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Large Time Behavior of Solutions to 3-D MHD System with Initial Data Near Equilibrium

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Abstract

Califano and Chiuderi (Phys Rev E 60 (PartB):4701–4707, 1999) conjectured that the energy of an incompressible Magnetic hydrodynamical system is dissipated at a rate that is independent of the ohmic resistivity. The goal of this paper is to mathematically justify this conjecture in three space dimensions provided that the initial magnetic field and velocity is a small perturbation of the equilibrium state $(e_3, 0)$. In particular, we prove that for such data, a 3-D incompressible MHD system without magnetic diffusion has a unique global solution. Furthermore, the velocity field and the difference between the magnetic field and e_3 decay to zero in both L^∞ and L^2 norms with explicit rates. We point out that the decay rate in the L^2 norm is optimal in sense that this rate coincides with that of the linear system. The main idea of the proof is to exploit Hörmander's version of the Nash–Moser iteration scheme, which is very much motivated by the seminar papers by Klainerman (Commun Pure Appl Math 33:43–101, 1980, Arch Ration Mech Anal 78:73–98, 1982, Long time behaviour of solutions to nonlinear wave equations. PWN, Warsaw, pp 1209–1215, 1984) on the long time behavior to the evolution equations.

1. Introduction

In this paper, we investigate the large time behavior of the global smooth solutions to the following three-dimensional incompressible magnetic hydrodynamical (or MHD in short) system with initial data being sufficiently close to the equilibrium state $(e_3, 0)$:

$$\begin{cases} \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = b \cdot \nabla b, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (b, u)|_{t=0} = (b_0, u_0) \quad \text{with} \quad b_0 = e_3 + \varepsilon\phi, \end{cases} \tag{1.1}$$

where $b = (b^1, b^2, b^3)$ denotes the magnetic field, $u = (u^1, u^2, u^3)$ and p stand for the velocity and scalar pressure of the fluid respectively. This MHD system (1.1) with zero diffusivity in the magnetic field equation can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions being extremely small. One may check the references [5,12,14,23] for more explanations of this system.

Whether or not there is dissipation for the magnetic field of (1.1) is a very important problem in the physics of plasmas. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at length scales which are much larger than the dissipative ones. It has been conjectured that in the two-dimensional MHD system, energy is dissipated at a rate that is independent of the ohmic resistivity [7]. In other words, the diffusivity for the magnetic field equation can be zero yet the whole system may still be dissipative. The goal of this paper is to rigorously justify this conjecture in three space dimensions provided that the initial data of (1.1) is a small perturbation of the equilibrium state $(e_3, 0)$.

Concerning the well-posedness issue of the system (1.1), CHEMIN et al. [11] proved the local well-posedness of (1.1) with initial data in the critical Besov spaces. Lin and the second author [25] proved the global well-posedness to a modified three-dimensional MHD system with initial data sufficiently close to the equilibrium state (see [26] for a simplified proof). LIN, XU and the second author [24] established the global well-posedness of (1.1) in 2-D provided that the initial data is near the equilibrium state $(e_d, 0)$ and the initial magnetic field, b_0, satisfies a sort of admissible condition, namely

$$\int_{\mathbb{R}} (b_0 - e_3)(Z(t, \alpha))\,\mathrm{d}t = 0 \quad \text{for all} \quad \alpha \in \mathbb{R}^{\mathrm{d}} \times \{0\}, \tag{1.2}$$

with $Z(t, \alpha)$ being determined by

$$\begin{cases} \dfrac{\mathrm{d}Z(t,\alpha)}{\mathrm{d}t} = b_0(Z(t,\alpha)), \\ Z(t,\alpha)|_{t=0} = \alpha \end{cases}$$

Similar results in three space dimensions were proved by Xu and the second author in [31].

In the 2-D case, the restriction (1.2) was removed by Ren, Wu, Xiang and Zhang in [28] by carefully exploiting the divergence structure of the velocity field. Moreover, the authors proved that

$$\begin{aligned} &\|\partial_{x_2}^k b(t)\|_{L^2} + \|\partial_{x_2}^k u(t)\|_{L^2} \\ &\quad \le C\langle t\rangle^{-\frac{1+2k}{4}+\epsilon} \quad \text{for any } \epsilon \in]0, 1/2[\quad \text{and} \quad k = 0, 1, 2, \end{aligned} \tag{1.3}$$

where $\langle t\rangle \overset{\text{def}}{=} \big(1 + t^2\big)^{\frac{1}{2}}$. A more elementary existence proof was also given by ZHANG [32]. Very recently, Abidi and the second author removed the restriction (1.2) in [1] for the 3-D MHD system. Moreover, if the initial magnetic field is equal to e_3 and with other technical assumptions, this solution decays to zero according to

$$\|u(t)\|_{H^2} + \|b(t) - e_3\|_{H^2} \leq C\langle t\rangle^{-\frac{1}{4}}. \tag{1.4}$$

Note that (1.4) corresponds to the critical case of (1.3), that is, $\epsilon = 0$ in (1.3).

This idea of considering the global well-posedness of MHD system with initial data close to the equilibrium state $(e_d, 0)$ goes back to the work of Bardos, Sulem and Sulem [2] for the global well-posedness of an ideal incompressible MHD system. In general, it is not known whether or not classical solutions of (1.1) can develop finite time singularities even in two dimensions. In the case when there is full magnetic diffusion in (1.1), one may check [16] for its local well-posedness in the classical Sobolev spaces and [29] for the global well-posedness of such a system in two space dimensions. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [8] (see also [9]) proved that such a system is globally well-posed for any data in $H^2(\mathbb{R}^2)$. Lately, He et al. [17] (see also [6] and [30]) justified the vanishing viscosity limit of the full diffusive MHD system to the solution constructed by Bardos et al. [2] for the ideal MHD system.

The main result of this paper is as follows:

Theorem 1.1. *Let* $e_3 = (0, 0, 1)$, $b_0 = e_3 + \varepsilon\phi$ *with* $\phi = (\phi_1, \phi_2, \phi_3) \in C_c^\infty$ *and* $\operatorname{div}\phi = 0$, *let* $u_0 \in W^{N_0,1} \cap H^{N_0}$ *for some integer* N_0 *sufficiently large. Then there exist sufficiently small positive constants* ε_0, c_0 *such that if*

$$\|u_0\|_{W^{N_0,1}} + \|u_0\|_{H^{N_0}} \leq c_0 \quad \textit{and} \quad \varepsilon \leq \varepsilon_0, \tag{1.5}$$

(1.1) *has a unique global solution* (b, u) *so that for any* $T > 0$, $b \in C^2([0, T]\times\mathbb{R}^3)$, $u \in C^2([0, T] \times \mathbb{R}^3)$. *Moreover, for some* $\kappa > 0$, *it holds that*

$$\begin{aligned} &\|u(t)\|_{W^{2,\infty}} \leq C_\kappa\langle t\rangle^{-\frac{5}{4}+\kappa}, \quad \|b(t) - e_3\|_{W^{2,\infty}} \leq C_\kappa\langle t\rangle^{-\frac{3}{4}+\kappa} \quad \textit{and} \\ &\|u(t)\|_{H^2} + \|b(t) - e_3\|_{H^2} \leq C\langle t\rangle^{-\frac{1}{2}}, \quad \|\nabla u(t)\|_{L^2} \leq C\langle t\rangle^{-1}. \end{aligned} \tag{1.6}$$

Let us remark that the above theorem recovers the global well-posedness result of the system (1.1) in [1]. Moreover, the bigger the integer N_0, the smaller the positive constant κ. The main idea of the proof here works in both two and three space dimensions. The L^∞ decay rates of the solution in (1.6) are completely new. The L^2 decay rates of the solution are optimal in the sense that these decay rates coincide with those of the linearized system (see Propositions 2.1 and 2.7 below), which greatly improves the rate given by (1.4). We can also work on the decay rates for the higher order derivatives of the solutions, but we choose not to pursue this direction here.

2. Structure and Strategies of the Proof

2.1. Lagrangian Formulation of (1.1)

As observed in the previous references [24,31], the linearized system of (1.1) around the equilibrium state $(e_3, 0)$ reads

$$\begin{cases} Y_{tt} - \Delta Y_t - \partial_3^2 Y = \boldsymbol{f} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ Y|_{t=0} = Y^{(0)}, \quad Y_t|_{t=0} = Y^{(1)}. \end{cases} \tag{2.1}$$

It is easy to calculate that this system has two different eigenvalues:

$$\lambda_1(\xi) = -\frac{|\xi|^2}{2} + \sqrt{\frac{|\xi|^4}{4} - \xi_3^2} \quad \text{and} \quad \lambda_2(\xi) = -\frac{|\xi|^2}{2} - \sqrt{\frac{|\xi|^4}{4} - \xi_3^2}. \tag{2.2}$$

The Fourier modes corresponding to $\lambda_2(\xi)$ decay like $e^{-t|\xi|^2}$. By contrast, the decay property of the Fourier modes corresponding to $\lambda_-(\xi)$ vary with the directions of ξ as

$$\lambda_1(\xi) = -\frac{2\xi_3^2}{|\xi|^2\left(1 + \sqrt{1 - \frac{4\xi_3^2}{|\xi|^4}}\right)} \to -1 \quad \text{as} \quad |\xi| \to \infty$$

only in the ξ_3 direction. This simple analysis shows that the dissipative properties of system (2.1) may be more complicated than that for the linearized system of the isentropic compressible Navier-Stokes system (see [13] for instance). Moreover, it is well-known that it is in general impossible to propagate the anisotropic regularities for the transport equation. This motivates us to use the Lagrangian formulation of the system (1.1).

Let us now recall the Lagrangian formulation of (1.1) from [1]. Letting (b, u) be a smooth enough solution of (1.1), we define:

$$\begin{aligned} &X(t, y) = y + \int_0^t u(t', X(t', y))\mathrm{d}t' \stackrel{\text{def}}{=} y + Y(t, y), \quad \boldsymbol{u}(t, y) \stackrel{\text{def}}{=} u(t, X(t, y)), \\ &\boldsymbol{b}(t, y) \stackrel{\text{def}}{=} b(t, X(t, y)), \quad \boldsymbol{p}(t, y) \stackrel{\text{def}}{=} p(t, X(t, y)), \quad \mathcal{A} \stackrel{\text{def}}{=} \left(Id + \nabla_y Y\right)^{-1} \quad \text{and} \\ &\nabla_Y \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y. \end{aligned} \tag{2.3}$$

Then $(Y, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{p})$ solves

$$\begin{cases} \boldsymbol{b}(t, y) = \partial_{b_0} X(t, y), \quad \nabla_Y \cdot \boldsymbol{b} = 0, \\ Y_{tt} - \Delta_y Y_t - \partial_{b_0}^2 Y = \partial_{b_0} b_0 + g, \\ Y_{|t=0} = Y^{(0)} = 0, \qquad Y_{t\,|t=0} = Y^{(1)} = u_0(y), \end{cases} \tag{2.4}$$

where

$$\begin{aligned} &g = \operatorname{div}_y\left[(\mathcal{A}\mathcal{A}^t - Id)\nabla_y Y_t\right] - \mathcal{A}^t \nabla_y \boldsymbol{p}, \quad \partial_{b_0} \stackrel{\text{def}}{=} b_0 \cdot \nabla_y, \quad \text{and} \\ &(\Delta_x p)(t, X(t, y)) = \sum_{i,j=1}^{3} \nabla_{Y^i} \nabla_{Y^j}\left(\partial_{b_0} X^i \partial_{b_0} X^j - Y_t^i Y_t^j\right)(t, y). \end{aligned} \tag{2.5}$$

In what follows, we assume that

$$\mathrm{supp}(b_0(x_\mathrm{h},\cdot)-e_3)\subset[0,K] \quad \text{and} \quad b_0^3\neq 0. \tag{2.6}$$

Due to the difficulty of the variable coefficients for the linearized system of (2.4), we shall use a Frobenius Theorem type argument to find a new coordinate system $\{z\}$ so that $\partial_{b_0}=\partial_{z_3}$. Towards this, let us define

$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}y_3}=\dfrac{b_0^1}{b_0^3}(y_1,y_2,y_3), \quad y_1|_{y_3=0}=w_1,\\ \dfrac{\mathrm{d}y_2}{\mathrm{d}y_3}=\dfrac{b_0^2}{b_0^3}(y_1,y_2,y_3), \quad y_2|_{y_3=0}=w_2,\\ y_3=w_3, \end{cases} \tag{2.7}$$

and

$$z_1=w_1,\quad z_2=w_2,\quad z_3=w_3+\int_0^{w_3}\Big(\frac{1}{b_0^3(y(w))}-1\Big)\,\mathrm{d}w_3'. \tag{2.8}$$

Then we have

$$\begin{aligned} &\partial_{b_0}f(y)=\frac{\partial f(y(w(z)))}{\partial z_3}, \quad \text{and} \quad \nabla_y=\nabla_Z=\mathcal{B}^t(z)\nabla_z \quad \text{with}\\ &\mathcal{B}(z)=\Big(\frac{\partial y(w(z))}{\partial z}\Big)^{-1}. \end{aligned} \tag{2.9}$$

It is easy to observe that

$$\begin{aligned} \mathcal{B}(z)=\Big(\frac{\partial y(w(z))}{\partial z}\Big)^{-1}&=\Big(\frac{\partial y(w(z))}{\partial w}\times\frac{\partial w(z)}{\partial z}\Big)^{-1}\\ &=\Big(\frac{\partial w(z)}{\partial z}\Big)^{-1}\Big(\frac{\partial y(w(z))}{\partial w}\Big)^{-1}=\Big(\frac{\partial z}{\partial w}\Big)\Big(\frac{\partial y(w(z))}{\partial w}\Big)^{-1}, \end{aligned}$$

yet it follows from (2.7) that

$$\begin{aligned} \Big(\frac{\partial y(w)}{\partial w}\Big)&=\begin{pmatrix}1 & 0 & \frac{b_0^1}{b_0^3}\\ 0 & 1 & \frac{b_0^2}{b_0^3}\\ 0 & 0 & 1\end{pmatrix}\\ &\quad+\begin{pmatrix}\int_0^{w_3}\frac{\partial}{\partial y_1}\big(\frac{b_0^1}{b_0^3}\big)\mathrm{d}y_3' & \int_0^{w_3}\frac{\partial}{\partial y_2}\big(\frac{b_0^1}{b_0^3}\big)\mathrm{d}y_3' & 0\\ \int_0^{w_3}\frac{\partial}{\partial y_1}\big(\frac{b_0^2}{b_0^3}\big)\mathrm{d}y_3' & \int_0^{w_3}\frac{\partial}{\partial y_2}\big(\frac{b_0^2}{b_0^3}\big)\mathrm{d}y_3' & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{\partial y_1}{\partial w_1} & \frac{\partial y_1}{\partial w_2} & \frac{\partial y_1}{\partial w_3}\\ \frac{\partial y_2}{\partial w_1} & \frac{\partial y_2}{\partial w_2} & \frac{\partial y_2}{\partial w_3}\\ \frac{\partial y_3}{\partial w_1} & \frac{\partial y_3}{\partial w_2} & \frac{\partial y_3}{\partial w_3}\end{pmatrix}\\ &\overset{\text{def}}{=}A_1(y(w))+A_2(y(w))\Big(\frac{\partial y(w)}{\partial w}\Big), \end{aligned} \tag{2.10}$$

which gives

$$\Big(\frac{\partial y(w)}{\partial w}\Big)=\big(Id-A_2(y(w))\big)^{-1}A_1(y(w)). \tag{2.11}$$

It is easy to observe that

$$\left(\frac{\partial z(w)}{\partial w}\right)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \int_0^{w_3}\frac{\partial}{\partial w_1}\left(\frac{1}{b_0^3(y(w))}\right)\mathrm{d}w_3' & \int_0^{w_3}\frac{\partial}{\partial w_2}\left(\frac{1}{b_0^3(y(w))}\right)\mathrm{d}y_3' & \frac{1}{b_0^3}\end{pmatrix}\overset{\text{def}}{=}A_3(w). \tag{2.12}$$

As a consequence, we obtain

$$\begin{aligned}y(w)&=(y_{\rm h}(w_{\rm h},w_3),w_3),\quad w(z)=(z_{\rm h},w_3(z)),\quad\text{and}\\ y(w(z))&=\big(y_{\rm h}(z_{\rm h},w_3(z)),w_3(z)\big),\\ \mathcal{B}(z)&=A_3(w(z))A_1^{-1}(y(w(z))\big(Id-A_2(w(z))\big),\end{aligned}\tag{2.13}$$

with the matrices A_1, A_2, A_3 being determined by (2.10) and (2.12), respectively.

For simplicity, let us abuse the notation that $Y(t,z)=Y(t,y(w(z)))$. Then the system (2.4) becomes

$$\begin{cases}Y_{tt}-\Delta_z Y_t-\partial_{z_3}^2 Y=\big(\nabla_Z\cdot\nabla_Z-\Delta_z\big)Y_t+\partial_{z_3}b_0(y(w(z)))+g(y(w(z))),\\ Y_{|t=0}=Y_0=0,\qquad Y_{t\,|t=0}=Y_1(z)=u_0(y(w(z))),\end{cases}\tag{2.14}$$

for g given by (2.4). Since $\partial_{z_3}b_0(y(w(z)))$ in the source term is a time independent function, we now introduce a smooth cut-off function $\eta(z_3)$ with $\eta(z_3)=\begin{cases}0, & z_3\geq 2+K,\\ 1, & -1\leq z_3\leq 1+K\\ 0, & z_3\leq -2,\end{cases}$ and a correction term $\tilde{Y}$ so that $Y=\tilde{Y}+\bar{Y}$ and

$$\begin{aligned}\widetilde{Y}(z)=\eta(z_3)\bigg(&\int_{-1}^{z_3}\big(e_3-b_0(y(w(z_{\rm h},z_3')))\big)\,\mathrm{d}z_3'\\ &-\int_{-1}^{K+1}\big(e_3-b_0(y(w(z_{\rm h},z_3')))\big)\,\mathrm{d}z_3'\bigg),\end{aligned}\tag{2.15}$$

which satisfies

$$\partial_{z_3}\tilde{Y}(z)=e_3-b_0(y(w(z))),\quad\text{and}\quad\partial_{z_3}\big(\partial_{z_3}\tilde{Y}+b_0(y(w(z)))\big)=0.\tag{2.16}$$

Then in view of (2.23), (2.24) and (2.30) of [1], $\bar{Y}$ solves

$$\begin{cases}\bar{Y}_{tt}-\Delta_z\bar{Y}_t-\partial_{z_3}^2\bar{Y}=f,\\ \bar{Y}_{|t=0}=\bar{Y}^{(0)}=-\tilde{Y},\qquad \bar{Y}_{t\,|t=0}=Y^{(1)},\end{cases}\tag{2.17}$$

with

$$\begin{aligned}&\mathcal{A}=\big(Id+\mathcal{B}^t\nabla_z\widetilde{Y}+\mathcal{B}^t\nabla_z\bar{Y}\big)^{-1},\quad\text{and}\\ &f=\mathcal{B}^t\nabla_z\cdot\big[(\mathcal{A}\mathcal{A}^t-Id)\mathcal{B}^t\nabla_z\bar{Y}_t\big]+\mathcal{B}^t\nabla_z\cdot(\mathcal{B}^t\nabla_z\bar{Y}_t)-\Delta_z\bar{Y}_t-(\mathcal{B}\mathcal{A})^t\nabla_z\boldsymbol{p},\\ &\nabla_z\boldsymbol{p}=-\nabla_z\Delta_z^{-1}\mathrm{div}_z\big(\det(\mathcal{B}^{-1})(\mathcal{B}\mathcal{A}\mathcal{A}^t\mathcal{B}^t-Id)\nabla_z\boldsymbol{p}\big)\\ &\qquad-\nabla_z\Delta_z^{-1}\mathrm{div}_z\big((\det(\mathcal{B}^{-1})Id-Id)\nabla_z\boldsymbol{p}\big)\\ &\qquad+\nabla_z\Delta_z^{-1}\mathrm{div}_z\Big(\mathcal{B}\mathcal{A}\mathrm{div}_z\Big(\det(\mathcal{B}^{-1})\mathcal{B}\mathcal{A}\big(\partial_3\bar{Y}\otimes\partial_3\bar{Y}-\bar{Y}_t\otimes\bar{Y}_t\big)\Big)\Big).\end{aligned}\tag{2.18}$$

2.2. The Proof of Theorem 1.1

Before presenting the main result for the system (2.17–2.18), let us first introduce notations of the norms: for $f : \mathbb{R}^3_y \to \mathbb{R}$, $u : \mathbb{R}^+ \times \mathbb{R}^3_y \to \mathbb{R}$, and $p \in [1, +\infty]$, $N \in \mathbb{N}$, we denote

$$\|f\|_{W^{N,p}} \stackrel{\text{def}}{=} \sum_{|\alpha|\leq N} \|D^\alpha_y f\|_{L^p} \quad \text{and} \quad \|u\|_{L^p;k,N} \stackrel{\text{def}}{=} \sup_{t>0}(1+t)^k \|u(t)\|_{W^{N,p}}.$$

In particular, when $p = 1$, $p = 2$ and $p = \infty$, we simplify the notations as

$$\begin{aligned} &|||f|||_N \stackrel{\text{def}}{=} \|f\|_{W^{N,1}}, \qquad \|f\|_N \stackrel{\text{def}}{=} \|f\|_{H^N}, \qquad |f|_N \stackrel{\text{def}}{=} \|f\|_{W^{N,\infty}} \\ &\text{and} \qquad \|u\|_{k,N} \stackrel{\text{def}}{=} \|u\|_{L^2;k,N}, \qquad |u|_{k,N} \stackrel{\text{def}}{=} \|u\|_{L^\infty;k,N}. \end{aligned} \tag{2.19}$$

Theorem 2.1. *There exist an integer L_0 and small constants $\eta, \varepsilon_0 > 0$ such that if*

$$|||(\bar{Y}^{(0)}, Y^{(1)})|||_{L_0} + \|(\bar{Y}^{(0)}, Y^{(1)})\|_{L_0} \leq \eta \quad \text{and} \quad \varepsilon \leq \varepsilon_0. \tag{2.20}$$

Then the system (2.17) has a unique global solution $\bar{Y} \in C^2([0,\infty); C^{N_1-4}(\mathbb{R}^3))$, where $N_1 = [(L_0 - 12)/2]$. Furthermore, for some $\kappa > 0$, there hold

$$|\partial_3 \bar{Y}|_{\frac{3}{4}-\kappa,2} + |\bar{Y}_t|_{\frac{5}{4}-\kappa,2} + |\bar{Y}|_{\frac{1}{4}-\kappa,2} \leq C_\kappa \eta, \tag{2.21}$$

and

$$\begin{aligned} &\||D|^{-1}(\partial_3 \bar{Y}, \bar{Y}_t)\|_{0,N_1+2} + \|\nabla \bar{Y}\|_{0,N_1+1} + \|(\bar{Y}_t, \partial_3 \bar{Y})\|_{\frac{1}{2},N_1+1} + \|\nabla \bar{Y}_t\|_{1,N_1-1} \\ &+ \|\bar{Y}_t\|_{L^2_t(H^{N_1+2})} + \left\|(\partial_3 \bar{Y}, \langle t\rangle^{\frac{1}{2}} \nabla \bar{Y}_t)\right\|_{L^2_t(H^{N_1+1})} + \|\bar{Y}_{tt}\|_{\frac{1}{2},N_1-2} \leq C. \end{aligned} \tag{2.22}$$

Admitting Theorem 2.1 for the time being, let us now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Indeed, in view of (2.3), one has

$$\begin{aligned} &Y_1(z) = u_0(y_{\rm h}(z_{\rm h}, w_3(z)), w_3(z)) \quad \text{and} \quad \boldsymbol{u}(t,y) = Y_t(t, y + Y(t,y)), \\ &\boldsymbol{b}(t,y) = b_0(y) + b_0(y) \cdot \nabla_y Y(t,y) \quad \text{with} \\ &Y(t, (y_{\rm h}(z_{\rm h}, w_3(z)), w_3(z))) = \widetilde{Y}(z) + \bar{Y}(t,z), \end{aligned} \tag{2.23}$$

with $\widetilde{Y}(z)$ and $\bar{Y}(t,z)$ being determined by (2.15) and (2.17) respectively.

In view of (2.10), (2.12) and (2.13), we get, by a similar proof to Lemma 4.3 of [1], that for any $N \in \mathbb{N}$,

$$|(\mathcal{B} - Id)|_N \leq C_N \varepsilon. \tag{2.24}$$

Thus, under the assumptions of (1.5), there holds (2.20). Then Theorem 2.1 ensures that the system (2.17–2.18) has a unique global classical solution $\bar{Y} \in C^2([0,\infty); C^{N_1-4}(\mathbb{R}^3))$, which verifies (2.21) and (2.22). In particular, it follows from (2.15) and (2.21) that

$$|\nabla_z Y|_{0,1} \leq |\partial_z \widetilde{Y}|_1 + |\nabla \bar{Y}|_{0,1} \leq C(\varepsilon + \eta),$$

which together with (2.23) ensures that $\boldsymbol{u} \in C^2([0,\infty)\times\mathbb{R}^3)$ and $\boldsymbol{b} \in C^2([0,\infty)\times\mathbb{R}^3)$. Furthermore, due to

$$\Big|\frac{\partial X}{\partial y} - Id\Big|_{0,1} = \big|{}^t\mathcal{B}\nabla_z Y|_{0,1} \le C(\varepsilon+\eta),$$

we deduce from (2.3) that $u \in C^2([0,\infty)\times\mathbb{R}^3)$ and $b \in C^2([0,\infty)\times\mathbb{R}^3)$, which verifies the system (1.1) thanks to the derivation at the beginning of Sect. 2.1.

On the other hand, by virtue of (2.16), we have

$$\boldsymbol{b}(t, y(w(z))) = b_0(y(w(z))) + \partial_3\widetilde{Y}(z) + \partial_3\bar{Y}(t,z) = e_3 + \partial_3\bar{Y}(t,z),$$

which together with (2.21), (2.22) and (2.23) implies that there holds (1.6). This completes the proof of Theorem 1.1. □

2.3. *Strategies of the Proof to Theorem 2.1*

Observing from the calculations in [1] that under the assumptions of Theorem 1.1, the matrix $\mathcal{B}$ given by (2.13) is sufficiently close to the identity matrix in the norms of $W^{N_0,1}$ and H^{N_0} as long as ε is sufficiently small. To avoid cumbersome calculation, here we just prove Theorem 2.1 for the system (2.1) with

$$\begin{aligned}
&\mathcal{A} = (Id + \nabla_y Y)^{-1}, \quad f = \nabla_y \cdot \big((\mathcal{A}\mathcal{A}^t - Id)\nabla_y Y_t\big) - \mathcal{A}^t\nabla_y \boldsymbol{p}, \quad \text{and}\\
&\boldsymbol{p} = -\Delta_y^{-1}\mathrm{div}_y\big((\mathcal{A}\mathcal{A}^t - Id)\nabla_y \boldsymbol{p}\big)\\
&\qquad + \Delta_y^{-1}\mathrm{div}_y\Big(\mathcal{A}\mathrm{div}_y\big(\mathcal{A}(\partial_{y_3}Y\otimes\partial_{y_3}Y - Y_t\otimes Y_t)\big)\Big),
\end{aligned} \tag{2.25}$$

which corresponds to $\mathcal{B} = Id$ in (2.17). The general case follows along the same lines.

Let us remark that the system (2.1) is not scaling, rotation and Lorentz invariant, so that Klainerman's vector field method [22] cannot be applied here. However, the ideas developed by Klainerman in the seminar papers [19–21] can be well adapted for this system. We now recall the classical result on the global well-posedness to some evolutionary system from [20]. Let us consider the following system:

$$\begin{cases} u_t - \mathcal{L}u = F(u, Du) \quad \text{with} \quad Du = (u_t, u_{x_1}, \dots, u_{x_d}),\\ u_{|t=0} = u_, \qquad Pu_0 = 0, \end{cases} \tag{2.26}$$

where $\mathcal{L} \stackrel{\text{def}}{=} \sum_{|\alpha|\le\gamma} a_\alpha D_x^\alpha$ with a_α being $r\times r$ matrices with constant entries. We have the following assumptions:

(1) $\mathcal{L}$ satisfies a dissipative condition of the following type: there exists a positive definite $r\times r$ matrix A such that

$$\text{either} \int_{\mathbb{R}^d} \Re(A\mathcal{L}f, f)\,\mathrm{d}x \le 0 \quad \text{or} \quad \int_{\mathbb{R}^d} \Re(A\mathcal{L}f, f)\,\mathrm{d}x \le -\|\nabla f\|_{L^2}^2$$

for any $f \in C_c^\infty$;

(2) $\Gamma(t)u_0$ is the solution of

$$\partial_t u - \mathcal{L}u = 0 \quad \text{and} \quad u(0, x) = u_0(x),$$

Such that there is a differential matrix P such that

$$|\Gamma(t)u_0|_0 \le C\langle t\rangle^{-k_0} |||u_0|||_d$$

for any $u_0 \in W^{d,1} \cap L^\infty$ that satisfies $Pu_0 = 0$;

(3) $AF_{u_t}, AF_{u_{x_i}}, i = 1, \cdots, d$, are symmetric matrices and F_{u_t} is independent of u_t. Moreover

$$|F(u, Du)| \le C(|u| + |Du|)^{p+1} \quad \text{for } |u| + |Du| \text{ sufficiently small;} \tag{2.27}$$

(4) p is an integer and F is a smooth function so that there holds

$$\frac{1}{p}\left(1 + \frac{1}{p}\right) < k_0. \tag{2.28}$$

Klainerman proved in [20] the following celebrated theorem:

Theorem 2.2. (Theorem 1 of [20]). *There exist an integer $N_0 > 0$ and a small constant $\eta > 0$ such that if*

$$|||u_0|||_{N_0} + \|u_0\|_{N_0} \le \eta,$$

(2.26) *has a unique solution $u \in C^1([0, T]; C^\gamma)$ for any $T > 0$. Moreover, the solution behaves, for t large, like*

$$|u(t, x)| = O\left(t^{-\frac{1+\varepsilon}{p}}\right) \quad as \quad t \to \infty \tag{2.29}$$

for some small $\varepsilon > 0$. Also,

$$\|u(t)\|_{L^2} = O(1) \quad as \quad t \to \infty. \tag{2.30}$$

Let us remark that due to the appearance of the double Riesz transform in the expression of f in (2.25), the source term f in (2.1) cannot satisfy the growth condition (2.27); secondly, even if we can assume the source term f is in quadratic growth of $(Y_t, \partial_3 Y)$, which corresponds to $p = 1$ in (2.27), the growth rate obtained in (3.2) below does not meet the requirement of (2.28). This makes it impossible to apply Theorem 2.2 for the system (2.1), yet by considering the specific anisotropic structure of the system (2.1), we can still succeed in applying the Nash–Moser scheme to establish the global existence as well as the large time behavior of solutions to (2.1–2.25).

Now we outline the proof of Theorem 2.1. According to the strategy in [19–21], the first step is to study the decay properties of the linear system

$$\begin{cases} \mathcal{Y}_{tt} - \Delta\mathcal{Y}_t - \partial_3^2\mathcal{Y} = 0, \\ \mathcal{Y}_{|t=0} = \mathcal{Y}_0, \qquad \mathcal{Y}_{t|t=0} = \mathcal{Y}_1. \end{cases} \tag{2.31}$$

Proposition 2.1. *Let $\mathcal{Y}(t)$ be a smooth enough solution of* (2.31)*. Given $\delta \in [0,1]$, $N \in \mathbb{N}$, there exist $C_{\delta,N}, C_N > 0$ such that:*

$$|\partial_3\mathcal{Y}|_{1,N} + |\partial_t\mathcal{Y}|_{\frac{3}{2}-\delta,N} + |\mathcal{Y}|_{\frac{1}{2},N} \leq C_{\delta,N}\big(\||D|^{2\delta}(\mathcal{Y}_0,\mathcal{Y}_1)\|_{L^1} + \||D|^{N+4}(\Delta\mathcal{Y}_0,\mathcal{Y}_1)\|_{L^1}\big); \tag{2.32}$$

$$\|(\partial_t\mathcal{Y},\partial_3\mathcal{Y})\|_{L^\infty_t(H^{N+1})} + \|\Delta\mathcal{Y}\|_{L^\infty_t(H^N)} + \|\nabla\partial_t\mathcal{Y}\|_{L^2_t(H^{N+1})} + \|\nabla\partial_3\mathcal{Y}\|_{L^2_t(H^N)} \leq C_N\left(\|(\partial_3\mathcal{Y}_0,\mathcal{Y}_1)\|_{N+1} + \|\Delta\mathcal{Y}_0\|_N\right); \tag{2.33}$$

$$\|\langle t\rangle^{\frac{1}{2}}(\partial_t\mathcal{Y},\partial_3\mathcal{Y})\|_{L^\infty_t(H^N)} + \|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t\mathcal{Y}\|_{L^2_t(H^N)} \leq C_N\big(\||D|^{-1}(\partial_3\mathcal{Y}_0,\mathcal{Y}_1)\|_{N+1} + \|\nabla\mathcal{Y}_0\|_N\big); \tag{2.34}$$

$$\|\langle t\rangle\Delta\partial_t\mathcal{Y}\|_{L^\infty_t(H^N)} \leq C_N\|(\Delta\mathcal{Y}_0,\mathcal{Y}_1)\|_{N+2}. \tag{2.35}$$

We emphasize here that the estimates of (2.32) and (2.33) are of anisotropic type, which means that the decay rates of the partial derivatives of the solution to (2.31) are different, which is consistent with the heuristic discussions at the beginning of Sect. 2. Moreover, the estimate of (2.32) is valid for $\delta = 0$. Similar estimates such as (2.34) and (2.35) were not proved in [19–21]; they are purely due to the special structure of the linearized system (2.31).

With the above proposition, we next turn to the decay estimates for the solutions of the following inhomogeneous equation of (2.31):

$$\begin{cases} Y_{tt} - \Delta Y_t - \partial_3^2 Y = g, \\ Y|_{t=0} = Y_t|_{t=0} = 0. \end{cases} \tag{2.36}$$

Proposition 2.2. *Let $\delta \in [0,1/4[$ and $\theta \in [1,\infty[$. We assume that $g(t) = 0$ if $t \geq \theta$. Then the solution Y to* (2.36) *verifies, for any $N \geq 0$,*

$$|\partial_3 Y|_{1,N} + |\partial_t Y|_{\frac{3}{2}-\delta,N} + |Y|_{\frac{1}{2},N} \leq C_{\delta,N} R_{N,\theta}(g), \tag{2.37}$$

where

$$R_{N,\theta}(g) \stackrel{def}{=} \|\!|g|\!\|_{L^1_t(\delta,N)} + \theta^{\frac{1}{2}}\left\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}g\right\|_{L^2_t(H^{N+3})} + \log\langle\theta\rangle\left\||D|^{-1}g\right\|_{\frac{3}{2}-\delta,N+3}, \tag{2.38}$$

where

$$\|\!|g|\!\|_{\delta,N} \stackrel{def}{=} \||D|^{2\delta}g\|_{L^1} + \||D|^{N+4}g\|_{L^1} \quad \textit{and} \quad \|\!|g|\!\|_{L^p_t(\delta,N)} = \left(\int_0^t \|\!|g(t')|\!\|^p_{\delta,N}\,dt'\right)^{\frac{1}{p}}. \tag{2.39}$$

The proof of the above propositions will be presented in Sect. 3.

The goal of Sect. 4 is to calculate the linearized system of (2.1), which reads

$$\begin{cases} X_{tt} - \Delta X_t - \partial_3^2 X = f'(Y;X) + g, \\ X|_{t=0} = X_t|_{t=0} = 0, \end{cases} \tag{2.40}$$

where $f'(Y;X) = f_0'(Y;X) + f_1'(Y;X) + f_2'(Y;X)$, and $f_0'(Y;X)$, $f_1'(Y;X)$ and $f_2'(Y;X)$ are determined respectively by (4.6) and (4.7). Furthermore, the second derivative of $f''(Y;X,W)$ will be presented in Sect. 4.2.

In Sect. 5, we shall derive the $\dot{W}^{2\delta,1} \cap \dot{W}^{N+4,1}$ and $\dot{H}^{N+1}$ estimates for the source term $f'(Y;X)$ in the linearized system (2.40), which will be used to derive the decay estimates for the solutions of (2.40). The main result reads as follows:

Proposition 2.3. *Let the functionals, $f_0'(Y;X)$, $f_1'(Y;X)$, $f_2'(Y;X)$, be given by (4.6) and (4.7) respectively, and the norm $|||\cdot|||_{\delta,N}$ be given by (2.39). Then under the assumptions that $\delta > 0$, and*

$$\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \le \delta_1 \quad \textit{and} \quad \|\nabla Y\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \le 1, \tag{2.41}$$

for some $\delta_1 > 0$ sufficiently small, we have

$$\begin{aligned} |||f_0'(Y;X)|||_{\delta,N} \le &\|\nabla Y\|_0 \|\nabla X_t\|_{N+6} + \|\nabla Y\|_{N+6}\|\nabla X_t\|_0 \\ &+ \|\nabla Y_t\|_0 \|\nabla X\|_{N+6} \\ &+ \big(\|\nabla Y_t\|_{N+6} + \|\nabla Y\|_{N+6}|\nabla Y_t|_0\big)\|\nabla X\|_0, \end{aligned} \tag{2.42}$$

and

$$|||f_1'(Y;X)|||_{\delta,N} \lesssim \mathfrak{f}_1(\partial_3 Y, \partial_3 X) \quad \textit{and} \tag{2.43}$$
$$|||f_2'(Y;X)|||_{\delta,N} \lesssim \mathfrak{f}_1(Y_t, X_t), \tag{2.44}$$

where the functional $\mathfrak{f}_1(\mathfrak{x},\mathfrak{y})$ is given by

$$\begin{aligned} \mathfrak{f}_1(\mathfrak{x},\mathfrak{y}) \stackrel{def}{=} &\|\mathfrak{x}\|_0\big(\|\mathfrak{y}\|_{N+6} + |\mathfrak{x}|_0\|\nabla X\|_{N+6}\big) + \|\mathfrak{y}\|_1\big(\|\mathfrak{x}\|_{N+6} + \|\nabla Y\|_{N+6}|\mathfrak{x}|_1\big) \\ &+ \big(\|\mathfrak{x}\|_{N+6} + \|\nabla Y\|_{N+6}\|\mathfrak{x}\|_3\big)|\mathfrak{x}|_1\|\nabla X\|_1. \end{aligned}$$

Proposition 2.4. *Under the assumption of Proposition 2.3, we have*

$$\begin{aligned} \||D|^{-1} f_0'(Y;X)\|_{N+1} \lesssim &|\nabla Y|_0\|\nabla X_t\|_{N+1} + |\nabla Y|_{N+1}\|\nabla X_t\|_0 \\ &+ |Y_t|_1\|\nabla X\|_{N+1} + \big(|Y_t|_{N+2} + |Y_t|_1|\nabla Y|_{N+1}\big)\|\nabla X\|_0, \end{aligned} \tag{2.45}$$

and

$$\||D|^{-1} f_1'(Y;X)\|_{N+1} \lesssim \mathfrak{f}_2(\partial_3 Y, \partial_3 X) \quad \textit{and} \tag{2.46}$$
$$\||D|^{-1} f_2'(Y;X)\|_{N+1} \lesssim \mathfrak{f}_2(Y_t, X_t), \tag{2.47}$$

where the functional $\mathfrak{f}_2(\mathfrak{x},\mathfrak{y})$ is given by

$$\begin{aligned} \mathfrak{f}_2(\mathfrak{x},\mathfrak{y}) \stackrel{def}{=} &\big(|\mathfrak{x}|_1^{\frac{4}{3}}\|\mathfrak{x}\|_0^{\frac{2}{3}} + |\mathfrak{x}|_1^2\big)\big(\|\nabla X\|_{N+1} + |\nabla Y|_{N+1}\|\nabla X\|_1\big) + |\mathfrak{x}|_0\|\mathfrak{y}\|_{N+1} \\ &+ \big(|\mathfrak{x}|_{N+1} + |\nabla Y|_{N+1}|\mathfrak{x}|_1\big)\|\mathfrak{y}\|_1 + \big(|\mathfrak{x}|_0^{\frac{1}{3}}\|\mathfrak{x}\|_0^{\frac{2}{3}} + |\mathfrak{x}|_0\big)|\mathfrak{x}|_{N+1}\|\nabla X\|_1. \end{aligned}$$

Let us remark that the Riesz transform does not map continuously from L^1 to L^1. Nevertheless due to (4.8) and (4.9), we cannot avoid estimates of this type. To overcome this difficulty, a natural replacement of $\dot{W}^{s,1}$ will be the Besov space $\dot{B}^s_{1,1}$, which satisfies

$$\|\nabla(-\Delta)^{-\frac{1}{2}}|D|^s(f)\|_{L^1} \lesssim \|f\|_{\dot{B}^s_{1,1}} \quad \forall\, s \in \mathbb{R}\,.$$

We now recall the precise definition of the Besov norms from, for instance [3].

Definition 2.1. Let us consider a smooth function φ on $\mathbb{R}$, the support of which is included in $[3/4, 8/3]$ such that

$$\forall \tau > 0\,, \ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1 \quad \text{and} \quad \chi(\tau) \stackrel{\text{def}}{=} 1 - \sum_{j\geq 0} \varphi(2^{-j}\tau) \in \mathcal{D}([0, 4/3]).$$

Let us define

$$\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), \quad \text{and} \quad S_j a = \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}).$$

Let (p, r) be in $[1, +\infty]^2$ and s in $\mathbb{R}$. We define the Besov norm by

$$\|a\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \big\|\big(2^{js}\|\Delta_j a\|_{L^p}\big)_j\big\|_{\ell^r(\mathbb{Z})}.$$

We remark that in the special case when $p = r = 2$, the Besov spaces $\dot{B}^s_{p,r}$ coincides with the classical homogeneous Sobolev spaces $\dot{H}^s$. Moreover, we have the following product laws (see Corollary 2.54 of [3]):

$$\|ab\|_{\dot{B}^s_{p,r}} \leq C\big(|a|_{L^\infty}\|b\|_{\dot{B}^s_{p,r}} + \|a\|_{\dot{B}^s_{p,r}}|b|_{L^\infty}\big) \tag{2.48}$$

for $s > 0$, $(p, r) \in [1, +\infty]^2$. Due to the product law (2.48), we need the index δ to be positive in Proposition 2.3.

