $(g,h) \in W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)$ be an initial datum. Let (u,ν) be the pair of a function $u \in W^{1,\infty}(\Omega_T)$ and a parametrized family $\nu = \{\nu_{(x,t)}\}_{(x,t)\in\Omega_T}$ of probability measures in $\mathbb{R}$ that are all supported in a compact interval $J \subset (-1,\infty)$ with the following property: for each $f = f(s) \in C(\mathbb{R})$,

$$\Omega_T \ni (x,t) \mapsto \langle \nu_{(x,t)}, f\rangle := \textstyle\int_{\mathbb{R}} f(s)\,d\nu_{(x,t)}(s) \text{ is measurable.}$$

Then the pair (u,ν) is called a *measure-valued solution* to (1.3), provided that the strain $u_x \geqq -1+\theta$ a.e. in Ω_T, for some constant $\theta > 0$, that for all $\varphi \in C_c^\infty(\Omega \times [0,T))$ one has

$$\int_{\Omega_T} (u_t\varphi_t - \langle \nu, \sigma\rangle\varphi_x)\,\mathrm{d}x\mathrm{d}t = -\int_0^1 h(x)\varphi(x,0)\,\mathrm{d}x, \tag{1.6}$$

and that (1.5) is satisfied.

We remark that if $u \in W^{1,\infty}(\Omega_T)$ is a weak solution to problem (1.3), then (u,ν) is a measure-valued solution to (1.3) by letting

$$\nu_{(x,t)} = \delta_{u_x(x,t)} \ \text{ for a.e. } (x,t) \in \Omega_T, \tag{1.7}$$

where δ_s denotes the point mass at each $s \in \mathbb{R}$. Motivated by this observation, we shall say that a measure-valued solution (u,ν) to (1.3) is a *weak solution* to (1.3) if (1.7) is satisfied.

We refer to the graph of the stress σ on the interval $(-1,s_1)$ as the α*-branch* and that of σ on (s_2,∞) as the β*-branch* (see Fig. 1). We also say that a weak solution u to (1.3) is in the α*-phase* (β*-phase*, resp.) at a point $(x,t) \in \Omega_T$ if $u_x(x,t) \in (-1,s_1)$ ($u_x(x,t) \in (s_2,\infty)$, resp.) and that these two phases are a *stable phase*. The graph of σ on $[s_1,s_2]$ will be called the *unstable branch* because, in elastostatics, displacements with values of their strain in $[s_1,s_2]$ can never satisfy the so-called *Weierstrass condition*, which serves as a necessary condition for *metastability* (see [25]). For this reason, we will only look for weak solutions u to problem (1.3) with strain–stress $(u_x,\sigma(u_x))$ lying in the α- or β-branch. Thus we introduce the following terminology:

Definition 1.3. For an initial datum $(g,h) \in W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)$, a weak solution u to (1.3) is called a *two-phase weak solution* provided that for a.e. $(x,t) \in \Omega_T$, u is in the α- or β-phase at (x,t) and that

$$\begin{cases} |\{(x,t) \in \Omega_T \mid u \text{ is in the } \alpha\text{-phase at } (x,t)\}| > 0, \\ |\{(x,t) \in \Omega_T \mid u \text{ is in the } \beta\text{-phase at } (x,t)\}| > 0. \end{cases}$$

The main existence result of the paper is Theorem 1.4 (below), which will be further strengthened and elaborated under a suitable setup in Section 2. Roughly speaking, obtained solutions behave like a hyperbolic evolution in some part of the space-time domain Ω_T and have a discontinuous strain in the other part of Ω_T with a sharp separation into the two stable phases. In the course of proving the existence of such solutions, we quantify phase separation by introducing the concept of *phase gauge* and show that the gauge of the solution to a certain modified hyperbolic problem can be *almost* carried over to the solutions for existence. Moreover, this notion of gauge plays a crucial role in extracting the formation of *microstructures* in the obtained solutions.

The main result in a simplified form below is a direct consequence of the elaborated main result, Theorem 2.1, so we do not include the proof of this theorem.

Theorem 1.4. (Main result: simplified) *Let σ satisfy Hypothesis (A) in Subsection 2.1 (see Fig. 1), and let $(g,h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)$ be such that $s_1^* < g'(x_0) < s_2^*$ for some $x_0 \in \Omega$ and that $g'(x) > -1$ for all $x \in \bar{\Omega}$. Then there exist a finite number $T > 0$, a function $u^* \in C^2(\bar{\Omega}_T)$, a compact set $D \subset (-1,\infty) \times \mathbb{R}$, three disjoint open sets $Q_1, Q_2, Q_3 \subset \Omega_T$ with $Q_2 \neq \emptyset$, $\partial Q_2 \cap (\Omega \times \{t=0\}) \neq \emptyset$, $u_x^*(\bar{Q}_2) \subset (s_1^*, s_2^*)$ and $\partial Q_1 \cap \partial Q_3 = \emptyset$, and a sequence $\{Q_2^k\}_{k\in\mathbb{N}}$ of disjoint open subsets of Q_2 with $|\cup_{k\in\mathbb{N}} Q_2^k| = |Q_2|$, $Q_2^1 \neq \emptyset$, and $\partial Q_2^1 \cap (\Omega \times \{t=0\}) \neq \emptyset$ such that for each $\varepsilon > 0$, there are infinitely many two-phase weak solutions $u \in W_{u^*}^{1,\infty}(\Omega_T)$ to problem (1.3) satisfying the following:*

(1) *$u = u^*$ on $\overline{Q_1 \cup Q_3}$,*
(2) *$\|u - u^*\|_{L^\infty(\Omega_T)} < \varepsilon$,*
(3) *$\nabla u \in D$ a.e. in Ω_T, where $\nabla := (\partial_x, \partial_t)$,*
(4) *for each $k \in \mathbb{N}$, there are four numbers $s_{a,+}^k, s_{b,+}^k \in (s_2, s_2^*)$ and $s_{a,-}^k, s_{b,-}^k \in (s_1^*, s_1)$, independent of ε and u, with*

$$s_{a,-}^k < s_{b,-}^k < \inf_{Q_2^k} u_x^* \leqq \sup_{Q_2^k} u_x^* < s_{a,+}^k < s_{b,+}^k$$

such that $|Q_{2,+}^{k,u}| > 0$, $|Q_{2,-}^{k,u}| > 0$ and $|Q_{2,+}^{k,u}| + |Q_{2,-}^{k,u}| = |Q_2^k|$, where

$$Q_{2,\pm}^{k,u} := \left\{ (x,t) \in Q_2^k \mid u_x(x,t) \in [s_{a,\pm}^k, s_{b,\pm}^k] \right\}.$$

To observe the formation of microstructures of such solutions u in Q_2, fix a decreasing sequence of positive reals $\varepsilon_j \to 0$. For each $j \in \mathbb{N}$, let u_j be a two-phase weak solution to (1.3) corresponding to $\varepsilon = \varepsilon_j$ in the above theorem. Let k be any positive integer such that $Q_2^k \neq \emptyset$. For a.e. $t \in (0,T)$ with $(Q_2^k)^t := \{x \in \Omega \mid (x,t) \in Q_2^k\} \neq \emptyset$, we see from (4) that for all $j \in \mathbb{N}$, $(u_j)_x(\cdot,t)$ is trapped in the two disjoint intervals $[s_{a,\pm}^k, s_{b,\pm}^k]$ a.e. in $(Q_2^k)^t$, while $u_x^*(\cdot,t)$ smoothly varies in the interval $[\inf_{Q_2^k} u_x^*, \sup_{Q_2^k} u_x^*]$ in $(Q_2^k)^t$. On the other hand, thanks to (2),

$$\sup_{(Q_2^k)^t} |u_j(\cdot,t) - u^*(\cdot,t)| < \varepsilon_j \to 0 \text{ as } j \to \infty.$$

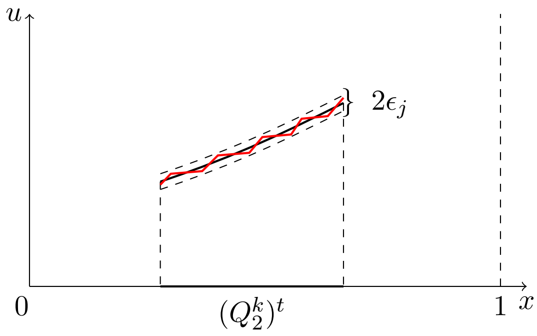

Fig. 2. Microstructure of $u_j(\cdot, t)$ and smooth $u^*(\cdot, t)$

Thus as the index $j \in \mathbb{N}$ increases, the $u_j(\cdot, t)$ become highly oscillatory near $u^*(\cdot, t)$ and exhibit finer and finer microstructures in the nonempty open set $(Q_2^k)^t \subset \Omega = (0, 1)$; see Fig. 2.

The previous argument applies to the case $k = 1$, since $Q_2^1 \neq \emptyset$. In this case, as $\partial Q_2^1 \cap (\Omega \times \{t = 0\}) \neq \emptyset$, we can certainly take a decreasing sequence of positive times $t_n \to 0$ such that the argument applies to each time $t = t_n$. We may say that the formation of microstructures of the u_j is *instantaneous* in this sense.

The existence and *non-uniqueness* of weak solutions to problem (1.3) have been generally accepted (especially, in the context of the Riemann problem) and actively studied in the field of solid mechanics. Such non-uniqueness has been usually understood to be arising from a constitutive deficiency in the theory of elastodynamics, reflecting the need to incorporate some additional relations (see, example, SLEMROD [50], ABEYARATNE AND KNOWLES [1] and TRUSKINOVSKY AND ZANZOTTO [53]).

Unfortunately, the existence of *global* weak solutions to problem (1.3) cannot be obtained in the course of proving the main result, Theorem 2.1, as it would require a global classical solution to some modified hyperbolic problem that serves as a certain subsolution in our proof, but such a global one might not exist due to a possible shock formation at a finite time. Thus it may be an interesting question to study whether global weak solutions to (1.3) can be achieved by another method.

We now introduce a motivational approach to attack problem (1.3) with σ as in Fig. 1. To solve equation (1.1) in the sense of distributions in Ω_T, suppose there exists a vector function $w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbb{R}^2)$ such that

$$v_x = u_t \quad \text{and} \quad v_t = \sigma(u_x) \quad \text{a.e. in } \Omega_T. \tag{1.8}$$

We remark that this formulation is motivated by the approach in [57] and different from the usual setup of conservation laws (1.2). For all $\varphi \in C_c^\infty(\Omega_T)$, we then have

$$\int_{\Omega_T} u_t \varphi_t \, dx dt = \int_{\Omega_T} v_x \varphi_t \, dx dt = \int_{\Omega_T} v_t \varphi_x \, dx dt = \int_{\Omega_T} \sigma(u_x) \varphi_x \, dx dt;$$

hence having (1.8) is sufficient to solve (1.1) in the sense of distributions in Ω_T. Equivalently, we can rewrite (1.8) as

$$\nabla w = \begin{pmatrix} u_x & u_t \\ v_x & v_t \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ v_x & \sigma(u_x) \end{pmatrix} \quad \text{a.e. in } \Omega_T,$$

where ∇ denotes the space-time gradient operator. Set

$$\Sigma_\sigma = \left\{ \begin{pmatrix} s & c \\ c & \sigma(s) \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \,\middle|\, s, c \in \mathbb{R} \right\}.$$

We can now recast (1.8) as a *homogeneous partial differential inclusion* with a *linear constraint* on the antidiagonal:

$$\nabla w(x,t) \in \Sigma_\sigma, \quad \text{a.e. } (x,t) \in \Omega_T.$$

We will solve this inclusion for a suitable subset K of Σ_σ to incorporate some detailed properties of weak solutions to (1.3).

Homogeneous differential inclusions of the form $\nabla w \in K \subset \mathbb{M}^{m\times n}$ were first encountered and successfully understood in the study of crystal microstructure by Ball and James [4], Chipot and Kinderlehrer [6] and with a constraint on a minor of ∇w by Müller and Šverák [38]. General inhomogeneous differential inclusions were studied by Dacorogna and Marcellini [10] using Baire's category method and by Müller and Sychev [40] using the method of convex integration; see also [33]. Moreover, the methods of differential inclusions have been applied to other important problems concerning elliptic systems [39], Euler equations [14], the porous media equation [8], active scalar equations [49], the Monge-Ampere equation [35], non-parabolic diffusions [27–30,57], ferromagnetism [56], and scalar conservation laws in 1-D [55].

The rest of the paper is organized as follows: Section 2 begins with precise structural assumptions on the stress $\sigma(s)$ corresponding to Fig. 1. Then the notion of a subsolution of a certain partial differential inclusion and its phase gauge is introduced and followed by the detailed statement of the main result, Theorem 2.1, along with some interesting corollaries. In Section 3, Theorem 2.1 is proved under the pivotal density result, Theorem 3.1. In Section 4, a major tool for proving the density result is established in a general form. Lastly, Section 5 carries out the proof of the density result, Theorem 3.1.

In closing this section, we fix some notation. Let m, n be positive integers. We denote by $\mathbb{M}^{m\times n}$ the space of $m \times n$ real matrices and by $\mathbb{M}^{n\times n}_{sym}$ that of symmetric $n \times n$ real matrices. We use $O(n)$ to denote the space of $n \times n$ orthogonal matrices. For a given matrix $M \in \mathbb{M}^{m\times n}$, we write M_{ij} for the component of M in the ith row and jth column and M^T for the transpose of M. For a bounded domain $U \subset \mathbb{R}^n$ and a function $w^* \in W^{m,p}(U)$ $(1 \leqq p \leqq \infty)$, we use $W^{m,p}_{w^*}(U)$ to denote the space of functions $w \in W^{m,p}(U)$ with boundary trace w^*. If $E \subset \mathbb{R}^n$ is measurable, $|E|$ denotes its n-dimensional Lebesgue measure.

2. Precise Statement of the Main Theorem

We begin this section with structural assumptions on the stress $\sigma(s)$ corresponding to Fig. 1. We then introduce the notion of a subsolution of a related partial differential inclusion and define its phase gauge (see [31]). Based on such assumptions and definitions, we give a detailed statement of the main result, Theorem 2.1. Also, some interesting byproducts are presented as Corollaries 2.3, 2.4 and 2.5, along with their proofs.

2.1. Hypothesis on the Stress σ

We impose the following conditions on the stress $\sigma : (-1, \infty) \to \mathbb{R}$ (see Fig. 1):

Hypothesis (A): There exist two numbers $s_2 > s_1 > -1$ with the following properties:

(a) $\sigma \in C^3((-1, s_1) \cup (s_2, \infty)) \cap C(-1, \infty)$;
(b) $\lim_{s \to -1^+} \sigma(s) = -\infty$;
(c) $\sigma(s_1) > \sigma(s_2)$, and $\sigma'(s) > 0$ for all $s \in (-1, s_1) \cup (s_2, \infty)$;
(d) There exist two numbers $c > 0$ and $s_1 + 1 > \rho > 0$ such that $\sigma'(s) \geqq c$ for all $s \in (-1, s_1 - \rho] \cup [s_2 + \rho, \infty)$;
(e) Let $s_1^* \in (-1, s_1)$ and $s_2^* \in (s_2, \infty)$ denote the unique numbers with $\sigma(s_1^*) = \sigma(s_2)$ and $\sigma(s_2^*) = \sigma(s_1)$, respectively.

For each $r \in (\sigma(s_2), \sigma(s_1))$, let $s_-(r) \in (s_1^*, s_1)$ and $s_+(r) \in (s_2, s_2^*)$ denote the unique numbers with $\sigma(s_\pm(r)) = r$. We may call the interval (s_1^*, s_2^*) the *phase transition zone* of problem (1.3) since the phase separation and formation of microstructures of weak solutions to (1.3) are observed to occur whenever the range of the initial strain g' overlaps with the interval (s_1^*, s_2^*) (see Theorem 2.1). Note that this zone (s_1^*, s_2^*) includes the interval $[s_1, s_2]$ for the unstable branch of σ.

2.2. Subsolution and Phase Gauge

Let r_a and r_b be any two numbers with $\sigma(s_2) < r_a < r_b < \sigma(s_1)$. We define some related sets (see Fig. 3):

$$\begin{aligned}
\tilde{K}_\pm^{r_a,r_b} &= \left\{(s, \sigma(s)) \in \mathbb{R}^2 \mid s_\pm(r_a) \leqq s \leqq s_\pm(r_b)\right\}, \\
\tilde{K}^{r_a,r_b} &= \tilde{K}_+^{r_a,r_b} \cup \tilde{K}_-^{r_a,r_b}, \\
\tilde{U}^{r_a,r_b} &= \left\{(s, r) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} r_a < r < r_b,\ 0 < \lambda < 1, \\ s = \lambda s_-(r) + (1-\lambda) s_+(r) \end{array}\right\}, \\
K = K^{r_a,r_b} &= \left\{\begin{pmatrix} s & c \\ c & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \;\middle|\; (s, r) \in \tilde{K}^{r_a,r_b},\ c \in \mathbb{R}\right\}, \\
U = U^{r_a,r_b} &= \left\{\begin{pmatrix} s & c \\ c & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \;\middle|\; (s, r) \in \tilde{U}^{r_a,r_b},\ c \in \mathbb{R}\right\}.
\end{aligned} \tag{2.1}$$

Note that $\tilde{K}_+^{r_a,r_b}$ and $\tilde{K}_-^{r_a,r_b}$ are the closed portion of the β- and α-branch of the stress σ, respectively, that is cut by the constant stress values $r = r_a$ and $r = r_b$. Also, $\tilde{U}^{r_a,r_b}$ is the open region, surrounded by $\tilde{K}_\pm^{r_a,r_b}$, $r = r_a$ and $r = r_b$.

