Arch. Rational Mech. Anal. 231 (2019) 1983–2005
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-018-1314-5

CrossMark

An Algebraic Reduction of the 'Scaling Gap' in the Navier–Stokes Regularity Problem

ZACHARY BRADSHAW, ASEEL FARHAT & ZORAN GRUJIĆ

Communicated by P. CONSTANTIN

Abstract

It is shown—within a mathematical framework based on the suitably defined scale of sparseness of the super-level sets of the positive and negative parts of the vorticity components, and in the context of a blow-up-type argument—that the ever-resisting 'scaling gap', that is, the scaling distance between a regularity criterion and a corresponding a priori bound (shortly, a measure of the super-criticality of the three dimensional Navier–Stokes regularity problem), can be reduced by an *algebraic factor*; since (independent) fundamental works of Ladyzhenskaya, Prodi and Serrin as well as Kato and Fujita in 1960s, all the reductions have been logarithmic in nature, regardless of the functional set up utilized. More precisely, it is shown that it is possible to obtain an a priori bound that is algebraically better than the energy-level bound, while keeping the corresponding regularity criterion at the same level as all the classical regularity criteria. The mathematics presented was inspired by morphology of the regions of intense vorticity/velocity gradients observed in computational simulations of turbulent flows, as well as by the physics of turbulent cascades and turbulent dissipation.

1. Prologue

The Navier–Stokes equations (NSE) describing the motion of three-dimensional (3D) incompressible, viscous, Newtonian fluids read

$$\partial_t u + (u \cdot \nabla)u = \nu \triangle u - \nabla p + f, \tag{1}$$

supplemented with the incompressibility condition $\nabla \cdot u = 0$. Here, the vector field u represents the velocity, the scalar field p the pressure, a positive constant ν is the viscosity, and the vector field f an external force.

Henceforth, for simplicity of the exposition, ν will be normalized to 1, f taken to be zero, and the spatial domain to be the whole space $\mathbb{R}^3$ (since this is an irreversible system, the temporal variable lives in the interval $(0, \infty)$). In this setting, the only relevant piece of data is the initial velocity configuration u_0.

Mathematical theory of the three dimensional NSE emerged from the groundbreaking work of Leray in the 1930s [24–26] in which he established global-in-time existence of weak solutions emanating from the initial data with arbitrary large energy and charted the paths for the study of local-in-time and global-in-time existence and uniqueness of strong solutions within the realms of arbitrary large and appropriately small initial data.

A central problem in the theory ever since has been the possibility of singularity formation in the system, and could be considered either from the point of view of possible singularity formation in a weak solution or from the point of view of possible finite time blow-up of a local-in-time strong solution. This problem is usually referred to as the Navier–Stokes regularity problem.

The Navier–Stokes regularity problem is supercritical, that is, there is a 'scaling gap'—with respect to the unique scaling transformation realizing the scaling-invariance of the equations—between any known regularity criterion and the corresponding a priori bound. (Note that if (u, p) is a solution to the three dimensional NSE on $\mathbb{R}^3 \times (0, \infty)$, then for any $\lambda > 0$, the rescaled pair (u_λ, p_λ) given by $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ and $p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$ is also a solution; in addition, this is the only such transformation).

If we focus on $L^\infty((0, T), X)$-type spaces, perhaps the most illustrative example is in the L^p-scale where the regularity class is $L^\infty((0, T), L^3)$ ([10]; for generalizations see [13,29]), and the corresponding a priori bound the Leray bound $L^\infty((0, T), L^2)$. Moreover, since 1960s, the scaling gap has been of the fixed 'size' in the sense that all the regularity classes are (at best) scaling-invariant while—in contrast—all the known a priori bounds are on the scaling level of the Leray bound.

Recall that arguably the most classical (scaling invariant) regularity criteria are the Ladyzhenskaya–Prodi–Serrin conditions given by $u \in L^{\frac{2g}{q-3}}((0, T), L^q)$, for $3 < q \leqq \infty$, the other endpoint being the aforementioned Escauriaza, Seregin, and Sverak class $L^\infty((0, T), L^3)$. In the vorticity realm, this designation goes to the BEALE, KATO, and MAJDA (BKM)-criterion [2], $\omega \in L^1((0, T), L^\infty)$ where ω is the vorticity of the fluid given by $\omega = \operatorname{curl} u$.

One might try to collect additional clues about possible singularity formation in the three dimensional NSE from the geometry of turbulent flows. One of the most prominent morphological signatures of this geometry—revealed in high-resolution computational simulations of high (and even moderate; see [31]) Reynolds number-flows—is spatial intermittency of the regions of intense vorticity and velocity gradients.

In order to better understand geometry of the flow mathematically, it is beneficial to study the vorticity–velocity formulation of the three dimensional NSE,

$$\partial_t \omega + (u \cdot \nabla)\omega = \triangle \omega + (\omega \cdot \nabla)u. \tag{2}$$

The incompressibility implies that u can be reconstructed from ω by solving $\triangle u = -\operatorname{curl} \omega$, leading to the Biot–Savart law, and closing the system.

Let us start by noting that conceivably a first work in mathematical analysis of the three dimensional NSE that was motivated by geometry of the flow was a work by Giga and Miyakawa [14] establishing global-in-time existence of small, measure-valued solutions modeling the vortex filaments (the space utilized is a scaling-invariant Morrey–Campanato space).

A conceptually related program aimed at mathematical modeling and analysis of coherent structures in fluid flows via the study of oscillating and localized solutions to the NSE was initiated by Meyer and his students (Brandolese, Cannone, Lemarie-Rieusset, Planchon,...); this brought in an array of scale-sensitive harmonic analysis tools such as wavelets, atomic decomposition and Littlewood–Paley analysis. A chapter in C.I.M.E. lecture notes by Meyer [28] provides insight into many of the main ideas and results related to this program.

The vortex-stretching term, $(\omega \cdot \nabla)u$, is responsible for a possibly unbounded growth of the vorticity magnitude in three dimensional; mechanically, the process of vortex stretching in conjunction with incompressibility of the fluid and conservation of the angular momentum amplifies the vorticity, formally, if we erase the vortex-stretching term in the equations, and the only nonlinear term left is the advection term $(u \cdot \nabla)\omega$, an a priori bound on the L^2-norm of the vorticity—which suffices to bootstrap to infinite smoothness—follows in one line ($(\omega \cdot \nabla)u$ is identically zero in 2D).

However, some caution is advised in labeling the vortex-stretching term as a clear-cut ‘bad guy’. While the process of vortex-stretching does amplify the vorticity, it also produces locally anisotropic small scales in the flow (diameter of a vortex tube, thickness of a vortex sheet, *etc.*), and this gives a chance to some form of a locally anisotropic diffusion to engage just on time to prevent the further amplification of the vorticity magnitude and possible singularity formation. As a matter of fact, Taylor [34] concluded his paper from the 1930s with the following thought on turbulent dissipation:

“It seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion.”

The stretching is also chiefly responsible for the phenomenon of local coherence of the vorticity direction prominent in coherent vortex structures (for example, in vortex tubes) which can be viewed as a local manifestation of the quasi low-dimensionality of the regions of intense fluid activity. This observation led to the rigorous mathematical study of locally anisotropic dissipation in the three dimensional NSE originating in the pioneering work by Constantin [7], where he presented a singular integral representation of the stretching factor in the evolution of the vorticity magnitude, featuring a geometric kernel depleted precisely by local coherence of the vorticity direction. This has been referred to as ‘geometric depletion of the nonlinearity’, and was a key in the proof of a fundamental result by Constantin and Fefferman [8], stating that as long as the vorticity direction (in the regions of intense vorticity) is Lipschitz-coherent, no finite time blow-up can occur.