The estimates of the second derivations of f_0, f_1 and f_2 can be listed as follows:

Proposition 2.5. *Let f_0'', f_1'', f_2'' be given by* (4.13) *and* (4.14) *respectively. Then under the assumption of* (2.41)*, we have*

$$\begin{aligned}
\||D|^{-1} f_0''(Y; X, W)\|_N &\lesssim |Y_t|_1\big(|\nabla X|_N\|\nabla W\|_0 + |\nabla X|_0\|\nabla W\|_N\big)\\
&+ \big((|Y_t|_{N+1} + |\nabla Y|_N|Y_t|_1)|\nabla X|_0 + |X_t|_{N+1}\big)\|\nabla W\|_0 + |\nabla X|_N\|\nabla W_t\|_0 \\
&+ |\nabla X|_0\|\nabla W_t\|_N + |\nabla Y|_N\big(|\nabla X|_0\|\nabla W_t\|_0 + |X_t|_1\|\nabla W\|_0\big) + |X_t|_1\|\nabla W\|_N,
\end{aligned} \tag{2.49}$$

and

$$\||D|^{-1} f_1''(Y; X, W)\|_N \lesssim \mathfrak{f}_3(\partial_3 Y, \partial_3 X, \partial_3 W) \quad \textit{and} \tag{2.50}$$

$$\||D|^{-1} f_2''(Y; X, W)\|_N \lesssim \mathfrak{f}_3(Y_t, X_t, W_t), \tag{2.51}$$

where the functional $\mathfrak{f}_3(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ *is given by*

$$\begin{aligned}
\mathfrak{f}_3(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \stackrel{def}{=} & (|\mathfrak{y}|_N + |\nabla Y|_N |\mathfrak{y}|_0) \|\mathfrak{z}\|_0 + |\mathfrak{y}|_0 \|\mathfrak{z}\|_N \\
& + \left(|\mathfrak{x}|_0 + |\mathfrak{x}|_1^{\frac{1}{3}} \|\mathfrak{x}\|_1^{\frac{2}{3}}\right) (|\nabla X|_N \|\mathfrak{z}\|_0 + |\nabla X|_0 \|\mathfrak{z}\|_N + |\mathfrak{y}|_1 \|\nabla W\|_N \\
& + (|\mathfrak{y}|_N + |\mathfrak{x}|_N |\nabla X|_1) \|\nabla W\|_1) + (|\mathfrak{x}|_N + |\nabla Y|_N |\mathfrak{x}|_1) |\nabla X|_1 \|\mathfrak{z}\|_1 \\
& + \left(|\mathfrak{x}|_1^{\frac{4}{3}} \|\mathfrak{x}\|_0^{\frac{2}{3}} + |\mathfrak{x}|_1^2\right) \big((|\nabla X|_N + |\nabla Y|_N |\nabla X|_1) \|\nabla W\|_1 \\
& + |\nabla X|_1 \|\nabla W\|_N\big) \\
& + \left(|\mathfrak{x}|_N + |\mathfrak{x}|_N^{\frac{1}{3}} \|\mathfrak{x}\|_N^{\frac{2}{3}} + |\nabla Y|_N \left(|\mathfrak{x}|_1 + |\mathfrak{x}|_0^{\frac{1}{3}} \|\mathfrak{x}\|_0^{\frac{2}{3}}\right)\right) |\mathfrak{y}|_1 \|\nabla W\|_1.
\end{aligned}$$

Remark 2.1. We mention that in the above inequalities, it is crucial to estimate the vector, X, by L^∞-norm. In Sect. 9, we shall deal with the estimate of the error term

$$e'_p = -\int_0^1 f''\big(Y_p + s(1 - S_p)Y_p; (1 - S_p)Y_p, X_p\big) \mathrm{d}s,$$

where the variable, $(1 - S_p)Y_p$, is "small" in the L^∞-norm, but only "bounded" in L^2-norm.

Proposition 2.6. *Let* f''_m, $m = 0, 1, 2$ *be given in* (4.13) *and* (4.14) *, the norm* $|||\cdot|||_{\delta,N}$ *be given by* (2.39)*. Then under the assumption of* (2.41)*, we have*

$$\begin{aligned}
|||f''_0(Y; X, W)|||_{\delta,N} \lesssim & |Y_t|_1 \big(\|\nabla X\|_{N+6} \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_{N+6}\big) \\
& + \big(\|\nabla Y_t\|_{N+6} + |Y_t|_1 \|\nabla Y\|_{N+6}\big) \big(|\nabla X|_0 \|\nabla W\|_0 + \|\nabla X\|_0 |\nabla W|_0\big) \\
& + \|\nabla X\|_0 \|\nabla W_t\|_{N+6} + \big(\|\nabla X\|_{N+6} + |\nabla X|_0 \|\nabla Y\|_{N+6}\big) \|\nabla W_t\|_0 \\
& + \|\nabla W\|_0 \|\nabla X_t\|_{N+6} + \big(\|\nabla W\|_{N+6} + |\nabla W|_0 \|\nabla Y\|_{N+6}\big) \|\nabla X_t\|_0,
\end{aligned} \tag{2.52}$$

and

$$|||f''_1(Y; X, W)|||_{\delta,N} \lesssim \mathfrak{f}_4(\partial_3 Y, \partial_3 X, \partial_3 W) \quad \textit{and} \tag{2.53}$$

$$|||f''_2(Y; X, W)|||_{\delta,N} \lesssim \mathfrak{f}_4(Y_t, X_t, W_t), \tag{2.54}$$

where the functional $\mathfrak{f}_4(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ *is given by*

$$\begin{aligned}
\mathfrak{f}_4(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \stackrel{def}{=} & \big(\|\mathfrak{z}\|_0 + |\mathfrak{x}|_0 \|\nabla W\|_0 + \|\mathfrak{x}\|_0 |\nabla W|_0\big) \|\mathfrak{y}\|_{N+6} \\
& + \big(\|\mathfrak{x}\|_0 |\nabla X|_0 + \|\mathfrak{y}\|_0\big) \|\mathfrak{z}\|_{N+6} \\
& + |\mathfrak{y}|_0 \|\mathfrak{z}\|_0 \|\nabla Y\|_{N+6} + |\mathfrak{x}|_0 \big(\|\mathfrak{z}\|_0 \|\nabla X\|_{N+6} + \|\nabla X\|_0 \|\mathfrak{z}\|_{N+6} \\
& + \|\mathfrak{y}\|_0 \|\nabla W\|_{N+6}\big) \\
& + \big(\|\mathfrak{x}\|_{N+6} + |\mathfrak{x}|_0 \|\nabla Y\|_{N+6}\big) \big(\|\nabla X\|_0 |\mathfrak{z}|_0 + |\nabla X|_1 \|\mathfrak{z}\|_1\big) \\
& + \big(\|\mathfrak{x}\|_{N+6} + \|\mathfrak{x}\|_3 \|\nabla Y\|_{N+6}\big) \big(|\mathfrak{y}|_1 \|\nabla W\|_1 + \|\mathfrak{y}\|_1 |\nabla W|_0\big) \\
& + |\mathfrak{x}|_1 \|\mathfrak{x}\|_3 \big(\|\nabla X\|_{N+6} (\|\nabla W\|_0 + |\nabla W|_0) + (\|\nabla X\|_0
\end{aligned}$$

$$+ |\nabla X|_0)\|\nabla W\|_{N+6}\big)$$
$$+ \big(|\mathfrak{x}|_1\|\mathfrak{x}\|_{N+6} + |\mathfrak{x}|_1\|\mathfrak{x}\|_3\|\nabla Y\|_{N+6}\big)\big(|\nabla X|_0\|\nabla W\|_1$$
$$+ \|\nabla X\|_1|\nabla W|_0\big).$$

The proofs of the above propositions are similar to those of Propositions 2.3 and 2.4. We skip the details here. Interested readers may check Sect. 9 of [15].

In Sect. 6, we investigate energy estimates for the solutions of the linearized equation (2.40).

Theorem 2.3. *Let Y be a smooth enough vector field and X be a smooth solution to the linearized equation* (2.40). *We assume that Y satisfies* (2.41) *and*

$$\|Y_t\|_{0,0} \le 1, \quad \textit{and} \quad |Y_t|_{0,1} \le 1. \tag{2.55}$$

Then for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathcal{E}_0(t) \le C_\varepsilon \|\langle t\rangle^{\frac{1+\varepsilon}{2}}|D|^{-1}g\|_{L^2_t(H^1)}E_\varepsilon(Y) \quad \textit{and} \quad \textit{for} \quad N \ge 1\\
&\mathcal{E}_N(t) \le C_{\varepsilon,N}\Big(\|\langle t\rangle^{\frac{1+\varepsilon}{2}}g\|_{L^2_t(H^N)} + \gamma_{\varepsilon,N+1}(Y)\|\langle t\rangle^{\frac{1+\varepsilon}{2}}|D|^{-1}g\|_{L^2_t(H^1)}\Big)E_\varepsilon(Y),
\end{aligned} \tag{2.56}$$

where

$$\begin{aligned}
\mathcal{E}_N(t) &\stackrel{def}{=} \||D|^{-1}(X_t, \partial_3 X)\|_{0,N+2} + \|\nabla X\|_{0,N+1} + \|X_t\|_{L^2_t(H^{N+2})}\\
&\quad + \|\partial_3 X\|_{L^2_t(H^{N+1})};\\
E_\varepsilon(Y) &\stackrel{def}{=} \exp\Big(C\big(|\partial_3 Y|^{\frac{4}{3}}_{\frac12+\varepsilon,1}\|\partial_3 Y\|^{\frac{2}{3}}_{L^2_t(L^2)} + |\partial_3 Y|^2_{\frac12+\varepsilon,1} + |Y_t|_{1+\varepsilon,2}\big)\Big),
\end{aligned} \tag{2.57}$$

and

$$\begin{aligned}
\gamma_{\varepsilon,N+1}(Y) &\stackrel{def}{=} 1 + |\partial_3 Y|_{\frac12+\varepsilon,N+1}\big(1 + |\partial_3 Y|_{\frac12+\varepsilon,1}\big) + |Y_t|_{1+\varepsilon,N+2}\\
&\quad + |\partial_3 Y|^{\frac13}_{\frac12+\varepsilon,1}\|\partial_3 Y\|^{\frac23}_{L^2_t(L^2)}\big(|\partial_3 Y|_{\frac12+\varepsilon,N+1} + |\nabla Y|_{0,N+1}\big)\\
&\quad + |\nabla Y|_{0,N+1}\big(1 + |\partial_3 Y|^2_{\frac12+\varepsilon,0} + |\partial_3 Y|_{\frac12+\varepsilon,0} + |Y_t|_{1+\varepsilon,1}\big).
\end{aligned} \tag{2.58}$$

We notice that when we perform the energy estimates for the derivatives of the solutions to (2.40), we are not able to treat the term $\nabla\cdot\big((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\big)$, which appears in $f_0'(Y;X)$ (see (4.6)) as a source term. Instead, we need to rewrite (2.40) as

$$X_{tt} - \nabla\cdot\partial_t\big(\mathcal{A}\mathcal{A}^t\nabla X\big) - \partial_3^2 X = \widetilde{f}'(Y;X) + g, \tag{2.59}$$

where $\widetilde{f}'(Y;X) = \widetilde{f_0}'(Y;X) + f_1'(Y;X) + f_2'(Y;X)$ with $f_m'(Y;X), m = 1,2,$ given by (4.7), and $\widetilde{f_0}'(Y;X)$ by

$$\widetilde{f_0}'(Y;X) = -\nabla\cdot\big(\mathcal{A}\big(\nabla X\mathcal{A} + \mathcal{A}^t(\nabla X)^t\big)\mathcal{A}^t\nabla Y_t\big) - \nabla\cdot\big(\partial_t(\mathcal{A}\mathcal{A}^t)\nabla X\big). \tag{2.60}$$

With the energy estimates obtained in Theorem 2.3, we can work on the time-weighted energy estimate for the solutions of (2.40).

Corollary 2.1. *Under the assumptions of Theorem 2.3, we have*

$$\mathcal{E}_0 + \|(X_t, \partial_3 X)\|_{\frac{1}{2},1} + \|\langle t\rangle^{\frac{1}{2}} \nabla X_t\|_{L^2_t(H^1)} \leq C_\varepsilon \|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L^2_t(H^1)} E_\varepsilon(Y), \tag{2.61}$$

and for $N \geq 1$,

$$\begin{aligned}&\mathcal{E}_N + \|(X_t, \partial_3 X)\|_{\frac{1}{2},N+1} + \|\langle t\rangle^{\frac{1}{2}} \nabla X_t\|_{L^2_t(H^{N+1})}\\ &\quad\leq C_{\varepsilon,N}\Big(\|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|_{L^2_t(H^N)} + \gamma_{\varepsilon,N+1}(Y)\|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L^2_t(H^1)}\Big) E_\varepsilon(Y). \end{aligned}\tag{2.62}$$

Proposition 2.7. *Under the assumptions of Theorem 2.3, we have for $N \geq 0$,*

$$\begin{aligned}\|\nabla X_t\|_{1,N} &\leq C_{\varepsilon,N}\big(\||D|^{-1} g\|_{1+\varepsilon,N+2} + \|\nabla Y\|_{N+2}\||D|^{-1} g\|_{1+\varepsilon,2}\big)\\ &\quad + C_{\varepsilon,N}\Big(\|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|_{L^2_t(H^{N+1})} + \gamma_{\varepsilon,N+2}(Y)\|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L^2_t(H^1)}\\ &\quad + \|\nabla Y\|_{0,N+2}\big(\|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|_{L^2_t(H^1)} + \gamma_{\varepsilon,2}(Y)\|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L^2_t(H^1)}\big)\Big) E_\varepsilon(Y).\end{aligned}\tag{2.63}$$

We emphasize that the decay estimates (2.63) cannot be obtained by energy estimate. In fact, we will have to exploit anisotropic Littlewood-Paley analysis and the dissipative properties of the linear system (2.1). The proof of Proposition 2.7 will be presented in Sect. 7, which is of independent interest.

Let us summarize that under the assumptions (2.41) and (2.55), and assuming

$$|\partial_3 Y|^{\frac{4}{3}}_{\frac{1}{2}+\varepsilon,1}\|\partial_3 Y\|^{\frac{2}{3}}_{L^2_t(L^2)} + |\partial_3 Y|^2_{\frac{1}{2}+\varepsilon,1} + |Y_t|_{1+\varepsilon,2} \leq 1, \tag{2.64}$$

we have the following energy estimates: for $N \geq 0$, (we make the convention $\|u\|_{k,-1} = 0$)

$$\begin{aligned}&\mathcal{E}_N + \|(X_t, \partial_3 X)\|_{\frac{1}{2},N+1} + \|\langle t\rangle^{\frac{1}{2}} \nabla X_t\|_{L^2_t(H^{N+1})} + \|\nabla X_t\|_{1,N-1}\\ &\quad\leq C_{\varepsilon,N}\Big(\||D|^{-1} g\|_{1+\varepsilon,N+1} + \|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|_{L^2_t(H^N)} + \widetilde{\gamma}_{\varepsilon,N+1}(Y)\big(\||D|^{-1} g\|_{1+\varepsilon,2}\\ &\quad + \|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L^2_t(H^1)}\big)\Big) \quad \text{with}\\ &\widetilde{\gamma}_{\varepsilon,N+1}(Y) \leq C\big(1 + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,N+1} + |Y_t|_{1+\varepsilon,N+2} + |\nabla Y|_{0,N+1} + \|\nabla Y\|_{0,N+1}\big).\end{aligned}\tag{2.65}$$

In Sect. 8, we shall present the estimates to the nonlinear source term $f(Y)$ given by (2.25).

With the preparations of the previous sections, we can now exploit the Nash–Moser iteration scheme to prove Theorem 2.1. In order to do so, we first recall some basic properties of the smoothing operator from [19,20]. Let $\chi(t) \in C^\infty(\mathbb{R}; [0, 1])$ be such that

$$\chi(t) = 1 \text{ for } t \leq \frac{1}{2}, \quad \chi(t) = 0 \text{ for } t \geq 1.$$

Define for $\theta \geq 1$, the (cutoff-in-time) operator

$$S^{(1)}(\theta)Y(t,y) \stackrel{\text{def}}{=} \chi\Big(\frac{t}{\theta}\Big)Y(t,y). \tag{2.66}$$

Then we have

$$|S^{(1)}(\theta)Y|_{k,N} \leq C_{k,s}\theta^{k-s}|Y|_{s,N}, \quad \text{if } \ k \geq s \geq 0$$

and

$$|\big(1 - S^{(1)}(\theta)\big)Y|_{s,N} \leq C_{k,s}\theta^{-(k-s)}|Y|_{k,N} \quad \text{if } \ k \geq s \geq 0.$$

For $\theta' \geq 1$, we define the usual mollifying operator $S^{(2)}(\theta')$ in the space variables by

$$S^{(2)}(\theta')Y(t,y) \stackrel{\text{def}}{=} \widehat{\varphi}\Big(\frac{D_y}{\theta'}\Big)Y(t,y) = (\theta')^3 \int_{\mathbb{R}^3} \varphi(\theta'(y-z))Y(t,z)\mathrm{d}z, \tag{2.67}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^3)$ satisfies

$$\widehat{\varphi}(\xi) = 1 \text{ for } |\xi| \leq \frac{1}{2}, \quad \widehat{\varphi}(\xi) = 0 \text{ for } |\xi| \geq 1,$$

so that

$$\int_{\mathbb{R}^3} \varphi(y)\mathrm{d}y = 1, \quad \int_{\mathbb{R}^3} y^\alpha \varphi(y)\mathrm{d}y = 0, \ \forall \, |\alpha| > 0.$$

We then have

$$|S^{(2)}(\theta')Y|_{k,N} \leq C_{N,M}(\theta')^{N-M}|Y|_{k,M} \quad \text{if } \ N \geq M \geq 0,$$

as well as

$$|\big(1 - S^{(2)}(\theta')\big)Y|_{k,M} \leq C_{N,M}(\theta')^{-(N-M)}|Y|_{k,N} \quad \text{if } N \geq M \geq 0.$$

Define the operator

$$S(\theta,\theta') \stackrel{\text{def}}{=} S^{(1)}(\theta)S^{(2)}(\theta'), \quad \text{for } \ \theta, \ \theta' \geq 1. \tag{2.68}$$

Then it follows that

$$\begin{aligned} &|S(\theta,\theta')Y|_{k,N} \leq C\theta^{k-s}(\theta')^{N-M}|Y|_{s,M}, \\ &\|\langle t\rangle^k S(\theta,\theta')g\|_{L^p_t(H^N)} \\ &\quad \leq C\theta^{k-s}(\theta')^{N-M}\|\langle t\rangle^s g\|_{L^p_t(H^M)} \quad \text{if } \ k \geq s \geq 0, \ N \geq M \geq 0. \end{aligned} \tag{2.69}$$

Moreover, due to

$$1 - S(\theta,\theta') = \big(1 - S^{(1)}(\theta)\big) + S^{(1)}(\theta)\big(1 - S^{(2)}(\theta')\big),$$

one has

$$\begin{aligned}|\big(1-S(\theta,\theta')\big)Y|_{s,M} &\leq C\theta^{-(k-s)}|Y|_{k,M} + C(\theta')^{-(N-M)}|Y|_{s,N},\\ \|\langle t\rangle^s\big(1-S(\theta,\theta')\big)g\|_{L^p_t(H^M)} &\leq C\theta^{-(k-s)}\|\langle t\rangle^k g\|_{L^p_t(H^M)}\\ &\quad + C(\theta')^{-(N-M)}\|\langle t\rangle^s g\|_{L^p_t(H^N)},\end{aligned} \tag{2.70}$$

provided that $k \geq s \geq 0,\ N \geq M \geq 0$.

Let us denote

$$\Phi(Y) \stackrel{\text{def}}{=} Y_{tt} - \Delta Y_t - \partial_3^2 Y - f(Y)$$

for f given by (2.25). Then we can write (2.1) equivalently as

$$\Phi(Y) = 0, \quad Y(0,y) = Y^{(0)}, \quad Y_t(0,y) = Y^{(1)}. \tag{2.71}$$

We aim to solve (2.71) via a Nash–Moser iteration scheme in Sect. 9.

Let us define Y_0 via

$$\begin{cases}\partial_{tt}Y_0 - \Delta\partial_t Y_0 - \partial_3^2 Y_0 = 0,\\ Y_0(0,y) = Y^{(0)}, \quad \partial_t Y_0(0,y) = Y^{(1)}.\end{cases} \tag{2.72}$$

Inductively, assume that we already determine Y_p. In order to define Y_{p+1}, we introduce a mollified version of $\Phi'(Y_p)$ as follows:

$$L_pX \stackrel{\text{def}}{=} \Phi'(S_pY_p)X = X_{tt} - \Delta X_t - \partial_3^2 X - f'(S_pY_p; X), \tag{2.73}$$

where S_p is the smoothing operator defined by

$$S_p = S(\theta_p, \theta_p'), \quad \text{with} \quad \theta_p = 2^p, \quad \theta_p' = \theta_p^{\bar\varepsilon} = 2^{\bar\varepsilon p}, \quad \text{and} \quad p \geq 0, \tag{2.74}$$

where $S(\theta,\theta')$ is defined in (2.68) and $\bar\varepsilon > 0$ is a small constant to be chosen later on. Then it follows from (2.69) and (2.70) that

$$\begin{aligned}|S_pY|_{k,N} &\leq C\theta_p^{k-s}\theta_p^{\bar\varepsilon(N-M)}|Y|_{s,M}\\ \|\langle t\rangle^k S_p g\|_{L^2_t(H^N)} &\leq C\theta_p^{k-s}\theta_p^{\bar\varepsilon(N-M)}\|\langle t\rangle^s g\|_{L^2_t(H^M)}\\ |\!|\!| S_p g |\!|\!|_{L^1_t(\delta,N)} &\leq C\theta_p^{\bar\varepsilon(N-M)}|\!|\!| g |\!|\!|_{L^1_t(\delta,M)},\end{aligned} \tag{S I}$$

and

$$\begin{aligned}|(1-S_p)Y|_{0,0} &\leq C_{k,N}\left(\theta_p^{-k}|Y|_{k,0} + \theta_p^{-\bar\varepsilon N}|Y|_{0,N}\right),\\ \|\langle t\rangle^s(1-S_p)g\|_{L^2_t(L^2)} &\leq C_{k,N}\left(\theta_p^{-(k-s)}\|\langle t\rangle^k g\|_{L^2_t(L^2)} + \theta_p^{-\bar\varepsilon N}\|\langle t\rangle^s g\|_{L^2_t(H^N)}\right)\end{aligned} \tag{S II}$$

for $k \geq s \geq 0$, $N \geq M \geq 0$, where the norm $|\!|\!| \cdot |\!|\!|_{L^1_t(\delta,N)}$ is given by (2.39).

Remark 2.2. According to Remark 4.1 below, we can write

$$f'(S_pY_p; X) = f_0'(S_pY_p; X) + f_1'(S_pY_p; X) + f_2'(S_pY_p; X)$$

where

$$\begin{aligned}
f_0'(S_pY_p; X) &= F_{0,U}'(S_p\nabla\partial_t Y_p, S_p\nabla Y_p)\nabla X_t + F_{0,V}'(S_p\nabla\partial_t Y_p, S_p\nabla Y_p)\nabla X,\\
f_1'(S_pY_p; X) &= F_U'(S_p\partial_3 Y_p, S_p\nabla Y_p)\partial_3 X + F_V'(S_p\partial_3 Y_p, S_p\nabla Y_p)\nabla X,\\
f_2'(S_pY_p; X) &= F_U'(S_p\partial_t Y_p, S_p\nabla Y_p)X_t + F_V'(S_p\partial_t Y_p, S_p\nabla Y_p)\nabla X,
\end{aligned}$$

where the functionals F_0', F' will be presented in Remark 4.1.

Following Hörmander's version of Nash–Moser Scheme [18] (see also Klainerman's seminar papers [19,20]), we define

$$Y_{p+1} = Y_p + X_p, \quad \text{with} \quad X_p = L_p^{-1} g_p, \tag{2.75}$$

where L_p^{-1} is a right inverse operator of L_p with zero initial data, that is: $X = L_p^{-1} g_p$ solves

$$\begin{cases} L_p X = g_p & \text{with } L_p \text{ given by (2.73)},\\ X(0,y) = 0, & X_t(0,y) = 0. \end{cases} \tag{2.76}$$

In order to prove the convergence of the scheme, we define

$$\begin{aligned}
&e_p' \overset{\text{def}}{=} \big(\Phi'(Y_p) - L_p\big)X_p, \quad e_p'' \overset{\text{def}}{=} \Phi(Y_{p+1}) - \Phi(Y_p) - \Phi'(Y_p)X_p, \quad \text{and}\\
&e_p \overset{\text{def}}{=} e_p' + e_p'',
\end{aligned} \tag{2.77}$$

from which we infer

$$\begin{aligned}
\Phi(Y_{p+1}) - \Phi(Y_p) &= \Phi'(Y_p)X_p + e_p'' = \Phi'(Y_p)L_p^{-1}g_p + e_p''\\
&= \big(\Phi'(Y_p) - L_p\big)L_p^{-1}g_p + g_p + e_p'' = e_p' + e_p'' + g_p.
\end{aligned}$$

As a result, it turns out that

$$\Phi(Y_{p+1}) - \Phi(Y_p) = e_p + g_p \quad \text{and} \quad \Phi(Y_{p+1}) - \Phi(Y_0) = \sum_{j=0}^{p}(e_j + g_j). \tag{2.78}$$

To achieve that the above limit is equal to $-\Phi(Y_0)$ as $p \to \infty$, we set

$$\sum_{j=0}^{p} g_j + S_p E_p = -S_p\Phi(Y_0) \quad \text{with} \quad E_p \overset{\text{def}}{=} \sum_{j=0}^{p-1} e_j. \tag{2.79}$$

The last relation defines g_p as follows:

$$\begin{aligned}
&g_0 = -S_0\Phi(Y_0), \quad \text{and}\\
&g_p = -(S_p - S_{p-1})E_{p-1} - S_p e_{p-1} - (S_p - S_{p-1})\Phi(Y_0).
\end{aligned} \tag{2.80}$$

Remark 2.3. By virtue of Remarks 2.2, 4.1 and 4.2, applying a Taylor formula to (2.77), we have

$$e'_p = -\int_0^1 f''\big(sY_p + (1-s)S_pY_p; (1-S_p)Y_p, X_p\big)ds, \quad \text{and}$$
$$e''_p = -\int_0^1 (1-s)f''\big(sY_{p+1} + (1-s)Y_p; X_p, X_p\big)ds,$$

where f'' should be understood in the way explained in Remark 4.2. Then we have

$$\begin{aligned} &e_p = e_{p,0} + e_{p,1} + e_{p,2}, \quad \text{with} \quad e_{p,m} \stackrel{\text{def}}{=} e'_{p,m} + e''_{p,m} \quad \text{and} \\ &e'_{p,m} \stackrel{\text{def}}{=} -\int_0^1 f''_m\big(sY_p + (1-s)S_pY_p; (1-S_p)Y_p, X_p\big)ds, \\ &e''_{p,m} \stackrel{\text{def}}{=} -\int_0^1 (1-s)f''_m\big(Y_p + sX_p; X_p, X_p\big)ds, \qquad m = 0, 1, 2. \end{aligned} \tag{2.81}$$

Let us fix the small constants ε, $\bar{\varepsilon}$ and $\delta > 0$ so that

$$\bar{\varepsilon} \le \frac{1}{20}, \quad \delta + 5\bar{\varepsilon} \le \frac{1}{4}, \quad \delta + \varepsilon + 4\bar{\varepsilon} \le \frac{1}{4}. \tag{2.82}$$

Let us take

$$\gamma = \frac{1}{4} - \bar{\varepsilon}, \quad \beta = \frac{1}{4} + \bar{\varepsilon}, \tag{2.83}$$

and $N_0 \in \mathbb{N}$ is chosen such that

$$\bar{\varepsilon}N_0 \ge \frac{1}{2} = \gamma + \beta. \tag{2.84}$$

In Sect. 9, we shall inductively prove the following statements:

Proposition 2.8. *Let $\delta_1 > 0$ be determined by Propositions 2.3, 2.4, 8.1, 8.2, 2.5, 2.6 and Theorem 2.3. Then for the constants $\beta, \gamma, N_0, \varepsilon, \bar{\varepsilon}$ and δ given by (2.82–2.84), for any $0 \le N \le N_0$, we have*

$$\begin{aligned} &\big\||D|^{-1}(\partial_3 X_p, \partial_t X_p)\big\|_{0,N+2} + \|\nabla X_p\|_{0,N+1} + \|(\partial_t X_p, \partial_3 X_p)\|_{\frac{1}{2},N+1} \\ &\quad + \|\partial_t X_p\|_{L^2_t(H^{N+2})} + \big\|(\partial_3 X_p, \langle t\rangle^{\frac{1}{2}}\nabla\partial_t X_p)\big\|_{L^2_t(H^{N+1})} \\ &\quad + \|\nabla\partial_t X_p\|_{1,N-1} \le \eta\theta_p^{-\beta+\bar{\varepsilon}N} \end{aligned} \tag{P1, p}$$

and

$$\begin{aligned} &|\partial_3 X_p|_{k,N} \le \eta\theta_p^{k-\frac{1}{2}-\gamma+\bar{\varepsilon}N} && \text{if } \frac{1}{2} \le k \le 1, \\ &|\partial_t X_p|_{k,N} \le \eta\theta_p^{k-(1-\delta)-\gamma+\bar{\varepsilon}N} && \text{if } 1-\delta \le k \le \frac{3}{2} - \delta, \\ &|X_p|_{k,N} \le \eta\theta_p^{k-\gamma+\bar{\varepsilon}N} && \text{if } 0 \le k \le \frac{1}{2} \end{aligned} \tag{P2, p}$$

and

$$\|\nabla Y_p\|_{L^\infty_t\left(\dot B^{\frac{3}{2}}_{2,1}\right)} \le \delta_1, \quad \|\nabla Y_p\|_{L^\infty_t\left(\dot B^{\frac{5}{2}}_{2,1}\right)} \le 1, \quad \|\partial_t Y_p\|_{0,0} \le 1, \quad |\partial_t Y_p|_{0,1} \le 1,$$

$$|\partial_3 Y_p|^{\frac{4}{3}}_{\frac{1}{2}+\varepsilon,1} \|\partial_3 Y_p\|^{\frac{2}{3}}_{L^2_t(L^2)} + |\partial_3 Y_p|^2_{\frac{1}{2}+\varepsilon,1} + |\partial_t Y_p|_{1+\varepsilon,2} \le 1. \tag{P3, p}$$

Recall the convention that $\|u\|_{k,-1} = 0$. We shall deduce the following propositions from Proposition 2.8:

Proposition 2.9. *Under the assumptions of Proposition 2.8, we have, for* $N \ge 0$*:*

$$\begin{aligned}
|S_{p+1}\partial_3 Y_{p+1}|_{k,N} &\le C_{k,N}\eta\theta_{p+1}^{k-\frac{1}{2}-\gamma+\bar\varepsilon N} && \text{if } k \ge \frac{1}{2}, \quad k-\frac{1}{2}-\gamma+\bar\varepsilon N \ge \bar\varepsilon,\\
|S_{p+1}\partial_t Y_{p+1}|_{k,N} &\le C_{k,N}\eta\theta_{p+1}^{k-(1-\delta)-\gamma+\bar\varepsilon N} && \text{if } k \ge 1-\delta,\ k-(1-\delta)-\gamma+\bar\varepsilon N \ge \bar\varepsilon,\\
|S_{p+1} Y_{p+1}|_{k,N} &\le C_{k,N}\eta\theta_{p+1}^{k-\gamma+\bar\varepsilon N} && \text{if } k \ge 0, \quad k-\gamma+\bar\varepsilon N \ge \bar\varepsilon;
\end{aligned} \tag{I) (i}$$

$$\begin{aligned}
\Delta_{p+1} &\overset{def}{=} \big\||D|^{-1}S_{p+1}(\partial_3 Y_{p+1}, \partial_t Y_{p+1})\big\|_{0,N+2} + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+1}\\
&\quad + \|S_{p+1}(\partial_t Y_{p+1}, \partial_3 Y_{p+1})\|_{\frac{1}{2},N+1}\\
&\quad + \big\|(S_{p+1}\partial_3 Y_{p+1}, \langle t\rangle^{\frac{1}{2}} S_{p+1}\nabla\partial_t Y_{p+1})\big\|_{L^2_t(H^{N+1})}\\
&\quad + \|S_{p+1}\partial_t Y_{p+1}\|_{L^2_t(H^{N+2})} + \|S_{p+1}\nabla\partial_t Y_{p+1}\|_{1,N-1}\\
&\le C_N\eta\theta_{p+1}^{-\beta+\bar\varepsilon N} \quad \text{if } -\beta+\bar\varepsilon N \ge \bar\varepsilon;
\end{aligned} \tag{I) (ii}$$

$$\begin{aligned}
|S_{p+1}\partial_3 Y_{p+1}|_{k,N} &\le C_{k,N}\eta && \text{if } k \ge \frac{1}{2}, \quad k-\frac{1}{2}-\gamma+\bar\varepsilon N \le -\bar\varepsilon,\\
|S_{p+1}\partial_t Y_{p+1}|_{k,N} &\le C_{k,N}\eta && \text{if } k \ge 1-\delta,\ k-(1-\delta)-\gamma+\bar\varepsilon N \le -\bar\varepsilon,\\
|S_{p+1} Y_{p+1}|_{k,N} &\le C_{k,N}\eta && \text{if } k \ge 0,\ k-\gamma+\bar\varepsilon N \le -\bar\varepsilon;
\end{aligned} \tag{II) (i}$$

$$\Delta_{p+1} \le C_N\eta \quad \text{if} \quad -\beta+\bar\varepsilon N \le -\bar\varepsilon; \tag{II) (ii}$$

$$\begin{aligned}
|(1-S_{p+1})\partial_3 Y_{p+1}|_{k,N} &\le C_{k,N}\eta\theta_{p+1}^{k-\frac{1}{2}-\gamma+\bar\varepsilon N} && \text{if } \frac{1}{2} \le k \le 1,\ N \le N_0,\\
|(1-S_{p+1})\partial_t Y_{p+1}|_{k,N} &\le C_{k,N}\eta\theta_{p+1}^{k-(1-\delta)-\gamma+\bar\varepsilon N} && \text{if } 1-\delta \le k \le \frac{3}{2}-\delta,\ N \le N_0,\\
|(1-S_{p+1}) Y_{p+1}|_{k,N} &\le C_{k,N}\eta\theta_{p+1}^{k-\gamma+\bar\varepsilon N} && \text{if } 0 \le k \le \frac{1}{2},\ N \le N_0.
\end{aligned} \tag{III}$$

Proposition 2.10. *Let* e_p, g_p *and* $R_{N,\theta}(g)$ *be given by* (2.77), (2.80) *and* (2.38) *respectively. Let* $\alpha \overset{def}{=} \frac{1}{2}-\delta-\bar\varepsilon > 0$. *Then there holds the following:*
(1) *Estimates for* e_p*:*

$$\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}e_p\|_{L^2_t(H^{N+1})} \lesssim \eta^2\theta_p^{k+\delta-\gamma-\beta+\bar\varepsilon(N+3)} \quad \text{if } 0 \le k \le \alpha,\ 0 \le N \le N_0-2, \tag{IV) (i}$$

$$\||D|^{-1}e_p\|_{1+k,N+1} \lesssim \eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \qquad \textit{if} \;\; 0 \le k \le \frac{1}{2}-\delta, \; N \le N_0-2, \qquad \text{(IV) (ii)}$$

$$\|\langle t\rangle^{\frac{1}{2}}e_p\|_{L_t^2(\delta,N)} \lesssim \eta^2\theta_p^{-\gamma+\bar{\varepsilon}(N+5)} \qquad \textit{if} \;\; 0 \le N \le N_0-6; \qquad \text{(IV) (iii)}$$

(2) *Estimates for* g_{p+1}*:*

$$\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}g_{p+1}\|_{L_t^2(H^{N+1})} \le C\eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \quad \textit{if}\, k \ge 0, \; N \ge 0, \qquad \text{(V) (i)}$$

$$\||D|^{-1}g_{p+1}\|_{1+k,N+1} \lesssim \eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \qquad \textit{if}\, k \ge 0, \; N \ge 0, \quad \text{(V) (ii)}$$

$$\|g_{p+1}\|_{L_t^1(N)} \le C\eta^2\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)} \qquad \textit{if} \; -\gamma+\bar{\varepsilon}(N+5) \ge \bar{\varepsilon}, \quad \text{(V) (iii)}$$

$$\|g_{p+1}\|_{L_t^1(N)} \le C\eta^2\theta_{p+1}^{\bar{\varepsilon}} \qquad \textit{if} \; -\gamma+\bar{\varepsilon}(N+5) \le -\bar{\varepsilon}; \quad \text{(V) (iv)}$$

(3) *Estimates for* $R_{N,\theta_{p+1}}(g_{p+1})$*:*

$$R_{N,\theta_{p+1}}(g_{p+1}) \le C\eta^2\theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N} \quad \textit{if} \; -\gamma+\bar{\varepsilon}(N+5) \ge \bar{\varepsilon}, \qquad \text{(VI) (i)}$$

$$R_{0,\theta_{p+1}}(g_{p+1}) \le C\eta^2\theta_{p+1}^{\frac{1}{2}-\gamma}. \qquad \text{(VI) (ii)}$$

The following interpolation lemma will be crucial in the proof of the above propositions, whose proof is exactly the same as that of Lemma 6.1 of [19], of which we omit the details here:

Lemma 2.1. (Interpolation lemma). *Let* $p \in [1,+\infty]$*,* $\theta \ge 1$ *and* $\bar{\varepsilon} > 0$*, which satisfy*

$$\beta > \bar{\varepsilon}, \quad k_0-\beta \ge \bar{\varepsilon}, \quad -\beta+\bar{\varepsilon}N_0 \ge \bar{\varepsilon}.$$

Assume that $u \in C^\infty([0,+\infty)\times\mathbb{R}^n)$ *satisfies*

$$\begin{aligned} \|u\|_{L_t^p(L^2)} &\le C\theta^{-\beta}, \\ \|\langle t\rangle^k u\|_{L_t^p(H^N)} &\le C\theta^{k-\beta+\bar{\varepsilon}N}, \; \textit{for}\; 0 \le k \le k_0, \; 0 \le N \\ &\le N_0 \; \textit{s.t}\; k-\beta+\bar{\varepsilon}N \ge \bar{\varepsilon}. \end{aligned} \tag{2.85}$$

Then for all $0 \le k \le k_0$*,* $0 \le N \le N_0$*,*

$$\|\langle t\rangle^k u\|_{L_t^p(H^N)} \le C_{k_0,N_0}\theta^{k-\beta+\bar{\varepsilon}N}.$$

Finally with the previous propositions, we shall prove the convergence of the approximate solutions constructed by (2.75) in Sect. 9.4, and this completes the proof of Theorem 2.1.

3. Decay Estimates of the Linear Equation

3.1. Decay Estimates for the Solution Operator

Following the strategy in [19,20], we first investigate the decay properties of the solutions to the linear equation (2.31) with $\mathcal{Y}_0 = 0$ and $\mathcal{Y}_1 = Y_1$. By taking Fourier transform to (2.31) with respect to y variables and solving the resulting ODE, we write

$$\mathcal{Y}(t, y) = \Gamma(t, D)Y_1 \quad \text{with} \quad \Gamma(t, \xi)=\frac{1}{\lambda_2(\xi) - \lambda_1(\xi)} \left(e^{t\lambda_2(\xi)} - e^{t\lambda_1(\xi)}\right), \tag{3.1}$$

where $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are given by (2.2).

