Arch. Rational Mech. Anal. 230 (2018) 49–123
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-018-1241-5

CrossMark

Decay of the Boltzmann Equation with the Specular Boundary Condition in Non-convex Cylindrical Domains

Chanwoo Kim & Donghyun Lee

Communicated by C. Mouhot

Abstract

The basic question about the existence, uniqueness, and stability of the Boltzmann equation in general non-convex domains with the specular reflection boundary condition has been widely open. In this paper, we consider cylindrical domains whose cross section is generally non-convex analytic bounded planar domain. We establish a global well-posedness and asymptotic stability of the Boltzmann equation with the specular reflection boundary condition. Our method consists of the delicate construction of ε-tubular neighborhoods of billiard trajectories which bounce infinitely many times or hit the boundary tangentially at some moment, and sharp estimates of the size of such neighborhoods.

Contents

1. Introduction . . . 50
  1.1. Uniform Upper Bound of Number of Bounces . . . 53
  1.2. Sticky Grazing Set . . . 54
  1.3. L^p–L^∞ Bootstrap and Double Iteration . . . 54
2. Domain Decomposition and Notations . . . 55
  2.1. Analytic Non-convex Domain and Notations for Trajectory . . . 55
  2.2. Decomposition of the Grazing Set and the Boundary $\partial\Omega$ . . . 57
3. L^∞ Estimate . . . 59
  3.1. Inflection Grazing Set . . . 59
  3.2. Dichotomy of Sticky Grazing . . . 74
  3.3. Grazing Set . . . 75
  3.4. Transversality and Double Duhamel Trajectory . . . 86
  3.5. Duhamel's Principle and L^∞ Estimate . . . 108
4. L^2-Coercivity via Contradiction Method . . . 114
5. Linear and Nonlinear Decay . . . 118
  5.1. Linear L^2 Decay . . . 118
  5.2. Nonlinear L^∞ Decay . . . 119
6. Appendix: Example of Sticky Grazing Point . . . 121
References . . . 122

1. Introduction

The Boltzmann equation is a fundamental mathematical model for dilute gases which undergo binary collisions. If there is no external force or self-consistent force, a probability density function $F(t, x, v) \geq 0$ is governed by

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F(0, x, v) = F_0(x, v), \tag{1.1}$$

where the position is $x \in U \subset \mathbb{R}^3$ and the velocity is $v \in \mathbb{R}^3$ at time $t \geqq 0$. The collision operator $Q(F_1, F_2)$ takes the form of

$$Q(F_1, F_2) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega)\big[F_1(u')F_2(v') - F_1(u)F_2(v)\big]\,\mathrm{d}\omega \mathrm{d}u,$$

where $u' = u + ((v-u)\cdot\omega)\omega$, $v' = v - ((v-u)\cdot\omega)\omega$. For the collision kernel $B(v-u, \omega)$, we choose the so-called hard potential model with angular cut-off: $B(v-u, \omega) = |v-u|^{\zeta} q_0(\theta)$ with $0 \leqq \zeta \leqq 1$ where $0 \leqq q_0(\theta) \lesssim |\cos\theta|$ and $\cos\theta = \frac{(v-u)\cdot\omega}{|v-u|}$.

When the gas contacts the boundary, we need to impose a boundary condition for F on ∂U, the boundary of the domain U. In this paper, we impose *the specular reflection boundary condition*, which is one of the most basic conditions:

$$F(t, x, v) = F(t, x, R_x v), \quad x \in \partial U, \tag{1.2}$$

where $R_x v := v - 2(\mathbf{n}(x)\cdot v)\mathbf{n}(x)$ and $\mathbf{n}(x)$ is the outward unit normal vector at $x \in \partial U$. Note that the global Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$ is an equilibrium state of (1.1) and satisfies (1.2).

Despite extensive developments in the study of the Boltzmann theory, many basic boundary problems, especially regarding the specular reflection BC with general domains, have remained widely open. In 1977, in [18], SHIZUTA and ASANO announced the global existence of the Boltzmann equation with the specular boundary condition in a smooth convex domain but without a complete proof. The first mathematical proof of such problem was given in [11] by Guo, but with an extra assumption that the boundary should be a level set of a *real analytic* function. Very recently the authors of this paper finally constructed a global unique solution and proved asymptotic stability of μ for general smooth convex domains (with or without external potentials) in [17], using a novel triple iteration method and sequence of geometric decompositions. This marks the complete resolution of a 40-years open question after an announcement [18].