We now consider the partial differential inclusion

$$\nabla w(x,t) \in K, \quad \text{a.e. } (x,t) \in Q, \tag{2.2}$$

where $Q \subset \mathbb{R}^2$ is a bounded open set and $w = (u, v) : Q \to \mathbb{R}^2$ is a Lipschitz function.

We say that a Lipschitz function $w : Q \to \mathbb{R}^2$ is a *subsolution* of differential inclusion (2.2) in Q if

$$\nabla w(x,t) \in K \cup U, \quad \text{a.e. } (x,t) \in Q,$$

and that it is a *strict subsolution* of (2.2) in Q if

$$\nabla w(x,t) \in U, \quad \text{a.e. } (x,t) \in Q.$$

Assume $Q \neq \emptyset$, and let $w = (u, v) : Q \to \mathbb{R}^2$ be a subsolution of differential inclusion (2.2). For a.e. $(x,t) \in Q$, we can define the quantity

$$Z_w^Q(x,t) = \frac{u_x(x,t) - s_-(v_t(x,t))}{s_+(v_t(x,t)) - s_-(v_t(x,t))} \in [0,1].$$

We then define the *phase gauge* of w over Q by

$$\Gamma_w^Q = \frac{1}{|Q|} \int_Q Z_w^Q(x,t)\, \mathrm{d}x\mathrm{d}t \in [0,1].$$

This gauge measures the tendency of the diagonal (u_x, v_t) of ∇w over Q towards the portion $\tilde{K}_+^{r_a,r_b}$ of the β-branch; in particular,

$$\Gamma_w^Q = 1 \iff (u_x, v_t) \in \tilde{K}_+^{r_a,r_b} \text{ a.e. in } Q,$$

and

$$\Gamma_w^Q = 0 \iff (u_x, v_t) \in \tilde{K}_-^{r_a,r_b} \text{ a.e. in } Q.$$

It is also easy to see that if w is a strict subsolution of (2.2), then $0 < \Gamma_w^Q < 1$.

If $u : Q \to \mathbb{R}$ is a Lipschitz solution of equation (1.1) in the sense of distributions in Q such that

$$u_x(x,t) \in [s_-(r_a), s_-(r_b)] \cup [s_+(r_a), s_+(r_b)], \quad \text{a.e. } (x,t) \in Q,$$

then, with a stream function $v : Q \to \mathbb{R}$ defined by

$$v_x = u_t \text{ a.e. in } Q,$$

which is unique up to a constant in each connected component of Q, the pair $w = (u, v)$ becomes a solution and thus a subsolution of differential inclusion (2.2). In this case, we define the *phase gauge* γ_u^Q of u over Q by

$$\gamma_u^Q = \Gamma_w^Q.$$

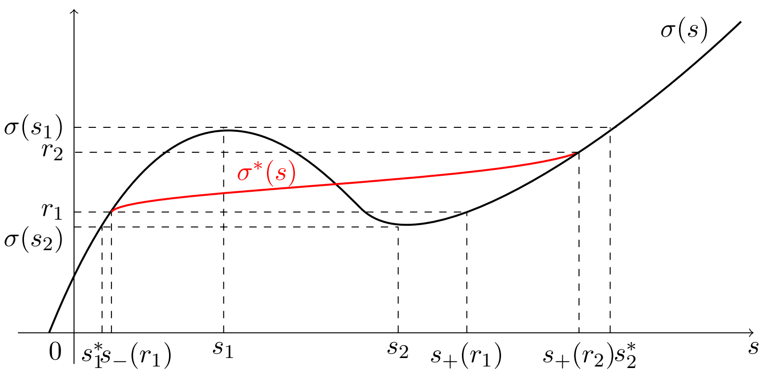

Fig. 3. The original $\sigma(s)$ and modified $\sigma^*(s)$

It is then easy to see that the following relations hold:

$$|Q_+^u| = \gamma_u^Q |Q| \quad \text{and} \quad |Q_-^u| = (1-\gamma_u^Q)|Q|,$$

where

$$Q_\pm^u := \{(x,t) \in Q \mid u_x(x,t) \in [s_\pm(r_a), s_\pm(r_b)]\}.$$

2.3. The Main Result

Prior to stating the main result of the paper in an elaborated form under Hypothesis (A), we set up suitable assumptions and definitions for the statement.

(Initial datum): We assume that the initial datum (g,h) to problem (1.3) satisfies

$$\begin{cases} (g,h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega), \\ g'(x) > -1 \text{ for all } x \in \bar{\Omega}, \\ s_1^* < g'(x_0) < s_2^* \text{ for some } x_0 \in \Omega. \end{cases} \tag{2.3}$$

(Modified hyperbolic problem): We fix any two numbers $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$ so that

$$s_-(r_1) < g'(x_0) < s_+(r_2).$$

Using elementary calculus, from Hypothesis (A), we can find a function $\sigma^* \in C^3(-1,\infty)$ (see Fig. 3) such that

$$\begin{cases} \sigma^*(s) = \sigma(s) \text{ for all } s \in (-1, s_-(r_1)] \cup [s_+(r_2), \infty), \\ (\sigma^*)'(s) \geqq c^* \text{ for all } s \in (-1,\infty), \text{ for some constant } c^* > 0, \\ \sigma^*(s) < \sigma(s) \text{ for all } s_-(r_1) < s \leqq s_-(r_2), \text{ and} \\ \sigma^*(s) > \sigma(s) \text{ for all } s_+(r_1) \leqq s < s_+(r_2). \end{cases} \tag{2.4}$$

(Function u^*): Thanks to [11, Theorem 5.2], there exists a finite number $T > 0$ such that the *modified* initial-boundary value problem

$$\begin{cases} u^*_{tt} = (\sigma^*(u^*_x))_x & \text{in } \Omega_T, \\ u^*(0,t) = u^*(1,t) = 0 & \text{for } t \in (0,T), \\ u^* = g,\ u^*_t = h & \text{on } \Omega \times \{t=0\} \end{cases} \tag{2.5}$$

admits a unique solution $u^* \in \bigcap_{k=0}^{3} C^k([0,T]; W_0^{3-k,2}(\Omega))$ with $u^*_x > -1$ on $\bar{\Omega}_T$, where $W_0^{0,2}(\Omega) := L^2(\Omega)$; then by the Sobolev embedding theorem, we have $u^* \in C^2(\bar{\Omega}_T)$. Let

$$\begin{cases} Q_1 = \{(x,t) \in \Omega_T \,|\, u^*_x(x,t) < s_-(r_1)\}, \\ Q_2 = \{(x,t) \in \Omega_T \,|\, s_-(r_1) < u^*_x(x,t) < s_+(r_2)\}, \\ Q_3 = \{(x,t) \in \Omega_T \,|\, u^*_x(x,t) > s_+(r_2)\}; \end{cases}$$

then

$$\begin{cases} \partial Q_1 \cap \Omega_0 \subset \{(x,0) \,|\, x \in \Omega,\ g'(x) \leqq s_-(r_1)\}, \\ \partial Q_2 \cap \Omega_0 \subset \{(x,0) \,|\, x \in \Omega,\ s_-(r_1) \leqq g'(x) \leqq s_+(r_2)\}, \\ \partial Q_3 \cap \Omega_0 \subset \{(x,0) \,|\, x \in \Omega,\ g'(x) \geqq s_+(r_2)\}, \end{cases} \tag{2.6}$$

and $\partial Q_1 \cap \partial Q_3 = \emptyset$, where $\Omega_0 := \Omega \times \{t=0\}$. As $s_-(r_1) < g'(x_0) = u^*_x(x_0,0) < s_+(r_2)$, we also have $Q_2 \neq \emptyset$ and $\partial Q_2 \cap (\Omega \times \{t=0\}) \neq \emptyset$.

(Separation of domain Q_2): Observe that

$$|\{(x,t) \in Q_2 \,|\, u^*_x(x,t) = s\}| = 0$$

for all but at most countably many $s \in (s_-(r_1), s_+(r_2))$. Fix any point $s_0 \in (s_1, s_2)$ so that

$$|\{(x,t) \in Q_2 \,|\, u^*_x(x,t) = s_0\}| = 0.$$

Let us write $s^{20} = s^{20*} = s_0$ and $r^{20} = \sigma^*(s^{20}) = \sigma^*(s^{20*}) = \sigma^*(s_0)$. Define recursively that

$$s^{2k*} = s_+(\sigma^*(s^{2(k-1)*})) \quad \text{for each } k \in \mathbb{N};$$

then $s^{20*} < s^{21*} < s^{22*} < \cdots < s_+(r_2)$ and $\lim_{k\to\infty} s^{2k*} = s_+(r_2)$.

Choose a point $s^{21} \in (s^{20*}, s^{21*})$ so that

$$|\{(x,t) \in Q_2 \,|\, u^*_x(x,t) = s^{21}\}| = 0,$$

and write $r^{21} = \sigma^*(s^{21})$. Then

$$s^{20} = s^{20*} < s^{21} < s^{21*} = s_+(\sigma^*(s^{20*})) = s_+(r^{20}) < s_+(r^{21}),$$

that is,

$$s^{20} < s^{21} < s_+(r^{20}) < s_+(r^{21}).$$

Note that $s^{22*} = s_+(\sigma^*(s^{21*})) > s_+(\sigma^*(s^{21})) = s_+(r^{21})$, and so

$$s^{21*} < s_+(r^{21}) < s^{22*}.$$

Next, choose a point $s^{22} \in (s^{21*}, s_+(r^{21}))$ (thus $s^{21*} < s^{22} < s^{22*}$) so that

$$|\{(x,t) \in Q_2 \mid u_x^*(x,t) = s^{22}\}| = 0,$$

and write $r^{22} = \sigma^*(s^{22})$. Then

$$s^{21} < s^{21*} < s^{22} < s_+(r^{21}) < s^{22*} = s_+(\sigma^*(s^{21*})) < s_+(r^{22}),$$

that is,

$$s^{21} < s^{22} < s_+(r^{21}) < s_+(r^{22}).$$

Note that $s^{23*} = s_+(\sigma^*(s^{22*})) > s_+(\sigma^*(s^{22})) = s_+(r^{22})$, and so

$$s^{22*} < s_+(r^{22}) < s^{23*}.$$

Having chosen $s^{21}, \dots, s^{2k}$ for some $k \geqq 2$ so that, with $r^{2j} := \sigma^*(s^{2j})$ ($j = 1, \dots, k$),

$$\begin{cases} s^{2(j-1)*} < s^{2j} < s^{2j*} \ \text{ for } j = 1, \dots, k, \\ |\{(x,t) \in Q_2 \mid u_x^*(x,t) = s^{2j}\}| = 0 \ \text{ for } j = 1, \dots, k, \\ s^{2(j-1)} < s^{2j} < s_+(r^{2(j-1)}) < s_+(r^{2j}) \ \text{ for } j = 1, \dots, k, \\ s^{2k*} < s_+(r^{2k}) < s^{2(k+1)*}, \end{cases}$$

choose a point $s^{2(k+1)} \in (s^{2k*}, s_+(r^{2k}))$ (thus $s^{2k*} < s^{2(k+1)} < s^{2(k+1)*}$) so that

$$|\{(x,t) \in Q_2 \mid u_x^*(x,t) = s^{2(k+1)}\}| = 0,$$

and write $r^{2(k+1)} = \sigma^*(s^{2(k+1)})$. Then

$$s^{2k} < s^{2k*} < s^{2(k+1)} < s_+(r^{2k}) < s^{2(k+1)*} = s_+(\sigma^*(s^{2k*})) < s_+(r^{2(k+1)}),$$

that is,

$$s^{2k} < s^{2(k+1)} < s_+(r^{2k}) < s_+(r^{2(k+1)}).$$

Note that $s^{2(k+2)*} = s_+(\sigma^*(s^{2(k+1)*})) > s_+(\sigma^*(s^{2(k+1)})) = s_+(r^{2(k+1)})$, and so

$$s^{2(k+1)*} < s_+(r^{2(k+1)}) < s^{2(k+2)*}.$$

By induction, we have constructed a sequence $\{s^{2k}\}_{k\in\mathbb{N}}$ of reals such that

$$\begin{cases} s_0 = s^{20} < s^{21} < s^{22} < \cdots < s_+(r_2), \quad \lim_{k\to\infty} s^{2k} = s_+(r_2), \\ |\{(x,t) \in Q_2 \mid u_x^*(x,t) = s^{2k}\}| = 0 \ \ \forall k \in \mathbb{N}, \\ s_-(r^{2(k-1)}) < s_-(r^{2k}) < s^{2(k-1)} < s^{2k} < s_+(r^{2(k-1)}) < s_+(r^{2k}) \ \ \forall k \in \mathbb{N}, \end{cases}$$

where $r^{2k} := \sigma^*(s^{2k})$ for each $k \in \mathbb{N}$. Similarly, we can construct a sequence $\{s^{1k}\}_{k\in\mathbb{N}}$ of reals such that

$$\begin{cases} s_0 = s^{10} > s^{11} > s^{12} > \cdots > s_-(r_1), \quad \lim_{k\to\infty} s^{1k} = s_-(r_1), \\ |\{(x,t) \in Q_2 \mid u_x^*(x,t) = s^{1k}\}| = 0 \ \ \forall k \in \mathbb{N}, \\ s_-(r^{1k}) < s_-(r^{1(k-1)}) < s^{1k} < s^{1(k-1)} < s_+(r^{1k}) < s_+(r^{1(k-1)}) \ \ \forall k \in \mathbb{N}, \end{cases}$$

where $r^{1k} := \sigma^*(s^{1k})$ for each $k \in \mathbb{N}$.

Now, we define, for each $k \in \mathbb{N}$,

$$\begin{cases} Q_2^{1k} = \{(x,t) \in Q_2 \mid s^{1k} < u_x^*(x,t) < s^{1(k-1)}\}, \\ Q_2^{2k} = \{(x,t) \in Q_2 \mid s^{2(k-1)} < u_x^*(x,t) < s^{2k}\}. \end{cases}$$

Then, by the above construction, $\{Q_2^{ik} \mid (i,k) \in \{1,2\} \times \mathbb{N}\}$ is a countable collection of pairwise disjoint open subsets of Q_2 whose union has measure $|Q_2|$. Let us write

$$\Lambda = \{(i,k) \in \{1,2\} \times \mathbb{N} \mid Q_2^{ik} \neq \emptyset\}.$$

Since $|Q_2| > 0$, we can guarantee that at least one of the sets Q_2^{ik} is nonempty; that is, $\Lambda \neq \emptyset$.

From the above construction, we check that $\exists (i_0, k_0) \in \Lambda$ such that

$$\partial Q_2^{i_0 k_0} \cap (\Omega \times \{t = 0\}) \neq \emptyset.$$

We only consider the case that $s_0 \leqq g'(x_0) < s_+(r_2)$. (The other case that $s_-(r_1) < g'(x_0) < s_0$ can be handled similarly.) In this case, there is a unique $k_0 \in \mathbb{N}$ such that $s^{2(k_0-1)} \leqq g'(x_0) = u_x^*(x_0, 0) < s^{2k_0}$. If $s^{2(k_0-1)} < g'(x_0) = u_x^*(x_0, 0) < s^{2k_0}$, we can take a small $r_0 > 0$ so that $u_x^*(\Omega_T \cap B_{r_0}(x_0, 0)) \subset (s^{2(k_0-1)}, s^{2k_0})$, and so $\Omega_T \cap B_{r_0}(x_0, 0) \subset Q_2^{2k_0}$, where $B_{r_0}(x_0, 0)$ is the open ball in $\mathbb{R}^2$ with center $(x_0, 0)$ and radius r_0. Thus,

$$\emptyset \neq (\Omega \cap (x_0 - r_0, x_0 + r_0)) \times \{t = 0\} \subset \partial Q_2^{2k_0} \cap (\Omega \times \{t = 0\}).$$

Next, assume $s^{2(k_0-1)} = g'(x_0) = u_x^*(x_0, 0)$. If $\exists r_0 > 0$ such that $u_x^*(\Omega_T \cap B_{r_0}(x_0, 0)) = \{s^{2(k_0-1)}\}$, then

$$|\{(x,t) \in Q_2 \mid u_x^*(x,t) = s^{2(k_0-1)}\}| > 0,$$

which is a contradiction to the above construction. Thus we can choose a sequence $\Omega_T \ni (x_n, t_n) \to (x_0, 0)$ such that either $s^{2(k_0-2)} < u_x^*(x_n, t_n) < s^{2(k_0-1)}$ $\forall n \in \mathbb{N}$ or $s^{2(k_0-1)} < u_x^*(x_n, t_n) < s^{2k_0}$ $\forall n \in \mathbb{N}$, and

$$\text{either } (x_0, 0) \in \partial Q_2^{2(k_0-1)} \text{ or } (x_0, 0) \in \partial Q_2^{2k_0}.$$

(Here, if $k_0 = 1$, then Q_2^{20} should be regarded as Q_2^{11}.)