The Lipschitz-coherence was later scaled down to $\frac{1}{2}$-Hölder [9], followed up by a complete spatiotemporal localization to an arbitrary small parabolic cylinder $B(x_0, 2R) \times (t_0 - (2R)^2, t_0)$ [17], and the construction of scaling-invariant, local, hybrid geometric-analytic regularity classes in which the coherence of the vorticity direction is balanced against the growth of the vorticity magnitude [18]. In particular, denoting the vorticity direction by ξ, the following scaling-invariant regularity class is included:

$$\int_{t_0-(2R)^2}^{t_0} \int_{B(x_0,2R)} |\omega(x,t)|^2 \rho^2_{\frac{1}{2},2R}(x,t)\,\mathrm{d}x\,\mathrm{d}t < \infty,$$

where

$$\rho_{\gamma,r}(x,t) = \sup_{y\in B(x,r),\, y\neq x} \frac{\left|\sin\angle\Big(\xi(x,t),\xi(y,t)\Big)\right|}{|x-y|^{\gamma}};$$

this is to be contrasted to an a priori bound obtained in [6],

$$\int_0^T \int_{\mathbb{R}^3} |\omega(x,t)||\nabla\xi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \leqq \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2\,\mathrm{d}x,$$

which exhibits the same scaling as the Leray class $L^\infty((0,T),L^2)$; another manifestation of the scaling gap.

Let us return to the spatial intermittency of the regions of intense vorticity, more specifically, to 'sparseness' of the vorticity super-level sets. This type of morphology motivated a recent work [19] where it was shown that as long as the vorticity super-level sets, cut at a fraction of the L^∞-norm, exhibit a suitably defined property of sparseness at the scale comparable to the radius of spatial analyticity measured in L^∞, no finite time blow-up can occur. (The proof was presented for the velocity formulation, and the argument for the vorticity formulation is completely analogous).

The concept of sparseness needed in the proof is essentially local existence of a sparse direction (the direction is allowed to depend on the point) at the scale of interest, and it will be referred to as '1D sparseness'. The precise definitions of 1D sparseness and a related concept of 'three dimensional sparseness' can be found at the beginning of Section 3, as well as a remark on their relationship. In the rest of this section, sparseness will refer to three dimensional sparseness; we copy the definition below for convenience of the reader (m^3 denotes the 3-dimensional Lebesgue measure).

Let S be an open subset of $\mathbb{R}^3$, $\delta \in (0,1)$, x_0 a point in $\mathbb{R}^3$, and $r \in (0,\infty)$. S is *three dimensional δ-sparse around x_0 at scale r* if

$$\frac{m^3\big(S\cap B(x_0,r)\big)}{m^3\big(B(x_0,r)\big)} \leqq \delta.$$

Since we are interested in the super-level sets of a vector field cut at a fraction of the L^∞-norm, at the scale comparable to some inverse power of the L^∞-norm, there are several parameters floating around: the fraction at which we cut (call it λ), the ratio of sparseness δ and a constant multiplying the inverse power of the L^∞-norm (call it $\frac{1}{c}$). More precisely, the class of functions of interest, X_α, is defined as follows:

Definition 1. For a positive exponent α, and a selection of parameters λ in (0, 1), δ in (0, 1) and $c_0 > 1$, the class of functions $X_\alpha(\lambda, \delta; c_0)$ consists of bounded, continuous functions $f : \mathbb{R}^3 \to \mathbb{R}^3$ subjected to the requirement that the super-level set

$$\left\{ x \in \mathbb{R}^3 : |f(x)| > \lambda \|f\|_\infty \right\} \tag{3}$$

is δ-sparse around any point x_0 at scale $\frac{1}{c} \frac{1}{\|f\|_\infty^\alpha}$, for some c, $\frac{1}{c_0} \leqq c \leqq c_0$.

It is instructive to make a brief comparison of the scale of classes X_α with the scale of Lorentz $L^{p,\infty}$ spaces defined by the requirement that the distribution function of f, the Lebesgue measure of the super-level set cut at level β, decays at least as the inverse p-power of β. There are two key differences:

(i) the definition of X_α involves a specific type of the super-level sets, cut at a fraction of the L^∞-norm, while the definition of $L^{p,\infty}$ involves all levels β;
(ii) the $L^{p,\infty}$ spaces are blind to any geometric properties of the super-level sets, and measure only the rate of decay of the volume, while the classes X_α can detect the scale of sparseness of the super-level sets of the highest interest.

Remark 2. It is worth noting that the same phenomenon will occur in all rearrangement-invariant (with respect to the Lebesgue measure) function classes, that is, the classes that do not differentiate between the functions having identical distribution functions. Typical examples are Lebesgue classes L^p, general Lorentz classes $L^{p,q}$ and Orlicz classes L^ϕ.

Taking a closer look, on one hand, it is plain that

$$f \in L^{p,\infty} \implies f \in X_\alpha \quad \text{for} \quad \alpha = \frac{p}{3} \tag{4}$$

(for a given selection of λ and δ, the tolerance parameter c_0 will depend on the $L^{p,\infty}$-norm of f), on the other hand, in the geometrically worst case scenario (from the point of view of sparseness), the whole super-level set being clumped into a single ball, being in X_α is consistent with being in $L^{3\alpha,\infty}$.

This provides a simple 'conversion rule' between the two scales, $\alpha = \frac{p}{3}$, determining the scaling signature of the classes X_α.

Let us decode the scaling distance between the regularity criterion established in [19] (in the vorticity formulation), and any corresponding a priori bounds.

Recall that the scale of sparseness required is comparable to the lower bound on the radius of spatial analyticity evaluated at a suitable time t prior to (possible) blow-up, namely, $\frac{1}{c} \frac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$, that is, one needs ω in $X_{\frac{1}{2}}$.

In contrast, the best a priori bound available is ω in $X_{\frac{1}{3}}$; this follows simply from the a priori bound on the L^1-norm of the vorticity [6] and Chebyshev's inequality (an $L \log L$-bound on the vorticity is available under a very weak assumption that the vorticity direction is in a local, logarithmically-weighted BMO space that allows for discontinuities [3]).

Applying the scaling conversion rule, $X_{\frac{1}{2}}$ corresponds to $L^{\frac{3}{2},\infty}$ and $X_{\frac{1}{3}}$ to $L^{1,\infty}$; since these are precisely the endpoint classes within the $L^{p,\infty}$-scale in the vorticity formulation, we arrive at yet another manifestation of the scaling gap (also, not surprising as the derivation of the a priori bound, ω in $X_{\frac{1}{3}}$, uses no geometric tools).

Back to the drawing board. Recall that computational simulations of turbulent flows indicate a high degree of local anisotropy present in the vorticity field, manifested through the formation of various coherent vortex structures (for example, vortex sheets and vortex tubes).

A simple but key observation is that a vector field exhibiting a high degree of local anisotropy is more likely to allow for a considerable discrepancy in sparseness between the full vectorial super-level sets and the super-level sets of the components. As a matter of fact, it is easy to construct locally anisotropic smooth vector fields for which the former are not sparse at any scale, while the latter can feature any predetermined scale of sparseness.

This is our primary motivation for replacing the scale of classes X_α by the scale of classes Z_α based on sparseness of the super-level sets of the locally selected positive and negative parts of the vectorial components f_i, $f_i^{\pm}$, for $i = 1, 2, 3$.

Definition 3. For a positive exponent α, and a selection of parameters λ in $(0, 1)$, δ in $(0, 1)$ and $c_0 > 1$, the class of functions $Z_\alpha(\lambda, \delta; c_0)$ consists of bounded, continuous functions $f : \mathbb{R}^3 \to \mathbb{R}^3$ subjected to the following local condition. For x_0 in $\mathbb{R}^3$, select the/a component $f_i^{\pm}$ such that $f_i^{\pm}(x_0) = |f(x_0)|$ (henceforth, the norm of a vector $v = (a, b, c)$, $|v|$, will be computed as $\max\{|a|, |b|, |c|\}$), and require that the set

$$\left\{x \in \mathbb{R}^3 : \ f_i^{\pm}(x) > \lambda \|f\|_\infty\right\}$$

is δ-sparse around x_0 at scale $\frac{1}{c}\frac{1}{\|f\|_\infty^\alpha}$, for some c, $\frac{1}{c_0} \leqq c \leqq c_0$. Enforce this for all x_0 in $\mathbb{R}^3$.

Remark 4. Shortly, we require (local) sparseness of the/a locally maximal component only.

(The scaling conversion rule remains the same, $\alpha = \frac{p}{3}$).

Going back to the vorticity, since this is potentially a much weaker condition, the two questions to examine are:

(a) is it possible to obtain an analogous regularity criterion at the same scale of sparseness (compared to the full vectorial super-level sets), or at all?

(b) is it possible to establish an a priori bound at a smaller scale of sparseness (compared to the full vectorial case), in particular, in the sense of modifying the scaling exponent α by an algebraic factor?