There were even fewer results on the Boltzmann equation for general *non-convex* domains with the specular boundary condition. An asymptotic stability of the global Maxwellian is established in [5], provided certain a priori strong Sobolev estimates can be verified. However, such strong estimates seem to fail especially when the domain is non-convex [9,10,13,16]. To the best of our knowledge, this paper is the first result on the global well-posedness and stability on the Boltzmann equation for any kind of non-convex domains with the specular boundary condition. One of the intrinsic difficulties of the Boltzmann equation in a non-convex domain is that the billiard trajectory is very complicated to control (for example infinite bouncing, grazing).

If F solves (1.1) and satisfies (1.2) then we have the total mass and energy conservation laws as

$$\iint_{U\times\mathbb{R}^3} F(t) = \iint_{U\times\mathbb{R}^3} F_0, \quad \iint_{U\times\mathbb{R}^3} \frac{|v|^2}{2} F(t) = \iint_{U\times\mathbb{R}^3} \frac{|v|^2}{2} F_0. \tag{1.3}$$

By normalization we assume that

$$\iint_{U\times\mathbb{R}^3} F_0(x,v) = \iint_{U\times\mathbb{R}^3} \mu, \quad \iint_{U\times\mathbb{R}^3} \frac{|v|^2}{2} F_0(x,v) = \iint_{U\times\mathbb{R}^3} \frac{|v|^2}{2} \mu. \tag{1.4}$$

On the other hand, for axis-symmetric domains, we have an angular momentum conservation law: if there exist a vector x_0 and an angular velocity ϖ such that

$$\{(x-x_0)\times\varpi\}\cdot\mathbf{n}(x) = 0 \quad \text{for all } x\in\partial U, \tag{1.5}$$

then we have a conservation of the angular momentum as

$$\iint_{U\times\mathbb{R}^3} \{(x-x_0)\times\varpi\}\cdot vF(t) = \iint_{U\times\mathbb{R}^3} \{(x-x_0)\times\varpi\}\cdot vF_0. \tag{1.6}$$

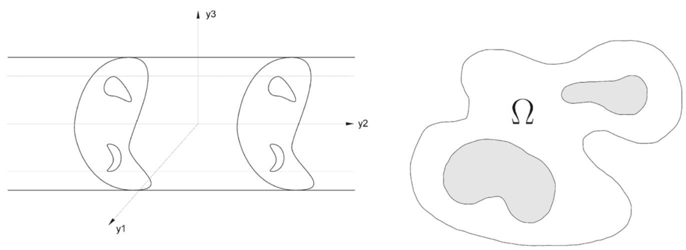
In this case we assume

$$\iint_{U\times\mathbb{R}^3} \{(x-x_0)\times\varpi\}\cdot vF_0(x,v) = 0. \tag{1.7}$$

In this paper, we consider a *periodic*-in-x_2 cylindrical domain with a non-convex analytic cross section. A domain U is given by, with fixed $H>0$,

$$U = \Omega\times(\mathbb{R}/H\mathbb{Z}), \quad (x_1,x_3)\in\Omega \text{ and } x_2\in\mathbb{R}/H\mathbb{Z} \ \text{ for } (x_1,x_2,x_3)\in U. \tag{1.8}$$

The domain U is periodic in x_2 with a period H. See Fig. 1. Denote the boundary of U as $\partial U := \partial\Omega\times(\mathbb{R}/H\mathbb{Z})$.



Periodic cylindrical domain Non-convex analytic cross section

Fig. 1. Periodic cylindrical domain with non-convex analytic cross section

Definition 1. Let $\Omega \subset \mathbb{R}^2$ be an open connected bounded domain and there exist simply connected subsets $\Omega_i \subset \mathbb{R}^2$, for $i = 0, 1, 2, \dots, M < \infty$ such that

$$\Omega = \Omega_0 \backslash \{\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_M\},$$

where

1. $\Omega_0 \supset\supset \Omega_i$ (compactly embedded) for all $i = 1, 2, \dots, M$, and $\partial\Omega_i \cap \partial\Omega_j = \emptyset$ for all $i \neq j$,
2. for each Ω_i, there is a closed regular analytic curve $\alpha_i : [a_i, b_i] \to \mathbb{R}^2$ such that $\partial\Omega_i$ is an image of α_i,
3. $\partial\Omega = \bigsqcup_{i=0}^{M} \partial\Omega_i$, where $\bigsqcup$ stands a disjoint union.

