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Nonconvex Model of Material Growth: Mathematical Theory

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Abstract

The model of volumetric material growth is introduced in the framework of finite elasticity. The new results obtained for the model are presented with complete proofs. The state variables include the deformations, temperature and the growth factor matrix function. The existence of global in time solutions for the quasistatic deformations boundary value problem coupled with the energy balance and the evolution of the growth factor is shown. The mathematical results can be applied to a wide class of growth models in mechanics and biology.

1. Introduction

Growth (resp. atrophy) refers to physical processes in which the material of a solid body increases (resp. decreases) its size by addition (resp. removal) of mass. The advantages and drawbacks of the existing growth models are discussed in the recent papers [24,30]. A first class of such models are kinematic models describing the evolution of the material growth towards a homeostatic state. These rely on the kinematic decomposition of the transformation gradient into a generally incompatible mapping and an elastic mapping; they were historically introduced in [35] and developed in [1,36,43,45]. Approaches analogous to elastoplasticity were then developed in a rational framework based on the second principle of thermodynamics for open systems, in order to identify the evolution laws of growth [12,28,31,33]. It is important to note the prominent role of Eshelby stress in relation to the material driving forces for growth [9,12,19,20]. The mathematical aspects of volumetric growth models are poorly investigated. The local existence and uniqueness results were established in [17,18]. We refer the reader to Taber [41], Cowin [10], Jones and Chapman, and Ambrosi et al. [3] for the state of the art in the domain. Some additional references and comments will be given below.

Mechanical background: thermoelastic material. In this Section we briefly discuss basic facts from finite elasticity theory. Throughout, we shall assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded reference domain with boundary $\partial\Omega$ of class C^∞ in the space of variable x. The state of an elastic material is characterized by a deformation field $\mathbf{u} = (u_i)_{1 \leqq i \leqq d} : \Omega \times [0, T] \to \mathbb{R}^d$ and the Kelvin temperature $\theta : \Omega \times [0, T] \to \mathbb{R}^+$. The deformation gradient $D\mathbf{u}$ is the Jacobi matrix of the mapping $\mathbf{u}$ with entries

$$(D\mathbf{u})_{ij}(x, t) = \partial_{x_j} u_i(x, t), \quad (x, t) \in \Omega \times [0, T].$$

The second derivative (Hessian) of the deformation field is the third order tensor $D^2\mathbf{u}$ with entries

$$(D^2\mathbf{u})_{ijk}(x, t) = \partial_{x_i}\partial_{x_j} u_k(x, t), \quad (x, t) \in \Omega \times [0, T].$$

We will assume that the material is hyperelastic and its properties are described by a specific free energy density $\Psi(D^2\mathbf{u}, D\mathbf{u}, \theta)$. In this case the entropy $v(D^2\mathbf{u}, D\mathbf{u}, \theta)$ and internal energy $e(D^2\mathbf{u}, D\mathbf{u}, \theta)$ are defined by

$$v = -\partial_\theta \Psi(D^2\mathbf{u}, D\mathbf{u}, \theta), \quad e = \Psi(D^2\mathbf{u}, D\mathbf{u}, \theta) - \theta\, \partial_\theta \Psi(D^2\mathbf{u}, D\mathbf{u}, \theta), \tag{1.1}$$

i.e., $e = \Psi + v\theta$. The presence of the second gradient of the deformation in the expression for the free energy density means that we deal with the strain gradient elasticity theory developed in the papers by TOUPIN [44], KOITER [27], and MINDLIN [32], see also FLECK AND HUTCHINSON [16]. The higher order effects are important for the modeling of laminated materials or materials with microstructure. Examples of such materials are the arterial walls and solid tumors containing cells and extracellular matrices. Moreover, in the theory of volumetric growth the second order terms are responsible for the mass diffusion. As was proved in [12], there can be no mass-diffusive effects in a first-order material. To allow for such effects it is necessary to include second-order gradients in the constitutive framework.

Typically, the free energy density can be represented as the sum of the strain-gradient energy $\mathcal{A}$, the bulk stored elastic energy W, and terms depending on the temperature only:

$$\Psi = \mathcal{A}(D^2\mathbf{u}) + \theta W(D\mathbf{u}) + c_1\theta - c_2\theta \log\theta. \tag{1.2}$$

In many applications the dependence of the free energy on $D^2\mathbf{u}$ is quadratic, i.e.

$$\mathcal{A}(D^2\mathbf{u}) = \sum_{i,j,m,n,p,q} a_{ijmnpq}\, \partial^2_{x_i x_j} u_p\ \partial^2_{x_m x_n} u_q.$$

As was shown in [32,44], for isotropic material the quadratic form $\mathcal{A}$ depends on five elastic constants a_i and admits the representation

$$\begin{aligned} \mathcal{A} = {} & a_1|\nabla \mathrm{div}\, \mathbf{u}|^2 + a_2 \Delta\mathbf{u} \cdot \nabla \mathrm{div}\, \mathbf{u} + a_3|\Delta\mathbf{u}|^2 + a_4|D^2\mathbf{u}|^2 \\ & + a_5 \sum_{ijk} \partial^2_{x_i x_j} u_k\ \partial^2_{x_k x_j} u_i. \end{aligned} \tag{1.3}$$

Notice that for incompressible solids $a_1 = a_2 = 0$. The simplest case of the strain gradient energy is the Falk model with $a_1 = a_2 = a_4 = a_5 = 0$. This model is widely accepted in the theory of solid-solid phase transitions (see [14,34,40]). In the Falk model the free energy density is

$$\Psi = \frac{\varepsilon}{2}|\Delta \mathbf{u}|^2 + \theta W(D\mathbf{u}) + c_1\theta - c_2\theta \log\theta. \tag{1.4}$$

Soft biological tissues experience large deformations, and their behavior is modeled by the finite elasticity theory. In contrast to the linear case, in the finite elasticity theory one can meet various forms of the stored elastic energy W. One important example is the Ogden material with the elastic energy density defined by the following relation:

$$W(D\mathbf{u}) = c_\alpha\big(W_\alpha(D\mathbf{u}) - d\big), \quad W_\alpha(D\mathbf{u}) = \sum_{i=1}^{d} \lambda_i^\alpha, \quad \alpha > 1/2, \tag{1.5}$$

where λ_i are the eigenvalues of the matrix $D\mathbf{u}^\top D\mathbf{u}$. More general form of the elastic energy of the Ogden material is the linear combination of the functions W_α,

$$W(D\mathbf{u}) = \sum_{i=1}^{N} c_{\alpha_i}(W_{\alpha_i} - d), \quad \alpha_i > 1/2. \tag{1.6}$$

The simplest case of the Ogden material is the Neo-Hookean material ($N = \alpha_1 = 1$) with the stored elastic energy

$$W(D\mathbf{u}) = c_1(|D\mathbf{u}|^2 - d). \tag{1.7}$$

Remark 1. The expression for the Ogden material stored energy density is often supplemented with an additional term $J(\det D\mathbf{u})$ which provides the positivity of the Jacobian of deformation field, see Ciarlet [8]. Here J is a convex function such that $J(s) \to \infty$ as $s \searrow 0$. We will not consider this case since it is poorly investigated, and the only available results are related to the existence of a minimizer of the stored energy functional Ball [6], Ciarlet [8].

There are various forms of the stored elastic energy proposed for different biological materials. We refer the reader to the paper by Fung [15] for details. Following [15], the elastic energy function for the vascular material for $d = 3$ can be taken in the form

$$W(D\mathbf{u}) = c_0(\exp(Q) - Q - 1) + c_1 q + c_2(III_C - d), \tag{1.8}$$

where

$$Q = \alpha_1(I_C - 3) + \alpha_2(II_C - 3) + \alpha_3(II_C - 3)^2,$$
$$q = \beta_1(I_C - 3) + \beta_2(II_C - 3) + \beta_3(II_C - 3)^2,$$

I_C, II_C, and III_C are the invariants of the matrix $D\mathbf{u}^\top D\mathbf{u}$.

Mechanical backgrounds: growing material. It seems that the first application of continuum mechanics to tissue growth is due to HSU [23], who considered homogeneous growth of linear viscoelastic materials. The fundamental contribution to the modern theory of volumetric material growth was made by SKALAK ET AL. [38] in the analytical description of the volumetrically distributed mass growth, and the mass growth by deposition or resorption on the surface. The paper [38] describes a kinematic model in which simultaneously occurring growth and deformation are considered as a composition of two mappings, one representing stress-free growth and the other representing the deformations of the tissue owing to forces acting on the tissue. This may be the first statement of the separability hypothesis which, following COWIN [10], can be formulated as follows: simultaneously occurring growth and deformation may be decomposed into a growth deformation and an elastic deformation associated with the instantaneous loading.

An important step toward the general analysis of finite volumetric growth of pseudo-elastic soft tissues was made by RODRIGUEZ ET AL. [35], who decomposed the total deformation gradient into its elastic and growth part. The hypothesis was extended [35] to a general three-dimensional theory of mechanically modulated volumetric growth for soft incompressible biological tissues. Rodriguez et al. rendered the mapping composition idea described in [38] as a composition of deformation gradient mappings. Notice that the overall growth deformation is represented by the deformation gradient $D\mathbf{u}$. Decomposition suggested by RODRIGUEZ ET AL. [35] is represented by

$$D\mathbf{u} = \mathbf{F}_e\,\mathbf{F}_g, \tag{1.9}$$

where $\mathbf{F}_g$ is a tensor representing the growth deformation named growth factor, and $\mathbf{F}_e$ represents an elastic deformation. Notice that the elastic timescale is much shorter than the timescale associated with growth [22]. It follows that the elastic deformation due to accommodation occurs instantaneously in response to the growth. Overall, the growth process can be written as a deformation gradient. Therefore, from the mechanical point of view $\mathbf{F}_e$ represents the true elastic deformation. From the mathematical point of view $\mathbf{F}_e$ is just the integrating factor which is necessary to maintain compatibility of the gradient $D\mathbf{u} = \mathbf{F}_e\mathbf{F}_g$. A general constitutive theory of the stress-modulated growth of soft tissues was developed by LUBARDA AND HOGER [29] and KLISCH ET AL. [26]. The work [29] provides an explicit representation of $\mathbf{F}_g$ for various material symmetries, and an incremental formulation for stress-modulated growth process. In particular, they considered in many details the case of the isotropic growth with $\mathbf{F}_g = w\mathbf{I}$ where $w(x,t)$ is a scalar. A theory of material growth (mass creation and resorption) was presented in EPSTEIN AND MAUGIN [12]. The extension of the theory to second order materials was given by CIARLETTA ET AL. [9]. The assumption that the tensor $\mathbf{F}_e$ is responsible for the elastic deformations along with the covariance principle leads to the following representation for the Helmholtz free energy density Ψ_g of the second order growing material, see [9],

$$\begin{aligned}\Psi_g(D^2\mathbf{u}, D\mathbf{u}, \theta, \mathbf{F}_g) &:= \det \mathbf{F}_g\ \Psi(\mathbf{Q}_e,\ D\mathbf{u}\,\mathbf{F}_g^{-1}, \theta)\\ &\equiv \det \mathbf{F}_g\ \Psi(\mathbf{Q}_e, \mathbf{F}_e, \theta).\end{aligned} \tag{1.10}$$

Here Ψ is the basic Helmholtz free energy of the original thermoelastic material, and $\mathbf{Q}_e$ is the third order tensor with components

$$(\mathbf{Q}_e)_{jki} = \partial_{x_\alpha} \partial_{x_\beta} u_i \ (\mathbf{F}_g^{-1})_{\alpha j} \ (\mathbf{F}_g^{-1})_{\beta k}. \tag{1.11}$$

The entropy of the growing material is defined by the relation

$$v_g = -\det \mathbf{F}_g \frac{\partial}{\partial \theta} \Psi(\mathbf{Q}_e, \ D\mathbf{u}\, \mathbf{F}_g^{-1}, \ \theta).$$

As shown in [9,12], the Clausius-Duhem inequality and the principle of independence of motions imply the following expressions for the second order Piola-Kirchhoff stress tensor $\mathbf{T}_f$, and the third order Piola-Kirchhoff hyperstress tensor $\mathbf{T}_s$:

$$\begin{aligned} (\mathbf{T}_f)_{ij} = & \quad \det \mathbf{F}_g \frac{\partial}{\partial (D\mathbf{u})_{ij}} \ \Psi(\mathbf{Q}_e, \ D\mathbf{u}\, \mathbf{F}_g^{-1}, \ \theta), \\ (\mathbf{T}_s)_{ijk} = & \quad \det \mathbf{F}_g \ \frac{\partial}{\partial (D^2\mathbf{u})_{ijk}} \Psi(\mathbf{Q}_e, D\mathbf{u}\mathbf{F}_g^{-1}, \ \theta). \end{aligned} \tag{1.12}$$

In the isotropic case with $\mathbf{F}_g = w\mathbf{I}$ we have $\mathbf{Q}_e = w^{-2} D^2\mathbf{u}$ which implies the following expression for the free energy density and the entropy of the growing material:

$$\Psi_g(\mathbf{u}, \theta, w) = w^d \mathcal{A}(w^{-2} D^2\mathbf{u}) + \theta w^d W(w^{-1} D\mathbf{u}) + c_1 w^d \theta - c_2 w^d \theta \log \theta, \tag{1.13a}$$

$$v_g = w^d v, \quad \text{where} \ \ v = -c_1 + c_2(1 + \log \theta) - W(w^{-1} D\mathbf{u}). \tag{1.13b}$$

In this case the stress tensors $\mathbf{T}_f$ and $\mathbf{T}_s$ are defined by the relations

$$\mathbf{T}_f = \theta w^{d-1} \, W'(w^{-1} D\mathbf{u}), \quad (T_s)_{ijk} = w^{d-4} a_{ijpqkl} \, \partial^2_{x_p x_q} u_l. \tag{1.13c}$$

Governing equations. Furthermore, we will consider elastic materials with the free energy density in the form (1.2). For such materials, the system of the governing equations for the temperature and the elastic deformations includes the momentum balance equation and the energy balance equation. Notice that the problem has three characteristic times. The first is the characteristic time τ_e of the elastic oscillations, which is proportional to the inverse sound speed. For soft material, like a rubber, $\tau_e \sim 10^{-2}$ s. The second characteristic time τ_h is the characteristic time of the heat transfer. For polymer materials, $\tau_h \sim 40$ s, see Boyard [7]. The third characteristic time τ_g is the characteristic time of growth of biological materials; it is about days or weeks. If we choose the basic time scale $\tau_g = 1$ day, then the ratio of characteristic times becomes

$$\tau_e^2 : \tau_h : \tau_g \ \sim 10^{-12} : 10^{-3} : 1.$$

Therefore we can neglect the inertial forces and take the momentum balance equation in the quasi static form

$$\operatorname{div} \big(\mathbf{T}_f - \operatorname{div} \mathbf{T}_s\big) + \mathbf{f} = 0, \tag{1.14}$$

where $\mathbf{f}$ is a given external force, the stress tensors are defined by relations (1.12). The energy balance equation for the thermoelastic material reads

$$\varepsilon \frac{\partial e_g}{\partial t} + \operatorname{div} \mathbf{q} = \varepsilon \mathbf{T}_f : \frac{\partial D\mathbf{u}}{\partial t} + \varepsilon \mathbf{T}_s \vdots \frac{\partial D^2 \mathbf{u}}{\partial t} + g. \tag{1.15}$$

Here $e_g = \Psi_g + \theta v_g$ is the internal energy of the growing material, g is an external heat source, $\varepsilon = \tau_h$. In view of the Fourier law we can take $\mathbf{q} = -\nabla \vartheta$. The energy balance equation can be rewritten in the equivalent form

$$\varepsilon \partial_t v_g = \theta^{-1} \Delta \theta + \theta^{-1} g.$$

In the case of the isotropic growth the momentum and energy balance equations can be equivalently rewritten in the form of the elliptic-parabolic system for $\mathbf{u}$ and θ,

$$\mathcal{L}_w(\mathbf{u}) - \operatorname{div} (w^{d-1} \theta W'(w^{-1} D\mathbf{u}) = \mathbf{f}, \tag{1.16a}$$

$$\varepsilon \frac{\partial}{\partial t} \big[w^d (c_2 - c_1 + c_2 \log \theta - W(w^{-1} D\mathbf{u})) \big]$$
$$= \Delta(\log \theta) + |\nabla(\log \theta)|^2 + \frac{g}{\theta}. \tag{1.16b}$$

Here c_i are the constants in expression (1.2), and the elliptic operator L_w is defined by the equality

$$L_w(u)_i = \partial^2_{x_n x_m} \big(w^{d-4} a_{nmpqij} \partial^2_{x_p x_q} u_j \big),$$

the notation $W'(\boldsymbol{\xi})$ stands for the matrix with entries

$$(W'(\boldsymbol{\xi}))_{ij} = \partial_{\xi_{ij}} W(\boldsymbol{\xi}). \tag{1.17}$$

Mechanical background: Growth rate. In order to obtain a closed system of equations for the deformation field $\mathbf{u}$, the temperature θ, and the growth factor $\mathbf{F}_g$, the momentum and energy balance equations should be supplemented with an extra equation for $\mathbf{F}_g$. The main idea is that the evolution of the growth factor is described by a nonconservative model. This model is based on the assumption that $\partial_t \mathbf{F}_g$ is a function of the deformation gradient, the temperature, and the growth factor. The specification of such a function is the most important question of the theory. Due to the lack of experimental data, this question requires careful theoretical analysis. It seems that the first step in this direction was taken by TABER AND EGGERS [42]. They considered the principle stretches λ_i associated with the growth factor and proposed that $\partial_t \lambda_i$ were proportional to the Cauchy stress in the artery wall. A comparable model was proposed by RODRIGUEZ ET AL. [37]. AMBROSI AND MOLLICA [4] developed an original theory of the tumor growth. They proposed that in the isotropic case with $\mathbf{F}_g = w\mathbf{I}$, the rate of growth is given by $\partial_t w \sim \exp(-(s/s_0)^2)(n - n_0)w$, where s is the trace of the stress tensor, and n is the nutrient concentration. LUBARDA AND HOGER [29] proposed an isotropic growth law which depends on whether the stress is tensile or compressive:

$$\partial_t w = k(w, \mathbf{T}) \mathrm{tr} \mathbf{T}.$$

Here $\mathbf{T}$ is the stress tensor, the coefficient k is defined by the equalities

$$k = \text{const.}\Big(\frac{w^+ - w}{w^+ - 1}\Big)^{m^+} \text{ for } \mathrm{tr}\mathbf{T} > 0,\ k = \text{const.}\Big(\frac{w - w^-}{1 - w^-}\Big)^{m^-} \text{ for } \mathrm{tr}\mathbf{T} < 0,$$

where $w^\pm$ and $m^\pm$ are some material constants.

The lack of biologically derived growth laws is the weak point of the current theories. One of the possible ways to cope with this problem is to develop the model consistent with the basic thermodynamical principles. The important step in this direction was taken in the seminal works by Epstein and Maugin [12], Di Carlo and Quiligotti [11], and Ambrosi and Guana [5]. The extension of the theory to the second order material was given by Ciarletta et al. [9]. It was showed by thermodynamical arguments that if the growth process is governed by some external forces, then the growth law for $\mathbf{F}_g$ can be derived as a rate equation involving those external forces. The process of growth in open systems leads to the generation of inhomogeneities, since the material points within the body do not grow at the same rate; these inhomogeneities lead in turn to residual stresses, and the modeling of their development in time fits within the Eshelby theory; the driving force for growth is identified as Eshelby stress [13]. In living tissues experiencing growth, the so-called material forces arising from Eshelby stress drive the evolution of growth at locations where mechanical stimulus is high, in order to promote a more homogeneous state. Notice that the Eshelby tensor arises in the framework of configurational mechanics which follows the pioneering ideas presented in Eshelby [13], who introduced the so-called Maxwell-tensor of elasticity as the driving force for the motion of an inclusion in this famous *Gedankenexperiment*. For growing material with the free energy density Ψ_g given by (1.10), the Eshelby tensor $\mathbf{b}$ is defined as follows, cf. [9],

$$\mathbf{b} = \Psi_g \mathbf{I} - \mathbf{F}_e^\top \frac{\partial \Psi_g}{\partial \mathbf{F}_e} - 2\left\{\frac{\partial \Psi_g}{\partial \mathbf{Q}_e} : \mathbf{Q}_e\right\}^\top. \tag{1.18}$$

Notice that the Eshelby tensor coincides, with the accuracy up to unessential multiplier, with the derivative of the free energy density with respect to $\mathbf{F}_g$. By analogy with definition (1.12) of stress tensors, $\mathbf{b}$ can be regarded as the stress tensor driven by inhomogeneities or by a change of configuration. In papers [5,9,11,12] it was observed that the Clausius-Duhem inequality yields the relation

$$\mathbf{b} : (\partial_t \mathbf{F}_g \mathbf{F}_g^{-1}) \leqq 0. \tag{1.19}$$

As noted in [9,12], inequality (1.19) and the covariance principle lead to the following evolution equation for the growth factor $\mathbf{F}_g$

$$\partial_t \mathbf{F}_g\ \mathbf{F}_g^{-1} = -c_0^+\,\mathrm{tr}\,\mathbf{b}\ \mathbf{I} - c_1^+\ \mathbf{b}, \tag{1.20}$$

where $c_k^+ = c_k^+(I_i, \theta)$ are nonnegative functions of the temperature and the invariants I_i of the Eshelby tensor. Notice that the right hand side of (1.20) is the only isotropic tensor function satisfying (1.19). In the case of the isotropic growth with $\mathbf{F}_g = w\mathbf{I}$ equation (1.20) becomes

$$d\,\partial_t w = -w\, c_0^+(\theta, \mathrm{tr}\,\mathbf{b})\ \mathrm{tr}\,\mathbf{b}. \tag{1.21}$$

It follows that the simplest version of the thermodynamically consistent evolution equation for the growth factor is the ordinary differential equation

$$\partial_t w = -w\, H(\mathrm{tr}\, \mathbf{b}), \tag{1.22}$$

where $H : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $H(s)s \geqq 0$.

Simplified problem. Equations (1.16) and (1.22) form the closed system of differential equations for the deformation field $\mathbf{u}$, the temperature θ, and the growth factor w. The mathematical analysis of this system encounters the following problems:

(a) The nonconvexity of the free energy density. For majority of nonlinear materials, the free energy density is a nonconvex function of the deformation gradient $D\mathbf{u}$. This leads to the multiplicity of solutions to the moment balance equations and spontaneous jumps of solutions to full system (1.16), (1.22).
(b) Compactness problem. The growth factor w serves as the coefficient in the principle part of the momentum balance equation (1.16a). On the other hand, w is coupled with the deformation gradient in a complicated manner via the evolution equation (1.22). In the general case only L^∞ estimates for w are admissible. These estimates are insufficient for applying the methods of the theory of elliptic equations to equation (1.16a).
(c) The high order nonlinearity $|\nabla(\log\theta)|^2$ in parabolic equation (1.16b) for $\log\theta$.

In this paper we focus on the problem (a). In order to cope with the other difficulties we replace equations (1.16) with a physically reasonable simplified system. First, we restrict our considerations by the Falk model with the strain gradient energy density $2^{-1}\varepsilon w^{d-4}|\Delta\mathbf{u}|^2$. As it will be shown in Section 2, in this case the hyperstresses $\varepsilon w^{d-4}\Delta\mathbf{u}$ have extra regularity properties which leads to estimates for the gradient of the growth factor.

A further simplification is the linearization of problem with respect to temperature near some equilibrium value θ_c. Without loss of generality we can assume that $\theta_c = 1$. This means that $\theta = 1 + \vartheta$. With this notation the temperature dependant terms in the expression (1.2) for the free energy density and in the energy balance equation (1.16b) become

$$\begin{aligned} c_1\theta - c_2\theta\log\theta &= (c_1 - c_2)(1+\vartheta) - \frac{c_2}{2}\vartheta^2 + o(\vartheta^3),\\ -c_1 + c_2 + c_2\log\theta &= -(c_1 - c_2) + \vartheta + o(\vartheta^2),\\ \Delta\log\theta &= \Delta\vartheta + o(\vartheta^2),\ |\nabla(\log\theta)|^2 = o(\vartheta^3). \end{aligned}$$

We assume that temperature deviation from the equilibrium is small and neglect the terms of order $o(\vartheta^2)$ in the energy balance equation (1.16b). Without loss of generality we may assume that $c_2 = 1$ and replace W by $W + c_1 - c_2$. After that we obtain the simplified version of equation (1.16b)

$$\varepsilon\partial_t\big(w^d(\vartheta - W(D(\mathbf{u}))\big) = \Delta\vartheta + (1-\vartheta)g. \tag{1.23}$$

Repeating these arguments and recalling formula (1.4) for the Falk energy we obtain the following simplified expression for the free energy density of the growing material:

$$\Psi_g(\mathbf{u}, \vartheta, w) = \frac{\varepsilon}{2} w^{d-4}|\Delta \mathbf{u}|^2 + (1+\vartheta) w^d W(w^{-1} D\mathbf{u}) - \frac{w^d \vartheta^2}{2}, \quad (1.24)$$

The corresponding entropy function is defined by equalities

$$v_g = w^d v, \quad \text{where} \quad v = \vartheta - W(w^{-1} D\mathbf{u}). \quad (1.25)$$

For simplicity we discard the external heat source g and take $\varepsilon = 1$. Next, recall that for the Falk strain-gradient energy density the coefficients a_{ijnmpq} in the principle part of (1.16a) are equal to $\delta_{ij}\delta_{mn}\delta_{pq}$. Combining momentum equation (1.16a) and energy balance equation (1.23) with the evolution equation (1.22) we arrive at the following system of differential equations which describe the isotropic volumetric growth of the thermoelastic material:

$$\varepsilon \Delta(w^{d-4}\Delta \mathbf{u}) - \operatorname{div}\big(w^{d-1}(1+\vartheta) W'(w^{-1} D\mathbf{u})\big) = \mathbf{f} \ \text{ in } \Omega \times (0, T) \quad (1.26a)$$

$$\partial_t (w^d v) = \Delta \vartheta \ \text{ in } \Omega \times (0, T), \quad (1.26b)$$

$$\partial_t w = -w\, H(\varphi) \ \text{ in } \Omega \times (0, T). \quad (1.26c)$$

Here ϑ is given by

$$\vartheta = v + W\big(w^{-1} D\mathbf{u}\big). \quad (1.26d)$$

Formulae (1.18) and (1.24) imply the following expression for the trace of the Eshelby tensor $\varphi := \operatorname{tr} \mathbf{b}$:

$$\begin{aligned} \varphi = {} & \frac{\varepsilon}{2}(d-4) w^{d-4}|\Delta \mathbf{u}|^2 + (1+\vartheta) w^d \big(d\, W(w^{-1} D\mathbf{u}) \\ & - W'(w^{-1} D\mathbf{u}) : (w^{-1} D\mathbf{u})\big) - \frac{d\, w^d}{2} \vartheta^2. \end{aligned} \quad (1.26e)$$

Obviously we have

$$\varphi = w\ \partial_w \Psi_g. \quad (1.26f)$$

Equations (1.26a)–(1.26c) should be supplemented with boundary and initial conditions. We take them in the form

$$\mathbf{u} = \mathbf{h}, \quad \Delta \mathbf{u} = 0, \quad \frac{\partial \vartheta}{\partial n} + \vartheta = 0 \ \text{ on } \partial\Omega \times (0, T), \quad (1.26g)$$

$$v\Big|_{t=0} = v_0, \quad w\Big|_{t=0} = w_0 \ \text{ in } \Omega, \quad (1.26h)$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$. The boundary condition for ϑ is the standard radiation condition. The boundary condition for the displacement means that the growing material is surrounded by the duct membrane whose shape is defined by the function $\mathbf{h}$.

Assumptions. The equations and the boundary and initial conditions (1.26) form a closed boundary value problem for the deformation field $\mathbf{u}$, entropy v, temperature ϑ, and the growth factor w. Furthermore we assume that the stored elastic energy W satisfies one of the following conditions:

H.1a The function W is in $C^2(\mathbb{R}^{d^2})$ and

$$0 \leqq W(\boldsymbol{\xi}) \leqq c(1+|\boldsymbol{\xi}|)^{\kappa},\ |W'(\boldsymbol{\xi})| \leqq c(1+|\boldsymbol{\xi}|)^{\kappa-1},$$
$$|W''(\boldsymbol{\xi})| \leqq c(1+|\boldsymbol{\xi}|)^{\kappa-2}, \tag{1.27}$$

where $\kappa \in [2, 3)$ for $d = 3$ and $\kappa \in [2, \infty)$ for $d = 2$.

The smoothness conditions are too restrictive for many real materials. In such a case we replace (**H.1a**) with the following algebraic condition:

H.1b The function W admits the representation

$$W(D\mathbf{u}) = \sum_{i=1}^{N} c_{\alpha_i} W_{\alpha_i}(D\mathbf{u}) + c_0, \tag{1.28}$$

where W_{α_i} are Lipschitz homogeneous functions such that

$$W_{\alpha_i}(w\boldsymbol{\xi}) = (w^2)^{\alpha_i} W_{\alpha_i}(\boldsymbol{\xi}) \text{ for all } w \in \mathbb{R}, \quad \boldsymbol{\xi} \in \mathbb{R}^{d^2}. \tag{1.29}$$
$$0 \leqq W_{\alpha_i}(\boldsymbol{\xi}) \leqq c|\boldsymbol{\xi}|^{2\alpha_i},\ |W'_{\alpha_i}(\boldsymbol{\xi})| \leqq c|\boldsymbol{\xi}|^{2\alpha_i-1}, \tag{1.30}$$

the exponents $\alpha_i \in [1, 3/2)$ for $d = 3$ and $\alpha_i \in [1, \infty)$ for $d = 2$.

Remark 2. If W satisfies condition (**H.1b**), then the free energy density and the trace of the Eshelby tensor depend algebraically on the growth factor w and admit the representation

$$\begin{aligned}\Psi_g = {} & \frac{\varepsilon}{2} w^{d-4}(\Delta\mathbf{u})^2 + (1+\vartheta)\sum_{i=1}^{N} c_{\alpha_i} w^{d-2\alpha_i} W_{\alpha_i}(D\mathbf{u}) \\ & - \frac{w^d}{2}\vartheta^2 + (1+\vartheta)c_0 w^d,\end{aligned} \tag{1.31}$$

$$\begin{aligned}\varphi = {} & \frac{\varepsilon(d-4)}{2} w^{d-4}(\Delta\mathbf{u})^2 + (1+\vartheta)\sum_{i=1}^{N} c_{\alpha_i}(d-2\alpha_i) w^{d-2\alpha_i} W_{\alpha_i}(D\mathbf{u}) \\ & - \frac{dw^d}{2}\vartheta^2 + (1+\vartheta)c_0 d w^d.\end{aligned} \tag{1.32}$$

The following lemma shows that the Ogden material satisfies condition (**H.1b**) for a suitable choice of the material constants α_i:

Lemma 1.1. *The stored elastic energy density of the Ogden material defined by relation* (1.6) *with exponents* $\alpha_i \in [1, 3/2)$ *for* $d = 3$ *and* $\alpha_i \in [1, \infty)$ *for* $d = 2$ *satisfies condition* (**H.1b**).