Affirmative answers to (a) and (b) are presented in Sections 3 (Theorem 19) and 4 (Theorem 26), respectively, yielding a regularity criterion in $Z_{\frac{1}{2}}$ and an *priori* bound in $Z_{\frac{2}{5}}$ (with the identical selection of the parameters λ, δ and c_0); since

Table 1. The scaling distance between regularity and energy classes is consistent across the table. Generally, a priori bounded quantities are in the energy class. Remarkably, Theorem 26 shows the a priori bound in the Z_α class is *algebraically better* than the energy class (the a priori bound in Z_α is derived in the context of a blow-up argument)

Regularity class	A priori bound	Energy class
$u \in L^\infty(0,T;L^3)$	$u \in L^\infty(0,T;L^2)$	$u \in L^\infty(0,T;L^2)$
$\omega \in L^\infty(0,T;L^{\frac{3}{2}})$	$\omega \in L^\infty(0,T;L^1)$	$\omega \in L^\infty(0,T;L^1)$
$\omega(\tau) \in X_{\frac{1}{2}}$ for a particular $\tau < T$	$\omega(\tau) \in X_{\frac{1}{3}}$ almost everywhere $\tau < T$	$\omega(\tau) \in X_{\frac{1}{3}}$ almost everywhere $\tau < T$
$\omega(\tau) \in Z_{\frac{1}{2}}$ for a particular $\tau < T$	$\omega(\tau) \in Z_{\frac{2}{5}}$ for $\tau < T$ whenever $\|\omega(\tau)\|_\infty$ is large enough	$\omega(\tau) \in Z_{\frac{1}{3}}$ almost everywhere $\tau < T$

$$\frac{1}{3} < \frac{2}{5} < \frac{1}{2},$$

this represents an algebraic reduction of the scaling gap in the Z_α-framework.

For a quick scaling comparison, several pointwise-in-time classes and their respective bounds are listed in Table 1.

Remark 5. Note that in the geometrically worst case scenario (for sparseness), ω being in $Z_{\frac{2}{5}}$ is consistent with ω being in $L^{\frac{6}{5},\infty}$ which is—in turn—at the same scaling level as $L^{\frac{6}{5}}$. Hence, a 'geometrically blind' scaling counterpart of our a priori bound in $Z_{\frac{2}{5}}$ would be the bound ω in $L^\infty((0,T), L^{\frac{6}{5}})$ which is well beyond state-of-the-art (ω in $L^\infty((0,T), L^1)$ [6]). A counterpart in the velocity realm would be the bound u in $L^\infty((0,T), L^{\frac{12}{5}})$, also well beyond state-of-the-art (this would represent an algebraic improvement of the Leray bound u in $L^\infty((0,T), L^2)$).

Remark 6. One should remark that this is not the first time that the exponent $\frac{2}{5}$ appeared in the study of formation of small scales in turbulent flows. In the context of the three dimensional inviscid dynamics, the BKM-type criteria reveal that the formation of infinite spatial gradients/infinitesimally-small spatial scales is necessary for a finite time blow-up. In particular (see a discussion in [7]), if we consider the case of approximately self-similar blow-up for the three dimensional Euler in the form

$$\omega(x,t) \approx \frac{1}{T-t}\Omega\left(\frac{x}{L(t)}\right),$$

then the relevant BKM-type necessary condition for the blow-up reads

$$\int_0^T \big(L(t)\big)^{-\frac{5}{2}}\,\mathrm{d}t = \infty,$$

and if we focus on the algebraic-type dependence

$$L(t) = L(0)\left(1 - \frac{t}{T}\right)^p,$$

then the blow-up can be sustained only when $p \geqq \frac{2}{5}$. Furthermore, since by dimensional analysis $\omega \approx \frac{1}{T-t}$, this scaling is fully consistent with the vorticity being in $Z_{\frac{2}{5}}$ near a possible singular time T, strongly hinting at the inviscid nature of $Z_{\frac{2}{5}}$.

Remark 7. It is not inconceivable that—at least partially—the discrepancy between the exponents $\frac{2}{5}$ and $\frac{1}{3}$ can be viewed as an indirect measure of the degree of local anisotropy of the vorticity field in the regions of intense fluid activity. It is worth noting that the same line of arguments, applied to the velocity itself, does not yield any reduction of the scaling gap [11]; this is consistent with the picture painted by computational simulations (and fluid mechanics) in which the regions of high velocity magnitude tend to be more isotropic (recall that in a suitable statistical sense, the K41 turbulence phenomenology—based on the velocity—is isotropic and homogeneous, while one way to rationalize the phenomenon of intermittency in turbulence is precisely via the coherent vortex structures).

2. Local-in-time Spatial Analyticity of the Vorticity in L^∞

One of the most compelling manifestations of diffusion in the three dimensional NSE is the instantaneous spatially-analytic smoothing of the rough initial data. An explicit, sharp lower bound on the radius of (spatial) analyticity of the solution emanating from an initial datum in L^p, for $3 < p < \infty$, was given in [16]; the method—inspired by the so-called Gevrey class-method introduced by Foias and Temam [12]—was based on complexifying the equations and tracking the boundary of the (locally-in-time expanding) domain of analyticity via solving a real system of PDEs satisfied by the real and the imaginary parts of the complexified solutions.

Since the Riesz transforms are not bounded on L^∞, to obtain an estimate in the case $p = \infty$ without a logarithmic correction requires a different argument (on the real level), see, for example, [15,20,23] in the real and the complex setting, respectively, in the realm of the velocity-pressure formulation of the three dimensional NSE.

The L^∞-argument within the vorticity–velocity description requires a modification, and we present a sketch here, including an estimate on the vortex-stretching term, for completeness. What follows is a modification of the exposition given in [23]. In addition to the initial vorticity ω_0 being bounded, a suitable decay of ω_0 at infinity will be required (we chose ω_0 in L^2 for convenience); however, it is worth noting that this is a 'soft assumption', that is, there will be no quantitative dependence on $\|\omega_0\|_2$ in the proof.

Theorem 8. (real setting) *Let the initial datum ω_0 be in $L^2 \cap L^\infty$. Then there exists a unique mild solution ω in $C_w\big([0,T], L^\infty\big)$ where $T \geqq \frac{1}{c}\frac{1}{\|\omega_0\|_\infty}$ for an absolute constant* $c > 0$.

Remark 9. Since L^∞ is not separable, the continuity of the heat semigroup at the initial time is guaranteed only in the weak sense; hence the appearance of the space $C_w\big([0,T], L^\infty\big)$ in the statement of the theorem. Alternatively, one could replace L^∞ with its (closed) subspace of the uniformly continuous functions, and have the strong continuity at the initial time.

Sketch of the proof. The vorticity–velocity formulation of the three dimensional NSE in the components is as follows:

$$\partial_t \omega_j - \triangle \omega_j + u_i \partial_i \omega_j = \omega_i \partial_i u_j, \quad j = 1, 2, 3, \tag{5}$$

where u can be recovered from ω by the Biot–Savart law,

$$u_j(x,t) = c \int \varepsilon_{j,k,l}\, \partial_{y_k} \frac{1}{|x-y|}\, \omega_l(y,t)\, \mathrm{d}y$$

($\varepsilon_{j,k,l}$ is the Levi–Civita symbol), making this a closed system for ω. Differentiation yields

$$\partial_i u_j(x,t) = c\, P.V. \int \varepsilon_{j,k,l} \frac{\partial^2}{\partial_{x_i}\partial_{y_k}} \frac{1}{|x-y|} \omega_l(y,t)\, \mathrm{d}y. \tag{6}$$

A key observation is that the kernel

$$\frac{\partial^2}{\partial_{x_i}\partial_{y_k}} \frac{1}{|x-y|}$$

is a classical Calderon–Zygmund kernel; hence,

$$\|\nabla u(\cdot,t)\|_{BMO} \leqq c\, \|\omega(\cdot,t)\|_{BMO}. \tag{7}$$

Since we are interested in the mild solutions, we rewrite the Equation (5) via the action of the heat semigroup,

$$\begin{aligned}\omega_j(x,t) = &\int G(x-y,t)(\omega_0)_j(y)\, \mathrm{d}y \\ &- \int_0^t \int G(x-y,t-s)\, u_i \partial_i \omega_j(y,s)\, \mathrm{d}y\, \mathrm{d}s \\ &+ \int_0^t \int G(x-y,t-s)\, \omega_i \partial_j u_j(y,s)\, \mathrm{d}y\, \mathrm{d}s,\end{aligned} \tag{8}$$

where G is the heat kernel.