(Strict subsolution w^*): We define

$$v^*(x,t) = \int_0^x h(z)\,dz + \int_0^t \sigma^*(u_x^*(x,\tau))\,d\tau \quad \forall (x,t) \in \Omega_T.$$

From (2.5), we see that $w^* := (u^*, v^*)$ satisfies

$$v_x^* = u_t^* \quad \text{and} \quad v_t^* = \sigma^*(u_x^*) \quad \text{in } \Omega_T. \tag{2.7}$$

This implies that $v^* \in C^2(\bar{\Omega}_T)$, hence $w^* \in C^2(\bar{\Omega}_T; \mathbb{R}^2)$. From (2.4) and (2.7), it also follows that w^* is a strict subsolution of differential inclusion (2.2) in Q_2,where r_a and r_b replaced by r_1 and r_2, respectively (see Fig. 3), so its phase gauge $\Gamma_{w^*}^{Q_2^{ik}}$ over Q_2^{ik} lies in the interval $(0, 1)$ whenever $(i,k) \in \Lambda$. In particular, the total phase gauge $\Gamma_{w^*}^{Q_2}$ belongs to $(0, 1)$.

Under Hypothesis (A) and the above setup, we are now ready to state the main result of the paper in a detailed fashion.

Theorem 2.1. (Main result: detailed) *For each* $\varepsilon > 0$, *there exist a number* $T_\varepsilon \in (0, T)$ *and infinitely many two-phase weak solutions* $u \in W^{1,\infty}_{u^*}(\Omega_T)$ *to problem* (1.3) *satisfying the following properties:*

(a) *Approximate initial rate of change:*

$$\|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} < \varepsilon,$$

where $\Omega_{T_\varepsilon} := \Omega \times (0, T_\varepsilon)$ *and* $H(x,t) := h(x)$*;*

(b) *Approximate properties:*
 (i) $\|u - u^*\|_{L^\infty(\Omega_T)} < \varepsilon$,
 (ii) $\|u_t\|_{L^\infty(\Omega_T)} < \|u^*_t\|_{L^\infty(\Omega_T)} + \varepsilon$*;*

(c) *Classical part:*
 (i) $u = u^*$ *on* $\overline{Q_1 \cup Q_3}$,
 (ii) $u_x(x,t) \begin{cases} \in (-1, s_-(r_1)) & \forall (x,t) \in Q_1, \\ > s_+(r_2) & \forall (x,t) \in Q_3; \end{cases}$

(d) *Phase separation in* Q_2^{ik}*: if* $(i,k) \in \Lambda$, *then*
 (i) $u_x(x,t) \in \begin{cases} [s_-(r^{1k}), s_-(r^{1(k-1)})] \cup [s_+(r^{1k}), s_+(r^{1(k-1)})] & \text{if } i = 1, \\ [s_-(r^{2(k-1)}), s_-(r^{2k})] \cup [s_+(r^{2(k-1)}), s_+(r^{2k})] & \text{if } i = 2, \end{cases}$ *for a.e.* $(x,t) \in Q_2^{ik}$,
 (ii) $|Q^{ik,u}_{2,+}| > 0$, $|Q^{ik,u}_{2,-}| > 0$ *and* $|Q^{ik,u}_{2,+}| + |Q^{ik,u}_{2,-}| = |Q_2^{ik}|$,
 where

$$Q^{ik,u}_{2,\pm} := \begin{cases} \{(x,t) \in Q_2^{1k} \mid u_x(x,t) \in [s_\pm(r^{1k}), s_\pm(r^{1(k-1)})]\} & \text{if } i = 1, \\ \{(x,t) \in Q_2^{2k} \mid u_x(x,t) \in [s_\pm(r^{2(k-1)}), s_\pm(r^{2k})]\} & \text{if } i = 2; \end{cases}$$

(e) *Total phase separation in* Q_2*:*
 (i) $u_x(x,t) \in (s_-(r_1), s_-(r_2)) \cup (s_+(r_1), s_+(r_2))$, *a.e.* $(x,t) \in Q_2$,
 (ii) $|Q^u_{2,+}| = \gamma_u^{Q_2}|Q_2|$, $|Q^u_{2,-}| = (1 - \gamma_u^{Q_2})|Q_2|$,
 (iii) $|\gamma_u^{Q_2} - \Gamma_{w^*}^{Q_2}| < \varepsilon$,
 where

$$Q^u_{2,\pm} := \{(x,t) \in Q_2 \mid u_x(x,t) \in (s_\pm(r_1), s_\pm(r_2))\};$$

(f) *Borderline:* $u_x(x,t) \in \{s_-(r_1), s_+(r_2)\}$, *a.e.* $(x,t) \in \Omega_T \setminus (\cup_{i=1}^3 Q_i)$.

Here several remarks are in order. In (d), it is important to note that for $(i,k) \in \Lambda$,

$$\begin{aligned} &s_-(r^{1k}) < s_-(r^{1(k-1)}) < s^{1k} < s^{1(k-1)} < s_+(r^{1k}) < s_+(r^{1(k-1)}) \text{ if } i = 1, \\ &s_-(r^{2(k-1)}) < s_-(r^{2k}) < s^{2(k-1)} < s^{2k} < s_+(r^{2(k-1)}) < s_+(r^{2k}) \text{ if } i = 2. \end{aligned}$$

Thus the (essential) ranges of u_x and u^*_x are non-overlapping in Q_2^{ik}. This and (b)(i) are the key facts that allow us to observe microstructures of two-phase weak solutions u if $\varepsilon > 0$ is small enough as carefully examined after the statement of Theorem 1.4.

Note that deformations of the elastic bar corresponding to the solutions u, $d(x,t) = u(x,t) + x$, satisfy

$$d_x(x,t) = u_x(x,t) + 1 \geqq \min_{\bar{\Omega}_T} u^*_x + 1 > -1 + 1 = 0, \quad \text{a.e. } (x,t) \in \Omega_T;$$

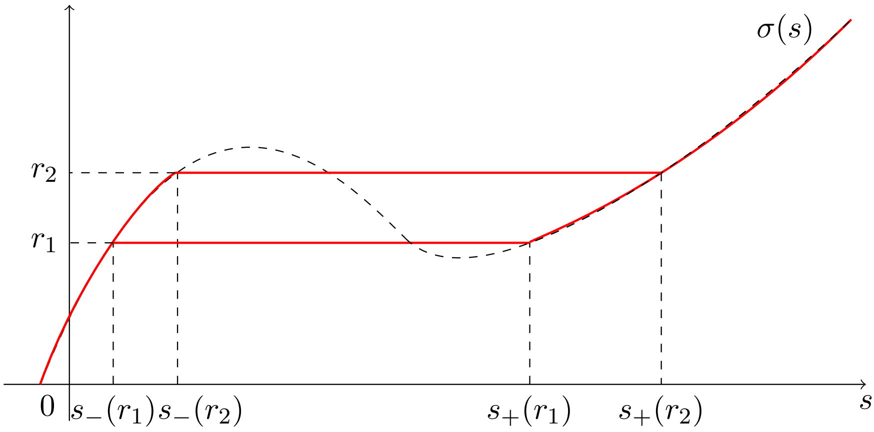

Fig. 4. Strain–stress hysteresis loop

this guarantees that for a.e. $t \in (0, T)$, the deformations $d : [0, 1] \times \{t\} \to [0, 1]$ are strictly increasing with $d(0, t) = 0$ and $d(1, t) = 1$. Moreover, for such a $t \in (0, T)$, $d(x, t)$ are smooth (as much as the initial displacement g) for the values of $x \in [0, 1]$ at which slope $d_x(x, t) \in (0, s_-(r_1) + 1) \cup (s_+(r_2) + 1, \infty)$ and Lipschitz a.e. on $[0, 1]$. In particular, these dynamic deformations fulfill a natural physical requirement of invertibility for the motion of an elastic bar not allowing interpenetration.

By (c), in Q_1, the solutions u are identical to u^* and thus a hyperbolic evolution in the α-phase below the threshold $s_-(r_1)$. Likewise, in Q_3, the u are equal to u^* and in the β-phase above $s_+(r_2)$. According to (c), (e) and (f), the phase separation of u into the two stable phases only occurs in $Q_2 \neq \emptyset$ with the proportion $\gamma_u^{Q_2}$, which can be arbitrarily close to the number $\Gamma_{w^*}^{Q_2} \in (0, 1)$. If $0 < \varepsilon < \min\{\Gamma_{w^*}^{Q_2}, 1 - \Gamma_{w^*}^{Q_2}\}$, we have $0 < \gamma_u^{Q_2} < 1$; that is, the phase of u in Q_2 is indeed separated into the α-phase on $(s_-(r_1), s_-(r_2))$ and the β-phase on $(s_+(r_1), s_+(r_2))$. Thus there is a formation of fine-scale *strain–stress hysteresis loop* in Q_2 (see Fig. 4).

2.4. Some Corollaries

In this subsection, we present some interesting byproducts of the main result, Theorem 2.1, along with their proofs, respectively.

For the first corollary, we define some terminologies. Let $C_0(\mathbb{R})$ denote the closure of the space $C_c(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support by the uniform norm. By the Riesz Representation Theorem, the dual of $C_0(\mathbb{R})$ can be identified with the space $\mathcal{M}(\mathbb{R})$ of signed Radon measures in $\mathbb{R}$ with finite mass via the pairing

$$\langle \mu, f \rangle = \int_{\mathbb{R}} f \, d\mu \quad (\mu \in \mathcal{M}(\mathbb{R}),\ f \in C_0(\mathbb{R})).$$

Let $U \subset \mathbb{R}^2$ be a bounded open set. A map $\mu = \mu_{(x,t)} : U \to \mathcal{M}(\mathbb{R})$ is called *weakly-⋆ measurable* if the function $(x,t) \mapsto \langle \mu_{(x,t)}, f \rangle$ is measurable for each $f \in C_0(\mathbb{R})$. We denote by $L^\infty_\star(U; \mathcal{M}(\mathbb{R}))$ the space of weakly-⋆ measurable maps $\mu : U \to \mathcal{M}(\mathbb{R})$ that are essentially bounded. Then $L^\infty_\star(U; \mathcal{M}(\mathbb{R}))$ is the dual of the separable Banach space $L^1(U; C_0(\mathbb{R}))$ via the pairing

$$\langle \mu, f \rangle = \int_U \langle \mu_{(x,t)}, f_{(x,t)} \rangle \, \mathrm{d}x\mathrm{d}t \quad (\mu \in L^\infty_\star(U; \mathcal{M}(\mathbb{R})),\ f \in L^1(U; C_0(\mathbb{R}));$$

for a proof of this, see, example, [20, p. 588]. We say that a sequence $\{\mu_j\}_{j\in\mathbb{N}}$ in $L^\infty_\star(U; \mathcal{M}(\mathbb{R}))$ *converges weakly-⋆* to a map $\mu \in L^\infty_\star(U; \mathcal{M}(\mathbb{R}))$ and write

$$\mu_j \overset{\star}{\rightharpoonup} \mu \quad \text{in } L^\infty_\star(U; \mathcal{M}(\mathbb{R})),$$

provided that for each $f \in L^1(U; C_0(\mathbb{R}))$, $\langle \mu_j, f \rangle \to \langle \mu, f \rangle$ in $\mathbb{R}$.

Motivated by [37, p. 43], we introduce the following definition:

Definition 2.2. A map $\nu \in L^\infty_\star(U; \mathcal{M}(\mathbb{R}))$ is called a $W^{1,\infty}$ *spatial-derivative Young measure* in $\mathbb{R}$ if there exist a sequence $u_j \in W^{1,\infty}(U)$ and a function $u \in W^{1,\infty}(U)$ such that

$$\begin{cases} u_j \overset{\star}{\rightharpoonup} u \ \text{ and } \ (u_j)_x \overset{\star}{\rightharpoonup} u_x & \text{in } L^\infty(U), \\ \delta_{(u_j)_x(\cdot)} \overset{\star}{\rightharpoonup} \nu & \text{in } L^\infty_\star(U; \mathcal{M}(\mathbb{R})). \end{cases}$$

In this case, ν is said to be *generated* by the sequence of spatial derivatives $(u_j)_x$.

We now state and prove the existence of a measure-valued solution (u^*, ν) to problem (1.3) with ν generated by a sequence of two-phase weak solutions u_j to (1.3) that develops finer and finer microstructures, where u^* is as in Theorem 2.1.

Corollary 2.3. (Measure-valued solution) *Assume all of the hypotheses of Theorem 2.1. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be any sequence of positive reals such that $\varepsilon_j \to 0$. For each $j \in \mathbb{N}$, let $u_j \in W^{1,\infty}_{u^*}(\Omega_T)$ be a two-phase weak solution to (1.3) corresponding to $\varepsilon = \varepsilon_j$ in Theorem 2.1. Then after passing to a subsequence if necessary, one has*

$$\begin{cases} u_j \to u^* & \text{in } L^\infty(\Omega_T), \\ \nabla u_j \overset{\star}{\rightharpoonup} \nabla u^* & \text{in } L^\infty(\Omega_T; \mathbb{R}^2), \\ \delta_{(u_j)_x(\cdot)} \overset{\star}{\rightharpoonup} \nu & \text{in } L^\infty_\star(\Omega_T; \mathcal{M}(\mathbb{R})) \end{cases} \tag{2.8}$$

for some map $\nu \in L^\infty_\star(\Omega_T; \mathcal{M}(\mathbb{R}))$ with the property that there exists a compact interval $J \subset (-1, \infty)$ such that for a.e. $(x,t) \in \Omega_T$, $\nu_{(x,t)}$ is a probability measure in $\mathbb{R}$ that is supported in J. Moreover, (u^*, ν) is a measure-valued solution to (1.3) that is not a weak solution itself. In particular, ν is a $W^{1,\infty}$ spatial-derivative Young measure in $\mathbb{R}$, generated by the (sub)sequence $\{(u_j)_x\}_{j\in\mathbb{N}}$.

Proof. Note from Theorem 2.1 that there exists a compact set $D \subset (-1, \infty) \times \mathbb{R}$ such that for all $j \in \mathbb{N}$, $\nabla u_j \in D$ a.e. in Ω_T. In particular, for some compact interval $J \subset (-1, \infty)$, for all $j \in \mathbb{N}$, $(u_j)_x \in J$ a.e. in Ω_T. Appealing to (b)(i) of Theorem 2.1 and to the Banach-Alaoglu Theorem, after passing to a subsequence if necessary, we obtain (2.8) for some map $\nu \in L^\infty_\star(\Omega_T; \mathcal{M}(\mathbb{R}))$ such that for a.e. $(x,t) \in \Omega_T$, $\nu_{(x,t)}$ is a probability measure in $\mathbb{R}$ that is supported in J. Thus ν is a $W^{1,\infty}$ spatial-derivative Young measure in $\mathbb{R}$, generated by the sequence $\{(u_j)_x\}_{j\in\mathbb{N}}$.

It remains to check that (u^*, ν) is a measure-valued solution to (1.3) that is not a weak solution. To do so, let $\varphi \in C_c^\infty(\Omega \times [0,T))$. Choose a function $\bar{\sigma} \in C_c(\mathbb{R})$ so that $\bar{\sigma} = \sigma$ on the interval $J \subset (-1,\infty)$. Let $f : \Omega_T \to C_0(\mathbb{R})$ be the map given by

$$f_{(x,t)}(s) = \bar{\sigma}(s)\varphi_x(x,t) \quad \text{for } (x,t) \in \Omega_T, s \in \mathbb{R}.$$

Then, clearly, $f \in L^1(\Omega_T; C_0(\mathbb{R}))$. Thus the weak-$\star$ convergence of $\delta_{(u_j)_x(\cdot)}$ to ν implies that as $j \to \infty$,

$$\int_{\Omega_T} \sigma((u_j)_x)\varphi_x \,\mathrm{d}x\mathrm{d}t = \langle \delta_{(u_j)_x(\cdot)}, f\rangle \to \langle \nu, f\rangle = \int_{\Omega_T} \langle \nu, \sigma\rangle \varphi_x \,\mathrm{d}x\mathrm{d}t.$$

Since $(u_j)_t$ converges weakly-$\star$ to u_t^*, we also have

$$\int_{\Omega_T} (u_j)_t\varphi_t \,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} u_t^*\varphi_t \,\mathrm{d}x\mathrm{d}t.$$

On the other hand, since u_j is a weak solution to (1.3), we have

$$\int_{\Omega_T} ((u_j)_t\varphi_t - \sigma((u_j)_x)\varphi_x) \,\mathrm{d}x\mathrm{d}t = -\int_0^1 h(x)\varphi(x,0)\,\mathrm{d}x.$$

Thus passing to the limit as $j \to \infty$, we obtain

$$\int_{\Omega_T} (u_t^*\varphi_t - \langle \nu, \sigma\rangle\varphi_x) \,\mathrm{d}x\mathrm{d}t = -\int_0^1 h(x)\varphi(x,0)\,\mathrm{d}x.$$

Since u^* satisfies (2.5), it now follows that (u^*, ν) is a measure-valued solution to (1.3). Also, note from Theorem 2.1 that for each $(i,k) \in \Lambda \neq \emptyset$,

$$(u_j)_x \not\to u_x^* \ \text{ in measure in } Q_2^{ik},$$

so it is easily checked that $\nu_{(x,t)} \neq \delta_{u_x^*(x,t)}$ in a set of positive measure in Q_2^{ik}. Therefore, (u^*, ν) is not a weak solution to (1.3). □

Secondly, we prove the local existence of weak solutions to problem (1.3) for all smooth initial data.