Proof. We begin with the observation that the stored elastic energy density for the Ogden material admits representation (1.28) with the functions W_{α_i} and the constant c_0 defined by

$$W_{\alpha_i}(D\mathbf{u}) = \sum_{k=1}^{d} \lambda_k^{\alpha_i}, \quad c_0 = -d\sum_{i=1}^{N} c_{\alpha_i}, \tag{1.33}$$

where λ_k are the eigenvalues of the nonnegative matrix $D\mathbf{u}^\top D\mathbf{u}$. Obviously $\lambda_k(wD\mathbf{u}) = w^2\lambda_k(D\mathbf{u})$ which implies identity (1.29). Notice that

$$|\lambda_k| \leqq |D\mathbf{u}^\top D\mathbf{u}| \leqq d^2|D\mathbf{u}|^2. \tag{1.34}$$

Next, consider the entries $D_j u_i$ of the matrix $D\mathbf{u}$ as independent variables. Fix an arbitrary (i, j) and the entries $D_n u_m$ with $(m, n) \neq (ij)$. In this setting the matrix $D\mathbf{u}^\top D\mathbf{u}$ becomes a quadratic function of the real variable $D_j u_i$, and the eigenvalues of this matrix can be regarded as a functions of $D_j u_i$. By the famous Rellich Theorem, see Kato [25] ch.2 Thm. 6.8, there is a complete collection of eigenvalues $\overline{\lambda}_k$, $1 \leqq k \leqq d$ of the matrix $D\mathbf{u}^\top D\mathbf{u}$ such that $\overline{\lambda}_k$ is continuously differentiable function of $D_j u_i$. Notice that the sequence of the eigenvalues $\overline{\lambda}_k$ is not ordered and their numeration depends on the choice of (i, j). It follows that each element of the ordered sequence of eigenvalues λ_k is a Lipschitz function of the entries of the matrix $D\mathbf{u}$. Hence the functions W_{α_i} satisfy the Lipshitz conditions for all $\alpha_i \geqq 1$. The derivative $\partial\lambda_k/\partial(D_j u_i)$ is defined by the equality, [25],

$$(D\mathbf{u}^\top D\mathbf{u} - \lambda_k\mathbb{I})\,\boldsymbol{\zeta} = \frac{\partial\lambda_k}{\partial(D_j u_i)}\boldsymbol{\eta}_k - \frac{\partial(D\mathbf{u}^\top D\mathbf{u})}{\partial(D_j u_i)}\boldsymbol{\eta}_k,$$

where $\boldsymbol{\eta}_k$ is the unit eigenvector corresponding to λ_k, $\boldsymbol{\zeta} \in \mathbb{R}^d$. Multiplying both sides of this equality by $\boldsymbol{\eta}_k$ we obtain

$$\left|\frac{\partial\lambda_k}{\partial(D_j u_i)}\right| = \left|\frac{\partial(D\mathbf{u}^\top D\mathbf{u})}{\partial(D_j u_i)}\,\boldsymbol{\eta}_k\cdot\boldsymbol{\eta}_k\right| \leqq \left|\frac{\partial(D\mathbf{u}^\top D\mathbf{u})}{\partial(D_j u_i)}\right| \leqq c|D\mathbf{u}|,$$

which along with (1.34) yields the estimate

$$\left|\frac{\partial(\lambda_k)^{\alpha_i}}{\partial(D_j u_i)}\right| \leqq c(\lambda_k)^{\alpha_i-1}|D\mathbf{u}| \leqq c|D\mathbf{u}|^{2\alpha_i-1} \quad \text{for} \quad \alpha_i \geqq 1. \tag{1.35}$$

Combining inequalities (1.34) and (1.35) we obtain the desired estimate (1.30). □

Remark 3. Since the functions W_{α_i} are invariant with respect to the permutations of the eigenvalues, the stored elastic energy of the Ogden material is a function of the class C^1. We will not use this fact.

Further we will assume that the function H in equation (1.26c) satisfies the following growth and monotonicity conditions

H.2 The function $H \in C^\infty(\mathbb{R})$ satisfies the conditions

$$H'(s) \geqq 0, \quad H(0) = 0, \quad |H(\varphi)| \leqq c, \quad |H'(\varphi)| \leqq c(1 + |\varphi|)^{-1}. \tag{1.36}$$

The boundedness of H prevents unlimited extension of the growing material and its collapse to a point. The monotonicity condition is due to more complex reasons. As it will be shown in Section 6, this condition prevents the fast oscillations in time of solutions to equations (1.26).

Finally we impose the following restrictions on the given data:

H.3 For simplicity we assume that $\partial_t \mathbf{h} = 0$, $\partial_t \mathbf{f} = 0$, and $|\Omega| = 1$. We also assume that the given data satisfy the conditions

$$\begin{aligned} &v_0 \in W^{1,2}(\Omega), \quad \mathbf{f} \in L^\infty(\Omega), \quad \mathbf{h} \in C^4(\Omega), \\ &w_0 \in W^{1,2}(\Omega), \quad 0 < c^{-1} < w_0 < c < \infty. \end{aligned} \tag{1.37}$$

Notice that the only physically reliable mass forces are the gravity force and the centrifugal force, which are independent of time.

Results. We are now in a position to formulate the main results of this paper. We are looking for a weak solution to problem (1.26), which is defined as follows:

Definition 1.1. (*Weak formulation*) Denote by $\mathcal{W}^{2,p}$, $1 < p < \infty$, the Banach space which consists of all functions $\mathbf{u} \in W_0^{1,p}(\Omega)$ such that

$$\|\mathbf{u}\|_{\mathcal{W}^{2,p}} = \Big(\int_\Omega |\Delta \mathbf{u}|^p \,\mathrm{d}x\Big)^{1/p} < \infty, \quad \mathbf{u} = 0 \text{ on } \partial\Omega.$$

The space $\mathcal{W}^{2,p}$ is topologically and algebraically isomorphic to the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. A tuple of functions $(\mathbf{u}, v, w, \varphi)$ is said to be a weak solution to problem (1.26) if the following apply:

(i) For a.e. $t \in (0, T)$ the function $\mathbf{u}(t) - \mathbf{h}$ belongs to the class $\mathcal{W}^{2,2} \cap W^{2,6}(\Omega)$,

$$\begin{gathered} v, w, \vartheta \in L^2(0, T; W^{1,2}(\Omega)), \quad w^{\pm 1} \in L^\infty(\Omega \times (0, T)) \\ v, \vartheta \in L^\infty(0, T; L^2(\Omega)). \end{gathered} \tag{1.38}$$

(ii) The function w satisfies equation (1.26c) and initial condition (1.26h). The temperature ϑ and the trace of the Eshelby tensor φ are connected with the growth factor w and the entropy v by the relation (1.26d).

(iii) The integral identity

$$\begin{aligned} &\int_0^T \int_\Omega (w^d v \partial_t \varsigma - \nabla\vartheta \cdot \nabla\varsigma)\,\mathrm{d}x\mathrm{d}t \\ &- \int_0^T \int_{\partial\Omega} \vartheta\varsigma \mathrm{d}s\mathrm{d}t + \int_\Omega w_0^d v_0 \varsigma(x, 0)\,\mathrm{d}x = 0 \end{aligned} \tag{1.39}$$

holds for all $\varsigma \in C^\infty(\Omega \times (0, T))$ vanishing on $\Omega \times \{t = T\}$.

(iv) The integral identity

$$\begin{aligned} &\int_\Omega \Big(\varepsilon w^{d-4} \Delta \mathbf{u}(t) \cdot \Delta \boldsymbol{\xi} \\ &\quad + (1 + \vartheta(t)) w^{d-1} W'(w^{-1}(t) D\mathbf{u}(t)) : D\boldsymbol{\xi} - \mathbf{f} \cdot \boldsymbol{\xi}\Big)\,\mathrm{d}x = 0 \end{aligned} \tag{1.40}$$

holds for all $\boldsymbol{\xi} \in C^2(\Omega)$ vanishing on $\partial\Omega$ and for almost all $t \in (0, T)$.

Definition 1.1 does not determine a solution to problem (1.26) in a unique way. Notice that there is a disparity between the unknown functions in equations (1.26). These equations involve time derivatives of the entropy v and of the growth factor w, i.e., v and w are evolutionary variables. The deformation vector field satisfies the estatic equation (1.26a). The properties of solutions to this equations are completely determined by the stored elastic energy density W. In nonlinear elasticity, W is polyconvex but it is not convex. Moreover, if the free energy density is of the form (1.24), then it is not convex and it is not bounded from below even if W is convex. It follows that in the general case the momentum balance equations represented by the integral identity (1.40) have multiple solutions. Hence, for given v and w there are many temperatures ϑ and traces φ satisfying relations (1.26d)–(1.26e), and the number of these quantities depends on the time variable which leads to spontaneous jumps of solutions in time. Due to the time scaling, this means that long periods of slow growth may alternate with the short inflation periods of the fast material growth. Such behavior was observed for aortic growth in blood vessels, where fast dynamics arises due to increased and decreased blood-flow rate, see [42,43]. In order to diminish this arbitrariness and to control the formation of jumps, it is necessary to supplement equations and boundary conditions (1.26) with additional selection rules. We intend to prove that such rules can be formulated as follows: introduce the functions

$$\Theta(D\mathbf{u}, v, w) = v + W(w^{-1}D\mathbf{u}), \quad V(D\mathbf{u}, \vartheta, w) = \vartheta - W(w^{-1}D\mathbf{u}). \tag{1.41}$$

Denote by $E = \Psi_g + V\vartheta w^d$ the density of the internal energy as a function of the temperature and growth factor, and denote by $\mathcal{E}$ the density of the internal energy as a function of the entropy and growth factor. Calculations show that

$$\begin{aligned} E &= \frac{\varepsilon}{2} w^{d-4}|\Delta \mathbf{u}|^2 + w^d W(w^{-1}D\mathbf{u}) + w^d \frac{\vartheta^2}{2}, \\ \mathcal{E} &= \frac{\varepsilon}{2} w^{d-4}|\Delta \mathbf{u}|^2 + w^d W(w^{-1}D\mathbf{u}) + \frac{1}{2} w^d \Theta(D\mathbf{u}, v, w)^2. \end{aligned} \tag{1.42}$$

We denote by $\mathbf{E}$ the total internal energy as a function of the displacements, temperature, and growth factor, and denote by $\boldsymbol{\mathcal{E}}$ the total internal energy as a function of the displacements, entropy, and growth factor, i.e.,

$$\mathbf{E}(\mathbf{u}, \vartheta, w) = \int_\Omega E(D^2\mathbf{u}, D\mathbf{u}, \vartheta, w)\, dx, \quad \boldsymbol{\mathcal{E}}(\mathbf{u}, v, w) = \int_\Omega \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v, w)\, dx. \tag{1.43}$$

Definition 1.2. (Work and marginal function) Introduce the functional

$$\boldsymbol{\mathcal{H}}(\mathbf{u}, v, w) = \boldsymbol{\mathcal{E}}(\mathbf{u}, v, w) - \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx. \tag{1.44}$$

We define the *marginal function* $\mathbf{M}$ of the functional $\boldsymbol{\mathcal{H}}$ by the relation

$$\mathbf{M}(v, w) = \inf_{\mathbf{u}-\mathbf{h} \in \mathcal{W}^{2,2}} \boldsymbol{\mathcal{H}}(\mathbf{u}, v, w, f). \tag{1.45}$$

Notice that $\mathbf{M}(v, w)$ is well defined if $v \in L^2(\Omega)$, $\mathbf{f} \in L^\infty(\Omega)$, and $w^{\pm 1} \in L^\infty(\Omega)$.

Definition 1.3. (*Selection principle 1*) Suppose a weak solution to problem (1.26) satisfies all conditions of Definition 1.1. We say that the deformation field $\mathbf{u}$ satisfies the first selection principle if $\mathcal{H}(\mathbf{u}(t), v(t), w(t)) = \mathbf{M}(v(t), w(t))$ for a.e. $t \in (0, T)$. In other words the deformation field $\mathbf{u}(t)$ is a minimizer of the functional $\mathcal{H}(\cdot, v(t), w(t))$.

Denote by Π the total dissipation rate

$$\Pi(\vartheta, \varphi) = \int_\Omega (|\nabla\vartheta|^2 + H(\varphi)\varphi)\,\mathrm{d}x + \int_{\partial\Omega} \vartheta^2\,\mathrm{d}s. \tag{1.46}$$

It is convenient to represent Π as the sum of two forms

$$\Pi = \Pi_0(\vartheta, \vartheta) + \Pi_1(H(\varphi), \varphi),$$
$$\Pi_0(\vartheta, \upsilon) = \int_\Omega \nabla\vartheta\nabla\upsilon\,\mathrm{d}x + \int_{\partial\Omega} \vartheta\upsilon\,\mathrm{d}s, \quad \Pi_1(\varphi, \psi) = \int_\Omega \psi\varphi\,\mathrm{d}x. \tag{1.47}$$

Definition 1.4. (**Admissible set**) For given $v \in W^{1,2}(\Omega)$ and strictly positive $w \in L^\infty(\Omega)$, denote by $\mathcal{P}(v, w)$ the set of all couples (ϑ, φ) with the following property: There is $\mathbf{u} \in \mathcal{W}^{2,2} + \mathbf{h}$ such that

$$\vartheta = v + W\big(w^{-1}D\mathbf{u}\big), \;\; \varphi = \varphi(D^2\mathbf{u}, D\mathbf{u}, \vartheta, w), \;\; \mathcal{H}(\mathbf{u}, v, w) = \mathbf{M}(v, w). \tag{1.48}$$

Definition 1.5. (Selection principle 2) Suppose a weak solution to problem (1.26) satisfies all conditions of Definition 1.1. We say that the functions ϑ and φ satisfy the second selection principle if the inequality

$$\Pi(\vartheta(t), \varphi(t)) \leqq \Pi_0(\tilde{\vartheta}, \vartheta(t)) + \Pi_1(H(\varphi(t)), \tilde{\varphi}) \tag{1.49}$$

holds for every $(\tilde{\vartheta}, \tilde{\varphi}) \in \mathcal{P}(v(t), w(t))$ and for a.e. $t \in (0, T)$.

The following theorem is the main result of this paper:

Theorem 1.1. *Assume that the stored elastic energy density W satisfies condition* (**H.1a**) *or condition* (**H.1b**). *Furthermore, assume that the function H and the initial and boundary data satisfy conditions* (**H.2**)–(**H.4**). *Then problem* (1.26) *has a weak solution which meets all requirements of Definition* 1.1. *For a. e. t and $v = v(t)$, $w = w(t)$, the functions $\mathbf{u}(t)$, $\vartheta(t)$, and $\varphi(t)$ satisfy the selection principles given by Definitions* 1.3 *and* 1.5.

Mathematical background. The proof of Theorem 1.1 is based on compactness and monotonicity arguments. There are three aspects of our method which deserve brief mention. The first is the implicit time discretization scheme for problem (1.26). Using this scheme we construct approximate solutions to this problem as saddle points of the "action" functional. The second aspect is the formulation of monotonicity inequalities for the sequence of the approximation solutions in terms of the marginal function. These inequalities allow us to eliminate the displacements from the further analysis and, by doing so, cope with the nonconvexity of the free energy with respect to the displacement field. The third aspect is the systematic application of the theory of sliced measures in Banach spaces to the problem of compactness of approximate solutions.

Organization of the paper. We now explain the organization of the paper. In Section 2 we employ the time discretization scheme in order to construct a sequence of approximate solutions $\mathbf{u}_N$, v_N, w_N, ϑ_N, and φ_N to problem (1.26). We deduce estimates for the approximate solutions. In particular, we show that the $\mathbf{u}_N$ are bounded in the space $L^\infty(0, T; W^{2,6}(\Omega))$ and that the strictly positive functions w_N are uniformly bounded from below and above. We also prove that the sequences ϑ_N and φ_N are bounded in the Lebesgue spaces $L^r(0, T; L^p(\Omega))$ and $L^s(0, T; L^q(\Omega))$ for all exponents satisfying inequalities (2.20).

In Section 3 we investigate the compactness properties of the approximate solutions. We show that the sequences v_N, and w_N contain subsequences, still denoted by v_N and w_N, such that v_N converges to some v in $L^r(0, T; L^p(\Omega))$ and w_N converges to some w a.e. in $\Omega \times (0, T)$. Moreover, in Section 3 we show that for every $\eta > 0$ there is a compact set $\mathcal{T}_\eta \subset (0, T)$ with $\mathrm{meas}((0, T)\setminus\mathcal{T}_\eta) < \eta$ such that the totality of the functions $(\vartheta_N(t), \varphi_N(t))$, $t \in \mathcal{T}_\eta$, belongs to a compact set $\Sigma_\eta \subset L^p(\Omega) \times L^q(\Omega)$.

Sections 4 and 5 are the heart of the paper. In Section 4 we derive the monotonicity relations. We prove that for a.e. $0 < t_1 < t_0 < T$, the approximate solutions satisfy the energy dissipation inequality

$$\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1)) + \limsup_{N\to\infty} \Big\{ \int_{t_1+T/N}^{t_0} \Pi_1(H(\varphi_N), \varphi_N) + \frac{1}{2}\int_{t_1+T/N}^{t_0} \Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}s + \frac{1}{2}\int_{t_1}^{t_0-T/N} \Pi(\overline{\vartheta}_N, \overline{\vartheta}_N)\,\mathrm{d}s\Big\} \leqq 0, \quad (1.50)$$

where the auxiliary functions $\overline{\vartheta}_N$ satisfy the conditions $\overline{\vartheta}_N - \vartheta_N \to 0$ in $L^2(0, T; L^2(\Omega))$ as $N \to \infty$. We also prove that the complementary inequality

$$\liminf_{t_1\nearrow t_0} \frac{1}{t_0 - t_1}\Big\{\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1))\Big\} + \Pi_0(\vartheta, \vartheta^*(t_0)) + \Pi_1(\varphi, H^*(t_0)) \geqq 0 \quad (1.51)$$

holds true for every $(\vartheta, \varphi) \in \boldsymbol{\mathcal{P}}(v(t_0), w(t_0))$. Here ϑ^*, H^* are weak limits of the sequences ϑ_N and $H(\varphi_N)$. Notice that the monotonicity relations (1.50) and (1.51) do not involve the displacement field. In Section 5 we obtain a representation for the weak limits of the sequences (ϑ_N, φ_N). We prove the existence of a measurable family of probability measures μ_t on the compact set $\Sigma_\eta \subset L^p(\Omega) \times L^q(\Omega)$ such that

$$\lim_{N\to\infty} \int_{\mathcal{T}_\eta}\int_\Omega F(t, \vartheta_N, \varphi_N)\,dx dt = \int_{\mathcal{T}_\eta}\Big\{\int_{\Sigma_\eta} F(t, \vartheta, \varphi) d\mu_t(\vartheta, \varphi)\Big\}\,\mathrm{d}t \quad (1.52)$$

for every continuous function $F : [0, T] \times \Sigma_\eta \to \mathbb{R}$. Here $\mathcal{T}_\eta$ is a compact set such that $\mathrm{meas}\,(0, T)\setminus\mathcal{T}_\eta \leqq \eta$, where η is an arbitrary positive number. Representation (1.52) has some advantages over the standard representation of weak limits via the Young measure, since F in (1.52) is a general nonlinear functional. It may be an integro-differential form like Π or a nonlinear integral operator. It is a remarkable fact that the support of μ_t is contained in the set $\boldsymbol{\mathcal{P}}(v(t), w(t))$ given by Definition

1.5. This means that for μ_t-almost every (ϑ, φ) there is a displacement field $\mathbf{u}$ such that $\mathbf{u}$, ϑ, and φ satisfy relations (1.48).

In Section 6 we use inequalities (1.50)–(1.51) and representation (1.52) in order to prove that μ_t is the Dirac measure concentrated at the point $(\vartheta^*, \varphi^*) \in L^p(\Omega) \times L^q(\Omega)$. This result yields the strong convergence of the sequences ϑ_N and φ_N. In Section 7 we prove that the limits ϑ^*, φ^*, v and w serve as a weak solution to problem (1.26). This completes the proof of Theorem 1.1.

2. Approximate Solutions: Time Discretization

In this section we construct sequences ϑ_N, v_N, $\mathbf{u}_N$, w_N, $N \geqq 1$, of approximate solutions to problem (1.26) by using time discretization. For given bounded functions w_{n-2}, w_{n-1}, v_{n-1}, and a vector field $\mathbf{f}$, we denote by $S_n(\vartheta, \mathbf{u})$ the integral functional

$$\mathbf{S}_n(\vartheta, \mathbf{u}) = \boldsymbol{\Psi}_g(\mathbf{u}, \vartheta, w_{n-1}) - \frac{\tau}{2}\Pi_0(\vartheta, \vartheta) + \int_\Omega \left(w_{n-2}^3 v_{n-1}\vartheta - \mathbf{f}\cdot\mathbf{u}\right) \mathrm{d}x, \quad (2.1)$$

where the free energy functional $\boldsymbol{\Psi}_g$ is given by (1.24) and the temperature energy dissipation rate Π_0 is given by (1.47). We are looking for the approximate solution to problem (1.26) in the form

$$\begin{aligned} \vartheta_N(x,t) &= \vartheta_n(x), \quad v_N(x,t) = v_n(x), \\ \mathbf{u}_N(x,t) &= \mathbf{u}_n(x), \quad w_N(x,t) = w_n(x,t) \end{aligned} \quad (2.2)$$

for

$$t \in \big((n-1)\tau, n\tau\big], \quad 1 \leqq n \leqq N, \quad \tau = TN^{-1}.$$

Set

$$w_n(x) = w_N(x, \tau n). \quad (2.3)$$

The functions ϑ_n, v_n, and $\mathbf{u}_n$ are defined by the following recurrence relations. We assume that v_0, w_0 are given by the initial data (1.26h) and

$$\theta_0 = \Theta(v_0, 0, w_0), \quad w_{-1} = 0. \quad (2.4)$$

If ϑ_{n-1}, v_{n-1}, and $w_{n-1} = w_N(x, \tau(n-1))$ are already determined for some $n \geqq 1$, we define ϑ_n and $\mathbf{u}_n$ as solutions to the variational problem

$$\mathbf{S}_n(\vartheta_n, \mathbf{u}_n) = \min_{\mathbf{u}-\mathbf{h}\in\mathcal{W}^{2,2}} \max_{\vartheta\in W_0^{1,2}} \mathbf{S}_n(\vartheta, \mathbf{u}). \quad (2.5)$$

Then we define v_n by

$$v_n = \vartheta_n - W(w_{n-1}^{-1} D\mathbf{u}_n) \equiv V(D\mathbf{u}_n, \vartheta_n, w_{n-1}). \quad (2.6)$$

Next, we define w_N on the interval $[\tau(n-1), \tau n]$ as a solution to the Cauchy problem

$$\begin{aligned}\partial_t w_N &= -H(\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N))w_N, \quad \tau(n-1) < t \leqq \tau n, \\ w_N(\tau(n-1)) &= w_{n-1}.\end{aligned} \tag{2.7}$$

Then we define w_n by (2.3) and repeat the process until $n = N = T/\tau$. Finally, we define the approximation φ_N of the trace of the Eshelby tensor by

$$\varphi_N = \varphi(D^2\mathbf{u}_N, D\mathbf{u}_N, \vartheta_N, w_N), \tag{2.8}$$

where φ is given by (1.26e). Notice that $\mathbf{u}_N$, ϑ_N and v_N are piecewise constant functions of the time variable. In contrast, w_N is a Lipschitz continuous function of t. Relations (2.4)–(2.7) form a closed system of recurrent equations for the definition of approximate solution. The next theorem asserts the existence of solutions to this system. In order to formulate this result, it is convenient to introduce the auxiliary functions

$$\overline{w}_N(t) = w_{n-1}, \quad \overline{\vartheta}_N = \overline{\vartheta}_n \text{ for } (n-1)\tau \leqq t < n\tau, \tag{2.9}$$

where $\overline{\vartheta}_n$ is a solution to the variational problem

$$\mathbf{S}_{n+1}(\mathbf{u}_n, \overline{\vartheta}_n) = \max_{\vartheta \in W^{1,2}(\Omega)} \mathbf{S}_{n+1}(\mathbf{u}_n, \vartheta). \tag{2.10}$$

Theorem 2.1. *Assume that the stored elastic energy density W satisfies condition* (**H.1a**) *or condition* (**H.1b**). *Furthermore, assume that the function H and the initial and boundary data satisfy conditions* (**H.2**)-(**H.4**). *Then there are $\tau_0 > 0$ and a positive constant c with the following properties. For every integer $N > T/\tau_0$, problem* (2.4)–(2.6) *has a solution satisfying*

$$\sup_t \int_\Omega \left(|\Delta \mathbf{u}_N|^2 + W(w_N^{-1} D\mathbf{u}_N) + |\vartheta_N|^2\right) \mathrm{d}x + \int_0^T \Pi(\vartheta_N, \varphi_N)\,\mathrm{d}t \leqq c, \tag{2.11}$$

$$0 < c^{-1} \leqq w_N(x,t) \leqq c, \quad |\partial_t w_N(x,t)| \leqq c \text{ a.e. in } \Omega \times [0,T], \tag{2.12}$$

$$|w_N - \overline{w}_N| \leqq c\tau, \quad \int_0^{T-\tau} \int_\Omega |\vartheta_N - \overline{\vartheta}_N|^2 \leqq c\tau. \tag{2.13}$$

Moreover, for every $0 \leqq t_1 < t_0 < T$, we have

$$\begin{aligned}\limsup_{N\to\infty} \Big\{ &\mathcal{H}_N(t_1) - \mathcal{H}_N(t_0) + \int_{t_1+\tau}^{t_0} \Pi_1(H(\varphi_N), \varphi_N) \\ &+ \frac{1}{2}\int_{t_1+\tau}^{t_0} \Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}s + \frac{1}{2}\int_{t_1+\tau}^{t_0-\tau} \Pi(\overline{\vartheta}_N, \overline{\vartheta}_N)\,\mathrm{d}s \Big\} \leqq 0.\end{aligned} \tag{2.14}$$

Here

$$\mathcal{H}_N(t) = \mathcal{E}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t)) - \int_\Omega \mathbf{f} \cdot \mathbf{u}_N(t)\,\mathrm{d}x, \tag{2.15}$$

the total internal energy $\mathcal{E}$ and the forms Π_i are given by (1.47), *and the trace of the Eshelby tensor φ_N is given by* (2.8).

Proof. The proof is in Appendix A. □

Theorem 2.1 implies that the functions $(w_N)^{\pm 1}$ are uniformly bounded and the functions $\mathbf{u}_N$, ϑ_N satisfy the estimates

$$\|\mathbf{u}_N\|_{L^\infty(0,T;\mathcal{W}^{2,2})} + \|\vartheta_N\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\vartheta_N\|_{L^\infty(0,T;L^2(\Omega))} \leqq c, \tag{2.16}$$

where c is independent of N. Now we use bootstrap arguments to obtain stronger estimates. In particular we estimate $\mathbf{u}_N$ in $L^\infty(0,T;W^{2,6}(\Omega))$ and estimate the derivatives of w_N. The corresponding result is

Theorem 2.2. *Under the assumptions of Theorem 2.1,*

$$\sup_{t\in[0,T]} \big(\|\mathbf{u}_N(t)\|_{W^{2,6}(\Omega)} + \|D\mathbf{u}_N(t)\|_{L^\infty(\Omega)}\big) \leqq c, \tag{2.17}$$

$$\sup_{t\in[0,T]} \|\overline{w}_N(t)^{-1}\,\Delta\mathbf{u}_N(t)\|_{W^{1,2}(\Omega)} \leqq c, \tag{2.18}$$

$$\sup_{t\in[0,T]} \|w_N(t)\|_{W^{1,2}(\Omega)} + \|\partial_t w_N\|_{L^2(0,T;W^{1,2}(\Omega))} \leqq c, \tag{2.19}$$

where the constant c is independent of N.

The following proposition gives estimates of the approximate solutions in Lebesgue spaces (it is a straightforward consequence of Theorem 2.2): introduce the exponent (r,p) and (s,q) satisfying the relations

$$\begin{aligned} &1 < s < \infty, \quad 1 < q < 6s/(6s-4),\\ &r = 2s, \quad p = 2q. \end{aligned} \tag{2.20}$$

It follows that

$$2 < r < \infty, \quad 1 < p < 6r/(3r-4).$$

Proposition 2.1. *Under the assumptions of Theorem 2.1,*

$$\|v_N\|_{L^\infty(0,T;L^2(\Omega)} + \|v_N\|_{L^2(0,T;W^{1,2}(\Omega)} \leqq c, \tag{2.21}$$

$$\|\vartheta_N\|_{L^r(0,T;L^p(\Omega)} + \|v_N\|_{L^r(0,T;L^p(\Omega)} \leqq c, \tag{2.22}$$

$$\|\varphi_N\|_{L^s(0,T;L^q(\Omega))} \leqq c, \tag{2.23}$$

where c is independent of N.

The rest of this section is devoted to the proof of Theorem 2.2 and Proposition 2.1.

Proof of Theorem 2.2. The proof falls into two steps.