Similarly as in the velocity-pressure description (see, for example, [16]), the estimates are performed on the sequence of smooth (entire in the spatial variable) global-in-time approximations.

$$\begin{aligned}\omega_j^{(n)}(x,t) = &\int G(x-y,t)(\omega_0)_j(y)\, \mathrm{d}y \\ &- \int_0^t \int G(x-y,t-s)\, u_i^{(n-1)} \partial_i \omega_j^{(n-1)}(y,s)\, \mathrm{d}y\, \mathrm{d}s \\ &+ \int_0^t \int G(x-y,t-s)\, \omega_i^{(n-1)} \partial_j u_j^{(n-1)}(y,s)\, \mathrm{d}y\, \mathrm{d}s,\end{aligned} \tag{9}$$

supplemented with

$$u_j^{(n)}(x,t) = c \int \varepsilon_{j,k,l}\, \partial_{y_k} \frac{1}{|x-y|}\, \omega_l^{(n)}(y,t)\, \mathrm{d}y.$$

The goal is to keep $\omega^{(n)}$ in L^∞ and $\nabla u^{(n)}$ in *BMO* (uniformly in n, locally-in-time), analogously to keeping $u^{(n)}$ in L^∞ and $p^{(n)}$ in BMO in the velocity-pressure formulation.

We present an estimate on the vortex-stretching term $\omega_i^{(n-1)}\partial_j u_j^{(n-1)}$, the term responsible for a potentially unrestricted growth of the vorticity magnitude, in some detail.

`An estimate on the vortex-stretching term.`

The main ingredients are: a pointwise estimate on the heat kernel G, $G(x,t) \leqq c\,\frac{\sqrt{t}}{\left(|x|+\sqrt{t}\right)^4}$ (see, for example, [27]), a property of a scalar-valued *BMO* function f featuring a suitable decay at infinity (for example, being in the closure of the test functions in the uniformly-local L^p for some p, $1 \leqq p < \infty$), $\int \frac{|f(x)|}{(|x|+1)^4}\,\mathrm{d}x \leqq c\,\|f\|_{BMO}$ (in general, one has to subtract a local average, for example, over a unit ball B, in which case the inequality takes the form $\int \frac{|f(x)-\frac{1}{|B|}\int_B f|}{(|x|+1)^4}\,\mathrm{d}x \leqq c\,\|f\|_{BMO}$ [33]), and the Calderon–Zygmund relation (6).

Combining the above ingredients implies the following string of inequalities:

$$\begin{aligned}
&\left|\int_0^t \int G(x-y,t-s)\,\omega_i^{(n)}(y,s)\,\partial_i u_j^{(n)}(y,s)\,\mathrm{d}y\,\mathrm{d}s\right| \\
&\quad\leqq \sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_\infty \int_0^t \int \frac{\sqrt{t-s}}{\left(|x-y|+\sqrt{t-s}\right)^4}\,|\partial_i u_j^{(n)}|\,\mathrm{d}y\,\mathrm{d}s \\
&\quad\leqq \sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_\infty \int_0^t \int \frac{1}{\left(|z|+1\right)^4}\,|\partial_i u_j^{(n)}|\,\mathrm{d}z\,\mathrm{d}s \\
&\quad\leqq c\,t \sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_\infty \sup_{s\in(0,t)} \|\partial_i u_j^{(n)}(s)\|_{BMO} \\
&\quad\leqq c\,t \sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_\infty \sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_{BMO} \\
&\quad\leqq c\,t \left(\sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_\infty\right)^2.
\end{aligned} \tag{10}$$

(Since $\omega^{(n)}(s)$ is in L^2, $\partial_i u_j^{(n)}(s)$ is in L^2 as well (by Calderon–Zygmund), and the first *BMO*-bound is available).

The estimate on the advection term can be absorbed in the above bound, and we arrive at

$$\sup_{s\in(0,t)} \|\omega^{(n+1)}(s)\|_\infty \leqq c\,\|\omega_0\|_\infty + c\,t\left(\sup_{s\in(0,t)} \|\omega^{(n)}(s)\|_\infty\right)^2,$$

which yields boundedness of the sequence on the time-interval of the desired length. A similar argument implies the contraction property in $C_w\left([0,T], L^\infty\right)$, concluding the proof.

The complexification argument is analogous to the complexification of the velocity-pressure formulation; for details, see [16].

The main idea is as follows. Recall that the functions in the sequence of approximations $(\omega^{(n)}, u^{(n)})$ are entire functions in the spatial variable for any $t > 0$. Then, denoting a point in $\mathbb{C}^3$ by $x + iy$, introduce a related sequence $(\Omega^{(n)}, U^{(n)})$ by

$$\Omega^{(n)}(x,t) = \omega^{(n)}(x + i\alpha t, t) \quad \text{and} \quad U^{(n)}(x,t) = u^{(n)}(x + i\alpha t, t),$$

where $\alpha > 0$ is an optimization parameter.

The real and the imaginary parts of each of the elements in the sequence $(\Omega^{(n)}, U^{(n)})$ satisfy a real system in which the principal linear parts are decoupled, and the only new type of terms generated by the complexification procedure are two lower-order (first order) linear terms coming from the chain rule and the Cauchy–Riemann equations in $\mathbb{C}^3$.

Local-in-time estimates on the system are then of the same type as the estimates on the real three dimensional NSE, resulting in expansion of the domain of analyticity—after the optimization in the parameter α—comparable to $\sqrt{t}$ (the only global-in-time property needed is for the real and imaginary parts of each $\Omega^{(n)}$, on each real-space 'slice', to be in L^2; this is, as in the real case, guaranteed by the assumption that ω_0 in L^2).

More precisely, we arrive at

Theorem 10. (complex setting) *Let the initial datum ω_0 be in $L^2 \cap L^\infty$, and M a constant larger than 1. Then there is a constant $c(M) > 1$ such that there exists a unique mild solution ω in $C_w\big([0,T], L^\infty\big)$ where $T \geqq \frac{1}{c(M)} \frac{1}{\|\omega_0\|_\infty}$, and for any t in $(0,T]$ the solution ω is the $\mathbb{R}^3$-restriction of a holomorphic function ω defined in the domain*

$$\Omega_t = \left\{ x + iy \in \mathbb{C}^3 : |y| < \frac{1}{\sqrt{c(M)}} \sqrt{t} \right\};$$

moreover, $\|\omega(t)\|_{L^\infty(\Omega_t)} \leqq M \, \|\omega_0\|_\infty$.

Remark 11. The above theorem can be localized in the spirit of [4]; this would provide a (local) lower bound on the radius of spatial analyticity in terms of the local quantities only (at the expense of a logarithmic correction).

3. Spatial Complexity of the Vorticity Components: A Regularity Criterion

Henceforth, m^n will denote the n-dimensional Lebesgue measure. Let us start with recalling the concepts of 1D and three dimensional sparseness suitable for our purposes [11,19].

Let S be an open subset of $\mathbb{R}^3$, $\delta \in (0,1)$, x_0 a point in $\mathbb{R}^3$, and $r \in (0,\infty)$.

Definition 12. S is *1D δ-sparse around x_0 at scale r* if there exists a unit vector $\mathbf{d}$ in S^2 such that

$$\frac{m^1\big(S \cap (x_0 - r\mathbf{d}, x_0 + r\mathbf{d})\big)}{2r} \leqq \delta.$$

Definition 13. S is *three dimensional δ-sparse around x_0 at scale r* if

$$\frac{m^3\big(S \cap B(x_0, r)\big)}{m^3\big(B(x_0, r)\big)} \leqq \delta.$$

Remark 14. On one hand, it is straightforward to check that, for any S, three dimensional δ-sparseness at scale r implies 1D $(\delta)^{\frac{1}{3}}$-sparseness at scale ρ, for some ρ in $(0, r]$, at any given pair (x_0, r). On the other hand, there are plenty of simple examples in which any attempt at the converse fails. However, if one is interested in sparseness of a set at scale r, uniformly in x_0 (as we will be), the difference is much less pronounced.

We will formulate our regularity criterion as a no-blow-up criterion for a solution ω belonging to $C\big([0, T^*), L^\infty\big)$ where T^* is the first (possible) blow-up time. In this setting, it is convenient to have the notion of an 'escape time'.