Corollary 2.4. (Existence) *For any initial datum* $(g,h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)$ *with* $g' > -1$ *on* $\bar{\Omega}$, *there exists a finite number* $T > 0$ *for which problem* (1.3) *has a weak solution.*

Proof. Let $(g,h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)$ be any given initial datum such that $g' > -1$ on $\bar{\Omega}$. If $g'(x_0) \in (s_1^*, s_2^*)$ for some $x_0 \in \Omega$, then the result follows immediately from Theorem 2.1.

Next, let us assume $g'(x) \notin (s_1^*, s_2^*)$ for all $x \in \bar{\Omega}$. We may only consider the case that $g'(x) \geqq s_2^*$ for all $x \in \bar{\Omega}$ as the other case can be shown similarly. Fix any two $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$, and choose a function $\sigma^* \in C^3(-1,\infty)$ in such a way that (2.4) is fulfilled. By [11, Theorem 5.2], there exists a finite number $\tilde{T} > 0$ such that the modified initial-boundary value problem (2.5), with T replaced by $\tilde{T}$, admits a unique solution $u^* \in \cap_{k=0}^3 C^k([0,\tilde{T}]; W_0^{3-k,2}(\Omega))$ with $u_x^* > -1$ on $\bar{\Omega}_{\tilde{T}}$. Now, choose a number $0 < T \leqq \tilde{T}$ so that $u_x^* \geqq s_+(r_2)$ on $\bar{\Omega}_T$. Then u^* itself is a classical and thus weak solution to problem (1.3). □

Lastly, we address the existence of local two-phase weak solutions to (1.3) that are all identical and smooth for a short period of time and then exhibit phase separation as well as microstructures for some smooth initial data.

Corollary 2.5. (Microstructures after a finite time) *Let $(g,h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)$ satisfy $g' > -1$ on $\bar{\Omega}$. Assume either* $\max_{\bar{\Omega}} g' \in (s_1^*, s_1)$ *or* $\min_{\bar{\Omega}} g' \in (s_2, s_2^*)$. *Then there exist finite numbers $T > T' > 0$ such that problem* (1.3) *admits infinitely many two-phase weak solutions that are all equal to some $u^* \in \cap_{k=0}^3 C^k([0,T']; W_0^{3-k,2}(\Omega))$ in $\Omega_{T'}$ and exhibit phase separation and microstructures from $t = T'$ as in Theorem* 2.1.

Proof. Let $(g,h) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)$ satisfy $\max_{\bar{\Omega}} g' \in (s_1^*, s_1)$ or $\min_{\bar{\Omega}} g' \in (s_2, s_2^*)$. Assume also that $g' > -1$ on $\bar{\Omega}$. We may only consider the case that $M := \max_{\bar{\Omega}} g' \in (s_1^*, s_1)$ as the other case can be handled in a similar way.

Fix any two numbers $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$ so that $s_-(r_1) > M$. Then take a function $\sigma^*(s) \in C^3(-1,\infty)$ satisfying (2.4). Using [11, Theorem 5.2], we can find a finite number $\tilde{T} > 0$ such that the modified problem (2.5), with T replaced by $\tilde{T}$, has a unique solution $u^* \in \cap_{k=0}^3 C^k([0,\tilde{T}]; W_0^{3-k,2}(\Omega))$ with $u_x^* > -1$ on $\bar{\Omega}_{\tilde{T}}$. Then choose a number $0 < T' \leqq \tilde{T}$ so small that $u_x^* \leqq s_-(r_1)$ on $\bar{\Omega}_{T'}$ and that $s_1^* < u_x^*(x_0, T')$ for some $x_0 \in \Omega$. With the initial datum $(u^*(\cdot,T'), u_t^*(\cdot,T')) \in W_0^{3,2}(\Omega) \times W_0^{2,2}(\Omega)$ at $t = T'$ such that $u_x^*(\cdot,T') > -1$ on $\bar{\Omega}$, we can apply Theorem 2.1 to obtain, for some finite number $T > T'$, infinitely many two-phase weak solutions $\tilde{u} \in W^{1,\infty}(\Omega \times (T',T))$ to the initial-boundary value problem

$$\begin{cases} \tilde{u}_{tt} = \sigma(\tilde{u}_x)_x & \text{in } \Omega \times (T',T), \\ \tilde{u}(0,t) = \tilde{u}(1,t) = 0 & \text{for } t \in (T',T), \\ \tilde{u} = u^*,\ \tilde{u}_t = u_t^* & \text{on } \Omega \times \{t = T'\} \end{cases}$$

satisfying the stated properties in the theorem. Then the glued functions $u = u^*\chi_{\Omega\times(0,T')} + \tilde{u}\chi_{\Omega\times[T',T)}$ are two-phase weak solutions to problem (1.3) fulfilling the required properties. □

3. Proof of the Main Theorem

In this section, we prove the main result, Theorem 2.1, with the help of the density result, Theorem 3.1, to be verified in Sections 4 and 5.

To start the proof, fix any $\varepsilon > 0$. For clarity, we divide the proof into several steps.

(Related matrix sets): Following notation (2.1), let

$$K_\beta = \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \,\middle|\, (s,r) \in \tilde{K}^{r_1,r_2},\ |b| \leqq \beta \right\},$$

and for each $k \in \mathbb{N}$, define the sets

$$K^{1k}_\beta = \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \,\middle|\, (s,r) \in \tilde{K}^{r^{1k},r^{1(k-1)}},\ |b| \leqq \beta \right\},$$
$$K^{2k}_\beta = \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \,\middle|\, (s,r) \in \tilde{K}^{r^{2(k-1)},r^{2k}},\ |b| \leqq \beta \right\},$$
$$U^{1k}_\beta = \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \,\middle|\, (s,r) \in \tilde{U}^{r^{1k},r^{1(k-1)}},\ |b| < \beta \right\},$$
$$U^{2k}_\beta = \left\{ \begin{pmatrix} s & b \\ b & r \end{pmatrix} \in \mathbb{M}^{2\times 2}_{sym} \,\middle|\, (s,r) \in \tilde{U}^{r^{2(k-1)},r^{2k}},\ |b| < \beta \right\},$$

where $\varepsilon' := \varepsilon/2$ and $\beta := \|u^*_t\|_{L^\infty(\Omega_T)} + \varepsilon'$ (irrelevant to the term "β-phase").

(Admissible class): We can choose a number $T_\varepsilon \in (0,T)$ so that

$$\|u^*_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} < \varepsilon'.$$

For each $(i,k) \in \Lambda$, let

$$\varepsilon^{ik}_* = \min\left\{ \frac{1}{2} \min\{\Gamma^{Q^{ik}_2}_{w^*}, 1 - \Gamma^{Q^{ik}_2}_{w^*}\}, \varepsilon' \right\} > 0.$$

The *admissible class* $\mathcal{A}$ is defined to be the set of all functions $w = (u,v) \in W^{1,\infty}_{w^*}(\Omega_T;\mathbb{R}^2) \cap C^2(\bar{\Omega}_T;\mathbb{R}^2)$ satisfying the following:

$$\begin{cases} \text{there exists a } \textit{finite} \text{ set } \Lambda_w \subset \Lambda \text{ such that } w = w^* \text{ in } \Omega_T \setminus (\cup_{(i,k)\in\Lambda_w} \bar{Q}^{ik}_w) \\ \quad \text{for some open sets } Q^{ik}_w \subset\subset Q^{ik}_2 \text{ with } (i,k) \in \Lambda_w \text{ and } |\partial Q^{ik}_w| = 0, \\ \nabla w(x,t) \in U^{ik}_\beta \ \ \forall (x,t) \in Q^{ik}_2, \forall (i,k) \in \Lambda, \\ |\Gamma^{Q^{ik}_2}_w - \Gamma^{Q^{ik}_2}_{w^*}| < \varepsilon^{ik}_* \ \ \forall (i,k) \in \Lambda, \\ \|u - u^*\|_{L^\infty(\Omega_T)} < \varepsilon',\ \|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} < \varepsilon'. \end{cases}$$

It is then easy to see from the definition of w^* that $w^* \in \mathcal{A} \neq \emptyset$. For each $\delta > 0$, we also define the δ-*subclass* $\mathcal{A}_\delta$ by

$$\mathcal{A}_\delta = \left\{ w \in \mathcal{A} \,\middle|\, \int_{Q_2} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \leqq \delta |Q_2| \right\}.$$

(Density result): One crucial step for the proof of Theorem 2.1 is the following density result whose proof appears in Section 5.

Theorem 3.1. *For each $\delta > 0$,*

$$\mathcal{A}_\delta \text{ is dense in } \mathcal{A} \text{ with respect to the } L^\infty(\Omega_T;\mathbb{R}^2)\text{-norm.}$$

(Baire's category method): Let $\mathcal{X}$ denote the closure of $\mathcal{A}$ in the space $L^\infty(\Omega_T;\mathbb{R}^2)$, so that $(\mathcal{X}, L^\infty)$ is a nonempty complete metric space. As the U_β^{ik} are uniformly bounded in $\mathbb{M}^{2\times 2}_{sym}$, $\mathcal{A}$ is also bounded in $W^{1,\infty}_{w^*}(\Omega_T;\mathbb{R}^2)$; thus it is easily checked that

$$\mathcal{X} \subset W^{1,\infty}_{w^*}(\Omega_T;\mathbb{R}^2).$$

Note that the space-time gradient operator $\nabla : \mathcal{X} \to L^1(\Omega_T;\mathbb{M}^{2\times 2})$ is a Baire-one map (see, example, [9, Proposition 10.17]), so by the Baire Category Theorem (see, example, [9, Theorem 10.15]), the set of points of discontinuity of the operator ∇, say $\mathcal{D}_\nabla$, is a set of the first category; thus the set of points at which ∇ is continuous, that is, $\mathcal{C}_\nabla := \mathcal{X}\setminus\mathcal{D}_\nabla$, is dense in $\mathcal{X}$.

(Completion of proof): Let us now confirm that for any function $w = (u, v) \in \mathcal{C}_\nabla$, its first component u is a two-phase weak solution to problem (1.3) satisfying (a)–(f). To this end, fix any $w = (u, v) \in \mathcal{C}_\nabla$.

(1.4) & (1.5): To verify (1.4), let $\varphi \in C_c^\infty(\Omega \times [0, T))$. From Theorem 3.1 and the density of $\mathcal{A}$ in $\mathcal{X}$, we can choose a sequence $w_j = (u_j, v_j) \in \mathcal{A}_{1/j}$ such that $w_j \to w$ in $\mathcal{X}$ as $j \to \infty$. As $w \in \mathcal{C}_\nabla$, we have $\nabla w_j \to \nabla w$ in $L^1(\Omega_T;\mathbb{M}^{2\times 2})$ and so pointwise a.e. in Ω_T after passing to a subsequence if necessary. By (2.7) and the definition of $\mathcal{A}$, we have $(v_j)_x = (u_j)_t$ in Ω_T and $(v_j)_x(x, 0) = v^*_x(x, 0) = u^*_t(x, 0) = h(x)$ $(x \in \Omega)$, so from the choice of the test function φ,

$$\begin{aligned}\int_{\Omega_T} (u_j)_t\varphi_t \,\mathrm{d}x\mathrm{d}t &= \int_{\Omega_T} (v_j)_x\varphi_t \,\mathrm{d}x\mathrm{d}t \\ &= -\int_{\Omega_T} (v_j)_{xt}\varphi \,\mathrm{d}x\mathrm{d}t - \int_0^1 (v_j)_x(x, 0)\varphi(x, 0)\,\mathrm{d}x \\ &= \int_{\Omega_T} (v_j)_t\varphi_x \,\mathrm{d}x\mathrm{d}t - \int_0^1 h(x)\varphi(x, 0)\,\mathrm{d}x,\end{aligned}$$

that is,

$$\int_{\Omega_T} ((u_j)_t\varphi_t - (v_j)_t\varphi_x)\,\mathrm{d}x\mathrm{d}t = -\int_0^1 h(x)\varphi(x, 0)\,\mathrm{d}x.$$

On the other hand, by the Dominated Convergence Theorem, we have

$$\int_{\Omega_T} ((u_j)_t\varphi_t - (v_j)_t\varphi_x)\,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} (u_t\varphi_t - v_t\varphi_x)\,\mathrm{d}x\mathrm{d}t,$$

thus

$$\int_{\Omega_T} (u_t\varphi_t - v_t\varphi_x)\,\mathrm{d}x\mathrm{d}t = -\int_0^1 h(x)\varphi(x, 0)\,\mathrm{d}x. \tag{3.1}$$

Also, by the Dominated Convergence Theorem,

$$\int_{Q_2} \operatorname{dist}(\nabla w_j(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \to \int_{Q_2} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t.$$

From the membership $w_j \in \mathcal{A}_{1/j}$, we have

$$\int_{Q_2} \operatorname{dist}(\nabla w_j(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \leqq \frac{|Q_2|}{j} \to 0,$$

so

$$\int_{Q_2} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t = 0.$$

Since K_β is closed, we must have

$$\nabla w(x,t) \in K_\beta, \quad \text{a.e. } (x,t) \in Q_2. \tag{3.2}$$

More specifically, if $(i,k) \in \Lambda$, then $\nabla w_j \in U_\beta^{ik}$ in Q_2^{ik} for each $j \in \mathbb{N}$, so that

$$\nabla w(x,t) \in K_\beta^{ik} \subset K_\beta, \quad \text{a.e. } (x,t) \in Q_2^{ik}. \tag{3.3}$$

From the membership $w_j \in \mathcal{A}_{1/j} \subset \mathcal{A}$,

$$\begin{cases} \text{there exists a finite set } \Lambda_{w_j} \subset \Lambda \text{ such that } w_j = w^* \text{ in } \Omega_T \setminus (\cup_{(i,k)\in\Lambda_{w_j}} \bar{Q}_{w_j}^{ik}) \\ \text{for some open sets } Q_{w_j}^{ik} \subset\subset Q_2^{ik} \text{ with } (i,k) \in \Lambda_{w_j} \text{ and } |\partial Q_{w_j}^{ik}| = 0, \end{cases} \tag{3.4}$$

and so $\nabla w_j = \nabla w^*$ in $\Omega_T \setminus (\cup_{(i,k)\in\Lambda_{w_j}} \bar{Q}_{w_j}^{ik})$; thus $\nabla w = \nabla w^*$ a.e. in $\Omega_T \setminus Q_2$. By (2.4), (2.7) and the definition of Q_2, we now have

$$v_t = \sigma^*(u_x^*) = \sigma(u_x) \quad \text{a.e. in } \Omega_T \setminus Q_2.$$

This together with (3.2) implies that $v_t = \sigma(u_x)$ a.e. in Ω_T. Reflecting this to (3.1), we have (1.4). With (2.5) and $w = w^*$ on $\partial\Omega_T$, we also have (1.5).