Step 1. First we prove estimates (2.17)–(2.18). We begin with the proof of these estimate in more complicated case $d = 3$. It follows from Conditions (**H.1a**) and (**H.1b**) that the matrix values function W' admits the estimates

$$W(\boldsymbol{\xi}) + W'(\boldsymbol{\xi}) : \boldsymbol{\xi} \leqq c(1 + |\boldsymbol{\xi}|)^{\varkappa}, \quad \varkappa \in [2,3).$$

The proof is based on the fact that the growth exponent $\varkappa \in [2, 3)$ in (1.27) is less than 3. Since the case $\varkappa = 2$ is trivial, we assume $\varkappa \in (2, 3)$. By the definition of the approximate solution, we have

$$\mathbf{S}_n(\mathbf{u}_n, \vartheta_n) = \min_{\mathbf{u}-\mathbf{h}\in\mathcal{W}^{2,2}} \mathbf{S}(\mathbf{u}, \vartheta_n).$$

It follows that

$$\lim_{\delta\to 0} \delta^{-1}\Big(\mathbf{S}_n(\mathbf{u}_n - \delta\mathbf{q}, \vartheta_n) - \mathbf{S}_n(\mathbf{u}_n, \vartheta_n)\Big) = 0$$

for every $\mathbf{q} \in \mathcal{W}^{2,2}$. This relation can be rewritten in the form

$$\int_\Omega \Big(\frac{\varepsilon}{w_{n-1}}\Delta\mathbf{u}_n\Delta\mathbf{q} + w_{n-1}^2(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n) : D\mathbf{q} - \mathbf{f}\cdot\mathbf{q}\Big)\,\mathrm{d}x = 0.$$

Now choose $\boldsymbol{\xi} \in L^2(\Omega)$ and set $\mathbf{q} = \Delta^{-1}\boldsymbol{\xi}$, where the inverse Δ^{-1} is defined as the solution to the Dirichlet problem

$$\Delta\big(\Delta^{-1}\boldsymbol{\xi}\big) = \boldsymbol{\xi} \ \text{ in } \Omega, \quad \Delta^{-1}\boldsymbol{\xi} = 0 \ \text{ on } \partial\Omega.$$

Thus we get

$$\int_\Omega \Big(\frac{\varepsilon}{w_{n-1}}\Delta\mathbf{u}_n - \Delta^{-1}\operatorname{div}\big(w_{n-1}^2(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n)\big) - \Delta^{-1}\mathbf{f}\Big)\cdot\boldsymbol{\xi}\,\mathrm{d}x = 0,$$

which yields the equation

$$\frac{\varepsilon}{2w_{n-1}}\Delta\mathbf{u}_n = \Delta^{-1}\operatorname{div}\Big\{w_{n-1}^2(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n)\Big\} + \Delta^{-1}\mathbf{f}. \tag{2.24}$$

Since $\mathbf{u}_n - \mathbf{h} \in \mathcal{W}^{2,2}$, we also have

$$\mathbf{u}_n = \mathbf{h} \ \text{ on } \partial\Omega. \tag{2.25}$$

It follows from the general theory of elliptic equations [21] that for all $\mathbf{q} \in W^{k-1,p}(\Omega)$, $k \geqq 1$, $p \in (1, \infty)$, we have

$$\|\Delta^{-1}\operatorname{div}\mathbf{q}\|_{W^{k,p}(\Omega)} \leqq c\|\mathbf{q}\|_{W^{k-1,p}(\Omega)}, \tag{2.26}$$

where c is independent of $\mathbf{q}$. Since the functions $w_N^{\pm 1}$ are uniformly bounded, it follows from the growth condition (1.27) that

$$|w_{n-1}^2(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n)| \leqq c(1+|\vartheta_n|)(1+|D\mathbf{u}_n|)^\gamma, \tag{2.27}$$

where $\gamma = \varkappa - 1 \in (1, 2)$. Now set $\beta_0 = 2$, $\alpha_0 = 6$. The energy estimate (2.16) implies

$$\|\Delta\mathbf{u}_n\|_{L^{\beta_0}(\Omega)} \leqq c, \quad \|D\mathbf{u}_n\|_{L^{\alpha_0}(\Omega)} \leqq c.$$

It follows from this, estimate (2.16), and the Hölder inequality that

$$\begin{aligned}&\|(1+|\vartheta_n|)(1+|D\mathbf{u}_n|)^\gamma\|_{L^{p_0}(\Omega)}\\ &\quad\leqq c(1+\|\vartheta_n\|_{L^2(\Omega)})(1+\|D\mathbf{u}_n\|_{L^6(\Omega)})^\gamma \leqq c,\end{aligned} \tag{2.28}$$

where ${p_0}^{-1} = 2^{-1} + \gamma 6^{-1} < 5/6$. Combining this estimate with (2.26) we arrive at the inequality

$$\|\Delta^{-1} \operatorname{div}\{w_{n-1}^2(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n)\}\|_{W^{1,p_0}(\Omega)} \leqq c.$$

Since the embedding $W^{1,p_0}(\Omega) \hookrightarrow L^{3p_0/(3-p_0)}(\Omega)$ is bounded, we conclude from this and (2.24)–(2.25) that

$$\|\Delta\mathbf{u}_n\|_{L^{\beta_1}(\Omega)} \leqq c, \quad \text{where} \quad \beta_1 = 3p_0/(3-p_0).$$

Since the embedding $W^{2,\beta_1}(\Omega) \hookrightarrow W^{1,3\beta_1/(3-\beta_1)}(\Omega)$ is bounded, we have

$$\|D\mathbf{u}_n\|_{L^{\alpha_1}(\Omega)} \leqq c, \quad \text{where} \quad \alpha_1 = 3\beta_1/(3-\beta_1).$$

Applying the Hölder inequality we arrive at

$$\begin{aligned}&\|(1+|\vartheta_n|)(1+|D\mathbf{u}_n|)^\gamma\|_{L^{p_1}(\Omega)}\\ &\quad \leqq c(1+\|\vartheta_n\|_{L^2(\Omega)})(1+\|D\mathbf{u}_n\|_{L^{\alpha_1}(\Omega)})^\gamma \leqq c,\end{aligned}$$

where ${p_1}^{-1} = 2^{-1} + \gamma {\alpha_1}^{-1}$. Arguing as before we conclude that

$$\|\Delta\mathbf{u}_n\|_{L^{\beta_k}(\Omega)} \leqq c, \quad \|D\mathbf{u}_n\|_{L^{\alpha_k}(\Omega)} \leqq c. \tag{2.29}$$

Here the sequences α_k, p_k, and β_k are defined by the recurrent relations

$$\alpha_k = 3\beta_{k-1}(3-\beta_{k-1})^{-1}, \beta_k = 3p_{k-1}(3-p_{k-1})^{-1}, \quad p_k^{-1} = 2^{-1} + \gamma\alpha_{k-1}^{-1}.$$

Estimates (2.29) hold provided $1 \leqq \beta_{k-1} < 3$ or equivalently $1 \leqq p_k < 3/2$. Notice that the quantities p_k are defined by the recurrent relations $p_k^{-1} = \gamma p_{k-1}^{-1} + 1/2 - 2\gamma/3$, which lead to the equality

$$p_k^{-1} = \gamma^k p_0^{-1} - (4\gamma-3)\frac{\gamma^k - 1}{6(\gamma-1)} = \frac{1}{6(\gamma-1)}\big(\gamma^{k+1}(\gamma-2) + (4\gamma-3)\big).$$

Since $\gamma \in (1,2)$, the sequence p_k^{-1} decreases and tends to $-\infty$ as $k \to \infty$. Hence there is a minimal k such that $p_{k-1} \leqq 3/2$ and $p_k > 3/2$. It follows that $\beta_{k+1} > 3$. In this case the embedding $\mathcal{W}^{2,\beta_{k+1}} \hookrightarrow L^\infty(\Omega)$ is bounded, which yields

$$\|D\mathbf{u}_n\|_{L^\infty(\Omega)} \leqq c, \quad \|(1+|\vartheta_n|)(1+|D\mathbf{u}_n|)^\gamma\|_{L^2(\Omega)} \leqq c. \tag{2.30}$$

We thus get

$$\|\Delta^{-1} \operatorname{div}\{w_{n-1}^2(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n)\}\|_{W^{1,2}(\Omega)} \leqq c.$$

From this and (2.24) we conclude that

$$\|w_{n-1}^{-1}\Delta\mathbf{u}_n\|_{W^{1,2}(\Omega)} \leqq c, \tag{2.31}$$

and hence that

$$\|\mathbf{u}_n\|_{W^{2,6}(\Omega)} \leqq c. \tag{2.32}$$

It remains to note that estimates (2.17) and (2.18) for $d = 3$ obviously follow from (2.30)–(2.32).

Let us consider the case $d = 2$. Arguing as before we conclude that

$$\frac{\varepsilon}{w_{n-1}^2}\Delta \mathbf{u}_n = \Delta^{-1}\operatorname{div}\Big\{w_{n-1}(1+\vartheta_n)W'(w_{n-1}^{-1}D\mathbf{u}_n)\Big\} + \Delta^{-1}\mathbf{f}. \tag{2.33}$$

Recall that $\mathbf{u}_n - \mathbf{h} \in \mathcal{W}^{2,2}$, which yields

$$\mathbf{u}_n = \mathbf{h} \ \text{ on } \partial\Omega. \tag{2.34}$$

Since the embedding $W^{2,2}(\Omega) \hookrightarrow L^\alpha(\Omega)$ is continuous for every $\alpha \in [1,\infty)$, the sequence $D\mathbf{u}_n$ is uniformly bounded in $L^\alpha(\Omega)$. It follows from this and conditions (**H.1a**), (**H.1b**) that the sequence $W'(w_{n-1}D\mathbf{u}_n)$ is bounded in $L^\alpha(\Omega)$ for all $\alpha \in [1,\infty)$. From this and the energy estimate (2.11) for ϑ_n we obtain

$$\|(1+|\vartheta_n|)W'(w_{n-1}D\mathbf{u}_n)\|_{L^{2-\beta}(\Omega)} \leqq c(\beta) \ \text{ for all } \ \beta \in (0,1]. \tag{2.35}$$

Combining this estimate with equation (2.33) and estimate (2.26) we arrive at the inequality

$$\|\Delta \mathbf{u}\|_{W^{1,2-\beta}(\Omega)} \leqq c(\beta). \tag{2.36}$$

Since the embedding $W^{1,2-\beta}(\Omega) \hookrightarrow L^{4/\beta-2}(\Omega)$ is bounded, we conclude from this that for $\beta < 1/2$

$$\|\mathbf{u}_n\|_{W^{2,6}(\Omega)} \leqq \|\Delta \mathbf{u}_n\|_{W^{1,2-\beta}(\Omega)} \leqq c(\beta), \quad \|D\mathbf{u}_n\|_{L^\infty(\Omega)} \leqq \|\mathbf{u}_n\|_{W^{2,6}(\Omega)} \leqq c.$$

Hence $\mathbf{u}_n$ satisfies inequalities (2.17). It remains to note that the boundedness of the sequence $D\mathbf{u}_n$ leads to the estimates

$$\begin{aligned}\|w_{n-1}^2\Delta \mathbf{u}_n\|_{W^{1,2}(\Omega)} &\leqq c\|\operatorname{div}\Delta^{-1}[(1+\vartheta_n)W'(w_{n-1}D\mathbf{u}_n)\|_{W^{1,2}(\Omega)}\\ &\leqq c\|(1+\vartheta_n)W'(w_{n-1}D\mathbf{u}_n)\|_{L^{2-\beta}(\Omega)} \leqq c,\end{aligned} \tag{2.37}$$

which yield (2.18).

Step 2. Now our task is to estimate w_N. Recall that w_N satisfies the ordinary differential equation and initial condition (2.7). Notice that the differentiability of w_N with respect to x follows from the differentiability of $\mathbf{u}_n, \vartheta_n$, and general results on the differentiability of solutions to ordinary differential equations with respect to parameters. It is necessary to prove (2.17). Differentiation of both sides of (2.7) with respect to x gives

$$\begin{aligned}&\partial_t(\nabla w_N) = -H(\varphi)\nabla w_N - H'(\varphi)w_N\nabla\varphi \text{ for } \tau(n-1) < t \leqq \tau n,\\ &\nabla w_N(\tau(n-1)) = \nabla w_{n-1}.\end{aligned} \tag{2.38}$$

If W satisfies condition (**H.1a**), then formula (1.26e) for φ implies

$$\begin{aligned}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N) &= \frac{\varepsilon}{2}(d-4)w_{n-1}^{8-2d}w_N^{d-4}\sigma_n^2\\ &+ w_N^d(1+\vartheta_n))\big(dW(w_N^{-1}D\mathbf{u}_n) - W'(w_N^{-1}D\mathbf{u}_n) : (w_N^{-1}D\mathbf{u}_n)\big) - \frac{d}{2}w_N^d\vartheta^2,\end{aligned} \tag{2.39}$$

where

$$\sigma_n = w_{n-1}^{-1} \Delta \mathbf{u}_n = (\overline{w}_N(t))^{-1} \Delta \mathbf{u}_N(t), \quad t \in ((n-1)\tau, n\tau]. \tag{2.40}$$

It follows that

$$\nabla \varphi = M_1 \nabla w_N + N_1 \nabla w_{n-1} + \mathbf{L}_1, \tag{2.41}$$

where

$$\begin{aligned} M_1 &= \frac{\varepsilon}{2}(d-4)^2 w_{n-1}^{8-2d} w_N^{d-5} \sigma_n^2 + R_1 - \frac{d^2}{2} w_N^{d-1} \vartheta^2, \\ N_1 &= -\varepsilon (d-4)^2 w_{n-1}^{7-2d} w_N^{d-4} \sigma_n^2, \\ \mathbf{L}_1 &= \varepsilon (d-4) w_{n-1}^{8-2d} w_N^{d-4} \sigma_n \nabla \sigma_n + (1+\vartheta_n) w^{d-1} \mathbf{P} \\ &\quad + w^d \big(d W(\boldsymbol{\xi}) - W'(\boldsymbol{\xi}) : \boldsymbol{\xi} \big) \nabla \vartheta_n - d w^d \vartheta_n \nabla \vartheta_n. \end{aligned} \tag{2.42}$$

Here the matrix–valued function $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = w_N^{-1} D \mathbf{u}_n, \tag{2.43}$$

and the scalar function R_1 and the vector function $\mathbf{P} = (P_i)_{1 \leqq i \leqq d}$ are given by

$$\begin{aligned} R_1 &= w_N^{d-1} (1+\vartheta_n) \Big(d^2 W(\boldsymbol{\xi}) + (1-2d) W'(\boldsymbol{\xi}) : \boldsymbol{\xi} + \frac{\partial^2 W(\boldsymbol{\xi})}{\partial \xi_{ij} \partial \xi_{pq}} \xi_{pq} \xi_{ij} \Big), \\ P_k &= \Big((d-1) \frac{\partial W(\boldsymbol{\xi})}{\partial \xi_{ij}} - \frac{\partial^2 W(\boldsymbol{\xi})}{\partial \xi_{ij} \partial \xi_{pq}} \xi_{pq} \Big) \frac{\partial^2 \mathbf{u}_i}{\partial x_j \partial x_k}. \end{aligned} \tag{2.44}$$

If W satisfies condition (**H.1b**), then formula (1.32) for φ implies

$$\begin{aligned} \varphi(D^2 \mathbf{u}_n, D \mathbf{u}_n, \vartheta_n, w_N) = {} & \frac{\varepsilon}{2}(d-4) w_{n-1}^{8-2d} w_N^{d-4} \sigma_n^2 \\ & + (1+\vartheta_n) \sum_{i=1}^{N} c_{\alpha_i} (d - 2\alpha_i) w_N^{d-2\alpha_i} W_{\alpha_i}(D \mathbf{u}_n) \\ & - \frac{d w_N^d}{2} \vartheta_n^2 + (1+\vartheta_n) c_0 d w_N^d. \end{aligned} \tag{2.45}$$

It follows that

$$\nabla \varphi = M_2 \nabla w_N + N_2 \nabla w_{n-1} + \mathbf{L}_2, \tag{2.46}$$

where

$$\begin{aligned} M_2 = {} & \frac{\varepsilon}{2}(d-4)^2 w_{n-1}^{8-2d} w_N^{d-5} \sigma_n^2 + (1+\vartheta_n) \sum_{i=1}^{N} c_{\alpha_i} (d - 2\alpha_i)^2 w_N^{d-1-2\alpha_i} W_{\alpha_i}(D \mathbf{u}_n) \\ & - \frac{d^2}{2} w_N^{d-1} \vartheta_n^2 + (1+\vartheta_n) c_0 d^2 w_N^{d-1} \\ N_2 = {} & N_1, \end{aligned}$$

$$\mathbf{L}_2 = \varepsilon(d-4)w_{n-1}^{8-2d}w_N^{d-4}\sigma_n\nabla\sigma_n - d w_N^{d-1}\vartheta_n\nabla\vartheta_n$$
$$+ (1+\vartheta_n)\sum_{i=1}^{N} c_{\alpha_i}(d-2\alpha_i)w_N^{d-2\alpha_i}W'_{\alpha_i}(D\mathbf{u}_n) : D\mathbf{u}_n\nabla\mathbf{u}$$
$$+ \left(\sum_{i=1}^{N} c_{\alpha_i}(d-2\alpha_i)w_N^{d-2\alpha_i}W_{\alpha_i}(D\mathbf{u}_n) + c_0 d w_N^d\right)\nabla\vartheta_n. \tag{2.47}$$

Since the functions w_N and $D\mathbf{u}_n$ are uniformly bounded, we have

$$|\boldsymbol{\xi}| \leqq c, \quad |R_1| \leqq c, \quad |\mathbf{P}_1| \leqq c|D^2\mathbf{u}_n|. \tag{2.48}$$

It follows that

$$\begin{aligned} &|\mathbf{M}_i| \leqq c(1+|\vartheta_n|^2+|\Delta\mathbf{u}_n|^2), \quad |\mathbf{N}_i| \leqq c|\Delta\mathbf{u}_n|, \\ &|\mathbf{L}_i| \leqq c(1+|\vartheta_n|)(1+|\nabla^2\mathbf{u}_n|) + c|\Delta\mathbf{u}_n||\nabla\sigma_n| + c(1+|\vartheta_n|)|\nabla\vartheta_n|. \end{aligned} \tag{2.49}$$

On the other hand, representations (2.39) and (2.45) imply

$$-\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N) \geqq c^{-1}(|\vartheta_n|^2 + |\Delta\mathbf{u}_n|^2) - c,$$

where $c > 0$ is independent of N. From this and $|H'(\varphi)| \leqq c(1+|\varphi|)^{-1}$ we conclude that

$$|H'(\varphi)| \leqq c(1+|\vartheta_n|^2+|\Delta\mathbf{u}_n|^2)^{-1}.$$

Combining this with (2.50) we arrive at

$$\begin{aligned} &|H'(\varphi)M_i| \leqq c, \quad |H'(\varphi)\mathbf{N}_i| \leqq c, \\ &|H'(\varphi)\mathbf{L}_i| \leqq c(1+|\nabla\sigma_n|+|\nabla\vartheta_n|+|\nabla^2\mathbf{u}_n|). \end{aligned} \tag{2.50}$$

Next, substituting (2.41) and (2.46) into (2.38) we obtain

$$\partial_t(\nabla w_N) = -\big(H(\varphi) + H'(\varphi)w_N M_i\big)\nabla w_N - w_N H'(\varphi)N_i)\nabla w_{n-1} - w_N H'(\varphi)\mathbf{L}_i,$$

which, along with (2.50), yields

$$\partial_t|\nabla w_N| \leqq |\partial_t(\nabla w_N)| \leqq c|\nabla w_N| + c|\nabla w_{n-1}| + cG_n. \tag{2.51}$$

Here

$$G_n = 1 + |\nabla^2\mathbf{u}_n| + |\nabla\vartheta_n| + |\nabla\sigma_n|. \tag{2.52}$$

Multiplying both sides of (2.51) by $\exp(-ct)$ we obtain

$$\partial_t(e^{-ct}|\nabla w_N|) \leqq ce^{-ct}(|\nabla w_{n-1}| + c(1+|\nabla^2\mathbf{u}_n|+|\nabla\vartheta_n|+|\nabla\sigma_n|). \tag{2.53}$$

Choose any $t \in [\tau(n-1), \tau n]$. Integrating this inequality over $[\tau(n-1), t]$ and multiplying the result by e^{ct}, we arrive at

$$\begin{aligned} |\nabla w_N(t)| &\leqq |\nabla w_{n-1}| + (e^{t-(n-1)\tau} - 1)\,(2|\nabla w_{n-1}| + G_n) \\ &\leqq |\nabla w_{n-1}|(1+c\tau) + c\,\tau\,G_n, \end{aligned}$$

where c is independent of N. Applying the Cauchy inequality we obtain

$$|\nabla w_N(t)|^2 \leqq |\nabla w_{n-1}|^2(1+c\tau)+c\tau G_n^2 \quad \text{for } \tau(n-1)\leqq t\leqq \tau n.$$

Integrating both sides over Ω and recalling estimates (2.17)–(2.18) we obtain

$$\begin{aligned}\int_\Omega |\nabla w_N(t)|^2\,\mathrm{d}x &\leqq (1+c\tau)\int_\Omega |\nabla w_{n-1}|^2\,\mathrm{d}x + c\tau\int_\Omega G_n^2\,\mathrm{d}x\\ &\leqq (1+c\tau)\int_\Omega |\nabla w_{n-1}|^2\,\mathrm{d}x + c\tau\int_\Omega (1+|\nabla\vartheta_n|^2)\,\mathrm{d}x.\end{aligned} \tag{2.54}$$

Since $w_N(n\tau)=w_n$, we conclude from this that

$$\int_\Omega |\nabla w_n|^2\,\mathrm{d}x \leqq (1+c\tau)\int_\Omega |\nabla w_{n-1}|^2\,\mathrm{d}x + c\tau\int_\Omega (1+|\nabla\vartheta_n|^2)\,\mathrm{d}x.$$

It follows that

$$\int_\Omega |\nabla w_n|^2\,\mathrm{d}x \leqq (1+c\tau)^n\int_\Omega |\nabla w_0|^2\,\mathrm{d}x + c\tau\sum_{k=0}^n (1+c\tau)^{n-k}\int_\Omega (1+|\nabla\vartheta_k|^2\,\mathrm{d}x.$$

In view of the relation $\tau = TN^{-1}$ we have

$$(1+c\tau)^n \leqq (1+c\tau)^N = \left\{(1+c\tau)^{1/(c\tau)}\right\}^{cT} \leqq e^{cT}.$$

Thus we get

$$\begin{aligned}\int_\Omega |\nabla w_n|^2\,\mathrm{d}x &\leqq e^{cT}\int_\Omega |\nabla w_0|^2\,\mathrm{d}x + ce^{cT}\tau\sum_{k=0}^n\int_\Omega (1+|\nabla\vartheta_k|^2)\,\mathrm{d}x\\ &= e^{cT}\int_\Omega |\nabla w_0|^2\,\mathrm{d}x + ce^{cT}\int_0^{n\tau}\int_\Omega (1+|\nabla\vartheta_k|^2)\,dxdt \leqq c\end{aligned} \tag{2.55}$$

for all $1\leqq n\leqq N$. Combining this result with (2.54) we obtain

$$\int_\Omega |\nabla w_N(t)|^2\,\mathrm{d}x \leqq c \quad \text{for all } t\in[0,T]. \tag{2.56}$$

From this, (2.55), and (2.51) we conclude that

$$\int_\Omega |\partial_t(\nabla w_N(t))|^2\,\mathrm{d}x \leqq c + c\int_\Omega |\nabla\vartheta_N|^2\,\mathrm{d}x$$

for $\tau(n-1) < t \leqq \tau n$. Noting that $\vartheta_N(t)=\vartheta_n$ on this interval, we obtain

$$\int_0^T\int_\Omega |\partial_t(\nabla w_N(t))|^2\,\mathrm{d}x\mathrm{d}t \leqq c + c\int_0^T\int_\Omega |\nabla\vartheta_N|^2\,\mathrm{d}x \leqq c. \tag{2.57}$$

It remains to note that the desired inequality (2.19) clearly follows from (2.56) and (2.57). □

Proof of Proposition 2.1. We first observe that estimate (2.21) obviously follows from (2.16)–(2.19) and the identity $v_N = \vartheta_N - W(w_N^{-1} D\mathbf{u}_N)$.

Let us prove estimate (2.22). Since the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ is bounded, the energy estimate (2.16) yields

$$\|\vartheta_N\|_{L^\infty(0,T;L^2(\Omega))} + \|\vartheta_N\|_{L^2(0,T;L^6(\Omega))} \leqq c.$$

By the interpolation inequality, for every $\alpha \in (0, 1)$ we have

$$\|\vartheta_N\|_{L^r(0,T;L^p(\Omega))} \leqq \|\vartheta_N\|^{1-\alpha}_{L^\infty(0,T;L^2(\Omega))} \|\vartheta_N\|^{\alpha}_{L^2(0,T;L^6(\Omega))} \leqq c, \tag{2.58}$$

where

$$\alpha/2 = 1/r, \quad (1-\alpha)/2 + \alpha/6 = 1/p.$$

Estimate (2.22) for ϑ_N obviously follows from (2.58). Repeating these arguments and using (2.21) we obtain (2.22) for v_N.

It remains to prove estimate (2.23). Recall representation (2.8) for φ_N. Since $w_n^{\pm 1}$ and $D\mathbf{u}_n$ are uniformly bounded, it follows from (2.8) that for almost every $t \in (0, T)$,

$$|\varphi_N(t)| \leqq c + c|\Delta \mathbf{u}_N(t)|^2 + c|\vartheta_N(t)|^2. \tag{2.59}$$

Notice that in view of (2.16) we have

$$\|(\Delta \mathbf{u}_N)^2\|_{L^s(0,T;L^q(\Omega))} \leqq c\|\Delta \mathbf{u}_N\|_{L^\infty(0,T;L^6(\Omega))} \leqq c. \tag{2.60}$$

Next, relation (2.20) yields $2s = r$ and $2q = p$. From this and estimate (2.58) we obtain

$$\|\vartheta_N^2\|_{L^s(0,T;L^q(\Omega))} \leqq c\|\vartheta_N\|^2_{L^r(0,T;L^p(\Omega))} \leqq c. \tag{2.61}$$

Combining (2.59)–(2.61) we arrive at estimate (2.23). □

3. Compactness

In Section 2 we proved the existence of approximate solutions $\mathbf{u}_N, \vartheta_N, v_N, w_N$ to problem (1.26). Our goal is to prove that this sequence has a limit point which is a weak solution to (1.26). Hence, the key question is the compactness of the set of approximate solutions in appropriate Banach spaces. In this section we give a preliminary analysis of this problem. Notice that among the thermodynamical and mechanical quantities in (1.26), only the entropy v and the growth factor w satisfy evolution equations. Therefore, the compactness properties of the sequences v_N and w_N can be established by applying the Dubinski-Lions Lemma. The corresponding result is given by the following theorem, which is the first main result of this section:

Theorem 3.1. *Let all conditions of Theorem 2.1 be satisfied. Then there is a subsequence of* (v_N, w_N)*, still denoted by* (v_N, w_N)*, and functions* v, w *with*

$$w^{\pm 1} \in L^\infty(0,T; L^\infty(\Omega)) \cap L^\infty(0,T; W^{1,2}(\Omega)),\ \partial_t w \in L^2(0,T; W^{1,2}(\Omega)),$$
$$v \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \tag{3.1}$$

such that

$$w_N^{\pm 1} \to w^{\pm 1} \ \text{ in } C(0,T; L^\alpha(\Omega)), \tag{3.2}$$
$$v_N \to v \ \text{ in } L^r(0,T; L^p(\Omega)) \tag{3.3}$$

as $N \to \infty$ *for all* $\alpha \in [1,\infty)$ *and all* (p, r) *satisfying* (2.20).

We cannot guarantee the strong convergence of the sequences $\mathbf{u}_N$, ϑ_N and φ_N since these functions have no smoothness with respect to the time variable. However, they have some smoothness with respect to the spatial variables. Hence we can expect that these functions map the interval $(0, T)$ onto some relatively compact set. The corresponding result is given by the following theorem, which is the second main result of this section:

Theorem 3.2. *Let exponents* s, q *and* r, p *satisfy conditions* (2.20). *Then for every* $\eta > 0$ *there is a compact set* $\mathcal{T}_\eta$ *with the following properties:*

(i) $\mathcal{T}_\eta \subset (0,T)$, $\text{meas}((0,T)\setminus\mathcal{T}_\eta) \leqq \eta$.
(ii) The mappings $v : \mathcal{T}_\tau \to L^p(\Omega)$ *and* $v_N : \mathcal{T}_\tau \to L^p(\Omega)$ *are continuous and*

$$v_N(t) \to v(t) \ \text{ in } L^p(\Omega) \text{ uniformly on } \mathcal{T}_\eta. \tag{3.4}$$

(iii) For every $\alpha \in [1,\infty)$,

$$w_N(t) \to w(t) \ \text{ in } L^\alpha(\Omega) \text{ uniformly on } \mathcal{T}_\eta. \tag{3.5}$$

(iv) The set

$$\mathfrak{T}(\eta) = \{(\vartheta_N(t), \varphi_N(t)) : N \geqq 1,\ t \in \mathcal{T}_\eta\}, \tag{3.6}$$

is relatively compact in $L^p(\Omega) \times L^q(\Omega)$.

The rest of this section is devoted to the proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. We begin by proving (3.2). It suffices to show that the sequence $\{w_N(t)\}_{N \geqq 1}$ is relatively compact in $C(0,T; L^\alpha(\Omega))$ for every $\alpha \in [1,\infty)$. Set

$$\mathfrak{M}_w = \{w_N(t) : t \in [0,T],\ N \geqq 1\}.$$

It follows from (2.17) that $\mathfrak{M}_w$ is bounded in $W^{1,2}(\Omega)$ and hence in $L^r(\Omega)$ for every $r \in [1,6)$. In particular, it is relatively compact in measure. On the other hand, inequality (2.17) yields the boundedness of $\mathfrak{M}_w$ in $L^\infty(\Omega)$. Hence $\mathfrak{M}_w$ is relatively compact in $L^\alpha(\Omega)$ for all $\alpha \in [1,\infty)$.

Next, it follows from estimate (2.19) that for $1 \leqq \alpha \leqq 6$, $h \in (0, T)$, and $0 \leqq t \leqq T - h$,

$$\|w_N(t+h) - w_N(t)\|_{L^\alpha(\Omega)} \leqq c\|w_N(t+h) - w_N(t)\|_{W^{1,2}(\Omega)}$$
$$\leqq c\left\| \int_t^{t+h} \partial_s w_N(s)\, \mathrm{d}s \right\| \leqq h^{1/2}\|\partial_t w_N(t)\|_{L^2(0,T;W^{1,2}(\Omega))} \leqq ch^{1/2}. \quad (3.7)$$

If $\alpha > 6$ we apply the interpolation inequality to obtain

$$\|w_N(t+h) - w_N(t)\|_{L^\alpha(\Omega)}$$
$$\leqq c\|w_N(t+h) - w_N(t)\|_{L^6(\Omega)}^{6/\alpha}\|w_N(t+h) - w_N(t)\|_{L^\infty(\Omega)}^{(\alpha-6)/\alpha}$$
$$\leqq c\|w_N(t+h) - w_N(t)\|_{L^6(\Omega)}^{6/\alpha} \leqq ch^{3/\alpha}. \quad (3.8)$$

Estimates (3.7) and (3.8) show that the sequence w_N is equicontinuous in $C(0, T; L^\alpha(\Omega))$. Recall that w_N takes values in the relatively compact set $\mathfrak{M}_w$. Application of the Ascoli Theorem completes the proof of (3.2). □

Our next task is to prove (3.3). Recall that $(\mathbf{u}_n, \vartheta_n)$ is a solution to the variational problem

$$\mathbf{S}_n(\mathbf{u}_n, \vartheta_n) = \max_{\vartheta \in W^{1,2}(\Omega)} \mathbf{S}_n(\mathbf{u}_n, \vartheta).$$

Calculation of the variation of $\mathbf{S}_n$ at the point ϑ_n leads to the linear elliptic boundary value problem for ϑ_n,

$$\begin{aligned} &-\tau\Delta\vartheta_n + w_{n-1}^d V(D\mathbf{u}_n, \vartheta_n, w_{n-1}) = w_{n-2}^d v_{n-1} \ \text{ in } \Omega, \\ &\partial_n\vartheta_n + \vartheta_n = 0 \ \text{ on } \partial\Omega. \end{aligned} \quad (3.9)$$

Here V is given by (1.41), i.e.,

$$V(D\mathbf{u}_n, \vartheta_n, w_{n-1}) = \vartheta_n - W(w_{n-1}^{-1} D\mathbf{u}_n) = v_n.$$

It follows that

$$v_n - v_{n-1} = w_{n-1}^{-d}(\tau\Delta\vartheta_n + R_n), \ \text{ where } \ R_n = (w_{n-2}^d - w_{n-1}^d)v_{n-1}. \quad (3.10)$$

Recall that $w_{n-2} = w_N((n-2)\tau)$ and $w_{n-1} = w_N((n-1)\tau)$. From this and (2.19) we conclude that $|w_{n-2}^d - w_{n-1}^d| \leqq c\tau$ and hence

$$|R_n| \leqq c\,\tau\,|v_{n-1}|. \quad (3.11)$$

In view of (2.19), we have $|w_{n-1}^{\pm 1}| \leqq c$ and $\|\nabla w_{n-1}\|_{L^2(\Omega)} \leqq c$. Fix $\lambda > 3$. Since the embedding $W_0^{1,\lambda}(\Omega) \hookrightarrow L^\infty(\Omega)$ is bounded, we have

$$\|w_{n-1}^{-d}\zeta\|_{W^{1,2}(\Omega)} \leqq c\|\zeta\|_{W_0^{1,\lambda}(\Omega)} \quad \text{for all } \zeta \in W_0^{1,\lambda}(\Omega). \quad (3.12)$$

Obviously $\zeta w_{n-1}^{-d} \in W_0^{1,2}(\Omega)$. Thus we get

$$\int_\Omega \zeta w_{n-1}^{-d}\Delta\vartheta_n\, \mathrm{d}x = -\int_\Omega \nabla(\zeta w_{n-1}^{-d})\nabla\vartheta_n\, \mathrm{d}x \leqq c\|\nabla\vartheta_n\|_{L^2(\Omega)}\|\zeta\|_{W_0^{1,\lambda}(\Omega)}.$$

This means that

$$\|w_{n-1}^{-1}\Delta\vartheta_n\|_{W^{-1,\mu}(\Omega)} \leqq c\|\nabla\vartheta_n\|_{L^2(\Omega)} \quad \text{for all } \mu = \lambda/(\lambda-1) \in (1, 3/2),$$

which along with (3.10)–(3.11) yields

$$\begin{aligned}\|v_n - v_{n-1}\|_{W^{-1,\mu}(\Omega)} &\leqq c\tau\|\nabla\vartheta_n\|_{L^2(\Omega)} + \|R_n\|_{L^\mu(\Omega)}\\ &\leqq c\tau\|\vartheta_n\|_{W^{1,2}(\Omega)} + c\tau\|v_{n-1}\|_{L^\mu(\Omega)}.\end{aligned}$$

Thus we get

$$\begin{aligned}\sum_{n=1}^{N}\|v_n - v_{n-1}\|_{W^{-1,\mu}(\Omega)} &\leqq c\tau\sum_{n=1}^{N}(\|\vartheta_n\|_{W^{1,2}(\Omega)} + \|v_{n-1}\|_{L^\mu(\Omega)})\\ &\leqq c\int_0^T(\|\vartheta_N(t)\|_{W^{1,2}(\Omega)} + \|v_N(t-\tau)\|_{L^\mu(\Omega)})\,\mathrm{d}t \leqq c.\end{aligned}$$

It follows that the total variation of the piecewise constant function $v_N : [0, T] \to W^{-1,\mu}(\Omega)$ is bounded by a constant c independent of N. Thus

$$\int_0^{T-h}\|v_N(t+h) - v_N(t)\|_{W^{-1,\mu}(\Omega)}\,\mathrm{d}t \leqq ch \quad \text{for } 0 < h < T. \tag{3.13}$$

On the other hand, estimate (2.21) yields

$$\int_T^{T-h}\|v_N(t)\|_{W^{1,2}(\Omega)} \leqq ch^{1/2} \quad \text{for } 0 < h < T. \tag{3.14}$$

As the embedding $W^{1,2}(\Omega) \hookrightarrow L^\mu(\Omega) \hookrightarrow W^{-1,\mu}(\Omega)$ is compact, Theorem 5 in [39] implies that the sequence v_N is relatively compact in $L^1(0, T; L^\mu(\Omega))$. Hence it is relatively compact in measure. On the other hand, in view of Proposition 2.1 this sequence is bounded in $L^r(0, T; L^p(\Omega))$ for all r and p satisfying (2.20). Since the set of admissible r and p is open, we conclude that the sequence v_N is relatively compact in $L^r(0, T; L^p(\Omega))$. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. Since $L^p(\Omega)$ is separable, the piecewise constant mappings $v_N : (0, T) \to L^p(\Omega)$ are strongly measurable on $(0, T)$. On the other hand, they converge strongly to v in $L^r(0, T; L^p(\Omega))$, Hence v_N converges to v in measure in $(0, T)$, and the sequence v_N meets all requirements of the Egoroff Theorem. We conclude that for every $\eta > 0$ there is a compact set $\mathcal{T}_\eta$ satisfying (i) and (ii). Item (iii) obviously follows from (3.3).