Definition 15. Let ω be in $C\big([0, T^*), L^\infty\big)$ where T^* is the first possible blow-up time. A time t in $(0, T^*)$ is an *escape time* if $\|\omega(\tau)\|_\infty > \|\omega(t)\|_\infty$ for any τ in (t, T^*).

Remark 16. It is a well-known fact that for any level $L > 0$, there exists a unique escape time $t(L)$; this follows from the local-in-time well-posedness in a straightforward manner. As a consequence, there are continuum many escape times in $(0, T^*)$.

As in the previous section, we assume that the initial datum ω_0 is (in addition) in L^2.

Then we can solve the three dimensional NSE locally-in-time, at any t in $(0, T^*)$, and uniqueness in conjunction with Theorem 10 will yield a lower bound on the radius of spatial analyticity near t. In particular, for an escape time t, we consider a temporal point $s = s(t)$ conforming to the requirement

$$s \text{ in } \left[t + \frac{1}{4c(M)\|\omega(t)\|_\infty}, t + \frac{1}{c(M)\|\omega(t)\|_\infty}\right]; \tag{11}$$

then, a lower bound on the radius of spatial analyticity at s is given by

$$\frac{1}{2c(M)\|\omega(t)\|_\infty^{\frac{1}{2}}}.$$

Moreover, since t is an escape time, this lower bound can be replaced by

$$\frac{1}{2c(M)\|\omega(s)\|_\infty^{\frac{1}{2}}},$$

that is, in this situation there is no need to worry about a time-lag—a lower bound on the analyticity radius at s is given in terms of the L^∞-norm of the solution at s.

The property of 1D sparseness will be assumed on the super-level sets of the positive and negative parts of the vorticity components ω_j at s (recall that for a scalar-valued function f, the positive and negative parts of f are given by $f^+(x) =$

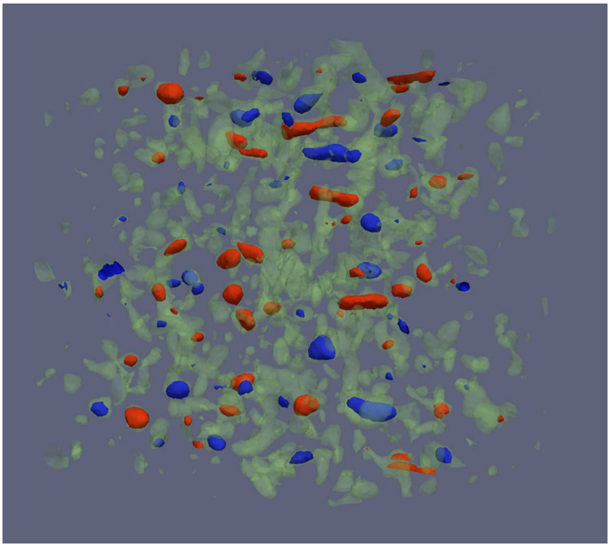

Fig. 1. The blue and the red regions are precisely the sets $V^{1,\pm}$, that is, the super-level sets of the positive and the negative parts of ω_1, to be contrasted to the totality of the region visualized which corresponds to the super-level set of the the full vectorial norm $\max\{|\omega_1|, |\omega_2|, |\omega_3|\}$ (from a snapshot of a simulation initialized with the low frequency noise-type initial data at the Reynolds number of approximately 10^4; courtesy of M. Mizstal, NBI) (colour figure online)

$\max(f(x), 0)$ and $f^-(x) = -\min(f(x), 0)$, respectively); more precisely, the sets of interests will be the sets $V_s^{j,\pm}$ defined by

$$V_s^{j,\pm} = \left\{ x \in \mathbb{R}^3 : \omega_j^\pm(x, s) > \frac{1}{2M} \|\omega(s)\|_\infty \right\} \tag{12}$$

(M is as in Theorem 10) (Fig. 1).

Since the proof of our regularity criterion is based on the harmonic measure maximum principle, we recall a version suitable for our purposes. Here, as in [19], we are concerned with the harmonic measure in the complex plane $\mathbb{C}$; for a suitable pair of sets (Ω, K), the harmonic measure of K with respect to Ω, evaluated at a point z, will be denoted by $h(z, \Omega, K)$. (For the basic properties of the harmonic measure in this setting, see, for example, [1,30]).

Proposition 17. [30] *Let Ω be an open, connected set in $\mathbb{C}$ such that its boundary has nonzero Hausdorff dimension, and let K be a Borel subset of the boundary. Suppose that u is a subharmonic function on Ω satisfying*

$$u(z) \leqq M, \quad \text{for} \quad z \in \Omega$$
$$\limsup_{z \to \zeta} u(z) \leqq m, \quad \text{for} \quad \zeta \in K.$$

Then

$$u(z) \leqq m\, h(z, \Omega, K) + M\big(1 - h(z, \Omega, K)\big) \quad \text{for} \quad z \in \Omega.$$

The last ingredient needed is an extremal property of the harmonic measure in the unit disk $\mathbb{D}$.

Proposition 18. [32] *Let λ be in $(0,1)$, K a closed subset of $[-1,1]$ such that $m^1(K)=2\lambda$, and suppose that the origin is in $\mathbb{D}\setminus K$. Then*

$$h(0,\mathbb{D},K) \geqq h(0,\mathbb{D},K_\lambda) = \frac{2}{\pi}\arcsin\frac{1-(1-\lambda)^2}{1+(1-\lambda)^2}$$

where $K_\lambda = [-1,-1+\lambda]\cup[1-\lambda,1]$.

At this point, it is beneficial to set a specific set of constants: let M be the solution to the equation $\frac{1}{2}h^* + (1-h^*)M = 1$ where $h^* = \frac{2}{\pi}\arcsin\frac{1-(\frac{3}{4})^{\frac{2}{3}}}{1+(\frac{3}{4})^{\frac{2}{3}}}$, and let $c(M)$ be as in Theorem 10 (note that $1 < M < \frac{3}{2}$).

Theorem 19. *Let ω be in $C\big([0,T^*),L^\infty\big)$ where T^* is the first possible blow-up time, and assume, in addition, that ω_0 is in L^2 (then, ω is automatically in $L^\infty\big((0,T^-),L^2\big)$ for any $0<T^-<T^*$).*

Let t be an escape time, and suppose that there exists a temporal point $s=s(t)$ as in (11) such that for any spatial point x_0, there exists a scale ρ, $0<\rho\leqq \frac{1}{2c(M)\|\omega(s)\|_\infty^{\frac{1}{2}}}$ and a direction $\mathbf{d}$ with the property that the super-level set $V_s^{j,\pm}$ delineated in (12) is 1D $(\frac{3}{4})^{\frac{1}{3}}$-sparse around x_0 at scale ρ; here, the pair $(j,\pm)$ is chosen according to the following selection criterion: $|\omega(x_0,s)| = \omega_j^{\pm}(x_0,s)$.

Then T^* is not a blow-up time.

Remark 20. Notice that—as far as the temporal intermittency goes—it is enough that the condition on local 1D sparseness holds at some $s(t)$ for a single escape time t (recall that there are continuum-many escape times; in particular, there are continuum-many escape times in any interval of the form $(T^*-\varepsilon,T^*)$).

Proof. (The argument to be presented is a refinement of the argument introduced in [19]).

Let t be an escape time and $s=s(t)$ satisfy the condition (11). The idea is to show that

$$\|\omega(s)\|_\infty \leqq \|\omega(t)\|_\infty, \tag{13}$$

utilizing the sparseness at s via the harmonic measure maximum principle; this would contradict the assumption that t is an escape and in turn the assumption that T^* is the first possible blow-up time.

Fix a spatial point x_0 and let $\mathbf{d}$ be the direction of 1D sparseness around x_0 postulated in the theorem. Since the three dimensional NSE are translationally and rotationally invariant, and the coordinate transformations would change neither the lower bound on the analyticity radius nor the L^∞-bound on the complexified solution (nor the sparseness), without loss of generality, assume that $x_0=0$ and $\mathbf{d}=(1,0,0)$. (The argument is completely local, that is, we are considering one spatial point x_0 at the time).