Proposition 3.1. *Given $\delta \in [0, 1[$ and $N \in \mathbb{N}$, there exists $C_{\delta,N} > 0$ such that there holds*

$$\begin{aligned}|\partial_3\Gamma(t)Y_1|_{1,N} + |\partial_3^2\Gamma(t)Y_1|_{\frac{3}{2},N} + |\partial_t\Gamma(t)Y_1|_{\frac{3}{2}-\delta,N} + |\Gamma(t)Y_1|_{\frac{1}{2},N}\\ \leq C_{\delta,N}\left\|(|D|^{2\delta}Y_1, |D|^{N+4}Y_1)\right\|_{L^1}.\end{aligned} \tag{3.2}$$

Proof. The estimate (3.2) for general $N \in \mathbb{N}$ follows from the case when $N = 0$. Due to the anisotropic properties of the eigenvalues $\lambda_1(\xi), \lambda_2(\xi)$, we shall split the frequency space into two parts: $\{\xi \in \mathbb{R}^3 : |\xi|^2 \geq 2|\xi_3|\}$ and $\{\xi \in \mathbb{R}^3 : |\xi|^2 < 2|\xi_3|\}$. When $|\xi|^2 \geq 2|\xi_3|$, let us denote $\alpha(\xi) \stackrel{\text{def}}{=} \sqrt{\frac{|\xi|^4}{4} - \xi_3^2}$. Then we have

$$\lambda_1(\xi) = -\frac{|\xi|^2}{2} + \alpha(\xi) \quad \text{and} \quad \lambda_2(\xi) = -\frac{|\xi|^2}{2} - \alpha(\xi),$$

and we write

$$\Gamma(t, \xi)\mathbf{1}_{|\xi|^2\geq2|\xi_3|} = e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)}\mathbf{1}_{|\xi|^2\geq2|\xi_3|}. \tag{3.3}$$

When $|\xi|^2 < 2|\xi_3|$, let us denote $\beta(\xi) \stackrel{\text{def}}{=} \sqrt{\xi_3^2 - \frac{|\xi|^4}{4}}$. Then we have

$$\lambda_1(\xi) = -\frac{|\xi|^2}{2} + i\beta(\xi) \quad \text{and} \quad \lambda_2(\xi) = -\frac{|\xi|^2}{2} - i\beta(\xi),$$

and we write

$$\Gamma(t, \xi)\mathbf{1}_{|\xi|^2<2|\xi_3|} = e^{-\frac{t}{2}|\xi|^2} \frac{\sin(t\beta(\xi))}{\beta(\xi)}\mathbf{1}_{|\xi|^2<2|\xi_3|}. \tag{3.4}$$

Next we handle the estimate of (3.2) term by term, below.
•*Estimates of $\|\partial_3\mathcal{Y}(t)\|_{L^\infty}$ and $\|\partial_3^2\mathcal{Y}(t)\|_{L^\infty}$.*

In view of (3.1), we deduce that

$$\begin{aligned}\|\partial_3\mathcal{Y}(t)\|_{L^\infty} &\leq \|\Gamma(t, \cdot)\xi_3\widehat{Y}_1(\cdot)\|_{L^1}\\ &= \int_{|\xi|^2\geq2|\xi_3|} e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)}|\xi_3\widehat{Y}_1(\xi)|\, d\xi\\ &\quad + \int_{|\xi|^2<2|\xi_3|} e^{-\frac{t}{2}|\xi|^2} \frac{|\sin(t\beta(\xi))|}{\beta(\xi)}|\xi_3\widehat{Y}_1(\xi)|\, d\xi \stackrel{\text{def}}{=} I_1 + I_2.\end{aligned} \tag{3.5}$$

It is easy to observe that

$$I_1 = \left(\int_{|\xi|\geq 3} + \int_{9>|\xi|^2\geq 2|\xi_3|}\right) e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)} \frac{1-e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)|\,\mathrm{d}\xi$$

and

$$\begin{aligned}
&\int_{|\xi|\geq 3} e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)} \frac{1-e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)|\,\mathrm{d}\xi \\
&\quad \leq \||\xi|^3 \widehat{Y_1}\|_{L^\infty} \int_{|\xi|\geq 3} e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}} \frac{|\xi_3|}{2\alpha(\xi)|\xi|^3}\,\mathrm{d}\xi \\
&\quad \leq 2\||\xi|^3 \widehat{Y_1}\|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_3^\infty e^{-t\cos^2\phi} \frac{1}{r\sqrt{r^2-4\cos^2\phi}} \sin\phi\cos\phi\,\mathrm{d}\phi\,\mathrm{d}r \\
&\quad \leq C\||\xi|^3 \widehat{Y_1}\|_{L^\infty} \int_0^1 e^{-t\tau} \int_3^\infty \frac{1}{r\sqrt{r^2-4\tau^2}}\,\mathrm{d}r\,\mathrm{d}\tau \\
&\quad \leq C\langle t\rangle^{-1} \||D|^3 Y_1\|_{L^1}.
\end{aligned}$$

Exactly along the same lines, we have

$$\begin{aligned}
&\int_{9>|\xi|^2\geq 2|\xi_3|} e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)} \frac{1-e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)|\,\mathrm{d}\xi \\
&\quad \leq 2\||\xi|\widehat{Y_1}\|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_{2\cos\phi}^3 e^{-t\cos^2\phi} \frac{\sin\phi\cos\phi}{\sqrt{r^2-4\cos^2\phi}} r\,\mathrm{d}r\,\mathrm{d}\phi \\
&\quad \leq C\||\xi|\widehat{Y_1}\|_{L^\infty} \int_0^1 e^{-t\tau} \int_{2\sqrt{\tau}}^3 \frac{r}{\sqrt{r^2-4\tau}}\,\mathrm{d}r\,\mathrm{d}\tau \\
&\quad \leq C\langle t\rangle^{-1} \||D|Y_1\|_{L^1}.
\end{aligned}$$

This proves

$$I_1 \leq C\langle t\rangle^{-1}\big(\||D|Y_1\|_{L^1} + \||D|^3 Y_1\|_{L^1}\big). \tag{3.6}$$

The estimate of I_2 is much simpler. By virtue of (3.5), we have

$$\begin{aligned}
I_2 &\leq 2\||\xi|\widehat{Y_1}\|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} e^{-\frac{t}{2}r^2} \frac{1}{\sqrt{4\cos^2\phi - r^2}} \sin\phi\cos\phi r\,\mathrm{d}r\,\mathrm{d}\phi \\
&\leq 2\||\xi|\widehat{Y_1}\|_{L^\infty} \int_0^1 e^{-\frac{t}{2}r^2} r \int_{\frac{r^2}{4}}^1 \frac{1}{\sqrt{4\tau - r^2}}\,\mathrm{d}\tau\,\mathrm{d}r \\
&\leq C\langle t\rangle^{-1} \||D|Y_1\|_{L^1}.
\end{aligned} \tag{3.7}$$

As a result, we achieve

$$\|\partial_3 \mathcal{Y}(t)\|_{L^\infty} \leq C\langle t\rangle^{-1}\big(\||D|Y_1\|_{L^1} + \||D|^3 Y_1\|_{L^1}\big). \tag{3.8}$$

Along the same lines as to the proof of (3.8), we infer

$$
\begin{aligned}
&\|\partial_3^2\mathcal{Y}(t)\|_{L^\infty}\\
&\quad\leq 2\||\xi|^4\widehat{Y_1}\|_{L^\infty}\int_0^{\frac{\pi}{2}}\int_3^\infty e^{-t\cos^2\phi}\frac{1}{r\sqrt{r^2-4\cos^2\phi}}\sin\phi\cos^2\phi\,\mathrm{d}\phi\,\mathrm{d}r\\
&\qquad+2\||\xi|^2\widehat{Y_1}\|_{L^\infty}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^3 e^{-t\cos^2\phi}\frac{\sin\phi\cos^2\phi}{\sqrt{r^2-4\cos^2\phi}}r\,\mathrm{d}r\,\mathrm{d}\phi\\
&\qquad+2\||\xi|\widehat{Y_1}\|_{L^\infty}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi} e^{-\frac{t}{2}r^2}\frac{1}{\sqrt{4\cos^2\phi-r^2}}\sin\phi\cos^2\phi r^2\,\mathrm{d}r\,\mathrm{d}\phi,
\end{aligned}
$$

so that for t large enough, there holds

$$
\begin{aligned}
&\|\partial_3^2\mathcal{Y}(t)\|_{L^\infty}\\
&\quad\leq Ct^{-\frac{1}{2}}\Big(\||\xi|^4\widehat{Y_1}\|_{L^\infty}\int_0^{\frac{\pi}{2}}\int_3^\infty e^{-\frac{t\cos^2\phi}{2}}\frac{1}{r\sqrt{r^2-4\cos^2\phi}}\sin\phi\cos\phi\,\mathrm{d}\phi\,\mathrm{d}r\\
&\qquad+\||\xi|^2\widehat{Y_1}\|_{L^\infty}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^3 e^{-\frac{t\cos^2\phi}{2}}\frac{\sin\phi\cos\phi}{\sqrt{r^2-4\cos^2\phi}}r\,\mathrm{d}r\,\mathrm{d}\phi\Big)\\
&\qquad+\||\xi|\widehat{Y_1}\|_{L^\infty}\int_0^1 e^{-\frac{t}{2}r^2}r^2\int_{\frac{r^2}{4}}^1\frac{1}{\sqrt{4\tau-r^2}}\,\mathrm{d}\tau\,\mathrm{d}r.
\end{aligned}
$$

This gives rise to

$$
\|\partial_3^2\mathcal{Y}(t)\|_{L^\infty}\leq C\langle t\rangle^{-\frac{3}{2}}\big(\||D|Y_1\|_{L^1}+\||D|^4Y_1\|_{L^1}\big). \tag{3.9}
$$

•*Estimate of $|\partial_t\mathcal{Y}(t)|_{L^\infty}$.*

It follows from (3.1) that

$$
\partial_t\Gamma(t,\xi)=\frac{1}{\lambda_2(\xi)-\lambda_1(\xi)}\left(\lambda_2(\xi)e^{t\lambda_2(\xi)}-\lambda_1(\xi)e^{t\lambda_1(\xi)}\right),
$$

so that one has

$$
\begin{aligned}
&\partial_t\Gamma(t,\xi)\mathbf{1}_{|\xi|^2\geq 2|\xi_3|}=e^{-t\left(\frac{|\xi|^2}{2}+\alpha(\xi)\right)}-e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)}\Big(\frac{|\xi|^2}{2}-\alpha(\xi)\Big)\frac{1-e^{-2t\alpha(\xi)}}{2\alpha(\xi)},\\
&\partial_t\Gamma(t,\xi)\mathbf{1}_{|\xi|^2<2|\xi_3|}=e^{-\frac{t}{2}|\xi|^2}\left(-\frac{|\xi|^2}{2}\frac{\sin(t\beta(\xi))}{\beta(\xi)}+\cos(t\beta(\xi))\right).
\end{aligned}
\tag{3.10}
$$

It is easy to observe that for any $d\in[0,1[$,

$$
\begin{aligned}
\int_{\mathbb{R}^3}e^{-\frac{t}{2}|\xi|^2}|\widehat{Y_1}(\xi)|\,\mathrm{d}\xi&\leq\||\xi|^{2\delta}\widehat{Y_1}\|_{L^\infty}\int_{\mathbb{R}^3}|\xi|^{-2\delta}e^{-\frac{t}{2}|\xi|^2}\,\mathrm{d}\xi\\
&\leq Ct^{-\left(\frac{3}{2}-\delta\right)}\||D|^{2\delta}Y_1\|_{L^1},
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} e^{-\frac{t}{2}|\xi|^2}|\widehat{Y_1}(\xi)|\,\mathrm{d}\xi \\
&\quad \le \||\xi|^{2\delta}\widehat{Y_1}\|_{L^\infty}\int_{|\xi|\le 1}|\xi|^{-2\delta}\,\mathrm{d}\xi + \||\xi|^4\widehat{Y_1}\|_{L^\infty}\int_{|\xi|>1}|\xi|^{-4}\,\mathrm{d}\xi \\
&\quad \le C\big(\||D|^{2\delta}Y_1\|_{L^1} + \||D|^4 Y_1\|_{L^1}\big).
\end{aligned}$$

This leads to

$$\int_{\mathbb{R}^3} e^{-\frac{t}{2}|\xi|^2}|\widehat{Y_1}(\xi)|\,\mathrm{d}\xi \le C\langle t\rangle^{-\left(\frac{3}{2}-\delta\right)}\big(\||D|^{2\delta}Y_1\|_{L^1} + \||D|^4 Y_1\|_{L^1}\big).$$

While similar to estimates of (3.6) and (3.7), we infer

$$\begin{aligned}
&\int_{|\xi|^2\ge 2|\xi_3|} e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}}\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}\frac{1-e^{-2t\alpha(\xi)}}{2\alpha(\xi)}|\widehat{Y_1}(\xi)|\,\mathrm{d}\xi \\
&\le 2\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{2\cos\phi}^\infty e^{-t\cos^2\phi}2\cos^2\phi\frac{|\widehat{Y_1}(\xi(r,\theta,\phi))|}{\sqrt{r^2-4\cos^2\phi}}\sin\phi r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\le 2\||\xi|^{2\delta}\widehat{Y_1}\|_{L^\infty}\int_0^1 e^{-t\tau}\sqrt{\tau}\int_{2\sqrt{\tau}}^3\frac{r^{1-2\delta}}{\sqrt{r^2-4\tau}}\,\mathrm{d}r\,\mathrm{d}\tau \\
&\quad + 2\||\xi|^2\widehat{Y_1}\|_{L^\infty}\int_0^1 e^{-t\tau}\sqrt{\tau}\int_3^\infty\frac{1}{r\sqrt{r^2-4\tau}}\,\mathrm{d}r\,\mathrm{d}\tau \\
&\le C\langle t\rangle^{-\frac{3}{2}}\big(\||D|^{2\delta}Y_1\|_{L^1} + \||D|^2 Y_1\|_{L^1}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|\xi|^2\le 2|\xi_3|} e^{-\frac{t}{4}|\xi|^2}\frac{|\xi_3|}{\beta(\xi)}|\widehat{Y_1}(\xi)|\,\mathrm{d}\xi \\
&\quad \le 2\||\xi|^{2\delta}\widehat{Y_1}\|_{L^\infty}\int_0^{\frac{\pi}{2}}\frac{2\sin\phi\cos\phi}{\sqrt{4\cos^2\phi-r^2}}\int_0^{2\cos\phi} e^{-\frac{t}{4}r^2}r^{2(1-\delta)}\,\mathrm{d}r\,\mathrm{d}\phi \\
&\quad \le C\langle t\rangle^{-\left(\frac{3}{2}-\delta\right)}\||\xi|^{2\delta}\widehat{Y_1}\|_{L^\infty} \le C\langle t\rangle^{-\left(\frac{3}{2}-\delta\right)}\||D|^{2\delta}Y_1\|_{L^1}.
\end{aligned}$$

Hence by virtue of (3.10), we obtain

$$\|\partial_t\mathcal{Y}(t)\|_{L^\infty} \le C\langle t\rangle^{-\left(\frac{3}{2}-\delta\right)}\big(\||D|^{2\delta}Y_1\|_{L^1} + \|D^4 Y_1\|_{L^1}\big). \tag{3.11}$$

•*Estimate of $\|\mathcal{Y}(t)\|_{L^\infty}$.*

Note that

$$\int_0^1\int_{2\tau}^3 e^{-t\tau^2}\frac{r^{\frac{1}{2}}}{\sqrt{r^2-4\tau^2}}\mathrm{d}r\,\mathrm{d}\tau \le \int_0^1 e^{-t\tau^2}\int_{2\tau}^3(r-2\tau)^{-\frac{1}{2}}\mathrm{d}r\,\mathrm{d}\tau \le C\langle t\rangle^{-\frac{1}{2}}.$$

We find

$$\int_{|\xi|^2\ge 2|\xi_3|} e^{-t\left(\frac{|\xi|^2}{2}-\alpha(\xi)\right)}\frac{1-e^{-2t\alpha(\xi)}}{2\alpha(\xi)}|\widehat{Y_1}(\xi)|\mathrm{d}\xi$$

$$
\begin{aligned}
&\le \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{2\cos\phi}^{\infty} e^{-t\cos^2\phi}\frac{|\widehat{Y}_1(\xi(r,\theta,\phi))|}{\frac{1}{2}r\sqrt{r^2-4\cos^2\phi}}r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= C\||\xi|^{\frac{1}{2}}\widehat{Y}_1\|_{L^\infty}\int_0^1\int_{2\tau}^3 e^{-t\tau^2}\frac{r^{\frac{1}{2}}}{\sqrt{r^2-4\tau^2}}\mathrm{d}r\,\mathrm{d}\tau\\
&\quad + C\||\xi|^2\widehat{Y}_1\|_{L^\infty}\int_0^1 e^{-t\tau^2}\int_3^\infty \frac{1}{r\sqrt{r^2-4\tau^2}}\mathrm{d}r\,\mathrm{d}\tau\\
&\le C\langle t\rangle^{-\frac{1}{2}}\big(\||D|^{\frac{1}{2}}Y_1\|_{L^1}+\||D|^2Y_1\|_{L^1}\big).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{|\xi|^2<2|\xi_3|} e^{-t\frac{|\xi|^2}{2}}\frac{\sin(t\beta(\xi))}{2\beta(\xi)}|\widehat{Y}_1(\xi)|\mathrm{d}\xi\\
&\quad\le \int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi} e^{-\frac{t}{2}r^2}\frac{|\widehat{Y}_1(\xi(r,\theta,\phi))|}{r\sqrt{4\cos^2\phi-r^2}}r^2\sin\phi\mathrm{d}r\,\mathrm{d}\phi\\
&\quad\le \||\xi|^{\frac{1}{2}}\widehat{Y}_1\|_{L^\infty}\int_0^2 e^{-\frac{t}{2}r^2}\int_{r/2}^1\frac{r^{\frac{1}{2}}}{\sqrt{4\tau^2-r^2}}\mathrm{d}\tau\,\mathrm{d}r\\
&\quad\le C\langle t\rangle^{-\frac{1}{2}}\||D|^{\frac{1}{2}}Y_1\|_{L^1}.
\end{aligned}
$$

As a result, by virtue of (3.3) and (3.4), it turns out that

$$
\|\mathcal{Y}(t)\|_{L^\infty}\le C\langle t\rangle^{-\frac{1}{2}}\big(\||D|^{\frac{1}{2}}Y_1\|_{L^1}+\|D^2Y_1\|_{L^1}\big). \tag{3.12}
$$

Then (3.8), together with (3.9), (3.11) and (3.12), imply the estimate (3.2) for $N=0$. □

Lemma 3.1. *For $N\in\mathbb{N}$, there exists $C_N>0$ such that for $t>0$,*

$$
\begin{aligned}
&\|t\Delta\partial_t\Gamma(t)Y_1\|_{L^\infty_t(H^N)}\le C_N\|Y_1\|_N\\
&\quad\textit{and}\quad \|t\nabla\partial_3^2\Gamma(t)Y_1\|_{L^\infty_t(H^N)}\le C_N\|Y_1\|_{N+1}.
\end{aligned}\tag{3.13}
$$

Proof. The two inequalities of (3.13) follow from the claim that

$$
t|\xi|^2\partial_t\Gamma(t,\xi)\in L^\infty_t(L^\infty_\xi),\quad \text{and}\quad \frac{t|\xi|}{1+|\xi|}|\xi_3|^2\Gamma(t,\xi)\in L^\infty_t(L^\infty_\xi). \tag{3.14}
$$

(1) When $|\xi|^2\ge 2|\xi_3|$, we separate the proof of (3.14) into the following two cases:

- If $\frac{\sqrt{3}}{4}|\xi|^2\le|\xi_3|\le\frac{1}{2}|\xi|^2$, we deduce from (3.10) that

$$
|\partial_t\Gamma(t,\xi)\mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2\le|\xi_3|\le\frac{1}{2}|\xi|^2}|\le e^{-t\frac{|\xi|^2}{4}}(1+|\xi|^2t),
$$

$$
|\xi_3^2\Gamma(t,\xi)\mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2\le|\xi_3|\le\frac{1}{2}|\xi|^2}\le t\xi_3^2e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}}\mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2\le|\xi_3|\le\frac{1}{2}|\xi|^2}|\le Ct\xi_3^2e^{-ct|\xi_3|}.
$$

As a result, we have

$$t|\xi|^2|\partial_t\Gamma(t,\xi)|\mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2\le|\xi_3|\le\frac{1}{2}|\xi|^2}\le C \quad\text{and}$$
$$\frac{t|\xi|}{1+|\xi|}|\xi_3|^2\Gamma(t,\xi)\mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2\le|\xi_3|\le\frac{1}{2}|\xi|^2}\le C.$$

- If $|\xi_3|\le\frac{\sqrt{3}}{4}|\xi|^2$, then $\frac{|\xi|^2}{4}\le\alpha(\xi)\le\frac{|\xi|^2}{2}$, we deduce from (3.10) that

$$|\partial_t\Gamma(t,\xi)|\mathbf{1}_{|\xi_3|\le\frac{\sqrt{3}}{4}|\xi|^2}\le e^{-t\frac{|\xi|^2}{2}}+e^{-t\frac{\xi_3^2}{|\xi|^2}}\frac{\xi_3^2}{|\xi|^4},$$
$$\xi_3^2|\Gamma(t,\xi)|\mathbf{1}_{|\xi_3|\le\frac{\sqrt{3}}{4}|\xi|^2}\le\frac{\xi_3^2}{\alpha(\xi)}e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}}\le C\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2}+\alpha(\xi)}},$$

so that there holds

$$t|\xi|^2|\partial_t\Gamma(t,\xi)|\xi_3|\mathbf{1}_{|\xi_3|\le\frac{\sqrt{3}}{4}|\xi|^2}\le t|\xi|^2e^{-t\frac{|\xi|^2}{2}}+te^{-t\frac{\xi_3^2}{|\xi|^2}}\frac{\xi_3^2}{|\xi|^2}\le C,$$
$$\frac{t|\xi|}{1+|\xi|}|\xi_3|^2\Gamma(t,\xi)\mathbf{1}_{|\xi_3|\le\frac{\sqrt{3}}{4}|\xi|^2}\le C.$$

(2) When $|\xi|^2>2|\xi_3|$, we infer from (3.10) that

$$|\partial_t\Gamma(t,\xi)|\mathbf{1}_{|\xi|^2>2|\xi_3|}\le e^{-t\frac{|\xi|^2}{2}}(|\xi|^2t+1),$$

which implies

$$t|\xi|^2|\partial_t\Gamma(t,\xi)\mathbf{1}_{|\xi|^2>2|\xi_3|}|\le C.$$

To prove the second estimate of (3.14), we further divide the region $\{|\xi|^2>2|\xi_3|\}$ into two parts:

- If $|\xi|^2\le\sqrt{3}|\xi_3|$, then we have $\frac{|\xi_3|}{2}\le\beta(\xi)\le|\xi_3|$, and it follows from (3.4) that

$$\xi_3^2|\Gamma(t,\xi)|\mathbf{1}_{|\xi|^2\le\sqrt{3}|\xi_3|}\le C|\xi_3|e^{-t\frac{|\xi|^2}{2}}\le\frac{C}{t|\xi|};$$

- When $\sqrt{3}|\xi_3|<|\xi|^2\le2|\xi_3|$, we have

$$\xi_3^2|\Gamma(t,\xi)|\mathbf{1}_{\sqrt{3}|\xi_3|<|\xi|^2\le2|\xi_3|}\le Ct|\xi_3|^2e^{-ct|\xi_3|}\le\frac{C}{t}.$$

By summarizing the above estimates, we obtain the second estimate of (3.14). This completes the proof of Lemma 3.1. □

3.2. Energy Estimates for the Linear Equation

Lemma 3.2. *Let $\mathcal{Y}(t)$ be a smooth enough solution of the linear equation (2.31) with initial data $(\mathcal{Y}_0, \mathcal{Y}_1)$. Then for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that there holds (2.33) and (2.34).*

Proof. Taking the L^2-inner product of the equation (2.31) with $\mathcal{Y}_t$ and $\mathcal{Y}_t - \frac{1}{4}\Delta\mathcal{Y} - \Delta\mathcal{Y}_t$, respectively, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathcal{Y}_t\|_0^2 + \|\partial_3\mathcal{Y}\|_0^2) + \|\nabla\mathcal{Y}_t\|_0^2 = 0$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathcal{Y}_t\|_1^2 + \|\partial_3\mathcal{Y}\|_1^2 + \frac{1}{4}\|\Delta\mathcal{Y}\|_0^2 - \frac{1}{4}(\mathcal{Y}_t|\Delta\mathcal{Y})_{L^2}\right)\\ &\quad + \frac{3}{4}\|\nabla\mathcal{Y}_t\|_0^2 + \|\Delta\mathcal{Y}_t\|_0^2 + \frac{1}{4}\|\nabla\partial_3\mathcal{Y}\|_0^2 = 0.\end{aligned}$$

Integrating the above equalities with respect to t gives rise to

$$\begin{aligned}&\|(\mathcal{Y}_t, \partial_3\mathcal{Y})\|_{L^\infty_t(L^2)} + \|\nabla\partial_t\mathcal{Y}\|_{L^2_t(L^2)} \le \|(\partial_3\mathcal{Y}_0, \mathcal{Y}_1)\|_0 \quad \text{and}\\ &\|(\mathcal{Y}_t, \partial_3\mathcal{Y})\|_{L^\infty_t(H^1)} + \|\Delta\mathcal{Y}\|_{L^\infty_t(L^2)} + \|\nabla\mathcal{Y}_t\|_{L^2_t(H^1)} + \|\nabla\partial_3\mathcal{Y}\|_{L^2_t(L^2)}\\ &\quad \le C\big(\|(\partial_3\mathcal{Y}_0, \mathcal{Y}_1)\|_1 + \|\Delta\mathcal{Y}_0\|_0\big).\end{aligned}$$

This proves (2.33) for $N = 0$. The general case with $N > 0$ follows similarly.

To show (2.34), we first get, by taking the H^N-inner product of the equation (2.31) with $\mathcal{Y}_t$, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathcal{Y}_t\|_N^2 + \|\partial_3\mathcal{Y}\|_N^2\big) + \|\nabla\mathcal{Y}_t\|_N^2 = 0,$$

so that for any nonegative $f(t) \in C^1([0, \infty[)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(f(t)\big(\|\mathcal{Y}_t\|_N^2 + \|\partial_3\mathcal{Y}\|_N^2\big)\Big) + 2f(t)\|\nabla\mathcal{Y}_t\|_N^2 = f'(t)\big(\|\mathcal{Y}_t\|_N^2 + \|\partial_3\mathcal{Y}\|_N^2\big).$$

Taking $f(t) = \langle t\rangle$ and integrating the resulting equality over $[0, t]$, we find

$$\begin{aligned}&\langle t\rangle\big(\|\mathcal{Y}_t(t)\|_N^2 + \|\partial_3\mathcal{Y}(t)\|_N^2\big) + 2\int_0^t \langle s\rangle\|\nabla\mathcal{Y}_t(s)\|_N^2\mathrm{d}s \le \|(\partial_3\mathcal{Y}_0, \mathcal{Y}_1)\|_N^2\\ &\quad + \int_0^t \big(\|\mathcal{Y}_t\|_N^2 + \|\partial_3\mathcal{Y}\|_N^2\big)\mathrm{d}s.\end{aligned} \tag{3.15}$$

However we have from (2.33) that

$$\|\mathcal{Y}_t\|_{L^2_t(H^{N+1})} + \|\partial_3\mathcal{Y}\|_{L^2_t(H^N)} \le C_N\big(\||D|^{-1}(\partial_3\mathcal{Y}_0, \mathcal{Y}_1)\|_{N+1} + \|\nabla\mathcal{Y}_0\|_N\big),$$

which together with (3.15) ensures (2.34). □

Recall that $\mathcal{Y}(t) = \Gamma(t)Y_1$ is the solution to (2.31) with initial data $(\mathcal{Y}_0, \mathcal{Y}_1) = (0, Y_1)$, so that one can deduce estimates for the operator Γ from the energy estimates (2.33) and (2.34). Indeed, combining (3.13) with (2.33) gives

$$\|\langle t\rangle \Delta\partial_t \Gamma(t)Y_1\|_{L^\infty_t(H^N)} + \|\langle t\rangle \partial_3^2\Gamma(t)Y_1\|_{L^\infty_t(H^N)} \leq C_N\|Y_1\|_{N+2}. \tag{3.16}$$

Let us remark that

$$\begin{aligned}\|Y\|_{L^\infty} &\leq \int |\widehat{Y}(\xi)|d\xi \leq \int_{|\xi|\leq 1} |\xi|^{-1}\cdot|\xi||\widehat{Y}(\xi)|d\xi + \int_{|\xi|>1}|\xi|^{-2}\cdot|\xi|^2|\widehat{Y}(\xi)|d\xi \\ &\leq C(\||D|Y\|_{L^2} + \||D|^2Y\|_{L^2}) \leq C\||D|Y\|_1.\end{aligned} \tag{3.17}$$

Summarizing (2.33), (3.16) and (3.17) then leads to

Corollary 3.1. *For $N \geq 0$, there exists $C_N > 0$ such that*

$$\begin{aligned}&\|\Gamma(t)Y_1\|_{L^\infty_t(W^{N,\infty})} \leq C_N\||D|^{-1}Y_1\|_{N+2},\\ &\|\partial_3\Gamma(t)Y_1\|_{L^2_t(W^{N,\infty})} \leq C_N\|Y_1\|_{N+2},\\ &\|\langle t\rangle\partial_t\Gamma(t)Y_1\|_{L^\infty_t(W^{N,\infty})} \leq C_N\||D|^{-1}Y_1\|_{N+3},\end{aligned} \tag{3.18}$$

where $\Gamma(t)$ is the solution operator given by (3.1).

Now we are in a position to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. (2.33) and (2.34) are already proved by Lemma 3.2, so it remains to deal with the estimates of (2.32) and (2.35). As a matter of fact, according to the definition of the solution operator $\Gamma(t)$ given by (3.1), we have

$$\mathcal{Y}(t) = \partial_t\Gamma(t)\mathcal{Y}_0 + \Gamma(t)(\mathcal{Y}_1 - \Delta\mathcal{Y}_0), \tag{3.19}$$

from which, with (3.2), we infer that for any $\delta \in]0, 1[$ and for $N \in \mathbb{N}$,

$$\begin{aligned}|\partial_3\mathcal{Y}|_{1,N} + |\partial_t\mathcal{Y}|_{\frac{3}{2}-\delta,N} + |\mathcal{Y}|_{\frac{1}{2},N} &\leq |\partial_3\partial_t\Gamma(t)\mathcal{Y}_0|_{1,N} + |\partial_t^2\Gamma(t)\mathcal{Y}_0|_{\frac{3}{2}-\delta,N}\\ &+|\partial_t\Gamma(t)\mathcal{Y}_0|_{\frac{1}{2},N} + C_N\big(\||D|^{2\delta}(\Delta\mathcal{Y}_0,\mathcal{Y}_1)\|_{L^1} + \||D|^{N+4}(\Delta\mathcal{Y}_0,\mathcal{Y}_1)\|_{L^1}\big).\end{aligned} \tag{3.20}$$

Notice that $\partial_t^2\Gamma(t)\mathcal{Y}_0 = \Delta\partial_t\Gamma(t)\mathcal{Y}_0 + \partial_3^2\Gamma(t)\mathcal{Y}_0$, so we get, by applying (3.2) once again, that

$$\begin{aligned}|\partial_3\partial_t\Gamma(t)\mathcal{Y}_0|_{1,N} &= |\partial_t\Gamma(t)\partial_3\mathcal{Y}_0|_{1,N} \leq C_N\big(\||D|^{2\delta}\partial_3\mathcal{Y}_0\|_{L^1} + \||D|^{N+4}\partial_3\mathcal{Y}_0\|_{L^1}\big),\\ |\partial_t^2\Gamma(t)\mathcal{Y}_0|_{\frac{3}{2}-\delta,N} &\leq |\Delta\partial_t\Gamma(t)\mathcal{Y}_0|_{\frac{3}{2}-\delta,N} + |\partial_3^2\Gamma(t)\mathcal{Y}_0|_{\frac{3}{2},N}\\ &\leq C_N\left(\||D|^{2\delta}\mathcal{Y}_0\|_{L^1} + \||D|^{N+6}\mathcal{Y}_0\|_{L^1}\right),\\ |\partial_t\Gamma(t)\mathcal{Y}_0|_{\frac{1}{2},N} &\leq C_N\left(\||D|^{2\delta}\mathcal{Y}_0\|_{L^1} + \||D|^{N+4}\mathcal{Y}_0\|_{L^1}\right).\end{aligned}$$

Inserting the above estimates into (3.20) leads to (2.32).

Finally notice that $\Delta\partial_t^2\Gamma(t)\mathcal{Y}_0 = \Delta^2\partial_t\Gamma(t)\mathcal{Y}_0 + \Delta\partial_3^2\Gamma(t)\mathcal{Y}_0$. Then by virtue of (3.16), we deduce

$$\begin{aligned}\|\langle t\rangle\Delta\partial_t\mathcal{Y}(t)\|_{L_t^\infty(H^N)} &\leq \|\langle t\rangle\Delta\partial_t^2\Gamma(t)\mathcal{Y}_0\|_{L_t^\infty(H^N)} \\ &\quad + \|\langle t\rangle\Delta\partial_t\Gamma(t)(\mathcal{Y}_1 - \Delta\mathcal{Y}_0)\|_{L_t^\infty(H^N)} \\ &\leq C_N\left(\|\Delta\mathcal{Y}_0\|_{N+2} + \|(\Delta\mathcal{Y}_0, \mathcal{Y}_1)\|_{N+2}\right).\end{aligned}$$

This proves (2.35), and thus we complete the proof of Proposition 2.1. □

3.3. Decay Estimates for the Inhomogeneous Equation

Proof of Proposition 2.2. In view of (3.1), we get, by applying Duhamel's principle to (2.36), that

$$Y(t) = \int_0^t \Gamma(t-s)g(s)\mathrm{d}s. \tag{3.21}$$

In what follows, we shall present the proof of (2.37) term by term.

•*Decay estimate of $\partial_3 Y$.*

We first separate the integral in (3.21) as

$$\begin{aligned}\partial_3 Y(t,y) &= \int_0^t \partial_3\Gamma(t-s)g(s)\mathrm{d}s \\ &= \int_0^{t/2} \partial_3\Gamma(t-s)g(s)\mathrm{d}s + \int_{t/2}^t \partial_3\Gamma(t-s)g(s)\mathrm{d}s.\end{aligned}$$

We deduce from (3.8) that

$$\begin{aligned}\langle t\rangle\int_0^{t/2}\left|\partial_3\Gamma(t-s)g(s)\right|_N \mathrm{d}s &\leq C_N\langle t\rangle\int_0^{t/2}\langle t-s\rangle^{-1}\||D|g(s)\||_{N+2}\,\mathrm{d}s \\ &\leq C_N\int_0^{t/2}\||D|g(s)\||_{N+2}\mathrm{d}s \leq C\||D|g\|_{L_t^1(W^{N+2,1})}.\end{aligned}$$

Meanwhile, it follows from the second inequality in (3.18) that

$$\begin{aligned}&\langle t\rangle\int_{t/2}^t\left|\partial_3\Gamma(t-s)g(s)\right|_N\mathrm{d}s \\ &\leq \langle t\rangle\Big(\int_{t/2}^t\|g(s)\|_{N+2}^2\mathrm{d}s\Big)^{\frac12} \leq C\theta^{\frac12}\|\langle t\rangle^{\frac12}g\|_{L_t^2(H^{N+2})}.\end{aligned}$$

Hence we achieve

$$|\partial_3 Y|_{1,N} \leq C_N\left(\||D|g\|_{L_t^1(W^{N+2,1})} + \theta^{\frac12}\|\langle t\rangle^{\frac12}g\|_{L_t^2(H^{N+2})}\right). \tag{3.22}$$

•*Decay estimate of Y_t.*

Noticing that $\Gamma(0) = 0$, we have

$$Y_t(t) = \int_0^t \partial_t\Gamma(t-s)g(s)\mathrm{d}s = \int_0^{t/2}\partial_t\Gamma(t-s)g(s)\mathrm{d}s + \int_{t/2}^t\partial_t\Gamma(t-s)g(s)\mathrm{d}s.$$

It follows from (3.11) that

$$
\begin{aligned}
&\langle t\rangle^{\frac{3}{2}-\delta}\int_0^{t/2}\left|\partial_t\Gamma(t-s)g(s)\right|_N \mathrm{d}s\\
&\quad\le C_N\langle t\rangle^{\frac{3}{2}-\delta}\int_0^{t/2}\langle t-s\rangle^{-\left(\frac{3}{2}-\delta\right)}\left(\||D|^{2\delta}g(s)\|_N+\|D^4g(s)\|_N\right)\mathrm{d}s\\
&\quad\le C_N\left(\||D|^{2\delta}g\|_{L^1_t(W^{N,1})}+\|D^4g\|_{L^2_t(W^{N,1})}\right).
\end{aligned}
$$

It follows from the third inequality in (3.18) that

$$
\begin{aligned}
\langle t\rangle^{\frac{3}{2}-\delta}\int_{t/2}^{t}\left|\partial_t\Gamma(t-s)g(s)\right|_N \mathrm{d}s &\le \int_{t/2}^{t}\langle t-s\rangle^{-1}\||D|^{-1}g(s)\|_{N+3}\langle s\rangle^{\frac{3}{2}-\delta}\mathrm{d}s\\
&\le C_N\log\langle\theta\rangle\|\langle t\rangle^{\frac{3}{2}-\delta}|D|^{-1}g\|_{L^\infty_t(H^{N+3})}\\
&\le C_N\log\langle\theta\rangle\left\||D|^{-1}g\right\|_{\frac{3}{2}-\delta,N+3}.
\end{aligned}
$$

As a result, it turs out that

$$
\begin{aligned}
|Y_t|_{\frac{3}{2}-\delta,N}\le C_N&\Big(\left(\||D|^{2\delta}g\|_{L^1_t(W^{N,1})}+\||D|^4g\|_{L^1_t(W^{N,1})}\right)\\
&+\log\langle\theta\rangle\left\||D|^{-1}g\right\|_{\frac{3}{2}-\delta,N+3}\Big).
\end{aligned}
\tag{3.23}
$$

•*Decay estimate of Y*.

As in the previous steps, we first split the integral (3.21) into two parts. For the integral from 0 to $t/2$, we use (3.12) to deduce that

$$
\begin{aligned}
&\langle t\rangle^{\frac{1}{2}}\int_0^{t/2}\left|\Gamma(t-s)g(s)\right|_N \mathrm{d}s\\
&\quad\le\langle t\rangle^{\frac{1}{2}}\int_0^{t/2}C_N\langle t-s\rangle^{-\frac{1}{2}}\left(\||D|^{\frac{1}{2}}g(s)\|_N+\||D|^2g(s)\|_N\right)\mathrm{d}s\\
&\quad\le C_N\left(\||D|^{\frac{1}{2}}g\|_{L^1_t(W^{N,1})}+\||D|^2g\|_{L^1_t(W^{N,1})}\right).
\end{aligned}
$$

For the integral from $t/2$ to t, we apply the first inequality of (3.18) to get

$$
\begin{aligned}
&\langle t\rangle^{\frac{1}{2}}\int_{t/2}^{t}\left|\Gamma(t-s)g(s)\right|_N \mathrm{d}s\\
&\quad\le C_N\langle t\rangle^{\frac{1}{2}}\int_{t/2}^{t}\||D|^{-1}g(s)\|_{N+2}\mathrm{d}s\\
&\quad\le C_N\langle t\rangle\left(\int_{t/2}^{t}\||D|^{-1}g(s)\|^2_{N+2}\mathrm{d}s\right)^{\frac{1}{2}}\le C_N\theta^{\frac{1}{2}}\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}g\|_{L^2_t(H^{N+2})}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
|Y|_{\frac{1}{2},N}\le C_N&\Big(\left(\||D|^{\frac{1}{2}}g\|_{L^1_t(W^{N,1})}+\||D|^2g\|_{L^1_t(W^{N,1})}\right)\\
&+\theta^{\frac{1}{2}}\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}g\|_{L^2_t(H^{N+2})}\Big).
\end{aligned}
\tag{3.24}
$$

By summarizing the estimates (3.22), (3.23) and (3.24), we complete the proof of (2.37). □

4. The Derivatives of f Given by (2.25)

4.1. Computation of $f'(Y; X)$

The goal of this subsection is to derive the linearized equations of the system (2.1-2.25). We first decompose the pressure function $\boldsymbol{p}$ given by (2.25) as $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ with

$$\boldsymbol{p}_1 \stackrel{\text{def}}{=} -\Delta^{-1}\text{div}\big((\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_1\big) + \Delta^{-1}\text{div}\Big(\mathcal{A}\text{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\Big) \tag{4.1}$$

$$\boldsymbol{p}_2 \stackrel{\text{def}}{=} -\Delta^{-1}\text{div}\big((\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_2\big) + \Delta^{-1}\text{div}\Big(\mathcal{A}\text{div}\big(\mathcal{A}(Y_t \otimes Y_t)\big)\Big). \tag{4.2}$$

Let us denote

$$f_0 \stackrel{\text{def}}{=} \nabla_y \cdot \big((\mathcal{A}\mathcal{A}^t - Id)\nabla_y Y_t\big), \quad f_1 \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y \boldsymbol{p}_1 \quad \text{and} \quad f_2 \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y \boldsymbol{p}_2. \tag{4.3}$$

Then the functional f given by (2.25) can be decomposed as $f_0 - f_1 + f_2$.

Before proceeding, let us recall that for a map $f : \mathcal{U} \to \mathcal{Y}$, where $\mathcal{U}$ is an open set of $\mathcal{X}$ and $\mathcal{X} \stackrel{\text{def}}{=} C^\infty([0, \infty[\times \mathbb{R}^3; \mathbb{R}^3)$, the differentiation of f at $Y \in \mathcal{U}$ along the direction $X \in \mathcal{X}$ is defined as

$$f'(Y; X) \stackrel{\text{def}}{=} \lim_{s\to 0} \frac{f(Y + sX) - f(Y)}{s} = \frac{d}{ds} f(Y + sX)|_{s=0}.$$

For $f \in C^\infty([0, +\infty) \times \mathbb{R}^3; M_{3\times 3}(\mathbb{R}))$, $g \in C^\infty([0, +\infty) \times \mathbb{R}^3; \mathbb{R}^3)$, we have

$$(fg)'(Y; X) = f'(Y; X)g(Y) + f(Y)g'(Y; X).$$

Then for $\mathcal{A}(Y) = (Id + \nabla Y)^{-1}$, we have

$$\mathcal{A}'(Y; X) = \mathcal{A}(-\nabla X)\mathcal{A}, \quad \text{and} \quad (\mathcal{A}^t)'(Y; X) = \mathcal{A}^t(-\nabla X)^t \mathcal{A}^t, \tag{4.4}$$

and thus

$$(\mathcal{A}\mathcal{A}^t - Id)'(Y; X) = \mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)^t \mathcal{A}^t. \tag{4.5}$$

As a result, we deduce that

$$\begin{aligned} f_0'(Y; X) &= \nabla \cdot \Big((\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\nabla Y_t\Big) + \nabla \cdot \big((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\big) \\ &= \nabla \cdot \Big(\big(\mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)^t\mathcal{A}^t\big)\nabla Y_t\Big) \\ &\quad + \nabla \cdot \big((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\big). \end{aligned} \tag{4.6}$$

For $m = 1, 2$, we have

$$\begin{aligned} f_m'(Y; X) &= (\mathcal{A}^t)'(Y; X)\nabla \boldsymbol{p}_m(Y) + \mathcal{A}^t\nabla \boldsymbol{p}_m'(Y; X) \\ &= -\mathcal{A}^t(\nabla X)^t\mathcal{A}^t(\nabla \boldsymbol{p}_m)(Y) + \mathcal{A}^t\nabla \boldsymbol{p}_m'(Y; X). \end{aligned} \tag{4.7}$$

Moreover, it follows from (4.1) that

$$\begin{aligned} \boldsymbol{p}_1'(Y;X) = \Delta^{-1}\mathrm{div}\Big(&- (\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_1'(Y;X) \\ &- \big(\mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)\mathcal{A}^t\big)\nabla \boldsymbol{p}_1(Y) \\ &+ \mathcal{A}\mathrm{div}\big((\mathcal{A}(-\nabla X)\mathcal{A})(\partial_3 Y \otimes \partial_3 Y)\big) \\ &+ (\mathcal{A}(-\nabla X)\mathcal{A})\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big) \\ &+ \mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\big)\Big). \end{aligned} \tag{4.8}$$

Similarly, it follows from (4.2) that

$$\begin{aligned} \boldsymbol{p}_2'(Y;X) = \Delta^{-1}\mathrm{div}\Big(&- (\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_2'(Y;X) - \big(\mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t \\ &+ \mathcal{A}\mathcal{A}^t(-\nabla X)\mathcal{A}^t\big)\nabla \boldsymbol{p}_2(Y) \\ &+ \mathcal{A}\mathrm{div}\big((\mathcal{A}(-\nabla X)\mathcal{A})(Y_t \otimes Y_t)\big) + (\mathcal{A}(-\nabla X)\mathcal{A})\mathrm{div}\big(\mathcal{A}(Y_t \otimes Y_t)\big) \\ &+ \mathcal{A}\mathrm{div}\big(\mathcal{A}(Y_t \otimes X_t + X_t \otimes Y_t)\big)\Big). \end{aligned} \tag{4.9}$$

The linearized equation of (2.1-2.25) then reads as (2.40).