Theorem 1. *Let $w(v) = (1+|v|)^\beta$ with $\beta > \frac{5}{2}$. Consider a periodic-in-x_2 cylindrical domain U defined in (1.8), with an analytic non-convex cross section Ω defined in Definition 1. We assume (1.4) and also assume (1.7) if U is axis-symmetric (1.5). Then, there exists $0 < \delta \ll 1$ such that if*

$$F_0 = \mu + \sqrt{\mu} f_0 \geqq 0 \quad \textit{and} \quad \|wf_0\|_\infty < \delta, \tag{1.9}$$

then (1.1) with (1.2) has a unique global-in-time solution $F(t) = \mu + \sqrt{\mu} f(t) \geqq 0$. Moreover, there exists $\lambda > 0$ such that

$$\sup_{t \geqq 0} e^{\lambda t} \|wf(t)\|_\infty \lesssim \|wf_0\|_\infty, \tag{1.10}$$

and conservation laws (1.3) hold. In the case of an axis-symmetric domain (1.5), we have an additional angular momentum conservation law (1.6).

The perturbation f satisfies

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f), \tag{1.11}$$

and $f(t, x, v) = f(t, x, R_x v)$ for $x \in \partial U$ where

$$Lf = -\frac{1}{\sqrt{\mu}}\big[Q(\mu, f\sqrt{\mu}) + Q(f\sqrt{\mu}, \mu)\big], \quad \Gamma(f, f) = \frac{1}{\sqrt{\mu}} Q(f\sqrt{\mu}, f\sqrt{\mu}). \tag{1.12}$$

The linear operator Lf can be decomposed into $Lf = \nu(v) f - Kf$ where the collisional frequency $\nu(v)$ is defined

$$\nu(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^\zeta q_0(\theta) \mu(u) \mathrm{d}\omega \mathrm{d}u, \quad 0 \leqq \zeta \leqq 1, \tag{1.13}$$

with $C_0 \langle v \rangle^\zeta \leqq \nu(v) \leqq C_1 \langle v \rangle^\zeta$ where $\langle v \rangle := \sqrt{1 + |v|^2}$ for some uniform $C_0, C_1 > 0$. The linear operator Kf is a compact operator on $L^2(\mathbb{R}^3_v)$ with kernel $\mathbf{k}(v, \cdot)$,

$$Kf(v) := \int_{\mathbb{R}^3} \mathbf{k}(v, u) f(u) \, \mathrm{d}u. \tag{1.14}$$

See [11] for the form of $\mathbf{k}(v, u)$.

Now we explain main ideas of the proof of the main theorem.

1.1. Uniform Upper Bound of Number of Bounces

Let us denote the characteristics $(X(s;t,x,v), V(s;t,x,v)) \in U \times \mathbb{R}^3$ at s, which start at position x with velocity v at time t. Also we use cycles $(t^k, x^k, v^k) = (t^k(x,v), x^k(x,v), v^k(x,v))$ to denote kth bouncing time, position, and velocity backward in time (See (2.4) for the precise definition). In contrast to the convex domain case, the characteristics $(X(s), V(s))$ can *graze* (hit the boundary tangentially) at some bouncing time t^k. We split such a grazing set $\{(x^k, v^k) : v^k \cdot \mathbf{n}(x^k) = 0\}$ into three categories depending on where $x^k \in \partial\Omega$ belongs to: convex grazing, concave grazing, and inflection grazing (x^k is an inflection point of $\partial\Omega$ for some k, and therefore x^{k+1} cannot be defined) in Definition 4. The following simplified lemma is the crucial tool to control the number of bounces:

Simplified version of Lemma 2 *If characteristics does not belong to the inflection grazing set then infinite bouncing cannot happen in a finite travel length.*

The analyticity of the boundary is crucial. One can construct an example of infinite number of bounces for finite travel length when the domain is smooth and convex [12].

We prove Lemma 2 via a contradiction argument. If it bounces infinitely many times in a finite travel length then we have a convergent sequence of boundary points $x^k \to x^\infty$. If x^∞ is concave part of the boundary then the trajectory cannot stay in this small neighborhood. If x^∞ is in convex part of the boundary, locally the boundary uniformly convex. Then we use the Velocity lemma [11,17] to exclude infinite bouncing in a finite travel length.