Let us focus on the complexification of the real coordinate x_1 (regarding the other two variables as parameters), and view $\omega(s)$ as an analytic, $\mathbb{C}^3$-valued function on the symmetric strip around the x_1-axis of the width of at least

$$\frac{1}{c(M)\|\omega(s)\|_\infty^{\frac{1}{2}}}.$$

Since ω is analytic, the real and the imaginary parts of the component functions ω_j are harmonic, and their positive and negative parts subharmonic in the strip; in particular, the assumption on the scale of sparseness ρ implies that they are subharmonic in the open disc

$$D_\rho = \{z \in \mathbb{C} : \ |z| < \rho\}.$$

Select a pair $(j, \pm)$ such that $|\omega(0, s)| = \omega_j^\pm(0, s)$. By the premise, the corresponding set $V_s^{j,\pm}$ is 1D $(\frac{3}{4})^{\frac{1}{3}}$-sparse around 0 at scale ρ (in the direction e_1).

Define a set K by

$$K = \overline{V_s^{j,\pm} \cap (-\rho, \rho)}$$

(the complement is taken within the interval $[-\rho, \rho]$), and note that $m^1(K) \geqq 2\rho\left(\frac{3}{4}\right)^{\frac{1}{2}}$.

There are two disjoint scenarios to consider. In the first one, 0 is in K. Then,

$$|\omega(0, s)| = \omega_j^\pm(0, s) \leqq \frac{1}{2M}\|\omega(s)\|_\infty \leqq \frac{1}{2}\|\omega(t)\|_\infty;$$

that is, the contribution of this particular x_0 is consistent with (13), and we can move on. In the second scenario, we can estimate the harmonic measure of K with respect to D_ρ at 0 utilizing the sparseness in conjunction with Proposition 18.

Recall that the harmonic measure is invariant with respect to any conformal map, and in particular with respect to the scaling map $\phi(z) = \frac{1}{\rho} z$; hence, Proposition 18 yields

$$h(0, D_\rho, K) \geqq \frac{2}{\pi} \arcsin \frac{1 - \left(\frac{3}{4}\right)^{\frac{2}{3}}}{1 + \left(\frac{3}{4}\right)^{\frac{2}{3}}} = h^*.$$

The harmonic measure maximum principle (Proposition 17) applied to the subharmonic function $\omega_j^\pm$ then implies the following bound:

$$\begin{aligned}\omega_j^\pm(0, s) &\leqq h^* \frac{1}{2}\|\omega(t)\|_\infty + (1 - h^*)\, M\|\omega(t)\|_\infty \\ &= \left(\frac{1}{2}h^* + (1 - h^*)M\right)\|\omega(t)\|_\infty = \|\omega(t)\|_\infty;\end{aligned}$$

the estimate on $\omega_j^\pm$ on the complement of K (within D_ρ) comes from the estimate on the complexified solution in Theorem 10, and the last equality from our choice of the parameters.

Consequently, the second scenario is also compatible with the inequality (13).

Since the point x_0 was an arbitrary point in $\mathbb{R}^3$, this concludes the proof. □

Remark 21. Note that the argument utilizing the sparseness via the harmonic measure maximum principle per se is completely local. It is the application of Theorem 10 that ascribes to Theorem 19 the uniformly local nature. To arrive to a completely local result, it suffices to localize the result presented in Theorem 10; c.f., Remark 11.

4. Spatial Complexity of the Vorticity Components: An A Priori Bound

Concepts of a set being r-mixed (or 'mixed to scale r') and r-semi-mixed appear in the study of mixing properties of incompressible (or nearly incompressible) flows, and in particular in the study of the optimal mixing strategies for passive scalars (for example, density of a tracer) advected by an incompressible velocity field (see, for example, [5,21]).

Definition 22. Let $r > 0$. An open set S is *r-semi-mixed* if there exists δ in $(0, 1)$ such that

$$\frac{m^3(S \cap B(x,r))}{m^3(B(x,r))} \leqq \delta$$

for every x in $\mathbb{R}^3$. If the complement of S is r-semi-mixed as well, then S is said to be *r-mixed.*

Remark 23. Recalling the definitions from the previous section, note that if the set S is r-semi-mixed (with the ratio δ), then it is three dimensional δ-sparse around every point x_0 in $\mathbb{R}^3$ at scale r.

The next lemma is a vector-valued version of a positive scalar-valued lemma in [21] where the application was to the density of a tracer; a vectorial Besov space version was given in [11] in the context of the $B^{-1}_{\infty,\infty}$-regularity criterion on the velocity. We present a detailed proof for completeness of the exposition.

Lemma 24. *Let $r \in (0, 1]$ and f a bounded, continuous vector-valued function on $\mathbb{R}^3$. Then, for any pair (λ, δ), λ in $(0, 1)$ and δ in $\left(\frac{1}{1+\lambda}, 1\right)$, there exists a constant $c^*(\lambda, \delta) > 0$ such that if*

$$\|f\|_{H^{-1}} \leqq c^*(\lambda, \delta)\, r^{\frac{5}{2}} \|f\|_\infty,$$

then each of the six super-level sets $S^{i,\pm}_\lambda = \left\{x \in \mathbb{R}^3 : f_i^\pm(x) > \lambda \|f\|_\infty\right\}$ is r-semi-mixed with the ratio δ. (H^{-1} denotes the dual of the Sobolev space H^1).

Remark 25. It is worth noting that there could be a significant discrepancy between semi-mixedness of the full vectorial super-level sets and the super-level sets of the components. As a matter of fact, the former may not be semi-mixed at all, while the latter may be r-semi-mixed at any predetermined scale. In particular, a duality argument of the type utilized in the lemma does not seem to lead to any quantifiable semi-mixedness of the full vectorial super-level sets.

Proof. Assume the opposite, that is, there exists an index i such that either $S_\lambda^{i,+}$ or $S_\lambda^{i,-}$ is not r-semi-mixed with the ratio δ. Suppose that it is $S_\lambda^{i,+}$ (if it were $S_\lambda^{i,-}$, the only modification would be to replace the function ϕ below with $-\phi$).

Consequently, there exists a spatial point x_0 such that

$$m^3\left(S_\lambda^{i,+}\cap B(x_0,r)\right) > \delta\, V_3\, r^3,$$

where V_3 denotes the volume of the unit ball in $\mathbb{R}^3$.

Let ϕ be an H^1-optimal, smooth radial (monotone) function, equal to 1 in $B(x_0,r)$, and vanishing outside $B(x_0,(1+\eta)r)$ for some $\eta > 0$ (the value to be determined). Then, by duality,

$$\|f\|_{H^{-1}} \geqq \frac{1}{\|\phi\|_{H^1}}\left|\int_{\mathbb{R}^3} f_i(x)\phi(x)\,\mathrm{d}x\right|. \tag{14}$$

An explicit calculation of the H^1-norm of ϕ yields

$$\|\phi\|_{H^1} \leqq c(\eta)\, r^{\frac{1}{2}} \tag{15}$$

for a suitable $c(\eta) > 0$ (recall that r is in $(0,1]$).

The objective is to obtain a lower bound on the numerator by performing a suitable domain-decomposition of the integral, that is,

$$\left|\int_{\mathbb{R}^3} f_i(x)\phi(x)\,\mathrm{d}x\right| \geqq \int_{\mathbb{R}^3} f_i(x)\phi(x)\,\mathrm{d}x \geqq I - |II| - |III|,$$

where

$$I = \int_{S_\lambda^{i,+}\cap B(x_0,r)} f_i(x)\phi(x)\,\mathrm{d}x,$$
$$II = \int_{B(x_0,r)\setminus S_\lambda^{i,+}} f_i(x)\phi(x)\,\mathrm{d}x$$

and

$$III = \int_{\left(B(x_0,(1+\eta)r)\setminus B(x_0,r)\right)} f_i(x)\phi(x)\,\mathrm{d}x.$$

It is transparent that

$$I = \int_{S_\lambda^{i,+}\cap B(x_0,r)} f_i(x)\,\mathrm{d}x = \int_{S_\lambda^{i,+}\cap B(x_0,r)} f_i^+(x)\,\mathrm{d}x$$
$$> \lambda\,\|f\|_\infty\, m^3\left(S_\lambda^{i,+}\cap B(x_0,r)\right) \geqq \lambda\,\delta\, V_3\, r^3\|f\|_\infty, \tag{16}$$

$$|II| = \left|\int_{B(x_0,r)\setminus S_\lambda^{i,+}} f_i(x),\ \mathrm{d}x\right| \leqq \|f\|_\infty\left(m^3(B(x_0,r) - m^3\left(S_\lambda^{i,+}\cap B(x_0,r)\right)\right)$$

$$\leqq \|f\|_\infty \left(V_3\, r^3 - \delta\, V_3\, r^3\right)$$
$$= (1-\delta)\; V_3\, r^3 \|f\|_\infty \tag{17}$$

and

$$\begin{aligned} |III| &\leqq \left| \int_{(B(x_0,(1+\eta)r)\setminus B(x_0,r))} f_i(x)\, \mathrm{d}x \right| \\ &\leqq \|f\|_\infty \left(m^3(B(x_0,(1+\eta)r) - m^3(B(x_0,r))\right) \\ &\leqq \left((1+\eta)^3 - 1\right)\; V_3\, r^3\, \|f\|_\infty . \end{aligned} \tag{18}$$