(a), (b), (c), (d), (e) & (f): From the membership $w_j \in \mathcal{A}_{1/j} \subset \mathcal{A}$, we have (3.4) and the following:

$$\begin{cases} \nabla w_j(x,t) \in U_\beta^{ik} \;\; \forall (x,t) \in Q_2^{ik}, \forall (i,k) \in \Lambda, \\ |\Gamma_{w_j}^{Q_2^{ik}} - \Gamma_{w^*}^{Q_2^{ik}}| < \varepsilon_*^{ik} \;\; \forall (i,k) \in \Lambda, \\ \|u_j - u^*\|_{L^\infty(\Omega_T)} < \varepsilon', \;\; \|(u_j)_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} < \varepsilon'. \end{cases}$$

Let $j \to \infty$. It then follows that

$$\begin{cases} w = w^* \text{ and } \nabla w = \nabla w^* \text{ a.e. in } \Omega_T \setminus Q_2, \\ \|u_t\|_{L^\infty(Q)} \leqq \beta = \|u_t^*\|_{L^\infty(\Omega_T)} + \varepsilon', \\ |\Gamma_w^{Q_2^{ik}} - \Gamma_{w^*}^{Q_2^{ik}}| \leqq \varepsilon_*^{ik} \;\; \forall (i,k) \in \Lambda, \\ \|u - u^*\|_{L^\infty(\Omega_T)} \leqq \varepsilon', \|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} \leqq \varepsilon'. \end{cases} \tag{3.5}$$

Thus, with $\varepsilon' = \varepsilon/2 < \varepsilon$, we see that (a), (b) and (c)(i) are satisfied. Since $u_x^* > -1$ on $\bar{\Omega}_T$, it follows from (3.5) and the definition of Q_1 and Q_3 that (c)(ii) holds. Also,

(f) follows from (3.5) and the definition of Q_1, Q_2 and Q_3. Note that (d)(i) is an immediate consequence of (3.3) from which (e)(i) follows. By the definition of the phase gauge, (e)(ii) always holds. For each $(i,k) \in \Lambda$, it follows from the definition of ε_*^{ik} and (3.5) that

$$0 < \Gamma_w^{Q_2^{ik}} < 1,$$

thus (d)(ii) holds. Finally, observe from (3.5) that

$$\begin{aligned} |\gamma_u^{Q_2} - \Gamma_{w^*}^{Q_2}| = |\Gamma_w^{Q_2} - \Gamma_{w^*}^{Q_2}| &= \left| \sum_{(i,k)\in\Lambda} \frac{|Q_2^{ik}|}{|Q_2|} \left(\Gamma_w^{Q_2^{ik}} - \Gamma_{w^*}^{Q_2^{ik}} \right) \right| \\ &\leqq \sum_{(i,k)\in\Lambda} \frac{|Q_2^{ik}|}{|Q_2|} |\Gamma_w^{Q_2^{ik}} - \Gamma_{w^*}^{Q_2^{ik}}| \leqq \sum_{(i,k)\in\Lambda} \frac{|Q_2^{ik}|}{|Q_2|} \varepsilon_*^{ik} \leqq \varepsilon' = \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

thus (e)(iii) is also true.

Infinitely many solutions: Having shown that the first component u of each pair $w = (u, v)$ in $\mathcal{C}_\nabla$ is a two-phase weak solution to problem (1.3) satisfying (a)–(f), it remains to show that $\mathcal{C}_\nabla$ has infinitely many elements and that no two different pairs in $\mathcal{C}_\nabla$ have the first components that are equal.

Suppose on the contrary that $\mathcal{C}_\nabla$ has finitely many elements. Then $w^* \in \mathcal{A} \subset \mathcal{X} = \bar{\mathcal{C}}_\nabla = \mathcal{C}_\nabla$, and so u^* itself is a weak solution to (1.3) satisfying (a)–(f); clearly, this is a contradiction. Thus $\mathcal{C}_\nabla$ has infinitely many elements. Next, we check that for any two $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in \mathcal{C}_\nabla$,

$$u_1 = u_2 \quad \Leftrightarrow \quad v_1 = v_2.$$

Suppose that $u_1 = u_2$ in Ω_T. As $(v_i)_x = (u_i)_t$ a.e. in Ω_T $(i = 1, 2)$, we have

$$(v_1)_x = (u_1)_t = (u_2)_t = (v_2)_x \ \text{ a.e. in } \Omega_T.$$

Since both v_1 and v_2 share the same trace v^* on $\partial\Omega_T$, it follows that $v_1 = v_2$ in Ω_T. The converse can be shown similarly. We can now conclude that there are infinitely many weak solutions to (1.3) satisfying (a)–(f).

The proof of Theorem 2.1 is now complete under the density result, Theorem 3.1, to be proved through Sections 4 and 5.

4. Rank-One Smooth Approximation Under Linear Constraint

In this section, we prepare the main tool, Theorem 4.1, for proving the density result, Theorem 3.1. Instead of presenting a special case that would be enough for our purpose, we present the following result in a generalized and refined form of [43, Lemma 2.1] that may be of independent interest (cf. [38, Lemma 6.2]):

Theorem 4.1. *Let $m, n \geqq 2$ be integers, and let $A, B \in \mathbb{M}^{m\times n}$ be such that* $\operatorname{rank}(A-B) = 1$*, hence*

$$A - B = a \otimes b = (a_i b_j)$$

for some non-zero vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ with $|b| = 1$. Let $L \in \mathbb{M}^{m\times n}$ satisfy

$$Lb \neq 0 \ \ in \ \ \mathbb{R}^m, \tag{4.1}$$

and let $\mathcal{L} : \mathbb{M}^{m\times n} \to \mathbb{R}$ be the linear map defined by

$$\mathcal{L}(\xi) = \sum_{1\leqq i\leqq m,\, 1\leqq j\leqq n} L_{ij}\xi_{ij} \quad \forall \xi \in \mathbb{M}^{m\times n}.$$

Assume $\mathcal{L}(A) = \mathcal{L}(B)$ and $0 < \lambda < 1$ is any fixed number. Then there exists a linear partial differential operator $\Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)$ satisfying the following properties:

(1) For any open set $U \subset \mathbb{R}^n$,

$$\Phi v \in C^{k-1}(U; \mathbb{R}^m) \ \ whenever \ \ k \in \mathbb{N} \ \ and \ \ v \in C^k(U; \mathbb{R}^m)$$

and

$$\mathcal{L}(\nabla\Phi v) = 0 \ \ in \ \ U \ \ \forall v \in C^2(U; \mathbb{R}^m);$$

(2) Let $U \subset \mathbb{R}^n$ be any bounded domain. For each $\tau > 0$, there exist a function $g = g_\tau \in C_c^\infty(U; \mathbb{R}^m)$ and two disjoint open sets $U_A, U_B \subset\subset U$ such that

(a) $\Phi g \in C_c^\infty(U; \mathbb{R}^m)$,
(b) $\operatorname{dist}(\nabla\Phi g, [-\lambda(A-B), (1-\lambda)(A-B)]) < \tau$ *in* U,
(c) $\nabla\Phi g(x) = \begin{cases} (1-\lambda)(A-B) & \forall x \in U_A, \\ -\lambda(A-B) & \forall x \in U_B, \end{cases}$
(d) $||U_A| - \lambda|U|| < \tau$, $||U_B| - (1-\lambda)|U|| < \tau$,
(e) $\|\Phi g\|_{L^\infty(U)} < \tau$,

where $[-\lambda(A-B), (1-\lambda)(A-B)]$ is the closed line segment in $\ker\mathcal{L} \subset \mathbb{M}^{m\times n}$ *joining $-\lambda(A-B)$ and $(1-\lambda)(A-B)$.*

Proof. We mainly follow and modify the proof of [43, Lemma 2.1] which is divided into three cases.

Set $r = \operatorname{rank}(L)$. By (4.1), we have $1 \leqq r \leqq m \wedge n =: \min\{m, n\}$.

(Case 1): Assume that the matrix L satisfies

$$L_{ij} = 0 \ \text{ for all } 1 \leqq i \leqq m,\ 1 \leqq j \leqq n \text{ but possibly the pairs}$$
$$(1,1), (1,2), \ldots, (1,n), (2,2), \ldots, (r,r) \text{ of } (i,j),$$

hence L is of the form

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1r} & \cdots & L_{1n} \\ & L_{22} & & & & \\ & & \ddots & & & \\ & & & L_{rr} & & \\ & & & & & \end{pmatrix} \in \mathbb{M}^{m\times n} \tag{4.2}$$

and that

$$A - B = a \otimes e_1 \text{ for some nonzero vector } a = (a_1, \dots, a_m) \in \mathbb{R}^m,$$

where each blank component in (4.2) is zero. From (4.1) and $\operatorname{rank}(L) = r$, it follows that the product $L_{11} \cdots L_{rr} \neq 0$. Since $0 = \mathcal{L}(A - B) = \mathcal{L}(a \otimes e_1) = L_{11}a_1$, we have $a_1 = 0$.

In this case, the linear map $\mathcal{L} : \mathbb{M}^{m\times n} \to \mathbb{R}$ is given by

$$\mathcal{L}(\xi) = \sum_{j=1}^{n} L_{1j}\xi_{1j} + \sum_{i=2}^{r} L_{ii}\xi_{ii}, \quad \xi \in \mathbb{M}^{m\times n}.$$

We will find a linear differential operator $\Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\mathcal{L}(\nabla\Phi v) \equiv 0 \quad \forall v \in C^2(\mathbb{R}^n; \mathbb{R}^m). \tag{4.3}$$

Thus our candidate for such a $\Phi = (\Phi^1, \dots, \Phi^m)$ is of the form

$$\Phi^i v = \sum_{1 \leqq k \leqq m,\, 1 \leqq l \leqq n} a^i_{kl} v^k_{x_l}, \tag{4.4}$$

where $1 \leqq i \leqq m$, $v \in C^1(\mathbb{R}^n; \mathbb{R}^m)$, and a^i_{kl}'s are real constants to be determined; then for $v \in C^2(\mathbb{R}^n; \mathbb{R}^m)$, $1 \leqq i \leqq m$, and $1 \leqq j \leqq n$,

$$\partial_{x_j}\Phi^i v = \sum_{1 \leqq k \leqq m,\, 1 \leqq l \leqq n} a^i_{kl} v^k_{x_l x_j}.$$

Rewriting (4.3) with this form of $\nabla\Phi v$ for $v \in C^2(\mathbb{R}^n; \mathbb{R}^m)$, we have

$$\begin{aligned}
0 &\equiv \sum_{1 \leqq k \leqq m,\, 1 \leqq j,l \leqq n} L_{1j} a^1_{kl} v^k_{x_l x_j} + \sum_{i=2}^{r} \sum_{1 \leqq k \leqq m,\, 1 \leqq l \leqq n} L_{ii} a^i_{kl} v^k_{x_l x_i} \\
&= \sum_{k=1}^{m} \Big(L_{11} a^1_{k1} v^k_{x_1 x_1} + \sum_{j=2}^{r} (L_{1j} a^1_{kj} + L_{jj} a^j_{kj}) v^k_{x_j x_j} + \sum_{j=r+1}^{n} L_{1j} a^1_{kj} v^k_{x_j x_j} \\
&\quad + \sum_{l=2}^{r} (L_{11} a^1_{kl} + L_{1l} a^1_{k1} + L_{ll} a^l_{k1}) v^k_{x_l x_1} + \sum_{l=r+1}^{n} (L_{11} a^1_{kl} + L_{1l} a^1_{k1}) v^k_{x_l x_1} \\
&\quad + \sum_{2 \leqq j < l \leqq r} (L_{1j} a^1_{kl} + L_{1l} a^1_{kj} + L_{jj} a^j_{kl} + L_{ll} a^l_{kj}) v^k_{x_l x_j} \\
&\quad + \sum_{2 \leqq j \leqq r,\, r+1 \leqq l \leqq n} (L_{1j} a^1_{kl} + L_{1l} a^1_{kj} + L_{jj} a^j_{kl}) v^k_{x_l x_j} \\
&\quad + \sum_{r+1 \leqq j < l \leqq n} (L_{1j} a^1_{kl} + L_{1l} a^1_{kj}) v^k_{x_l x_j} \Big).
\end{aligned}$$

Should (4.3) hold, it is thus sufficient to solve the following algebraic system for each $k = 1, \ldots, m$ (after adjusting the letters for some indices):

$$L_{11}a^1_{k1} = 0, \tag{4.5}$$

$$L_{1j}a^1_{kj} + L_{jj}a^j_{kj} = 0 \qquad \forall j = 2, \ldots, r, \tag{4.6}$$

$$L_{11}a^1_{kj} + L_{1j}a^1_{k1} + L_{jj}a^j_{k1} = 0 \qquad \forall j = 2, \ldots, r, \tag{4.7}$$

$$L_{1l}a^1_{kj} + L_{1j}a^1_{kl} + L_{ll}a^l_{kj} + L_{jj}a^j_{kl} = 0 \qquad \begin{array}{l}\forall j = 3, \ldots, r,\\ \forall l = 2, \ldots, j-1,\end{array} \tag{4.8}$$

$$L_{1j}a^1_{kj} = 0 \qquad \forall j = r+1, \ldots, n, \tag{4.9}$$

$$L_{11}a^1_{kj} + L_{1j}a^1_{k1} = 0 \qquad \forall j = r+1, \ldots, n, \tag{4.10}$$

$$L_{1l}a^1_{kj} + L_{1j}a^1_{kl} + L_{ll}a^l_{kj} = 0 \qquad \begin{array}{l}\forall j = r+1, \ldots, n,\\ \forall l = 2, \ldots, r,\end{array} \tag{4.11}$$

$$L_{1l}a^1_{kj} + L_{1j}a^1_{kl} = 0 \qquad \begin{array}{l}\forall j = r+2, \ldots, n,\\ \forall l = r+1, \ldots, j-1.\end{array} \tag{4.12}$$

Although these systems have infinitely many solutions, we will solve those in a way for a later purpose that the matrix $(a^j_{k1})_{2 \leqq j,k \leqq m} \in \mathbb{M}^{(m-1)\times(m-1)}$ fulfills

$$a^j_{21} = a_j \quad \forall j = 2, \ldots, m, \quad \text{and} \quad a^j_{k1} = 0 \quad \text{otherwise.} \tag{4.13}$$

First, we let the coefficients a^i_{kl} $(1 \leqq i, k \leqq m,\ 1 \leqq l \leqq n)$ that do not appear in systems (4.5)–(4.12) $(k = 1, \ldots, m)$ be zero with an exception that we set $a^j_{21} = a_j$ for $j = r+1, \ldots, m$ to reflect (4.13). Secondly, for $1 \leqq k \leqq m,\ k \neq 2$, let us take the trivial (that is, zero) solution of system (4.5)–(4.12). Lastly, we take $k = 2$ and solve system (4.5)–(4.12) as follows with (4.13) satisfied. Since $L_{11} \neq 0$, we set $a^1_{21} = 0$. Then (4.5) is satisfied. Thus we set

$$a^j_{21} = -\frac{L_{11}}{L_{jj}}a^1_{2j}, \quad a^1_{2j} = -\frac{L_{jj}}{L_{11}}a_j \quad \forall j = 2, \ldots, r,$$

then (4.7) and (4.13) hold. Next, set

$$a^j_{2j} = -\frac{L_{1j}}{L_{jj}}a^1_{2j} = \frac{L_{1j}}{L_{11}}a_j \quad \forall j = 2, \ldots, r,$$

then (4.6) is fulfilled. Set

$$a^l_{2j} = -\frac{L_{1l}a^1_{2j} + L_{1j}a^1_{2l}}{L_{ll}} = \frac{L_{1l}L_{jj}a_j + L_{1j}L_{ll}a_l}{L_{ll}L_{11}}, \quad a^j_{2l} = 0$$

for $j = 3, \ldots, r$ and $l = 2, \ldots, j-1$. Then (4.8) holds. Set

$$a^1_{2j} = 0 \quad \forall j = r+1, \ldots, n,$$

then (4.9) and (4.10) are satisfied. Lastly, set

$$a^1_{2j} = 0, \quad a^l_{2j} = -\frac{L_{1j}}{L_{ll}} a^1_{2l} = \frac{L_{1j}}{L_{11}} a_l \quad \forall j = r+1, \dots, n, \ \forall l = 2, \dots, r,$$

so that (4.11) and (4.12) hold. In summary, we have determined the coefficients a^i_{kl} $(1 \leqq i, k \leqq m,\ 1 \leqq l \leqq n)$ in such a way that system (4.5)–(4.12) holds for each $k = 1, \dots, m$ and that (4.13) is also satisfied. Therefore, (1) follows from (4.3) and (4.4).