The last case is that x^∞ is an inflection point of the boundary (analytic bounded boundary has only finitely many inflection points); see Fig. 2. Then x^k has to converge to x^∞ through the convex part of the boundary. Since the boundary is analytic a profile of $\partial\Omega$ near inflection points can be approximated by a polynomial with vanishing curvature at the inflection point. For a curvature vanishing we obtain $|x^{k-1} - x^k| \leqq |x^k - x^{k+1}|$ which is contradiction to the hypothesis $x^k \to x^\infty$.

Based on Lemma 2 we derive a uniform number of bounces for given finite travel length away from $\mathfrak{IB}$: a ε-tubular neighborhood of all characteristics graze the boundary which is carefully constructed in Section 3.1.

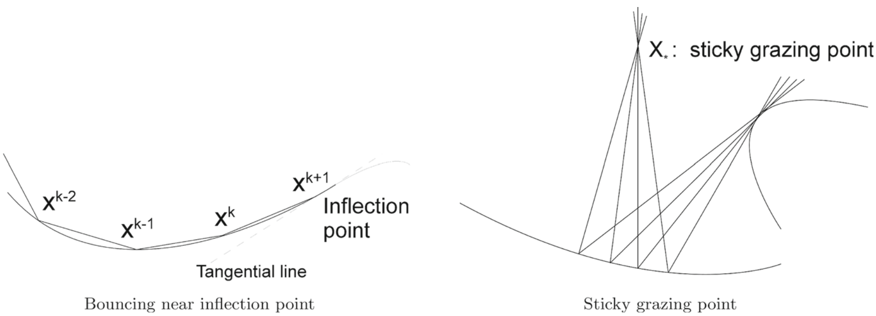


Fig. 2. Bouncing near inflection and Sticky grazing point $\mathcal{SG}$

1.2. Sticky Grazing Set

We consider the characteristics that graze the concave part of boundary at some backward time. Since the billiard map is measure-preserving such set has measure zero in the phase space. However such a "soft" estimate in the phase space is not good enough for our purpose. What we need is for *every* x (not almost every x) to estimate the size of velocity (and the size has to be small) whose characteristics graze the concave part of boundary. It turns out that there could exist *sticky grazing points* $x \in cl(\Omega)$ such that for non-small set of $v \in \mathbb{R}^3$ the characteristics $(X(s;t,x,v), V(s;t,x,v))$ grazes the concave part of boundary at some backward time s. See the Appendix for the construction of such an example and the second picture in Fig. 2. The crucial observation is that from the rigidity of the analytic function such a sticky grazing point can exist uniquely once the number of bounces and the concave part are fixed. Thanks to Section 1.1 and Lemma 2 we have a uniform bound of bounces away from some (small) neighborhood $\mathfrak{IB}$. Moreover the bounded analytic domain has finite parts of the concave part. Therefore all possible sticky grazing points, denoted by $\mathcal{SG}$, are finite at most.

Simplified version of Lemma 1 Away from $\mathfrak{IB}$ and ε neighborhood of $\mathcal{SG}$, the number of bounces is uniformly bounded and uniformly non-grazing as

$$|v^k(x,v)\cdot \mathbf{n}(x^k(x,v))| > \delta > 0, \quad \text{for all } k.$$

1.3. L^p–L^∞ Bootstrap and Double Iteration

Equipped with Lemma 1 and Lemma 2, we illustrate how to estimate the solution of the Boltzmann equation. For the sake of simplicity we consider linearized Boltzmann equation,

$$\partial_t f + v\cdot\nabla_x f + \nu(v) f = Kf. \tag{1.15}$$

To apply the L^p–L^∞ bootstrap argument of [11,17], we aim to derive

$$\|f\|_{L^\infty} \lesssim \|f_0\|_{L^\infty} + \int_0^T \|f\|_{L^2}. \tag{1.16}$$

Let us explain our scheme with a simplified version of (1.15) $\partial_t f + v\cdot\nabla_x f + f = \int_{|u|\leqq N} f\,\mathrm{d}u$. Duhamel's principle gives

$$f(t,x,v) = e^{-t} f_0(x,v) + \int_0^t e^{-(t-s)} \int_{|u|\leqq N} f(s, X(s;t,x,v),u)\,\mathrm{d}u\mathrm{d}s.$$