Collecting the estimates (14), (15) and (16)–(18), it follows that

$$\|f\|_{H^{-1}} > c^*(\eta)\, r^{\frac{5}{2}}\, \|f\|_\infty\, (\lambda\delta + \delta - (1+\eta)^3).$$

Since $\delta > \frac{1}{1+\lambda}$ is postulated, the equation $(1+\eta)^3 = \frac{\delta(1+\lambda)+1}{2}$ has a unique solution $\eta = \eta(\lambda, \delta)$; this choice of η yields

$$\|f\|_{H^{-1}} > c^*(\lambda, \delta)\, r^{\frac{5}{2}}\, \|f\|_\infty \tag{19}$$

with $c^*(\lambda, \delta) = c^*(\eta)\, \frac{\delta(1+\lambda)-1}{2}$ (which is positive since $\delta > \frac{1}{1+\lambda}$). This produces a contradiction. □

The next theorem is a simple consequence of the above lemma and a careful choice of parameters throughout the paper; it is designated a theorem because of its significance.

Theorem 26. *Let u be a Leray solution (a global-in-time weak solution satisfying the global energy inequality), and assume that ω is in $C\big((0, T^*), L^\infty\big)$ for some $T^* > 0$. Then for any τ in $(0, T^*)$ for which the scale r^* defined below is less or equal to one, the super-level sets*

$$V_\tau^{j,\pm} = \left\{ x \in \mathbb{R}^3 : \omega_j^\pm(x,\tau) > \frac{1}{2M} \|\omega(\tau)\|_\infty \right\}$$

are three dimensional $\frac{3}{4}$-sparse around any spatial point x_0 at scale

$$r^* = c(\|u_0\|_2)\, \frac{1}{\|\omega(\tau)\|_\infty^{\frac{2}{5}}}$$

where $c(\|u_0\|_2)$ is a constant depending only on the energy at time 0 (M is the parameter set preceding the statement of Theorem 19; recall that $1 < M < \frac{3}{2}$).

Proof. Notice that our choice of parameters implies that

$$\frac{3}{4} > \frac{1}{1 + \frac{1}{2M}},$$

and Lemma 24 is applicable.

Since we are on the whole space, an efficient way to estimate $\|\omega(\tau)\|_{H^{-1}}$ is by switching to the Fourier space where one is required to estimate integrals of the form

$$I_{i,j} = \int \frac{1}{1+|\xi|^2}\, |\widehat{\partial_i u_j}(\xi,\tau)|^2 \, d\xi.$$

This is plain since

$$I_{i,j} \leqq \int \frac{|\xi_i|^2}{1+|\xi|^2}\, |\widehat{u_j}(\xi,\tau)|^2 \, d\xi \leqq \|\widehat{u(\tau)}\|_2^2 = \|u(\tau)\|_2^2 \leqq \|u_0\|_2^2$$

(by the energy inequality).

Consequently, in order to satisfy all the assumptions in the lemma, it suffices to postulate

$$c\,\|u_0\|_2 \leqq c^*\left(\frac{1}{1+\frac{1}{2M}}, \frac{3}{4}\right) r^{\frac{5}{2}}\, \|\omega(\tau)\|_\infty,$$

which forces the choice of the scale of sparseness r^* in the theorem. □

Remark 27. For our purposes, the restriction $r^* \leqq 1$ in the theorem is irrelevant since we are only interested in temporal points τ leading to a possible blow-up time. It is also more of a feature than a bug since the formation of small scales is a characteristic of the fully nonlinear regime.

Remark 28. Some effort has been made to assure that both the cut-off levels for the super-level sets and the ratios of sparseness in Theorem 19 and Theorem 26 are identical, $\lambda = \frac{1}{2M}$ and $\delta = \frac{3}{4}$. Moreover, since the constants featured in either of the two scales of sparseness are either absolute (after our choice of parameters was made) constants or the constant depending only on the initial energy of the solution, the tolerance parameter c_0 depends only on the initial energy. Shortly, the regularity condition is given in the class $Z_{\frac{1}{2}}\left(\frac{1}{2M}, \frac{3}{4}; c_0(\|u_0\|_2)\right)$, and the a priori bound in the class $Z_{\frac{2}{5}}\left(\frac{1}{2M}, \frac{3}{4}; c_0(\|u_0\|_2)\right)$.

5. Epilogue

The authors' goal was to present a mathematical framework—the scale of classes Z_α—in which one could break the archetypal three dimensional NS scaling barrier in the context of a blow-up-type argument. More precisely—in this context—we showed that the a priori bound can be shifted by an algebraic factor, from $Z_{\frac{1}{3}}$, which corresponds to the classical energy-level a priori bounds, to $Z_{\frac{2}{5}}$; recall that the regularity class in this setting is $Z_{\frac{1}{2}}$, which corresponds to the classical regularity criteria, and can be viewed as a statement that the radius of spatial analyticity is a *bona fide* diffusion scale.

Parallel to the continued mathematical efforts, there is an effort directed at gaining some insight into a 'typical value' of α by performing fully resolved computational simulations of turbulent flows, as well as harvesting data from the Johns

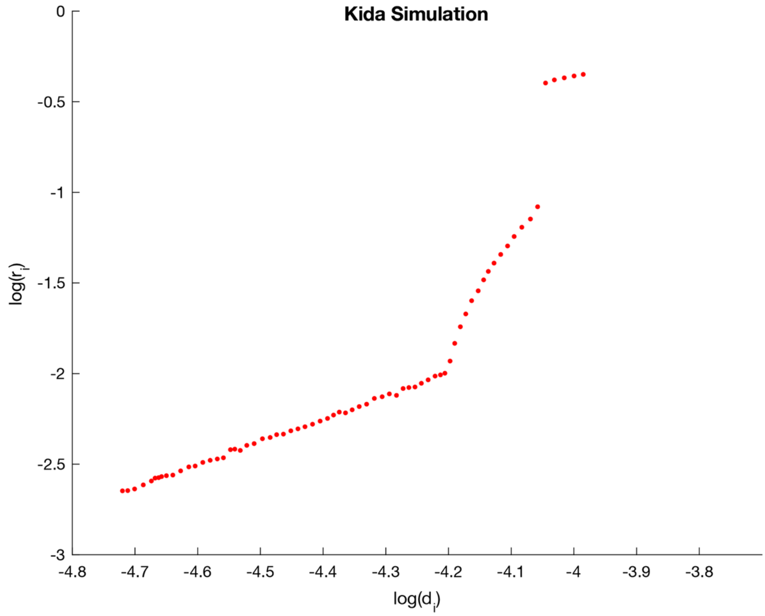

Fig. 2. Log–log plot of the scale of three dimensional sparseness (of the vorticity components) $r_i = r_i(t_i)$ versus the diffusion scale $d_i = \frac{\nu^{\frac{1}{2}}}{\|\omega(t_i)\|_\infty^{\frac{1}{2}}}$ from a simulation initialized with the Kida vortex [22] initial condition at the Reynolds number of approximately 10^4. Originating the flow at the Kida vortex initial condition causes $\|\omega(t)\|_\infty$ to increase sharply before reaching the maximum and slumping, that is, it produces a burst of $\|\omega(t)\|_\infty$ which is as close to 'modeling a singularity' in the three dimensional NS flows as feasible. In the plot, the time runs from right to left; (approximately) the first third of the time-slices are sourced from the initial time-interval in which the flow still has 'too much memory' of the initial condition, and the last two thirds of the time slices are sourced from the time-interval leading to the peak. As illustrated in the plot—in the time-interval leading to the peak—a power-type relation of the form $r \approx d^\alpha$ settles in; after proper rescaling, it is revealed that the slope of the line is approximately equal to $\frac{6}{5}$, that is, $r \approx d^{\frac{6}{5}}$. This implies that the solution in view (leading to the peak) is approximately in $Z_{\frac{3}{5}}$. Recall that the energy-level bound is $Z_{\frac{1}{3}}$, our a priori bound is $Z_{\frac{2}{5}}$, and what is needed for our no blow-up criterion is at least $Z_{\frac{1}{2}}$. Hence, the plot indicates that further improvements of the Z_α a priori bounds—even past the critical class $Z_{\frac{1}{2}}$—might indeed be possible (courtesy of M. Mizstal, NBI and J. Rafner, NBI and Aarhus)

Hopkins Turbulence Databases (JHTDB), and trying to identify the scaling properties of the scale of sparseness of the super-level sets of the positive and negative parts of the vorticity components with respect to the diffusion scale $\frac{\nu^{\frac{1}{2}}}{\|\omega\|_\infty^{\frac{1}{2}}}$. This has been a joint project with the complexity group of J. Mathiesen at the Niels Bohr Institute (NBI) in Copenhagen, and in particular, with M. Mizstal, and the ScienceAtHome group at Aarhus University led by J. Sherson; J. Rafner, of both the NBI and Aarhus is the project coordinator.