In order to prove (iv) notice that in view of (2.6) and (2.9) we have

$$\vartheta_N(t) = v_N(t) + W(\overline{w}_N(t)^{-1}D\mathbf{u}_N(t)), \tag{3.15}$$

where $\overline{w}_N(t)$ is defined by (2.9). It follows from (3.3) that

$$\|\overline{w}_N - w_N\|_{C(0,T;L^\alpha(\Omega))} \to 0 \quad \text{uniformly in } N \quad \text{as } N \to \infty. \tag{3.16}$$

Now choose a sequence $t_m \in \mathcal{T}_\eta$, $m \geqq 1$. After passing to a subsequence we may assume that $t_m \to t_0 \in \mathcal{T}_\eta$ as $m \to \infty$. It follows from (3.3) and (3.16) that

$\overline{w}_N(t_m) \to w(t_0)$ in $L^\alpha(\Omega)$ as $m, N \to \infty$. After passing to a subsequence we may assume that

$$\overline{w}_N(t_m, x) \to w(t_0, x) \text{ a.e. in } \Omega. \tag{3.17}$$

Next, it follows from (2.17) that the sequence $\mathbf{u}_n(t_m)$ is uniformly bounded in $W^{2,6}(\Omega)$. Recall that the embedding $W^{2,6}(\Omega) \hookrightarrow C^1(\Omega)$ is compact. Hence, after passing to a subsequence we may assume that $D\mathbf{u}_N(t_m)$ converges uniformly in Ω. Recalling (3.16) we deduce that $W(\overline{w}_N(t_m)^{-1} D\mathbf{u}_N(t_m))$ converges in measure in Ω. Since the functions $W(\overline{w}_N(t_m)^{-1} D\mathbf{u}_N(t_m))$ are bounded, it follows that this sequence converges in $L^p(\Omega)$. On the other hand, (ii) implies that $v_N(t_m)$ converges to $v(t_0)$ in $L^p(\Omega)$. From this and (3.15) we find that $\vartheta_N(t_m)$ converges in $L^p(\Omega)$ as $(m, N) \to \infty$. Hence, *every sequence* $\vartheta_N(t_m)$ *contains a subsequence which converges in* $L^p(\Omega)$. Next, in view of (2.8) and (1.26e),

$$\begin{aligned} \varphi_N = {} & \frac{\varepsilon}{2}(d-4) w_N^{d-4} \overline{w}_N^{8-2d} \big(\overline{w}_N^{d-4} |\Delta \mathbf{u}_N|\big)^2 - \frac{d\varepsilon}{2} \overline{w}_N^d \vartheta_N^2 \\ & + w_N^d (1 + \vartheta_N) \big\{ d W(w_N^{-1} D\mathbf{u}_N) - w_N^{-1} W'(w_N^{-1} D\mathbf{u}_N) : D\mathbf{u}_N \big\}. \end{aligned} \tag{3.18}$$

Consider now $\varphi_N(t_m)$. By (2.18) the sequence $\overline{w}_N^{d-4} \Delta \mathbf{u}_N(t_m)$ is bounded in $W^{1,2}(\Omega)$. Hence, after passing to a subsequence, we may assume that this sequence converges a.e. in Ω. We have proved that $\overline{w}_N(t_m)$, $\vartheta_n(t_m)$ and $D\mathbf{u}_N(t_m)$ converge a.e. in Ω as $(N, m) \to \infty$. Hence $\varphi_N(t_m)$ converges a.e. in Ω.

Next, since $w_N^{\pm 1}$ and $D\mathbf{u}_N$ are uniformly bounded, relation (3.16) implies that for every $t \in (0, T)$,

$$|\varphi_N(t)| \leqq c + c|\Delta \mathbf{u}_N(t)|^2 + c|\vartheta_N(t)|^2. \tag{3.19}$$

Notice that in view of (2.20) we have

$$\| |\Delta \mathbf{u}_N(t)|^2 \|_{L^q(\Omega)} \leqq c \|\Delta \mathbf{u}_N\|_{L^\infty(0,T;L^6(\Omega))} \leqq c. \tag{3.20}$$

We have already proved that the sequence $\vartheta_N(t_m)$ is relatively compact in $L^p(\Omega)$. From this and $2q = p$ we conclude that the sequence $\vartheta_N(t_m)^2$ is relatively compact in $L^q(\Omega)$. Recalling (3.19) and (3.20) we conclude that the sequence $\varphi_N(t_m)$ is bounded in $L^q(\Omega)$ for all q satisfying (2.20). Since this sequence converges in measure in Ω and the set of admissible q is open, we conclude that $\varphi_N(t_m)$ converges strongly in $L^q(\Omega)$. Thus we prove that for every $t_m \in \mathcal{T}_\eta$, the sequence $\varphi_N(t_m)$ contains a subsequence which converges in $L^q(\Omega)$. Hence the set $\mathfrak{T}(\eta)$ is relatively compact in $L^p(\Omega) \times L^q(\Omega)$. This completes the proof of Theorem 3.2. □

4. Marginal Function: Energy Dissipation Inequalities

In this section we deduce the energy dissipation inequalities which play a crucial role in the further analysis. In Sections 2 and 3 we have built the sequence of approximate solutions $\mathbf{u}_N$, ϑ_N, w_N, and v_N to problem (1.26) and investigated their properties. In particular, we have proved the strong convergence of the evolutionary variables v_N and w_N. Now we start a long sequence in order to prove the convergence of ϑ_N and φ_N. Our tool is the monotonicity method, which is based on a careful analysis of the energy dissipation inequality and works well for problems with a convex free energy functional. In our case the main difficulty is that the free energy density is a nonconvex function of the displacement vector field **u**; however, it is a concave function of the temperature ϑ. Moreover, the right hand side of equation (1.26c) for the growth factor w is a monotone function of the trace φ of the material Eshelby tensor. The idea is to eliminate the displacement vector field and to focus on the sequences ϑ_N and φ_N. The key observation is the following: Substitute the approximate solution into expressions (1.42)–(1.43) for the internal energy we get the approximate value of the total internal energy $\mathcal{E}_N$ as a real valued function of the time variable. Since the free energy and the internal energy depend on the displacement vector field **u**, it is hard to expect that the sequence $\mathcal{E}_N$ converges for a fixed t. It is a remarkable fact of the theory is that the sequence of internal energies converges almost everywhere on $(0, T)$ and its limit can be expressed in terms of a marginal function depending only on the evolutionary variables v and w. This fact immediately leads to the desired energy dissipation inequality. Recall Definition 1.2 for the functional $\mathcal{H}$ and the marginal function **M**. Now we are in a position to formulate the first main result of this section.

Theorem 4.1. *Let all conditions of Theorem 3.1 be satisfied and v, w be the limits of v_N and w_N defined by Theorem 3.1. Then*

$$\mathcal{H}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t)) \to \mathbf{M}(v(t), w(t)) \text{ as } N \to \infty \text{ for a.e. } t \in (0, T). \tag{4.1}$$

Moreover,

$$\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1)) + \limsup_{N\to\infty} \Big\{ \int_{t_1+\tau}^{t_0} \Pi_1(H(\varphi_N), \varphi_N)\, \mathrm{d}s \\ + \frac{1}{2} \int_{t_1+\tau}^{t_0} \Pi_0(\vartheta_N, \vartheta_N)\, \mathrm{d}s + \frac{1}{2} \int_{t_1+\tau}^{t_0-\tau} \Pi(\overline{\vartheta}_N, \overline{\vartheta}_N)\, \mathrm{d}s \Big\} \leqq 0. \tag{4.2}$$

for a.e. $0 < t_1 < t_0 < T$. *Here the energy dissipation rate Π is given by (1.47), the function $\overline{\vartheta}_N$ is given by (2.9), (2.10).*

Inequality (4.2) estimates $\mathbf{M}(t_0) - \mathbf{M}(t_1)$ from above. Our next task is to estimate this difference from below. We will thus obtain an estimate which is complementary to the energy dissipation inequality. Such estimates are essential ingredients of the monotonicity method.

In order to formulate the corresponding result we introduce some notation. In view of Proposition 2.1 after passing to a subsequence we may assume that there are functions

$$\vartheta^* \in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \\ \varphi^* \in L^s(0, T; L^q(\Omega)), \quad H^* \in L^\infty(\Omega \times (0, T)) \tag{4.3}$$

such that

$$\begin{aligned}
&\vartheta_N \to \vartheta^* \ \text{ weakly in } L^r(0, T; L^p(\Omega)) \text{ and in } L^2(0, T; W^{1,2}(\Omega)), \\
&\varphi_N \to \varphi^* \ \text{ weakly in } L^s(0, T; L^q(\Omega)), \\
&H(\varphi_N) \to H^* \ \text{ star weakly in } L^\infty(\Omega \times (0, T)).
\end{aligned} \tag{4.4}$$

Here (r, p) and (s, q) are arbitrary exponents satisfying (2.20). Since the spaces $W^{1,2}(\Omega)$ and $L^\alpha(\Omega)$, $1 \leqq \alpha < \infty$, are separable, the mappings $v, \vartheta^* : (0, T) \to W^{1,2}(\Omega)$ and $H^* : (0, T) \to L^\alpha(\Omega)$ are strongly measurable. It follows that there exists a set $\mathcal{L}$ of full measure in $(0, T)$ such that for all $t_0, t_1 \in \mathcal{L}$ we have

$$\begin{aligned}
&\frac{1}{t_0 - t_1} \int_{t_1}^{t_0} \big(\|v(t_0) - v(s)\|_{W^{1,2}(\Omega)} + \|\vartheta^*(t_0) - \vartheta^*(s)\|_{W^{1,2}(\Omega)} \\
&\quad + \|H^*(t_0) - H^*(s)\|_{L^\alpha(\Omega)} \big)\, \mathrm{d}s \to 0 \ \text{ as } t \nearrow t_0.
\end{aligned} \tag{4.5}$$

For every $\eta > 0$, the set $\mathcal{L}$ contains a compact subset $\mathcal{L}_\eta$ with $\operatorname{meas}([0, T]\setminus\mathcal{L}_\eta) < \eta/2$. Next, it follows from the Lusin theorem that there is a compact set $\mathcal{C}_\eta \subset [0, T]$ such that $\operatorname{meas}([0, T]\setminus\mathcal{C}_\eta) < \eta/2$ and

$$\begin{aligned}
&\lim_{t_1 \to t_0,\, t_i \in \mathcal{C}_\eta} \big(\|v(t_0) - v(t_1)\|_{W^{1,2}(\Omega)} + \|\vartheta^*(t_0) - \vartheta^*(t_1)\|_{W^{1,2}(\Omega)} \\
&\quad + \|H^*(t_0) - H^*(t_1)\|_{L^\alpha(\Omega)} \big) = 0.
\end{aligned} \tag{4.6}$$

Theorem 4.2. *Let $t_0, t_1 \in \mathcal{L}_\eta \cap \mathcal{C}_\eta$. Furthermore, assume that $\mathbf{u} \in W^{2,6}(\Omega)$ is a minimizer of the functional $\boldsymbol{\mathcal{H}}(\cdot, v(t_0), w(t_0))$, i.e.,*

$$\boldsymbol{\mathcal{H}}(\mathbf{u}, v(t_0), w(t_0)) = \mathbf{M}(v(t_0), w(t_0)). \tag{4.7}$$

and

$$\begin{aligned}
\vartheta &= v(t_0) - W(w(t_0)^{-1} D\mathbf{u}) \in W^{1,2}(\Omega), \\
\varphi &= \varphi(D^2\mathbf{u}, D\mathbf{u}, \vartheta, w(t_0)) \in L^q(\Omega).
\end{aligned} \tag{4.8}$$

Then

$$\begin{aligned}
&\liminf_{t_1 \nearrow t_0} \frac{1}{t_0 - t_1} \Big\{ \mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1)) \Big\} \\
&\quad + \Pi_0(\vartheta, \vartheta^*(t_0)) + \Pi_1(\varphi, H^*(t_0)) \geqq 0.
\end{aligned} \tag{4.9}$$

Here the bilinear forms Π_i are given by (1.47).

The rest of the section is devoted to the proof of Theorems 4.1 and 4.2.

Proof of Theorem 4.1. *Proof of* (4.1). Recall representations (2.2) and (2.3) of the approximate solution in terms of $\mathbf{u}_n$, v_n, ϑ_n, and w_n. Fix $\mathbf{u}$ such that $\mathbf{u} - \mathbf{h} \in \mathcal{W}^{2,2}$ and define $\vartheta^* \in W^{1,2}(\Omega)$ as a solution to the variational problem

$$\mathbf{S}_n(\vartheta^*, \mathbf{u}) = \max_{\vartheta \in W^{1,2}(\Omega)} \mathbf{S}_n(\vartheta, \mathbf{u}),$$

where S_n is given by (2.1). It follows from (2.5) that

$$\mathbf{S}_n(\vartheta_n, \mathbf{u}_n) \leqq \mathbf{S}_n(\vartheta^*, \mathbf{u}). \tag{4.10}$$

Note that

$$\mathbf{S}_n(\vartheta_n, \mathbf{u}_n) = \sup_{\vartheta \in W^{1,2}(\Omega)} S_n(\vartheta, \mathbf{u}_n).$$

It now follows from Lemma A.1 in the Appendix that

$$\begin{aligned} \mathbf{S}_n(\vartheta_n, \mathbf{u}_n) &= \mathbf{E}(\mathbf{u}_n, \vartheta_n, w_{n-1}) - \int_\Omega \mathbf{f} \cdot \mathbf{u}_n \, \mathrm{d}x + \frac{\tau}{2} \Pi_0(\vartheta_n, \vartheta_n) \\ &\equiv \boldsymbol{\mathcal{H}}(\mathbf{u}_n, v_n, w_{n-1}) + \frac{\tau}{2} \Pi_0(\vartheta_n, \vartheta_n). \end{aligned} \tag{4.11}$$

Next, expressions (1.24) and (2.1) for Ψ_g and $\mathbf{S}_n$ imply

$$\begin{aligned} \mathbf{S}_n(\vartheta^*, \mathbf{u}) = \int_\Omega \Psi_g(D^2\mathbf{u}, D\mathbf{u}, \vartheta^*, w_{n-1}) \, \mathrm{d}x + \int_\Omega (w^d_{n-2} v_{n-1} \vartheta^* - \mathbf{f} \cdot \mathbf{u}) \, \mathrm{d}x \\ - \frac{\tau}{2} \Pi_0(\vartheta^*, \vartheta^*). \end{aligned} \tag{4.12}$$

Now set

$$\Theta = \Theta(D\mathbf{u}, v_n, w_{n-1}) \equiv v_n + W(w_{n-1} D\mathbf{u}). \tag{4.13}$$

Obviously we have

$$V(D\mathbf{u}, \Theta, w_{n-1}) \equiv \Theta - W(w_{n-1} D\mathbf{u}) = v_n \tag{4.14}$$

and

$$\frac{\partial}{\partial \Theta} \Psi_g(D^2\mathbf{u}, D\mathbf{u}, \Theta, w_{n-1}) \equiv -w^d_{n-1} V(D\mathbf{u}, \Theta, w_{n-1}) = -w^d_{n-1} v_n. \tag{4.15}$$

Since Ψ_g is a concave function of the temperature, we have

$$\begin{aligned} &\Psi_g(D^2\mathbf{u}, D\mathbf{u}, \vartheta^*, w_{n-1}) \\ &\quad \leqq \Psi_g(D^2\mathbf{u}, D\mathbf{u}, \Theta, w_{n-1}) + \frac{\partial}{\partial \Theta} \Psi_g(D^2\mathbf{u}, D\mathbf{u}, \Theta, w_{n-1})(\vartheta^* - \Theta). \end{aligned}$$

Substituting this in the right hand side of (4.12) and using (4.15) we arrive at

$$\begin{aligned} \mathbf{S}_n(\vartheta^*, \mathbf{u}) \leqq \int_\Omega \Psi_g(D^2\mathbf{u}, D\mathbf{u}, \Theta, w_{n-1}) + w^d_{n-1} v_n \Theta) \, \mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \\ + \int_\Omega (w^d_{n-2} v_{n-1} - w^d_{n-1} v_n) \vartheta^* \, \mathrm{d}x - \frac{\tau}{2} \Pi_0(\vartheta^*, \vartheta^*). \end{aligned} \tag{4.16}$$

Next, (4.13) and expression (1.42) for the density of the internal energy give the identity

$$\Psi_g(D^2\mathbf{u}, D\mathbf{u}, \Theta, w_{n-1}) + w^d_{n-1} v_n \Theta = \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_n, w_{n-1}),$$

which along with (1.44) implies

$$\int_{\Omega} \big(\Psi_g(D^2\mathbf{u}, D\mathbf{u}, \Theta, w_{n-1}) + w^d_{n-1} v_n \Theta - \mathbf{f}\cdot\mathbf{u}\big)\,\mathrm{d}x = \mathcal{H}(\mathbf{u}, v_n, w_{n-1}). \tag{4.17}$$

Multiplying both sides of (3.9) by ϑ^* and integrating the result over Ω we obtain

$$\int_{\Omega} \big(w^d_{n-2} v_{n-1} - w^d_{n-1} v_n\big)\vartheta_n\,\mathrm{d}x = \tau \Pi_0(\vartheta_n, \vartheta^*). \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.16) we obtain

$$\mathbf{S}_n(\mathbf{u}, \vartheta^*) \leqq \mathcal{H}(\mathbf{u}, v_n, w_{n-1}) + \tau \Pi_0(\vartheta_n, \vartheta^*) - \frac{\tau}{2}\Pi_0(\vartheta^*, \vartheta^*). \tag{4.19}$$

Combining (4.10) with (4.11) and (4.19) we arrive at

$$\mathcal{H}(\mathbf{u}_n, v_n, w_{n-1}) \leqq \mathcal{H}(\mathbf{u}, v_n, w_{n-1}) - \frac{\tau}{2}\Pi_0(\vartheta_n - \vartheta^*, \vartheta_n - \vartheta^*)$$

for all integers $n \in [1, N]$. Recalling the definition (2.2) of $\mathbf{u}_N$ and v_N and the definition (2.9) of $\overline{w}_N$ we deduce that

$$\mathcal{H}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t)) \leqq \mathcal{H}(\mathbf{u}, v_N(t), \overline{w}_N(t)) \tag{4.20}$$

for all $t \in (0, T)$ and all $\mathbf{u} \in \mathcal{W}^{2,2}$. By (3.3) there exists a set $\mathcal{Q}$ of full measure in $[0, T]$ such that for every $t \in \mathcal{Q}$,

$$v_N(t) \to v(t) \;\text{ in } L^p(\Omega), \quad \overline{w}_N(t) \to w(t) \;\text{ in } L^\alpha(\Omega).$$

Letting $N \to \infty$ in (4.20), we obtain

$$\limsup_{N\to\infty} \mathcal{H}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t)) \leqq \mathcal{H}(\mathbf{u}, v(t), w(t))$$

for all $\mathbf{u} \in \mathcal{W}^{2,2} + \mathbf{h}$, which along with the definition of the marginal function $\mathbf{M}$ gives

$$\limsup_{N\to\infty} \mathcal{H}(\mathbf{u}_N(t), v_N(t)) \leqq \mathbf{M}(v(t), w(t)) \;\text{ for all } t \in \mathcal{Q}.$$

It remains to prove that

$$\liminf_{N\to\infty} \mathcal{H}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t)) \geqq \mathbf{M}(v(t), w(t)) \;\text{ for all } t \in \mathcal{T}_\eta. \tag{4.21}$$

To this end, we fix $\eta > 0$ and $t \in \mathcal{T}_\eta$, where $\mathcal{T}_\eta$ is given by Theorem 3.2. Next, choose a sequence N_k such that

$$\liminf_{N\to\infty} \mathcal{H}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t)) = \lim_{N_k\to\infty} \mathcal{H}(\mathbf{u}_{N_k}(t), v_{N_k}(t), \overline{w}_{N_k}(t)).$$

Since $\mathbf{u}_N(t)$ is bounded in $W^{2,6}(\Omega)$, we can assume, after passing to a subsequence, that there is $\mathbf{u}^* \in W^{2,6}(\Omega)$ such that

$$\mathbf{u}_{N_k}(t) \to \mathbf{u}^* \;\text{ weakly in } W^{2,6}(\Omega), \;\; \mathbf{u}_{N_k}(t) \to \mathbf{u}^* \;\text{ strongly in } C^1(\overline{\Omega}).$$

Letting $N_k \to \infty$ and using (3.4) and (3.5) we obtain

$$\mathcal{H}(\mathbf{u}^*, v(t), w(t)) \leqq \lim_{N_k\to\infty} \mathcal{H}(\mathbf{u}_{N_k}(t), v_{N_k}(t), \overline{w}_{N_k}(t)).$$

On the other hand,

$$\mathbf{M}(v(t), w(t)) \leqq \mathcal{H}(\mathbf{u}^*, v(t), w(t)),$$

which yields (4.21). Hence the desired relation (4.21) holds for every $t \in \mathcal{T}_\eta$. Letting $\eta \to 0$ we conclude that it holds for a.e. $t \in (0, T)$. This completes the proof of (4.1). □

In order to prove (4.2) it suffices to note that the desired inequality obviously follows from (2.14) and (4.1).

Proof of Theorem 4.2. The proof is based on the following lemma: □

Lemma 4.1. *Let $\varsigma \in L^\infty(0, T; W^{1,2}(\Omega))$, $\partial_t\varsigma \in L^\infty(\Omega)$ and $\varsigma(t) = 0$ in a neighborhood of T. Then*

$$\int_0^T \int_\Omega (w^d v\partial_t\varsigma - \nabla\vartheta^*\nabla\varsigma)dxdt - \int_0^T \int_{\partial\Omega} \vartheta^*\varsigma\,\mathrm{d}s + \int_\Omega w_0^d v_0\varsigma(0)\,\mathrm{d}x = 0. \tag{4.22}$$

Moreover,

$$\int_\Omega (w(t_0)^d v(t_0) - w(t_1)^d v(t_1))\eta\,\mathrm{d}x + \int_{t_1}^{t_0} \int_\Omega \nabla\vartheta^*\nabla\eta\,dxdt + \int_{t_1}^{t_0} \int_{\partial\Omega} \vartheta^*\eta\,dsdt \tag{4.23}$$

for all $\eta \in W^{1,2}(\Omega)$ and all $t_0, t_1 \in \mathcal{L}_\eta \cap \mathcal{C}_\eta$.

Proof. The variation of the functional $S_n(\vartheta, \mathbf{u}_n)$ at the critical point $\vartheta = \vartheta_n$ leads to the equality

$$\frac{1}{\tau}\left\{\overline{w}_N^d(t)v_N(t) - \overline{w}_N^d(t-\tau)v_N(t-\tau)\right\} - \Delta\vartheta_N(t) = 0 \tag{4.24}$$

for $t \in (0, T)$. Notice that

$$w_N(t-\tau) = w_{-1} = w_0, \quad v_N(t-\tau) = v_0 \quad \text{for } t \in (0, \tau].$$

Multiplying both sides of (4.24) by ς and integrating the result over $\Omega \times (0, T)$, we obtain

$$\int_0^T \int_\Omega \left\{w_N(t)^d v_n(t)\frac{\varsigma(t+\tau) - \varsigma(t)}{\tau} - \nabla\vartheta^*\nabla\varsigma\right\} dxdt - \int_0^T \int_{\partial\Omega} \vartheta^*\varsigma\,\mathrm{d}s = \int_\Omega w_0^d v_0\varsigma(0)\,\mathrm{d}x = 0.$$

Letting $N \to \infty$ and using (3.3) and (4.4) we arrive at (4.22). Next choose $\eta \in W_0^{1,2}(\Omega)$ and $t_i \in \mathcal{L}_\eta \cap \mathcal{C}_\eta$. Then choose a compactly supported continuous function ζ such that

$$\zeta = 1 \ \text{ for } t \in (t_1, t_0 - \delta), \quad \zeta = 0 \ \text{ for } t \in (-\infty, t_1 - \delta] \cup [t_0, \infty),$$

and ζ is linear on the intervals $(t_1 - \delta, t_1)$ and $(t_0 - \delta, t_0)$. Substituting $\varsigma = \zeta\eta$ into (4.22), letting $\delta \to 0$, and using (4.5) we obtain (4.23). □

Let us turn to the proof of Theorem 4.2. We assume that $t_1, t_0 \in \mathcal{L}_\eta \cap \mathcal{C}_\eta$. By abuse of notation we will write v_i and w_i instead of $v(t_i)$ and $w(t_i)$. By the definition of the marginal function, we have $\mathbf{M}(v_1, w_1) \leqq \boldsymbol{\mathcal{H}}(\mathbf{u}, v_1, w_1)$, which leads to

$$\begin{aligned} \mathbf{M}(v_0, w_0) - \mathbf{M}(v_1, w_1) &\geqq \boldsymbol{\mathcal{H}}(\mathbf{u}, v_0, w_0) - \boldsymbol{\mathcal{H}}(\mathbf{u}, v_1, w_1) \\ &= \boldsymbol{\mathcal{E}}(\mathbf{u}, v_0, w_0) - \boldsymbol{\mathcal{E}}(\mathbf{u}, v_1, w_1). \end{aligned} \tag{4.25}$$

Here the total internal energy functional $\boldsymbol{\mathcal{E}}$ has the integral representation by (1.43) with the integrand $\mathcal{E}$ given by (1.42). The Taylor formula implies

$$\begin{aligned} &\mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0) - \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_1, w_1) = \partial_v \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(v_0 - v_1) \\ &\quad + \partial_w \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(w_0 - w_1) + R_1 + R_2 + R_3, \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} R_1 &= -(w_0 - w_1)^2 \frac{1}{2} \int_0^1 \partial_w^2 \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, \lambda w_1 + (1-\lambda) w_0)\, \mathrm{d}\lambda, \\ R_2 &= -(w_0 - w_1)(v_0 - v_1) \int_0^1 \partial_v \partial_w \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, \lambda w_1 + (1-\lambda) w_0)\, \mathrm{d}\lambda, \\ R_3 &= -(v_0 - v_1)^2 \frac{1}{2} \int_0^1 \partial_v^2 \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, \lambda v_1 + (1-\lambda) v_0, w_1)\, \mathrm{d}\lambda. \end{aligned} \tag{4.27}$$

Now the task is to let $t_1 \to t_0$ in expansion (4.26). Our considerations are based on the following lemma:

Lemma 4.2. *Let $\eta \in W^{1,2}(\Omega)$ and $g \in L^\beta(\Omega)$, $\beta > 1$. Then*

$$\begin{aligned} &\frac{1}{t_0 - t_1} \int_\Omega \eta (w(t_0)^d v(t_0) - w(t_1)^d v(t_1))\, \mathrm{d}x + \int_\Omega \nabla \vartheta^*(t_0) \nabla \eta\, \mathrm{d}x \\ &\quad + \int_{\partial\Omega} \vartheta^*(t_0) \eta\, \mathrm{d}s \to 0 \ \ \text{as } t_1 \to t_0, \end{aligned} \tag{4.28}$$

and

$$\frac{1}{t_0 - t_1} \int_\Omega g(x)(w(t_0) - w(t_1))\, \mathrm{d}x + \int_\Omega g H^*(t_0) w(t_0)\, \mathrm{d}x \to 0 \ \ \text{as } t_1 \to t_0. \tag{4.29}$$