Remark 4.1. Let $V \in C^\infty([0,+\infty) \times \mathbb{R}^3; M_{3\times 3}(\mathbb{R}))$ and $U \in C^\infty([0,+\infty) \times \mathbb{R}^3; \mathbb{R}^3)$, we denote $h(V) \stackrel{\text{def}}{=} (Id + V)^{-1}$, and

$$F_0(U,V) \stackrel{\text{def}}{=} \nabla \cdot \Big(\big(h(V)h(V)^t - Id\big)U\Big), \qquad F(U,V) \stackrel{\text{def}}{=} h(V)^t \boldsymbol{q}(U,V) \quad \text{with}$$

$$\boldsymbol{q} \stackrel{\text{def}}{=} -\Delta^{-1}\mathrm{div}\big((h(V)h(V)^t - Id)\nabla \boldsymbol{q}\big) + \Delta^{-1}\mathrm{div}\Big(h(V)\mathrm{div}\big(h(V)(U \otimes U)\big)\Big).$$

Then f_0, f_1, f_2 defined by (4.3) can be written as

$$f_0 = F_0(\nabla Y_t, \nabla Y), \quad f_1 = F(\partial_3 Y, \nabla Y) \quad \text{and} \quad f_2 = F(Y_t, \nabla Y),$$

and hence f_0', f_1' and f_2' read

$$\begin{aligned} f_0'(Y;X) &= F_{0,U}'(\nabla Y_t, \nabla Y)\nabla X_t + F_{0,V}'(\nabla Y_t, \nabla Y)\nabla X, \\ f_1'(Y;X) &= F_U'(\partial_3 Y, \nabla Y)\partial_3 X + F_V'(\partial_3 Y, \nabla Y)\nabla X, \\ f_2'(Y;X) &= F_U'(Y_t, \nabla Y)X_t + F_V'(Y_t, \nabla Y)\nabla X, \end{aligned}$$

where the functionals $F_{0,U}'(U,V)$, $F_{0,V}'(U,V)$, $F_U'(U,V)$ and $F_V'(U,V)$ are given by

$$\begin{aligned} F_{0,U}'(U,V)\dot{U} &= \nabla \cdot \Big(\big(h(V)h(V)^t - Id\big)\dot{U}\Big), \\ F_{0,V}'(U,V)\dot{V} &= \nabla \cdot \Big(\big((h'(V)\dot{V})h(V)^t + h(V)(h'(V)\dot{V})^t\big)U\Big), \\ F_U'(U,V)\dot{U} &= h(V)^t \boldsymbol{q}_U'(U,V)\dot{U} \quad \text{and} \quad F_V'(U,V)\dot{V} = \big(h'(V)\dot{V}\big)^t \boldsymbol{q}(U,V) \\ &\quad + h(V)^t \boldsymbol{q}_V'(U,V)\dot{V}, \end{aligned}$$

and

$$
\begin{aligned}
h'(V)\dot{V} &= (Id+V)^{-1}(-\dot{V})(Id+V)^{-1},\\
\boldsymbol{q}'_U(U,V)\dot{U} &= -\Delta^{-1}\mathrm{div}\Big((h(V)h(V)^t - Id)\nabla\boldsymbol{q}'_U(U,V)\dot{U}\\
&\qquad - h(V)\mathrm{div}\big(h(V)(U\otimes\dot{U}+\dot{U}\otimes U)\big)\Big)\\
\boldsymbol{q}'_V(U,V)\dot{V} &= -\Delta^{-1}\mathrm{div}\Big((h(V)h(V)^t - Id)\nabla\boldsymbol{q}'_V(U,V)\dot{V}\\
&\qquad - (h'(V)\dot{V})\mathrm{div}\big(h(V)(U\otimes U)\big)\\
&\qquad - h(V)\mathrm{div}\big((h'(V)\dot{V})(U\otimes U)\big) + \big((h'(V)\dot{V})h(V)^t\\
&\qquad + h(V)(h'(V)\dot{V})^t\big)\nabla\boldsymbol{q}\Big).
\end{aligned}
$$

4.2. *Computation of* $f''(Y; X, W)$

In order to estimate the error that has arisen in the Nash–Moser iteration scheme, we need the second derivatives of f. Towards this, let us recall the product rule

$$
\begin{aligned}
(fg)''(Y;X,W) &= f''(Y;X,W)g(Y) + f(Y)g''(Y;X,W)\\
&\quad + f'(Y;X)g'(Y;W) + f'(Y;W)g'(Y;X).
\end{aligned}
\tag{4.10}
$$

It is easy to observe from (4.4) that

$$
\mathcal{A}''(Y;X,W) = \mathcal{A}(\nabla X)\mathcal{A}(\nabla W)\mathcal{A} + \mathcal{A}(\nabla W)\mathcal{A}(\nabla X)\mathcal{A}. \tag{4.11}
$$

Then applying the product rule (4.10) as well as (4.4) gives

$$
\begin{aligned}
(\mathcal{A}\mathcal{A}^t - Id)''(Y;X,W) &= \mathcal{A}(\nabla X)\mathcal{A}(\nabla W)\mathcal{A}\mathcal{A}^t + \mathcal{A}(\nabla W)\mathcal{A}(\nabla X)\mathcal{A}\mathcal{A}^t\\
&\quad + \mathcal{A}\mathcal{A}^t(\nabla X)^t\mathcal{A}^t(\nabla W)^t\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(\nabla W)^t\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\\
&\quad + \mathcal{A}(\nabla X)\mathcal{A}\mathcal{A}^t(\nabla W)^t\mathcal{A}^t + \mathcal{A}(\nabla W)\mathcal{A}\mathcal{A}^t(\nabla X)^t\mathcal{A}^t.
\end{aligned}
\tag{4.12}
$$

Recall that f_0 is given by (4.3); we deduce from (4.10) that

$$
\begin{aligned}
f_0''(Y;X,W) &= \nabla\cdot\Big((\mathcal{A}\mathcal{A}^t - Id)''(Y;X,W)\nabla Y_t\Big)\\
&\quad + \nabla\cdot\big((\mathcal{A}\mathcal{A}^t - Id)'(Y;X)\nabla W_t\big)\\
&\quad + \nabla\cdot\big((\mathcal{A}\mathcal{A}^t - Id)'(Y;W)\nabla X_t\big).
\end{aligned}
\tag{4.13}
$$

Similarly, for $f_m(Y) = \mathcal{A}^t\nabla\boldsymbol{p}_m$, $m = 1, 2$, we have

$$
\begin{aligned}
f_m''(Y;X,W) &= (\mathcal{A}^t)''(Y;X,W)\nabla\boldsymbol{p}_m(Y) + \mathcal{A}^t\nabla\big(\boldsymbol{p}_m''(Y;X,W)\big)\\
&\quad + (\mathcal{A}^t)'(Y;X)\nabla\big(\boldsymbol{p}_m'(Y;W)\big) + (\mathcal{A}^t)'(Y;W)\nabla\big(\boldsymbol{p}_m'(Y;X)\big).
\end{aligned}
\tag{4.14}
$$

Then in view of (4.8) and (4.9), to obtain the expression of $f_m''(Y; X, W)$, $m = 1, 2$, it remains to calculate $\boldsymbol{p}_m''(Y; X, W)$, $m = 1, 2$. Indeed, it follows from (4.1), (4.2) and (4.10) that

$$\begin{aligned}
\boldsymbol{p}_1''(Y; X, W) = -\Delta^{-1}\mathrm{div}\Big(&(\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_1''(Y; X, W)\\
&+ (\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\nabla \boldsymbol{p}_1(Y)\big)\\
&+ (\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\nabla \boldsymbol{p}_1'(Y; W)\\
&+ (\mathcal{A}\mathcal{A}^t - Id)'(Y; W)\nabla \boldsymbol{p}_1'(Y; X)\\
&- \mathcal{A}\mathrm{div}\big[\mathcal{A}(\partial_3 X \otimes \partial_3 W + \partial_3 W \otimes \partial_3 X)\\
&+ \mathcal{A}''(Y; X, W)(\partial_3 Y \otimes \partial_3 Y)\\
&+ \mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y)\big)\\
&+ \mathcal{A}'(Y; W)(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\big]\\
&- \mathcal{A}'(Y; X)\mathrm{div}\big[\mathcal{A}'(Y; W)(\partial_3 Y \otimes \partial_3 Y)\\
&+ \mathcal{A}(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y)\big]\\
&- \mathcal{A}'(Y; W)\mathrm{div}\big[\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y)\\
&+ \mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\big]\\
&- \mathcal{A}''(Y; X, W)\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\Big)
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\boldsymbol{p}_2''(Y; X, W) = -\Delta^{-1}\mathrm{div}\Big(&(\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_2''(Y; X, W)\\
&+ (\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\nabla \boldsymbol{p}_2(Y)\\
&+ (\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\nabla \boldsymbol{p}_2'(Y; W)\\
&+ (\mathcal{A}\mathcal{A}^t - Id)'(Y; W)\nabla \boldsymbol{p}_2'(Y; X)\\
&- \mathcal{A}\mathrm{div}\big[\mathcal{A}(X_t \otimes W_t + W_t \otimes X_t) + \mathcal{A}''(Y; X, W)(Y_t \otimes Y_t)\\
&+ \mathcal{A}'(Y; X)(Y_t \otimes W_t + W_t \otimes Y_t)\\
&+ \mathcal{A}'(Y; W)(Y_t \otimes X_t + X_t \otimes Y_t)\big]\\
&- \mathcal{A}'(Y; X)\mathrm{div}\big[\mathcal{A}(Y_t \otimes W_t + W_t \otimes Y_t) + \mathcal{A}'(Y; W)(Y_t \otimes Y_t)\big]\\
&- \mathcal{A}'(Y; W)\mathrm{div}\big[\mathcal{A}(Y_t \otimes X_t + X_t \otimes Y_t) + \mathcal{A}'(Y; X)(Y_t \otimes Y_t)\big]\\
&- \mathcal{A}''(Y; X, W)\mathrm{div}\big(\mathcal{A}(Y_t \otimes Y_t)\big)\Big).
\end{aligned} \tag{4.16}$$

Remark 4.2. In view of Remark 4.1, f_m'' can be written as

$$\begin{aligned}
f_0''(Y; X, W) &= F_{0,UU}''(\nabla Y_t, \nabla Y)\nabla X_t \cdot \nabla W_t + F_{0,UV}''(\nabla Y_t, \nabla Y)\nabla X_t \cdot \nabla W\\
&\quad + F_{0,VU}''(\nabla Y_t, \nabla Y)\nabla W_t \cdot \nabla X + F_{0,VV}''(\nabla Y_t, \nabla Y)\nabla X \cdot \nabla W,\\
f_1''(Y; X, W) &= F_{UU}''(\partial_3 Y, \nabla Y)\partial_3 X \cdot \partial_3 W + F_{UV}''(\partial_3 Y, \nabla Y)\partial_3 X \cdot \nabla W\\
&\quad + F_{VU}''(\partial_3 Y, \nabla Y)\nabla X \cdot \partial_3 W + F_{VV}''(\partial_3 Y, \nabla Y)\nabla X \cdot \nabla W,\\
f_2''(Y; X, W) &= F_{UU}''(Y_t, \nabla Y)X_t \cdot W_t + F_{UV}''(Y_t, \nabla Y)X_t \cdot \nabla W\\
&\quad + F_{VU}''(Y_t, \nabla Y)\nabla X \cdot W_t + F_{VV}''(Y_t, \nabla Y)\nabla X \cdot \nabla W,
\end{aligned}$$

where $F''_{0,UU}(U,V)\dot U_1\cdot\dot U_2=0$, and

$$\begin{aligned}
F''_{0,UV}(U,V)\dot U\cdot\dot V &= \nabla\cdot\Big(\big((h'(V)\dot V)h(V)^t+h(V)(h'(V)\dot V)^t\big)\dot U\Big)\\
&= F''_{0,VU}(U,V)\dot V\cdot\dot U,\\
F''_{0,VV}(U,V)\dot V_1\cdot\dot V_2 &= \nabla\cdot\Big(\big((h''(V)\dot V_1\cdot\dot V_2)h(V)^t+h(V)(h''(V)\dot V_1\cdot\dot V_2)^t+\\
&\quad+\big((h'(V)\dot V_1)(h'(V)\dot V_2)^t+(h'(V)\dot V_2)(h'(V)\dot V_1)^t\big)U\Big) \quad \text{with}\\
h''(V)\dot V_1\cdot\dot V_2 &= (Id+V)^{-1}(-\dot V_1)(Id+V)^{-1}(-\dot V_2)(Id+V)^{-1}\\
&\quad+(Id+V)^{-1}(-\dot V_2)(Id+V)^{-1}(-\dot V_1)(Id+V)^{-1};
\end{aligned}$$

and

$$\begin{aligned}
F''_{UU}(U,V)\dot U_1\cdot\dot U_2 &= h(V)^t\boldsymbol{q}''_{UU}(U,V)\dot U_1\cdot\dot U_2,\\
F''_{UV}(U,V)\dot U\cdot\dot V &= h(V)^t\boldsymbol{q}''_{UV}(U,V)\dot U\cdot\dot V+(h'(V)\dot V)^t\boldsymbol{q}'_U(U,V)\dot U\\
&= F''_{VU}(U,V)\dot V\cdot\dot U,\\
F''_{VV}(U,V)\dot V_1\cdot\dot V_2 &= \big(h''(V)\dot V_1\cdot\dot V_2\big)^t\boldsymbol{q}(U,V)+h(V)^t\boldsymbol{q}''_{VV}(U,V)\dot V_1\cdot\dot V_2\\
&\quad+(h'(V)\dot V_1)^t\boldsymbol{q}'_V(U,V)\dot V_2+(h'(V)\dot V_2)^t\boldsymbol{q}'_V(U,V)\dot V_1,
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{q}''_{UU}(U,V)\dot U_1\cdot\dot U_2 &= -\Delta^{-1}\mathrm{div}\Big((h(V)h(V)^t-Id)\nabla\boldsymbol{q}''_{UU}(U,V)\dot U_1\cdot\dot U_2\\
&\quad-h(V)\mathrm{div}\big(h(V)(\dot U_1\otimes\dot U_2+\dot U_2\otimes\dot U_1)\big)\Big),\\
\boldsymbol{q}''_{UV}(U,V)\dot U\cdot\dot V &= -\Delta^{-1}\mathrm{div}\Big((h(V)h(V)^t-Id)\nabla\boldsymbol{q}''_{UV}(U,V)\dot U\cdot\dot V\\
&\quad+\big((h'(V)\dot V)h(V)^t+h(V)(h'(V)\dot V)^t\big)\nabla\boldsymbol{q}'_U(U,V)\dot U\\
&\quad-(h'(V)\dot V)\mathrm{div}\big(h(V)(\dot U\otimes U+U\otimes\dot U)\big)\\
&\quad-h(V)\mathrm{div}\big((h'(V)\dot V)(\dot U\otimes U+U\otimes\dot U)\big)\Big)\\
&= \boldsymbol{q}''_{VU}(U,V)\dot V\cdot\dot U,
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{q}''_{VV}(U,V)\dot V_1\cdot\dot V_2 = &-\Delta^{-1}\mathrm{div}\Big(\big((h'(V)\dot V_2)h(V)^t\\
&+h(V)(h'(V)\dot V_2)^t\big)\nabla\boldsymbol{q}'_V(U,V)\dot V_1\\
&+(h(V)h(V)^t-Id)\nabla\boldsymbol{q}''_{VV}(U,V)\dot V_1\cdot\dot V_2\\
&-(h''(V)\dot V_1\cdot\dot V_2)\mathrm{div}\big(h(V)(U\otimes U)\big)\\
&+\big((h'(V)\dot V_1)h(V)^t+h(V)(h'(V)\dot V_1)^t\big)\nabla\boldsymbol{q}'_V(U,V)\dot V_2\\
&-(h'(V)\dot V_1)\mathrm{div}\big((h'(V)\dot V_2)(U\otimes U)\big)\\
&+\big((h''(V)\dot V_1\cdot\dot V_2)h(V)^t+h(V)(h''(V)\dot V_1\cdot\dot V_2)^t\\
&-(h'(V)\dot V_2)\mathrm{div}\big((h'(V)\dot V_1)(U\otimes U)\big)
\end{aligned}$$

$$+ (h'(V)\dot{V}_1)(h'(V)\dot{V}_2)^t + (h'(V)\dot{V}_2)(h'(V)\dot{V}_1)^t\Big)\nabla \boldsymbol{q}$$
$$- h(V)\mathrm{div}\Big((h''(V)\dot{V}_1 \cdot \dot{V}_2)(U \otimes U)\Big)\Big).$$

5. The Estimates of $f'(Y; X)$

5.1. The Estimate of $\|\|f'(Y; X)\|\|_{\delta,N}$

The main result of this subsection is listed in Proposition 2.3. As we explained in Sect. 2, the main idea is to use the norm of the homogeneous Besov spaces $\dot{B}^s_{1,1}$ to replace the norm of the classical Sobolev spaces $\dot{W}^{s,1}$. In order to do so, we need not only the product law (2.48), but also the following one:

Lemma 5.1. *For any* $s > 0$, *there holds*

$$\|ab\|_{\dot{B}^s_{1,1}} \leq C\left(\min\big(|a|_0\|b\|_{\dot{B}^s_{1,1}}, \|a\|_0\|b\|_{\dot{B}^s_{2,1}}\big) + \|a\|_{\dot{B}^s_{2,1}}\|b\|_0\right). \tag{5.1}$$

Proof. We first get, by applying Bony's decomposition [4], that

$$ab = T_a b + R'(a, b) \quad \text{with}$$
$$T_a b = \sum_{j\in\mathbb{Z}} S_{j-1} a \Delta_j b \quad \text{and} \quad R'(a, b) = \sum_{j\in\mathbb{Z}} \Delta_j a S_{j+2} b.$$

Due to the support properties to the Fourier transform of the terms in $T_a b$, we have

$$\|\dot{\Delta}_j(T_a b)\|_{L^1} \leq \sum_{|j'-j|\leq 4} |S_{j'-1}a|_0 \|\dot{\Delta}_{j'} b\|_{L^1} \lesssim d_j 2^{-js}|a|_0\|b\|_{\dot{B}^s_{1,1}},$$

where $(d_j)_{j\in\mathbb{Z}}$ is a non-negative generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}} d_j = 1$.

Along the same lines, we also have

$$\|\dot{\Delta}_j(T_a b)\|_{L^1} \leq \sum_{|j'-j|\leq 4} \|S_{j'-1}a\|_0 \|\dot{\Delta}_{j'} b\|_0 \lesssim d_j 2^{-js}\|a\|_0\|b\|_{\dot{B}^s_{2,1}},$$

and

$$\begin{aligned}\|\dot{\Delta}_j(R'(a, b))\|_{L^1} &\leq \sum_{j'\geq j-N_0} \|\dot{\Delta}_{j'} a\|_0 \|S_{j'+2} b\|_0 \\ &\leq \sum_{j'\geq j-N_0} d_{j'} 2^{-j's}\|a\|_{\dot{B}^s_{2,1}}\|b\|_0 \lesssim d_j 2^{-js}\|a\|_{\dot{B}^s_{2,1}}\|b\|_0,\end{aligned}$$

where in the last step, we used the fact that $s > 0$. By summing up the above inequalities, we arrive at (5.1). □

Notice that $\mathcal{A}(\nabla Y) = (Id + \nabla Y)^{-1}$, so we write

$$\mathcal{A}\mathcal{A}^t - Id = (\mathcal{A} - Id)(\mathcal{A} - Id)^t + (\mathcal{A} - Id) + (\mathcal{A} - Id)^t,$$
$$\mathcal{A} - Id = \sum_{n=1}^{\infty}(-1)^n(\nabla Y)^n.$$

Thus, under the assumption of (2.41), for $s > 0$, we get, by applying (2.48), that

$$\begin{aligned}\|\mathcal{A}f\|_{\dot{B}^s_{p,r}} &\lesssim (1 + |\mathcal{A} - Id|_0)\|f\|_{\dot{B}^s_{p,r}} + \|\mathcal{A} - Id\|_{\dot{B}^s_{p,r}}|f|_0 \\ &\lesssim \|f\|_{\dot{B}^s_{p,r}} + \|\nabla Y\|_{\dot{B}^s_{p,r}}|f|_0.\end{aligned} \tag{5.2}$$

Along the same lines, we get, by applying (5.1), that

$$\|\mathcal{A}f\|_{\dot{B}^s_{1,1}} \lesssim \|f\|_{\dot{B}^s_{1,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}\|f\|_0. \tag{5.3}$$

5.1.1. Estimate of $\|f_0'(Y; X)\|_{\dot{B}^s_{1,1}}$ In view of (4.6), we have

$$\|f_0'(Y; X)\|_{\dot{B}^s_{1,1}} \le \big\|\mathcal{A}\big(\nabla X\mathcal{A} + \mathcal{A}^t(\nabla X)^t\big)\mathcal{A}^t\nabla Y_t\big\|_{\dot{B}^{s+1}_{1,1}} + \|(\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\|_{\dot{B}^{s+1}_{1,1}}.$$

It follows from (2.41) and (5.1) that

$$\|(\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\|_{\dot{B}^{s+1}_{1,1}} \lesssim \|\nabla Y\|_0\|\nabla X_t\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\|\nabla X_t\|_0.$$

While applying (5.3) gives

$$\|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_{\dot{B}^{s+1}_{1,1}} \lesssim \|\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_{\dot{B}^{s+1}_{1,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\|\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_0,$$

it follows from (2.48) and (5.1) that

$$\begin{aligned}\|\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_{\dot{B}^{s+1}_{1,1}} &\lesssim \|\nabla X\|_0\|\mathcal{A}\mathcal{A}^t\nabla Y_t\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla X\|_{\dot{B}^{s+1}_{2,1}}\|\mathcal{A}\mathcal{A}^t\nabla Y_t\|_0 \\ &\lesssim \|\nabla X\|_0\big(\|\nabla Y_t\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}|\nabla Y_t|_0\big) \\ &\quad + \|\nabla X\|_{\dot{B}^{s+1}_{2,1}}\|\nabla Y_t\|_0,\end{aligned}$$

so that it holds that

$$\begin{aligned}\|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_{\dot{B}^{s+1}_{1,1}} &\lesssim \|\nabla Y_t\|_0\|\nabla X\|_{\dot{B}^{s+1}_{2,1}} \\ &\quad + \big(\|\nabla Y_t\|_{\dot{B}^{s+1}_{2,1}} + |\nabla Y_t|_0\|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\big)\|\nabla X\|_0.\end{aligned}$$

The same estimate holds for $\|\mathcal{A}\mathcal{A}^t(-\nabla X)^t\mathcal{A}^t\nabla Y_t\|_{\dot{B}^{s+1}_{1,1}}$. As a result, we obtain

$$\begin{aligned}\|f_0'(Y; X)\|_{\dot{B}^s_{1,1}} \le{}& \|\nabla Y\|_0\|\nabla X_t\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\|\nabla X_t\|_0 \\ &+ \|\nabla Y_t\|_0\|\nabla X\|_{\dot{B}^{s+1}_{2,1}} + \big(\|\nabla Y_t\|_{\dot{B}^{s+1}_{2,1}} \\ &+ \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}|\nabla Y_t|_0\big)\|\nabla X\|_0.\end{aligned} \tag{5.4}$$

5.1.2. Estimate of $\|f_m'(Y;X)\|_{\dot{B}^s_{1,1}}$, $m=1,2$ In view of (4.7), we have

$$\|f_m'(Y;X)\|_{\dot{B}^s_{1,1}} \le \|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} + \|\mathcal{A}^t\nabla(\boldsymbol{p}_m'(Y;X))\|_{\dot{B}^s_{1,1}}.$$

Applying (5.3) gives

$$\|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} \lesssim \|(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}\|(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_0,$$
$$\|\mathcal{A}^t\nabla(\boldsymbol{p}_m'(Y;X))\|_{\dot{B}^s_{1,1}} \lesssim \|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_{\dot{B}^s_{1,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}\|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_0.$$

Applying (2.48) and (5.1) leads to

$$\|(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} \lesssim \|\nabla X\|_0(\|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{2,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}|\nabla \boldsymbol{p}_m|_0) + \|\nabla X\|_{\dot{B}^s_{2,1}}\|\nabla \boldsymbol{p}_m\|_0,$$

which yields

$$\|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} \lesssim \|\nabla \boldsymbol{p}_m\|_0\|\nabla X\|_{\dot{B}^s_{2,1}} + \big(\|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{2,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}|\nabla \boldsymbol{p}_m|_0\big)\|\nabla X\|_0.$$

Hence we have

$$\|f_m'(Y;X)\|_{\dot{B}^s_{1,1}} \lesssim \|\nabla \boldsymbol{p}_m\|_0\|\nabla X\|_{\dot{B}^s_{2,1}} + \big(\|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{2,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}|\nabla \boldsymbol{p}_m|_0\big)\|\nabla X\|_0 + \|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_{\dot{B}^s_{1,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}\|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_0. \tag{5.5}$$

It remains to handle the estimates of

$$\|\nabla \boldsymbol{p}_m\|_0, \quad \|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{2,1}}, \quad \|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_{\dot{B}^s_{1,1}} \quad \text{and} \quad \|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_0.$$

•Estimate of $\|\nabla \boldsymbol{p}_m\|_0$.

We first deduce from (4.1) that

$$\begin{aligned}\|\nabla \boldsymbol{p}_1\|_0 &\le |\mathcal{A}\mathcal{A}^t - Id|_0\|\nabla \boldsymbol{p}_1\|_0 + |\mathcal{A}|_0\|\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\|_{\dot{H}^1}\\ &\le |\mathcal{A}\mathcal{A}^t - Id|_0\|\nabla \boldsymbol{p}_1\|_0 + |\mathcal{A}|_0\left(1 + \|\mathcal{A} - Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\right)\|\partial_3 Y\otimes\partial_3 Y\|_{\dot{H}^1}.\end{aligned}$$

Due to the assumption (2.41), one has

$$|\mathcal{A}\mathcal{A}^t - Id|_0 \lesssim |\nabla Y|_0 \lesssim \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \le \delta_1,$$

so we infer

$$\|\nabla \boldsymbol{p}_1\|_0 \lesssim \|\partial_3 Y\otimes\partial_3 Y\|_{\dot{H}^1} \lesssim |\partial_3 Y|_0\|\partial_3 Y\|_1. \tag{5.6}$$

Similarly, we have

$$\|\nabla \boldsymbol{p}_2\|_0 \lesssim |Y_t|_0\|Y_t\|_1. \tag{5.7}$$

•Estimates of $\|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{2,1}}$ for $s > 0$.

We start with the estimate of $\|\nabla \boldsymbol{p}_m\|_{\dot{B}^{\frac{3}{2}}_{2,1}}$. Indeed by (4.1), one has

$$\|\nabla \boldsymbol{p}_1\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\nabla \boldsymbol{p}_1\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_{\dot{B}^{\frac{3}{2}}_{2,1}},$$

from which (2.41) and the product law (2.48) infer

$$\begin{aligned}\|\nabla \boldsymbol{p}_1\|_{\dot{B}^{\frac{3}{2}}_{2,1}} &\lesssim \big(1 + \|\mathcal{A} - Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\big)\|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \\ &\lesssim (1 + |\mathcal{A} - Id|_0)\|\partial_3 Y \otimes \partial_3 Y\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\mathcal{A} - Id\|_{\dot{B}^{\frac{5}{2}}_{2,1}} |\partial_3 Y \otimes \partial_3 Y|_0.\end{aligned}$$

As a result, by virtue of (2.41), it transpires that

$$\|\nabla \boldsymbol{p}_1\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim |\partial_3 Y|_0\big(\|\partial_3 Y\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\nabla Y\|_{\dot{B}^{\frac{5}{2}}_{2,1}} |\partial_3 Y|_0\big) \lesssim |\partial_3 Y|_0\|\partial_3 Y\|_3. \tag{5.8}$$

In general, for $s > 0$, we deduce from (4.1) that

$$\begin{aligned}\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{2,1}} \lesssim |\mathcal{A}\mathcal{A}^t - Id|_0\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{2,1}} + \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}^s_{2,1}} |\nabla \boldsymbol{p}_1|_0 \\ + \|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_{\dot{B}^s_{2,1}},\end{aligned}$$

from which, with (2.41), we infer

$$\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{2,1}} \lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}^s_{2,1}} |\nabla \boldsymbol{p}_1|_0 + \|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_{\dot{B}^s_{2,1}}.$$

It follows however from the product law (5.2) that

$$\begin{aligned}\|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_{\dot{B}^s_{2,1}} &\lesssim \|\partial_3 Y \otimes \partial_3 Y\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}} |\partial_3 Y \otimes \partial_3 Y|_0 \\ &\quad + \|\nabla Y\|_{\dot{B}^s_{2,1}}\big(|\partial_3 Y|_1|\partial_3 Y|_0 + |\nabla Y|_1|\partial_3 Y|_0^2\big),\end{aligned}$$

which together with (2.41) and (5.8) ensures that

$$\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{2,1}} \lesssim |\partial_3 Y|_0 \left(\|\partial_3 Y\|_{\dot{B}^{s+1}_{2,1}} + \big(\|\nabla Y\|_{\dot{B}^s_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\big)\|\partial_3 Y\|_3\right). \tag{5.9}$$

Along exactly the same lines, we have

$$\|\nabla \boldsymbol{p}_2\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim |Y_t|_0\|Y_t\|_3 \quad \text{and} \tag{5.10}$$

$$\|\nabla \boldsymbol{p}_2\|_{\dot{B}^s_{2,1}} \lesssim |Y_t|_0 \left(\|Y_t\|_{\dot{B}^{s+1}_{2,1}} + \big(\|\nabla Y\|_{\dot{B}^s_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\big)\|Y_t\|_3\right). \tag{5.11}$$

•Estimate of $\|\nabla \boldsymbol{p}'_m(Y; X)\|_0$.

We first deduce from (4.8) that

$$\begin{aligned}\|\nabla \boldsymbol{p}'_1(Y; X)\|_0 \lesssim \delta_1\|\nabla p'_1(Y; X)\|_0 &+ \left\|\mathcal{A}\left(\nabla X\mathcal{A} + \mathcal{A}^t\nabla X\right)\mathcal{A}^t\nabla \boldsymbol{p}_1\right\|_0 \\ &+ \|\mathcal{A}\mathrm{div}\big(\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_0 \\ &+ \|\mathcal{A}\nabla X\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_0 \\ &+ \|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\big)\|_0.\end{aligned} \tag{5.12}$$

We observe that

$$\|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla \boldsymbol{p}_1\|_0 \lesssim \|\nabla X\|_{L^6}\|\nabla \boldsymbol{p}_1\|_{L^3},$$

yet it follows by a similar derivation of (5.6) that

$$\|\nabla \boldsymbol{p}_1\|_{L^3} \lesssim \|\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\|_{W^{1,3}} \lesssim |\partial_3 Y|_1\|\partial_3 Y\|_{L^3} \lesssim |\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}, \tag{5.13}$$

so that

$$\|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla \boldsymbol{p}_1\|_0 \le \|\nabla X\|_{L^6}\|\nabla \boldsymbol{p}_1\|_{L^3} \lesssim |\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}\|\nabla X\|_1.$$

Let us handle the remaining terms in (5.12). Indeed with the assumption (2.41), a direct calculation shows that

$$\begin{aligned}
&\|\mathcal{A}\mathrm{div}\big(\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\|_0 \lesssim \|\mathcal{A}\nabla X\|_1|\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)|_1 \lesssim |\partial_3 Y|_1^2\|\nabla X\|_1,\\
&\|\mathcal{A}\nabla X\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\|_0 \lesssim \|\nabla X\|_0|\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)|_1 \lesssim |\partial_3 Y|_1^2\|\nabla X\|_0,\\
&\|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 X+\partial_3 X\otimes\partial_3 Y)\big)\|_0 \lesssim \|\partial_3 Y\otimes\partial_3 X\|_1 \lesssim |\partial_3 Y|_1\|\partial_3 X\|_1.
\end{aligned}$$

Substituting the above estimates into (5.12) leads to

$$\|\nabla \boldsymbol{p}_1'(Y;X)\|_0 \lesssim \big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2\big)\|\nabla X\|_1 + |\partial_3 Y|_1\|\partial_3 X\|_1. \tag{5.14}$$

The same procedure gives rise to

$$\|\nabla \boldsymbol{p}_2\|_{L^3} \lesssim |Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}; \quad \text{and} \tag{5.15}$$

$$\|\nabla \boldsymbol{p}_2'(Y;X)\|_0 \lesssim \big(|Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}} + |Y_t|_1^2\big)\|\nabla X\|_1 + |Y_t|_1\|X_t\|_1. \tag{5.16}$$

•Estimate of $\|\nabla \boldsymbol{p}_m'(Y;X)\|_{\dot B^s_{1,1}}$ with $s>0$.

For any $s>0$, we deduce from (4.8) that

$$\begin{aligned}
&\|\nabla \boldsymbol{p}_1'(Y;X)\|_{\dot B^s_{1,1}}\\
&\quad\lesssim \big\|(\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_1'(Y;X)\big\|_{\dot B^s_{1,1}} + \big\|\mathcal{A}\big(\nabla X\mathcal{A}+\mathcal{A}^t\nabla X\big)\mathcal{A}^t\nabla \boldsymbol{p}_1\big\|_{\dot B^s_{1,1}}\\
&\qquad + \big\|\mathcal{A}\mathrm{div}\big(\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\big\|_{\dot B^s_{1,1}} + \big\|\mathcal{A}\nabla X\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\big\|_{\dot B^s_{1,1}}\\
&\qquad + \big\|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 X+\partial_3 X\otimes\partial_3 Y)\big)\big\|_{\dot B^s_{1,1}}.
\end{aligned} \tag{5.17}$$

It follows from (5.1) that

$$\begin{aligned}
&\big\|(\mathcal{A}\mathcal{A}^t - Id)\nabla \boldsymbol{p}_1'(Y;X)\big\|_{\dot B^s_{1,1}}\\
&\quad\lesssim \delta_1\|\nabla \boldsymbol{p}_1'(Y;X)\|_{\dot B^s_{1,1}} + \|\nabla Y\|_{\dot B^s_{2,1}}\|\nabla \boldsymbol{p}_1'(Y;X)\|_0.
\end{aligned}$$

Applying (5.2) and (5.1) gives

$$\begin{aligned}
&\big\|\mathcal{A}\big(\nabla X\mathcal{A}+\mathcal{A}^t(\nabla X)^t\big)\mathcal{A}^t\nabla \boldsymbol{p}_1\big\|_{\dot B^s_{1,1}}\\
&\quad\lesssim \|\nabla \boldsymbol{p}_1\|_0\|\nabla X\|_{\dot B^s_{2,1}} + \big(\|\nabla \boldsymbol{p}_1\|_{\dot B^s_{2,1}} + \|\nabla Y\|_{\dot B^s_{2,1}}|\nabla \boldsymbol{p}_1|_0\big)\|\nabla X\|_0,
\end{aligned}$$

and

$$
\begin{aligned}
&\big\|\mathcal{A}\mathrm{div}\big(\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\big\|_{\dot B^s_{1,1}}\\
&\quad\lesssim \|\nabla X\|_{\dot B^{s+1}_{2,1}}\|\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\|_0+\|\nabla X\|_0\|\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\|_{\dot B^{s+1}_{2,1}}\\
&\qquad+\|\nabla Y\|_{\dot B^{s+1}_{2,1}}\|\nabla X\|_0|\partial_3 Y|_0^2+\|\nabla Y\|_{\dot B^s_{2,1}}\|\nabla X\|_1|\partial_3 Y|_1|\partial_3 Y|_0\\
&\quad\lesssim |\partial_3 Y|_0\|\partial_3 Y\|_0\|\nabla X\|_{\dot B^{s+1}_{2,1}}\\
&\qquad+|\partial_3 Y|_0\Big(\|\partial_3 Y\|_{\dot B^{s+1}_{2,1}}+(\|\nabla Y\|_{\dot B^{s+1}_{2,1}}+\|\nabla Y\|_{\dot B^s_{2,1}})|\partial_3 Y|_1\Big)\|\nabla X\|_1.
\end{aligned}
$$

Exactly along the same lines, we find that

$$
\begin{aligned}
\big\|\mathcal{A}\nabla X\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\big\|_{\dot B^s_{1,1}}&\lesssim |\partial_3 Y|_0\|\partial_3 Y\|_{\dot H^1}\|\nabla X\|_{\dot B^s_{2,1}}\\
&+|\partial_3 Y|_0\Big(\|\partial_3 Y\|_{\dot B^{s+1}_{2,1}}+(\|\nabla Y\|_{\dot B^{s+1}_{2,1}}+\|\nabla Y\|_{\dot B^s_{2,1}})|\partial_3 Y|_1\Big)\|\nabla X\|_0
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 X+\partial_3 X\otimes\partial_3 Y)\big)\big\|_{\dot B^s_{1,1}}\\
&\quad\lesssim \|\partial_3 Y\|_0\|\partial_3 X\|_{\dot B^{s+1}_{2,1}}+\|\partial_3 Y\|_{\dot B^{s+1}_{2,1}}\|\partial_3 X\|_0\\
&\qquad+\|\nabla Y\|_{\dot B^{s+1}_{2,1}}|\partial_3 Y|_0\|\partial_3 X\|_0+\|\nabla Y\|_{\dot B^s_{2,1}}|\partial_3 Y|_1\|\partial_3 X\|_1.
\end{aligned}
$$

Substituting the above estimates into (5.17) and using the estimates (5.6), (5.8), (5.9) and (5.14), we obtain

$$
\begin{aligned}
&\|\nabla \boldsymbol{p}_1'(Y;X)\|_{\dot B^s_{1,1}}\lesssim \mathfrak{g}_1(\partial_3 Y,\partial_3 X)\quad\text{with}\\
&\qquad\mathfrak{g}_1(\mathfrak{x},\mathfrak{y})\overset{\text{def}}{=}\|\mathfrak{x}\|_0\|\mathfrak{y}\|_{\dot B^{s+1}_{2,1}}+|\mathfrak{x}|_0\big(\|\mathfrak{x}\|_0\|\nabla X\|_{\dot B^{s+1}_{2,1}}+\|\mathfrak{x}\|_1\|\nabla X\|_{\dot B^s_{2,1}}\big)\\
&\qquad\qquad+\big(\|\mathfrak{x}\|_{\dot B^{s+1}_{2,1}}+(\|\nabla Y\|_{\dot B^{s+1}_{2,1}}+\|\nabla Y\|_{\dot B^s_{2,1}})|\mathfrak{x}|_1\big)\|\mathfrak{y}\|_1\\
&\qquad\qquad+|\mathfrak{x}|_1\big(\|\mathfrak{x}\|_{\dot B^{s+1}_{2,1}}+(\|\nabla Y\|_{\dot B^{s+1}_{2,1}}+\|\nabla Y\|_{\dot B^s_{2,1}})\|\mathfrak{x}\|_3\big)\|\nabla X\|_1.
\end{aligned}\tag{5.18}
$$

The same procedure gives rise to

$$
\|\nabla \boldsymbol{p}_2'(Y;X)\|_{\dot B^s_{1,1}}\lesssim\mathfrak{g}_1(Y_t,X_t).\tag{5.19}
$$

Inserting the estimates (5.6), (5.8), (5.9), (5.14) and (5.18) into (5.5) for $m=1$ yields

$$
\|f_1'(Y;X)\big)\|_{\dot B^s_{1,1}}\lesssim\mathfrak{g}_1(\partial_3 Y,\partial_3 X).\tag{5.20}
$$

By inserting the estimates (5.7), (5.10), (5.11), (5.15), (5.16) and (5.19) into (5.5) for $m=2$, we obtain

$$
\|f_2'(Y;X)\big)\|_{\dot B^s_{1,1}}\lesssim\mathfrak{g}_1(Y_t,X_t).\tag{5.21}
$$

Let us now complete the proof of Proposition 2.3.

Proof of Proposition 2.3. Note that for $s_1 < s < s_2$ and $\alpha = \frac{s_2 - s}{s_2 - s_1}$, one has

$$\|f\|_{\dot B^s_{2,1}} \le C\left(\frac{1}{s-s_1}+\frac{1}{s_2-s}\right)\|f\|^{\alpha}_{\dot H^{s_1}}\|f\|^{1-\alpha}_{\dot H^{s_2}}.$$

In particular, for $s > 0$, this yields

$$\|f\|_{\dot B^s_{2,1}} \le C\big(\|f\|_0 + \|f\|_{\dot H^{[s]+1}}\big) \le C\|f\|_{[s]+1}. \tag{5.22}$$

On the other hand, recalling (2.39), we deduce from (5.4) that

$$\begin{aligned}\|\!|\!| f_0'(Y;X)|\!|\!|_{\delta,N} &\lesssim \|\nabla Y\|_0\big(\|\nabla X_t\|_{\dot B^{2\delta+1}_{2,1}} + \|\nabla X_t\|_{\dot B^{N+5}_{2,1}}\big)\\ &+ \big(\|\nabla Y\|_{\dot B^{2\delta+1}_{2,1}} + |\nabla Y\|_{\dot B^{N+5}_{2,1}}\big)\|\nabla X_t\|_0 + \|\nabla Y_t\|_0\big(\|\nabla X\|_{\dot B^{2\delta+1}_{2,1}} + \|\nabla X\|_{\dot B^{N+5}_{2,1}}\big)\\ &+ \big(\|\nabla Y_t\|_{\dot B^{2\delta+1}_{2,1}} + \|\nabla Y_t\|_{\dot B^{N+5}_{2,1}} + (\|\nabla Y\|_{\dot B^{2\delta+1}_{2,1}} + \|\nabla Y\|_{\dot B^{N+5}_{2,1}})|\nabla Y_t|_0\big)\|\nabla X\|_0,\end{aligned}$$

which together with (5.22) ensures (2.42). Along the same lines, we deduce (2.43) and (2.44) from (5.20) and (5.21), respectively. This completes the proof of Proposition 2.3. □

5.2. *The Estimate of* $\|\,|D|^{-1}f'(Y;X)\|_N$

The purpose of this subsection is to prove Proposition 2.4. We split its proof into the following steps:

5.2.1. The Estimate of $\|\,|D|^{-1}f_0'(Y;X)\|_N$ We first deduce from (4.6) that

$$\|\,|D|^{-1}f_0'(Y;X)\|_N \lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\|_N + \|\mathcal{A}\big(\nabla X\mathcal{A} + \mathcal{A}^t(\nabla X)^t\big)\mathcal{A}^t\nabla Y_t\|_N. \tag{5.23}$$

Applying Moser-type inequality and using (2.41) gives

$$\begin{aligned}&\|(\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\|_N \lesssim |\nabla Y|_0\|\nabla X_t\|_N + |\nabla Y|_N\|\nabla X_t\|_0,\\ &\|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_N \lesssim |\nabla Y_t|_0\|\nabla X\|_N + \big(|\nabla Y_t|_N + |\nabla Y_t|_0|\nabla Y|_N\big)\|\nabla X\|_0.\end{aligned}$$

Substituting the above estimates into (5.23) leads to (2.45).