To prove (2), without loss of generality, we can assume $U = (0,1)^n \subset \mathbb{R}^n$. Let $\tau > 0$ be given. Let $u = (u^1, \dots, u^m) \in C^\infty(U; \mathbb{R}^m)$ be a function to be determined. Suppose that u depends only on the first variable $x_1 \in (0,1)$. We wish to have

$$\nabla \Phi u(x) \in \{-\lambda a \otimes e_1, (1-\lambda) a \otimes e_1\}$$

for all $x \in U$ except in a set of small measure. Since $u(x) = u(x_1)$, it follows from (4.4) that for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$,

$$\Phi^i u = \sum_{k=1}^m a^i_{k1} u^k_{x_1}; \quad \text{thus} \quad \partial_{x_j} \Phi^i u = \sum_{k=1}^m a^i_{k1} u^k_{x_1 x_j}.$$

As $a^1_{k1} = 0$ for $k = 1, \dots, m$, we have $\partial_{x_j} \Phi^1 u = \sum_{k=1}^m a^1_{k1} u^k_{x_1 x_j} = 0$ for $j = 1, \dots, n$. We first set $u^1 \equiv 0$ in U. Then from (4.13), it follows that for $i = 2, \dots, m$,

$$\partial_{x_j} \Phi^i u = \sum_{k=2}^m a^i_{k1} u^k_{x_1 x_j} = a^i_{21} u^2_{x_1 x_j} = a_i u^2_{x_1 x_j} = \begin{cases} a_i u^2_{x_1 x_1} & \text{if } j = 1, \\ 0 & \text{if } j = 2, \dots, n. \end{cases}$$

As $a_1 = 0$, we thus have that for $x \in U$,

$$\nabla \Phi u(x) = (u^2)''(x_1) a \otimes e_1.$$

For irrelevant components of u, we simply take $u^3 = \cdots = u^m \equiv 0$ in U. Lastly, for a number $\delta > 0$ to be chosen later, we choose a function $u^2(x_1) \in C^\infty_c(0,1)$ such that there exist two disjoint open sets $I_1, I_2 \subset\subset (0,1)$ satisfying $||I_1| - \lambda| < \tau/2$, $||I_2| - (1-\lambda)| < \tau/2$, $\|u^2\|_{L^\infty(0,1)} < \delta$, $\|(u^2)'\|_{L^\infty(0,1)} < \delta$, $-\lambda \leqq (u^2)''(x_1) \leqq 1 - \lambda$ for $x_1 \in (0,1)$, and

$$(u^2)''(x_1) = \begin{cases} 1 - \lambda & \text{if } x_1 \in I_1, \\ -\lambda & \text{if } x_1 \in I_2. \end{cases}$$

In particular,

$$\nabla \Phi u(x) \in [-\lambda a \otimes e_1, (1-\lambda) a \otimes e_1] \ \ \forall x \in U. \tag{4.14}$$

We now choose an open set $U'_\tau \subset\subset U' := (0,1)^{n-1}$ with $|U' \setminus U'_\tau| < \tau/2$ and a function $\eta \in C^\infty_c(U')$ so that

$$0 \leqq \eta \leqq 1 \ \text{ in } \ U', \quad \eta \equiv 1 \ \text{ in } U'_\tau, \quad \text{and} \ \ |\nabla^i_{x'} \eta| < \frac{C}{\tau^i} \ \ (i = 1, 2) \ \text{ in } U',$$

where $x' = (x_2, \dots, x_n) \in U'$ and the constant $C > 0$ is independent of τ. Now, we define $g(x) = \eta(x')u(x_1) \in C_c^\infty(U; \mathbb{R}^m)$. Set $U_A = I_1 \times U'_\tau$ and $U_B = I_2 \times U'_\tau$. Clearly, (a) follows from (1). As $g(x) = u(x_1) = u(x)$ for $x \in U_A \cup U_B$, we have

$$\nabla\Phi g(x) = \begin{cases} (1-\lambda)a \otimes e_1 & \text{if } x \in U_A, \\ -\lambda a \otimes e_1 & \text{if } x \in U_B, \end{cases}$$

hence (c) holds. Also,

$$||U_A| - \lambda|U|| = ||U_A| - \lambda| = ||I_1||U'_\tau| - \lambda| = ||I_1| - |I_1||U' \backslash U'_\tau| - \lambda| < \tau,$$

and likewise

$$||U_B| - (1-\lambda)|U|| < \tau,$$

so (d) is satisfied. Note that for $i = 1, \dots, m$,

$$\begin{aligned} \Phi^i g = \Phi^i(\eta u) &= \sum_{1 \leqq k \leqq m,\, 1 \leqq l \leqq n} a^i_{kl}(\eta u^k)_{x_l} = \eta\Phi^i u + \sum_{1 \leqq k \leqq m,\, 1 \leqq l \leqq n} a^i_{kl}\eta_{x_l}u^k \\ &= \eta\Phi^i u + u^2 \sum_{l=1}^n a^i_{2l}\eta_{x_l} = \eta a^i_{21}u^2_{x_1} + u^2 \sum_{l=1}^n a^i_{2l}\eta_{x_l}, \end{aligned}$$

so

$$\|\Phi g\|_{L^\infty(U)} \leqq C \max\{\delta, \delta\tau^{-1}\} < \tau$$

if $\delta > 0$ is chosen small enough, and (e) holds. Next, for $i = 1, \dots, m$ and $j = 1, \dots, n$,

$$\partial_{x_j}\Phi^i g = \eta_{x_j} a^i_{21}u^2_{x_1} + \eta\partial_{x_j}\Phi^i u + u^2_{x_j}\sum_{l=1}^n a^i_{2l}\eta_{x_l} + u^2 \sum_{l=1}^n a^i_{2l}\eta_{x_l x_j},$$

hence from (4.14),

$$\operatorname{dist}(\nabla\Phi g, [-\lambda a \otimes e_1, (1-\lambda)a \otimes e_1]) \leqq C \max\{\delta\tau^{-1}, \delta\tau^{-2}\} < \tau \quad \text{in } U,$$

if δ is sufficiently small. Thus (b) is fulfilled.

(Case 2): Assume that $L_{i1} = 0$ for all $i = 2, \dots, m$, that is,

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ 0 & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & L_{m2} & \cdots & L_{mn} \end{pmatrix} \in \mathbb{M}^{m \times n} \tag{4.15}$$

and

$$A - B = a \otimes e_1 \quad \text{for some nonzero vector } a \in \mathbb{R}^m.$$

Then by (4.1), we have $L_{11} \neq 0$.

Set

$$\hat{L} = \begin{pmatrix} L_{22} & \cdots & L_{2n} \\ \vdots & \ddots & \vdots \\ L_{m2} & \cdots & L_{mn} \end{pmatrix} \in \mathbb{M}^{(m-1)\times(n-1)}.$$

As $L_{11} \neq 0$ and $\operatorname{rank}(L) = r$, we must have $\operatorname{rank}(\hat{L}) = r - 1$. Using the singular value decomposition theorem, there exist two matrices $\hat{W} \in O(m-1)$ and $\hat{V} \in O(n-1)$ such that

$$\hat{W}^T \hat{L} \hat{V} = \operatorname{diag}(\sigma_2, \dots, \sigma_r, 0, \dots, 0) \in \mathbb{M}^{(m-1)\times(n-1)}, \tag{4.16}$$

where $\sigma_2, \dots, \sigma_r$ are the positive singular values of $\hat{L}$. Define

$$W = \begin{pmatrix} 1 & 0 \\ 0 & \hat{W} \end{pmatrix} \in O(m), \quad V = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V} \end{pmatrix} \in O(n). \tag{4.17}$$

Let $L' = W^T L V$, $A' = W^T A V$, and $B' = W^T B V$. Let $\mathcal{L}' : \mathbb{M}^{m\times n} \to \mathbb{R}$ be the linear map given by

$$\mathcal{L}'(\xi') = \sum_{1 \leqq i \leqq m,\, 1 \leqq j \leqq n} L'_{ij}\xi'_{ij} \quad \forall \xi' \in \mathbb{M}^{m\times n}.$$

Then from (4.15), (4.16) and (4.17), it is straightforward to check the following:

$$\begin{cases} A' - B' = a' \otimes e_1 \text{ for some nonzero vector } a' \in \mathbb{R}^m, \\ L'e_1 \neq 0,\ \mathcal{L}'(A) = \mathcal{L}'(B), \text{ and} \\ L' \text{ is of the form (4.2) in Case 1 with } \operatorname{rank}(L') = r. \end{cases}$$

Thus we can apply the result of Case 1 to find a linear operator $\Phi' : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)$ satisfying the following:

(1') For any open set $U' \subset \mathbb{R}^n$,

$$\Phi' v' \in C^{k-1}(U'; \mathbb{R}^m) \ \text{ whenever } \ k \in \mathbb{N} \ \text{ and } \ v' \in C^k(U'; \mathbb{R}^m)$$

and

$$\mathcal{L}'(\nabla \Phi' v') = 0 \ \text{ in } \ U' \ \text{ for all } \ v' \in C^2(U'; \mathbb{R}^m);$$

(2') Let $U' \subset \mathbb{R}^n$ be any bounded domain. For each $\tau > 0$, there exist a function $g' = g'_\tau \in C_c^\infty(U'; \mathbb{R}^m)$ and two disjoint open sets $U'_{A'}, U'_{B'} \subset\subset U'$ such that

(a') $\Phi' g' \in C_c^\infty(U'; \mathbb{R}^m)$,
(b') $\operatorname{dist}(\nabla \Phi' g', [-\lambda(A' - B'), (1-\lambda)(A' - B')]) < \tau$ in U',
(c') $\nabla \Phi' g'(x) = \begin{cases} (1-\lambda)(A' - B') & \forall x \in U'_{A'}, \\ -\lambda(A' - B') & \forall x \in U'_{B'}, \end{cases}$
(d') $||U'_{A'}| - \lambda|U'|| < \tau$, $||U'_{B'}| - (1-\lambda)|U'|| < \tau$,
(e') $\|\Phi' g'\|_{L^\infty(U')} < \tau$.

For $v \in C^1(\mathbb{R}^n;\mathbb{R}^m)$, let $v' \in C^1(\mathbb{R}^n;\mathbb{R}^m)$ be defined by $v'(y) = W^T v(Vy)$ for $y \in \mathbb{R}^n$. We define $\Phi v(x) = W\Phi' v'(V^T x)$ for $x \in \mathbb{R}^n$, so that $\Phi v \in C^0(\mathbb{R}^n;\mathbb{R}^m)$. Then it is straightforward to check that properties (1') and (2') of Φ' imply respective properties (1) and (2) of the linear operator $\Phi : C^1(\mathbb{R}^n;\mathbb{R}^m) \to C^0(\mathbb{R}^n;\mathbb{R}^m)$.

(Case 3): Finally, we consider the general case that A, B and L are as in the statement of the theorem. As $|b| = 1$, there exists an $R \in O(n)$ such that $R^T b = e_1 \in \mathbb{R}^n$. Also there exists a symmetric (Householder) matrix $P \in O(m)$ such that the matrix $L' := PLR$ has the first column parallel to $e_1 \in \mathbb{R}^m$. Let

$$A' = PAR \quad \text{and} \quad B' = PBR.$$

Then $A' - B' = a' \otimes e_1$, where $a' = Pa \neq 0$. Note also that $L'e_1 = PLRR^t b = PLb \neq 0$. Define $\mathcal{L}'(\xi') = \sum_{i,j} L'_{ij}\xi'_{ij}$ $(\xi' \in \mathbb{M}^{m\times n})$; then $\mathcal{L}'(A') = \mathcal{L}(A) = \mathcal{L}(B) = \mathcal{L}'(B')$. Thus by the result of Case 2, there exists a linear operator $\Phi' : C^1(\mathbb{R}^n;\mathbb{R}^m) \to C^0(\mathbb{R}^n;\mathbb{R}^m)$ satisfying (1') and (2') above.

For $v \in C^1(\mathbb{R}^n;\mathbb{R}^m)$, let $v' \in C^1(\mathbb{R}^n;\mathbb{R}^m)$ be defined by $v'(y) = Pv(Ry)$ for $y \in \mathbb{R}^n$, and define $\Phi v(x) = P\Phi' v'(R^T x) \in C^0(\mathbb{R}^n;\mathbb{R}^m)$. Then it is easy to check that the linear operator $\Phi : C^1(\mathbb{R}^n;\mathbb{R}^m) \to C^0(\mathbb{R}^n;\mathbb{R}^m)$ satisfies (1) and (2) by (1') and (2') in a manner similar as to Case 2. □

5. Proof of the Density Result

In this final section, we prove Theorem 3.1, which plays a pivotal role in the proof of the main result, Theorem 2.1.

To start the proof, fix any $\delta > 0$ and choose any $w = (u, v) \in \mathcal{A}$ so that $w \in W^{1,\infty}_{w^*}(\Omega_T;\mathbb{R}^2) \cap C^2(\bar{\Omega}_T;\mathbb{R}^2)$ satisfies the following:

$$\begin{cases} \text{there exists a finite set } \Lambda_w \subset \Lambda \text{ such that } w = w^* \text{ in } \Omega_T \setminus (\cup_{(i,k)\in\Lambda_w} \bar{Q}^{ik}_w) \\ \quad \text{for some open sets } Q^{ik}_w \subset\subset Q^{ik}_2 \text{ with } (i,k) \in \Lambda_w \text{ and } |\partial Q^{ik}_w| = 0, \\ \nabla w(x,t) \in U^{ik}_\beta \quad \forall (x,t) \in Q^{ik}_2, \forall (i,k) \in \Lambda, \\ |\Gamma^{Q^{ik}_2}_w - \Gamma^{Q^{ik}_2}_{w^*}| < \varepsilon^{ik}_* \quad \forall (i,k) \in \Lambda, \\ \|u - u^*\|_{L^\infty(\Omega_T)} < \varepsilon', \ \|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} < \varepsilon'. \end{cases} \tag{5.1}$$

Let $\eta > 0$. Our goal is to construct a function $w_\eta = (u_\eta, v_\eta) \in \mathcal{A}_\delta$ with $\|w - w_\eta\|_{L^\infty(\Omega_T)} < \eta$; that is, a function $w_\eta \in W^{1,\infty}_{w^*}(\Omega_T;\mathbb{R}^2) \cap C^2(\bar{\Omega}_T;\mathbb{R}^2)$ with the following properties:

$$\begin{cases} \text{there exists a finite set } \Lambda_{w_\eta} \subset \Lambda \text{ such that } w_\eta = w^* \text{ in } \Omega_T \setminus (\cup_{(i,k)\in\Lambda_{w_\eta}} \bar{Q}^{ik}_{w_\eta}) \\ \quad \text{for some open sets } Q^{ik}_{w_\eta} \subset\subset Q^{ik}_2 \text{ with } (i,k) \in \Lambda_{w_\eta} \text{ and } |\partial Q^{ik}_{w_\eta}| = 0, \\ \nabla w_\eta(x,t) \in U^{ik}_\beta \quad \forall (x,t) \in Q^{ik}_2, \forall (i,k) \in \Lambda, \\ |\Gamma^{Q^{ik}_2}_{w_\eta} - \Gamma^{Q^{ik}_2}_{w^*}| < \varepsilon^{ik}_* \quad \forall (i,k) \in \Lambda, \\ \|u_\eta - u^*\|_{L^\infty(\Omega_T)} < \varepsilon', \ \|(u_\eta)_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} < \varepsilon', \ \|w - w_\eta\|_{L^\infty(\Omega_T)} < \eta, \\ \int_{Q_2} \operatorname{dist}(\nabla w_\eta(x,t), K_\beta)\, dx dt \leqq \delta |Q_2|. \end{cases} \tag{5.2}$$

For clarity, we divide the proof into several steps.

(Step 0): Choose a finite set $\Lambda_{w_\eta} \subset \Lambda$ with $\Lambda_w \subset \Lambda_{w_\eta}$ so large that

$$\sum_{(i,k)\in\Lambda\setminus\Lambda_{w_\eta}} \int_{Q_2^{ik}} \operatorname{dist}(\nabla w(x,t), K_\beta)\, \mathrm{d}x\mathrm{d}t \leqq \frac{\delta}{n}|Q_2|, \tag{5.3}$$

where $n \in \mathbb{N}$ is a constant to be determined later. For each $(i,k) \in \Lambda_{w_\eta}\setminus\Lambda_w$, let us take $Q_w^{ik} = \emptyset$.