Proof. In view of (4.23), we have

$$\begin{aligned}
&\frac{1}{t_0-t_1}\int_\Omega \eta(x)(w(t_0)^d v(t_0) - w(t_1)^d v(t_1))\,\mathrm{d}x \\
&\quad + \int_\Omega \nabla\vartheta^*(t_0)\nabla\eta\, dxdt + \int_{\partial\Omega} \vartheta^*(t_0)\eta\,\mathrm{d}s \\
&= \frac{1}{t_0-t_1}\int_{t_1}^{t_0}\Big\{\int_\Omega (\nabla\vartheta^*(t_0)-\nabla\vartheta^*(t))\nabla\eta\, dxdt \\
&\quad + \int_{\partial\Omega}(\vartheta^*(t_0)-\vartheta^*(t))\eta\,\mathrm{d}s\Big\}\,\mathrm{d}t
\end{aligned} \tag{4.30}$$

for all $\eta \in W^{1,2}(\Omega)$. Since the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is bounded, it follows from (4.5) that

$$\begin{aligned}
&\frac{1}{t_0-t_1}\int_{t_1}^{t_0}\Big|\int_\Omega (\nabla\vartheta^*(t_0)-\nabla\vartheta^*(t))\nabla\eta\, dxdt + \int_{\partial\Omega}(\vartheta^*(t_0)-\vartheta^*(t))\eta\,\mathrm{d}s\Big|\mathrm{d}t \\
&\leqq c\|\eta\|_{W^{1,2}(\Omega)}\frac{1}{t_0-t_1}\int_{t_1}^{t_0} c\|\vartheta^*(t_0)-\vartheta(t)\|_{W^{1,2}(\Omega)}\,\mathrm{d}t \to 0 \ \text{ as } t_1\to t_0,
\end{aligned}$$

which obviously yields (4.28). Next, we have

$$w_1 - w_0 = -\int_{t_1}^{t_0} H^*(s)w(s)\,\mathrm{d}s \ \text{ and } \ |w_1-w_0| \leqq c(t_0-t_1). \tag{4.31}$$

We thus get

$$\begin{aligned}
&\frac{1}{t_0-t_1}\int_\Omega g(x)(w(t_0)-w(t_1))\,\mathrm{d}x \\
&= -\int_\Omega g(x)H^*(t_0)w_0\,\mathrm{d}x + \frac{1}{t_0-t_1}\int_{t_1}^{t_0}\int_\Omega g(x)(H^*(t_0)-H^*(t))w_0\, dxdt \\
&\quad + \frac{1}{t_0-t_1}\int_{t_1}^{t_0}\int_\Omega g(x)H^*(t)(w_0-w_1)\, dxdt.
\end{aligned} \tag{4.32}$$

Next, for $\alpha > \beta/(\beta-1)$ we have

$$\begin{aligned}
&\frac{1}{t_0-t_1}\Big|\int_{t_1}^{t_0}\int_\Omega g(x)(H^*(t_0)-H^*(t))w_0\,\mathrm{d}x\Big|\,\mathrm{d}t \\
&\leqq \frac{c\|g\|_{L^q(\Omega)}}{t_0-t_1}\int_{t_1}^{t_0}\|H^*(t_0)-H^*(t)\|_{L^\alpha(\Omega)}\,\mathrm{d}t \to 0
\end{aligned}$$

as $t_1 \to t_0$. Now, estimate (4.31) implies

$$\begin{aligned}
&\frac{1}{t_0-t_1}\int_{t_1}^{t_0}\int_\Omega |g(x)H^*(t)(w_0-w_1)|\, dxdt \\
&\leqq c(t_0-t_1)\int_\Omega |g(x)|\,\mathrm{d}x \to 0 \text{ as } t_1 \to t_0.
\end{aligned}$$

Combining this with (4.32) we arrive at (4.29). □

Let us turn to the proof of Theorem 4.2. Our first task is to estimate the quantities R_i in expansion (4.26). Since w_i and $|D\mathbf{u}|$ are bounded, it follows from formula (1.42) for $\mathcal{E}$ that

$$|\partial_w^2 \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, \lambda w_1 + (1-\lambda)w_0)| \leqq c(|v_0|^2 + |\Delta \mathbf{u}|^2|). \tag{4.33}$$

From this and inequality (4.31) we obtain

$$\begin{aligned}(t_0 - t_1)^{-1} \int_\Omega |R_1|\,\mathrm{d}x &\leqq c(t_0 - t_1) \int_\Omega (|v_0|^2 + |\Delta \mathbf{u}|^2|)\,\mathrm{d}x \\ &\leqq c(t_0 - t_1) \to 0 \ \text{ as } t_1 \to t_0.\end{aligned} \tag{4.34}$$

Let us estimate R_2. It follows from the boundedness of w_i and $D\mathbf{u}$ that

$$|\partial_v \partial_w \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, \lambda w_1 + (1-\lambda)w_0)| \leqq c(1 + |v_0|),$$

which along with (4.31) yields

$$(t_0 - t_1)^{-1}|R_2| \leqq c(|v_0| + 1)|v_0 - v_1|. \tag{4.35}$$

Next, (4.6) implies

$$\|v_0 - v_1\|_{W^{1,2}(\Omega)} \to 0, \quad \|v_1\|_{W^{1,2}(\Omega)} \to \|v_0\|_{W^{1,2}(\Omega)} \tag{4.36}$$

as $t_1 \nearrow t_0$. From this we obtain

$$(t_0 - t_1)^{-1} \int_\Omega |R_2|\,\mathrm{d}x \leqq c\|v_0\|_{L^2(\Omega)} \|v_0 - v_1\|_{L^2(\Omega)} \to 0 \ \text{ as } t_1 \to t_0. \tag{4.37}$$

It remains to estimate R_3. To this end, notice that

$$\partial_v^2 \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, \lambda v_1 + (1-\lambda)v_0, w_1) = w_1^d.$$

Thus we get

$$R_3 = -\frac{1}{2}(v_0 - v_1)(w_0^d v_0 - w_1^d v_1) + \frac{1}{2}(v_0 - v_1)(w_0^d - w_1^d)v_0 = I_1 + I_2.$$

We have

$$(t_0 - t_1)^{-1}|I_2| \leqq c(t_0 - t_1)^{-1}|v_0||v_1 - v_0||w_1 - w_0| \leqq c(1 + |v_0|)|v_1 - v_0|.$$

Hence I_2 admits estimate (4.35). Arguing as in the proof of (4.37) we obtain

$$(t_0 - t_1)^{-1} \int_\Omega |I_2|\,\mathrm{d}x \to 0 \ \text{ as } t_1 \to t_0.$$

Next, applying Lemma 4.2 with $\eta = v_0 - v_1$ and noting that

$$\begin{aligned}&\Big| \int_\Omega \nabla \vartheta^*(t_0) \nabla \eta\, dx dt + \int_{\partial\Omega} \vartheta^*(t_0)\eta\,\mathrm{d}s \Big| \\ &\quad \leqq c\|\vartheta^*(t_0)\|_{W^{1,2}(\Omega)} \|v_0 - v_1\|_{W^{1,2}(\Omega)} \to 0\end{aligned}$$

as $t_1 \to t_0$, we obtain

$$(t_0 - t_1)^{-1} \int_\Omega |I_1| \, \mathrm{d}x \to 0 \ \text{ as } t_1 \to t_0,$$

and hence

$$(t_0 - t_1)^{-1} \int_\Omega |R_3| \, \mathrm{d}x \to 0 \ \text{ as } t_1 \to t_0. \tag{4.38}$$

Thus we have proved that the limits of all second order terms in the Taylor expansion (4.26) equal zero. In order to find the limits of the first order terms, notice that in view of (1.42) we have

$$\begin{aligned} \partial_v \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(v_0 - v_1) &= w_0^d \vartheta, \\ \partial_w \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(v_0 - v_1) &= w_0^{-1}\varphi + d w_0^{d-1} v_0 \vartheta, \end{aligned}$$

where ϑ and φ are given by (4.8). It follows that

$$\begin{aligned} &\partial_v \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(v_0 - v_1) + \partial_w \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(w_0 - w_1) \\ &\quad = (w_0^d v_0 - w_1^d v_1)\vartheta + w_0^{-1}(w_0 - w_1)\varphi + (w_1^d - w_0^d)(v_1 - v_0)\vartheta \\ &\qquad + v_0\big(w_1^d - w_0^d - d w_0^{d-1}(w_1 - w_0)\big)\vartheta. \end{aligned} \tag{4.39}$$

Now set $\eta = \vartheta \in W^{1,2}(\Omega)$ and $g = w_0^{-1}\varphi \in L^q(\Omega)$. Applying Lemma 4.2 we obtain

$$\begin{aligned} &\frac{1}{t_0 - t_1} \int_\Omega \Big((w_0^d v_0 - w_1^d v_1)\vartheta + w_0^{-1}(w_0 - w_1)\varphi \Big) \, \mathrm{d}x \to \\ &- \int_\Omega \nabla\vartheta^*(t_0) \nabla\vartheta \, \mathrm{d}x - \int_{\partial\Omega} \vartheta^*(t_0)\vartheta \, \mathrm{d}s - \int_\Omega \varphi H^*(t_0) \, \mathrm{d}x \ \text{ as } t_1 \to t_0. \end{aligned} \tag{4.40}$$

Next, (4.31) and (4.36) imply

$$\begin{aligned} \frac{1}{t_0 - t_1} \int_\Omega |(w_1^d - w_0^d)(v_1 - v_0)\vartheta| \, \mathrm{d}x &\leqq c \int_\Omega |(v_1 - v_0)\vartheta| \, \mathrm{d}x \\ &\leqq c\|\vartheta\|_{W^{1,2}(\Omega)} \|v_0 - v_1\|_{W^{1,2}(\Omega)} \to 0 \ \text{ as } t_1 \to t_0. \end{aligned} \tag{4.41}$$

Finally, we apply estimate (4.31) to obtain

$$\begin{aligned} &\frac{1}{t_0 - t_1} \int_\Omega |v_0\big(w_1^d - w_0^d - d w_0^{d-1}(w_1 - w_0)\big)\vartheta| \, \mathrm{d}x \\ &\quad \leqq \frac{c}{t_0 - t_1} \int_\Omega |\vartheta v_0|(w_1 - w_0)^2 \, \mathrm{d}x \leqq c(t_0 - t_1) \to 0 \text{ as } t_1 \to t_0. \end{aligned} \tag{4.42}$$

Combining (4.40)–(4.42) with identity (4.39) we arrive at

$$\begin{aligned} &\frac{1}{t_0 - t_1} \int_\Omega \Big\{ \partial_v \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(v_0 - v_1) \\ &\quad + \partial_w \mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0)(w_0 - w_1) \Big\} \, \mathrm{d}x \\ &\quad \to - \int_\Omega \nabla\vartheta^*(t_0) \nabla\vartheta \, \mathrm{d}x - \int_{\partial\Omega} \vartheta^*(t_0)\vartheta \, \mathrm{d}s - \int_\Omega \varphi H^*(t_0) \, \mathrm{d}x \end{aligned} \tag{4.43}$$

as $t_1 \to t_0$. Substituting this relation along with the limiting relations (4.34), (4.37), (4.38) for the second order remainders R_i into the Taylor expansion (4.26) we obtain

$$\frac{1}{t_0 - t_1}\Big(\boldsymbol{\mathcal{E}}(\mathbf{u}, v_0, w_0) - \boldsymbol{\mathcal{E}}(\mathbf{u}, v_1, w_1)\Big)$$
$$\equiv \frac{1}{t_0 - t_1}\int_\Omega \Big\{\mathcal{E}(D^2\mathbf{u}, D\mathbf{u}, v_0, w_0) - \mathcal{E}(D^2\mathbf{u}D\mathbf{u}, v_1, w_1)\Big\}\,\mathrm{d}x$$
$$\to -\int_\Omega \nabla\vartheta^*(t_0)\nabla\vartheta\,\mathrm{d}x - \int_{\partial\Omega}\vartheta^*(t_0)\vartheta\,\mathrm{d}s - \int_\Omega \varphi H^*(t_0)w_0\,\mathrm{d}x$$

as $t_1 \to t_0$. This result along with (4.25) implies the desired relation (4.9).

5. Sliced Measures in Banach Spaces

In this section we develop a theory of sliced measures in Banach spaces. Using this theory we will prove the strong convergence of the sequences ϑ_N and φ_N. For technical reasons, it is convenient to introduce the following notation. Fix exponents s, q and r, p satisfying relations (2.20) and set

$$X = L^p(\Omega) \times L^q(\Omega).$$

Further we will denote by $\boldsymbol{\omega}_N$ and $\boldsymbol{\omega}^*$ the couples

$$\boldsymbol{\omega}_N = (\vartheta_N, \varphi_N), \quad \boldsymbol{\omega}^* = (\vartheta^*, \varphi^*), \tag{5.1}$$

where the approximate solutions ϑ_N, φ_N are defined by Theorem 2.1, and the weak limits ϑ^*, φ^* are given by (4.4). Next, recall the definitions of the compact set $\mathcal{T}_\eta$ in Theorem 3.2 and of the compact sets $\mathcal{L}_\eta, \mathcal{C}_\eta$ in Theorem 4.2. Choose $\eta > 0$ and set

$$\mathcal{F}_\eta = \mathcal{T}_\eta \cap \mathcal{C}_\eta \cap \mathcal{L}_\eta, \quad \mathfrak{F}_\eta = \big\{\boldsymbol{\omega}_N(t) : t \in \mathcal{F}_\eta,\ N \geqq 1\big\}, \quad \Sigma_\eta = \mathrm{cl}\,\mathfrak{F}_\eta. \tag{5.2}$$

In view of Theorem 3.2 the set $\mathfrak{F}_\eta$ is relatively compact in X and the set Σ_η is compact in X. The following theorem gives the desired representation for the weak limits of the sequences ϑ_N and φ_N:

Theorem 5.1. *There exists a Borel measure ν on $\mathcal{F}_\eta \times \Sigma_\eta$ and a subsequence of $\boldsymbol{\omega}_N$, still denoted by $\boldsymbol{\omega}_N$, with the following properties. For every $F \in C(\mathcal{F}_\eta \times \Sigma_\eta)$ we have*

$$\lim_{N\to\infty}\int_{\mathcal{F}_\eta}\int_\Omega F(t, \boldsymbol{\omega}_N)\,dxdt = \int_{\mathcal{F}_\eta\times\Sigma_\eta} F(t, \boldsymbol{\omega})\,d\nu(t, \boldsymbol{\omega}). \tag{5.3}$$

Moreover, there is a measurable family of Borel probability measures μ_t, $t \in \mathcal{F}_\eta$, on Σ_η such that

$$\int_{\mathcal{F}_\eta\times\Sigma_\eta} F(t, \boldsymbol{\omega})\,d\nu(t, \boldsymbol{\omega}) = \int_{\mathcal{T}_\eta}\Big\{\int_{\Sigma_\eta} F(t, \boldsymbol{\omega})d\mu_t(\boldsymbol{\omega})\Big\}\,\mathrm{d}t. \tag{5.4}$$

There is a set $\mathcal{F} \subset \mathcal{F}_\eta$ of full measure such that

$$\lim_{n\to\infty} \frac{1}{\operatorname{meas}(I_n \cap \mathcal{F}_\eta)} \int_{(I_n\cap\mathcal{F}_\eta)\times\Sigma_\eta} F(\boldsymbol{\omega})\, d\nu(\boldsymbol{\omega}) = \int_{\Sigma_\eta} F(\boldsymbol{\omega})\, d\mu_{t_0}(\boldsymbol{\omega}) \quad (5.5)$$

for all $t_0 \in \mathcal{F}$, for all continuous functions $F : \Sigma_\eta \to \mathbb{R}$, and for all intervals $I_n = [t_n, t_0]$ such that $t_n \to t_0$.

The following theorem specifies the structure of the support of the measure μ_t (recall that, in view of Theorem 2.17, there is a constant c_0 independent of t and N such that):

$$\|\mathbf{u}_N(t)\|_{W^{2,6}(\Omega)} \leqq c_0 \quad \text{for all } N \geqq 1 \text{ and } t \in [0, T]. \quad (5.6)$$

Theorem 5.2. *There is a set $\mathcal{D}$ of full measure in $\mathcal{F}_\eta$ with the following property: for every $t_0 \in \mathcal{D}$ and $\boldsymbol{\omega} = (\vartheta, \varphi) \in \operatorname{supp} \mu_{t_0}$, there is $\mathbf{u} \in W^{2,6}(\Omega)$ such that $\|\mathbf{u}\|_{W^{2,6}(\Omega)} \leqq c_0$ and*

$$\vartheta = v(t_0) + W(w(t_0)^{-1} D\mathbf{u}), \quad \varphi = \varphi(D^2\mathbf{u}, D\mathbf{u}, \vartheta, w(t_0)), \quad (5.7)$$

$$\boldsymbol{\mathcal{H}}(\mathbf{u}, v(t_0), w(t_0)) = \mathbf{M}(v(t_0), w(t_0)). \quad (5.8)$$

Here $\varphi(D^2\mathbf{u}, D\mathbf{u}, \vartheta, w(t))$ is given by (1.26e), and the functionals $\boldsymbol{\mathcal{H}}$, $\mathbf{M}$ are given by (1.44) and (1.45).

The rest of this section is devoted to the proof of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Observe that the space $C(\mathcal{F}_\eta \times \Sigma_\eta)$ is separable; let F_k, $k \geqq 1$, be a dense set in it. Applying the diagonal process we may assume that there is a subsequence of $\boldsymbol{\omega}_N$, still denoted by $\boldsymbol{\omega}_N$, such that the limit

$$\lim_{N\to\infty} \int_{\mathcal{F}_\eta} \int_\Omega F_k(t, \boldsymbol{\omega}_N)\, dx dt =: \overline{F}_k$$

exists for every $k \geqq 1$. Since the set $\{F_k\}$ is dense in $C(\mathcal{F}_\eta \times \Sigma_\eta)$, the limit

$$\lim_{N\to\infty} \int_{\mathcal{T}_\eta} \int_\Omega F(t, \boldsymbol{\omega}_N)\, dx dt =: \overline{F}$$

exists for every $F \in C(\mathcal{F}_\eta \times \Sigma_\eta)$. Obviously the quantity $\overline{F}$ linearly depends on F and satisfies

$$|\overline{F}| \leqq \|F\|_{C(\mathcal{F}_\eta\times\Sigma_\eta)}, \quad \overline{F} \geqq 0 \text{ for } F \geqq 0.$$

Hence the mapping $F \mapsto \overline{F}$ define a continuous functional on $C(\mathcal{F}_\eta \times \Sigma_\eta)$. By the Riesz representation theorem, there exists a nonnegative Borel measure ν on $\mathcal{F}_\eta \times \Sigma_\eta$ such that

$$\overline{F} = \int_{\mathcal{F}_\eta\times\Sigma_\eta} F(t, \boldsymbol{\omega})\, d\nu(t, \boldsymbol{\omega}).$$

This leads to representation (5.3). If $F = F(t)$ is independent of $\boldsymbol{\omega}$, we have

$$\int_{\mathcal{F}_\eta} F(t)\,\mathrm{d}t = \int_{\mathcal{F}_\eta\times\Sigma_\eta} F(t)\,d\nu(t,\boldsymbol{\omega}).$$

This means that the projection of the measure ν on $\mathcal{F}_\eta$ coincides with the restriction of the Lebesgue measure to $\mathcal{F}_\eta$. Hence we can apply the disintegration theorem (see [2]) to obtain representation (5.4). It remains to note that (5.5) is a standard result of the theory of measure derivatives.

Proof of Theorem 5.2. The proof falls into three steps.
Step 1. Consider the following construction. By (4.1) the piecewise constant functions $\boldsymbol{\mathcal{H}}(\mathbf{u}_N(t), v_N(t), \overline{w}_N(t))$ converge to $\mathbf{M}(v(t), w(t))$ as $N \to \infty$ for a.e. $t \in \mathcal{F}_\eta$. Applying the Egoroff and Lusin theorems we conclude that for every $\delta > 0$ there is a set $\mathcal{G}_\delta \subset \mathcal{F}_\eta$ such that $\mathrm{meas}(\mathcal{F}_\eta\backslash\mathcal{G}_\delta) < \delta$ and

$$\boldsymbol{\mathcal{H}}(\mathbf{u}_N, v_N, \overline{w}_N) \to \mathbf{M}(v, w) \text{ in } C(\mathcal{G}_\delta). \tag{5.9}$$

Step 2. Choose $\delta > 0$ and let t_0 be a Lebesgue point of $\mathcal{G}_\delta$. Next, choose $\boldsymbol{\omega}_0 = (\vartheta_0, \varphi_0) \in \mathrm{supp}\,\mu_{t_0} \subset X$. Let us prove that there are sequences N_k and $t_k \in \mathcal{G}_\delta$ such that $t_k \nearrow t_0$ and $N_k \to \infty$ as $k \to \infty$, and

$$\boldsymbol{\omega}_{N_k}(t_k) \to \boldsymbol{\omega}_0 \text{ in } X \text{ as } k \to \infty. \tag{5.10}$$

In other words, we have to prove that

$$\limsup_{\min\{N^{-1}, t_0 - t\}\to 0} \|\boldsymbol{\omega}_N(t) - \boldsymbol{\omega}_0\|_X^{-1} = \infty \tag{5.11}$$

for $t \in \mathcal{G}_\delta$ and $t \leqq t_0$. Suppose that (5.11) is false. Then there are $m > 0$ and $\varepsilon > 0$ such that

$$\|\boldsymbol{\omega}_N(t) - \boldsymbol{\omega}_0\|_X \geqq \varepsilon \text{ for } N \geqq m,\ 0 \leqq t_0 - t \leqq m^{-1},\ t \in \mathcal{G}_\delta. \tag{5.12}$$

Choose a continuous nonnegative function $g : X \to \mathbb{R}$ such that

$$g(\boldsymbol{\omega}) = 0 \quad \text{for } \|\boldsymbol{\omega} - \boldsymbol{\omega}_0\|_X \geqq \varepsilon \quad \text{and} \quad g(\boldsymbol{\omega}) = 1 \quad \text{for } \|\boldsymbol{\omega} - \boldsymbol{\omega}_0\|_X \leqq \varepsilon/2.$$

It follows from (5.12) that

$$g(\boldsymbol{\omega}_N(t)) = 0 \text{ for } N \geqq m,\ 0 \leqq t_0 - t \leqq m^{-1},\ t \in \mathcal{G}_\delta. \tag{5.13}$$

Now choose $n \geqq m$ and set $I_n = [t_0 - 1/n, t_0]$. It follows from (5.13) that

$$g(\boldsymbol{\omega}_N(t)) = 0 \text{ for } t \in I_n \cap \mathcal{G}_\delta \text{ and } N \geqq m.$$

Noting that $I_n \subset (I_n\backslash\mathcal{G}_\delta) \cup (I_n \cap \mathcal{G}_\delta)$ we obtain

$$\int_{I_n\cap\mathcal{F}_\eta} g(\boldsymbol{\omega}_N(t))\,\mathrm{d}t \leqq \int_{(I_n\backslash\mathcal{G}_\delta)} g(\boldsymbol{\omega}_N(t))\,\mathrm{d}t \leqq \mathrm{meas}(I_n\backslash\mathcal{G}_\delta) \equiv \sigma_n \tag{5.14}$$

for all sufficiently large N. Letting $N \to \infty$ and recalling (5.3) we arrive at

$$\int_{(I_n \cap \mathcal{F}_\eta)\times\Sigma_\eta} g(\boldsymbol{\omega})\, d\nu(t,\boldsymbol{\omega}) = \lim_{N\to\infty} \int_{(I_n\cap\mathcal{F}_\eta)} g(\boldsymbol{\omega}_N(t))\, \mathrm{d}t \leqq \sigma_n. \tag{5.15}$$

Since t_0 is a Lebesgue point of $\mathcal{G}_\delta$, we have $\lim_{n\to\infty} n\sigma_n = 0$. Combining this with (5.2) and (5.5) we obtain

$$\int_{\Sigma_\eta} g(\boldsymbol{\omega})\, d\mu_{t_0}(\boldsymbol{\omega}) = \lim_{n\to\infty} n \int_{(I_n\cap\mathcal{F}_\eta)\times\Sigma_\eta} g(\boldsymbol{\omega})\, d\nu(t,\boldsymbol{\omega}) = 0.$$

Since g is positive in a neighborhood of $\boldsymbol{\omega}_0$, this equality contradicts the inclusion $\boldsymbol{\omega}_0 \in \operatorname{supp} \mu_{t_0}$, thus proving (5.10).

Step 3. Let N_k and $t_k \in \mathcal{G}_\delta$ satisfy condition (5.10). It follows from definition (2.2), (2.8) of the approximate solution that

$$\begin{aligned} \vartheta_{N_k}(t_k) &= v_{N_k}(t_k) + W(\overline{w}_{N_k}(t_k)^{-1} D\mathbf{u}_{N_k}(t_k)), \\ \varphi_{N_k}(t_k) &= \varphi(D^2\mathbf{u}_{N_k}(t), D\mathbf{u}_{N_k}(t), \vartheta_{N_k}(t), w_{N_k}(t)). \end{aligned} \tag{5.16}$$

Since $\mathcal{G}_\delta \subset \mathcal{T}_\eta$, relations (3.4) and (3.5) in Theorem 3.2 imply

$$v_{N_k} \to v \ \text{ in } C(\mathcal{G}_\delta; L^p(\Omega)), \quad \overline{w}_{N_k}^{\pm 1}, w_{N_k}^{\pm 1} \to w^{\pm 1} \ \text{ in } C(\mathcal{G}_\delta; L^\alpha(\Omega)) \tag{5.17}$$

for every $\alpha \in [1,\infty)$ and every p satisfying (2.20). In particular, these relations hold for every $p \in [1, 6)$. Moreover the mappings $v : \mathcal{G}_\delta \to L^p(\Omega)$ and $v : \mathcal{G}_\delta \to L^\alpha(\Omega)$ are continuous. It follows that

$$v_{N_k}(t_k) \to v(t_0) \ \text{ in } L^p(\Omega), \quad \overline{w}_{N_k}(t_k)^{\pm 1}, w_{N_k}(t_k)^{\pm 1} \to w(t_0)^{\pm 1} \ \text{ in } L^\alpha(\Omega). \tag{5.18}$$

After passing to a subsequence we may assume that

$$v_{N_k}(t_k) \to v(t_0),\ \overline{w}_{N_k}(t_k)^{\pm 1} \to w(t_0)^{\pm 1},\ w_{N_k}(t_k)^{\pm 1}(t) \to w(t_0)^{\pm 1} \text{ a.e. in } \Omega. \tag{5.19}$$

Next, estimates (2.17) and (2.18) in Theorem 2.2 imply

$$\|\mathbf{u}_{N_k}(t_k)\|_{W^{2,6}(\Omega)} \leqq c_0, \quad \|\overline{w}_{N_K}(t_k)^{-1}\, \Delta\mathbf{u}_{N_k}(t_k)\|_{W^{1,2}(\Omega)} \leqq c. \tag{5.20}$$

Notice that the embeddings $W^{2,6}(\Omega) \hookrightarrow C^1(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ are compact. Since the functions $\overline{w}_{N_k}(t_k)^{\pm 1}$ are uniformly bounded and converge to $\overline{w}(t_0)^{\pm 1}$, we can assume after passing to a subsequence that

$$\begin{aligned} &\Delta\mathbf{u}_{N_k}(t_k) \to \Delta\mathbf{u} \text{ weakly in } L^6(\Omega), \quad D\mathbf{u}_{N_k}(t_k) \to D\mathbf{u} \text{ in } C(\Omega), \\ &\Delta\mathbf{u}_{N_k}(t_k) \to \Delta\mathbf{u} \ \text{ a.e. in } \Omega \end{aligned} \tag{5.21}$$

for some $\mathbf{u} \in W^{2,6}(\Omega)$ satisfying (5.20). Letting $k \to \infty$ in identities (5.16) and using relations (5.19) and (5.21) we arrive at

$$\begin{aligned} &\vartheta_{N_k}(t_k) \to v(t_0) + W(w(t_0)^{-1} D\mathbf{u}) \equiv \tilde{\vartheta} \ \text{ a.e. in } \Omega, \\ &\varphi_{N_k}(t_k) \to \varphi(D^2\mathbf{u}, D\mathbf{u}, \tilde{\vartheta}, w(t_0)) \ \text{ a.e. in } \Omega. \end{aligned} \tag{5.22}$$

On the other hand, relations (5.10) imply

$$\boldsymbol{\omega}_{N_k}(t_k) = (\vartheta_{N_k}(t_k), \varphi_{N_k}(t_k)) \to \boldsymbol{\omega}(t_0) = (\vartheta_0, \varphi_0) \ \text{ in } X \ \text{ as } k \to \infty.$$

It follows from this and (5.22) that

$$\vartheta_0 = v(t_0) + W(w(t_0)^{-1} D\mathbf{u}), \quad \varphi_0 = \varphi(D^2\mathbf{u}, D\mathbf{u}, \vartheta_0, w(t_0)), \tag{5.23}$$

which gives the desired relation (5.7). It remains to prove that $\mathbf{u}$ is a minimizer of the functional $\mathcal{H}(\cdot, v(t_0), w(t_0))$. Notice that

$$\begin{aligned} &\mathcal{H}(\mathbf{u}_{N_k}(t_k), v_{N_k}(t_k), w_{N_k}(t_k)) \\ &= \int_\Omega \Big(\frac{\varepsilon}{2} w_{N_k}(t_k)^{d-4} |\Delta \mathbf{u}_{N_k}(t_k)|^2 + w_{N_k}(t_k)^d W(w_{N_k}(t_k)^{-1} D\mathbf{u}_{N_k}(t_k))\Big)\, \mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega \Big(w_{N_k}(t_k)^d (v_{N_k}(t_k) + W(w_{N_k}(t_k)^{-1} D\mathbf{u}_{N_k}(t_k))^2 - \mathbf{f}\cdot \mathbf{u}_{N_k}(t_k)\Big)\, \mathrm{d}x. \end{aligned} \tag{5.24}$$

From this and relations (5.17), (5.21) we conclude that

$$\begin{aligned} &\liminf_{N\to\infty} \int_\Omega w_{N_k}(t_k)^{d-4} |\Delta \mathbf{u}_{N_k}(t_k)|^2\, \mathrm{d}x \geqq \int_\Omega w(t_0)^{d-4} |\Delta \mathbf{u}|^2\, \mathrm{d}x, \\ &\lim_{N\to\infty} \int_\Omega (v_{N_k}(t_k) + (w_{N_k}(t_k))^d W(w_{N_k}(t_k)^{-1} D\mathbf{u}_{N_k}(t_k))^2\, \mathrm{d}x \\ &\quad = \int_\Omega (v(t_0) + w(t_0)^d W(w(t_0)^{-1} D\mathbf{u})^2\, \mathrm{d}x. \end{aligned}$$

Thus we get

$$\mathcal{H}(\mathbf{u}, v(t_0), w(t_0)) \leqq \liminf_{k\to\infty} \mathcal{H}(\mathbf{u}_{N_k}(t_k), v_{N_k}(t_k), w_{N_k}(t_k)).$$

On the other hand, relations (5.9) yield

$$\liminf_{k\to\infty} \mathcal{H}(\mathbf{u}_{N_k}(t_k), v_{N_k}(t_k), w_{N_k}(t_k)) = \mathbf{M}(v(t_0), w(t_0)).$$

Combining these results we arrive at the inequality

$$\mathcal{H}(\mathbf{u}, v(t_0), w(t_0)) \leqq \mathbf{M}(v(t_0), w(t_0)). \tag{5.25}$$

Hence $\mathbf{u}$ is a minimizer of $\mathcal{H}(\cdot, v(t_0), w(t_0))$ and we have $\mathcal{H}(\mathbf{u}, v(t_0), w(t_0)) = \mathbf{M}(v(t_0), w(t_0))$. It follows that the desired relation (5.8) holds for all $t_0 \in \mathcal{G}_\delta$. Letting $\delta \to 0$ we conclude that (5.8) is fulfilled for a.e. $t_0 \in \mathcal{F}_\eta$. This completes the proof of Theorem 5.2.