5.2.2. L^2**-estimates for** $f_m'(Y;X)$ We shall divide the proof of (2.46) and (2.47) into the following steps:

(i) Estimates of $\|\,|D|^{-1}f_m'(Y;X)\|_0$.
By virtue of (4.7), we have

$$\|\,|D|^{-1}f_m'(Y;X)\|_0 \le \big\|\,|D|^{-1}\mathcal{A}^t(\nabla X)^t\mathcal{A}^t(\nabla \boldsymbol{p}_m)(Y)\big\|_0 + \big\|\mathcal{A}^t\nabla \boldsymbol{p}_m'(Y;X)\big\|_0. \tag{5.24}$$

It follows from the law of products in Besov spaces and the imbedding $L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow \dot H^{-1}(\mathbb{R}^3)$ that

$$\begin{aligned}\|\,|D|^{-1}\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_0 &\le \big(1 + \|\mathcal{A} - Id\|_{\dot B^{\frac{3}{2}}_{2,1}}\big)\|(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot H^{-1}}\\ &\le C\|\nabla X\|_0\|\nabla \boldsymbol{p}_m\|_{L^3},\end{aligned} \tag{5.25}$$

from which, with (5.13) and (5.15), we infer

$$\||D|^{-1}\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_1\|_0 \leq C|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}\|\nabla X\|_0,$$
$$\||D|^{-1}\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_2\|_0 \leq C|Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}\|\nabla X\|_0.$$

Similarly, we get, by applying the law of products in Besov spaces, that

$$\||D|^{-1}\mathcal{A}^t\nabla \boldsymbol{p}'_m(Y;X)\|_0 \lesssim \big(1+\|\mathcal{A}-Id\|_{\mathcal{B}^{\frac{3}{2}}_{2,1}}\big)\|\nabla \boldsymbol{p}'_m(Y;X)\|_{\dot{H}^{-1}}.$$

To deal with the estimate of $\|\nabla \boldsymbol{p}'_m(Y;X)\|_{\dot{H}^{-1}}$, we deduce from (4.8) and a similar derivation of (5.25) that

$$\begin{aligned}\|\boldsymbol{p}'_1(Y;X)\|_0 &\lesssim \|\mathcal{A}\mathcal{A}^t-Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\nabla \boldsymbol{p}'_1(Y;X)\|_{\dot{H}^{-1}}\\ &\quad+\|\nabla X\|_0\|\nabla \boldsymbol{p}_1\|_{L^3}+\Big(1+\|\mathcal{A}-Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\Big)\\ &\quad\times\Big(\|\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\|_0\\ &\quad+\|\nabla X\|_0\|\mathcal{A}\,\mathrm{div}(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y))\|_{L^3}\\ &\quad+\|\mathcal{A}\partial_3 Y\otimes\partial_3 X\|_0\Big)\\ &\lesssim \delta_1\|\boldsymbol{p}'_1(Y;X)\|_0+\big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}+|\partial_3 Y|_0^2\big)\|\nabla X\|_0\\ &\quad+|\partial_3 Y|_0\|\partial_3 X\|_0,\end{aligned}$$

which together with (2.41) ensures that

$$\|\boldsymbol{p}'_1(Y;X)\|_0 \lesssim \left(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}+|\partial_3 Y|_0^2\right)\|\nabla X\|_0+|\partial_3 Y|_0\|\partial_3 X\|_0. \tag{5.26}$$

Exactly along the same lines, we deduce from (4.9) that

$$\|\boldsymbol{p}'_2(Y;X)\|_0 \lesssim \left(|Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}+|Y_t|_0^2\right)\|\nabla X\|_0+|Y_t|_0\|X_t\|_0. \tag{5.27}$$

Inserting the above estimates into (5.24) leads to

$$\||D|^{-1}f'_1(Y;X)\|_0 \lesssim \left(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}+|\partial_3 Y|_0^2\right)\|\nabla X\|_0+|\partial_3 Y|_0\|\partial_3 X\|_0, \tag{5.28}$$

$$\||D|^{-1}f'_2(Y;X)\|_0 \lesssim \left(|Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}+|Y_t|_0^2\right)\|\nabla X\|_0+|Y_t|_0\|X_t\|_0. \tag{5.29}$$

(ii) Estimates of $\|f'_m(Y;X)\|_{\dot{H}^k}$ for $k\geq 0$. By (4.7) we have

$$\|f'_m(Y;X)\|_{\dot{H}^k} \leq \|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{H}^k}+\|\mathcal{A}^t\nabla\big(\boldsymbol{p}'_m(Y;X)\big)\|_{\dot{H}^k}. \tag{5.30}$$

• Estimates for $\|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{H}^k}$.

We get, by applying Moser-type inequalities, that

$$\|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{\dot{H}^k}$$

$$\lesssim \|\mathcal{A}^t(\nabla X)^t\|_{L^6}\|D^k(\mathcal{A}^t\nabla \boldsymbol{p}_m)\|_{L^3} + \|D^k(\mathcal{A}^t(\nabla X)^t)\|_{L^6}\|\mathcal{A}^t\nabla \boldsymbol{p}_m\|_{L^3}$$
$$\lesssim \|D^k\nabla X\|_{L^6}\|\nabla \boldsymbol{p}_m\|_{L^3} + \|\nabla X\|_{L^6}\left(\|D^k\nabla \boldsymbol{p}_m\|_{L^3} + |D^k\mathcal{A}|_0\|\nabla \boldsymbol{p}_m\|_{L^3}\right).$$

Here and in all that follows, we always denote that $D^k = \sum_{|\alpha|=k}\partial^\alpha$.

In view of (4.1), applying Moser-type inequalities yields

$$\|D^k\nabla \boldsymbol{p}_1\|_{L^3} \le |\mathcal{A}\mathcal{A}^t - Id|_0\|D^k\nabla \boldsymbol{p}_1\|_{L^3} + |\mathcal{A}\mathcal{A}^t - Id|_k\|\nabla \boldsymbol{p}_1\|_{L^3}$$
$$+ \|D^{k+1}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\|_{L^3},$$

from which, with (2.41), we infer

$$\|D^k\nabla \boldsymbol{p}_1\|_{L^3} \lesssim |\nabla Y|_k\|\nabla \boldsymbol{p}_1\|_{L^3} + \|D^{k+1}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\|_{L^3}.$$

It is easy to observe that

$$\|D^{k+1}\big(\mathcal{A}(\partial_3 Y\otimes\partial_3 Y)\big)\|_{L^3} \lesssim \|D^{k+1}(\partial_3 Y\otimes\partial_3 Y)\|_{L^3} + |D^{k+1}\mathcal{A}|_0\|\partial_3 Y\otimes\partial_3 Y\|_{L^3}$$
$$\lesssim |\partial_3 Y|_{k+1}\|\partial_3 Y\|_{L^3} + |\nabla Y|_{k+1}|\partial_3 Y|_0\|\partial_3 Y\|_{L^3},$$

which together with (5.13) ensures that

$$\|D^k\nabla \boldsymbol{p}_1\|_{L^3} \le \left(|\partial_3 Y|_{k+1}|\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{k+1}|\partial_3 Y|_0^{\frac{4}{3}}\right)\|\partial_3 Y\|_0^{\frac{2}{3}},$$

and hence, we obtain

$$\begin{aligned}\|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_1\|_{\dot H^k} &\le |\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}\|\nabla X\|_{\dot H^{k+1}}\\ &\quad + \left(|\partial_3 Y|_{k+1}|\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{k+1}|\partial_3 Y|_0^{\frac{4}{3}}\right)\|\partial_3 Y\|_0^{\frac{2}{3}}\|\nabla X\|_{\dot H^1}.\end{aligned} \tag{5.31}$$

By the same procedure, we can show that

$$\|D^k\nabla \boldsymbol{p}_2\|_{L^3} \le \left(|Y_t|_{k+1}|Y_t|_0^{\frac{1}{3}} + |\nabla Y|_{k+1}|Y_t|_0^{\frac{4}{3}}\right)\|Y_t\|_0^{\frac{2}{3}}$$

and

$$\begin{aligned}\|\mathcal{A}^t(\nabla X)^t\mathcal{A}^t\nabla \boldsymbol{p}_2\|_{\dot H^k} &\le |Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}\|\nabla X\|_{\dot H^{k+1}}\\ &\quad + \left(|Y_t|_{k+1}|Y_t|_0^{\frac{1}{3}} + |\nabla Y|_{k+1}|Y_t|_0^{\frac{4}{3}}\right)\|Y_t\|_0^{\frac{2}{3}}\|\nabla X\|_{\dot H^1}.\end{aligned} \tag{5.32}$$

Furthermore, it holds that

$$\|\nabla \boldsymbol{p}_1\|_{W^{N,3}} \le \left(|\partial_3 Y|_{N+1}|\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{N+1}|\partial_3 Y|_0^{\frac{4}{3}}\right)\|\partial_3 Y\|_0^{\frac{2}{3}}, \tag{5.33}$$

$$\|\nabla \boldsymbol{p}_2\|_{W^{N,3}} \le \left(|Y_t|_{N+1}|Y_t|_0^{\frac{1}{3}} + |\nabla Y|_{N+1}|Y_t|_0^{\frac{4}{3}}\right)\|Y_t\|_0^{\frac{2}{3}}. \tag{5.34}$$

- Estimates of $\|\mathcal{A}^t\nabla\left(\boldsymbol{p}_m'(Y;X)\right)\|_{\dot H^k}$.

Applying Moser-type inequality gives

$$\|\mathcal{A}^t\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_{\dot H^k}\le\|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_{\dot H^k}+|\mathcal{A}^t-Id|_k\|\nabla\big(\boldsymbol{p}_m'(Y;X)\big)\|_0,\tag{5.35}$$

yet in view of (4.8), we have

$$\begin{aligned}&\|\nabla\boldsymbol{p}_1'(Y;X)\|_{\dot H^k}\\&\quad\lesssim\|(\mathcal{A}\mathcal{A}^t-Id)\nabla\boldsymbol{p}_1'(Y;X)\|_{\dot H^k}+\|\mathcal{A}\big(\nabla X\mathcal{A}+\mathcal{A}^t\nabla X\big)\mathcal{A}^t\nabla\boldsymbol{p}_1\|_{\dot H^k}\\&\qquad+\|\mathcal{A}\mathrm{div}\big(\mathcal{A}\nabla X\mathcal{A}(\partial_3Y\otimes\partial_3Y)\big)\|_{\dot H^k}+\|\mathcal{A}\nabla X\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3Y\otimes\partial_3Y)\big)\|_{\dot H^k}\\&\qquad+\|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3Y\otimes\partial_3X+\partial_3X\otimes\partial_3Y)\big)\|_{\dot H^k}.\end{aligned}$$

It follows from a similar derivation of (5.31) that

$$\begin{aligned}\|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla\boldsymbol{p}_1\|_{\dot H^k}&\le|\partial_3Y|_1^{\frac43}\|\partial_3Y\|_0^{\frac23}\|\nabla X\|_{\dot H^{k+1}}\\&\quad+\big(|\partial_3Y|_{k+1}|\partial_3Y|_0^{\frac13}+|\nabla Y|_{k+1}|\partial_3Y|_0^{\frac43}\big)\|\partial_3Y\|_0^{\frac23}\|\nabla X\|_{\dot H^1},\end{aligned}$$

and we get, by applying Moser-type inequality, that

$$\|(\mathcal{A}\mathcal{A}^t-Id)\nabla\boldsymbol{p}_1'(Y;X)\|_{\dot H^k}\le C\delta_1\|\nabla\boldsymbol{p}_1'(Y;X)\|_{\dot H^k}+|\nabla Y|_k\|\nabla\boldsymbol{p}_1'(Y;X)\|_0$$

and

$$\begin{aligned}&\|\mathcal{A}\mathrm{div}\big(\mathcal{A}\nabla X\mathcal{A}(\partial_3Y\otimes\partial_3Y)\big)\|_{\dot H^k}\\&\quad\lesssim|\partial_3Y|_0^2\|\nabla X\|_{\dot H^{k+1}}+\big(|\partial_3Y|_{k+1}|\partial_3Y|_0+|\nabla Y|_{k+1}|\partial_3Y|_0^2\big)\|\nabla X\|_0\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{A}\nabla X\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3Y\otimes\partial_3Y)\big)\|_{\dot H^k}&\lesssim|\partial_3Y|_1|\partial_3Y|_0\|\nabla X\|_{\dot H^k}\\&\quad+\big(|\partial_3Y|_{k+1}|\partial_3Y|_0+|\nabla Y|_{k+1}|\partial_3Y|_0^2\big)\|\nabla X\|_0,\end{aligned}$$

and finally,

$$\begin{aligned}\|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3Y\otimes\partial_3X+\partial_3X\otimes\partial_3Y)\big)\|_{\dot H^k}&\lesssim|\partial_3Y|_0\|\partial_3X\|_{\dot H^{k+1}}\\&\quad+\big(|\partial_3Y|_{k+1}+|\nabla Y|_{k+1}|\partial_3Y|_0\big)\|\partial_3X\|_0.\end{aligned}$$

As a result, by virtue of (5.14), we have

$$\begin{aligned}&\|\nabla\boldsymbol{p}_1'(Y;X)\|_{\dot H^k}\lesssim\mathfrak{g}_2(\partial_3Y,\partial_3X)\quad\text{with}\\&\mathfrak{g}_2(\mathfrak{x},\mathfrak{y})\overset{\text{def}}{=}\big(|\mathfrak{x}|_1^{\frac43}\|\mathfrak{x}\|_0^{\frac23}+|\mathfrak{x}|_1^2\big)\big(\|\nabla X\|_{\dot H^{k+1}}+|\nabla Y|_{k+1}\|\nabla X\|_1\big)+|\mathfrak{x}|_0\|\mathfrak{y}\|_{\dot H^{k+1}}\\&\quad+\big(|\mathfrak{x}|_{k+1}+|\nabla Y|_{k+1}|\mathfrak{x}|_1\big)\|\mathfrak{y}\|_1+|\mathfrak{x}|_{k+1}\big(|\mathfrak{x}|_0^{\frac13}\|\mathfrak{x}\|_0^{\frac23}+|\mathfrak{x}|_0\big)\|\nabla X\|_1.\end{aligned}\tag{5.36}$$

Substituting the above estimate and (5.14) into (5.35) for $m=1$ shows that $\|\mathcal{A}^t\nabla\big(\boldsymbol{p}_1'(Y;X)\big)\|_{\dot H^k}$ shares the same estimate as above.

Similarly, we can show that

$$\|\nabla\boldsymbol{p}_2'(Y;X)\|_{\dot H^k}\lesssim\mathfrak{g}_2(Y_t,X_t).\tag{5.37}$$

Substituting the above estimate and (5.16) into (5.35) for $m=2$ shows that $\|\mathcal{A}^t\nabla\big(\boldsymbol{p}_2'(Y;X)\big)\|_{\dot H^k}$ shares the same estimate as above.

Let us now turn to the estimates of $\|f_1'(Y;X)\|_{\dot{H}^k}$ and $\|f_2'(Y;X)\|_{\dot{H}^k}$. As a matter of fact, by inserting (5.31) and (5.36) into (5.30) for $m=1$, we achieve

$$\|f_1'(Y;X)\|_{\dot{H}^k} \lesssim \mathfrak{g}_2(\partial_3 Y, \partial_3 X). \tag{5.38}$$

Similarly, by inserting (5.32) and (5.37) into (5.30) for $m=2$, we obtain

$$\|f_2'(Y;X)\|_{\dot{H}^k} \lesssim \mathfrak{g}_2(Y_t, X_t). \tag{5.39}$$

Now we are in a position to complete the proof of Proposition 2.4.

Proof of Proposition 2.4. It remains to prove (2.46) and (2.47). Indeed, combining (5.28) with (5.38), we obtain (2.46), while combining (5.29) with (5.39) leads to (2.47). This completes the proof of Proposition 2.4. □

6. Energy Estimates for the Linearized Equation

The goal of this section is to present the proof of Theorem 2.3.

6.1. First-Order Energy Estimates

Let us first carry out the estimate of $\mathcal{E}_0(t)$ (2.56).

•Estimate of $\|\nabla X\|_0$.

We first get, by taking L^2 as the inner product of (2.40) with X, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\nabla X\|_0^2+(X_t|X)_{L^2}\Big)+\|\partial_3 X\|_0^2-\|X_t\|_0^2=(f'(Y;X)+g|X)_{L^2}. \tag{6.1}$$

It follows by taking as the L^2 inner product of (2.40) with $(-\Delta)^{-1}X_t$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\||D|^{-1}X_t\|_0^2+\||D|^{-1}\partial_3 X\|_0^2\Big)+\|X_t\|_0^2=\big((-\Delta)^{-1}(f'(Y;X)+g)|X_t\big)_{L^2}.$$

Summing up the above equality with $\frac{1}{4}\times$(6.1) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\big(\||D|^{-1}X_t\|_0^2+\||D|^{-1}\partial_3 X\|_0^2+\frac{1}{4}\|\nabla X\|_0^2\big)+\frac{1}{4}(X_t|X)_{L^2}\Big)\\&\quad+\frac{3}{4}\|X_t\|_0^2+\frac{1}{4}\|\partial_3 X\|_0^2=\big(|D|^{-1}(f'(Y;X)+g)|\frac{1}{4}|D|X+|D|^{-1}X_t\big)_{L^2}.\end{aligned} \tag{6.2}$$

It is easy to observe that

$$\begin{aligned}&\big|\big(|D|^{-1}\nabla\cdot\big(\mathcal{A}(\nabla X\mathcal{A}+\mathcal{A}^t(\nabla X)^t)\mathcal{A}^t\big)\nabla Y_t\big)|\frac{1}{4}|D|X+|D|^{-1}X_t\big)_{L^2}\big|\\&\quad\le C|\nabla Y_t|_0\|\nabla X\|_0\big(\|\nabla X\|_0+\||D|^{-1}X_t\|_0\big)\end{aligned}$$

and

$$\begin{aligned}&\big(|D|^{-1}\nabla\cdot\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X_t\big)||D|X\big)_{L^2}=-\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X_t|\nabla X\big)_{L^2}\\&\quad=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X|\nabla X\big)_{L^2}+\int_{\mathbb{R}^3}\partial_t(\mathcal{A}\mathcal{A}^t)|\nabla X|^2\,dx\end{aligned}$$

and

$$\begin{aligned}&\big|\big(|D|^{-1}\nabla\cdot\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X_t\big)\big||D|^{-1}X_t\big)_{L^2}\big|\\&\qquad\lesssim\|\mathcal{A}\mathcal{A}^t-Id\|_{\dot B^{\frac32}_{2,1}}\|\nabla X_t\|_{\dot H^{-1}}\|X_t\|_0\le C\delta_1\|X_t\|_0^2.\end{aligned}$$

Hence in view of (4.6), under the assumption of (2.41), by taking δ_1 so small that $C\delta_1\le\frac14$, we obtain

$$\begin{aligned}&\big|\big(|D|^{-1}f_0'(Y;X)|\frac14|D|X+|D|^{-1}X_t\big)_{L^2}+\frac18\frac{\mathrm{d}}{\mathrm{d}t}\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X|\nabla X\big)_{L^2}\big|\\&\le C|\nabla Y_t|_0\|\nabla X\|_0\big(\|\nabla X\|_0+\||D|^{-1}X_t\|_0\big)+\frac14\|X_t\|_0^2.\end{aligned}\tag{6.3}$$

By virtue of (5.28) and (5.29), we have

$$\begin{aligned}&\big|\big(|D|^{-1}(f_1'(Y;X)+f_2'(Y;X))|\frac14|D|X+|D|^{-1}X_t\big)_{L^2}\big|\\&\le\frac18\big(\|X_t\|_0^2+\|\partial_3X\|_0^2\big)+C\Big(|\partial_3Y|_1^{\frac43}\|\partial_3Y\|_0^{\frac23}+|\partial_3Y|_0^2\\&\quad+|Y_t|_1^{\frac43}\|Y_t\|_0^{\frac23}+|Y_t|_0^2\Big)\big(\|\nabla X\|_0^2+\||D|^{-1}X_t\|_0^2\big).\end{aligned}\tag{6.4}$$

Inserting (6.3) and (6.4) into (6.2) gives rise to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac12\big(\||D|^{-1}X_t\|_0^2+\||D|^{-1}\partial_3X\|_0^2+\frac14\big(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X\big)_{L^2}\big)+\frac14(X_t|X)_{L^2}\Big)\\&\quad+\frac18\big(\|X_t\|_0^2+\|\partial_3X\|_0^2\big)\le\||D|^{-1}g\|_0\big(\|\nabla X\|_0+\||D|^{-1}X_t\|_0\big)\\&\quad+C\Big(|\partial_3Y|_1^{\frac43}\|\partial_3Y\|_0^{\frac23}+|\partial_3Y|_0^2+|Y_t|_1\Big)\big(\|\nabla X\|_0^2+\||D|^{-1}X_t\|_0^2\big),\end{aligned}\tag{6.5}$$

by applying the assumption (2.55).

On the other hand, since $\mathcal{A}\mathcal{A}^t$ is a positive definite matrix ($|\mathcal{A}\mathcal{A}^t-Id|_0\le C\delta_1\le\frac14$), it holds that

$$\big(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X\big)_{L^2}\ge(1-C\delta_1)\|\nabla X\|_0^2\ge\frac34\|\nabla X\|_0^2,$$

so that one has

$$\begin{aligned}&\frac12\big(\||D|^{-1}X_t\|_0^2+\||D|^{-1}\partial_3X\|_0^2+\frac14\big(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X\big)_{L^2}\big)+\frac14(X_t|X)_{L^2}\\&\quad\ge\frac14\||D|^{-1}X_t\|_0^2+\frac12\||D|^{-1}\partial_3X\|_0^2+\frac1{32}\|\nabla X\|_0^2.\end{aligned}\tag{6.6}$$

•<u>Estimate of $\|X_t\|_0$.</u>

Multiplying (2.40) by X_t and integrating the resulting equality over $\mathbb{R}^3$, we get

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|X_t\|_0^2+\|\partial_3X\|_0^2\big)+\|\nabla X_t\|_0^2=\big(f'(Y;X)+g\big|X_t\big)_{L^2}.$$

In view of (4.6), we infer

$$\left|\left(f_0'(Y;X)|X_t\right)_{L^2}\right| \leq C|\nabla Y_t|_0^2\|\nabla X\|_0^2 + \frac{1}{4}\|\nabla X_t\|_0^2,$$

while it follows from (5.28) to (5.29) that

$$\begin{aligned}&\left|\left(f_1'(Y;X) + f_2'(Y;X)|X_t\right)_{L^2}\right|\\ &\leq C\left((|\partial_3 Y|_0\|\partial_3 X\|_0 + |Y_t|_0\|X_t\|_0) + \left(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}\right.\right.\\ &\quad\left.\left.+|\partial_3 Y|_1^2 + |Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}} + |Y_t|_1^2\right)\|\nabla X\|_0\right)\|\nabla X_t\|_0.\end{aligned}$$

As a result, thanks to the assumption (2.55), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|X_t\|_0^2 + \|\partial_3 X\|_0^2\right) + \|\nabla X_t\|_0^2\\ &\leq C\left(|\partial_3 Y|_1^{\frac{8}{3}}\|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4 + |Y_t|_1^2\right)\|\nabla X\|_0^2\\ &\quad + C\left(|\partial_3 Y|_0^2\|\partial_3 X\|_0^2 + |Y_t|_0^2\|X_t\|_0^2\right) + 4\||D|^{-1}g\|_0^2.\end{aligned} \tag{6.7}$$

•<u>Estimate of $\|\nabla X_t\|_0$.</u>

By taking L^2 as the inner product of (2.40) with $-\Delta X_t$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla X_t\|_{L^2}^2 + \|\nabla\partial_3 X\|_0^2\right) + \|\Delta X_t\|_0^2 = -\left(f'(Y;X) + g|\Delta X_t\right)_{L^2}. \tag{6.8}$$

It is easy to observe from (2.41) and (4.6) that

$$\|f_0'(Y;X)\|_0 \leq \frac{1}{4}\|\Delta X_t\|_0 + |\nabla Y_t|_1\|\nabla X\|_1. \tag{6.9}$$

Then by substituting the estimates (6.9), (5.38) and (5.39) into (6.8) and using the assumptions (2.41) and (2.55), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla X_t\|_{L^2}^2 + \|\nabla\partial_3 X\|_0^2\right) + \|\Delta X_t\|_0^2\\ &\leq C\left(\left(|Y_t|_2 + |\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2\right)\|\nabla X\|_1\right.\\ &\quad\left.+|\partial_3 Y|_1\|\partial_3 X\|_1 + |Y_t|_1\|X_t\|_1 + \|g\|_0\right)\|\Delta X_t\|_0,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla X_t\|_{L^2}^2 + \|\nabla\partial_3 X\|_0^2\right) + \|\Delta X_t\|_0^2\\ &\leq C\left(|Y_t|_2^2 + |\partial_3 Y|_1^{\frac{8}{3}}\|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4\right)\|\nabla X\|_1^2\\ &\quad + C\left(|\partial_3 Y|_1^2\|\partial_3 X\|_1^2 + |Y_t|_1^2\|X_t\|_1^2\right) + \|g\|_0^2.\end{aligned} \tag{6.10}$$

• <u>The estimate of $\|\nabla X\|_{\dot{H}^1}$.</u>

In this step, we shall use the equivalent formulation, (2.59), of (2.40). We first get, by taking L^2 as the inner product of (2.59) with $-\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)$, that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_0^2+\big(\partial_3^2X|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{L^2}-\big(X_{tt}|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{L^2}\\&\quad=-\big(\widetilde{f}'(Y;X)+g|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{L^2}.\end{aligned}$$

By using integration by parts, one has

$$\begin{aligned}&\big(X_{tt}|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{L^2}=-\frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla X_t|\mathcal{A}\mathcal{A}^t\nabla X\big)_{L^2}+\big(\nabla X_t|\partial_t(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{L^2},\\&\big(\partial_3^2X|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{L^2}=\big(\nabla\partial_3X|\mathcal{A}\mathcal{A}^t\nabla\partial_3X)\big)_{L^2}+\big(\nabla\partial_3X|\partial_3\big(\mathcal{A}\mathcal{A}^t\big)\nabla X\big)_{L^2}.\end{aligned}$$

Since $\mathcal{A}\mathcal{A}^t$ is a positive definitive matrix, we infer

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_0^2+\big(\nabla X_t|\mathcal{A}\mathcal{A}^t\nabla X\big)_{L^2}\Big)+\frac{1}{2}\|\nabla\partial_3X\|_0^2\\&\leq 2\|\nabla X_t\|_0^2+\frac{1}{4}\|\nabla\partial_3X\|_0^2+C\big(|\nabla Y_t|_0^2+|\partial_3\nabla Y|_0^2\big)\|\nabla X\|_0^2\\&\quad-\big(\widetilde{f}'(Y;X)+g|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{L^2},\end{aligned}\tag{6.11}$$

yet under the assumption of (2.41), it is easy to observe from (2.60) that

$$\|\widetilde{f}_0'(Y;X)\|_0\leq C|\nabla Y_t|_1\|\nabla X\|_1,$$

whereas it follows from (5.38) and (5.39) that

$$\begin{aligned}&\big|\big(f_1'(Y;X)+f_2'(Y;X)|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{L^2}\big|\lesssim\Big(|\partial_3Y|_1\|\partial_3X\|_1+|Y_t|_1\|X_t\|_1\\&\quad+\big(|\partial_3Y|_1^{\frac{4}{3}}\|\partial_3Y\|_0^{\frac{2}{3}}+|\partial_3Y|_1^2+|Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}+|Y_t|_1^2\big)\|\nabla X\|_1\Big)\|\nabla X\|_1.\end{aligned}$$

Inserting the above estimates into (6.11) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_0^2+\big(\nabla X_t|\mathcal{A}\mathcal{A}^t\nabla X\big)_{L^2}\Big)\\&\quad+\frac{1}{8}\|\nabla\partial_3X\|_0^2\leq 3\|X_t\|_1^2+\frac{1}{20}\|\partial_3X\|_0^2\\&\quad+\|g\|_0\|\nabla X\|_1\\&\quad+C\big(|\partial_3Y|_1^2+|Y_t|_1^2+|\partial_3Y|_1^{\frac{4}{3}}\|\partial_3Y\|_0^{\frac{2}{3}}+|Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}}\big)\|\nabla X\|_1^2.\end{aligned}\tag{6.12}$$

Let us denote

$$\begin{aligned}E_0(t)&\stackrel{\text{def}}{=}\frac{1}{2}\Big(\||D|^{-1}X_t\|_{H^2}^2+\||D|^{-1}\partial_3X\|_2^2+\frac{1}{4}\big(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X\big)_{L^2}\Big)\\&\quad+\frac{1}{4}(X_t|X)_{L^2}+\frac{1}{48}\Big(\frac{1}{2}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_0^2+\big(\nabla X_t|\mathcal{A}\mathcal{A}^t\nabla X\big)_{L^2}\Big).\end{aligned}\tag{6.13}$$

Then by summing up the inequalities (6.5), (6.7), (6.10) and $\frac{1}{48}\times$(6.12), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E_0(t)+\frac{1}{16}\|X_t\|_2^2+\frac{1}{384}\|\partial_3 X\|_1^2\leq +\||D|^{-1}g\|_1^2\\&+C\big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}+|\partial_3 Y|_1^2+|Y_t|_2\big)\\&\times\big(\|\nabla X\|_1^2+\||D|^{-1}X_t\|_0^2+\|\partial_3 X\|_1^2+\|X_t\|_1^2\big)\\&+\||D|^{-1}g\|_1\big(\|\nabla X\|_1+\||D|^{-1}X_t\|_0\big)\end{aligned}\tag{6.14}$$

Notice that

$$\big(\nabla X_t|\mathcal{A}\mathcal{A}^t\nabla X\big)_{L^2}\geq-\|X_t\|_0^2-\frac{1}{4}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_0^2$$

and

$$\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_0\geq\|\nabla X\|_{\dot H^1}-\|(\mathcal{A}\mathcal{A}^t-Id)\nabla X\|_{\dot H^1}\geq(1-C\delta_1)\|\nabla X\|_{\dot H^1},$$

so we deduce from from (6.6) and (6.13) that

$$E_0(t)\geq\frac{1}{16^2}\Big(\||D|^{-1}X_t\|_2^2+\||D|^{-1}\partial_3 X\|_2^2+\|\nabla X\|_1^2\Big).\tag{6.15}$$

Hence, for any $\varepsilon>0$, we deduce from (6.14) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E_0(t)+\frac{1}{16}\|X_t\|_2^2+\frac{1}{384}\|\partial_3 X\|_1^2\leq\langle t\rangle^{1+\varepsilon}\||D|^{-1}g\|_1^2\\&+C_\varepsilon\Big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}}+|\partial_3 Y|_1^2+|Y_t|_2+\langle t\rangle^{-(1+\varepsilon)}\Big)E_0(t).\end{aligned}\tag{6.16}$$

Applying Gronwall's inequality yields for any $\varepsilon>0$ that

$$\begin{aligned}&E_0(t)+\frac{1}{16}\|X_t\|^2_{L^2_t(H^2)}+\frac{1}{384}\|\partial_3 X\|^2_{L^2_t(H^1)}\leq C_\varepsilon\Big(\int_0^t\langle s\rangle^{1+\varepsilon}\||D|^{-1}g(s)\|_1^2\mathrm{d}s\Big)\\&\times\exp C\big(|\partial_3 Y|^{\frac{4}{3}}_{\frac{1}{2}+\varepsilon,1}\|\partial_3 Y\|^{\frac{2}{3}}_{L^2_t(L^2)}+|\partial_3 Y|^2_{\frac{1}{2}+\varepsilon,1}+|Y_t|_{1+\varepsilon,2}\big),\end{aligned}$$

which together with (6.15) ensures the first inequality of (2.56).

6.2. Higher-Order Energy Estimates

In this subsection, we shall derive the estimates for

$$\dot E_{k+1}(t)\overset{\text{def}}{=}\|\partial_3 X\|^2_{\dot H^{k+1}}+\|X_t\|^2_{\dot H^{k+1}}+\|\nabla X\|^2_{\dot H^{k+1}}\quad\text{for }k\geq 0.\tag{6.17}$$

We first get, by taking the $\dot H^{k+1}$-inner product of (2.40) with X_t, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|X_t\|^2_{\dot H^{k+1}}+\|\partial_3 X\|^2_{\dot H^{k+1}}\big)+\|X_t\|^2_{\dot H^{k+2}}=\big(f'(Y;X)+g\big|X_t\big)_{\dot H^{k+1}},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|X_t\|^2_{\dot H^{k+1}}+\|\partial_3 X\|^2_{\dot H^{k+1}}\big)+\|X_t\|^2_{\dot H^{k+2}}\le\|f'(Y;X)\|^2_{\dot H^k}+\|g\|^2_{\dot H^k}. \tag{6.18}$$

In view of (4.6), it follows from Moser-type inequality that

$$\begin{aligned}\|f_0'(Y;X)\|_{\dot H^k}\lesssim{}&|\nabla Y_t|_0\|\nabla X\|_{\dot H^{k+1}}+\big(|\nabla Y_t|_{k+1}+|\nabla Y_t|_0|\nabla Y|_{k+1}\big)\|\nabla X\|_0\\&+|\nabla Y|_0\|\nabla X_t\|_{\dot H^{k+1}}+|\nabla Y|_{k+1}\|\nabla X_t\|_0,\end{aligned}\tag{6.19}$$

from which, with (5.38), (5.39) and the assumption (2.55), we infer

$$\begin{aligned}&\|f'(Y;X)\|_{\dot H^k}\\&\lesssim|\partial_3 Y|_0\|\partial_3 X\|_{\dot H^{k+1}}+|Y_t|_0\|X_t\|_{\dot H^{k+1}}+|\nabla Y|_0\|X_t\|_{\dot H^{k+2}}\\&\quad+\big(|\partial_3 Y|_1^{\frac43}\|\partial_3 Y\|_0^{\frac23}+|\partial_3 Y|_1^2+|Y_t|_1\big)\big(\|\nabla X\|_{\dot H^{k+1}}+|\nabla Y|_{k+1}\|\nabla X\|_1\big)\\&\quad+\big(|\partial_3 Y|_{k+1}+|\nabla Y|_{k+1}|\partial_3 Y|_1\big)\|\partial_3 X\|_1+\big(|Y_t|_{k+1}+|\nabla Y|_{k+1}|Y_t|_1\big)\|X_t\|_1\\&\quad+|\nabla Y|_{k+1}\|\nabla X_t\|_0+\big(|\partial_3 Y|_{k+1}\big(|\partial_3 Y|_0^{\frac13}\|\partial_3 Y\|_0^{\frac23}+|\partial_3 Y|_1\big)+|Y_t|_{k+2}\big)\|\nabla X\|_1.\end{aligned}\tag{6.20}$$

Inserting (6.20) into (6.18), and using assumption (2.41) so that $|\nabla Y|_0\le\delta_1$, we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|X_t\|^2_{\dot H^{k+1}}+\|\partial_3 X\|^2_{\dot H^{k+1}}\big)+\frac34\|X_t\|^2_{\dot H^{k+2}}\lesssim|\partial_3 Y|_0^2\|\partial_3 X\|^2_{\dot H^{k+1}}+|Y_t|_0^2\|X_t\|^2_{\dot H^{k+1}}\\&\quad+\|g\|^2_{\dot H^k}+\big(|\partial_3 Y|_1^{\frac83}\|\partial_3 Y\|_0^{\frac43}+|\partial_3 Y|_1^4+|Y_t|_1^2\big)\big(\|\nabla X\|^2_{\dot H^{k+1}}+|\nabla Y|^2_{k+1}\|\nabla X\|_1^2\big)\\&\quad+\big(|\partial_3 Y|^2_{k+1}+|\nabla Y|^2_{k+1}|\partial_3 Y|_1^2\big)\|\partial_3 Y\|_1^2+\big(|Y_t|^2_{k+1}+|\nabla Y|^2_{k+1}|Y_t|_1^2\big)\|X_t\|_1^2\\&\quad+|\nabla Y|^2_{k+1}\|\nabla X_t\|_0^2+\big(|\partial_3 Y|^2_{k+1}\big(|\partial_3 Y|_0^{\frac23}\|\partial_3 Y\|_0^{\frac43}+|\partial_3 Y|_1^2\big)+|Y_t|^2_{k+2}\big)\|\nabla X\|_1^2.\end{aligned}\tag{6.21}$$

Secondly, by taking the $\dot H^k$-inner product of (2.59) with $-\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)$, we obtain

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|^2_{\dot H^k}+\big(\partial_3^2 X|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{\dot H^k}\\&\quad-\big(X_{tt}|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{\dot H^k}=-\big(\widetilde{f}'(Y;X)+g|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{\dot H^k}.\end{aligned}\tag{6.22}$$

By using integration by parts, one has

$$-\big(X_{tt}|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{\dot H^k}=-\frac{\mathrm{d}}{\mathrm{d}t}\big(X_t|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot H^k}-\big(\nabla X_t|\partial_t(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot H^k}$$

and

$$\begin{aligned}\big|\big(\nabla X_t|\partial_t(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot H^k}\big|\le\|X_t\|_{\dot H^{k+1}}\bigg(&\frac32\|\nabla X_t\|_{\dot H^k}+|\nabla Y|_k\|\nabla X_t\|_0\\&+|\nabla Y_t|_0\|\nabla X\|_{\dot H^k}+|\nabla Y_t|_k\|\nabla X\|_0\bigg),\end{aligned}$$

so that we arrive at

$$\begin{aligned}&\big|\big(X_{tt}|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{\dot H^k}-\frac{\mathrm{d}}{\mathrm{d}t}\big(X_t|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot H^k}\big|\\&\quad\le 2\|X_t\|^2_{\dot H^{k+1}}+C_k\big(|\nabla Y_t|_0^2\|\nabla X\|^2_{\dot H^k}+|\nabla Y|_k^2\|\nabla X_t\|_0^2+|\nabla Y_t|_k^2\|\nabla X\|_0^2\big).\end{aligned}$$

Similarly, again by using integration by parts, one has

$$\big(\partial_3^2X|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\big)_{\dot H^k}=\big(\nabla\partial_3X|\mathcal{A}\mathcal{A}^t\nabla\partial_3X\big)_{\dot H^k}+\big(\nabla\partial_3X|\partial_3\big(\mathcal{A}\mathcal{A}^t\big)\nabla X\big)_{\dot H^k}.$$

Since $|\mathcal{A}\mathcal{A}^t-Id|_0\le C\delta_1\le\frac14$, due to (2.41), applying Moser-type inequality gives

$$\big(\nabla\partial_3X|\mathcal{A}\mathcal{A}^t\nabla\partial_3X\big)_{\dot H^k}\ge\frac12\|\nabla\partial_3X\|^2_{\dot H^k}-C_k|\nabla Y|_k^2\|\nabla\partial_3X\|_0^2$$

and

$$\begin{aligned}\big|\big(\nabla\partial_3X|\partial_3\big(\mathcal{A}\mathcal{A}^t\big)\nabla X\big)_{\dot H^k}\big|\le\frac14\|\nabla\partial_3X\|^2_{\dot H^k}+C_k\Big(&|\partial_3Y|_1^2\|\nabla X\|^2_{\dot H^k}\\&+|\partial_3Y|^2_{k+1}\|\nabla X\|_0^2\Big),\end{aligned}$$

so that it holds that

$$\begin{aligned}\big(\nabla\partial_3X|\mathcal{A}\mathcal{A}^t\nabla\partial_3X\big)_{\dot H^k}\ge\frac14\|\nabla\partial_3X\|^2_{\dot H^k}-C_k\big(&|\nabla Y|_k^2\|\nabla\partial_3X\|_0^2\\&+|\partial_3Y|_1^2\|\nabla X\|^2_{\dot H^k}+|\partial_3Y|^2_{k+1}\|\nabla X\|_0^2\big).\end{aligned}$$

Inserting the above estimates into (6.22) gives rise to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac12\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|^2_{\dot H^k}-\big(X_t|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot H^k}\Big)+\frac14\|\partial_3X\|^2_{\dot H^{k+1}}\\&\quad\le 2\|X_t\|^2_{\dot H^{k+1}}+\big(|\partial_3Y|_0^2+|Y_t|_0^2\big)\|\nabla X\|^2_{\dot H^{k+1}}\\&\qquad+C_k|\nabla Y|_k^2\big(\|\nabla\partial_3X\|_0^2+\|\nabla X_t\|_0^2\big)\\&\qquad+C_k(|\partial_3Y|^2_{k+1}+|Y_t|^2_{k+2})\|\nabla X\|_0^2+\big(\|\widetilde{f}'(Y;X)\|_{\dot H^k}\\&\qquad+\|g\|_{\dot H^k}\big)\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_{\dot H^k}.\end{aligned}\tag{6.23}$$

We remark that

$$\begin{aligned}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)-\Delta X\|_{\dot H^k}&\le|\mathcal{A}\mathcal{A}^t-Id|_0\|\nabla X\|_{\dot H^{k+1}}\\&\quad+C_k|\mathcal{A}\mathcal{A}^t-Id|_{k+1}\|\nabla X\|_0\\&\le\frac12\|\nabla X\|_{\dot H^{k+1}}+C_k|\nabla Y|_{k+1}\|\nabla X\|_0.\end{aligned}\tag{6.24}$$

Moreover, in view of (2.60), we have

$$\|\widetilde{f}_0'(Y;X)\|_{\dot H^k}\lesssim|Y_t|_1\|\nabla X\|_{\dot H^{k+1}}+\big(|Y_t|_{k+2}+|Y_t|_1|\nabla Y|_{k+1}\big)\|\nabla X\|_0,\tag{6.25}$$

which together with (5.38) and (5.39), ensures that

$$\begin{aligned}\|\widetilde{f}'(Y;X)\|_{\dot{H}^k} &\lesssim |\partial_3 Y|_0\|\partial_3 X\|_{\dot{H}^{k+1}} + |Y_t|_0\|X_t\|_{\dot{H}^{k+1}} + \big(|\partial_3 Y|_{k+1}\\ &\quad + |\nabla Y|_{k+1}|\partial_3 Y|_1\big)\|\partial_3 X\|_1\\ &\quad + \big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2 + |Y_t|_1\big)\big(\|\nabla X\|_{\dot{H}^{k+1}}\\ &\quad + |\nabla Y|_{k+1}\|\nabla X\|_1\big) + \|X_t\|_1\big(|Y_t|_{k+1}\\ &\quad + |\nabla Y|_{k+1}|Y_t|_1\big)\\ &\quad + \big(|\partial_3 Y|_{k+1}\big(|\partial_3 Y|_0^{\frac{1}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1\big) + |Y_t|_{k+2}\big)\|\nabla X\|_1.\end{aligned} \tag{6.26}$$

Inserting the above inequalities into (6.23) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\nabla\cdot\big(\mathcal{A}\mathcal{A}^t\nabla X\big)\|_{\dot{H}^k}^2 - \big(X_t|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot{H}^k}\Big) + \frac{1}{8}\|\partial_3 X\|_{\dot{H}^{k+1}}^2\\ &\le 3\|X_t\|_{\dot{H}^{k+1}}^2 + \langle t\rangle^{1+\varepsilon}\|g\|_{\dot{H}^k}^2 + C_k\big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2 + |Y_t|_1\\ &\quad + \langle t\rangle^{-(1+\varepsilon)}\big)\|\nabla X\|_{\dot{H}^{k+1}}^2\\ &\quad + C_k\big(\big(|\partial_3 Y|_{k+1}^2 + |\nabla Y|_{k+1}^2|\partial_3 Y|_1^2\big)\langle t\rangle^{1+\varepsilon} + |\nabla Y|_k^2\big)\|\partial_3 X\|_1^2\\ &\quad + C_k\big(\big(|Y_t|_{k+1}^2 + |\nabla Y|_{k+1}^2|Y_t|_1^2\big)\langle t\rangle^{1+\varepsilon} + |\nabla Y|_k^2\big)\|X_t\|_1^2\\ &\quad + C_k\Big\{\big(|\partial_3 Y|_{k+1}^2\big(|\partial_3 Y|_0^{\frac{2}{3}}\|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^2\big) + |Y_t|_{k+2}^2\big)\langle t\rangle^{1+\varepsilon} + |\partial_3 Y|_{k+1}^2\\ &\quad + |\nabla Y|_{k+1}^2\big(\big(|\partial_3 Y|_1^{\frac{8}{3}}\|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4 + |Y_t|_1^2\big)\langle t\rangle^{1+\varepsilon} + \langle t\rangle^{-(1+\varepsilon)}\big)\big)\Big\}\|\nabla X\|_1^2.\end{aligned} \tag{6.27}$$

Let us introduce

$$\begin{aligned}\dot{D}_{k+1}(t) &\stackrel{\text{def}}{=} \|X_t\|_{\dot{H}^{k+1}}^2 + \|\partial_3 X\|_{\dot{H}^{k+1}}^2 + \frac{1}{2}\|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\|_{\dot{H}^k}^2\\ &\quad - \big(X_t|\nabla\cdot(\mathcal{A}\mathcal{A}^t\nabla X)\big)_{\dot{H}^k}.\end{aligned} \tag{6.28}$$

Then it follows from (6.24) that

$$\dot{D}_{k+1}(t) \ge \frac{1}{8}\dot{E}_{k+1}(t) - C_{k+1}\|X_t\|_0^2 - C_{k+1}|\nabla Y|_{k+1}^2\|\nabla X\|_0^2, \tag{6.29}$$

with $\dot{E}_{k+1}(t)$ being given by (6.17).