(Step 1): Fix any $(i,k) \in \Lambda_{w_\eta}$. Choose a nonempty open set $G_1^{ik} \subset\subset Q_2^{ik}\setminus\partial Q_w^{ik}$ with $|\partial G_1^{ik}| = 0$ so that

$$\int_{(Q_2^{ik}\setminus\partial Q_w^{ik})\setminus G_1^{ik}} \operatorname{dist}(\nabla w(x,t), K_\beta^{ik})\, \mathrm{d}x\mathrm{d}t \leqq \frac{\delta}{n}|Q_2^{ik}|. \tag{5.4}$$

Since $\nabla w \in U_\beta^{ik}$ on $\bar{G}_1^{ik}$, we have $\|u_t\|_{L^\infty(G_1^{ik})} < \beta$; with this and (5.1), we can fix a number $\theta^{ik} > 0$ such that

$$\theta^{ik} < \min\left\{\begin{matrix} \varepsilon_*^{ik} - |\Gamma_w^{Q_2^{ik}} - \Gamma_{w^*}^{Q_2^{ik}}|,\ \varepsilon' - \|u-u^*\|_{L^\infty(\Omega_T)}, \\ \varepsilon' - \|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})},\ \beta - \|u_t\|_{L^\infty(G_1^{ik})} \end{matrix}\right\}. \tag{5.5}$$

Let us write

$$S_M^{ik} = \max_{r_a^{ik} \leqq r \leqq r_b^{ik}} (s_+(r) - s_-(r)) \text{ and } S_m^{ik} = \min_{r_a^{ik} \leqq r \leqq r_b^{ik}} (s_+(r) - s_-(r)) > 0, \tag{5.6}$$

where

$$\begin{cases} r_a^{1k} := r^{1k} \\ r_b^{1k} := r^{1(k-1)} \end{cases} \text{if } i = 1 \text{ and } \begin{cases} r_a^{2k} := r^{2(k-1)} \\ r_b^{2k} := r^{2k} \end{cases} \text{if } i = 2.$$

By the uniform continuity of $s_\pm$ on $[r_a^{ik}, r_b^{ik}]$, we can find a $\kappa^{ik} > 0$ such that

$$|s_\pm(r) - s_\pm(\tilde{r})| < \min\left\{\frac{\theta^{ik} S_m^{ik}}{n}, \frac{\theta^{ik}(S_m^{ik})^2}{n S_M^{ik}}\right\} \tag{5.7}$$

whenever $r, \tilde{r} \in [r_a^{ik}, r_b^{ik}]$ and $|r - \tilde{r}| < \kappa^{ik}$. For each $\mu > 0$, let

$$\begin{aligned} G_2^{ik,\mu} &= \{(x,t) \in G_1^{ik} \mid \operatorname{dist}((u_x(x,t), v_t(x,t)), \partial\tilde{U}^{r_a^{ik}, r_b^{ik}}) > \mu\}, \\ H_2^{ik,\mu} &= \{(x,t) \in G_1^{ik} \mid \operatorname{dist}((u_x(x,t), v_t(x,t)), \partial\tilde{U}^{r_a^{ik}, r_b^{ik}}) < \mu\}, \\ F_2^{ik,\mu} &= \{(x,t) \in G_1^{ik} \mid \operatorname{dist}((u_x(x,t), v_t(x,t)), \partial\tilde{U}^{r_a^{ik}, r_b^{ik}}) = \mu\}. \end{aligned}$$

Since $\lim_{\mu\to 0^+} |H_2^{ik,\mu}| = 0$, we can find a number $\nu^{ik} > 0$ with

$$\nu^{ik} < \min\left\{\frac{\delta}{n}, \frac{\theta^{ik} S_m^{ik}}{n}, \frac{\theta (S_m^{ik})^2}{n S_M^{ik}}, \frac{s_+(r_a^{ik}) - s_-(r_b^{ik})}{2}\right\} \tag{5.8}$$

such that

$$\int_{H_2^{ik,\nu^{ik}}} \operatorname{dist}(\nabla w(x,t), K_\beta^{ik})\,\mathrm{d}x\mathrm{d}t \leqq \frac{\delta}{n}|Q_2^{ik}|, \quad G_2^{ik,\nu^{ik}} \neq \emptyset \text{ and } |F_2^{ik,\nu^{ik}}| = 0. \tag{5.9}$$

We write $G_2^{ik} = G_2^{ik,\nu^{ik}}$ and $H_2^{ik} = H_2^{ik,\nu^{ik}}$. We also define

$$\tilde{U}_\pm^{ik} = \left\{ (s,r) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} r_a^{ik} < r < r_b^{ik},\ 0 < \lambda < 1, \\ s = \lambda(s_\pm(r) \mp \nu^{ik}) + (1-\lambda)s_\pm(r) \end{array} \right\} \subset \tilde{U}^{r_a^{ik}, r_b^{ik}}$$

and take

$$d^{ik} = \min\left\{ \min_{r_a^{ik} \leqq r \leqq r_b^{ik}} \operatorname{dist}\Big(\big(s_+(r) - \frac{\nu^{ik}}{2}, r\big), \tilde{K}_+^{r_a^{ik}, r_b^{ik}}\Big), \right.$$
$$\left. \min_{r_a^{ik} \leqq r \leqq r_b^{ik}} \operatorname{dist}\Big(\big(s_-(r) + \frac{\nu^{ik}}{2}, r\big), \tilde{K}_-^{r_a^{ik}, r_b^{ik}}\Big) \right\} > 0; \tag{5.10}$$

it is then easy to see that

$$d^{ik} \leqq \frac{\nu^{ik}}{2}. \tag{5.11}$$

Choose finitely many disjoint open squares $D_1^{ik}, \ldots, D_{N^{ik}}^{ik} \subset G_2^{ik}$, parallel to the axes, such that

$$\int_{G_2^{ik} \setminus (\cup_{j=1}^{N^{ik}} D_j^{ik})} \operatorname{dist}(\nabla w(x,t), K_\beta^{ik})\,\mathrm{d}x\mathrm{d}t \leqq \frac{\delta}{n}|Q_2^{ik}|. \tag{5.12}$$

(Step 2): Dividing the squares $D_1^{ik}, \ldots, D_{N^{ik}}^{ik}$ into smaller disjoint sub-squares if necessary, we can assume that

$$|\nabla w(x,t) - \nabla w(\tilde{x},\tilde{t})| < \min\left\{\frac{d^{ik}}{n}, \kappa^{ik}\right\} =: c^{ik} \tag{5.13}$$

whenever $(x,t), (\tilde{x},\tilde{t}) \in \bar{D}_j^{ik}$ and $j \in \{1, \ldots, N^{ik}\}$.

Now, fix an index $j \in \{1, \ldots, N^{ik}\}$. Let (x_j^{ik}, t_j^{ik}) denote the center of the square D_j^{ik} and write

$$(s_j^{ik}, r_j^{ik}) = (u_x(x_j^{ik}, t_j^{ik}), v_t(x_j^{ik}, t_j^{ik})) \in \tilde{U}^{r_a^{ik}, r_b^{ik}};$$

then $\operatorname{dist}((s_j^{ik}, r_j^{ik}), \partial\tilde{U}^{r_a^{ik}, r_b^{ik}}) > \nu^{ik}$, and so $(s_j^{ik}, r_j^{ik}) \notin \tilde{U}_\pm^{ik}$. Let $\alpha_j^{ik} > 0$ and $\beta_j^{ik} > 0$ be the numbers given by

$$s_j^{ik} + \beta_j^{ik} = s_+(r_j^{ik}) - \frac{\nu^{ik}}{2}, \quad s_j^{ik} - \alpha_j^{ik} = s_-(r_j^{ik}) + \frac{\nu^{ik}}{2}; \tag{5.14}$$

then $(s_j^{ik} + \beta_j^{ik}, r_j^{ik}) \in \tilde{U}_+^{ik}$ and $(s_j^{ik} - \alpha_j^{ik}, r_j^{ik}) \in \tilde{U}_-^{ik}$.

To apply Theorem 4.1 in the square D_j^{ik}, let

$$A_j^{ik} = \begin{pmatrix} s_j^{ik} - \alpha_j^{ik} & b_j^{ik} \\ b_j^{ik} & r_j^{ik} \end{pmatrix} \text{ and } B_j^{ik} = \begin{pmatrix} s_j^{ik} + \beta_j^{ik} & b_j^{ik} \\ b_j^{ik} & r_j^{ik} \end{pmatrix},$$

where $b_j^{ik} := u_t(x_j^{ik}, t_j^{ik})$; then $|b_j^{ik}| \leqq \|u_t\|_{L^\infty(G_1^{ik})}$ and

$$A_j^{ik} - B_j^{ik} = \begin{pmatrix} -\alpha_j^{ik} - \beta_j^{ik} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\alpha_j^{ik} - \beta_j^{ik} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $\mathcal{L} : \mathbb{M}^{2\times 2} \to \mathbb{R}$ be the linear map defined by

$$\mathcal{L}(\xi) = -\xi_{21} + \xi_{12} \quad \forall \xi \in \mathbb{M}^{2\times 2}$$

with its associated matrix $L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; then

$$\mathcal{L}(A_j^{ik}) = \mathcal{L}(B_j^{ik})(= 0) \text{ and } C_j^{ik} = \lambda_j^{ik} A_j^{ik} + (1 - \lambda_j^{ik}) B_j^{ik}$$

with $C_j^{ik} := \nabla w(x_j^{ik}, t_j^{ik})$ and $\lambda_j^{ik} = \frac{\beta_j^{ik}}{\alpha_j^{ik} + \beta_j^{ik}} \in (0, 1)$. By Theorem 4.1, there exists a linear operator $\Phi_j^{ik} : C^1(\mathbb{R}^2; \mathbb{R}^2) \to C^0(\mathbb{R}^2; \mathbb{R}^2)$ satisfying properties (1) and (2) in the statement of the theorem with $A = A_j^{ik}$, $B = B_j^{ik}$ and $\lambda = \lambda_j^{ik}$. In particular, for the square $D_j^{ik} \subset \mathbb{R}^2$ and a number $\tau^{ik} > 0$ with

$$\tau^{ik} < \min\left\{ c^{ik}, \eta, \frac{\theta^{ik}}{n}, \frac{\theta^{ik}|Q_2^{ik}|}{nN^{ik}}, \frac{\delta|Q_2^{ik}|}{nS_M^{ik}N^{ik}} \right\}, \tag{5.15}$$

we can find a function $g_j^{ik} \in C_c^\infty(D_j^{ik}; \mathbb{R}^2)$ and two disjoint open sets $D_{A_j^{ik}}, D_{B_j^{ik}} \subset\subset D_j^{ik}$ such that

$$\begin{cases} \text{(a) } (\varphi_j^{ik}, \psi_j^{ik}) := \Phi_j^{ik} g_j^{ik} \in C_c^\infty(D_j^{ik}; \mathbb{R}^2),\ \mathcal{L}(\nabla \Phi_j^{ik} g_j^{ik}) = 0 \text{ in } D_j^{ik}, \\ \text{(b) } \operatorname{dist}(\nabla \Phi_j^{ik} g_j^{ik}, [-\lambda_j^{ik}(A_j^{ik} - B_j^{ik}), (1 - \lambda_j^{ik})(A_j^{ik} - B_j^{ik})]) < \tau^{ik} \text{ in } D_j^{ik}, \\ \text{(c) } \nabla \Phi_j^{ik} g_j^{ik}(x) = \begin{cases} (1 - \lambda_j^{ik})(A_j^{ik} - B_j^{ik}), & x \in D_{A_j^{ik}}, \\ -\lambda_j^{ik}(A_j^{ik} - B_j^{ik}), & x \in D_{B_j^{ik}}, \end{cases} \\ \text{(d) } ||D_{A_j^{ik}}| - \lambda_j^{ik}|D_j^{ik}|| < \tau^{ik},\ ||D_{B_j^{ik}}| - (1 - \lambda_j^{ik})|D_j^{ik}|| < \tau^{ik}, \\ \text{(e) } \|\Phi_j^{ik} g_j^{ik}\|_{L^\infty(D_j^{ik})} < \tau^{ik}. \end{cases} \tag{5.16}$$

We finally define

$$w_\eta = w + \sum_{(i,k)\in\Lambda_{w_\eta}} \sum_{j=1}^{N^{ik}} \chi_{D_j^{ik}} \Phi_j^{ik} g_j^{ik} \text{ in } \Omega_T.$$

(Step 3): In this final step, let us check that $w_\eta = (u_\eta, v_\eta)$ is indeed a desired function satisfying (5.2). Since this step is rather long, we further divide it into several substeps.

(Substep 3-1): We begin with some properties that are relatively easy to check.

It is clear from (5.1) and the construction above that

$$w_\eta \in W^{1,\infty}_{w^*}(\Omega_T;\mathbb{R}^2) \cap C^2(\bar{\Omega}_T;\mathbb{R}^2). \tag{5.17}$$

For each $(i,k) \in \Lambda_{w_\eta}$, set $Q^{ik}_{w_\eta} = Q^{ik}_w \cup (\cup_{j=1}^{N^{ik}} D^{ik}_j)$. Then we clearly have that

$$Q^{ik}_{w_\eta} \subset\subset Q^{ik}_2 \text{ and } |\partial Q^{ik}_{w_\eta}| = 0. \tag{5.18}$$

By the definition of w_η, we also have

$$w_\eta = w = w^* \text{ in } \Omega_T \setminus (\cup_{(i,k)\in\Lambda_{w_\eta}} \bar{Q}^{ik}_{w_\eta}). \tag{5.19}$$

Next, we check that for each $(i,k) \in \Lambda$,

$$\nabla w_\eta \in U^{ik}_\beta \;\text{ in } Q^{ik}_2. \tag{5.20}$$

If $(i,k) \in \Lambda\setminus\Lambda_{w_\eta}$, we have from the definition of w_η and (5.1) that $\nabla w_\eta = \nabla w \in U^{ik}_\beta$ in Q^{ik}_2. Now, let $(i,k) \in \Lambda_{w_\eta}$. If we take

$$n \geqq \frac{S^{ik}_m}{4}, \tag{5.21}$$

then from (5.5), (5.8) and (5.11), we have $\frac{d^{ik}}{2} \leqq \frac{\nu^{ik}}{4} < \frac{\theta^{ik} S^{ik}_m}{4n} < \beta - \|u_t\|_{L^\infty(G^{ik}_1)}$. Since $|b^{ik}_j| \leqq \|u_t\|_{L^\infty(G^{ik}_1)}$ and $\operatorname{dist}((s^{ik}_j, r^{ik}_j), \partial\tilde{U}^{r^{ik}_a, r^{ik}_b}) > \nu^{ik}$ $(j = 1,\dots,N^{ik})$, it thus follows from (5.10), (5.14), and the definition of A_i and B_i that

$$[A^{ik}_j, B^{ik}_j]_{d^{ik}/2} \subset U^{ik}_\beta,$$

where $[A^{ik}_j, B^{ik}_j]_{d^{ik}/2}$ is the $\frac{d^{ik}}{2}$-neighborhood of the closed line segment $[A^{ik}_j, B^{ik}_j]$ in the space $\mathbb{M}^{2\times2}_{sym}$. In turn, with (5.1), (5.13), (5.15) and (5.16)(a)(b), we have

$$\nabla w_\eta = \nabla w + \nabla\Phi^{ik}_j g^{ik}_j \in [A^{ik}_j, B^{ik}_j]_{d^{ik}/2} \;\text{ in } D^{ik}_j$$

if we let

$$n \geqq 4; \tag{5.22}$$

in this case, (5.20) holds.

For each $(i,k) \in \Lambda_{w_\eta}$, by (5.5), (5.15), (5.16)(b)(e) and the zero antidiagonal of $A^{ik}_j - B^{ik}_j$ $(j = 1,\dots,N^{ik})$, we have

$$\begin{aligned}
&\|u_\eta - u^*\|_{L^\infty(\Omega_T)} \leqq \|u - u^*\|_{L^\infty(\Omega_T)} + \tau^{ik} < \|u - u^*\|_{L^\infty(\Omega_T)} + \frac{\theta^{ik}}{n} < \varepsilon',\\
&\|(u_\eta)_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} \leqq \|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} + \tau^{ik} < \|u_t - H\|_{L^\infty(\Omega_{T_\varepsilon})} + \frac{\theta^{ik}}{n} < \varepsilon',\\
&\|w - w_\eta\|_{L^\infty(\Omega_T)} = \Bigg\| \sum_{(i,k)\in\Lambda_{w_\eta}} \sum_{j=1}^{N^{ik}} \chi_{D^{ik}_j} \Phi^{ik}_j g^{ik}_j \Bigg\|_{L^\infty(\Omega_T)} < \tau^{ik} < \eta.
\end{aligned} \tag{5.23}$$

(Substep 3-2): Here, we show that

$$|\Gamma_{w_\eta}^{Q_2^{ik}} - \Gamma_{w^*}^{Q_2^{ik}}| < \varepsilon_*^{ik} \quad \forall (i,k) \in \Lambda. \tag{5.24}$$

If $(i,k) \in \Lambda \setminus \Lambda_{w_\eta}$, then $\Gamma_{w_\eta}^{Q_2^{ik}} = \Gamma_{w^*}^{Q_2^{ik}}$, so we assume that $(i,k) \in \Lambda_{w_\eta}$.