6. Strong Convergence of Temperature and Eshelby Tensor

In this section we employ the results obtained in Sections 4 and 5 in order to prove the strong convergence of sequences ϑ_N and φ_N. This result is given by

Theorem 6.1. *Let exponents* (r, p) *and* (s, q) *satisfy inequalities* (2.20) *and* (ϑ^*, φ^*) *be defined by* (4.4). *Then*

$$\vartheta_N \to \vartheta^* \text{in } L^r(0, T; L^p(\Omega)), \quad \varphi_N \to \varphi^* \text{ in } L^s(0, T; L^q(\Omega)). \tag{6.1}$$

For almost every $t \in (0, T)$, *there is a function* $\mathbf{u} \in \mathcal{W}^{2,2}(\Omega) + \mathbf{h}$ *such that* $\|\mathbf{u}\|_{W^{2,6}(\Omega)} \leqq c_0$ *and*

$$\vartheta^*(t) = v(t) + W(w(t)^{-1}D\mathbf{u}), \quad \varphi^*(t) = \varphi(D^2\mathbf{u}, D\mathbf{u}, \vartheta^*(t), w(t)), \tag{6.2}$$

$$\boldsymbol{\mathcal{H}}(\mathbf{u}, v(t), w(t)) = \mathbf{M}(v(t), w(t)), \tag{6.3}$$

i.e., $\mathbf{u}$ *is a minimizer of the functional* $\boldsymbol{\mathcal{H}}(\cdot, v(t), w(t))$.

The rest of the section is devoted to the proof of Theorem 6.1. We split the proof into the sequence of lemmas. First we prove that the dissipation energy rate Π given by (1.46) is integrable with respect to the measure μ_t. Notice that μ_t is defined on the compact subset Σ_η of space $X = L^p(\Omega) \times L^q(\Omega)$, while Π is defined on the space $W^{1,2}(\Omega) \times L^q(\Omega)$. The energy dissipation rate Π can be considered as a discontinuous unbounded functional defined on the dense subspace of X. However, we intend to prove that $\Pi(\vartheta, \varphi)$ is integrable over the measure μ_t. The proof is based on the special approximation of Π which is defined as follows: recall decomposition (1.47) of Π

$$\Pi(\vartheta, \varphi) = \Pi_0(\vartheta, \vartheta) + \Pi_1(H(\varphi), \varphi), \tag{6.4}$$

where the bilinear forms Π_i are given by

$$\Pi_0(\vartheta, \upsilon) = \int_\Omega \nabla\vartheta \nabla\upsilon \, \mathrm{d}x + \int_{\partial\Omega} \vartheta\upsilon \, \mathrm{d}s, \quad \Pi_1(\psi, \varphi) = \int_\Omega \psi\varphi \, \mathrm{d}x. \tag{6.5}$$

In view of the general theory of the second order elliptic equations, the spectral problem

$$-\Delta\zeta = \lambda\zeta \text{ in } \Omega, \quad \partial_n\zeta + \zeta = 0 \text{ on } \partial\Omega \tag{6.6}$$

has a countable set of eigenvalues $\lambda_k > 0, k \geqq 1$, and eigenfunctions $\zeta_k \in W^{2,2}(\Omega)$, $k \geqq 1$ The eigenfunctions form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $W^{1,2}(\Omega)$ Every element $\vartheta \in L^2(\Omega)$ admits the representation

$$\vartheta = \sum_k \vartheta_k\zeta_k, \quad \vartheta_k = \int_\Omega \vartheta\zeta_k \, \mathrm{d}x. \tag{6.7}$$

In particular, the Bessel identity implies the relations

$$\|\vartheta\|^2_{L^2(\Omega)} = \sum_k |\vartheta_k|^2, \quad \Pi_0(\vartheta, \vartheta) = \sum_k \lambda_k|\vartheta_k|^2. \tag{6.8}$$

Now set

$$\Pi^{(n)}(\vartheta, \varphi) = \Pi_0^{(n)}(\vartheta, \vartheta) + \Pi_1(H(\varphi), \varphi), \tag{6.9}$$

where

$$\Pi_0^{(n)}(\vartheta, \upsilon) = \Pi_0(P_n\vartheta, P_n\upsilon), \quad P_n\vartheta = \sum_{k=1}^{n} \vartheta_k \zeta_k. \tag{6.10}$$

For every $\upsilon \in W^{1,2}(\Omega)$ and $\psi \in L^\infty(\Omega)$ define the linear forms

$$\begin{aligned} \Gamma(\vartheta, \varphi) &:= \Pi_0(\vartheta, \upsilon) + \Pi_1(\psi, \varphi), \\ \Gamma^{(n)}(\vartheta, \varphi) &:= \Pi_0(P_n\vartheta, P_n\upsilon) + \Pi_1(\psi, \varphi). \end{aligned} \tag{6.11}$$

The following Lemma describe the properties of Π and Γ and their approximations:

Lemma 6.1. *The functions $\Pi, \Gamma : W^{1,2}(\Omega) \times L^1(\Omega) \to \mathbb{R}$ and $\Pi^{(n)}, \Gamma^{(n)} : L^1(\Omega) \times L^1(\Omega) \to \mathbb{R}$ are continuous. In particular, $\Pi^{(n)}$ and $\Gamma^{(n)}$ are continuous on the Banach space X. For every $\boldsymbol{\omega} = (\vartheta, \varphi) \in X$ we have*

$$\Pi^{(n)}(\boldsymbol{\omega}) \nearrow \Pi(\boldsymbol{\omega}) \text{ when } \vartheta \in W^{1,2}(\Omega) \text{ and } \Pi^{(n)}(\boldsymbol{\omega}) \nearrow \infty \text{ otherwise}, \tag{6.12}$$

$$\Gamma^{(n)}(\boldsymbol{\omega}) \to \Gamma(\boldsymbol{\omega}) \text{ when } \vartheta \in W^{1,2}(\Omega) \tag{6.13}$$

as $n \to \infty$.

Proof. The continuity of functions Π, Γ is obvious. The continuity of $\Pi^{(n)}, \Gamma^{(n)} : L^1(\Omega) \times L^1(\Omega) \to \mathbb{R}$ obviously follows from the representations

$$\Pi^{(n)}(\boldsymbol{\omega}) = \sum_{k=1}^{n} \lambda_k \Big(\int_\Omega \vartheta \zeta_k \, dx \Big)^2 + \int_\Omega H(\varphi)\varphi \, dx,$$

$$\Gamma^{(n)} = \int_\Omega (\upsilon^{(n)}\vartheta + \psi\varphi) \, dx, \quad \upsilon^{(n)} = \sum_{1}^{n} \lambda_k \psi_k \zeta_k \in C(\Omega).$$

Since $\Pi_0(\vartheta, \vartheta)$ determines the norm in $W^{1,2}(\Omega)$, relation (6.12) obviously follows from representations (6.9) and (6.10). Since ζ_k form the orthogonal basis in $W^{1,2}(\Omega)$, the sequence $P_n\vartheta$ converges ϑ in $W^{1,2}(\Omega)$, which along with (6.11) yields (6.13). □

The next Lemma constitutes the differentiability of the marginal function.

Lemma 6.2. *There is a set $\mathcal{Q}$ of the full measure in $(0, T)$ with the following properties: for every $t_0 \in \mathcal{D}$ we have*

$$\frac{1}{t_0 - t}(\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t), w(t)) \to \mathbf{M}'(t_0) \in (-\infty, 0], \tag{6.14}$$

as $t \nearrow t_0$ and $t \in \mathcal{Q}$.

Proof. Since Π_0 is nonnegative, inequality (4.2) in Theorem 4.1 implies that

$$\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t), w(t)) \leqq 0$$

for almost all $0 < t < t_0 < T$. In other words, there is a set $\mathcal{Q}_1$ of full measure in $[0, T]$ such that this inequality holds true for all $t < t_0$ in this set. Hence the function $\mathbf{M}(v(t), w(t))$ decreases on $\mathcal{Q}_1$. Obviously, it can be extended to a decreasing function to the whole interval $[0, T]$. Hence there is a set $\mathcal{Q} \subset \mathcal{Q}_1$ of full measure in $(0, T)$ such that the extended function has the non-positive finite derivative $M'(t_0)$ at every point of $\mathcal{Q}$. □

Without loss of generality we can assume that $\mathcal{Q}$ contains the set $\mathcal{D}$ given by Theorem 5.2. To this end, it suffices to replace $\mathcal{D}$ by $\mathcal{D} \cap \mathcal{Q}$. Thus we can assume that the marginal function is differentiable on $\mathcal{D}$. The following lemma constitutes the integrability of the functions Π and Γ with respect to the measure μ_t:

Lemma 6.3. *For every $t_0 \in \mathcal{D}$ and all $\upsilon \in W^{1,2}(\Omega)$, $\psi \in L^\infty(\Omega)$, the functions*

$$(\vartheta, \varphi) \mapsto \Pi(\vartheta, \varphi) \;\; \text{and} \;\; (\vartheta, \varphi) \mapsto \Gamma(\vartheta, \varphi) \tag{6.15}$$

are integrable with respect to the measure μ_{t_0} given by Theorem 5.1. Moreover, we have

$$\int_{\Sigma_\eta} \Pi(\omega)\, d\mu_{t_0}(\boldsymbol{\omega}) \leqq -\mathbf{M}'(t_0). \tag{6.16}$$

Proof. Choose $t_1, t_0 \in \mathcal{D}$ with $0 < t_1 < t_0$. Recall definition (2.9) of the function $\overline{\vartheta}_N$. Since $0 \leqq \Pi_0^{(n)} \leqq \Pi_0$, we have

$$\begin{aligned} &\int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N, \vartheta_N) + \int_{(t_1+\tau, t_0-\tau)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\overline{\vartheta}_N, \overline{\vartheta}_N)) \\ &\quad \leqq \frac{1}{2}\int_{(t_1+\tau, t_0)} \Pi_0(\vartheta_N, \vartheta_N) + \frac{1}{2}\int_{(t_1+\tau, t_0-\tau)\cap} \Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)). \end{aligned}$$

It follows from this and inequality (4.2) in Theorem 4.1 that

$$\begin{aligned} &\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1)) + \limsup_{N\to\infty} \Bigg\{ \int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_1(H(\varphi_N), \varphi_N) \\ &\quad + \frac{1}{2}\int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N, \vartheta_N)\, \mathrm{d}s + \frac{1}{2}\int_{(t_1+\tau, t_0-\tau)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\overline{\vartheta}_N, \overline{\vartheta}_N)\, \mathrm{d}s \Bigg\} \leqq 0. \end{aligned} \tag{6.17}$$

Notice that the quadratic form $\Pi_0^{(n)} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is continuous. Next, estimate (2.13) in Theorem 2.1 implies that

$$\int_0^{t_0} \|\vartheta_N - \overline{\vartheta}_N\|^2_{L^2(\Omega)}\, \mathrm{d}t \to 0 \;\text{ as } N \to \infty.$$

In particular, we have

$$\int_0^{t_0} \Pi_0^{(n)}(\vartheta_N - \overline{\vartheta}_N, \vartheta_N - \overline{\vartheta}_N)\,\mathrm{d}t \to 0 \text{ as } N \to \infty.$$

Since the quadratic form $\Pi^{(n)}$ is nonnegative, the Cauchy inequality implies the estimate

$$\begin{aligned}&\Pi_0^{(n)}(\overline{\vartheta}_N, \overline{\vartheta}_N) - \Pi_0^{(n)}(\vartheta_N, \vartheta_N)\\ &\quad \leqq \delta \Pi_0(\vartheta_N, \vartheta_N) + \delta^{-1} \Pi_0^{(n)}(\vartheta_N - \overline{\vartheta}_N, \vartheta_N - \overline{\vartheta}_N).\end{aligned}$$

It follows from this and energy estimate (2.12) that

$$\int_0^{t_0} |\Pi_0^{(n)}(\overline{\vartheta}_N, \overline{\vartheta}_N) - \Pi_0^{(n)}(\vartheta_N, \vartheta_N)|\mathrm{d}t \leqq c\delta + \int_0^{t_0} \Pi_0^{(n)}(\vartheta_N - \overline{\vartheta}_N, \vartheta_N - \overline{\vartheta}_N).$$

Letting $N \to \infty$ we obtain

$$\limsup_{N\to\infty} \int_0^{t_0} |\Pi_0^{(n)}(\overline{\vartheta}_N, \overline{\vartheta}_N) - \Pi_0^{(n)}(\vartheta_N, \vartheta_N)|\mathrm{d}t \leqq c\delta \to 0 \text{ as } \delta \to 0.$$

It follows from this that

$$\lim_{N\to\infty} \int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} (\Pi_0^{(n)}(\overline{\vartheta}_N, \overline{\vartheta}_N) - \Pi_0^{(n)}(\vartheta_N, \vartheta_N))\mathrm{d}t = 0. \tag{6.18}$$

Combining this relation with (6.17) we arrive at the inequality

$$\begin{aligned}&\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1)) + \limsup_{N\to\infty} \Big\{ \int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_1(H(\varphi_N), \varphi_N)\mathrm{d}s\\ &+ \frac{1}{2}\int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N, \vartheta_N)\,\mathrm{d}s + \frac{1}{2}\int_{(t_1+\tau, t_0-\tau)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N, \vartheta_N)\,\mathrm{d}s\Big\} \leqq 0.\end{aligned} \tag{6.19}$$

In view of (5.2) and Theorem 3.2 the functions $\boldsymbol{\omega}_N(t)$, $t \in \mathcal{F}_\eta$, belong to the set $\mathfrak{F}_(\eta)$ which is relatively compact in X. Hence for a fixed n, the functions $\Pi^{(n)}(\vartheta_N(t), \varphi_N(t))$ are uniformly bounded on $\mathcal{F}_\eta$. Hence the functions $\Pi_0^{(n)}(\vartheta_N, \vartheta_N)(t)$ and $\Pi_1(H(\varphi_N), \varphi_N)$ are bounded on $\mathcal{F}_\eta$ uniformly in N. Since $\tau \to 0$ as $N \to \infty$, we have

$$\begin{aligned}&\limsup_{N\to\infty} \Big\{ \int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_1(H(\varphi_N), \varphi_N) + \frac{1}{2}\int_{(t_1+\tau, t_0)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N, \vartheta_N)\,\mathrm{d}s\\ &\quad + \frac{1}{2}\int_{(t_1+\tau, t_0-\tau)\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N, \vartheta_N)\,\mathrm{d}s\Big\} = \limsup_{N\to\infty} \int_{(t_1, t_0)\cap\mathcal{F}_\eta} \Pi^{(n)}(\vartheta_N, \varphi_N)\,\mathrm{d}t.\end{aligned}$$

Here we use the identity

$$\Pi_0^{(n)}(\vartheta_N, \vartheta_N) + \Pi_1(H(\varphi_N), \varphi_N) = \Pi^{(n)}(\vartheta_N, \varphi_N).$$

Combining this result with (6.19) we arrive at the inequality

$$\limsup_{N\to\infty} \int_{(t_1,t_0)\cap\mathcal{F}_\eta} \Pi^{(n)}(\vartheta_N(t), \varphi_N(t))\,\mathrm{d}t \leqq -\{\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1))\}.$$

In view of Lemma 6.13 $\Pi^{(n)}(\boldsymbol{\omega})$ is continuous in X. Hence we can apply Theorem 5.1 to obtain

$$\lim_{N\to\infty} \int_{[t_1,t_0]\cap\mathcal{F}_\eta} \Pi^{(n)}(\vartheta_N, \varphi_N)\,\mathrm{d}t = \int_{[t_0,t_1]\cap\mathcal{F}_\eta\times\Sigma_\eta} \Pi^{(n)}(\boldsymbol{\omega})\,d\nu(\boldsymbol{\omega}),$$

where $\boldsymbol{\omega} = (\vartheta, \varphi)$. Thus we get

$$\int_{[t_0,t_1]\cap\mathcal{F}_\eta\times\Sigma_\eta} \Pi^{(n)}(\boldsymbol{\omega})\,d\nu \leqq -\{\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1))\}. \tag{6.20}$$

Since $t_0 \in \mathcal{D} \subset \mathcal{F}$ we can apply the relation (5.5) in Theorem 5.1 to obtain

$$\lim_{t_1\to t_0} \frac{1}{t_0 - t_1} \int_{[t_0,t_1]\cap\mathcal{T}_\eta\times\Sigma_\eta} \Pi^{(n)}(\boldsymbol{\omega})\,d\nu = \int_{\Sigma_\eta} \Pi^{(n)}(\boldsymbol{\omega})\,d\mu_{t_0}(\boldsymbol{\omega}). \tag{6.21}$$

On the other hand, Lemma 6.3 yields

$$\lim_{t_1\to t_0} \frac{1}{t_0 - t_1}\{\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1))\} = \mathbf{M}'(t_0). \tag{6.22}$$

Combining relations (6.21) and (6.22) with inequality (6.20) we obtain

$$\int_{\Sigma_\eta} \Pi^{(n)}(\omega)\,d\mu_{t_0}(\boldsymbol{\omega}) \leqq -\mathbf{M}'(t_0). \tag{6.23}$$

Relation (6.12) in Lemma 6.1 implies that the sequence $\Pi^{(n)}(\boldsymbol{\omega})$ increases and converges to $\Pi(\boldsymbol{\omega})$. Letting $n \to \infty$ in (6.23) and applying the Fatou theorem we conclude that the function Π is integrable with respect to the measure μ_{t_0} and satisfies inequality (6.16).

Let us prove the integrability of Γ. Choose $\upsilon \in W^{1,2}(\Omega)$ and $\psi \in L^\infty(\Omega)$. It follows from the Cauchy inequality that

$$\Pi_0(\vartheta, \upsilon) \leqq \Pi_0(\vartheta, \vartheta)^{1/2}\Pi_0(\upsilon, \upsilon)^{1/2} \leqq c\Pi_0(\vartheta, \vartheta)^{1/2} \leqq c\Pi(\vartheta, \varphi)^{1/2}.$$

It follows from this and the representation (6.9) that

$$|\Gamma(\vartheta, \varphi)| \leqq c(1 + \Pi(\vartheta, \varphi)).$$

Hence Γ has the integrable majorant. Since $\vartheta \in W^{1,2}(\Omega)$ almost everywhere on the support μ_{t_0}, it follows that the continuous functions $\Gamma^{(n)} \to \Gamma$ μ_{t_0}-almost everywhere. Hence the function Γ is measurable and integrable withy respect to μ_{t_0}. □

The following lemma plays a key role in the proof of Theorem 6.1:

Lemma 6.4. *The inequality*

$$\int_{\Sigma_\eta} \Pi(\vartheta, \varphi)\, d\mu_{t_0} \leqq \int_{\Sigma_\eta} \Big(\Pi_0(\vartheta, \vartheta^*(t_0)) + \Pi_1(H^*(t_0), \varphi)\Big)\, d\mu_{t_0} \tag{6.24}$$

holds true for every $t_0 \in \mathcal{D}$. Here ϑ^* and H^* are the weak limits of ϑ_N and φ_N defined by (4.4).

Proof. Choose $t_0 \in \mathcal{D} \subset \mathcal{F}_\eta$. Recall that $\mathcal{F}_\eta \in \mathcal{C}_\eta$, where $\mathcal{C}_\eta$ is given by (4.6). It follows from 4.6 that $\vartheta^*(t_0) \in W^{1,2}(\Omega)$. By virtue of Theorem 5.2, every element $\boldsymbol{\omega} = (\vartheta, \varphi) \in \operatorname{supp} \mu_{t_0}$ has representation (5.7). Moreover, it follows from Lemma 6.3 that $\vartheta \in W^{1,2}(\Omega)$ μ_{t_0}- almost everywhere. Hence $\boldsymbol{\omega}$ meets all requirements of Theorem 4.2 μ_{t_0}- almost everywhere. In view of relation (4.9) in this theorem the inequality

$$\begin{aligned} &\Pi_0(\vartheta, \vartheta^*(t_0)) + \Pi_1(H^*(t_0), \varphi) \\ &\quad \geqq -\liminf_{t_1 \nearrow t_0} \frac{1}{t_0 - t_1}\Big\{\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1))\Big\} \end{aligned}$$

holds true for all $t_1 \in \mathcal{D}$. From this and relation (6.14) in Lemma 6.2 we obtain

$$\Pi_0(\vartheta, \vartheta^*(t_0)) + \Pi_1(H^*(t_0), \varphi) \geqq -\mathbf{M}'(t_0).$$

for μ_{t_0}- a.e. (ϑ, φ). Integrating both sides of this inequality with respect to the probability measure μ_{t_0} we obtain

$$\int_{\Sigma_\eta} \Big(\Pi_0(\vartheta, \vartheta^*(t_0)) + \Pi_1(H^*(t_0), \varphi)\Big)\, d\mu_{t_0}(\boldsymbol{\omega}) \geqq -\mathbf{M}'(t_0).$$

Combining this result with (6.16) we arrive at desired inequality (6.24). □

We are now in a position to prove that the μ_{t_0} is the Dirac measure.

Lemma 6.5. *Let $t_0 \in \mathcal{D}$ be a Lebesgue point of $\mathcal{D}$. Then the measure μ_{t_0} is the Dirac measure concentrated at point $\boldsymbol{\omega}^* = (\vartheta^*(t_0), \varphi^*(t_0))$. Moreover, the functions $\vartheta^*(t_0)$ and $\varphi^*(t_0)$ admit representations* (6.2) *and* (6.3).

Proof. First we prove the identities

$$\int_{\Sigma_\eta} \Pi_0(\vartheta, \vartheta^*(t_0))\, d\mu_{t_0} = \Pi_0(\vartheta^*(t_0), \vartheta^*(t_0)), \tag{6.25}$$

$$\int_{\Sigma_\eta} \Pi_1(H^*(t_0), \varphi)\, d\mu_{t_0} = \Pi_1(H^*(t_0), \varphi^*(t_0)) = \int_{\Sigma_\eta} \Pi_1(H(\varphi), \varphi^*(t_0))\, d\mu_{t_0}, \tag{6.26}$$

$$\int_{\Sigma_\eta} \Pi_1(H(\varphi), \varphi)\, d\mu_{t_0} = \Pi_1((H\varphi)^*(t_0),\ 1). \tag{6.27}$$

In order to prove (6.25) notice that the function $\Pi_0^{(n)}$ given by (6.9) admits the representation

$$\Pi_0^{(n)}(\vartheta, \vartheta^*(t_0)) = \int_\Omega \upsilon^{(n)} \vartheta \,\mathrm{d}x, \quad \upsilon^{(n)} = \sum_1^n \lambda_k c_k \zeta_k \in C(\Omega),$$

where c_k are the Fourier coefficients of $\vartheta^*(t_0)$ in the basis ζ_k. Since the sequence ϑ_N converges to ϑ^* weakly in $L^r(0, T; L^p(\Omega)$ with $r > 1$ and $p > 2$ we have

$$\lim_{N\to\infty} \int_{[t_1,t_0]\cap\mathcal{F}_\eta} \Pi_0^{(n)}(\vartheta_N(t), \vartheta^*(t_0)) \,\mathrm{d}t = \int_{[t_1,t_0]\cap\mathcal{F}_\eta} \int_\Omega \upsilon^{(n)} \vartheta^*(t)\, dx dt. \tag{6.28}$$

In view of Theorem 4.4 the function ϑ^* belongs to the space $C(\mathcal{F}_\eta; L^p(\Omega))$. Since t_0 is a Lebesgue point of $\mathcal{D} \subset \mathcal{F}_\eta$, we conclude from this that

$$\lim_{t_1\to t_0} \frac{1}{t_0 - t_1} \int_{[t_1,t_0]\cap\mathcal{F}_\eta} \int_\Omega \upsilon^{(n)} \vartheta^*(t)\, dx dt$$
$$= \int_\Omega \upsilon^{(n)} \vartheta^*(t_0) \,\mathrm{d}x = \Pi_0^{(n)}(\vartheta^*(t_0), \vartheta^*(t_0)). \tag{6.29}$$

On the other hand, relation (6.28) implies that $\Pi_0^{(n)}(\vartheta, \vartheta^*(t_0))$ is a continuous function of ϑ on X Hence we can apply relation (5.3) in Theorem 5.1 to obtain

$$\lim_{N\to\infty} \int_{[t_1,t_0]\cap\mathcal{F}_\eta]} \Pi_0^{(n)}(\vartheta_N(t), \vartheta^*(t_0)) \,\mathrm{d}t = \int_{[t_1,t_0]\cap\mathcal{F}_\eta\times\Sigma_\eta} \Pi_0^{(n)}(\vartheta, \vartheta^*(t_0))\, d\nu(\boldsymbol{\omega}). \tag{6.30}$$

Next, relation (5.5) in Theorem 5.1 yields

$$\lim_{t_1\to t_0} \frac{1}{t_0 - t_1} \int_{[t_1,t_0]\cap\mathcal{F}_\eta\times\Sigma_\eta} \Pi_0^{(n)}(\vartheta, \vartheta^*(t_0))\, d\nu(\boldsymbol{\omega})$$
$$= \int_{\Sigma_\eta} \Pi_0^{(n)}(\vartheta, \vartheta^*(t_0))\, d\mu_{t_0}(\boldsymbol{\omega}). \tag{6.31}$$

Combining (6.28)–(6.31) we finally arrive at

$$\int_{\Sigma_\eta} \Pi_0^{(n)}(\vartheta, \vartheta^*(t_0))\, d\mu_{t_0}(\boldsymbol{\omega}) = \Pi_0^{(n)}(\vartheta^*(t_0), \vartheta^*(t_0)). \tag{6.32}$$

Recall that Π_0 is integrable with respect to measure μ_{t_0} and $\Pi_0^{(n)} \nearrow \Pi_0$ in $W^{1,2}(\Omega)$. Notice that $\vartheta \in W^{1,2}(\Omega)$ for μ_{t_0} almost every point (ϑ, φ). Letting $n \to \infty$ in (6.32) and applying the Fatou theorem we arrive at (6.25). Now our task is to prove the first equality in (6.26). Recall that $\varphi_N \to \varphi^*$ weakly in $L^s(0, T; L^q(\Omega))$ and $H^* \in L^\infty(\Omega)$. Thus we get

$$\lim_{N\to\infty} \int_{[t_1,t_0]\cap\mathcal{F}_\eta} \Pi_1(H^*(t_0), \varphi_N(t)) \,\mathrm{d}t = \int_{[t_1,t_0]\cap\mathcal{F}_\eta} \int_\Omega H^*(t_0) \varphi^*(t)\, dx dt.$$

On the other hand, relation (6.28) implies that $\varPi_1(H^*(t_0), \varphi)$ is a continuous function of φ on X. Hence we can apply relations (5.3) in Theorem 5.1 to obtain

$$\lim_{N\to\infty}\int_{[t_1,t_0]\cap\mathcal{T}_\eta} \varPi_1(H^*(t_0), \varphi_N(t))\,\mathrm{d}t = \int_{([t_1,t_0]\cap\mathcal{T}_\eta)\times\Sigma_\eta} \varPi_1(H^*(t_0), \varphi)\,d\nu(\boldsymbol{\omega}).$$

Thus we get

$$\int_{[t_1,t_0]\cap\mathcal{F}_\eta}\int_\Omega H^*(t_0)\varphi^*(t)\,dx\,dt = \int_{([t_1,t_0]\cap\mathcal{T}_\eta)\times\Sigma_\eta} \varPi_1(H^*(t_0), \varphi)\,d\nu(\boldsymbol{\omega}). \tag{6.33}$$

Since the mapping $\mathcal{F}_\eta \ni t \to \varphi^*(t) \in L^q(\Omega)$ is continuous and t_0 is a Lebesgue point of $\mathcal{F}_\eta$, we have

$$\lim_{t_1\to t_0}\frac{1}{t_0-t_1}\int_{[t_1,t_0]\cap\mathcal{T}_\eta}\int_\Omega H^*(t_0)\varphi^*(t)\,dx\,dt = \int_\Omega H^*(t_0)\varphi^*(t_0)\mathrm{d}x = \varPi_1(H^*(t_0), \varphi^*(t_0)). \tag{6.34}$$

Next, relation (5.5) in Theorem 5.1 implies

$$\lim_{t_1\to t_0}\frac{1}{t_0-t_1}\int_{([t_1,t_0]\cap\mathcal{T}_\eta)\times\Sigma_\eta} \varPi_1(H^*(t_0), \varphi)\,d\nu(\boldsymbol{\omega}) = \int_{\Sigma_\eta} \varPi_1(H^*(t_0), \varphi)d\mu_{t_0}(\boldsymbol{\omega}). \tag{6.35}$$

Combining (6.33)–(6.35) we arrive at the first equality in (6.26). Arguing as before we obtain

$$\lim_{N\to\infty}\int_{[t_1,t_0]\cap\mathcal{F}_\eta} \varPi_1(H(\varphi_N), \varphi^*(t_0)\,\mathrm{d}t = \int_{[t_1,t_0]\cap\mathcal{F}_\eta}\int_\Omega H^*(t_0)\varphi^*(t)\,dx\,dt$$

and

$$\lim_{N\to\infty}\int_{[t_1,t_0]\cap\mathcal{T}_\eta} \varPi_1(H(\varphi_N), \varphi^*(t_0))\,\mathrm{d}t = \int_{([t_1,t_0]\cap\mathcal{T}_\eta)\times\Sigma_\eta} \varPi_1(H(\varphi), \varphi^*(t_0))\,d\nu(\boldsymbol{\omega}),$$

which leads to

$$\int_{[t_1,t_0]\cap\mathcal{F}_\eta}\int_\Omega H^*(t_0)\varphi^*(t)\,dx\,dt = \int_{([t_1,t_0]\cap\mathcal{T}_\eta)\times\Sigma_\eta} \varPi_1(H(\varphi), \varphi^*(t_0))\,d\nu(\boldsymbol{\omega}). \tag{6.36}$$

Relation (5.5) in Theorem 5.1 implies

$$\lim_{t_1\to t_0}\frac{1}{t_0-t_1}\int_{([t_1,t_0]\cap\mathcal{T}_\eta)\times\Sigma_\eta} \varPi_1(H(\varphi), \varphi^*(t_0))\,d\nu(\boldsymbol{\omega}) = \int_{\Sigma_\eta} \varPi_1(H(\varphi), \varphi^*(t_0))d\mu_{t_0}(\boldsymbol{\omega}). \tag{6.37}$$

Combining (6.34), (6.36), and (6.37) we obtain the second equality in (6.26). The proof of inequality (6.27) is similar.