Hence by summing up (6.21) and (6.27), and then integrating the resulting inequality over $[0,t]$ and using (6.29), we achieve

$$\begin{aligned}&\dot{E}_{k+1}(t) + \int_0^t \big(\frac{1}{2}\|X_t\|_{\dot{H}^{k+2}}^2 + \frac{1}{8}\|\partial_3 X\|_{\dot{H}^{k+1}}^2\big)\,\mathrm{d}s\\ &\le 8\dot{D}_{k+1}(t) + \|X_t\|_{0,0}^2 + |\nabla Y|_{0,k+1}^2\|\nabla X\|_{0,0}^2\\ &\quad + \int_0^t \big(\frac{1}{2}\|X_t\|_{\dot{H}^{k+2}}^2 + \frac{1}{8}\|\partial_3 X\|_{\dot{H}^{k+1}}^2\big)\,\mathrm{d}s\\ &\lesssim \int_0^t \big(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2 + |Y_t|_1 + \langle s\rangle^{-(1+\varepsilon)}\big)\dot{E}_{k+1}(s)\,\mathrm{d}s\\ &\quad + \|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(\dot{H}^k)}^2 + \gamma_{\varepsilon,k+1}(Y)^2\mathcal{E}_0^2(t),\end{aligned} \tag{6.30}$$

where $\mathcal{E}_0(t)$ is given by (2.56) and $\gamma_{\varepsilon,k+1}(Y)$ by (2.58). Applying Gronwall's inequality to (6.30) and using (2.56), we obtain

$$\dot{E}_{k+1}(t) \leq C_{\varepsilon,k}\left(\|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|^2_{L^2_t(\dot{H}^k)} + \gamma_{\varepsilon,k+1}(Y)^2 \|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|^2_{L^2_t(H^1)}\right) E_\varepsilon(Y),$$

from which, along with (6.30), we infer

$$\begin{aligned}
&\|(X_t, \partial_3 X, \nabla X)\|_{L^\infty_t(\dot{H}^{k+1})} + \|X_t\|_{L^2_t(\dot{H}^{k+2})} + \|\partial_3 X\|_{L^2_t(\dot{H}^{k+1})} \\
&\quad \leq C_{\varepsilon,k}\big(\|\langle t\rangle^{\frac{1+\varepsilon}{2}} g\|_{L^2_t(\dot{H}^k)} + \gamma_{\varepsilon,k+1}(Y)\|\langle t\rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L^2_t(H^1)}\big) E_\varepsilon(Y).
\end{aligned} \tag{6.31}$$

Summing up the above inequality with respect to k leads to (2.57). This completes the proof of Theorem 2.3.

Now let us turn to the proof of Corollary 2.1.

Proof of Corollary 2.1. By summing up (6.7) and (6.10), and then multiplying the resulting inequality by $\langle t\rangle$ and integrating the above inequality over $[0, t]$, we find

$$\begin{aligned}
&\langle t\rangle\big(\|X_t\|_1^2 + \|\partial_3 X\|_1^2\big) + \int_0^t \langle s\rangle \|\nabla X_t\|_1^2 \mathrm{d}s \leq \|X_t\|^2_{L^2_t(H^1)} \\
&\quad + \big(1 + |\partial_3 Y|^2_{\frac{1}{2},1}\big)\|\partial_3 X\|^2_{L^2_t(H^1)} \\
&\quad + C\|\langle t\rangle^{\frac{1}{2}} |D|^{-1} g\|^2_{L^2_t(H^1)} + C\big(|\partial_3 Y|^{\frac{8}{3}}_{\frac{1}{2}+\varepsilon,1} \|\partial_3 Y\|^{\frac{4}{3}}_{L^2_t(L^2)} + |\partial_3 Y|^4_{\frac{1}{2}+\varepsilon,1} \\
&\quad + |Y_t|^2_{1+\varepsilon,2}\big)\mathcal{E}_0^2(t).
\end{aligned}$$

Then (2.61) follows from (2.56).

Similarly, we get, by multiplying (6.21) by $\langle t\rangle$ and integrating the inequality over $[0, t]$ and taking the square root of the resulting inequality, that

$$\begin{aligned}
&\langle t\rangle^{\frac{1}{2}}\big(\|(X_t, \partial_3 X)\|_{\dot{H}^{k+1}}\big) + \Big(\frac{3}{4}\int_0^t \langle s\rangle \|X_t\|^2_{\dot{H}^{k+2}} \mathrm{d}s\Big)^{\frac{1}{2}} \lesssim \|\langle t\rangle^{\frac{1}{2}} g\|_{L^2_t(\dot{H}^k)} \\
&\quad + \big(1 + |Y_t|_{\frac{1}{2},0}\big)\|X_t\|_{L^2_t(\dot{H}^{k+1})} + \big(1 + |\partial_3 Y|_{\frac{1}{2},0}\big)\|\partial_3 X\|_{L^2_t(\dot{H}^{k+1})} \\
&\quad + |\nabla Y|_{0,k+1}\|\langle t\rangle^{\frac{1}{2}} \nabla X_t\|_{L^2_t(L^2)} \\
&\quad + \big(|\partial_3 Y|^{\frac{4}{3}}_{\frac{1}{2}+\varepsilon,1}\|\partial_3 Y\|^{\frac{2}{3}}_{L^2_t(L^2)} + |\partial_3 Y|^2_{\frac{1}{2}+\varepsilon,1} + |Y_t|_{1+\varepsilon,1}\big)\big(\|\nabla X\|_{L^\infty_t(\dot{H}^{k+1})} \\
&\quad + |\nabla Y|_{0,k+1}\|\nabla X\|_{L^\infty_t(H^1)}\big) \\
&\quad + \|\partial_3 Y\|_{L^2_t(L^2)}\big(|\nabla Y|_{0,k+1}|\partial_3 Y|_{\frac{1}{2},1} + |\partial_3 Y|_{\frac{1}{2},k+1}\big) \\
&\quad + \big(|Y_t|_{\frac{1}{2},k+1} + |\nabla Y|_{0,k+1}|Y_t|_{\frac{1}{2},1}\big)\|X_t\|_{L^2_t(L^2)} \\
&\quad + \big(|\partial_3 Y|_{\frac{1}{2}+\varepsilon,k+1}\big(|\partial_3 Y|^{\frac{1}{3}}_{\frac{1}{2}+\varepsilon,0}\|\partial_3 Y\|^{\frac{2}{3}}_{L^2_t(L^2)} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}\big) \\
&\quad + |Y_t|_{1+\varepsilon,k+2}\big)\|\nabla X\|_{L^\infty_t(H^1)}.
\end{aligned}$$

Then (2.62) follows from (2.57) and (2.61), and this completes the proof of Corollary 2.1. □

7. Energy Decay for ∇X_t

The main idea for proving Proposition 2.7 is to use the following proposition:

Proposition 7.1. *Let X be a smooth enough solution of*

$$\begin{cases} X_{tt} - \Delta X_t - \partial_3^2 X = \nabla \cdot \big((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\big) + \hbar \overset{def}{=} f \\ X(0) = 0 \quad and \quad X_t(0) = 0, \end{cases} \tag{7.1}$$

on $[0, T]$. *Then under the assumption that*

$$\|\nabla Y\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} < \delta_1, \tag{7.2}$$

we have, for any $t \in [0, T]$ and any $\varepsilon > 0$, that

$$\begin{aligned} t\|\nabla X_t(t)\|_{L^2} &\leq C_\varepsilon \left(\sup_{s\in[0,t]} \|s^{1+\varepsilon}|D|^{-1}\hbar\|_{L^2} + \sup_{s\in[0,t]} \|s^{1+\varepsilon}|D|\hbar\|_{L^2} \right) \\ &\leq C_\varepsilon \||D|^{-1}\hbar\|_{1+\varepsilon,2}. \end{aligned} \tag{7.3}$$

Moreover, we have, for $k \in \mathbb{N}$, that

$$\begin{aligned} &t\|\nabla X_t(t)\|_{\dot{H}^k} \\ &\leq C_{\varepsilon,k} \left(\big(\delta_1 + \|D^k \nabla Y\|_{L_t^\infty\big(\dot{B}^{\frac{3}{2}}_{2,1}\big)}\big)\||D|^{-1}\hbar\|_{1+\varepsilon,2} + \||D|^{-1}\hbar\|_{1+\varepsilon,k+2} \right). \end{aligned} \tag{7.4}$$

Admitting this proposition for the time being, we present the proof of Proposition 2.7.

Proof of Proposition 2.7. In our situation, (2.40),

$$\hbar = \nabla \cdot \Big(\mathcal{A}\big((-\nabla X)\mathcal{A} + \mathcal{A}^t(-\nabla X)^t\big)\mathcal{A}^t \nabla Y_t \Big) - f_1'(Y; X) + f_2'(Y; X) + g.$$

We infer from (5.23), (2.46) and (2.47) that for $k \geq 0$,

$$\begin{aligned} &\||D|^{-1}\hbar\|_{1+\varepsilon,k+2} \\ &\quad\lesssim |\partial_3 Y|_{\frac{1}{2}+\varepsilon,0}\|\partial_3 X\|_{\frac{1}{2},k+2} + |Y_t|_{\frac{1}{2}+\varepsilon,0}\|X_t\|_{\frac{1}{2},k+2} + \||D|^{-1}g\|_{1+\varepsilon,k+2} \\ &\qquad + \big(|\partial_3 Y|^{\frac{4}{3}}_{\frac{1}{2}+\varepsilon,1}\|\partial_3 Y\|^{\frac{2}{3}}_{\frac{1}{2},0} + |\partial_3 Y|^2_{\frac{1}{2}+\varepsilon,1} + |Y_t|_{1+\varepsilon,1}\big)\|\nabla X\|_{0,k+2} \\ &\qquad + \gamma_{\varepsilon,k+2}(Y)(\|\partial_3 X\|_{\frac{1}{2},1} + \|X_t\|_{\frac{1}{2},1} + \|\nabla X\|_{0,1}), \end{aligned} \tag{7.5}$$

where $\gamma_{\varepsilon,k+2}(Y)$ is given in (2.58). Proposition 2.7 then follows from Proposition 7.1, (7.5), Corollary 2.1 and the fact that $\|D^k\nabla Y\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} \leq \|\nabla Y\|_{0,k+2}$. □

In order to prove Proposition 7.1 we need to exploit the tool of anisotropic Littlewood-Paley analysis. Similar to the dyadic operators Δ_j, and S_j given by Definition 2.1, let us recall the dyadic operators in the x_3 variable:

$$\Delta_\ell^{\rm v} a \stackrel{\rm def}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), \quad \text{and} \quad S_\ell^{\rm v} a \stackrel{\rm def}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}). \tag{7.6}$$

Let us also recall the following anisotropic type Besov norm from [24,25]:

Definition 7.1. Let $s_1, s_2 \in \mathbb{R}$, $r \in [1, \infty]$ and $a \in \mathcal{S}_h'(\mathbb{R}^3)$, we define the norm

$$\|a\|_{\mathcal{B}_r^{s_1,s_2}} \stackrel{\rm def}{=} \left(2^{js_1}2^{\ell s_2}\|\Delta_j\Delta_\ell^{\rm v} a\|_{L^2}\right)_{\ell^r(\mathbb{Z}^2)}.$$

In particular, when $r = 2$, we denote $\|a\|_{\dot{H}^{s_1,s_2}} \stackrel{\rm def}{=} \|a\|_{\mathcal{B}_2^{s_1,s_2}} = \big\||D|^{s_1}|D_{x_3}|^{s_2}a\big\|_{L^2}$.

In order to obtain a better description of the regularizing effect for the transport-diffusion equation, we will use an anisotropic version of the Chemin-Lerner type norm (see [3] for instance).

Definition 7.2. Let $(r, q) \in [1, +\infty]^2$ and $T \in (0, +\infty]$. We define the norm $\widetilde{L}_T^q(\mathcal{B}_r^{s_1,s_2}(\mathbb{R}^3))$ by

$$\|u\|_{\widetilde{L}_T^q(\mathcal{B}_r^{s_1,s_2})} \stackrel{\rm def}{=} \Big(\sum_{(j,\ell)\in\mathbb{Z}^2}\big(2^{js_1}2^{\ell s_2}\|\Delta_j\Delta_\ell^{\rm v} u\|_{L_T^q(L^2)}\big)^r\Big)^{\frac{1}{r}},$$

with the usual change if $r = \infty$.

For the convenience of the readers, we recall the following Bernstein type lemma from [3,10,27]:

Lemma 7.1. *Let $\mathfrak{B}_{\rm h}$ (resp. $\mathfrak{B}_{\rm v}$) be a ball of $\mathbb{R}^2$ (resp. $\mathbb{R}$), and $\mathcal{C}_{\rm h}$ (resp. $\mathcal{C}_{\rm v}$) a ring of $\mathbb{R}^2$ (resp. $\mathbb{R}$), and let $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$. Then it holds that:*
if the support of $\widehat{a}$ is included in $2^k\mathfrak{B}_{\rm h}$, then

$$\|\partial_{\rm h}^\alpha a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_1})} \lesssim 2^{k\left(|\alpha|+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)}\|a\|_{L_{\rm h}^{p_2}(L_{\rm v}^{q_1})};$$

if the support of $\widehat{a}$ is included in $2^\ell\mathfrak{B}_{\rm v}$, then

$$\|\partial_3^\beta a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_1})} \lesssim 2^{\ell\left(\beta+\left(\frac{1}{q_2}-\frac{1}{q_1}\right)\right)}\|a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_2})};$$

if the support of $\widehat{a}$ is included in $2^k\mathcal{C}_{\rm h}$, then

$$\|a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_1})} \lesssim 2^{-kN}\max_{|\alpha|=N}\|\partial_{\rm h}^\alpha a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_1})};$$

if the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_{\rm v}$, then

$$\|a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_1})} \lesssim 2^{-\ell N}\|\partial_3^N a\|_{L_{\rm h}^{p_1}(L_{\rm v}^{q_1})}.$$

Let us now turn to the proof of Proposition 7.1.

Proof of Proposition 7.1. The proof of this lemma is motivated by the proof of Proposition 4.1 of [24,31]. By applying the operator $\Delta_j\Delta_\ell^{\mathrm{v}}$ to (7.1) and then taking the L^2 inner product of the resulting equation with $\Delta_j\Delta_\ell^{\mathrm{v}}X_t$, we write

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\partial_3X\|_{L^2}^2\big)+\|\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2\\=\big(\Delta_j\Delta_\ell^{\mathrm{v}}f\,|\,\Delta_j\Delta_\ell^{\mathrm{v}}X_t\big)_{L^2}.\end{aligned}\tag{7.7}$$

Along the same lines, one has

$$\begin{aligned}(\Delta_j\Delta_\ell^{\mathrm{v}}X_{tt}\,|\,\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X)-\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2-\|\partial_3\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2\\=(\Delta_j\Delta_\ell^{\mathrm{v}}f\,|\,\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X).\end{aligned}$$

Notice that

$$(\Delta_j\Delta_\ell^{\mathrm{v}}X_{tt}\,|\,\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X)=\frac{\mathrm{d}}{\mathrm{d}t}(\Delta_j\Delta_\ell^{\mathrm{v}}X_t\,|\,\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X)+\|\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2,$$

so that we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac12\|\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2-(\Delta_j\Delta_\ell^{\mathrm{v}}X_t\,|\,\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X)\Big)\\-\|\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2+\|\partial_3\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2=-(\Delta_j\Delta_\ell^{\mathrm{v}}f\,|\,\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X).\end{aligned}\tag{7.8}$$

By summing up (7.7) with $\frac14$ of (7.8), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}g_{j,\ell}^2(t)+\frac34\|\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2+\frac14\|\partial_3\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2\\=\Big(\Delta_j\Delta_\ell^{\mathrm{v}}f\;|\;\Delta_j\Delta_\ell^{\mathrm{v}}X_t-\frac14\Delta\Delta_j\Delta_\ell^{\mathrm{v}}X\Big),\end{aligned}\tag{7.9}$$

where

$$\begin{aligned}g_{j,\ell}^2(t)\overset{\text{def}}{=}\frac12\bigg(\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t(t)\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\partial_3X(t)\|_{L^2}^2+\frac14\|\Delta_j\Delta_\ell^{\mathrm{v}}\Delta X(t)\|_{L^2}^2\Big)\\-\frac14\big(\Delta_j\Delta_\ell^{\mathrm{v}}X_t(t)|\Delta_j\Delta_\ell^{\mathrm{v}}\Delta X(t)\bigg).\end{aligned}$$

It is easy to observe that

$$g_{j,\ell}^2(t)\sim\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t(t)\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\partial_3X(t)\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\Delta X(t)\|_{L^2}^2.\tag{7.10}$$

Now, according to the heuristic analysis presented at the beginning of Section 2, we split the frequency analysis into two case.

• When $j\le\frac{\ell+1}{2}$

In this case, one has

$$g_{j,\ell}^2(t)\sim\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t(t)\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\partial_3X(t)\|_{L^2}^2,$$

and Lemma 7.1 implies that

$$\frac{3}{4}\|\nabla\Delta_j\Delta_\ell^{\rm v}X_t\|_{L^2}^2+\frac{1}{4}\|\partial_3\nabla\Delta_j\Delta_\ell^{\rm v}X\|_{L^2}^2\geq c2^{2j}\left(\|\Delta_j\Delta_\ell^{\rm v}X_t\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\rm v}\partial_3X\|_{L^2}^2\right).$$

Hence it follows from (7.9) that

$$\frac{\rm d}{{\rm d}t}g_{j,\ell}(t)+c2^{2j}g_{j,\ell}(t)\leq\|\Delta_j\Delta_\ell^{\rm v}f(t)\|_{L^2},$$

which, in particular, implies that

$$g_{j,\ell}(t)\leq\int_0^t e^{-c(t-s)2^{2j}}\|\Delta_j\Delta_\ell^{\rm v}f(s)\|_{L^2}\,{\rm d}s, \tag{7.11}$$

and

$$2^j\|\Delta_j\Delta_\ell^{\rm v}X_t\|_{L^1_t(L^2)}\lesssim 2^{-j}\|\Delta_j\Delta_\ell^{\rm v}f\|_{L^1_t(L^2)}. \tag{7.12}$$

Now let us turn to the estimate of $\|\Delta_j\Delta_\ell^{\rm v}f\|_{L^1_t(L^2)}$. Indeed it follows by the law of product in the anisotropic Besov spaces (see Lemma 3.3 of [31]) that

$$\begin{aligned}\left\|(\mathcal{A}\mathcal{A}^t-Id)\nabla X_t\right\|_{L^1_t(\dot H^{0,0})}&\lesssim\|(\mathcal{A}\mathcal{A}^t-Id)\|_{L^\infty_t\left(\mathcal{B}_1^{1,\frac12}\right)}\|\nabla X_t\|_{\widetilde L^1_t(\dot H^{0,0})}\\&\lesssim\|(\mathcal{A}\mathcal{A}^t-Id)\|_{L^\infty_t\left(\dot B^{\frac32}_{2,1}\right)}\|\nabla X_t\|_{\widetilde L^1_t(\dot H^{0,0})}\\&\lesssim\delta_1\|\nabla X_t\|_{\widetilde L^1_t(\dot H^{0,0})},\end{aligned} \tag{7.13}$$

where we use the fact that $\dot B^{\frac32}_{2,1}(\mathbb{R}^3)\hookrightarrow\mathcal{B}_1^{1,\frac12}$ (one may check Lemma 3.2 of [25,31] for details). Hence we obtain

$$2^{-j}\|\Delta_j\Delta_\ell^{\rm v}f\|_{L^1_t(L^2)}\lesssim c_{j,\ell}\delta_1\|\nabla X_t\|_{\widetilde L^1_t(\dot H^{0,0})}+\|\Delta_j\Delta_\ell^{\rm v}|D|^{-1}\hbar\|_{L^1_t(L^2)}, \tag{7.14}$$

where $(c_{j,\ell})_{j,\ell\in\mathbb{Z}^2}$ is a generic element of $\ell^2(\mathbb{Z}^2)$ so that $\sum_{j,\ell\in\mathbb{Z}^2}c_{j,\ell}^2=1$.

It follows from Lemma 7.1 and (7.11) that

$$\begin{aligned}&2^jt\|\Delta_j\Delta_\ell^{\rm v}X_t(t)\|_{L^2}\\&\lesssim\int_0^t 2^{2j}(t-s)e^{-c(t-s)2^{2j}}\left\|\Delta_j\Delta_\ell^{\rm v}\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X_t\big)(s)\right\|_{L^2}{\rm d}s\\&\quad+\int_0^t 2^{2j}e^{-c(t-s)2^{2j}}\left\|\Delta_j\Delta_\ell^{\rm v}\big((\mathcal{A}\mathcal{A}^t-Id)s\nabla X_t\big)(s)\right\|_{L^2}{\rm d}s\\&\quad+t\Big(\int_0^{\frac t2}+\int_{\frac t2}^t\Big)2^je^{-c(t-s)2^{2j}}\|\Delta_j\Delta_\ell^{\rm v}\hbar(s)\|_{L^2}\,{\rm d}s.\end{aligned} \tag{7.15}$$

By virtue of (7.13), we have

$$\int_0^t 2^{2j}(t-s)e^{-c(t-s)2^{2j}}\left\|\Delta_j\Delta_\ell^{\rm v}\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X_t\big)(s)\right\|_{L^2}{\rm d}s$$

$$\lesssim \big\|\Delta_j\Delta_\ell^{\mathrm{v}}\big((\mathcal{A}\mathcal{A}^t-Id)\nabla X_t\big)\big\|_{L^1_t(L^2)}\lesssim c_{j,\ell}\delta_1\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}.$$

Along the same lines, we have

$$\begin{aligned}&\int_0^t 2^{2j}e^{-c(t-s)2^{2j}}\big\|\Delta_j\Delta_\ell^{\mathrm{v}}\big((\mathcal{A}\mathcal{A}^t-Id)s\nabla X_t\big)(s)\big\|_{L^2}\,\mathrm{d}s\\&\quad\lesssim\big\|\Delta_j\Delta_\ell^{\mathrm{v}}\big((\mathcal{A}\mathcal{A}^t-Id)s\nabla X_t\big)\big\|_{L^\infty_t(L^2)}\\&\quad\lesssim c_{j,\ell}\|(\mathcal{A}\mathcal{A}^t-Id)\|_{L^\infty_t\left(\mathcal{B}_1^{1,\frac12}\right)}\|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}\lesssim c_{j,\ell}\delta_1\|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}.\end{aligned}\tag{7.16}$$

Meanwhile, it is easy to observe from Lemma 7.1 that

$$\begin{aligned}&t\int_0^{\frac t2}2^je^{-c(t-s)2^{2j}}\|\Delta_j\Delta_\ell^{\mathrm{v}}\hbar(s)\|_{L^2}\,\mathrm{d}s\\&\quad\lesssim\int_0^{\frac t2}(t-s)2^{2j}e^{-c(t-s)2^{2j}}\|\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar(s)\|_{L^2}\,\mathrm{d}s\\&\quad\lesssim\|\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar\|_{L^1_t(L^2)},\end{aligned}$$

and

$$\begin{aligned}&t\int_{\frac t2}^t 2^je^{-c(t-s)2^{2j}}\|\Delta_j\Delta_\ell^{\mathrm{v}}\hbar(s)\|_{L^2}\,\mathrm{d}s\\&\quad\lesssim\int_{\frac t2}^t\langle t-s\rangle^{-1}\big(2^{-j}+2^j\big)\|s\Delta_j\Delta_\ell^{\mathrm{v}}\hbar(s)\|_{L^2}\,\mathrm{d}s\\&\quad\lesssim\int_{\frac t2}^t\langle t-s\rangle^{-1}\left(\|s\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar(s)\|_{L^2}+\|s\Delta_j\Delta_\ell^{\mathrm{v}}|D|\hbar(s)\|_{L^2}\right)\mathrm{d}s.\end{aligned}$$

Substituting the above estimates into (7.15) leads to

$$\begin{aligned}&2^jt\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t(t)\|_{L^2}\\&\quad\lesssim c_{j,\ell}\delta_1\big(\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}+\|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}\big)+\|\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar\|_{L^1_t(L^2)}\\&\qquad+\int_{\frac t2}^t\langle t-s\rangle^{-1}\left(\|s\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar(s)\|_{L^2}+\|s\Delta_j\Delta_\ell^{\mathrm{v}}|D|\hbar(s)\|_{L^2}\right)\mathrm{d}s\end{aligned}\tag{7.17}$$

for all (j,ℓ) satisfying $j\leq\frac{\ell+1}{2}$.

• <u>When $j>\frac{\ell+1}{2}$</u>

In this case, we have

$$g_{j,\ell}^2(t)\sim\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t(t)\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\Delta X(t)\|_{L^2}^2,$$

and Lemma 7.1 implies that

$$\begin{aligned}&\frac34\|\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2+\frac14\|\partial_3\nabla\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2\\&\quad\geq c\left(2^{2j}\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2+2^{2j}2^{2\ell}\|\Delta_j\Delta_\ell^{\mathrm{v}}X\|_{L^2}^2\right)\\&\quad\geq c\frac{2^{2\ell}}{2^{2j}}\left(\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^2}^2+\|\Delta_j\Delta_\ell^{\mathrm{v}}\Delta X\|_{L^2}^2\right).\end{aligned}$$

Then we deduce from (7.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{j,\ell}(t) + c2^{2(\ell-j)} g_{j,\ell}(t) \le \|\Delta_j \Delta_\ell^{\mathrm{v}} f(t)\|_{L^2},$$

which implies that

$$g_{j,\ell}(t) \le \int_0^t e^{-c(t-s)2^{2(\ell-j)}} \|\Delta_j \Delta_\ell^{\mathrm{v}} f(s)\|_{L^2}\,\mathrm{d}s \tag{7.18}$$

and

$$2^{2\ell} \|\Delta_j \Delta_\ell^{\mathrm{v}} X\|_{L^1_t(L^2)} \lesssim \|\Delta_j \Delta_\ell^{\mathrm{v}} f\|_{L^1_t(L^2)}. \tag{7.19}$$

On the other hand, we get, by taking L^2 inner product of (7.1) with $\Delta_j \Delta_\ell^{\mathrm{v}} X_t$, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta_j \Delta_\ell^{\mathrm{v}} X_t\|_{L^2}^2 + \|\nabla \Delta_j \Delta_\ell^{\mathrm{v}} X_t\|_{L^2}^2 = \big(\partial_3^2 \Delta_j \Delta_\ell^{\mathrm{v}} X + \Delta_j \Delta_\ell^{\mathrm{v}} f \mid \Delta_j \Delta_\ell^{\mathrm{v}} X_t\big)_{L^2},$$

from which, with Lemma 7.1, we infer

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta_j \Delta_\ell^{\mathrm{v}} X_t(t)\|_{L^2} + c2^{2j} \|\Delta_j \Delta_\ell^{\mathrm{v}} X_t(t)\|_{L^2} \\ &\lesssim 2^{2\ell} \|\Delta_j \Delta_\ell^{\mathrm{v}} X(t)\|_{L^2} + \|\Delta_j \Delta_\ell^{\mathrm{v}} f(t)\|_{L^2}, \end{aligned}$$

so that it holds that

$$\begin{aligned} 2^j \|\Delta_j \Delta_\ell^{\mathrm{v}} X_t(t)\|_{L^2} \lesssim\ & 2^{2\ell+j} \int_0^t e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^{\mathrm{v}} X(s)\|_{L^2}\,\mathrm{d}s \\ & + 2^j \int_0^t e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^{\mathrm{v}} f(s)\|_{L^2}\,\mathrm{d}s. \end{aligned} \tag{7.20}$$

Then we deduce from (7.19) that for $j > \frac{\ell+1}{2}$

$$\begin{aligned} 2^j \|\Delta_j \Delta_\ell^{\mathrm{v}} X_t\|_{L^1_t(L^2)} &\lesssim 2^{2\ell-j} \|\Delta_j \Delta_\ell^{\mathrm{v}} X\|_{L^1_t(L^2)} + 2^{-j} \|\Delta_j \Delta_\ell^{\mathrm{v}} f\|_{L^1_t(L^2)} \\ &\lesssim 2^{-j} \|\Delta_j \Delta_\ell^{\mathrm{v}} f\|_{L^1_t(L^2)}. \end{aligned} \tag{7.21}$$

Moreover, in this case, it follows from Lemma 7.1 and (7.18) that

$$\begin{aligned} &2^{2\ell+j} t \int_0^t e^{-c2^{2j}(t-s)} \|\Delta_j \Delta_\ell^{\mathrm{v}} X(s)\|_{L^2}\,\mathrm{d}s \\ &\lesssim 2^{2\ell-j} t \|\Delta_j \Delta_\ell^{\mathrm{v}} X\|_{L^\infty_t(L^2)} \\ &\lesssim 2^{2\ell-3j} t \|\Delta_j \Delta_\ell^{\mathrm{v}} \Delta X\|_{L^\infty_t(L^2)} \lesssim 2^{2\ell-3j} t \|g_{j,\ell}\|_{L^\infty_t} \\ &\lesssim 2^{2\ell-3j} t \int_0^t e^{-c(t-s)2^{2(\ell-j)}} \|\Delta_j \Delta_\ell^{\mathrm{v}} f(s)\|_{L^2}\,\mathrm{d}s, \end{aligned}$$

from which, in a proof similar to that of (7.17), we infer

$$\begin{aligned}
&2^{2\ell+j}t\int_0^t e^{-c2^{2j}(t-s)}\|\Delta_j\Delta_\ell^{\mathrm{v}}X(s)\|_{L^2}\,\mathrm{d}s\\
&\quad\lesssim c_{j,\ell}\delta_1\big(\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}+\|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}\big)\\
&\qquad+\|\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar\|_{L^1_t(L^2)}\\
&\qquad+\int_{\frac{t}{2}}^t\langle t-s\rangle^{-1}\big(\|s\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar(s)\|_{L^2}+\|s\Delta_j\Delta_\ell^{\mathrm{v}}|D|\hbar(s)\|_{L^2}\big)\,\mathrm{d}s.
\end{aligned}\tag{7.22}$$

Here we use the fact $j\geq\ell-N_0$ for some fixed integer N_0 in the operator $\Delta_j\Delta_\ell^{\mathrm{v}}$.

By virtue of (7.20) and (7.22), we get, by a similar derivation of (7.17), that (7.17) holds for all $(j,\ell)\in\mathbb{Z}^2$. Furthermore, in view of (7.12)-(7.21), we obtain for all $(j,\ell)\in\mathbb{Z}^2$ that

$$2^j\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|_{L^1_t(L^2)}\lesssim 2^{-j}\|\Delta_j\Delta_\ell^{\mathrm{v}}f\|_{L^1_t(L^2)}.\tag{7.23}$$

Inserting (7.14) into (7.23) gives rise to

$$\begin{aligned}
\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}&=\Big(\sum_{j,\ell\in\mathbb{Z}^2}2^{2j}\|\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|^2_{L^1_t(L^2)}\Big)^{\frac12}\\
&\leq C\delta_1\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}+C\Big(\sum_{j,\ell\in\mathbb{Z}^2}\|\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar\|^2_{L^1_t(L^2)}\Big)^{\frac12}\\
&\leq C\delta_1\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}+C\int_0^t\Big(\sum_{j,\ell\in\mathbb{Z}^2}\|\Delta_j\Delta_\ell^{\mathrm{v}}|D|^{-1}\hbar(s)\|^2_{L^2}\Big)^{\frac12}\mathrm{d}s\\
&\leq C\left(\delta_1\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}+\||D|^{-1}\hbar\|_{L^1_t(L^2)}\right).
\end{aligned}$$

In particular, by taking δ_1 to be sufficiently small in (7.2), we conclude that

$$\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}\leq C\||D|^{-1}\hbar\|_{L^1_t(L^2)}.\tag{7.24}$$

Along the same lines, we deduce from (7.17) that

$$\begin{aligned}
\|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}&=\Big(\sum_{j,\ell\in\mathbb{Z}^2}2^{2j}\|t\Delta_j\Delta_\ell^{\mathrm{v}}X_t\|^2_{L^\infty_t(L^2)}\Big)^{\frac12}\\
&\leq C\bigg(\delta_1\Big(\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}+\|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}\Big)+\||D|^{-1}\hbar\|_{L^1_t(L^2)}\\
&\qquad+\int_{\frac{t}{2}}^t\langle t-s\rangle^{-1}\Big(\|s|D|^{-1}\hbar(s)\|_{L^2}+\|s|D|\hbar(s)\|_{L^2}\Big)\,\mathrm{d}s\bigg).
\end{aligned}\tag{7.25}$$

Thus, by taking that δ_1 is small enough in (7.2), we obtain

$$\begin{aligned} t\|\nabla X_t(t)\|_{L^2} &\leq \|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})} \\ &\leq C\Big(\||D|^{-1}\hbar\|_{L^1_t(L^2)} \\ &\quad + \int_{\frac{t}{2}}^t \langle t-s\rangle^{-1}\big(\|s|D|^{-1}\hbar(s)\|_{L^2} + \|s|D|\hbar(s)\|_{L^2}\big)\,\mathrm{d}s\Big) \\ &\leq C_\varepsilon\big(\sup_{s\in[0,t]}\|s^{1+\varepsilon}|D|^{-1}\hbar\|_{L^2} + \sup_{s\in[0,t]}\|s^{1+\varepsilon}|D|\hbar\|_{L^2}\big), \end{aligned} \tag{7.26}$$

which leads to (7.3).

The proof of the general estimates in (7.4) follow along the same lines. Indeed for any $k \geq 1$, we have

$$\begin{aligned} &\big\|D^k\big((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\big)\big\|_{\widetilde{L}^1_t(\dot{H}^{0,0})} \\ &\lesssim C_k \sum_{k_1+k_2=k}\|D^{k_1}(\mathcal{A}\mathcal{A}^t - Id)\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\|D^{k_2}\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})} \\ &\lesssim C_k \sum_{k_1+k_2=k}\|D^{k_1}\nabla Y\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\|D^{k_2}\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})}, \end{aligned}$$

from which, along with a similar derivation of (7.24), we inductively infer that

$$\begin{aligned} &\|D^k\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})} \leq C\||D|^{k-1}\hbar\|_{L^1_t(L^2)} \\ &+ C_k \sum_{k_1+\cdots+k_\ell=k}\|D^{k_1}\nabla Y\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\cdots\|D^{k_\ell}\nabla Y\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\||D|^{-1}\hbar\|_{L^1_t(L^2)}. \end{aligned}$$

Hence by applying the interpolation inequality, which says that

$$\|D^{k_i}\nabla Y\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)} \lesssim \|\nabla Y\|^{1-k_i/k}_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\|D^k\nabla Y\|^{k_i/k}_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)} \quad \text{for } 0 \leq k_i \leq k,$$

and assumption (7.2), we obtain

$$\begin{aligned} &\|D^k\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})} \\ &\leq C_k\Big(\big(\delta_1 + \|D^k\nabla Y\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\big)\||D|^{-1}\hbar\|_{L^1_t(L^2)} + \||D|^{k-1}\hbar\|_{L^1_t(L^2)}\Big). \end{aligned} \tag{7.27}$$

It follows from a similar derivation of (7.25) that

$$\begin{aligned} &\|tD^k\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})} \\ &\leq C_k\Big(\||D|^{k-1}\hbar\|_{L^1_t(L^2)} + \delta_1\big(\|D^k\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})} + \|tD^k\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}\big) \\ &\quad + \big(\delta_1 + \|D^k\nabla Y\|_{L^\infty_t\left(\dot{B}^{\frac{3}{2}}_{2,1}\right)}\big)\big(\|\nabla X_t\|_{\widetilde{L}^1_t(\dot{H}^{0,0})} + \|t\nabla X_t\|_{\widetilde{L}^\infty_t(\dot{H}^{0,0})}\big) \\ &\quad + \int_{\frac{t}{2}}^t \langle t-s\rangle^{-1}\big(\|s|D|^{k-1}\hbar(s)\|_{L^2} + \|s|D|^{k+1}\hbar(s)\|_{L^2}\big)\,\mathrm{d}s\Big). \end{aligned}$$

Thus (7.4) follows from (7.27) and the argument in (7.26). This completes the proof of Proposition 7.1. □

8. Estimates of the source term $f(Y)$

In this section, we shall present the estimates to the nonlinear source term $f(Y)$ determined by (2.25).