Recalling the definition of the phase gauge operator $\Gamma_{\cdot}^{Q_2^{ik}}$ over Q_2^{ik} from Subsection 2.2, we have

$$\begin{aligned}
\Gamma_{w_\eta}^{Q_2^{ik}} - \Gamma_{w}^{Q_2^{ik}} &= \frac{1}{|Q_2^{ik}|}\int_{Q_2^{ik}} (Z_{w_\eta}^{Q_2^{ik}}(x,t) - Z_{w}^{Q_2^{ik}}(x,t))\,\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{|Q_2^{ik}|}\sum_{j=1}^{N^{ik}} \int_{D_j^{ik}} (Z_{w+\Phi_j^{ik} g_j^{ik}}^{Q_2^{ik}}(x,t) - Z_{w}^{Q_2^{ik}}(x,t))\,\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{|Q_2^{ik}|}\sum_{j=1}^{N^{ik}} \left(\int_{D_j^{ik}\setminus(D_{A_j^{ik}} \cup D_{B_j^{ik}})} + \int_{D_{A_j^{ik}}} + \int_{D_{B_j^{ik}}} \right) (Z_{w+\Phi_j^{ik} g_j^{ik}}^{Q_2^{ik}} - Z_{w}^{Q_2^{ik}}).
\end{aligned}$$

Here and below, let $j \in \{1, \dots, N^{ik}\}$. First, we easily get from (5.16)(d) that

$$\left| \int_{D_j^{ik}\setminus(D_{A_j^{ik}} \cup D_{B_j^{ik}})} (Z_{w+\Phi_j^{ik} g_j^{ik}}^{Q_2^{ik}} - Z_{w}^{Q_2^{ik}}) \right| \leqq 2|D_j^{ik}\setminus(D_{A_j^{ik}} \cup D_{B_j^{ik}})| < 4\tau^{ik}. \tag{5.25}$$

Next, it follows from (5.6), (5.7), (5.8), (5.11), (5.13), (5.14) and (5.16)(c) that in $D_{A_j^{ik}}$,

$$\begin{aligned}
|Z_{w+\Phi_j^{ik} g_j^{ik}}^{Q_2^{ik}}| &= \left| \frac{u_x + (\varphi_j^{ik})_x - s_-(v_t + (\psi_j^{ik})_t)}{s_+(v_t + (\psi_j^{ik})_t) - s_-(v_t + (\psi_j^{ik})_t)} \right| \leqq \frac{|u_x - \alpha_j^{ik} - s_-(v_t)|}{S_m} \\
&\leqq \frac{|u_x - s_j^{ik}| + |s_j^{ik} - \alpha_j^{ik} - s_-(r_j^{ik})| + |s_-(r_j^{ik}) - s_-(v_t)|}{S_m^{ik}} \\
&\leqq \frac{c^{ik} + \frac{\nu^{ik}}{2} + \frac{\theta^{ik} S_m^{ik}}{n}}{S_m^{ik}} \leqq \frac{\frac{1+n}{2n}\nu^{ik} + \frac{\theta^{ik} S_m^{ik}}{n}}{S_m^{ik}} \\
&\leqq \frac{\frac{1+n}{2n}\frac{\theta^{ik} S_m^{ik}}{n} + \frac{\theta^{ik} S_m^{ik}}{n}}{S_m^{ik}} = \frac{1+3n}{2n^2}\theta^{ik},
\end{aligned}$$

and so

$$\left| \int_{D_{A_j^{ik}}} Z_{w+\Phi_j^{ik} g_j^{ik}}^{Q_2^{ik}} \right| \leqq \frac{1+3n}{2n^2}\theta^{ik} |D_{A_j^{ik}}| \leqq \frac{1+3n}{2n^2}\theta^{ik} |D_j^{ik}|. \tag{5.26}$$

Lastly, we estimate the quantity

$$\Big|\int_{D_{B_j^{ik}}} Z^{Q_2^{ik}}_{w+\Phi_j^{ik} g_j^{ik}} - \int_{D_{A_j^{ik}} \cup D_{B_j^{ik}}} Z^{Q_2^{ik}}_{w}\Big|.$$

We first consider

$$\begin{aligned}
&\Big|\int_{D_{B_j^{ik}}} Z^{Q_2^{ik}}_{w+\Phi_j^{ik} g_j^{ik}}(x,t)\,\mathrm{d}x\mathrm{d}t - \frac{\alpha_j^{ik}}{\alpha_j^{ik}+\beta_j^{ik}}|D_j^{ik}|\Big| \\
&\leqq \Big|\int_{D_{B_j^{ik}}} \Big(\frac{u_x+(\varphi_j^{ik})_x - s_-(v_t+(\psi_j^{ik})_t)}{s_+(v_t+(\psi_j^{ik})_t) - s_-(v_t+(\psi_j^{ik})_t)} \\
&\qquad - \frac{s_j^{ik}+\beta_j^{ik}-s_-(r_j^{ik})}{s_+(v_t+(\psi_j^{ik})_t) - s_-(v_t+(\psi_j^{ik})_t)}\Big)\,\mathrm{d}x\mathrm{d}t\Big| \\
&\quad + \Big|\int_{D_{B_j^{ik}}} \Big(\frac{s_j^{ik}+\beta_j^{ik}-s_-(r_j^{ik})}{s_+(v_t+(\psi_j^{ik})_t) - s_-(v_t+(\psi_j^{ik})_t)} - \frac{s_j^{ik}+\beta_j^{ik}-s_-(r_j^{ik})}{s_+(r_j^{ik})-s_-(r_j^{ik})}\Big)\,\mathrm{d}x\mathrm{d}t\Big| \\
&\quad + \Big|\frac{s_j^{ik}+\beta_j^{ik}-s_-(r_j^{ik})}{s_+(r_j^{ik})-s_-(r_j^{ik})}|D_{B_j^{ik}}| - (1-\lambda_j^{ik})|D_j^{ik}|\Big| =: I_1 + I_2 + I_3.
\end{aligned}$$

From (5.6), (5.7), (5.8), (5.11), (5.13), (5.14) and (5.16)(c)(d), we have

$$\begin{aligned}
I_1 &\leqq \int_{D_{B_j^{ik}}} \Big|\frac{u_x+\beta_j^{ik}-s_-(v_t)}{s_+(v_t)-s_-(v_t)} - \frac{s_j^{ik}+\beta_j^{ik}-s_-(r_j^{ik})}{s_+(v_t)-s_-(v_t)}\Big|\,\mathrm{d}x\mathrm{d}t \\
&\leqq \int_{D_{B_j^{ik}}} \frac{c^{ik}+\frac{\theta^{ik}S_m^{ik}}{n}}{S_m^{ik}}\,\mathrm{d}x\mathrm{d}t \leqq \frac{1+2n}{2n^2}\theta^{ik}|D_{B_j^{ik}}|, \\
I_2 &\leqq \int_{D_{B_j^{ik}}} \frac{|s_j^{ik}+\beta_j^{ik}-s_-(r_j^{ik})|(|s_+(v_t)-s_+(r_j^{ik})|+|s_-(v_t)-s_-(r_j^{ik})|)}{(s_+(v_t)-s_-(v_t))(s_+(r_j^{ik})-s_-(r_j^{ik}))}\,\mathrm{d}x\mathrm{d}t \\
&\leqq \frac{(\alpha_j^{ik}+\beta_j^{ik}+\frac{\nu^{ik}}{2})\frac{2\theta^{ik}(S_m^{ik})^2}{nS_M^{ik}}}{(S_m^{ik})^2}|D_{B_j^{ik}}| \leqq \frac{S_M^{ik}\frac{2\theta^{ik}(S_m^{ik})^2}{nS_M^{ik}}}{(S_m^{ik})^2}|D_{B_j^{ik}}| = \frac{2}{n}\theta^{ik}|D_{B_j^{ik}}|, \\
I_3 &\leqq \Big|\frac{s_j^{ik}+\beta_j^{ik}-s_+(r_j^{ik})}{s_+(r_j^{ik})-s_-(r_j^{ik})}\Big||D_{B_j^{ik}}| + ||D_{B_j^{ik}}| - (1-\lambda_j^{ik})|D_j^{ik}|| \\
&\leqq \frac{\nu^{ik}}{2S_m^{ik}}|D_{B_j^{ik}}| + \tau^{ik} \leqq \frac{1}{2n}\theta^{ik}|D_{B_j^{ik}}| + \tau^{ik}.
\end{aligned}$$

Next, we concern ourselves with

$$\begin{aligned}
&\Big|\int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}} Z_w^{Q_2^{ik}}(x,t)\,\mathrm{d}x\mathrm{d}t - \frac{\alpha_j^{ik}}{\alpha_j^{ik}+\beta_j^{ik}}|D_j^{ik}|\Big| \\
&\leqq \Big|\int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}} \Big(\frac{u_x - s_-(v_t)}{s_+(v_t)-s_-(v_t)} - \frac{\alpha_j^{ik}}{s_+(v_t)-s_-(v_t)}\Big)\,\mathrm{d}x\mathrm{d}t\Big| \\
&\quad + \Big|\int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}} \Big(\frac{\alpha_j^{ik}}{s_+(v_t)-s_-(v_t)} - \frac{\alpha_j^{ik}}{s_+(r_j^{ik})-s_-(r_j^{ik})}\Big)\,\mathrm{d}x\mathrm{d}t\Big| \\
&\quad + \Big|\frac{\alpha_j^{ik}}{s_+(r_j^{ik})-s_-(r_j^{ik})}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| - \frac{\alpha_j^{ik}}{\alpha_j^{ik}+\beta_j^{ik}}|D_j^{ik}|\Big| =: I_4+I_5+I_6.
\end{aligned}$$

From (5.6), (5.7), (5.8), (5.11), (5.13), (5.14) and (5.16)(d), it follows that

$$\begin{aligned}
I_4 &\leqq \int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}} \frac{|u_x - s_j^{ik}| + |s_-(r_j^{ik}) - s_-(v_t)| + |s_j^{ik} - s_-(r_j^{ik}) - \alpha_j^{ik}|}{s_+(v_t)-s_-(v_t)}\,\mathrm{d}x\mathrm{d}t \\
&\leqq \frac{\frac{\nu^{ik}}{2n} + \frac{\theta^{ik}S_m^{ik}}{n} + \frac{\nu^{ik}}{2}}{S_m^{ik}}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| \leqq \frac{\frac{1+n}{2n}\frac{\theta^{ik}S_m^{ik}}{n} + \frac{\theta^{ik}S_m^{ik}}{n}}{S_m^{ik}}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| \\
&= \frac{1+3n}{2n^2}\theta^{ik}|D_{A_j^{ik}}\cup D_{B_j^{ik}}|, \\
I_5 &\leqq \int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}} \frac{\alpha_j^{ik}(|s_+(v_t) - s_+(r_j^{ik})| + |s_-(v_t) - s_-(r_j^{ik})|)}{(s_+(v_t)-s_-(v_t))(s_+(r_j^{ik})-s_-(r_j^{ik}))}\,\mathrm{d}x\mathrm{d}t \\
&\leqq \frac{2\alpha_j^{ik}\frac{\theta^{ik}(S_m^{ik})^2}{nS_M^{ik}}}{(S_m^{ik})^2}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| \leqq \frac{2}{n}\theta^{ik}|D_{A_j^{ik}}\cup D_{B_j^{ik}}|, \\
I_6 &\leqq \Big|\frac{\alpha_j^{ik}}{s_+(r_j^{ik})-s_-(r_j^{ik})} - \frac{\alpha_j^{ik}}{\alpha_j^{ik}+\beta_j^{ik}}\Big||D_{A_j^{ik}}\cup D_{B_j^{ik}}| \\
&\quad + \frac{\alpha_j^{ik}}{\alpha_j^{ik}+\beta_j^{ik}}\big||D_{A_j^{ik}}\cup D_{B_j^{ik}}| - |D_j^{ik}|\big| \\
&\leqq \frac{\alpha_j^{ik}\nu^{ik}}{(S_m^{ik})^2}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| + 2\tau^{ik} \leqq \frac{\alpha_j^{ik}\frac{\theta^{ik}(S_m^{ik})^2}{nS_M^{ik}}}{(S_m^{ik})^2}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| + 2\tau^{ik} \\
&\leqq \frac{1}{n}\theta^{ik}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| + 2\tau^{ik}.
\end{aligned}$$

Gathering the above estimates on $I_1, \ldots, I_6$, we get

$$\begin{aligned}\Big|\int_{D_{B_j^{ik}}} Z^{Q_2^{ik}}_{w+\Phi_j^{ik} g_j^{ik}} - \int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}} Z^{Q_2^{ik}}_{w}\Big| &\leqq \Big(\frac{1+2n}{2n^2}+\frac{2}{n}+\frac{1}{2n}\Big)\theta^{ik}|D_{B_j^{ik}}| \\ &\quad + \Big(\frac{1+3n}{2n^2}+\frac{2}{n}+\frac{1}{n}\Big)\theta^{ik}|D_{A_j^{ik}}\cup D_{B_j^{ik}}| + 3\tau^{ik} \\ &\leqq \frac{1+8n}{n^2}\theta^{ik}|D_j^{ik}| + 3\tau^{ik}.\end{aligned}$$

Combining this with (5.25) and (5.26), we obtain from (5.15) that

$$\begin{aligned}|\Gamma^{Q_2^{ik}}_{w_\eta} - \Gamma^{Q_2^{ik}}_{w}| &\leqq \frac{1}{|Q_2^{ik}|}\sum_{j=1}^{N^{ik}}\Big(4\tau^{ik} + \frac{1+3n}{2n^2}\theta^{ik}|D_j^{ik}| + \frac{1+8n}{n^2}\theta^{ik}|D_j^{ik}| + 3\tau^{ik}\Big) \\ &\leqq \frac{3+19n}{2n^2}\theta^{ik} + \frac{7\tau^{ik}N^{ik}}{|Q_2^{ik}|} \\ &\leqq \Big(\frac{3+19n}{2n^2}+\frac{7}{n}\Big)\theta^{ik} = \frac{3+33n}{2n^2}\theta^{ik} < \theta^{ik}\end{aligned}$$

if we take

$$n \geqq 17; \tag{5.27}$$

in this case, (5.5) implies that

$$|\Gamma^{Q_2^{ik}}_{w_\eta} - \Gamma^{Q_2^{ik}}_{w^*}| \leqq |\Gamma^{Q_2^{ik}}_{w_\eta} - \Gamma^{Q_2^{ik}}_{w}| + |\Gamma^{Q_2^{ik}}_{w} - \Gamma^{Q_2^{ik}}_{w^*}| < \theta^{ik} + |\Gamma^{Q_2^{ik}}_{w} - \Gamma^{Q_2^{ik}}_{w^*}| < \varepsilon^{ik}_*.$$

Thus (5.24) is satisfied.

(Substep 3-3): In this final substep, we show that

$$\int_{Q_2} \operatorname{dist}(\nabla w_\eta(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \leqq \delta|Q_2|. \tag{5.28}$$

Note that

$$\begin{aligned}\int_{Q_2} \operatorname{dist}(\nabla w_\eta(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t &= \sum_{(i,k)\in\Lambda\setminus\Lambda_{w_\eta}} \int_{Q_2^{ik}} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \\ &+ \sum_{(i,k)\in\Lambda_{w_\eta}} \int_{(Q_2^{ik}\setminus\partial Q_w^{ik})\setminus G_1^{ik}} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \\ &+ \sum_{(i,k)\in\Lambda_{w_\eta}} \int_{H_2^{ik}} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t \\ &+ \sum_{(i,k)\in\Lambda_{w_\eta}} \int_{G_2^{ik}\setminus(\cup_{j=1}^{N^{ik}} D_j^{ik})} \operatorname{dist}(\nabla w(x,t), K_\beta)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$+\sum_{(i,k)\in\Lambda_{w_\eta}}\sum_{j=1}^{N^{ik}}\int_{D_j^{ik}}\operatorname{dist}(\nabla w(x,t)+\nabla\Phi_j^{ik}g_j^{ik}(x,t),K_\beta)\,\mathrm{d}x\mathrm{d}t$$
$$=: J_1+J_2+J_3+J_4+J_5.$$

Observe from (5.6), (5.8), (5.11), (5.13), (5.14) and (5.16)(c)(d) that for $(i,k)\in\Lambda_{w_\eta}$ and $j\in\{1,\dots,N^{ik}\}$,

$$\begin{aligned}\int_{D_j^{ik}}\operatorname{dist}(\nabla w+\nabla\Phi_j^{ik}g_j^{ik},K_\beta)\,\mathrm{d}x\mathrm{d}t &= \int_{D_{A_j^{ik}}\cup D_{B_j^{ik}}}\operatorname{dist}(\nabla w+\nabla\Phi_j^{ik}g_j^{ik},K_\beta)\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{D_j^{ik}\setminus(D_{A_j^{ik}}\cup D_{B_j^{ik}})}\operatorname{dist}(\nabla w+\nabla\Phi_j^{ik}g_j^{ik},K_\beta)\,\mathrm{d}x\mathrm{d}t\\ &\leqq\Big(\frac{\nu^{ik}}{2}+\frac{d^{ik}}{n}\Big)|D_{A_j^{ik}}\cup D_{B_j^{ik}}|+S_M^{ik}|D_j^{ik}\setminus(D_{A_j^{ik}}\cup D_{B_j^{ik}})|\\ &\leqq\frac{1+n}{2n}\nu^{ik}|D_j^{ik}|+2\tau^{ik}S_M^{ik}\leqq\frac{1+n}{2n^2}\delta|D_j^{ik}|+\frac{2}{n}\delta\frac{|Q_2^{ik}|}{N^{ik}}.\end{aligned}$$

Thus

$$J_5\leqq\frac{1+n}{2n^2}\delta|Q_2|+\frac{2}{n}\delta|Q_2|=\frac{1+5n}{2n^2}\delta|Q_2|,$$

and with (5.3), (5.4), (5.9) and (5.12), we have

$$\begin{aligned}J_1+J_2+J_3+J_4+J_5&\leqq\Big(\frac1n+\frac1n+\frac1n+\frac1n+\frac{1+5n}{2n^2}\Big)\delta|Q_2|\\ &=\frac{1+13n}{2n^2}\delta|Q_2|\leqq\delta|Q_2|\end{aligned}$$

if we let

$$n\geqq 7. \tag{5.29}$$

In this case, (5.28) is fulfilled.

We now fix a constant $n\in\mathbb{N}$ so that $n\geqq\max\{\frac{S_M}{4},17\}$, where

$$S_M:=\max_{r_1\leqq r\leqq r_2}(s_+(r)-s_-(r)).$$

Then (5.21), (5.22), (5.27) and (5.29) hold. Thus (5.20), (5.24) and (5.28) are satisfied. These facts, together with (5.17), (5.18) and (5.23), are precisely (5.2), as desired.

The proof of the density result, Theorem 3.1, is now complete.

Acknowledgements. The authors would like to thank the anonymous referees for many helpful suggestions and corrections that greatly improved the presentation of the paper. S. KIM was supported by the National Research Foundation of Korea under the Grant Number NRF-2018R1C1B6001308. Y. KOH was supported by the National Research Foundation of Korea under the Grant Number NRF-2016R1D1A1B03932049.

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(*Received July 31, 2017 / Accepted October 5, 2018*)
Published online October 16, 2018