It remains to show that μ_{t_0} is the Dirac measure. Using identities (6.25)–(6.26) we rewrite the right hand side of inequality (6.24) in the form

$$\int_{\Sigma_\eta}\Big(\Pi_0(\vartheta,\vartheta^*(t_0))+\Pi_1(H^*(t_0),\varphi)\Big)\,d\mu_{t_0}=-\int_{\Sigma_\eta}\Pi_0(\vartheta^*(t_0),\vartheta^*(t_0))\,d\mu_{t_0}$$
$$+2\int_{\Sigma_\eta}\Pi_0(\vartheta,\vartheta^*(t_0))\,d\mu_{t_0}+\Pi_1(H^*(t_0),\varphi^*(t_0)).$$

Next, using identity (6.27) and recalling that μ_{t_0} is a probability measure we rewrite the left hand side of (6.24) in the form

$$\int_{\Sigma_\eta}\Pi(\vartheta,\varphi)\,d\mu_{t_0}=\int_{\Sigma_\eta}\Pi_0(\vartheta,\vartheta)\,d\mu_{t_0}+\Pi_1((H\varphi)^*(t_0),\ 1).$$

Substituting these results in (6.24) we arrive at the important inequality

$$\int_{\Sigma_\eta}\Pi_0(\vartheta-\vartheta^*(t_0),\vartheta-\vartheta^*(t_0))\,d\mu_{t_0}$$
$$+\Pi_1((H\varphi)^*(t_0),1)-\Pi_1(H^*(t_0),\varphi^*(t_0)))\leqq 0.$$

Since $\Pi_0\geqq 0$, it follows that

$$\begin{aligned}&\Pi_1((H\varphi)^*(t_0),\ 1)-\Pi_1(H^*(t_0),\varphi^*t_0))\\&\quad\equiv\int_\Omega((H\varphi)^*(x,t_0)-H^*(x,t_0),\varphi^*(x,t_0)))\,\mathrm{d}x\leqq 0\text{ for all }t_0\in\mathcal{D}.\end{aligned}\tag{6.38}$$

Let us prove that $H^*(t_0)=H(\varphi^*(t_0))$. The proof of this fact is based on the representation of the weak limits in terms of the Young measure. Notice that $\varphi_N\to\varphi^*$ star weakly in $L^\infty(\mathcal{F}_\eta;L^q(\Omega))$, $q>1$. Since the function H is bounded and continuous, it follows from the fundamental theorem on the Young measures that there is a measurable family of probability measures $\sigma_{x,t}$ such that

$$\varphi^*=\int_{\mathbb{R}}\lambda\,d\sigma_{x,t}(\lambda),\ \ H^*=\int_{\mathbb{R}}H(\lambda)\,d\sigma_{x,t}(\lambda),\ \ (H\varphi)^*=\int_{\mathbb{R}}H(\lambda)\lambda\,d\sigma_{x,t}(\lambda)$$

almost everywhere in $\Omega\times\mathcal{F}_\eta$. Since $\sigma_{x,t}$ is a probability measure, we have

$$\begin{aligned}(H\varphi)^*(x,t_0)-H^*(x,t_0),\varphi^*(x,t_0)&=\int_{\mathbb{R}}(H(\lambda)\lambda-\overline{\lambda}\ \overline{H})\,d\sigma_{x,t}(\lambda)\\&=\int_{\mathbb{R}}(H(\lambda)-\overline{H})(\lambda-\overline{\lambda})\,d\sigma_{x,t}\\&=\int_{\mathbb{R}}H(\lambda)(\lambda-\overline{\lambda})\,d\sigma_{x,t}\\&=\int_{\mathbb{R}}(H(\lambda)-H(\overline{\lambda}))(\lambda-\overline{\lambda})\,d\sigma_{x,t}\end{aligned}$$

a.e. in $\Omega \times \mathcal{F}_\eta$. It follows from this and (6.38) that

$$\int_{\mathbb{R}} (H(\lambda) - H(\overline{\lambda}))(\lambda - \overline{\lambda})\, d\sigma_{x,t} \leqq 0$$

almost everywhere in $\Omega \times \mathcal{F}_\eta$. Since the function H is strictly monotone, it is possible if and only if $\sigma_{x,t}$ is the Dirac measure for a.e. $(x, t) \in \Omega \times \mathcal{F}_\eta$. From this and general theory of the Young measures we conclude that $\varphi_N \to \varphi$ in measure in $\Omega \times \mathcal{F}_\eta$. It follows that $H^*(t_0) = H(\varphi^*(t_0)$ for a.e. $t \in \mathcal{F}_\eta$. Thus we get

$$\begin{aligned}\Pi_1(H^*(t_0), \varphi) = \Pi_1(H(\varphi^*(t_0)), \varphi), \quad \Pi_1(H^*(t_0), \varphi^*(t_0))\\ = \Pi_1(H(\varphi^*(t_0)), \varphi^*(t_0)).\end{aligned}$$

From this and (6.26) we obtain

$$\begin{aligned}\int_{\Sigma_\eta} \Pi_1(H^*(t_0), \varphi)\, d\mu_{t_0} = \int_{\Sigma_\eta} \big\{\Pi_1(H(\varphi^*(t_0)), \varphi) + \Pi_1(H(\varphi), \varphi^*(t_0))\big\}\, d\mu_{t_0}\\ - \int_{\Sigma_\eta} \Pi_1(H(\varphi^*(t_0)), \varphi^*(t_0))\big\}\, d\mu_{t_0}.\end{aligned} \tag{6.39}$$

On the other hand, equality (6.25) implies

$$\begin{aligned}\int_{\Sigma_\eta} \Pi_0(\vartheta, \vartheta^*(t_0))\, d\mu_{t_0} = 2\int_{\Sigma_\eta} \Pi_0(\vartheta, \vartheta^*(t_0))\, d\mu_{t_0}\\ - \int_{\Sigma_\eta} \Pi_0(\vartheta^*(t_0), \vartheta^*(t_0))\, d\mu_{t_0}.\end{aligned} \tag{6.40}$$

Substituting (6.39) and (6.40) into inequality (6.24) we may rewrite this inequality in the equivalent form

$$\begin{aligned}&\int_{\Sigma_\eta} \Pi_0(\vartheta - \vartheta^*(t_0), \vartheta - \vartheta^*(t_0))\, d\mu_{t_0}\\ &\quad + \int_{\Sigma_\eta} \Pi_1((H(\varphi) - H(\varphi^*(t_0)))(\varphi - \varphi^*(t_0))\, d\mu_{t_0} \leqq 0.\end{aligned}$$

Notice that the integrands in the left hand side of this inequality are nonnegative and equal zero if and only if $\vartheta = \vartheta^*(t_0)$ and $\varphi = \varphi^*(t_0)$. Hence μ_{t_0} is the Dirac measure concentrated in $(\vartheta^*(t_0), \varphi^*(t_0))$. This completes the proof of Lemma 6.1. In remains to note that in view of Theorem 5.2 representations (5.7) and (5.8) hold for every element of supp μ_{t_0}. Hence they hold for $(\vartheta^*(t_0), \varphi^*(t_0))$ which yields (6.2) and (6.3). □

Finally we prove the strong convergence of the sequences ϑ_N and φ_N.

Lemma 6.6. *Let exponents* (r, p) *and* (s, q) *satisfy condition* (2.20). *Then we have*

$$\vartheta_N \to \vartheta^* \text{in } L^r(0, T; L^p(\Omega)), \quad \varphi_N \to \varphi^* \text{ in } L^s(0, T; L^q(\Omega)). \tag{6.41}$$

Proof. Choose $\gamma > 1$ satisfying the inequality $\gamma < \min\{r, p, s, q\}$ and notice that the mapping

$$F : (\vartheta, \varphi) \to \int_\Omega (|\vartheta|^\gamma + |\varphi|^\gamma)\,\mathrm{d}x$$

is continuous on X. Lemma 6.5 implies

$$\int_{\Sigma_\eta} F(\vartheta, \varphi)\, d\mu_{t_0} = F(\vartheta^*(t_0), \varphi^*(t_0)) = \int_\Omega (|\vartheta^*(x, t_0)|^\gamma + |\varphi^*(x, t_0)|^\gamma)\,\mathrm{d}x. \tag{6.42}$$

for a.e. $t_0 \in \mathcal{F}_\eta$. Applying Theorem 5.1 we obtain that

$$\begin{aligned}\lim_{N\to\infty} \int_{\mathcal{F}_\eta} \int_\Omega (|\vartheta_N|^\gamma + |\varphi|^\gamma)\, dxdt &= \lim_{N\to\infty} \int_{\mathcal{F}_\eta} F(\vartheta_N(t), \varphi_N(t)), dxdt \\ = \int_{\mathcal{F}_\eta} \Big\{ \int_{\Sigma_\eta} F(\vartheta, \varphi) d\mu_t \Big\}\,\mathrm{d}t &= \int_{\mathcal{F}_\eta} \int_\Omega (|\vartheta^*(x, t)|^\gamma + |\varphi^*(t)|^\gamma)\, dxdt.\end{aligned} \tag{6.43}$$

Recall that the sequence (ϑ_N, φ_N) converges to (ϑ^*, φ^*) weakly in $L^\gamma(\Omega \times (0, T))$. Since F is strictly convex, it follows from this and (6.43) that $(\vartheta_N, \varphi_N) \to (\vartheta^*, \varphi^*)$ in $L^\gamma(\Omega \times \mathcal{F}_\eta)$. In particular, the sequence (ϑ_N, φ_N) converges in measure in $\Omega \times \mathcal{F}_\eta$. Letting $\eta \to 0$ we conclude that this sequence converges to (ϑ^*, φ^*) in measure in $\Omega \times (0, T)$. It follows from (4.4) that the sequence ϑ_N is bounded in $L^r(0, T; L^p(\Omega))$ and the sequence φ_n is bounded in $L^s(0, T; L^p(\Omega))$ for all exponents (r, p) and (s, q) satisfying the inequalities (2.20). Since these sequences converge in measure and the set of admissible exponents (r, p) and (s, q) is open, we conclude that $\vartheta_N \to \vartheta^*$ in $L^r(0, T; L^p(\Omega))$ and $\varphi_N \to \varphi^*$ strongly in $L^s(0, T; L^q(\Omega))$. □

It remains to note that Theorem 6.1 is a straightforward consequence of Lemmas 6.1 and 6.6.

7. Proof of Theorem 1.1

In this section we complete the proof of the main Theorem 1.1. Let us consider the sequence of the approximate solutions ϑ_n, v_n and w_N defined by Theorem 2.1. We begin with the observation that Theorems 3.1 and 6.1 imply the relations

$$\begin{aligned} &w_N^{\pm 1} \to w^{\pm 1} \text{ in } C(0, T; L^\alpha(\Omega)), \\ &(v_N, \vartheta_N) \to (v, \vartheta^*) \text{ in } L^r(0, T; L^p(\Omega)), \\ &\varphi_N \equiv \varphi(D^2\mathbf{u}_N, D\mathbf{u}_N, \vartheta_N, w_N) \to \varphi^* \text{ in } L^s(0, T; L^q(\Omega)),\end{aligned} \tag{7.1}$$

which hold true for all $\alpha \in [1, \infty)$ and for all (p, r), (s, q) satisfying (2.20). The limits satisfy the conditions

$$\begin{aligned} &w^{\pm 1} \in L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; W^{1,2}(\Omega)),\ \partial_t w \in L^2(0, T; W^{1,2}(\Omega)), \\ &v, \vartheta^* \in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).\end{aligned} \tag{7.2}$$

Moreover, Theorem 6.1 implies that for almost every $t \in (0, T)$, there is a function $\mathbf{u}(t) \in \mathcal{W}^{2,2} + \mathbf{h}$ such that $\|\mathbf{u}(t)\|_{W^{2,6}(\Omega)} \leqq c_0$ and

$$\vartheta^*(t) = v(t) + W(w(t)^{-1} D\mathbf{u}(t)), \quad \varphi^*(t) = \varphi(D^2\mathbf{u}(t), D\mathbf{u}(t), \vartheta^*(t), w(t)), \tag{7.3}$$

$$\mathcal{H}(\mathbf{u}(t), v(t), w(t)) = \mathbf{M}(v(t), w(t)). \tag{7.4}$$

Let us prove that the functions $v, w, \mathbf{u}, \vartheta^*$ meet all requirements of Definition 1.1 and serve as a weak solution to problem (1.26). It suffices to prove that these functions satisfy equations (1.26c), (1.26h) and integral identities (1.39) and (1.40). Notice that equation (2.7) in the definition of the approximate solution yields

$$\partial_t w_N = -H(\varphi(D^2\mathbf{u}_N, D\mathbf{u}_N, \vartheta_n, w_N))w_N, \quad 0 < t \leqq T, \quad w_N(0) = w_0.$$

Letting $N \to \infty$ and using relations (7.1)–(7.2) we conclude that w satisfies equation and initial condition (1.26c) and (1.26h). Next, integral identity (4.22) implies that w and ϑ^* satisfy integral identity (1.39) with ϑ replaced by ϑ^*. In view of the definition (1.45) of the marginal functional the vector field $\mathbf{u}(t)$ is a minimizer of the functional $\mathcal{H}(\mathbf{u}, v(t), w(t))$. Hence, the equality

$$\lim_{\lambda \to 0} \lambda^{-1}\Big(\mathcal{H}(\mathbf{u} + \lambda\boldsymbol{\xi}, v(t), w(t)) - \mathcal{H}(\mathbf{u}, v(t), w(t))\Big) = 0 \tag{7.5}$$

holds for every function $\boldsymbol{\xi}$ vanishing at $\partial\Omega$. Recall that the functional $\mathcal{H}$ is defined by

$$\mathcal{H}(\mathbf{u}, v(t), w(t))\Big) = \mathcal{E}(\mathbf{u}, v(t), w(t))\Big) + \int_\Omega \mathbf{u} \cdot \mathbf{f}\, \mathrm{d}x,$$

where the integral functional $\mathcal{E}$ is given by (1.43). Substituting the expressions for $\mathcal{H}$ into (7.5) we obtain the integral identity

$$\int_\Omega \Big(\varepsilon w(t)^{d-4} \Delta\mathbf{u}(t) \cdot \Delta\boldsymbol{\xi} \\ + w(t)^{d-1}\Big(1 + \Theta(D\mathbf{u}(t), v(t))\Big) W'(w^{-1}(t) D\mathbf{u}(t)) : D\boldsymbol{\xi} + f \cdot \boldsymbol{\xi}\Big)\, \mathrm{d}x = 0.$$

Noting that $\vartheta^* = \Theta(D\mathbf{u}(t), v(t))$ we conclude that $\mathbf{u}$, w and ϑ^* satisfy integral identity (1.40). Next, notice that in view of (7.4) the deformation field $\mathbf{u}(t)$ satisfies the first selection principle given by Definition 1.3.

It remains to prove that $\vartheta^*(t)$ and $\varphi^* = \varphi(D^2\mathbf{u}(t), D\mathbf{u}(t), \vartheta^*(t), w(t)$ satisfy the second selection principle formulated in Definition 1.5. To this end choose a minimizer $\tilde{\mathbf{u}} \in \mathcal{W}^{2,2} + \mathbf{h}$ of the functional $\mathcal{H}(\cdot, \mathbf{v}(t_0), w(t_0))$ and set

$$\tilde{\vartheta} = v(t_0) + W(w(t_0)^{-1} D\tilde{\mathbf{u}}), \quad \tilde{\varphi} = \varphi(D^2\tilde{\mathbf{u}}, D\tilde{\mathbf{u}}, \tilde{\vartheta}, w(t)).$$

It follows that $(\tilde{\vartheta}, \tilde{\varphi}) \in \mathcal{P}(v(t_0), w(t_0))$, where the set $\mathcal{P}(v, w)$ is given by Definition 1.4. Notice that the function $\tilde{\mathbf{u}}$ meets all requirements of Theorem 4.2 with

$\mathbf{u}$ replaced by $\tilde{\mathbf{u}}$. Recall that the function $\mathbf{M}(v(t_0), w(t_0))$ is differentiable at a.e. point $t_0 \in (0, T)$. From this and relation (4.9) in Theorem 4.2 we obtain

$$\Pi_0(\tilde{\vartheta}, \vartheta^*(t_0)) + \Pi_1(\tilde{\varphi}, H^*(t_0)) \geqq -\mathbf{M}'(v(t_0), w(t_0)) \quad \text{for a.e. } t_0 \in (0, T). \tag{7.6}$$

Obviously we have $\vartheta_N \to \vartheta^*$ weakly in $L^2(0, T; W^{1,2}(\Omega))$. Let us consider the sequence of the functions $\overline{\vartheta}_N$ given by (2.9)–(2.10). In view of Theorem 2.1 they are bounded in $L^2(t_1, t_0; W^{1,2}(\Omega))$ and $\vartheta_N - \overline{\vartheta}_N \to 0$ in $L^2(t_1, t_0; L^2(\Omega))$. It follows that $\overline{\vartheta}_N \to \vartheta^*$ weakly in $L^2(t_1, t_0; W^{1,2}(\Omega)$. Thus we get

$$\liminf_{N\to\infty} \frac{1}{2}\Big\{ \int_{t_1+\tau}^{t_0} \Pi_0(\vartheta_N, \vartheta_N)\, \mathrm{d}s + \int_{t_1+\tau}^{t_0-\tau} \Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)\, \mathrm{d}s \Big\} \geqq \int_{t_1}^{t_0} \Pi_0(\vartheta^*, \vartheta^*). \tag{7.7}$$

It obviously follows from (7.1) that

$$\lim_{N\to\infty} \int_{t_1+\tau}^{t_0} \Pi_1(H(\vartheta_N), \vartheta_N)\, \mathrm{d}s = \int_{t_1}^{t_0} \Pi_1(H(\vartheta^*), \vartheta^*)\, \mathrm{d}s. \tag{7.8}$$

Letting $N \to \infty$ in relation (4.2) in Theorem 4.1, and using (7.7) and (7.8) we get the inequality

$$\frac{1}{t_0 - t_1}\big(\mathbf{M}(v(t_0), w(t_0)) - \mathbf{M}(v(t_1), w(t_1))\big) + \frac{1}{t_0 - t_1} \int_{t_1}^{t_0} \Pi(\vartheta^*, \varphi^*)\, \mathrm{d}t \leqq 0.$$

Letting $t_1 \to t_0$ we arrive at the estimate

$$\Pi(\vartheta^*(t_0), \varphi^*(t_0)) \leqq -\mathbf{M}'(v(t_0), w(t_0)) \quad \text{for a.e. } t_0 \in (0, T).$$

Combining this estimate with (7.6) we conclude that ϑ^* and φ^* satisfy inequality (1.49). Hence ϑ^* and φ^* satisfy the second selection principle. This completes the proof of Theorem 1.1.

8. Conclusion

In the paper we consider the quasi-stationary mathematical model describing the volumetric growth of soft tissues. The model is based on the strain gradient theory of the nonlinear thermoelastic material and takes into account the surface and mass diffusion effects. The main ingredients of the modeling of the growth process are the multiplicative decomposition of the deformation gradient as the product of the growth factor and elastic deformation tensor, and the thermodynamically consistent nonconservative model for the description of the growth rate. Because of the complexity of the problem, we assume in addition that the strain gradient energy density is taken in the simplest Falk form, the temperature is close to an equilibrium value, and the growth is isotropic. The main peculiarity of the problem is that the momentum balance equation may have multiple solutions at every moment, and the number of such solutions may change when time increases.

This leads to possible spontaneous jumps of the deformations and the temperature. Due to the time scaling, this means that periods of slow growth can alternate with periods of fast material growth (the inflation phenomenon). In other words, the whole system exhibits the fast-slow dynamics behavior. We prove the existence of solutions satisfying the additional selection principles which control the possible multiplicity of solutions and the formation of jumps. The first principle states that at every moment the deformation field minimize the total internal energy for fixed entropy and growth factor. The second principle states that among all admissible material stresses, related to the Eshelby material tensor, and all admissible temperature fields, the system chooses the material stresses and temperature fields which minimize the total energy dissipation rate. This principle is close to the Prigogin minimum entropy production principle. Finally notice that our work is the first attempt to develop the non-local rigorous mathematical theory for the mathematical models of the volumetric material growth. Many important questions still remain unsolved; among these are the mathematical theory of the volumetric growth with two growth factors satisfying the covariance principle, and the full theory of the volumetric growth which takes into account nutrition transport, angiogenesis, and cell proliferation.

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A. Proof of Theorem 2.1

Step 1. First we prove the solvability of problem (2.4)–(2.6). Our task is to show that there exist functions $\mathbf{u}_n$, ϑ_n, w_n, and v_n, $1 \leqq n \leqq N$, satisfying (2.5)–(2.6). We proceed by the induction principle. Assume that

$$v_k \in L^2(\Omega), \quad \vartheta_k \in W^{1,2}(\Omega)), \quad w_k^{\pm 1} \in L^\infty(\Omega), \quad \mathbf{u}_k - \mathbf{h} \in \mathcal{W}^{2,2}$$

are defined for all $k \leqq n-1$. We aim to show that there are $(\mathbf{u}_n, v_n, \vartheta_n, w_n^{\pm 1})$ satisfying (2.5)–(2.6). We begin with the observation that the functional $W^{1,2}(\Omega) \ni \vartheta \to \mathbf{S}_n(\mathbf{u}, \vartheta)$ is strictly concave, continuous, and bounded from above for every $\mathbf{u} \in W^{2,2}(\Omega)$. Hence there exists a unique $\vartheta_n(\mathbf{u}) \in W^{1,2}(\Omega)$ such that

$$\mathbf{S}_n(\mathbf{u}, \vartheta_n(\mathbf{u})) = \max_{\vartheta \in W_0^{1,2}(\Omega)} \mathbf{S}_n(\mathbf{u}, \vartheta). \tag{A.1}$$

The following lemma gives the explicit expression for left hand side of this relation:

Lemma A.1. *Let* $\mathbf{u} - \mathbf{h} \in \mathcal{W}^{2,2}$ *and* $\vartheta_n = \vartheta_n(\mathbf{u})$. *Then we have*

$$\mathbf{S}_n(\mathbf{u}, \vartheta_n) = \mathbf{E}(\mathbf{u}, \vartheta_n, w_{n-1}) + \frac{\tau}{2}\Pi_0(\vartheta_n, \vartheta_n) - \int_\Omega \mathbf{f} \cdot \mathbf{u}\, \mathrm{d}x. \tag{A.2}$$

Proof. Calculation of the variation of $\mathbf{S}_n$ at the point $\vartheta_n = \vartheta_n(\mathbf{u})$ leads to the linear elliptic boundary boundary value problem for ϑ_n

$$\begin{aligned} &-\tau\Delta\vartheta_n + w^d_{n-1}\vartheta_n = w^d_{n-2}v_{n-1} + w^d_{n-1}W(w^{-1}_{n-1}D\mathbf{u}) \text{ in } \Omega, \\ &\partial_n\vartheta_n + \vartheta_n = 0 \text{ on } \partial\Omega. \end{aligned} \tag{A.3}$$

Since w_{n-1}, w_{n-2} are uniformly bounded and $W(w^{-1}_{n-1}D\mathbf{u}), v_{n-1} \in L^2(\Omega)$, it follows from the general theory of elliptic boundary value problems that problem (A.3) has a unique solution $\vartheta_n = \vartheta_n(\mathbf{u}) \in W^{2,2}(\Omega)$ and the mapping

$$W^{2,2}(\Omega) \ni \mathbf{u} \to \vartheta_n(\mathbf{u}) \in W^{2,2}(\Omega) \tag{A.4}$$

is continuous. Multiplying both sides of (A.3) by ϑ_n and integrating the result by parts we arrive at the identity

$$\int_\Omega w^d_{n-2}v_{n-1}\vartheta_n\,\mathrm{d}x = \int_\Omega w^d_{n-1}V(D\mathbf{u}, \vartheta_n, w_{n-1})\vartheta_n\,\mathrm{d}x + \tau\Pi_0(\vartheta_n, \vartheta_n).$$

Combining this result with the expression (2.1) for $\mathbf{S}_n$ and noting that

$$\boldsymbol{\Psi}_g(\mathbf{u}, \vartheta_n, w_{n-1}) = \mathbf{E}(\mathbf{u}, \vartheta_n, w_{n-1}) + \int_\Omega w^d_{n-1}V(D\mathbf{u}, \vartheta_n, w_{n-1})\vartheta_n\,\mathrm{d}x$$

we arrive at (A.2). □

It follows from (A.2) and the continuity of the mapping (A.4) that the functional

$$W^{2,2}(\Omega) \ni \mathbf{u} \to \mathbf{S}_n(\mathbf{u}, \vartheta_n(\mathbf{u})) \in \mathbb{R}$$

is the sum of the strictly convex and weakly continuous parts. It is obviously bounded from below. Hence there is $\mathbf{u}_n \in W^{2,2}(\Omega)$ such that

$$\mathbf{S}_n(\mathbf{u}_n, \vartheta_n(\mathbf{u}_n)) = \min_{\mathbf{u}-\mathbf{h}\in\mathcal{W}^{2,2}} \mathbf{S}_n(\mathbf{u}, \vartheta_n(\mathbf{u})) = \min_{\mathbf{u}-\mathbf{h}\in\mathcal{W}^{2,2}} \max_{\vartheta\in W^{1,2}} \mathbf{S}_n(\mathbf{u}, \vartheta). \tag{A.5}$$

Thus we prove the existence of functions $(\mathbf{u}_n, \vartheta_n)W^{2,2}(\Omega) \times W^{1,2}(\Omega)$ satisfying (2.5).

Now our task is to find w_n. We begin with the observation that $\mathbf{u}_n$ and ϑ_n are independent of t. From this and equality (1.26e) we conclude that for a.e. $x \in \Omega$, the functions $\varphi(\vartheta_n, D^2\mathbf{u}_n, \mathbf{u}_n, w)$ and $H(\varphi(\vartheta_n, D^2\mathbf{u}_n, \mathbf{u}_n, w))$ are continuously differentiable with respect to w on the interval $(0, \infty)$. Since the functions $w^{\pm 1}_{n-1}$ are uniformly bounded, the Cauchy problem

$$\partial_t w = -H(\varphi(\vartheta_n, D^2\mathbf{u}_n, D\mathbf{u}_n, w)), \quad w((n-1)\tau) = w_{n-1},$$

has a unique solution defined in a neighborhood of $(n-1)\tau$. Moreover, since the function H is uniformly bounded and $t \in (0, T)$, this solution admits the estimates

$$\begin{aligned} &(\|w_{n-1}\|_{L^\infty(\Omega)} + \|w^{-1}_{n-1}\|_{L^\infty(\Omega)})^{-1}e^{-CT} \leqq w(t) \\ &\leqq (\|w_{n-1}\|_{L^\infty(\Omega)} + \|w^{-1}_{n-1}\|_{L^\infty(\Omega)})e^{-Ct}, \end{aligned}$$

where $C = \sup |H|$. Hence it can be extended to the interval $((n-1)\tau, n\tau]$. Denote this extension as $w_n(x,t)$. By construction it satisfies (2.7). Let us show that w_n is uniformly bounded from below and above. To this end, notice that the function $w_N(x,t)$ given by (2.2) is defined on the interval $(0, n\tau]$ and satisfies the equation and initial condition

$$\partial_t w_N = -H(\varphi(\vartheta_N, D^2\mathbf{u}_N, D\mathbf{u}_n, w_N)) \ \text{ for } t \in (0, n\tau),\ w(0) = w_0.$$

Since $|H| \leqq C$, the function w_N satisfies the inequalities

$$\begin{aligned} 0 < c^{-1} \leqq w_N(x,t) \leqq c < \infty \ \text{ for } 0 \leqq t \leqq n\tau, \\ |\partial_t w_N(x,t)| \leqq c < \infty \ \text{ for } 0 \leqq t \leqq n\tau. \end{aligned} \tag{A.6}$$

Thus we find $\mathbf{u}_n$, ϑ_n and w_n satisfying (2.5)–(2.7). It remains to note that v_n is given by the formula (2.6). In view of growth condition (**H.1a**) and (**H.1b**), it follows from the embedding theorem that $W(w_n^{-1} D\mathbf{u}_n) \in L^2(\Omega)$ and hence $v_n \in L^2(\Omega)$. Applying the induction principle we conclude that problem (2.4)–(2.6) has a solution

$$\begin{aligned} \mathbf{u}_N \in L^\infty(0,T; W^{2,2}(\Omega)), \quad \vartheta_N \in L^\infty(0,T; W^{2,2}(\Omega)), \\ w_N^{\pm 1} \in L^\infty(0,T; L^\infty(\Omega)), \quad v_N \in L^\infty(0,T; L^2(\Omega)). \end{aligned} \tag{A.7}$$

Moreover, in view of (A.6), the growth factor w_N satisfies the inequalities

$$0 < c^{-1} \leqq w_N(x,t) \leqq c < \infty, \quad |\partial_t w_N(x,t)| \leqq c < \infty \ \text{ for } 0 \leqq t \leqq T. \tag{A.8}$$

Step 2. Our next task is to derive the energy estimate (2.11). First we derive the auxiliary inequality (A.18) which leads to the desired energy estimate. The proof of this inequality is purely algebraic. We begin with the observation that

$$\mathbf{S}_n(\mathbf{u}_n, \vartheta_n) = \min_{\mathbf{u}-\mathbf{h} \in \mathcal{W}^{2,2}} \max_{\vartheta \in W^{1,2}} \mathbf{S}_n(\mathbf{u}, \vartheta) \leqq \max_{\vartheta \in W_0^{1,2}} \mathbf{S}_n(\mathbf{u}_{n-1}, \vartheta) = \mathbf{S}_n(\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}), \tag{A.9}$$

where $\overline{\vartheta}_{n-1}$ is a solution to the variational problem

$$\mathbf{S}_n(\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}) = \max_{\vartheta \in W^{1,2}(\Omega)} \mathbf{S}_n(\mathbf{u}_{n-1}, \vartheta). \tag{A.10}$$

Next, notice that problem (A.10) is a particular case of variational problem (A.1) with $\mathbf{u} = \mathbf{u}_{n-1}$. Arguing as before we conclude that variational problem (A.10) has a unique solution $\overline{\vartheta}_{n-1} \in W^{2,2}(\Omega)$. Expression (2.1) for $\mathbf{S}_n$ implies

$$\begin{aligned} \mathbf{S}_n(\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}) = \boldsymbol{\Psi}_g(\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}, w_{n-1}) - \frac{\tau}{2} \Pi_0(\overline{\vartheta}_{n-1}, \overline{\vartheta}_{n-1}) \\ + \int_\Omega \left(w_{n-2}^d v_{n-1} \overline{\vartheta}_{n-1} - \mathbf{f} \cdot \mathbf{u}_{n-1} \right) \mathrm{d}x. \end{aligned} \tag{A.11}$$

Notice that the integrand Ψ_g is a concave function of ϑ, which leads to the inequality

$$\Psi_g(D^2\mathbf{u}_{n-1}, D\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}, w_{n-1}) \leqq \Psi_g(D^2\mathbf{u}_{n-1}, D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) + \partial_\vartheta \Psi_g(D^2\mathbf{u}_{n-1}, D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1})(\overline{\vartheta}_{n-1} - \vartheta_{n-1}).$$

Noting that

$$\partial_\vartheta \Psi_g(D^2\mathbf{u}_{n-1}, D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) = -V(D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}),$$
$$E(D^2\mathbf{u}_{n-1}, D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) = \Psi_g(D^2\mathbf{u}_{n-1}, D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) + w_{n-1}^3 V(D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1})\vartheta_{n-1},$$

and recalling representation (1.43) for the internal energy $\mathbf{E}$, we obtain

$$\boldsymbol{\Psi}_g(\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}, w_{n-1}) \leqq \mathbf{E}(\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) - \int_\Omega w_{n-1}^d V(D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1})\overline{\vartheta}_{n-1}\, dx.$$

Substituting this estimate into (A.11) and recalling that $v_{n-1} = V(D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-2})$ we obtain

$$\mathbf{S}_n(\mathbf{u}_{n-1}, \overline{\vartheta}_{n-1}) \leqq \mathbf{E}(\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) - \frac{\tau}{2}\Pi_0(\overline{\vartheta}_{n-1}, \overline{\vartheta}_{n-1}) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_{n-1}\, dx + \int_\Omega Q_{n-2,n-1}\vartheta_{n-1}^*\, dx, \tag{A.12}$$

where

$$Q_{n-2,n-1} = w_{n-2}^d V(D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-2}) - w_{n-1}^d V(D\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}). \tag{A.13}$$