•The estimate of $|||f(Y)|||_{\delta,N}$

Proposition 8.1. *Let the functionals* f_0, f_1, f_2 *be given in* (4.3) *and the norm* $|||\cdot|||_{\delta,N}$ *by* (2.39). *Then under the assumption of* (2.41), *we have:*

$$|||f_0(Y)|||_{\delta,N} \lesssim \|\nabla Y\|_0\|\nabla Y_t\|_{N+6} + \|\nabla Y\|_{N+6}\|\nabla Y_t\|_0; \tag{8.1}$$

$$|||f_1(Y)|||_{\delta,N} \lesssim \|\partial_3 Y\|_0\|\partial_3 Y\|_{N+6} + \|\nabla Y\|_{N+6}|\partial_3 Y|_0\|\partial_3 Y\|_1; \tag{8.2}$$

$$|||f_2(Y)|||_{\delta,N} \lesssim \|Y_t\|_0\|Y_t\|_{N+6} + \|\nabla Y\|_{N+6}|Y_t|_0\|Y_t\|_1. \tag{8.3}$$

Proof. As in Sect. 4, we shall deal with the estimate of $f(Y)$ by the norm of the homogeneous Besov space $\dot{B}^s_{1,1}$ instead of the one in the homogeneous Sobolev space $\dot{W}^{s,1}$. Indeed, in view of (4.3), we get, by applying the law of products (5.1), that for $s > 0$,

$$\|f_0(Y)\|_{\dot{B}^s_{1,1}} \lesssim \|(\mathcal{A}^t\mathcal{A} - Id)\nabla Y_t\|_{\dot{B}^{s+1}_{1,1}} \lesssim \|\nabla Y\|_0\|\nabla Y_t\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^{s+1}_{2,1}}\|\nabla Y_t\|_0.$$

We then have that (8.1) follows from the above inequality and the interpolation inequality (5.22). Along the same lines, we deduce from (4.3) that

$$\begin{aligned}\|f_m(Y)\|_{\dot{B}^s_{1,1}} &\lesssim (1 + |\mathcal{A}^t - Id|_0)\|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} + \|\mathcal{A} - Id\|_{\dot{B}^s_{2,1}}\|\nabla \boldsymbol{p}_m\|_0\\ &\lesssim \|\nabla \boldsymbol{p}_m\|_{\dot{B}^s_{1,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}\|\nabla \boldsymbol{p}_m\|_0.\end{aligned}$$

However, it follows from (4.1) that

$$\begin{aligned}\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{1,1}} \lesssim\ &\delta_1\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{1,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}}\|\nabla \boldsymbol{p}_1\|_0 + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{B}^{s+1}_{1,1}}\\ &+ \|\nabla Y\|_{\dot{B}^s_{2,1}}\|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{H}^1},\end{aligned}$$

which, together with (5.6), implies

$$\|\nabla \boldsymbol{p}_1\|_{\dot{B}^s_{1,1}} \lesssim \|\partial_3 Y\|_0\|\partial_3 Y\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}\cap\dot{B}^{s+1}_{2,1}}|\partial_3 Y|_0\|\partial_3 Y\|_1.$$

As a result, we have that

$$\|f_1(Y)\|_{\dot{B}^s_{1,1}} \lesssim \|\partial_3 Y\|_0\|\partial_3 Y\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}\cap\dot{B}^{s+1}_{2,1}}|\partial_3 Y|_0\|\partial_3 Y\|_1.$$

Similarly, we have

$$\|f_2(Y)\|_{\dot{B}^s_{1,1}} \lesssim \|Y_t\|_0\|Y_t\|_{\dot{B}^{s+1}_{2,1}} + \|\nabla Y\|_{\dot{B}^s_{2,1}\cap\dot{B}^{s+1}_{2,1}}|Y_t|_0\|Y_t\|_1.$$

Then (8.2) and (8.3) then follow from the above estimates and the interpolation inequality (5.22). This completes the proof of Proposition 8.1. □

•The estimate of $\||D|^{-1}f(Y)\|_{N+1}$

Proposition 8.2. *Under the same assumptions of Proposition 8.1, we have*

$$\||D|^{-1}f_0(Y)\|_{N+1} \lesssim |\nabla Y|_0\|\nabla Y_t\|_{N+1} + |\nabla Y|_{N+1}\|\nabla Y_t\|_0; \tag{8.4}$$

$$\||D|^{-1}f_1(Y)\|_{N+1} \lesssim |\partial_3 Y|_0\|\partial_3 Y\|_{N+1} + |\nabla Y|_{N+1}|\partial_3 Y|_0\|\partial_3 Y\|_1; \tag{8.5}$$

$$\||D|^{-1}f_2(Y)\|_{N+1} \lesssim |Y_t|_0\|Y_t\|_{N+1} + |\nabla Y|_{N+1}|Y_t|_0\|Y_t\|_1. \tag{8.6}$$

Proof. In view of (4.3), we get, by applying Moser type inequality, that

$$\||D|^{-1}f_0(Y)\|_N \leq \|(\mathcal{A}^t\mathcal{A} - Id)\nabla Y_t\|_N \lesssim |\nabla Y|_0\|\nabla Y_t\|_N + |\nabla Y|_N\|\nabla Y_t\|_0,$$

which gives (8.4). Meanwhile, again by (4.3) and the law of products in Besov spaces, one has

$$\||D|^{-1}f_m(Y)\|_0 \lesssim \big(1 + \|\mathcal{A}^t - Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\big)\|\nabla \boldsymbol{p}_m\|_{\dot{H}^{-1}},$$

yet it follows from (4.1) that

$$\|\nabla \boldsymbol{p}_1\|_{\dot{H}^{-1}} \lesssim \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\nabla \boldsymbol{p}_1\|_{\dot{H}^{-1}} + (1 + \|\mathcal{A} - Id\|_{\dot{B}^{\frac{3}{2}}_{2,1}})\|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_0,$$

from which, with the assumption (2.41), we infer

$$\||D|^{-1}f_1(Y)\|_0 \lesssim \|\nabla \boldsymbol{p}_1\|_{\dot{H}^{-1}} \lesssim |\partial_3 Y|_0\|\partial_3 Y\|_0. \tag{8.7}$$

Similarly, we have

$$\||D|^{-1}f_2(Y)\|_0 \lesssim \|\nabla \boldsymbol{p}_2\|_{\dot{H}^{-1}} \lesssim |Y_t|_0\|Y_t\|_0. \tag{8.8}$$

For $N \geq 0$, we deduce from (4.3) that

$$\|f_1(Y)\|_N \lesssim \|\nabla \boldsymbol{p}_1\|_N + |\nabla Y|_N\|\nabla \boldsymbol{p}_1\|_0,$$

and it follows from (4.1) that

$$\|\nabla \boldsymbol{p}_1\|_N \lesssim |\nabla Y|_0\|\nabla \boldsymbol{p}_1\|_N + |\nabla Y|_N\|\nabla \boldsymbol{p}_1\|_0 + \|\mathcal{A}\mathrm{div}\big(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\big)\|_N,$$

which, together with (2.41) and (5.6), ensures that

$$\|\nabla \boldsymbol{p}_1\|_N \lesssim |\partial_3 Y|_0\|\partial_3 Y\|_{N+1} + |\nabla Y|_{N+1}|\partial_3 Y|_0\|\partial_3 Y\|_1.$$

As a result,

$$\|f_1(Y)\|_N \lesssim |\partial_3 Y|_0\|\partial_3 Y\|_{N+1} + |\nabla Y|_{N+1}|\partial_3 Y|_0\|\partial_3 Y\|_1. \tag{8.9}$$

The same procedure for $f_2(Y)$ yields

$$\|f_2(Y)\|_N \lesssim |Y_t|_0\|Y_t\|_{N+1} + |\nabla Y|_{N+1}|Y_t|_0\|Y_t\|_1. \tag{8.10}$$

(8.5) and (8.6) follow from (8.7)-(8.10). This completes the proof of Proposition 8.2. □

9. The Proof of Theorem 2.1

The goal of this section is to prove Theorem 2.1 by using the Nash–Moser scheme. The key ingredients are the uniform estimates of the approximate solutions obtained in Propositions 2.8, 2.9 and 2.10, which we will prove by induction in what follows.

9.1. The Estimates of Y_0

Recall that Y_0 solves the linear equation (2.72). Let $\bar{N}_0 = N_0+6$, for $\eta \in]0,1[$, we choose the initial data $(Y^{(0)}, Y^{(1)})$ such that (2.20) holds for $L_0 = N_0 + 12$. Then we get, by applying (2.32) of Proposition 2.1, that

$$\begin{aligned}&|\partial_3 Y_0|_{1,\bar{N}_0} + |\partial_t Y_0|_{\frac{3}{2}-\delta,\bar{N}_0} + |Y_0|_{\frac{1}{2},\bar{N}_0}\\ &\quad\le C_{N_0}\big(\||D|^{2\delta}(Y^{(0)}, Y^{(1)})\|_{L^1} + \||D|^{\bar{N}_0+4}(|D|^2 Y^{(0)}, Y^{(1)})\|_{L^1}\big) \le \eta.\end{aligned} \tag{9.1}$$

Note that

$$\||D|^{-1}h\|_0 \le \Big(\int_{|\xi|\le 1}\frac{1}{|\xi|^2}|\hat{h}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} + \|h\|_0 \le |\hat{h}|_0 + \|h\|_0 \le \|h\|_{L^1} + \|h\|_0,$$

so that we get, by applying (2.33), (2.34) and (2.35) of Proposition 2.1, that

$$\begin{aligned}&\||D|^{-1}(\partial_3 Y_0, \partial_t Y_0)\|_{0,\bar{N}_0+2} + \|\nabla Y_0\|_{0,\bar{N}_0+1} + \|\nabla\partial_t Y_0\|_{1,\bar{N}_0-1}\\ &\quad + \|(\partial_t Y_0, \partial_3 Y_0)\|_{\frac{1}{2},\bar{N}_0+1} + \|\partial_t Y_0\|_{L^2_t(H^{\bar{N}_0+2})}\\ &\quad + \big\|(\partial_3 Y_0, \langle t\rangle^{\frac{1}{2}}\nabla\partial_t Y_0)\big\|_{L^2_t(H^{\bar{N}_0+1})}\\ &\le C_{\bar{N}_0}\||D|^{-1}(\partial_3 Y^{(0)}, Y^{(1)}, \Delta Y^{(0)})\|_{\bar{N}_0+2}\\ &\le C_{\bar{N}_0}\big(\|(\partial_3 Y^{(0)}, Y^{(1)}, \Delta Y^{(0)})\|_{L^1} + \|(\partial_3 Y^{(0)}, Y^{(1)}, \Delta Y^{(0)})\|_{\bar{N}_0+1}\big) \le \eta.\end{aligned} \tag{9.2}$$

By virtue of (9.1) and (9.2), we deduce from Proposition 8.2 that

$$\begin{aligned}&\|\langle t\rangle|D|^{-1}f(Y_0)\|_{L^2_t(H^{N_0+1})} \lesssim |\partial_3 Y_0|_{1,0}\|\partial_3 Y_0\|_{L^2_t(H^{N_0+1})}\\ &\quad + |\partial_t Y_0|_{1,0}\|\partial_t Y_0\|_{L^2_t(H^{N_0+1})}\\ &\quad + |\nabla Y_0|_{\frac{1}{2},0}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t Y_0\|_{L^2_t(H^{N_0+1})} + |\nabla Y_0|_{\frac{1}{2},N_0+1}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t Y_0\|_{L^2_t(L^2)}\\ &\quad + |\nabla Y_0|_{0,N_0+1}\big(|\partial_3 Y_0|_{1,0}\|\partial_3 Y_0\|_{L^2_t(H^1)} + |\partial_t Y_0|_{1,0}\|\partial_t Y_0\|_{L^2_t(H^1)}\big) \lesssim C_{N_0}\eta^2,\end{aligned} \tag{9.3}$$

and

$$\begin{aligned}&\||D|^{-1}f(Y_0)\|_{\frac{3}{2},N_0+1} \lesssim |\partial_3 Y_0|_{1,0}\|\partial_3 Y_0\|_{\frac{1}{2},N_0+1} + |\partial_t Y_0|_{1,0}\|\partial_t Y_0\|_{\frac{1}{2},N_0+1}\\ &\quad + |\nabla Y_0|_{\frac{1}{2},0}\|\nabla\partial_t Y_0\|_{1,N_0+1} + |\nabla Y_0|_{\frac{1}{2},N_0+1}\|\nabla\partial_t Y_0\|_{1,0}\\ &\quad + |\nabla Y_0|_{0,N_0+1}\big(|\partial_3 Y_0|_{1,0}\|\partial_3 Y_0\|_{\frac{1}{2},1} + |\partial_t Y_0|_{1,0}\|\partial_t Y_0\|_{\frac{1}{2},1}\big) \lesssim C_{N_0}\eta^2.\end{aligned} \tag{9.4}$$

Similarly, we deduce from Proposition 8.1 and (9.1) and (9.2) that

$$\begin{aligned}
&\|\langle t\rangle^{\frac12} f(Y_0)\|_{L^2_t(\delta,N_0)} \lesssim \|\nabla Y_0\|_{0,0}\|\langle t\rangle^{\frac12}\nabla\partial_t Y_0\|_{L^2_t(H^{N_0+6})}\\
&+\|\nabla Y_0\|_{0,N_0+6}\|\langle t\rangle^{\frac12}\nabla\partial_t Y_0\|_{L^2_t(L^2)}\\
&+\|\partial_3 Y_0\|_{\frac12,0}\|\partial_3 Y_0\|_{L^2_t(H^{N_0+6})}+\|\partial_t Y_0\|_{\frac12,0}\|\partial_t Y_0\|_{L^2_t(H^{N_0+6})}\\
&+\|\nabla Y_0\|_{0,N_0+6}\big(|\partial_3 Y_0|_{\frac12,0}\|\partial_3 Y_0\|_{L^2_t(H^1)}+|\partial_t Y_0|_{\frac12,0}\|\partial_t Y_0\|_{L^2_t(H^1)}\big)\lesssim C_{N_0}\eta^2.
\end{aligned}\tag{9.5}$$

9.2. The Proof of Proposition 2.9 and Proposition 2.10 from Proposition 2.8

Let us assume that

$$(\mathrm{P1},\ j),\ (\mathrm{P2},\ j),\ (\mathrm{P3},\ j) \text{ of Proposition 2.8 hold for } j\le p. \tag{9.6}$$

We are going to prove Proposition 2.9 and Proposition 2.10.

Proof of Proposition 2.9. Notice from (2.75) that

$$|\partial_3 Y_{p+1}|_{k,N}\le|\partial_3 Y_0|_{k,N}+\sum_{j=0}^{p}|\partial_3 X_j|_{k,N},$$

which, together with (9.1) and (P2, j) with $j\le p$, ensures that for $\frac12\le k\le 1$, $0\le N\le N_0$,

$$\begin{aligned}
&|\partial_3 Y_{p+1}|_{k,N}\le C\eta\theta_{p+1}^{k-\frac12-\gamma+\bar\varepsilon N}, && \text{if } k-\frac12-\gamma+\bar\varepsilon N\ge\bar\varepsilon,\\
&|\partial_3 Y_{p+1}|_{k,N}\le C\eta, && \text{if } k-\frac12-\gamma+\bar\varepsilon N\le-\bar\varepsilon.
\end{aligned}\tag{9.7}$$

For $\hat k\overset{\text{def}}{=}\min(k,1)$, $\hat N\overset{\text{def}}{=}\min(N,N_0)$, we observe from the property (S I) of smoothing operator S_{p+1} that

$$\begin{aligned}
&|S_{p+1}\partial_3 Y_{p+1}|_{k,N}\le C|\partial_3 Y_{p+1}|_{k,N}\quad\text{for}\quad \frac12\le k\le 1, 0\le N\le N_0,\\
&|S_{p+1}\partial_3 Y_{p+1}|_{k,N}\\
&\quad\le C_{k,N}\theta_{p+1}^{\max(0,k-\hat k)}\theta_{p+1}^{\bar\varepsilon\max(0,N-\hat N)}|\partial_3 Y_{p+1}|_{\hat k,\hat N}\quad\text{for}\quad k\ge 1\ \text{ or }\ N\ge N_0;
\end{aligned}$$

the first inequalities of (I)(i) and (II)(i) of Proposition 2.9 then follow from (9.7).

Along the same lines as the proof of (9.7), we have:

- for $1-\delta\le k\le\frac32-\delta$, $0\le N\le N_0$,

$$\begin{aligned}
&|\partial_t Y_{p+1}|_{k,N}\le C\eta\theta_{p+1}^{k-(1-\delta)-\gamma+\bar\varepsilon N}, && \text{if } k-(1-\delta)-\gamma+\bar\varepsilon N\ge\bar\varepsilon,\\
&|\partial_t Y_{p+1}|_{k,N}\le C\eta, && \text{if } k-(1-\delta)-\gamma+\bar\varepsilon N\le-\bar\varepsilon;
\end{aligned}\tag{9.8}$$

- for $0 \le k \le \frac{1}{2}$, $0 \le N \le N_0$,

$$
\begin{aligned}
&|Y_{p+1}|_{k,N} \le C\eta\theta_{p+1}^{k-\gamma+\bar{\varepsilon}N}, \quad \text{if } k-\gamma+\bar{\varepsilon}N \ge \bar{\varepsilon},\\
&|Y_{p+1}|_{k,N} \le C\eta, \qquad \text{if } k-\gamma+\bar{\varepsilon}N \le -\bar{\varepsilon}.
\end{aligned} \tag{9.9}
$$

Then other inequalities in (I)(i) and (II)(i) of Proposition 2.9 follow.

(I)(ii) and (II)(ii) of Proposition 2.9 follow from property (S I) of the mollifying operator and the following fact:

$$
\begin{aligned}
&\big\||D|^{-1}(\partial_3 Y_{p+1}, \partial_t Y_{p+1})\big\|_{0,N+2} + \|\nabla Y_{p+1}\|_{0,N+1} + \|\partial_t Y_{p+1}\|_{L^2_t(H^{N+2})}\\
&\quad + \|(\partial_t Y_{p+1}, \partial_3 Y_{p+1})\|_{\frac{1}{2},N+1} + \|\nabla\partial_t Y_{p+1}\|_{1,N-1}\\
&\quad + \big\|(\partial_3 Y_{p+1}, \langle t\rangle^{\frac{1}{2}}\nabla\partial_t Y_{p+1})\big\|_{L^2_t(H^{N+1})}\\
&\le \begin{cases} C\eta\theta_{p+1}^{-\beta+\bar{\varepsilon}N}, & \text{for } -\beta+\bar{\varepsilon}N \ge \bar{\varepsilon},\ N \le N_0,\\ C\eta, & \text{for } -\beta+\bar{\varepsilon}N \le -\bar{\varepsilon},\ N \le N_0, \end{cases}
\end{aligned} \tag{9.10}
$$

which is a direct consequence of (P1, j) of Proposition 2.8 for $j \le p$ and (9.2).

Finally let us prove (III) of Proposition 2.9. Indeed it follows from property (S II) of S_{p+1} that

$$
|(1-S_{p+1})\partial_3 Y_{p+1}|_{\frac{1}{2},0} \le C\big(\theta_{p+1}^{-\frac{1}{2}}|\partial_3 Y_{p+1}|_{1,0} + \theta_{p+1}^{-\bar{\varepsilon}N_0}|\partial_3 Y_{p+1}|_{\frac{1}{2},N_0}\big).
$$

Due to (2.83) and (2.84), there hold $\frac{1}{2}-\gamma \ge \bar{\varepsilon}$ and $-\gamma+\bar{\varepsilon}N_0 \ge \bar{\varepsilon}$, so that we can apply (9.7) to deduce that

$$
|(1-S_{p+1})\partial_3 Y_{p+1}|_{\frac{1}{2},0} \le C\eta\big(\theta_{p+1}^{-\frac{1}{2}}\theta_{p+1}^{\frac{1}{2}-\gamma} + \theta_{p+1}^{-\bar{\varepsilon}N_0}\theta_{p+1}^{-\gamma+\bar{\varepsilon}N_0}\big) \le C\eta\theta_{p+1}^{-\gamma}. \tag{9.11}
$$

Using (9.7) once again gives rise to

$$
|(1-S_{p+1})\partial_3 Y_{p+1}|_{1,N} \le C|\partial_3 Y_{p+1}|_{1,N} \le \eta\theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N} \qquad \text{for } \ 0 \le N \le N_0, \tag{9.12}
$$

$$
|(1-S_{p+1})\partial_3 Y_{p+1}|_{k,N_0} \le C|\partial_3 Y_{p+1}|_{k,N_0} \le \eta\theta_{p+1}^{k-\frac{1}{2}-\gamma+\bar{\varepsilon}N_0} \quad \text{for } \ \frac{1}{2} \le k \le 1. \tag{9.13}
$$

Interpolating between (9.11), (9.12) and (9.13) leads to

$$
|(1-S_{p+1})\partial_3 Y_{p+1}|_{k,N} \le C\eta\theta_{p+1}^{k-\frac{1}{2}-\gamma+\bar{\varepsilon}N}, \quad \text{for all } \frac{1}{2} \le k \le 1,\ 0 \le N \le N_0.
$$

The other two inequalities in (III) of Proposition 2.9 can be proved by the same procedure. This completes the proof of Proposition 2.9. □

Let us now turn to the proof of Proposition 2.10.

Proof of Proposition 2.10. We shall divide the proof of this proposition in a number of steps:

Step 1. The Proof of (IV) of Proposition 2.10. The proof of (IV) will be based on the following lemmas:

Lemma 9.1. *Let* $e'_{p,j}, e''_{p,j}$, *for* $j = 0, 1, 2$, *be given by* (2.81). *Then under the assumption of* (9.6), *one has*

$$\|\langle t\rangle^{\frac{1}{2}+k}|D|^{-1}(e''_{p,1}+e''_{p,2})\|_{L^2_t(H^{N+1})} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \le k \le \frac{1}{2}, 0 \le N \le N_0 - 1; \tag{9.14}$$

$$\|\langle t\rangle^{\frac{1}{2}+k}|D|^{-1}(e'_{p,1}+e'_{p,2})\|_{L^2_t(H^{N+1})} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \le k \le \frac{1}{2}, 0 \le N \le N_0 - 1; \tag{9.15}$$

$$\|\langle t\rangle^{\frac{1}{2}+k}|D|^{-1}e''_{p,0}\|_{L^2_t(H^{N+1})} \lesssim \eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \quad \text{if } 0 \le k \le \alpha, 0 \le N \le N_0 - 2; \tag{9.16}$$

$$\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}e'_{p,0}\|_{L^2_t(H^{N+1})} \lesssim \eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \quad \text{if } 0 \le k \le \alpha, 0 \le N \le N_0 - 2. \tag{9.17}$$

Lemma 9.2. *Under the assumption of Lemma 9.1, one has*

$$\||D|^{-1}(e''_{p,1}+e''_{p,2})\|_{1+k,N+1} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \le k \le \frac{1}{2}, 0 \le N \le N_0 - 1; \tag{9.18}$$

$$\||D|^{-1}(e'_{p,1}+e'_{p,2})\|_{1+k,N+1} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \le k \le \frac{1}{2}, 0 \le N \le N_0 - 1; \tag{9.19}$$

$$\||D|^{-1}e''_{p,0}\|_{1+k,N+1} \lesssim \eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \quad \text{if } 0 \le k \le \frac{1}{2} - \delta, N \le N_0 - 2; \tag{9.20}$$

$$\||D|^{-1}e'_{p,0}\|_{1+k,N+1} \lesssim \eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \quad \text{if } 0 \le k \le \frac{1}{2} - \delta, N \le N_0 - 2. \tag{9.21}$$

Lemma 9.3. *Under the assumption of Lemma 9.1, for* $0 \le N \le N_0 - 6$, *there hold*

$$|||(e''_{p,1}+e''_{p,2})|||_{L^1_t(\delta,N)} \lesssim \eta^2\theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)}; \tag{9.22}$$

$$|||(e'_{p,1}+e'_{p,2})|||_{L^1_t(\delta,N)} \lesssim \eta^2\theta_p^{-\gamma+\bar{\varepsilon}(N+5)}; \tag{9.23}$$

$$|||\langle t\rangle^{\frac{1}{2}}e''_{p,0}|||_{L^2_t(\delta,N)} \lesssim \eta^2\theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)}; \tag{9.24}$$

$$|||\langle t\rangle^{\frac{1}{2}}e'_{p,0}|||_{L^2_t(\delta,N)} \lesssim \eta^2\theta_p^{-\gamma+\bar{\varepsilon}(N+5)}. \tag{9.25}$$

We shall postpone the proof of the above lemmas to Appendix 10. It is easy to observe that (IV) (i) follows from Lemma 9.1, (IV) (ii) from Lemma 9.2, and (IV) (iii) from Lemma 9.3.

Step 2. The proof of (V) of Proposition 2.10. Recall (2.80) that

$$g_{p+1} = -(S_{p+1} - S_p)E_p - S_{p+1}e_p + (S_{p+1} - S_p)f(Y_0).$$

In another paper, we shall handle the above term by term.

•Estimates of $S_{p+1}e_p$

It follows from (IV) of Proposition 2.10 and property (S I) that for $k \geq 0$ and $N \geq 0$,

$$\begin{aligned}
&\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}S_{p+1}e_p\|_{L^2_t(H^{N+1})} \lesssim \eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)};\\
&\||D|^{-1}S_{p+1}e_p\|_{1+k,N+1} \lesssim \eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)};\\
&|||\langle t\rangle^{\frac{1}{2}}S_{p+1}e_p|||_{L^2_t(\delta,N)} \lesssim \eta^2\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+5)}.
\end{aligned} \tag{9.26}$$

Notice that the operator S_{p+1} contains a cutoff in the variable t of size θ_{p+1}, so that

$$|||S_{p+1}e_p|||_{L^1_t(\delta,N)} \lesssim (\log\theta_{p+1})^{\frac{1}{2}}|||\langle t\rangle^{\frac{1}{2}}S_{p+1}e_p|||_{L^2_t(\delta,N)} \lesssim \eta^2\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)}. \tag{9.27}$$

•Estimates for $(S_{p+1} - S_p)E_p$

We first deduce from (IV) (i) of Proposition 2.10 that for $0 \leq k \leq \alpha$ and $0 \leq N \leq N_0 - 2$,

$$\begin{aligned}
\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}E_p\|_{L^2_t(H^{N+1})} &\leq \sum_{j=0}^{p-1}\|\langle t\rangle^{k+\frac{1}{2}}D|^{-1}e_j\|_{L^2_t(H^{N+1})}\\
&\lesssim \begin{cases} C\eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3) \geq \bar{\varepsilon};\\ C\eta^2, & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3) \leq -\bar{\varepsilon}.\end{cases}
\end{aligned} \tag{9.28}$$

In particular, due to the choice of parameters (2.83) and (2.84), it holds that

$$\frac{1}{2} - \gamma - \beta + 2\bar{\varepsilon} \geq \bar{\varepsilon}, \quad -\gamma - \beta + \bar{\varepsilon}(N_0+1) \geq \bar{\varepsilon}. \tag{9.29}$$

We deduce from (9.28) and the property (S II) of $1 - S_p$ that

$$\begin{aligned}
&\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}(S_{p+1} - S_p)E_p\|_{L^2_t(H^1)}\\
&\lesssim \theta_p^{-\alpha}\|\langle t\rangle^{\frac{1}{2}+\alpha}|D|^{-1}E_p\|_{L^2_t(H^1)} + \theta_p^{-\bar{\varepsilon}(N_0-1)}\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}E_p\|_{L^2_t(H^{N_0-1})}\\
&\lesssim \eta^2\big(\theta_p^{-\alpha}\theta_p^{\alpha+\delta-\gamma-\beta+3\bar{\varepsilon}} + \theta_p^{-\bar{\varepsilon}(N_0-1)}\theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N_0+1)}\big) \lesssim \eta^2\theta_{p+1}^{\delta-\gamma-\beta+3\bar{\varepsilon}}.
\end{aligned} \tag{9.30}$$

On the other hand, for $k \leq \alpha$, $N \leq N_0 - 2$ with $k + \delta - \gamma - \beta + \bar{\varepsilon}(N+3) \geq \bar{\varepsilon}$, we have

$$\begin{aligned}
\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}(S_{p+1} - S_p)E_p\|_{L^2_t(H^{N+1})} &\lesssim \|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}E_p\|_{L^2_t(H^{N+1})}\\
&\lesssim \eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}.
\end{aligned} \tag{9.31}$$

Interpolating between (9.30) and (9.31), we conclude that

$$\|\langle t\rangle^{k+\frac{1}{2}}|D|^{-1}(S_{p+1}-S_p)E_p\|_{L^2_t(H^{N+1})}\lesssim\eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \tag{9.32}$$

for $0\le k\le\alpha$ and $0\le N\le N_0-2$. This, together with property (S I) of S_p, ensures that (9.32) holds for any $k\ge 0$, $N\ge 0$.

Similarly we infer from (IV) (ii) of Proposition 2.10 that for $0\le k\le\frac{1}{2}-\delta$, $0\le N\le N_0-2$,

$$\begin{aligned}&\||D|^{-1}E_p\|_{1+k,N+1}\\&\le\sum_{j=0}^{p-1}\||D|^{-1}e_j\|_{1+k,N+1}\\&\lesssim\begin{cases}C\eta^2\theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}, & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)\ge\bar{\varepsilon};\\ C\eta^2, & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)\le-\bar{\varepsilon}.\end{cases}\end{aligned} \tag{9.33}$$

Then due to (9.29), we deduce from (9.33) and the property (S II) of $1-S_p$ that

$$\begin{aligned}&\||D|^{-1}(S_{p+1}-S_p)E_p\|_{1,1}\\&\lesssim\theta_p^{-\frac{1}{2}+\delta}\||D|^{-1}E_p\|_{\frac{3}{2}-\delta,1}+\theta_p^{-\bar{\varepsilon}(N_0-1)}\||D|^{-1}E_p\|_{1,N_0-1}\\&\lesssim\eta^2\left(\theta_p^{-\frac{1}{2}+\delta}\theta_p^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}}+\theta_p^{-\bar{\varepsilon}(N_0-1)}\theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N_0+1)}\right)\\&\lesssim\eta^2\theta_{p+1}^{\delta-\gamma-\beta+2\bar{\varepsilon}}.\end{aligned} \tag{9.34}$$

On the other hand, for $k\le\frac{1}{2}-\delta$, $N\le N_0-2$ such that $k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)\ge\bar{\varepsilon}$, we get

$$\begin{aligned}&\||D|^{-1}(S_{p+1}-S_p)E_p\|_{1+k,N+1}\lesssim\||D|^{-1}E_p\|_{1+k,N+1}\\&\lesssim\eta^2\theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}.\end{aligned} \tag{9.35}$$

Interpolating between the inequalities (9.34) and (9.35), we achieve (9.35) for any $0\le k\le\frac{1}{2}-\delta$, $0\le N\le N_0-2$. This, together with the property (S I) of S_p, ensures that (9.35) holds for any $k\ge 0$ and $N\ge 0$.

It follows from (IV) (iii) of Proposition 2.10 that for $N\le N_0-6$,

$$\begin{aligned}\|\langle t\rangle^{\frac{1}{2}}E_p\|_{L^2_t(\delta,N)}&\le\sum_{j=0}^{p-1}\|\langle t\rangle^{\frac{1}{2}}e_j\|_{L^2_t(\delta,N)}\lesssim\eta^2\sum_{j=0}^{p-1}\theta_j^{-\gamma+\bar{\varepsilon}(N+5)}\\&\lesssim\begin{cases}\eta^2\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+5)} & \text{if } -\gamma+\bar{\varepsilon}(N+5)\ge\bar{\varepsilon};\\ \eta^2, & \text{if } -\gamma+\bar{\varepsilon}(N+5)\le-\bar{\varepsilon},\end{cases}\end{aligned}$$

which together with the property (S I) and compact support of mollifying operator ensures that for any $N\ge 0$,

$$\|(S_{p+1}-S_p)E_p\|_{L^1_t(\delta,N)}\lesssim\begin{cases}\eta^2\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)}, & \text{if } -\gamma+\bar{\varepsilon}(N+5)\ge\bar{\varepsilon};\\ \eta^2\theta_{p+1}^{\bar{\varepsilon}}, & \text{if } -\gamma+\bar{\varepsilon}(N+5)\le-\bar{\varepsilon}.\end{cases} \tag{9.36}$$

•**Estimates for** $(S_{p+1}-S_p)f(Y_0)$

Recalling (9.29), we get, by applying (S II) and (9.3), that

$$\begin{aligned}
&\|\langle t\rangle^{\frac12}|D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{L^2_t(H^1)}\\
&\lesssim \theta_{p+1}^{-\frac12}\|\langle t\rangle|D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{L^2_t(H^1)}\\
&\quad+\theta_{p+1}^{-\bar\varepsilon N_0}\|\langle t\rangle^{\frac12}|D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{L^2_t(H^{N_0+1})}\\
&\lesssim \eta^2\big(\theta_{p+1}^{-\frac12}+\theta_{p+1}^{-\bar\varepsilon N_0}\big)\lesssim \eta^2\theta_{p+1}^{-\gamma-\beta+\bar\varepsilon},
\end{aligned}$$

whereas for $k\le\frac12$ and $N\le N_0$ with $k-\gamma-\beta+\bar\varepsilon(N+3)\ge\bar\varepsilon$, we deduce from (9.3) that

$$\begin{aligned}
\|\langle t\rangle^{k+\frac12}|D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{L^2_t(H^{N+1})}&\lesssim\|\langle t\rangle|D|^{-1}f(Y_0)\|_{L^2_t(H^{N_0+1})}\\
&\lesssim\eta^2\le\eta^2\theta_{p+1}^{k-\gamma-\beta+\bar\varepsilon(N+3)}.
\end{aligned}$$

Interpolating the above two inequalities gives rise to

$$\|\langle t\rangle^{k+\frac12}|D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{L^2_t(H^{N+1})}\le\eta^2\theta_{p+1}^{k-\gamma-\beta+\bar\varepsilon(N+3)}$$

for all $0\le k\le\frac12, 0\le N\le N_0$. This, together with the property (S I) of S_{p+1}, ensures that

$$\|\langle t\rangle^{k+\frac12}|D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{L^2_t(H^{N+1})}\le\eta^2\theta_{p+1}^{k-\gamma-\beta+\bar\varepsilon(N+3)}\tag{9.37}$$

for all $k\ge0$ and $N\ge0$.

Along the same lines, it follows from (9.4) that for $k\ge0,\ N\ge0$,

$$\||D|^{-1}(S_{p+1}-S_p)f(Y_0)\|_{1+k,N+1}\le\eta^2\theta_{p+1}^{k-\gamma-\beta+\bar\varepsilon(N+3)}.\tag{9.38}$$

It further follows from (9.5) that if $-\gamma+\bar\varepsilon(N+5)\le-\bar\varepsilon$ (implying $N\le N_0$),

$$|||(S_{p+1}-S_p)f(Y_0)|||_{L^1_t(\delta,N)}\lesssim(\log\theta_{p+1})^{\frac12}|||\langle t\rangle^{\frac12}f(Y_0)|||_{L^2_t(\delta,N_0)}\lesssim\eta^2\theta_{p+1}^{\bar\varepsilon},$$

and if $-\gamma+\bar\varepsilon(N+5)\ge\bar\varepsilon$, one has

$$\begin{aligned}
|||(S_{p+1}-S_p)f(Y_0)|||_{L^1_t(\delta,N)}&\lesssim(\log\theta_{p+1})^{\frac12}\theta_{p+1}^{\bar\varepsilon\max(N-N_0,0)}|||\langle t\rangle^{\frac12}f(Y_0)|||_{L^2_t(\delta,N_0)}\\
&\lesssim\eta^2\theta_{p+1}^{-\gamma+\bar\varepsilon(N+6)},
\end{aligned}$$

by using (S I) and the fact that $\bar\varepsilon(N_0+5)\ge\gamma$. Along with (9.26), (9.27), (9.32), (9.35), (9.36), (9.37) and (9.38), we complete the proof of (V).

Step 3. The proof of (VI) of Proposition 2.10.

In the case when $-\gamma+\bar\varepsilon(N+5)\ge\bar\varepsilon$, we deduce from (V)(i), (V)(ii) and (V)(iii) of Proposition 2.10 that

$$R_{N,\theta_{p+1}}(g_{p+1})=|||g_{p+1}|||_{L^1_t(\delta,N)}+\theta_{p+1}^{\frac12}\big\|\langle t\rangle^{\frac12}|D|^{-1}g_{p+1}\big\|_{L^2_t(H^{N+3})}$$

$$
\begin{aligned}
&+\log\langle\theta_{p+1}\rangle \| |D|^{-1} g_{p+1} \|_{\frac{3}{2}-\delta, N+3} \\
&\lesssim \eta^2 \left(\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)} + \theta_{p+1}^{\frac{1}{2}+\delta-\gamma-\beta+\bar{\varepsilon}(N+5)} + \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+5)} \right) \\
&\lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N},
\end{aligned}
$$

provided that

$$
6\bar{\varepsilon} \leq \frac{1}{2} \text{ and } \quad \beta \geq \delta + 5\bar{\varepsilon}, \tag{9.39}
$$

which are satisfied due to (2.83) and (2.82).

On the other hand, since $-\gamma + 6\bar{\varepsilon} \leq -\bar{\varepsilon}$, we deduce from (V)(i), (V)(ii) and (V)(iv) of Proposition 2.10 that

$$
\begin{aligned}
R_{0,\theta_{p+1}}(g_{p+1}) = &\| g_{p+1} \|_{L^1_t(\delta,0)} + \theta_{p+1}^{\frac{1}{2}} \left\| \langle t\rangle^{\frac{1}{2}} |D|^{-1} g_{p+1} \right\|_{L^2_t(H^3)} \\
&+ \log\langle\theta_{p+1}\rangle \| |D|^{-1} g_{p+1} \|_{\frac{3}{2}-\delta,3}, \\
&\lesssim \eta^2 \big(\theta_{p+1}^{\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}+\delta-\gamma-\beta+5\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+5\bar{\varepsilon}} \big) \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma},
\end{aligned}
$$

due to (9.39) and $\frac{1}{2} - \gamma \geq \bar{\varepsilon}$. This finishes the proof of (VI) of Proposition 2.10 and hence the whole of Proposition 2.10. □

9.3. The Proof of Proposition 2.8 from Proposition 2.9 and Proposition 2.10

Let us assume in this subsection that

$$
\text{both Proposition 2.9 and Proposition 2.10 are valid.} \tag{9.40}
$$

We are going to prove (P1, $p+1$), (P2, $p+1$) and (P3, $p+1$), that is, that Proposition 2.8 is valid for $p+1$.

Proof of Proposition 2.8. We shall divide that proof into the several steps.
Step 1. The proof of (P3, $p+1$) of Proposition 2.8.

(P3, $p+1$) is a direct consequence of (9.7), (9.8), (9.9), (9.10) and the choice of parameters (see (2.83) and (2.82)):

$$
\beta \geq 3\bar{\varepsilon}, \quad C\eta \leq \delta_1, \quad \gamma \geq \delta + \varepsilon + 3\bar{\varepsilon}.
$$

Step 2. The proof of (P1, $p+1$) of Proposition 2.8.

Recall that $X = X_{p+1}$ solves

$$
X_{tt} - \Delta X_t - \partial_3^2 X = f'(S_{p+1} Y_{p+1}; X) + g_{p+1}. \tag{9.41}
$$

Due to (P3, $p+1$), the hypotheses of Theorem 2.3 and (2.64) are satisfied, so we can apply the energy estimate (2.65) to the system (9.41). When $N \geq 0$ with

$-\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon$ and $-\beta+\bar\varepsilon N\ge\bar\varepsilon$, we deduce from (I) (i), (ii) of Proposition 2.9 that

$$\begin{aligned}\widetilde{\gamma}_{\varepsilon,N+1}(S_{p+1}Y_{p+1}) \lesssim{}& |S_{p+1}\partial_3 Y_{p+1}|_{\frac12+\varepsilon,N+1}+|S_{p+1}\partial_t Y_{p+1}|_{1+\varepsilon,N+2}\\ &+|S_{p+1}\nabla Y_{p+1}|_{0,N+1}\\ &+\|S_{p+1}\nabla Y_{p+1}\|_{0,N+1}+1\lesssim \theta_{p+1}^{-\gamma+\varepsilon+\delta+\bar\varepsilon(N+2)}+\theta_{p+1}^{-\beta+\bar\varepsilon N}.\end{aligned}$$

Then in this case, we get, by applying the energy estimate (2.65) to system (9.41) and using (V) (i), (V) (ii) of Proposition 2.10, that

$$\begin{aligned}&\big\||D|^{-1}(\partial_3 X_{p+1},\partial_t X_{p+1})\big\|_{0,N+2}+\|\nabla X_{p+1}\|_{0,N+1}\\ &\quad+\|(\partial_t X_{p+1},\partial_3 X_{p+1})\|_{\frac12,N+1}\\ &\quad+\|\partial_t X_{p+1}\|_{L^2_t(H^{N+2})}+\big\|(\partial_3 X_{p+1},\langle t\rangle^{\frac12}\nabla\partial_t X_{p+1})\big\|_{L^2_t(H^{N+1})}\\ &\quad+\|\nabla\partial_t X_{p+1}\|_{1,N-1}\\ &\le C_{\varepsilon,N}\Big(\||D|^{-1}g_{p+1}\|_{1+\varepsilon,N+1}+\|\langle t\rangle^{\frac{1+\varepsilon}{2}}g_{p+1}\|_{L^2_t(H^N)}\\ &\quad+\widetilde{\gamma}_{\varepsilon,N+1}(S_{p+1}Y_{p+1})\big(\||D|^{-1}g\|_{1+\varepsilon,2}+\|\langle t\rangle^{\frac{1+\varepsilon}{2}}|D|^{-1}g_{p+1}\|_{L^2_t(H^1)}\big)\Big)\\ &\lesssim \eta^2\theta_{p+1}^{\delta-\gamma-\beta+\bar\varepsilon N}\big(\theta_{p+1}^{\varepsilon+2\bar\varepsilon}+\theta_{p+1}^{\frac{\varepsilon}{2}+3\bar\varepsilon}+(\theta_{p+1}^{-\gamma+\varepsilon+\delta+2\bar\varepsilon}+\theta_{p+1}^{-\beta})\theta_{p+1}^{\varepsilon+3\bar\varepsilon}\big)\lesssim\eta\theta_{p+1}^{-\beta+\bar\varepsilon N},\end{aligned}\tag{9.42}$$

provided that $\gamma\ge\delta+\varepsilon+3\bar\varepsilon$ which is satisfied due to (2.83) and (2.82). Along the same lines, we have

$$\begin{aligned}&\big\||D|^{-1}(\partial_t X_{p+1},\partial_3 X_{p+1})\big\|_{0,2}+\|\nabla X_{p+1}\|_{0,1}\\ &\quad+\|(\partial_t X_{p+1},\partial_3 X_{p+1})\|_{\frac12,1}+\|\partial_t X_{p+1}\|_{L^2_t(H^2)}\\ &\quad+\big\|\big(\partial_3 X_{p+1},\langle t\rangle^{\frac12}\nabla\partial_t X_{p+1}\big)\big\|_{L^2_t(H^1)}\\ &\quad\le C_\varepsilon\|\langle t\rangle^{\frac{1+\varepsilon}{2}}|D|^{-1}g_{p+1}\|_{L^2_t(H^1)}\lesssim\eta\theta_{p+1}^{-\beta}\quad\text{and}\\ &\|\nabla\partial_t X_{p+1}\|_{1,0}\le C_\varepsilon\Big(\||D|^{-1}g_{p+1}\|_{1+\varepsilon,2}+\|\langle t\rangle^{\frac{1+\varepsilon}{2}}|D|^{-1}g_{p+1}\|_{L^2_t(H^2)}\Big)\\ &\quad\le\eta\theta_{p+1}^{-\beta+\bar\varepsilon}.\end{aligned}\tag{9.43}$$

By interpolating the inequalities (9.42) and (9.43), we achieve (P1, $p+1$) for $N\ge 0$.

Step 3. The proof of (P2, $p+1$) of Proposition 2.8.

Notice that by definition $S_{p+1}Y_{p+1}=0$ and $g_{p+1}=0$ for $t\ge\theta_{p+1}$. In order to apply Proposition 2.2 to the equation (9.41), it remains to estimate $R_{N,\theta_{p+1}}\big(f'(S_{p+1}Y_{p+1};X_{p+1})\big)$ given by (2.38).

•The estimate of $\|\!|f'(S_{p+1}Y_{p+1};X_{p+1})|\!\|_{L^1_t(\delta,N)}$

It follows from (2.43) that

$$
\begin{aligned}
&|||f_1'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^1_t(\delta,N)} \lesssim \|S_{p+1}\partial_3 Y_{p+1}\|_{L^2_t(H^1)}\big(\|\partial_3 X_{p+1}\|_{L^2_t(H^{N+6})}\\
&\quad + |S_{p+1}\partial_3 Y_{p+1}|_{\frac12+\bar\varepsilon,0}\|\nabla X_{p+1}\|_{0,N+6}\big) + |S_{p+1}\partial_3 Y_{p+1}|_{\frac12+\bar\varepsilon,1}\|\nabla X_{p+1}\|_{0,1}\\
&\quad \times \big(\|S_{p+1}\partial_3 Y_{p+1}\|_{L^2_t(H^{N+6})} + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6}\|S_{p+1}\partial_3 Y_{p+1}\|_{L^2_t(H^3)}\big)\\
&\quad + \big(\|S_{p+1}\partial_3 Y_{p+1}\|_{L^2_t(H^{N+6})}\\
&\quad + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6}|S_{p+1}\partial_3 Y_{p+1}|_{\frac12+\bar\varepsilon,1}\big)\|\partial_3 X_{p+1}\|_{L^2_t(H^1)},
\end{aligned}
$$

from which, with (P1, $p+1$), (II) of Proposition 2.9 and the fact that $\beta \geq 6\bar\varepsilon$, we infer

$$
|||f_1'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^1_t(\delta,0)} \lesssim \eta\theta_{p+1}^{-\beta+5\bar\varepsilon}.
$$

For $-\beta+\bar\varepsilon(N+5)\geq\bar\varepsilon$, it follows from (I) (II) of proposition 2.9 and (P1, $p+1$) that

$$
|||f_1'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^1_t(\delta,N)} \lesssim \eta\theta_{p+1}^{-\beta+\bar\varepsilon(N+5)}.
$$

We have that $f_2'(S_{p+1}Y_{p+1};X_{p+1})$ can be handled along the same lines.