The simulations performed and the databases utilized cover both decaying turbulence (for example, initialized with the Kida vortex or with the frequency-localized noise) and forced isotropic turbulence; in all simulations/databases, the spatial domain is a periodic box.

The data was collected and analyzed in three different regimes: deeply in the inertial range, deeply in the dissipation range, and in the regime of the most interest in the studies of possible singularity formation in the three dimensional NSE, the regime of transition between the two.

The results will be detailed in a separate publication. Here, we would like to report that there are indications of a statistically significant dependence between the two scales (the geometric scale of sparseness and the analytic diffusion scale) in all three regimes; however, the transitional regime has been the one signified by the strong evidence of a power law dependence with the range of exponents implying that the solution in view stabilizes in a class Z_α, for some $\alpha > \frac{1}{2}$, indicating no obstruction to furthering the rigorous Z_α-theory presented here. Figure 2 provides an example of this phenomenon.

Acknowledgements. The work of A.F. is supported in part by the National Science Foundation Grant DMS-1418911. The work of Z.G. is supported in part by the National Science Foundation Grant DMS-1515805 and the Lundbeck Foundation Grant R217-2016-446 (a collaborative grant with the NBI and Aarhus). Z.G. thanks the Program in Applied and Computational Mathematics at Princeton for their hospitality in Spring 2017 when the paper was finalized. The authors thank the referee for the constructive comments.

References

1. Ahlfors, L.: *Conformal Invariants: Topics in Geometric Function Theory*. AMS Chelsea, Providence, RI, 2010
2. Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3D Euler equations. *Commun. Math. Phys.* **94**, 61, 1984
3. Bradshaw, Z., Grujić, Z.: A spatially localized L log L estimate on the vorticity in the 3D NSE. *Indiana Univ. Math. J.* **64**, 433, 2015
4. Bradshaw, Z., Grujić, Z., Kukavica, I.: Local analyticity radii of solutions to the 3D Navier–Stokes equations with locally analytic forcing. *J. Differ. Equ.* **259**, 3955, 2015
5. Bressan, A.: A lemma and a conjecture on the cost of rearrangements. *Rend. Sem. Mat. Univ. Padova* **110**, 97, 2003
6. Constantin, P.: Navier–Stokes equations and area of interfaces. *Commun. Math. Phys.* **129**, 241, 1990
7. Constantin, P.: Geometric statistics in turbulence. *SIAM Rev.* **36**, 73, 1994
8. Constantin, P., Fefferman, C.: Direction of vorticity and the problem of global regularity for the Navier–Stokes equations. *Indiana Univ. Math. J.* **42**, 775, 1993
9. da Veiga, H.B., Berselli, L.C.: On the regularizing effect of the vorticity direction in incompressible viscous flows. *Differ. Integral Eqs.* **15**, 345, 2002
10. Escauriaza, L., Seregin, G., Shverak, V.: $L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk.* **58**, 3, 2003
11. Farhat, A., Grujić, Z., Leitmeyer, K.: The space $B^{-1}_{\infty,\infty}$, volumetric sparseness, and 3D NSE. *J. Math. Fluid Mech.* **19**, 515, 2017
12. Foias, C., Temam, R.: Gevrey class regularity for the solutions of the Navier–Stokes equations. *J. Funct. Anal.* **87**, 359, 1989

13. Gallagher, I., Koch, G., Planchon, F.: Blow-up of critical Besov norms at a potential Navier–Stokes singularity. *Commun. Math. Phys.* **343**, 1, 2016
14. Giga, Y., Miyakawa, T.: Navier–Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces. *Commun. Partial Differ. Equ.* **14**, 577, 1989
15. Giga, Y., Inui, K., Matsui, S.: On the Cauchy problem for the Navier–Stokes equations with nondecaying initial data. *Hokkaido University Preprint Series in Mathematics*, 1998
16. Grujić, Z., Kukavica, I.: Space analyticity for the Navier–Stokes and related equations with initial data in L^p. *J. Funct. Anal.* **152**, 447, 1998
17. Grujić, Z.: Localization and geometric depletion of vortex-stretching in the 3D NSE. *Commun. Math. Phys.* **290**, 861, 2009
18. Grujić, Z., Guberović, R.: Localization of analytic regularity criteria on the vorticity and balance between the vorticity magnitude and coherence of the vorticity direction in the 3D NSE. *Commun. Math. Phys.* **298**, 407, 2010
19. Grujić, Z.: A geometric measure-type regularity criterion for solutions to the 3D Navier–Stokes equations. *Nonlinearity* **26**, 289, 2013
20. Guberović, R.: Smoothness of Koch–Tataru solutions to the Navier–Stokes equations revisited. *Discrete Contin. Dyn. Syst.* **27**, 231, 2010
21. Iyer, G., Kiselev, A., Xu, X.: Lower bound on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows. *Nonlinearity* **27**, 973, 2014
22. Kida, S.: Three-dimensional periodic flows with high symmetry. *J. Phys. Soc. Jpn.* **54**, 2132, 1985
23. Kukavica, I.: On local uniqueness of weak solutions of the Navier–Stokes system with bounded initial data. *J. Differ. Equ.* **194**, 39, 2003
24. Leray, J.: Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.* **12**, 1, 1933
25. Leray, J.: Essai sur les mouvements plans d'un liquide visqueux que limitent de parois. *J. Math. Pures Appl.* **13**, 331, 1934
26. Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63**, 193, 1934
27. Lemarie-Rieusset, P.G.: *Recent Developments in the Navier–Stokes Problem*. Chapman and Hall/CRC, London, 2002
28. Meyer, Y.: Oscillating patterns in some nonlinear evolution equations. *Lecture Notes in Mathematics* **1871**, *C. I. M. E. Foundation Subseries: Mathematical Foundation of Turbulent Viscous Flows*, Springer, 2003
29. Phuc, N.C.: The Navier–Stokes equations in nonendpoint borderline Lorentz spaces. *J. Math. Fluid Mech.* **17**, 4, 2015
30. Ransford, T.: *Potential Theory in the Complex Plane*. Cambridge University Press, Cambridge, 1995
31. Schumacher, J., Scheel, J., Krasnov, D., Donzis, D., Yakhot, V., Sreenivasan, K.R.: Small-scale universality in fluid turbulence. *PNAS* **111**, 10961, 2014
32. Solynin, A.Y.: Ordering of sets, hyperbolic metrics, and harmonic measure. *J. Math. Sci.* **95**, 2256, 1999
33. Stein, E.M.: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, 1993
34. Taylor, G.I.: Production and dissipation of vorticity in a turbulent fluid. *Proc. R. Soc. Lond. A* **164**, 15, 1937

Zachary Bradshaw
Department of Mathematics,
University of Arkansas,
Fayetteville,
USA.
e-mail: zb002@uark.edu

and

Aseel Farhat
Department of Mathematics,
Florida State University,
Tallahassee,
USA.
e-mail: aseelfarhat842@gmail.com

and

Zoran Grujić
Department of Mathematics,
University of Virginia,
Charlottesville,
USA.
e-mail: zg7c@virginia.edu

(*Received April 9, 2018 / Accepted September 14, 2018*)
Published online September 21, 2018