On the other hand, representation (A.2) implies

$$\mathbf{S}_n(\mathbf{u}_n, \vartheta_n) = \mathbf{E}(\mathbf{u}_n, \vartheta_n, , w_{n-1}) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_n\, dx + 2^{-1}\tau\Pi_0(\vartheta_n, \vartheta_n).$$

Substituting this identity and inequality (A.12) in inequality (A.9) we get

$$\begin{aligned}&\mathbf{E}(\mathbf{u}_n\vartheta_n, w_n) - \mathbf{E}(\mathbf{u}_{n-1}\vartheta_{n-1}, w_{n-1}) + \frac{\tau}{2}\big(\Pi_0(\vartheta_n, \vartheta_n) + \Pi_0(\overline{\vartheta}_{n-1}, \overline{\vartheta}_{n-1})\big)\\ &\leqq \mathbf{E}(\mathbf{u}_n, \vartheta_n, w_n) - \mathbf{E}(\mathbf{u}_n, \vartheta_n, w_{n-1})\\ &\quad + \int_\Omega \mathbf{f}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\, dx + \int_\Omega Q_{n-2,n-1}\overline{\vartheta}_{n-1}\, dx.\end{aligned} \tag{A.14}$$

On the other hand, the identity

$$\mathbf{E}(\mathbf{u}, \vartheta, w) = \boldsymbol{\Psi}_g(\vartheta, \mathbf{u}, w) + \int_\Omega w^d \vartheta V(D\mathbf{u}, \vartheta, w)\, dx$$

implies the representation

$$\begin{aligned}&\mathbf{E}(\mathbf{u}_n, \vartheta_n, w_n) - \mathbf{E}(\mathbf{u}_n, \vartheta_n, w_{n-1}) + \int_\Omega Q_{n-2,n-1}\overline{\vartheta}_{n-1}\,\mathrm{d}x\\ &\quad = \mathbf{A}_n - \mathbf{A}_{n-1} + \mathbf{B}_n + \mathbf{C}_n,\end{aligned} \tag{A.15}$$

where

$$\begin{aligned}&\mathbf{A}_n = -\int_\Omega Q_{n-1,n}\vartheta_n\,\mathrm{d}x, \quad \mathbf{B}_n = \int_\Omega Q_{n-2,n-1}(\overline{\vartheta}_{n-1} - \vartheta_{n-1})\,\mathrm{d}x,\\ &\mathbf{C}_n = \boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_n) - \boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_{n-1}).\end{aligned} \tag{A.16}$$

Substituting (A.15) into (A.14) we finally obtain

$$\begin{aligned}&\mathbf{E}(\mathbf{u}_n, \vartheta_n, w_n) - \mathbf{E}(\mathbf{u}_{n-1}, \vartheta_{n-1}, w_{n-1}) + \frac{\tau}{2}\big(\Pi_0(\vartheta_n, \vartheta_n) + \Pi_0(\overline{\vartheta}_{n-1}, \overline{\vartheta}_{n-1})\big)\\ &\quad \leqq \int_\Omega \mathbf{f}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\,\mathrm{d}x + \mathbf{A}_n - \mathbf{A}_{n-1} + \mathbf{B}_n + \mathbf{C}_n.\end{aligned}$$

Summing both sides with respect to n and noting that

$$\begin{aligned}&\tau\sum_{n=l}^{m}\Pi_0(\vartheta_n, \vartheta_n) = \int_{\tau l}^{\tau m}\Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t, \quad \tau\sum_{n=l}^{m-1}\Pi_0(\overline{\vartheta}_n, \overline{\vartheta}_n)\\ &\qquad \leqq \int_{\tau l}^{\tau(m-1)}\Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)\mathrm{d}t,\end{aligned}$$

we conclude that for all integers $0 \leqq l < m \leqq N$,

$$\begin{aligned}&\Big\{\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_m) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_m\,\mathrm{d}x\Big\} - \Big\{\mathbf{E}(\mathbf{u}_l, \vartheta_l, w_l) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_l\,\mathrm{d}x\Big\}\\ &\quad + \frac{1}{2}\int_{\tau l}^{\tau m}\Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t + \frac{1}{2}\int_{\tau l}^{\tau(m-1)}\Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)\mathrm{d}t \leqq \mathbf{A}_m - \mathbf{A}_l\\ &\quad + \sum_{n=l+1}^{m}(\mathbf{B}_n + \mathbf{C}_n).\end{aligned} \tag{A.17}$$

Setting $l = 0$ in (A.17) we finally arrive at the estimate

$$\begin{aligned}&\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_m) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_m\,\mathrm{d}x + \frac{1}{2}\int_0^{\tau m}\Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t\\ &\quad \leqq |\mathbf{A}_m| + \sum_{n=1}^{m}(|\mathbf{B}_n| + \mathbf{C}_n) + c.\end{aligned} \tag{A.18}$$

Step 3. Now our task is to estimate the right hand side of (A.18). Introduce the quantities

$$\mathbf{I}_n = \int_\Omega \Big(|\Delta\mathbf{u}_n|^2 + W(w_{n-1}^{-1}D\mathbf{u}_n) + \vartheta_n^2\Big)\,\mathrm{d}x, \quad n \geqq 0. \tag{A.19}$$

It obviously follows from formula (1.43) for $\mathbf{E}$ and the inequality $W \geqq 0$ that

$$c^{-1}\,\mathbf{I}_n \leqq \mathbf{E}(\mathbf{u}_n, \vartheta_n, w_m) \leqq c\,\mathbf{I}_n. \tag{A.20}$$

In view of conditions (**H.1a**) and (**H.1b**), The elastic stored energy satisfies the estimate

$$W(\boldsymbol{\xi}) \leqq c(1 + |\boldsymbol{\xi}|)^{\kappa}, \tag{A.21}$$

where $\kappa < 3$ for $d = 3$ and $\kappa < \infty$ for $d = 2$. Since the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,2d}(\Omega)$ is bounded and $\mathbf{u} = \mathbf{h}$ on $\partial\Omega$, we have

$$\int_\Omega |D\mathbf{u}_n|^{2\kappa}\mathrm{d}x \leqq c\Big(1 + \int_\Omega |\Delta\mathbf{u}_n|^2\,\mathrm{d}x\Big)^{\kappa} \leqq c(\mathbf{I}_n^{\kappa} + 1), \quad \int_\Omega |\vartheta_n|^2\,\mathrm{d}x \leqq c\mathbf{I}_n. \tag{A.22}$$

Here the constant c is independent of n. Our first task is to estimate $\mathbf{A}_m$ in terms of $\mathbf{I}_m$. Expression (A.16) for $\mathbf{A}_m$ implies

$$|\mathbf{A}_m| \leqq \int_\Omega |\vartheta_m||Q_{m-1,m}w_m^d|\,\mathrm{d}x. \tag{A.23}$$

It follows from (A.13) and (1.41) that

$$\begin{aligned}&|Q_{m-1,m}|\\ &\quad\leqq |\vartheta_m||w_m^d - w_{m-1}^d| + |w_m^d W(w_m^{-1}D\mathbf{u}_m) - w_{m-1}^d W(w_{m-1}^{-1}D\mathbf{u}_m)|.\end{aligned}$$

In view of (A.8) we have

$$|w_n|^{\pm 1} + |w_{n-1}|^{\pm 1} \leqq c, \quad |w_n - w_{n-1}| \leqq c\tau \ \text{ for all } 1 \leqq n \leqq N. \tag{A.24}$$

From this and (A.21) we obtain the estimate

$$|Q_{m-1,m}| \leqq c\tau|\vartheta_m| + c\tau(|D\mathbf{u}_m|^{\kappa} + 1), \tag{A.25}$$

which along with the Cauchy inequality implies

$$|\mathbf{A}_m| \leqq (c\tau + \delta)\int_\Omega |\vartheta_m|^2\,\mathrm{d}x + c(\delta)\tau^2\int_\Omega (|D\mathbf{u}_m|^{2\kappa} + 1)\,\mathrm{d}x.$$

Here δ is an arbitrary positive number. Combining this and (A.22) we arrive at the desired estimate for $\mathbf{A}_m$:

$$|\mathbf{A}_m| \leqq (c\tau + \delta)\mathbf{I}_m + c(\delta)\tau^2(1 + \mathbf{I}_m^{\kappa}). \tag{A.26}$$

The derivation of the estimate for $\mathbf{B}_n$ is based on the following lemma:

Lemma A.2. *The estimate*

$$\tau\Pi_0(\vartheta_{n-1} - \overline{\vartheta}_{n-1}) + \int_\Omega |\vartheta_{n-1} - \overline{\vartheta}_{n-1}|^2\,\mathrm{d}x \leqq c\tau(1 + \mathbf{I}_n)^{\kappa} \tag{A.27}$$

holds true for all $1 \leqq n \leqq N$. *Here the constant* c *is independent of* n *and* N.

Proof. The variation of the functional $S_n(\mathbf{u}_{n-1}, \vartheta)$ at the extremal point $\vartheta = \overline{\vartheta}_{n-1}$ leads to the following equations for $\overline{\vartheta}_{n-1}$:

$$\begin{aligned} &-\tau\Delta\overline{\vartheta}_{n-1} + w^d_{n-1}\overline{\vartheta}_{n-1} = w^d_{n-2}v_{n-1} - w^d_{n-1}W(w^{-1}_{n-1}D\mathbf{u}_{n-1} \text{ in } \Omega, \\ &\partial_n\overline{\vartheta}_{n-1} + \overline{\vartheta}_{n-1} = 0. \end{aligned} \tag{A.28}$$

In view of (A.3) we have

$$\begin{aligned} &-\tau\Delta\vartheta_{n-1} + w^d_{n-2}\vartheta_{n-1} = w^d_{n-3}v_{n-2} + w^d_{n-2}W(w^{-1}_{n-2}D\mathbf{u}_{n-1}) \text{ in } \Omega, \\ &\partial_n\vartheta_{n-1} + \vartheta_{n-1} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{A.29}$$

Notice that equation (A.29) can be written in the equivalent form

$$w^d_{n-2}v_{n-1} - w^d_{n-3}v_{n-2} = \tau\Delta\vartheta_{n-1}. \tag{A.30}$$

It follows from (A.28)–(A.30) that the function $\zeta = \overline{\vartheta}_{n-1} - \vartheta_{n-1}$ satisfies the equations and boundary conditions as follows:

$$-\tau\Delta\zeta + w^d_{n-1}\zeta = Q_{n-2,n-1} + \tau\Delta\vartheta_{n-1}, \quad \partial_n\zeta + \zeta = 0 \text{ on } \partial\Omega, \tag{A.31}$$

where $Q_{n-2,n-1}$ is given by (A.13). Recall that $\vartheta_{n-1} \in W^{2,2}(\Omega)$. Multiplying both sides of this equation by ζ, integrating by parts and using the Cauchy inequality we obtain

$$\begin{aligned} \tau\Pi_0(\zeta,\zeta) + c\int_\Omega |\zeta|^2\,\mathrm{d}x &\leqq \tau\Pi_0(\zeta,\vartheta_{n-1}) \\ &+ \delta\int_\Omega |\zeta|^2\,\mathrm{d}x + \delta^{-1}\int_\Omega |Q_{m-2,m-1}|^2\,\mathrm{d}x, \end{aligned} \tag{A.32}$$

where the positive constant c is independent of n, g, $\mathbf{f}$. Notice that

$$\tau\Pi_0(\zeta,\vartheta_{n-1}) \leqq \delta\Pi_0(\zeta,\zeta) + \delta^{-1}\Pi_0(\vartheta_{n-1},\vartheta_{n-1}). \tag{A.33}$$

Choosing $\delta \leqq \min\{c/4, 1/2\}$ we arrive at the estimate

$$\tau\Pi_0(\zeta,\zeta) + \int_\Omega |\zeta|^2\,\mathrm{d}x \leqq c\int_\Omega |Q_{m-2,m-1}|^2\,\mathrm{d}x + c\Pi_0(\vartheta_{n-1},\vartheta_{n-1}). \tag{A.34}$$

Next, inequality (A.25) and estimate (A.22) imply that

$$\int_\Omega |Q_{m-2,m-1}|^2\,\mathrm{d}x \leqq c\tau^2(1+\mathbf{I}_{n-1})^\kappa. \tag{A.35}$$

Substituting this estimate into (A.34) we obtain desired estimate (A.27). □

Next, formula (A.16) for the quantity $\mathbf{B}_n$ along with the Cauchy inequality yields the estimate

$$\begin{aligned} |\mathbf{B}_n| &\leqq \delta^{-1}\int_\Omega |Q_{n-2,n-1}|^2\,\mathrm{d}x + \delta\int_\Omega (\vartheta_{n-1} - \overline{\vartheta}_{n-1})^2\,\mathrm{d}x \\ &\leqq c\tau^2\delta^{-1}(1+\mathbf{I}^\kappa_{n-1}) + c\delta\tau\Pi_0(\vartheta_{n-1},\vartheta_{n-1}), \end{aligned}$$

where δ is an arbitrary positive number. Summing both sides with respect to n we arrive at the desired inequality:

$$\sum_{n=1}^{m} |\mathbf{B}_n| \leqq c\tau^2\delta^{-1}\sum_{n=1}^{m}(1+\mathbf{I}_{n-1})^{\kappa} + c\delta\int_0^{\tau m} \Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t. \tag{A.36}$$

Our next task is to estimate $\mathbf{C}_n$. To this end notice that

$$\begin{aligned}\mathbf{C}_n &\equiv \boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_n) - \boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_{n_1})\\ &= \int_0^1\int_\Omega \frac{\partial \Psi_g}{\partial w}(\vartheta_n, D^2\mathbf{u}_n, D\mathbf{u}_n, \eta(x,s))(w_n - w_{n-1})\,dxds,\end{aligned}$$

where

$$\eta(x,s) = sw_n + (1-s)w_{n-1}, \quad s\in[0,1]. \tag{A.37}$$

Identity (1.18) yields

$$\frac{\partial \Psi_g}{\partial w}(\vartheta_n, D^2\mathbf{u}_n, D\mathbf{u}_n, \eta(x,s)) = \frac{1}{\eta(x,s)}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \eta(s)).$$

In view of (2.7) we have

$$w_n - w_{n-1} = -\int_{\tau n-1}^{\tau n} H\big(\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N(t))\big)\, w_N(t)\,dt.$$

Combining the obtained results we arrive at the identity

$$\begin{aligned}&\boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_n) - \boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_{n-1})\\ &\quad = -\int_{\tau(n-1)}^{\tau n}\int_0^1\int_\Omega \frac{1}{\eta(s)}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \eta(s))\\ &\qquad \times H\big(\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N(t))\big)\, w_N(t)\,dxds\mathrm{d}t.\end{aligned}$$

Recalling the identity

$$\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N(t)) = \varphi_N(t) \ \text{ for } t\in(\tau(n-1), \tau n],$$

we can rewrite this relation in the form

$$\begin{aligned}&\boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_n) - \boldsymbol{\Psi}_g(\vartheta_n, \mathbf{u}_n, w_{n_1})\\ &\quad = -\int_{\tau(n-1)}^{\tau n}\int_\Omega H(\varphi_N(\lambda))\,\varphi_N(\lambda)\,\mathrm{d}x d\lambda + \mathbf{R}_n,\end{aligned} \tag{A.38}$$

where

$$\begin{aligned}\mathbf{R}_n &= \int_{\tau(n-1)}^{\tau n}\int_0^1\int_\Omega \mathbf{P}(x,s,t)H(\varphi_N(t))w_N(t)\,dxds\mathrm{d}t,\\ \mathbf{P} &= \frac{1}{w_N(t)}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, w_N(t)) - \frac{1}{\eta(s)}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \eta(s)).\end{aligned} \tag{A.39}$$

Let us estimate $\mathbf{R}_n$. We begin with the observation that H is bounded and w_N is uniformly bounded from below and above. Thus we get

$$\mathbf{R}_n \leqq \int_{\tau(n-1)}^{\tau n} \int_0^1 \int_\Omega |\mathbf{P}(x,s,t)|\, dxds\mathrm{d}t. \tag{A.40}$$

Next, we have for $s \in [0,1]$ and $t \in [\tau(n-1), \tau n]$,

$$\mathbf{P}(x,s,t) = (w_N(t) - \eta(s)) \int_0^1 \frac{\partial}{\partial \varsigma}\Big(\frac{1}{\varsigma(\lambda)}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma(\lambda))\Big)\,\mathrm{d}\lambda,$$

where

$$\varsigma = \lambda w_N(t) + (1-\lambda)\eta(s) \in [w_N(t), \eta(s)].$$

The rest of the proof is based on the following lemma:

Lemma A.3. *Let $\varsigma : \Omega \to \mathbb{R}$ satisfies the inequalities $\|\varsigma^{\pm 1}\|_{L^\infty(\Omega)} \leqq c_1$. Then there is a constant $c(c_1)$ independent of n such that*

$$\int_\Omega \Big(|\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n,, \vartheta_n, \varsigma)| + |\partial_\varsigma \varphi(D^2\mathbf{u}_n, D\mathbf{u}_n,, \vartheta_n, \varsigma)|\Big)\,\mathrm{d}x \leqq c(1+\mathbf{I}_n)^\kappa, \tag{A.41}$$

$$\int_\Omega |\partial_\varsigma E(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)|\,\mathrm{d}x \leqq c(1+\mathbf{I}_n)^\kappa. \tag{A.42}$$

Proof. Recall that

$$\begin{aligned}\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma) &= \frac{\varepsilon}{2}\varsigma^{d-4}|\Delta\mathbf{u}_n|^2 - \varsigma^d\frac{\vartheta_n^2}{2}\\ &+ \varsigma^d(1+\vartheta_n)\big\{dW(\varsigma^{-1}D\mathbf{u}_n) - \varsigma^{-1}W'(\varsigma^{-1}D\mathbf{u}_n) : D\mathbf{u}_n\big\}.\end{aligned}$$

The growth condition (A.21) implies

$$\begin{aligned}&\big|\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)| + |\partial_\varsigma\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)\big|\\ &\leqq \big(|\Delta\mathbf{u}_n|^2 + |\vartheta_n|^2 + (1+|\vartheta_n|)(1+|D\mathbf{u}_n|)^\kappa\big)\\ &\leqq c(1+|\Delta\mathbf{u}_n|^2 + |\vartheta_n|^2) + c|D\mathbf{u}_n|^{2\kappa}.\end{aligned}$$

Integrating these inequalities over Ω and using estimate (A.22) we obtain desired estimate (A.41). Next, it follows from the expression (1.42) for the density of the internal energy E that

$$|\partial_\varsigma E(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)\big| \leqq c\big(|\Delta\mathbf{u}_n|^2 + |\vartheta_n|^2 + |D\mathbf{u}_n|^\kappa + 1\big).$$

Arguing as before we arrive at (2.11). $\square$

We are now in a position to derive the estimate for $\mathbf{C}_n$. It follows from (A.37) that

$$|\eta - w_N(t)| \leqq |\eta - w_{n-1}| + |w_N(t) - w_{n-1}| \leqq c\tau \quad \text{for } t \in [(n-1)\tau n\tau].$$

and

$$0 < w_N^{\pm 1} \leqq c, \quad 0 < \eta^{\pm 1} \leqq c, \quad 0 < \varsigma^{\pm 1} \leqq c,$$

Applying inequality (A.41) in Lemma A.3 we obtain

$$\int_\Omega |\mathbf{P}(x, s, t)|\, \mathrm{d}x \leqq c\tau \int_0^1 \int_\Omega \Big| \frac{\partial}{\partial \varsigma}\Big(\frac{1}{\varsigma(\lambda)} \varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma(\lambda))\Big)\Big|\, \mathrm{d}x\mathrm{d}\lambda$$
$$\leqq c\tau \int_0^1 \int_\Omega \big(|\varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)| + |\partial_\varsigma \varphi(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)|\big)\, dx d\lambda$$
$$c\tau \int_0^1 (1 + \mathbf{I}_n)^\kappa\, \mathrm{d}\lambda = c\tau(1 + \mathbf{I}_n)^\kappa.$$

Combining this result with (A.40) we arrive at the inequality $|\mathbf{R}_n| \leqq c\tau^2(1+\mathbf{I}_n)^\kappa$. Substituting this inequality into (A.38) and recalling the expression (A.16) for $\mathbf{C}_n$ we finally obtain the desired estimate for $\mathbf{C}_n$:

$$\mathbf{C}_n \leqq -\int_{\tau(n-1)}^{\tau n} \int_\Omega H(\varphi_N(t))\varphi_N(t)\, dx dt + c\tau^2(1 + \mathbf{I}_n)^\kappa. \tag{A.43}$$

Summing both the sides of this inequality with respect to n and recalling expression (1.47) for the form Π_1 we arrive at the estimate

$$\sum_{n=1}^m \mathbf{C}_N \leqq -\int_0^{\tau m} \Pi_1(H(\varphi_n),\ \varphi_n) + c\tau^2 \sum_{n=1}^m (1 + \mathbf{I}_n)^\kappa. \tag{A.44}$$

Substituting (A.26), (A.36), and (A.44) into (A.18) we get the inequality

$$\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_m) - \int_\Omega \mathbf{f} \cdot \mathbf{u}_m\, \mathrm{d}x + (1/2 - c\delta) \int_0^{\tau m} \Pi_0(\vartheta_N, \vartheta_N)$$
$$+ \int_0^{\tau m} \Pi_1(H(\varphi_N), \varphi_N)\, \mathrm{d}t \leqq c(\delta)\tau^2 \sum_{n=1}^m (1 + \mathbf{I}_n)^\kappa + c(\delta).$$

Noting that

$$\int_\Omega \mathbf{f} \cdot \mathbf{u}_m\, \mathrm{d}x \leqq \delta \int_\Omega |\Delta \mathbf{u}_m|^2\, \mathrm{d}x + c\delta^{-1} \leqq \delta I_m + c\delta^{-1},$$

we obtain

$$\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_m) - \delta \mathbf{I}_m + (1/2 - c\delta) \int_0^{\tau m} \Pi_0(\vartheta_N, \vartheta_N)$$
$$+ \int_0^{\tau m} \Pi_1(H(\varphi_N), \varphi_N)\, \mathrm{d}t \leqq c(\delta)\tau^2 \sum_{n=1}^m (1 + \mathbf{I}_n)^\kappa + c(\delta).$$

Recalling estimate (A.20) and choosing δ sufficiently small we finally arrive at the inequality

$$\mathbf{I}_m + \int_0^{\tau m} \Pi(\vartheta_N, \varphi_N)\mathrm{d}t \leqq c\tau^2 \sum_{n=1}^{m} (1+\mathbf{I}_n)^\kappa + c, \tag{A.45}$$

where the energy dissipation rate Π is defined by (1.46). Let us estimate the right hand side of this equality. Set

$$\mathbf{I}_N(t) = \mathbf{I}_n \quad \text{for } (n-1)\tau < t \leqq n\tau, \quad \mathbf{J}_N(t) = \int_0^t \mathbf{I}_N^\kappa \,\mathrm{d}s.$$

Since Π is nonnegative, estimate (A.45) implies

$$\mathbf{I}_N(t) \leqq c + c\tau \int_0^t \mathbf{I}_N(s)^\kappa \,\mathrm{d}s, \quad \text{for } 0 < t \leqq T. \tag{A.46}$$

It follows that on the interval $(0, T)$ the function $\mathbf{J}_N$ satisfies the inequality

$$\frac{d}{\mathrm{d}t}\mathbf{J}_N \leqq c(1+\tau \mathbf{J}_N)^\kappa.$$

Obviously $\mathbf{J}_N(t) \leqq \sigma(t)$, where σ is a solution to the Cauchy problem

$$\frac{d}{\mathrm{d}t}\sigma = c(1+\tau\sigma)^\kappa, \quad \sigma(0) = 0 \ \text{ given by } \ \sigma(t) = \frac{1}{\tau}\Big((1+(1-\kappa)c\tau t)^{\frac{1}{1-\kappa}} - 1\Big).$$

For $\tau \leqq 1/(2(\kappa-1)cT)$, the function σ is defined and uniformly bounded on the interval $[0, T]$. This yields the estimate $\mathbf{J}_N \leqq \sigma \leqq c$. Combining this result with (A.46) we obtain the estimate

$$\mathbf{I}_n \leqq c \ \text{ for } 0 \leqq n \leqq N, \tag{A.47}$$

which along with (A.45) yields energy estimate (2.11). It remains to prove inequality (2.14). We begin with the observation that inequalities (A.47) and (A.26) implies the estimate

$$|\mathbf{A}_n| \leqq c(\delta+\tau) + c(\delta)\tau^2. \tag{A.48}$$

Next, inequalities (A.47), (A.36), and (2.11) imply the estimate

$$\sum_{n=1}^{N} |\mathbf{B}_n| \leqq c\tau^2\delta^{-1} \sum_{l+1}^{N} 1 + c\delta \int_0^{\tau N} \Pi_0(\vartheta_N, \vartheta_N)\mathrm{d}t \leqq c\tau\delta^{-1} + c\delta. \tag{A.49}$$

In its turn, inequalities (A.47), (A.36), and (2.11) imply

$$\sum_{l+1}^{m} \mathbf{C}_n \leqq -\int_{\tau l}^{\tau m} \int_\Omega H(\varphi_N(t))\varphi_N(t)\, dxdt + c\tau^2. \tag{A.50}$$

Substituting (A.49)–(A.50) into (A.17) and recalling the definition (1.46) of the energy dissipation rate Π we obtain

$$\begin{aligned}&\Big\{\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_m) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_m\,\mathrm{d}x\Big\} - \Big\{\mathbf{E}(\mathbf{u}_l, \vartheta_l, w_l) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_l\,\mathrm{d}x\Big\}\\ &\quad+\frac{1}{2}\int_{\tau l}^{\tau m}\Pi_0(\vartheta_N,\vartheta_N)\,\mathrm{d}t + \frac{1}{2}\int_{\tau l}^{\tau(m-1)}\Pi_0(\overline{\vartheta}_N,\overline{\vartheta}_N)\mathrm{d}t\\ &\quad+\int_{l\tau}^{m\tau}\Pi_1(H(\varphi_N),\varphi_N)\mathrm{d}t \leqq c\delta + c(\delta)\tau.\end{aligned}\tag{A.51}$$

Let us show that we can replace w_m and w_l in the left hand side of this inequality by w_{m-1} and w_{l-1}. To this end, notice that for every integer $n \in [1, N]$, we have

$$\begin{aligned}&\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_m) - \mathbf{E}(\mathbf{u}_m, \vartheta_m, w_{m-1})\\ &\quad= \int_0^1\int_\Omega \partial_\varsigma E(D^2\mathbf{u}_m, D\mathbf{u}_m, \vartheta_m, \varsigma)(w_m - w_{m-1})\,\mathrm{d}x\mathrm{d}s,\end{aligned}$$

where $\varsigma(s) = sw_n + (1-s)w_{n-1}$ satisfies the inequalities $\varsigma^{\pm1} \leqq c$. From this, the inequality $|w_n - w_{n-1}| \leqq c\tau$, and estimate (A.42) in Lemma A.3 we obtain

$$\begin{aligned}|\mathbf{E}(\mathbf{u}_n, \vartheta_n, w_n) - \mathbf{E}(\mathbf{u}_n, \vartheta_n, w_{n-1})| &\leqq c\tau\int_0^1\int_\Omega |\partial_\varsigma E(D^2\mathbf{u}_n, D\mathbf{u}_n, \vartheta_n, \varsigma)|\,\mathrm{d}x\mathrm{d}s\\ &\leqq c\tau(1+\mathbf{I}_n)^\kappa \leqq c\tau.\end{aligned}$$

Combining this result with (A.51) and noting that $\mathbf{E}(\mathbf{u}_m, \vartheta_m, w_{m-1}) = \boldsymbol{\mathcal{E}}(\mathbf{u}_m, v_m, w_{m-1})$ we obtain

$$\begin{aligned}&\Big\{\boldsymbol{\mathcal{E}}(\mathbf{u}_m, v_m, w_{m-1}) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_m\,\mathrm{d}x\Big\} - \Big\{\boldsymbol{\mathcal{E}}(\mathbf{u}_l, v_l, w_{l-1}) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_l\,\mathrm{d}x\Big\}\\ &\quad+\frac{1}{2}\int_{\tau l}^{\tau m}\Pi_0(\vartheta_N,\vartheta_N)\,\mathrm{d}t + \frac{1}{2}\int_{\tau l}^{\tau(m-1)}\Pi_0(\overline{\vartheta}_N,\overline{\vartheta}_N)\mathrm{d}t\\ &\quad+\int_{l\tau}^{m\tau}\Pi_1(H(\varphi_N),\varphi_N)\mathrm{d}t \leqq c\delta + c(\delta)\tau.\end{aligned}$$

It follows from the definition (2.15) of the functional $\boldsymbol{\mathcal{H}}_N$ that

$$\boldsymbol{\mathcal{H}}_N(t) = \boldsymbol{\mathcal{E}}(\mathbf{u}_n, v_n, w_{n-1}) - \int_\Omega \mathbf{f}\cdot\mathbf{u}_n\,\mathrm{d}x \quad \text{for } t \in ((n-1)\tau, n\tau].$$

Thus we get

$$\begin{aligned}&\boldsymbol{\mathcal{H}}_N(m\tau) - \boldsymbol{\mathcal{H}}_N(l\tau) + \frac{1}{2}\int_{\tau l}^{\tau m}\Pi_0(\vartheta_N,\vartheta_N)\,\mathrm{d}t\\ &\quad+\frac{1}{2}\int_{\tau l}^{\tau(m-1)}\Pi_0(\overline{\vartheta}_N,\overline{\vartheta}_N)\mathrm{d}t + \int_{l\tau}^{m\tau}\Pi_1(H(\varphi_N),\varphi_N)\mathrm{d}t \leqq c\delta + c(\delta)\tau.\end{aligned}\tag{A.52}$$

Now fix $t_0 > t_1$ from the interval $(0, T)$. For every N, choose l and m such that $t_1 \in ((l-1)\tau, l\tau]$ and $t_0 \in ((m-1)\tau, m\tau)$. We have

$$\frac{1}{2}\int_{\tau l}^{\tau m} \Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t + \frac{1}{2}\int_{\tau l}^{\tau(m-1)} \Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)\mathrm{d}t + \int_{l\tau}^{m\tau} \Pi_1(H(\varphi_N), \varphi_N)\mathrm{d}t$$
$$\geqq \frac{1}{2}\int_{t_1+\tau}^{t_0} \Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t$$
$$+ \frac{1}{2}\int_{t_1+\tau}^{t_0-\tau} \Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)\mathrm{d}t + \int_{t_1+\tau}^{t_0} \Pi_1(H(\varphi_N), \varphi_N)\mathrm{d}t.$$

Notice that $\mathcal{H}_N(t)$ is constant on every interval $((n-1)\tau, n\tau]$, $1 \leqq n \leqq N$. From this and (A.52) we conclude that

$$\mathcal{H}_N(t_0) - \mathcal{H}_N(t_1) + \frac{1}{2}\int_{t_1+\tau}^{t_0} \Pi_0(\vartheta_N, \vartheta_N)\,\mathrm{d}t$$
$$+ \frac{1}{2}\int_{t_1+\tau}^{t_0-\tau} \Pi_0(\overline{\vartheta}_N, \overline{\vartheta}_N)\mathrm{d}t + \int_{t_1+\tau}^{t_0} \Pi_1(H(\varphi_N), \varphi_N)\mathrm{d}t \leqq c\delta + c(\delta)\tau.$$

Letting $N \to \infty$ and then $\delta \to 0$ we obtain the desired relation (2.14).

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