For $f_0'(S_{p+1}Y_{p+1};X_{p+1})$, we deduce from (2.42) that

$$
\begin{aligned}
&|||\langle t\rangle^{\frac12} f_0'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^2_t(\delta,N)} \leq \|S_{p+1}\nabla Y_{p+1}\|_{0,0}\|\langle t\rangle^{\frac12}\nabla\partial_t X_{p+1}\|_{L^2_t(H^{N+6})}\\
&\quad + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6}\|\langle t\rangle^{\frac12}\nabla\partial_t X_{p+1}\|_{L^2_t(L^2)}\\
&\quad + \|\langle t\rangle^{\frac12}S_{p+1}\nabla\partial_t Y_{p+1}\|_{L^2_t(L^2)}\|\nabla X_{p+1}\|_{0,N+6}\\
&\quad + \big(\|\langle t\rangle^{\frac12}S_{p+1}\nabla\partial_t Y_{p+1}\|_{L^2_t(H^{N+6})}\\
&\quad + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6}|S_{p+1}\nabla\partial_t Y_{p+1}|_{1+\bar\varepsilon,0}\big)\|\nabla X_{p+1}\|_{0,0}.
\end{aligned}
$$

Notice that $f_0'(S_{p+1}Y_{p+1};X_{p+1})$ is supported in $\{0\leq t\leq\theta_{p+1}\}$ so that

$$
\begin{aligned}
&|||f_0'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^1_t(\delta,N)}\\
&\lesssim (\log\theta_{p+1})^{\frac12}|||\langle t\rangle^{\frac12} f_0'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^2_t(\delta,N)},
\end{aligned}
$$

which together with (P1, $p+1$) and (II) of Proposition 2.9, ensures that

$$
|||f'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^1_t(\delta,0)} \lesssim \eta\theta_{p+1}^{-\beta+6\bar\varepsilon} \quad \text{and} \tag{9.44}
$$

$$
|||f'(S_{p+1}Y_{p+1};X_{p+1})|||_{L^1_t(\delta,N)} \lesssim \eta\theta_{p+1}^{-\beta+\bar\varepsilon(N+6)} \quad \text{if } -\beta+\bar\varepsilon(N+5)\geq\bar\varepsilon. \tag{9.45}
$$

•The estimate of $\|\langle t\rangle^{\frac12}|D|^{-1}f'(S_{p+1}Y_{p+1};X_{p+1})\|_{L^2_t(H^{N+1})}$

It follows from (2.46) that

$$\begin{aligned}
&\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}f_1'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{L_t^2(H^{N+1})}\\
&\quad\le |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2},0}\big(\|\partial_3 X_{p+1}\|_{L_t^2(H^{N+1})}\\
&\qquad + |S_{p+1}\nabla Y_{p+1}|_{0,N+1}\|\partial_3 X_{p+1}\|_{L_t^2(H^1)}\big)\\
&\qquad + \big(\|\nabla X_{p+1}\|_{0,N+1} + |S_{p+1}\nabla Y_{p+1}|_{0,N+1}\|\nabla X_{p+1}\|_{0,1}\big)\\
&\qquad \times\bigg(|S_{p+1}\partial_3 Y_{p+1}|^{\frac{4}{3}}_{\frac{1}{2}+\bar\varepsilon,1}\|S_{p+1}\partial_3 Y_{p+1}\|^{\frac{2}{3}}_{L_t^2(L^2)}\\
&\qquad + |S_{p+1}\partial_3 Y_{p+1}|^2_{\frac{1}{2}+\bar\varepsilon,1}\bigg) + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2},N+1}\\
&\qquad \times\bigg(\|\partial_3 X_{p+1}\|_{L_t^2(H^1)} + \bigg(|S_{p+1}\partial_3 Y_{p+1}|^{\frac{1}{3}}_{\frac{1}{2}+\bar\varepsilon,0}\|S_{p+1}\partial_3 Y_{p+1}\|^{\frac{2}{3}}_{L_t^2(L^2)}\\
&\qquad + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar\varepsilon,0}\bigg)\|\nabla X_{p+1}\|_{0,1}\bigg),
\end{aligned}$$

which together with (II) of Proposition 2.9 and (P1, $p+1$), ensures that

$$\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}f_1'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{L_t^2(H^3)} \lesssim \eta\theta_{p+1}^{-\beta+2\bar\varepsilon}.$$

For N satisfying $-\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon$, we deduce from (I) of Proposition 2.9 and (P1, $p+1$) that

$$\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}f_1'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{L_t^2(H^{N+1})} \lesssim \eta\theta_{p+1}^{-\beta+\bar\varepsilon N}.$$

We note that $f_2'(S_{p+1}Y_{p+1};\,X_{p+1})$ can be treated similarly.

For $f_0'(S_{p+1}Y_{p+1};\,X_{p+1})$, by virtue of (2.45), we get

$$\begin{aligned}
&\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}f_0'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{L_t^2(H^{N+1})}\\
&\quad\lesssim |S_{p+1}\nabla Y_{p+1}|_{0,0}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t X_{p+1}\|_{L_t^2(H^{N+1})}\\
&\qquad + |S_{p+1}\nabla Y_{p+1}|_{0,N+1}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t X_{p+1}\|_{L_t^2(L^2)}\\
&\qquad + |S_{p+1}\partial_t Y_{p+1}|_{1+\bar\varepsilon,1}\|\nabla X_{p+1}\|_{0,N+1}\\
&\qquad + \big(|S_{p+1}\partial_t Y_{p+1}|_{1+\bar\varepsilon,N+2}\\
&\qquad + |S_{p+1}\partial_t Y_{p+1}|_{1+\bar\varepsilon,1}|S_{p+1}\nabla Y_{p+1}|_{0,N+1}\big)\|\nabla X_{p+1}\|_{0,0}.
\end{aligned}$$

As a result,

$$\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}f'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{L_t^2(H^3)} \lesssim \eta\theta_{p+1}^{-\beta+2\bar\varepsilon} \quad\text{and} \tag{9.46}$$

$$\|\langle t\rangle^{\frac{1}{2}}|D|^{-1}f'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{L_t^2(H^{N+1})} \lesssim \eta\theta_{p+1}^{-\beta+\bar\varepsilon N} \quad\text{if } -\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon. \tag{9.47}$$

•**The Estimate of** $\||D|^{-1}f'(S_{p+1}Y_{p+1};\,X_{p+1})\|_{\frac{3}{2}-\delta,N+1}$

By virtue of (2.46), we have

$$
\begin{aligned}
&\||D|^{-1} f_1'(S_{p+1}Y_{p+1}; X_{p+1})\|_{\frac{3}{2},N+1}\\
&\quad\le |S_{p+1}\partial_3 Y_{p+1}|_{1,0}\big(|S_{p+1}\nabla Y_{p+1}|_{0,N+1}\|\partial_3 X_{p+1}\|_{\frac{1}{2},1}\\
&\qquad+\|\partial_3 X_{p+1}\|_{\frac{1}{2},N+1}\big)\\
&\qquad+\big(|S_{p+1}\partial_3 Y_{p+1}|^{\frac{4}{3}}_{\frac{7}{8},1}\|S_{p+1}\partial_3 Y_{p+1}\|^{\frac{2}{3}}_{\frac{1}{2},0}+|S_{p+1}\partial_3 Y_{p+1}|^{2}_{\frac{3}{4},1}\big)\\
&\qquad\times\big(\|\nabla X_{p+1}\|_{0,N+1}\\
&\qquad+|S_{p+1}\nabla Y_{p+1}|_{0,N+1}\|\nabla X_{p+1}\|_{0,1}\big)+|S_{p+1}\partial_3 Y_{p+1}|_{1,N+1}\|\partial_3 X_{p+1}\|_{\frac{1}{2},1}\\
&\qquad+\big(|S_{p+1}\partial_3 Y_{p+1}|_{\frac{7}{8},N+1}|S_{p+1}\partial_3 Y_{p+1}|^{\frac{1}{3}}_{\frac{7}{8},0}\|S_{p+1}\partial_3 Y_{p+1}\|^{\frac{2}{3}}_{\frac{1}{2},0}\\
&\qquad+|S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},N+1}|S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},0}\big)\|\nabla X_{p+1}\|_{0,1}.
\end{aligned}
$$

Noticing from (2.83) that $\frac{1}{4}-\gamma\ge\bar{\varepsilon}$, we get, by applying (II) (i) of Proposition 2.9, that

$$
\begin{aligned}
&|S_{p+1}\partial_3 Y_{p+1}|_{\frac{7}{8},0}\le \eta\theta_{p+1}^{\frac{3}{8}-\gamma},\quad |S_{p+1}\partial_3 Y_{p+1}|_{1,0}\lesssim \eta\theta_{p+1}^{\frac{1}{2}-\gamma},\\
&|S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},0}\le \eta\theta_{p+1}^{\frac{1}{4}-\gamma}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\||D|^{-1} f_1'(S_{p+1}Y_{p+1}; X_{p+1})\|_{\frac{3}{2},3}\\
&\quad\lesssim \eta^2\left(\theta_{p+1}^{\frac{1}{2}-\gamma-\beta+3\bar{\varepsilon}}+\theta_{p+1}^{\frac{1}{2}-\frac{4}{3}\gamma-\beta+\frac{10}{3}\bar{\varepsilon}}+\theta_{p+1}^{\frac{1}{2}-2\gamma-\beta+3\bar{\varepsilon}}\right)\\
&\quad\lesssim \eta^2\theta_{p+1}^{\frac{1}{2}-\gamma-\beta+3\bar{\varepsilon}},
\end{aligned}
\tag{9.48}
$$

provided that $\frac{1}{3}\gamma+\frac{2}{3}\beta\ge\frac{1}{3}\bar{\varepsilon}$, which is the case due to (2.83) and (2.82).

For N with $-\gamma+\bar{\varepsilon}(N+1)\ge\bar{\varepsilon}$, we deduce from (I) (i) of Corollary 2.9 that

$$
\begin{aligned}
&|S_{p+1}\partial_3 Y_{p+1}|_{\frac{7}{8},N+1}\le \eta\theta_{p+1}^{\frac{3}{8}-\gamma+\bar{\varepsilon}(N+1)},\quad |S_{p+1}\partial_3 Y_{p+1}|_{1,N+1}\lesssim \eta\theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}(N+1)},\\
&|S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},N+1}\le \eta\theta_{p+1}^{\frac{1}{4}-\gamma+\bar{\varepsilon}(N+1)},\quad |S_{p+1}\nabla Y_{p+1}|_{0,N+1}\lesssim \eta\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+1)},
\end{aligned}
$$

which, together with (P1, $p+1$), ensures that

$$
\||D|^{-1} f_1'(S_{p+1}Y_{p+1}; X_{p+1})\|_{\frac{3}{2},N+1}\lesssim \eta^2\theta_{p+1}^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+1)}. \tag{9.49}
$$

Similar estimates as to above hold for f_2'.

To deal with the term $f_0'(S_{p+1}Y_{p+1}; X_{p+1})$, we get, by applying (2.45), that

$$
\begin{aligned}
&\||D|^{-1} f_0'(S_{p+1}Y_{p+1}; X_{p+1})\|_{\frac{3}{2}-\delta,N+1}\lesssim |S_{p+1}\nabla Y_{p+1}|_{\frac{1}{2}-\delta,0}\|\nabla\partial_t X_{p+1}\|_{1,N+1}\\
&\quad+|S_{p+1}\nabla Y_{p+1}|_{\frac{1}{2}-\delta,N+1}\|\nabla\partial_t X_{p+1}\|_{1,0}+|S_{p+1}\partial_t Y_{p+1}|_{\frac{3}{2}-\delta,1}\|\nabla X_{p+1}\|_{0,N+1}\\
&\quad+\big(|S_{p+1}\partial_t Y_{p+1}|_{\frac{3}{2}-\delta,N+2}+|S_{p+1}\partial_t Y_{p+1}|_{\frac{3}{2}-\delta,1}|S_{p+1}\nabla Y_{p+1}|_{0,N+1}\big)\|\nabla X_{p+1}\|_{0,0}.
\end{aligned}
$$

Then along the same lines as to proof of (9.48) and (9.49), we can show that

$$\||D|^{-1} f'(S_{p+1}Y_{p+1}; X_{p+1})\|_{\frac{3}{2}-\delta,3} \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+4\bar{\varepsilon}}, \tag{9.50}$$

and for N with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, it holds that

$$\||D|^{-1} f'(S_{p+1}Y_{p+1}; X_{p+1})\|_{\frac{3}{2}-\delta,N+1} \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+2)}. \tag{9.51}$$

Moreover, we can prove in the same way that

$$\begin{aligned} &\||D|^{-1} f'(S_pY_p; X_p)\|_{1,1} \lesssim \eta^2 \theta_p^{-\beta+2\bar{\varepsilon}}, \\ &\||D|^{-1} f'(S_pX_p; X_p)\|_{1,N+1} \lesssim \eta^2 \theta_p^{-\beta+\bar{\varepsilon}(N+2)} \quad \text{for } -\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}. \end{aligned} \tag{9.52}$$

Recalling (2.38), we get, by summarizing the estimates (9.44), (9.46) and (9.50), that

$$\begin{aligned} R_{0,\theta_{p+1}}\big(f'(S_{p+1}Y_{p+1}; X_{p+1})\big) &\lesssim \eta^2\big(\theta_{p+1}^{-\beta+6\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\beta+2\bar{\varepsilon}} \\ &\quad + (\log\theta_{p+1})\theta_{p+1}^{\frac{1}{2}-\gamma-\beta+4\bar{\varepsilon}}\big) \lesssim \eta^2\theta_{p+1}^{\frac{1}{2}-\gamma}, \end{aligned}$$

provided that

$$\beta + \frac{1}{2} \geq \gamma + 6\bar{\varepsilon}, \quad \beta \geq \gamma + 2\bar{\varepsilon}, \quad \beta \geq 5\bar{\varepsilon}, \tag{9.53}$$

which is the case here due to (2.83) and (2.82).

Due to (9.53), $-\beta + \bar{\varepsilon}(N_0+5) \geq \bar{\varepsilon}$ and $-\gamma + \bar{\varepsilon}(N_0+2) \geq \bar{\varepsilon}$, by summarizing the estimates (9.45), (9.47) and (9.51), we achieve

$$\begin{aligned} &R_{N_0,\theta_{p+1}}\big(f'(S_{p+1}Y_{p+1}; X_{p+1})\big) \\ &\quad\lesssim \eta^2\theta_{p+1}^{\bar{\varepsilon}(N_0+2)}\big(\theta_{p+1}^{-\beta+4\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\beta} + (\log\theta_{p+1})\theta_{p+1}^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}}\big) \lesssim \eta^2\theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N_0}. \end{aligned}$$

Now we apply Proposition 2.2 and (VI) of Proposition 2.10 to (9.41) to get

$$\begin{aligned} &|\partial_3 X_{p+1}|_{1,0} + |X_{p+1,t}|_{\frac{3}{2}-\delta,0} + |X_{p+1}|_{\frac{1}{2},0} \\ &\quad \leq R_{0,\theta_{p+1}}\big(f'(S_{p+1}Y_{p+1}; X_{p+1})\big) + R_{0,\theta_{p+1}}(g_{p+1}) \leq C\eta^2\theta_{p+1}^{\frac{1}{2}-\gamma} \end{aligned}$$

and

$$\begin{aligned} &|\partial_3 X_{p+1}|_{1,N_0} + |X_{p+1,t}|_{\frac{3}{2}-\delta,N_0} + |X_{p+1}|_{\frac{1}{2},N_0} \\ &\quad \leq R_{N_0,\theta_{p+1}}\big(f'(S_{p+1}Y_{p+1}; X_{p+1})\big) + R_{N_0,\theta_{p+1}}(g_{p+1}) \leq C\eta^2\theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N_0}. \end{aligned}$$

Interpolating the above two inequalities gives, for all $0 \leq N \leq N_0$,

$$|\partial_3 X_{p+1}|_{1,N} + |X_{p+1,t}|_{\frac{3}{2}-\delta,N} + |X_{p+1}|_{\frac{1}{2},N} \lesssim \eta\theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N}. \tag{9.54}$$

It follows from Sobolev embedding and (P1, $p+1$) that for any $0 \le N \le N_0$,

$$
\begin{aligned}
|X_{p+1}|_{0,N} &\lesssim \|\nabla X_{p+1}\|_{0,N+1} \le \eta\theta_{p+1}^{-\beta+\bar\varepsilon N} \le \eta\theta_{p+1}^{-\gamma+\bar\varepsilon N},\\
|\partial_3 X_{p+1}|_{\frac12,N} &\lesssim \|\partial_3 X_{p+1}\|_{\frac12,N+2} \le \eta\theta_{p+1}^{-\beta+\bar\varepsilon(N+1)} \le \eta\theta_{p+1}^{-\gamma+\bar\varepsilon N},\\
|\partial_t X_{p+1}|_{1-\delta,N} &\lesssim \|\nabla\partial_t X_{p+1}\|_{1,N+1} \le \eta\theta_{p+1}^{-\beta+\bar\varepsilon(N+2)} \le \eta\theta_{p+1}^{-\gamma+\bar\varepsilon N},
\end{aligned} \tag{9.55}
$$

provided that $\beta \ge \gamma + 2\bar\varepsilon$, which is satisfied due to (2.83).

By interpolating the inequalities (9.54) and (9.55), we arrive at (P2, $p+1$). This completes the proof of Proposition 2.8 for $p+1$. □

9.4. The Proof of Theorem 2.1

The goal of this subsection is to prove the convergence of the approximate solutions $\{Y_p\}$ constructed via (2.75) in some appropriate norms, which in particular ensures Theorem 2.1.

Proof of Theorem 2.1. We infer from (2.76), (9.52), (P1) of Proposition 2.8 and (V) of Proposition 2.10 that

$$
\begin{aligned}
\|\partial_{tt} X_p\|_{\frac12,0} &\le \|(\Delta\partial_t X_p, \partial_3^2 X_p)\|_{\frac12,0} + \|f'(S_pY_p; X_p)\|_{\frac12,0} + \|g_p\|_{\frac12,0}\\
&\le C\eta\theta_p^{-\beta+2\bar\varepsilon},\\
\|\partial_{tt} X_p\|_{\frac12,N} &\le C\eta\theta_p^{-\beta+\bar\varepsilon(N+2)}, \qquad \text{for } -\gamma+\bar\varepsilon(N+1) \ge \bar\varepsilon.
\end{aligned} \tag{9.56}
$$

Interpolating the above two inequalities leads to

$$
\|\partial_{tt} X_p\|_{\frac12,N} \le C\eta\theta_p^{-\beta+\bar\varepsilon(N+2)}, \quad \forall\, N \ge 0. \tag{9.57}
$$

Due to the choices of the parameters in (2.83) and (2.82), it follows from (P2) of Proposition 2.8 that

$$
\begin{aligned}
&\sum_{p=0}^{\infty} |\partial_3 Y_{p+1} - \partial_3 Y_p|_{\frac34-4\bar\varepsilon,2} = \sum_{p=0}^{\infty} |\partial_3 X_p|_{\frac34-4\bar\varepsilon,2} \le \eta \sum_{p=0}^{\infty} \theta_p^{-\bar\varepsilon} < +\infty,\\
&\sum_{p=0}^{\infty} |\partial_t Y_{p+1} - \partial_t Y_p|_{\frac54-\delta-4\bar\varepsilon,2} = \sum_{p=0}^{\infty} |\partial_t X_p|_{\frac54-\delta-4\bar\varepsilon,2} \le \eta \sum_{p=0}^{\infty} \theta_p^{-\bar\varepsilon} < +\infty,\\
&\sum_{p=0}^{\infty} |Y_{p+1} - Y_p|_{\frac14-4\bar\varepsilon,2} = \sum_{p=0}^{\infty} |X_p|_{\frac14-4\bar\varepsilon,2} \le \eta \sum_{p=0}^{\infty} \theta_p^{-\bar\varepsilon} < +\infty.
\end{aligned}
$$

Similarly, taking $N_0 = [1/2\bar\varepsilon] + 1$ and $N_1 \stackrel{\text{def}}{=} [N_0/2]$, we deduce from (P2) of Proposition 2.8 and (9.56) that

$$
\sum_{p=0}^{\infty} \Big(\big\| |D|^{-1} \big(\partial_3 Y_{p+1} - \partial_3 Y_p, \partial_t Y_{p+1} - \partial_t Y_p\big) \big\|_{0,N_1+2} + \|\nabla Y_{p+1} - \nabla Y_p\|_{0,N_1+1}
$$

$$
\begin{aligned}
&+\big\|\big(\partial_3 Y_{p+1}-\partial_3 Y_p, \langle t\rangle^{\frac12}(\nabla\partial_t Y_{p+1}-\nabla\partial_t Y_p)\big)\big\|_{L^2_t(H^{N_1+1})}\\
&+\|\partial_t Y_{p+1}-\partial_t Y_p\|_{L^2_t(H^{N_1+2})}\\
&+\Big\|\Big(\partial_t Y_{p+1}-\partial_t Y_p, \partial_3 Y_{p+1}-\partial_3 Y_p\Big)\Big\|_{\frac12,N_1+1}+\|\nabla\partial_t Y_{p+1}-\nabla\partial_t Y_p\|_{1,N_1-1}\\
&+\|\partial_{tt} Y_{p+1}-\partial_{tt} Y_p\|_{\frac12,N_1-2}\Big)\le \eta\sum_{p=0}^{\infty}\theta_p^{-\beta+\bar\varepsilon N_1}\le \eta\sum_{p=0}^{\infty}\theta_p^{-\bar\varepsilon}<+\infty.
\end{aligned}
$$

This ensures the existence of $Y\in C^2([0,+\infty);C^{N_1-4}(\mathbb{R}^3))$ such that

$$
|\partial_3 Y-\partial_3 Y_p|_{\frac34-4\bar\varepsilon,2}+|Y_t-\partial_t Y_p|_{\frac54-\delta-4\bar\varepsilon,2}+|Y-Y_p|_{\frac14-4\bar\varepsilon,2}\to 0 \tag{9.58}
$$

and

$$
\begin{aligned}
&\big\||D|^{-1}\big(\partial_3 Y-\partial_3 Y_p, Y_t-\partial_t Y_p\big)\big\|_{0,N_1+2}\\
&\quad+\|\nabla Y-\nabla Y_p\|_{0,N_1+1}+\|\partial_t Y-\partial_t Y_p\|_{L^2_t(H^{N_1+2})}\\
&\quad+\big\|\big(\partial_3 Y-\partial_3 Y_p, \langle t\rangle^{\frac12}(\nabla\partial_t Y-\nabla\partial_t Y_p)\big)\big\|_{L^2_t(H^{N_1+1})}+\|\nabla\partial_t Y-\nabla\partial_t Y_p\|_{1,N_1-1}\\
&\quad+\big\|\big(\partial_t Y-\partial_t Y_p, \partial_3 Y-\partial_3 Y_p\big)\big\|_{\frac12,N_1+1}\\
&\quad+\|\partial_{tt} Y-\partial_{tt} Y_p\|_{\frac12,N_1-2}\to 0,\quad \text{as } p\to+\infty,
\end{aligned} \tag{9.59}
$$

which ensures (2.21) and (2.22).

Next we show that Y is the solution to (2.71). As a matter of fact, we first observe from (2.78) and (2.79) that

$$
\Phi(Y_{p+1})-\Phi(Y_0)=\sum_{j=0}^{p}e_j+\sum_{j=0}^{p}g_j=E_p+e_p-S_pE_p-S_p\Phi(Y_0),
$$

which implies

$$
\Phi(Y_{p+1})=e_p+(1-S_p)E_p+(1-S_p)\Phi(Y_0),
$$

from which, with (9.34), (9.38) and (IV) of Proposition 2.10, we infer

$$
\begin{aligned}
\|\Phi(Y_{p+1})\|_{1,0}&\le\|e_p\|_{1,0}+\|(1-S_p)E_p\|_{1,0}+\|(1-S_p)f(Y_0)\|_{1,0}\\
&\le C\theta_{p+1}^{\delta-\gamma-\beta+2\bar\varepsilon}.
\end{aligned} \tag{9.60}
$$

Next, we show that $\Phi(Y_{p+1})\to\Phi(Y)$ as $p\to+\infty$ in the norm $\|\cdot\|_{1,0}$. Indeed denoting $\widetilde\Box\overset{\text{def}}{=}\partial_t^2-\Delta\partial_t-\partial_3^2$, one has

$$
\|\Phi(Y)-\Phi(Y_{p+1})\|_{1,0}\le\|\widetilde\Box(Y-Y_{p+1})\|_{1,0}+\|f(Y)-f(Y_{p+1})\|_{1,0}. \tag{9.61}
$$

Using a Taylor formula, applying (2.45), (2.46) and (2.47), and using (9.58) and (9.59), we have

$$\begin{aligned}\|f(Y)-f(Y_{p+1})\|_{1,0} &\leq \int_0^1 \|f'\big((1-s)Y_{p+1}+sY;\,Y-Y_{p+1}\big)\|_{1,0}\mathrm{d}s\\ &\lesssim C\Big(\|\partial_3 Y-\partial_3 Y_{p+1}\|_{\frac{1}{2},1}+\|Y_t-\partial_t Y_{p+1}\|_{\frac{1}{2},1}\\ &\quad+\|\nabla Y_t-\nabla\partial_t Y_{p+1}\|_{1,1}\\ &\quad+\|\nabla Y-\nabla Y_{p+1}\|_{0,1}\Big)\to 0,\ \text{as } p\to+\infty.\end{aligned}$$

On the other hand, recalling from (2.76) that

$$\widetilde{\Box}X_p = f'(S_pY_p;\,X_p)+g_p,$$

we get, by applying (P1) of Proposition 2.8, (II) of Proposition 2.9 and (V)(ii) of Proposition 2.10, that

$$\begin{aligned}\|\widetilde{\Box}X_p\|_{1,0} &\leq \|f'(S_pY_p;\,X_p)\|_{1,0}+\|g_p\|_{1,0}\\ &\lesssim C\big(\|\partial_3 X_p\|_{\frac{1}{2},1}+\|\partial_t X_p\|_{\frac{1}{2},1}+\|\nabla\partial_t X_p\|_{1,1}+\|\nabla X_p\|_{0,1}\big)+\|g_p\|_{1,0}\\ &\lesssim C\theta_p^{-\beta+2\bar{\varepsilon}}+\theta_p^{\delta-\gamma-\beta+2\bar{\varepsilon}}.\end{aligned}$$

Consequently, we achieve

$$\|\widetilde{\Box}(Y-Y_{p+1})\|_{1,0}\leq\sum_{j=p+1}^{\infty}\|\widetilde{\Box}X_j\|_{1,0}\leq C\sum_{j=p+1}^{\infty}\theta_j^{-\beta+2\bar{\varepsilon}}\to 0,\quad \text{as } p\to\infty. \tag{9.62}$$

We then deduce from (9.61) and (9.62) that

$$\|\Phi(Y)-\Phi(Y_{p+1})\|_{1,0}\to 0\quad\text{as } p\to\infty,$$

which together with (9.60) implies $\Phi(Y)=0$. Finally, for each p, we have

$$Y_p(0,y)=Y^{(0)},\quad \partial_t Y_p(0,y)=Y^{(1)}(y),$$

therefore,

$$Y(0,y)=Y^{(0)},\quad Y_t(0,y)=Y^{(1)}(y),$$

and thus Y is the desired classical solution to (2.71). This ends the proof of Theorem 2.1. □

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Conflict of interest

The authors declare that there are no potential conflicts of interest.

Appendix A: The Proof of Lemmas 9.1, 9.2 and 9.3

The goal of this appendix is to present the proof of Lemmas 9.1, 9.2 and 9.3. Notice that the estimates for $e'_{p,2}$, $e''_{p,2}$ are the same as (or even better than) those for $e'_{p,1}$, $e''_{p,1}$, so that we only perform the estimates for the latter in what follows.

A.1: The Proof of Lemma 9.1

Since the proofs of (9.14-9.17) are very much similar, here we only present a detailed estimate to (9.14). Interested readers may check Sect. A.1 of [15] for the proof of the remaining inequalities.

In view of (2.81), we get, by applying (2.50) (with $Y \simeq Y_p + Y_{p+1}$, $X = W = X_p$), that for $N \geq 0$,

$$
\begin{aligned}
&\|\langle t\rangle^{\frac12}|D|^{-1}e''_{p,1}\|_{L^2_t(H^{N+1})} \lesssim |\partial_3 X_p|_{\frac12,N+1}\|\partial_3 X_p\|_{L^2_t(L^2)}\\
&+|\partial_3 X_p|_{\frac12,0}\|\partial_3 X_p\|_{L^2_t(H^{N+1})}\\
&+\sum_{j=p}^{p+1}\Big\{\Big(|\nabla Y_j|_{0,N+1}|\partial_3 X_p|_{\frac12,0}\\
&+\big(|\partial_3 Y_j|_{\frac12,N+1}+|\nabla Y_j|_{0,N+1}|\partial_3 Y_j|_{\frac12,1}\big)|\nabla X_p|_{0,1}\Big)\|\partial_3 X_p\|_{L^2_t(H^1)}\\
&+\big(|\partial_3 Y_j|_{\frac12+\bar\varepsilon,1}+|\partial_3 Y_j|^{\frac13}_{\frac12+\bar\varepsilon,1}\|\partial_3 Y_j\|^{\frac23}_{\frac12,1}\big)\Big(|\nabla X_p|_{0,N+1}\|\partial_3 X_p\|_{L^2_t(L^2)}\\
&+|\nabla X_p|_{0,0}\|\partial_3 X_p\|_{L^2_t(H^{N+1})}\\
&+|\partial_3 X_p|_{\frac12,1}\|\nabla X_p\|_{0,N+1}+\big(|\partial_3 X_p|_{\frac12,N+1}+|\nabla Y_j|_{0,N+1}|\partial_3 X_p|_{\frac12,1}\\
&+|\partial_3 Y_j|_{\frac12+\bar\varepsilon,0}|\nabla X_p|_{0,1}\big)\|\nabla X_p\|_{0,1}\Big)\\
&+\big(|\partial_3 Y_j|^{\frac43}_{\frac12+\bar\varepsilon,1}\|\partial_3 Y_j\|^{\frac23}_{\frac12,0}+|\partial_3 Y_j|^2_{\frac12+\bar\varepsilon,1}\big)\Big(|\nabla X_p|_{0,1}\big(\|\nabla X_p\|_{0,N+1}\\
&+|\nabla Y_j|_{0,N+1}\|\nabla X_p\|_{0,1}\big)+|\nabla X_p|_{0,N+1}\|\nabla X_p\|_{0,1}\Big)\\
&+\big(|\partial_3 Y_j|_{\frac12+\bar\varepsilon,N+1}+|\partial_3 Y_j|^{\frac13}_{\frac12+\bar\varepsilon,N+1}\|\partial_3 Y_j\|^{\frac23}_{\frac12,N+1}\big)|\partial_3 X_p|_{\frac12,1}\|\nabla X_p\|_{0,1}\Big\}.
\end{aligned}
$$

A similar estimate holds for $\|\langle t\rangle|D|^{-1}e''_{p,1}\|_{L^2_t(H^{N+1})}$, with $|\partial_3 X_p|_{\frac12,l}$ above being replaced by $|\partial_3 X_p|_{1,l}$ and $|\nabla X_p|_{0,l}$ by $|\nabla X_p|_{\frac12,l}$.

It follows from (9.7), (9.9) and (9.10) that

$$
|\partial_3 Y_{p+1}|_{\frac12+\bar\varepsilon,1} \leq C\eta, \quad |\nabla Y_{p+1}|_{0,1} \leq C\eta \quad \text{since } \gamma \geq 3\bar\varepsilon; \tag{A.1}
$$

$$
\|\partial_3 Y_{p+1}\|_{\frac12,1} \leq C\eta \qquad \text{since } \beta \geq \bar\varepsilon. \tag{A.2}
$$

As a result, applying (P1, p) and (P2, p), it turns out that

$$
\begin{aligned}
&\|\langle t\rangle^{\frac12}|D|^{-1}e''_{p,1}\|_{L^2_t(H^1)} \lesssim \eta^2\theta_p^{-\gamma-\beta+\bar\varepsilon}, \quad \text{and}\\
&\|\langle t\rangle|D|^{-1}e''_{p,1}\|_{L^2_t(H^1)} \lesssim \eta^2\theta_p^{\frac12-\gamma-\beta+\bar\varepsilon}.
\end{aligned}
$$

Interpolating between the above two inequalities gives rise to

$$\|\langle t\rangle^{\frac12+k}|D|^{-1}e''_{p,1}\|_{L^2_t(H^1)} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar\varepsilon} \quad \text{if } 0\le k\le \frac12. \tag{A.3}$$

For $0\le N\le N_0-1$ such that $-\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon$ and $-\beta+\bar\varepsilon N\ge\bar\varepsilon$, we deduce from (9.7), (9.9) and (9.10) that

$$\begin{aligned}&|\partial_3 Y_{p+1}|_{\frac12+\bar\varepsilon,N+1}\le C\eta\theta_{p+1}^{-\gamma+\bar\varepsilon(N+2)}, \quad |\nabla Y_{p+1}|_{0,N+1}\le C\eta\theta_{p+1}^{-\gamma+\bar\varepsilon(N+1)}\\ &\|\partial_3 Y_{p+1}\|_{\frac12,N+1}\le C\eta\theta_{p+1}^{-\beta+\bar\varepsilon N}.\end{aligned} \tag{A.4}$$

Therefore, for such N, it holds that

$$\begin{aligned}&\|\langle t\rangle^{\frac12}|D|^{-1}e''_{p,1}\|_{L^2_t(H^{N+1})}\lesssim \eta^2\theta_p^{-\gamma-\beta+\bar\varepsilon(N+1)},\\ &\|\langle t\rangle|D|^{-1}e''_{p,1}\|_{L^2_t(H^{N+1})}\lesssim \eta^2\theta_p^{\frac12-\gamma-\beta+\bar\varepsilon(N+1)}.\end{aligned}$$

Interpolating the above two inequalities, we obtain for $0\le k\le\frac12$ and $N\le N_0-1$ such that $-\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon$ and $-\beta+\bar\varepsilon N\ge\bar\varepsilon$,

$$\|\langle t\rangle^{\frac12+k}|D|^{-1}e''_{p,1}\|_{L^2_t(H^{N+1})}\lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar\varepsilon(N+1)}. \tag{A.5}$$

Interpolating between (A.3) and (A.5) leads to (9.14).

A.2: The Proof of Lemma 9.2

As in the previous lemma, here we present the detailed proof of (9.19). One may check Sect. A.2 of [15] for the proofs of the remaining inequalities.
Applying (2.50) to $e'_{p,1}$ determined by (2.81) gives that for $N\ge 0$,

$$\begin{aligned}\||D|^{-1}e'_{p,1}\|_{\frac32,N+1}&\lesssim |(1-S_p)\partial_3 Y_p|_{1,0}\Big(\|\partial_3 X_p\|_{\frac12,N+1}+|\nabla Y_p|_{0,N+1}\|\partial_3 X_p\|_{\frac12,0}\Big)\\ &+|(1-S_p)\partial_3 Y_p|_{1,N+1}\|\partial_3 X_p\|_{\frac12,0}+\Big(|\partial_3 Y_p|_{\frac12,1}+|\partial_3 Y_p|_{\frac12,1}^{\frac13}\|\partial_3 Y_p\|_{\frac12,1}^{\frac23}\Big)\\ &\times\Big(|(1-S_p)\nabla Y_p|_{\frac12,N+1}\|\partial_3 X_p\|_{\frac12,0}\\ &+|(1-S_p)\nabla Y_p|_{\frac12,1}\big(\|\partial_3 X_p\|_{\frac12,N+1}+|\partial_3 Y_p|_{\frac12,N+1}\|\nabla X_p\|_{0,1}\big)\\ &+|(1-S_p)\partial_3 Y_p|_{1,N+1}\|\nabla X_p\|_{0,1}+|(1-S_p)\partial_3 Y_p|_{1,1}\big(\|\nabla X_p\|_{0,N+1}\\ &+|\nabla Y_p|_{0,N+1}\|\nabla X_p\|_{0,1}\big)\Big)\\ &+\big(|\partial_3 Y_p|_{\frac12,N+1}+|\nabla Y_p|_{0,N+1}|\partial_3 Y_p|_{\frac12,1}\big)|(1-S_p)\nabla Y_p|_{\frac12,1}\|\partial_3 X_p\|_{\frac12,1}\\ &+\big(|\partial_3 Y_p|_{\frac12,N+1}+|\partial_3 Y_p|_{\frac12,N+1}^{\frac13}\|\partial_3 Y_p\|_{\frac12,N+1}^{\frac23}\big)|(1-S_p)\partial_3 Y_p|_{1,1}\|\nabla X_p\|_{0,1}\\ &+\big(|\partial_3 Y_p|_{\frac12,1}^{\frac43}\|\partial_3 Y_p\|_{\frac12,0}^{\frac23}+|\partial_3 Y_p|_{\frac12,1}^2\big)\Big(|(1-S_p)\nabla Y_p|_{\frac12,N+1}\|\nabla X_p\|_{0,1}\\ &+|(1-S_p)\nabla Y_p|_{\frac12,1}\big(\|\nabla X_p\|_{0,N+1}+|\nabla Y_p|_{0,N+1}\|\nabla X_p\|_{0,1}\big)\Big).\end{aligned}$$

A similar estimate holds for $\||D|^{-1}e'_{p,1}\|_{1,N+1}$, with $|(1-S_p)\partial_3 X_p|_{1,l}$ and $|(1-S_p)\nabla X_p|_{\frac{1}{2},l}$ above being replaced by $|(1-S_p)\partial_3 Y_p|_{\frac{1}{2},l}$ and $|(1-S_p)\nabla X_p|_{0,l}$, respectively.

Hence we deduce from (A.1) that

$$\||D|^{-1}e'_{p,1}\|_{1,1} \lesssim \eta^2\theta_p^{-\gamma-\beta+\bar\varepsilon}, \quad \||D|^{-1}e'_{p,1}\|_{\frac{3}{2},1} \lesssim \eta^2\theta_p^{\frac{1}{2}-\gamma-\beta+\bar\varepsilon}.$$

Interpolating the above two inequalities yields

$$\||D|^{-1}e'_{p,1}\|_{1+k,1} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar\varepsilon} \quad \text{for } 0\le k\le \frac{1}{2}. \tag{A.6}$$

For $N\le N_0-1$ satisfying $-\beta+\bar\varepsilon N\ge\bar\varepsilon$ and $-\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon$, (A.4) holds, so we infer that

$$\begin{aligned}\||D|^{-1}e'_{p,1}\|_{1,N+1} &\lesssim \eta^2\theta_p^{-\gamma-\beta+\bar\varepsilon(N+1)},\\ \||D|^{-1}e'_{p,1}\|_{\frac{3}{2},N+1} &\lesssim \eta^2\theta_p^{\frac{1}{2}-\gamma-\beta+\bar\varepsilon(N+1)}.\end{aligned}$$

Interpolating the above inequalities leads to

$$\||D|^{-1}e'_{p,1}\|_{1+k,N+1} \lesssim \eta^2\theta_p^{k-\gamma-\beta+\bar\varepsilon(N+1)} \tag{A.7}$$

for $0\le k\le\frac{1}{2}$, $N\le N_0-1$ such that $-\beta+\bar\varepsilon N\ge\bar\varepsilon$ and $-\gamma+\bar\varepsilon(N+1)\ge\bar\varepsilon$. We then conclude the proof of (9.19) by interpolating between (A.6) and (A.7).

A.3: The Proof of Lemma 9.3

Here we present the detailed proof of (9.24). Interested readers may check Sect. A.3 for the proof of the remaining inequalities.

Applying (2.52) to $e''_{p,0}$ gives

$$\begin{aligned}\|\langle t\rangle^{\frac{1}{2}}e''_{p,0}\|_{L^2_t(\delta,N)} &\lesssim \|\nabla X_p\|_{0,0}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t X_p\|_{L^2_t(H^{N+6})}\\ &+\|\nabla X_p\|_{0,N+6}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t X_p\|_{L^2_t(L^2)}\\ &+\sum_{j=p}^{p+1}\Bigg(|\partial_t Y_j|_{1+\bar\varepsilon,1}\|\nabla X_p\|_{0,N+6}\|\nabla X_p\|_{0,0}\\ &+\|\nabla Y_j\|_{0,N+6}|\nabla X_p|_{0,0}\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t X_p\|_{L^2_t(L^2)}\\ &+\big(\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t Y_j\|_{L^2_t(H^{N+6})}+\|\nabla Y_j\|_{0,N+6}|\partial_t Y_j|_{1+\bar\varepsilon,1}\big)|\nabla X_p|_{0,0}\|\nabla X_p\|_{0,0}\Bigg).\end{aligned}$$

Again due to $\beta\ge7\bar\varepsilon$, we deduce from (9.10) that

$$\|\nabla Y_{p+1}\|_{0,6}+\|\langle t\rangle^{\frac{1}{2}}\nabla\partial_t Y_{p+1}\|_{L^2_t(H^6)}\le C\eta. \tag{A.8}$$

As a result,

$$|||\langle t\rangle^{\frac{1}{2}} e''_{p,0}|||_{L^2_t(\delta,0)} \lesssim \eta^2\theta_p^{-\beta-\gamma+5\bar{\varepsilon}}.$$

In the case for when $N \leq N_0 - 6$ with $-\beta + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$, it follows from (9.10) that

$$\|\nabla Y_{p+1}\|_{0,N+6} + \|\langle t\rangle^{\frac{1}{2}} \nabla\partial_t Y_{p+1}\|_{L^2_t(H^{N+6})} \leq C\eta\theta_p^{-\beta+\bar{\varepsilon}(N+5)}, \tag{A.9}$$

so that in this case, we have

$$|||\langle t\rangle^{\frac{1}{2}} e''_{p,0}|||_{L^2_t(\delta,N)} \lesssim \eta^2\theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)}.$$

Then (9.24) follows by interpolating the above inequalities.

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