Applying this formula again (double iteration) to $f(s, X(s;t,x,v),u)$, we get

$$\begin{aligned} f(t,x,v) &= \textit{initial datum's contributions} + O\left(\frac{1}{N}\right) \\ &\quad + \int_0^t e^{-(t-s)} \int_0^{s-\varepsilon} e^{-(s-s')} \iint_{|u|\leqq N, |u'|\leqq N} \\ &\qquad f(s', X(s'; s, X(s;t,x,v),u),u')\,\mathrm{d}u'\,\mathrm{d}u\mathrm{d}s'\mathrm{d}s. \end{aligned}$$

The key step is to prove that the change of variables from u to $X(s'; s, X(s; t, x, v), u)$ is valid. We apply geometric decomposition of trajectories in [17] and use the fact that the characteristics is trivial in x_2 as $\frac{\partial X(s')}{\partial u_2} = -(t-s)$ to verify such change of variables away from $\mathfrak{IB}$ and ε neighborhood of $\mathcal{SG}$. The size of $\mathfrak{IB}$ in v is small but $\mathcal{SG}$ could be large in v. For $\mathcal{SG}$ we use temporal integration to exclude such cases.

2. Domain Decomposition and Notations

2.1. Analytic Non-convex Domain and Notations for Trajectory

Throughout this paper, cross section Ω is a connected and bounded open subset in $\mathbb{R}^2$. In this section, we denote the spatial variable $x = (x_1, x_3) \in cl(\Omega) \subset \mathbb{R}^2$, where $cl(\Omega)$ denotes the closure of Ω in the standard topology of $\mathbb{R}^2$, and the velocity variable $v = (v_1, v_3) \in \mathbb{R}^2$. We also define standard inner product using dot product notation: $a \cdot b := (a_1, a_3) \cdot (b_1, b_3) = a_1 b_1 + a_3 b_3$.

The cross section boundary $\partial\Omega$ is a local image of some smooth regular curve. More precisely, for each $x \in \partial\Omega$, there exists $r > 0$ and $\delta_1 < 0 < \delta_2$ and a curve $\alpha := (\alpha_1, \alpha_3) : \{\tau \in \mathbb{R} : \delta_1 < \tau < \delta_2\} \to \mathbb{R}^2$ such that

$$\partial\Omega \cap B(x, r) = \{\alpha(\tau) \in \mathbb{R}^2 : \tau \in (\delta_1, \delta_2)\}, \tag{2.1}$$

where $B(x, r) := \{y \in \mathbb{R}^2 : |y-x| < r\}$ and $|\dot{\alpha}(\tau)| = [(\dot{\alpha}_1(\tau))^2 + (\dot{\alpha}_3(\tau))^2]^{1/2} := \left[\left(\frac{\mathrm{d}\alpha_1(\tau)}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\alpha_3(\tau)}{\mathrm{d}\tau}\right)^2\right]^{1/2} \neq 0$, for all $\tau \in (\delta_1, \delta_2)$. Without loss of generality, we can assume that $\alpha(\tau)$ is regularly parametrized curve, that is, $|\dot{\alpha}(\tau)| = 1$. For a smooth regularized curve $\alpha(\tau) = (\alpha_1(\tau), \alpha_3(\tau)) \in \mathbb{R}^2$, we define the *signed curvature* of α at τ by

$$\kappa(\tau) := \ddot{\alpha}(\tau) \cdot \mathbf{n}(\alpha(\tau)) = \ddot{\alpha}_1(\tau)\dot{\alpha}_3(\tau) - \dot{\alpha}_1(\tau)\ddot{\alpha}_3(\tau), \tag{2.2}$$

where $\mathbf{n}(\alpha(\tau)) = (\dot{\alpha}_3(\tau), -\dot{\alpha}_1(\tau))$ is outward unit normal vector on $\alpha(\tau) \in \partial\Omega$.

Meanwhile, we assume that the curvature of $\partial\Omega$ is uniformly bounded from above, so (2.1) should be understood as simply connected curve, that is, we can choose sufficiently small $r > 0$ so that $\partial\Omega \cap B(x, r)$ is simply connected curve for all $x \in \partial\Omega$. Throughout this paper, we assume that a local parametrization of boundary satisfies (2.1) as a simply connected curve.

We define convexity and concavity of α by the sign of κ :

Definition 2. Let $\Omega \subset \mathbb{R}^2$ be an open connected bounded subset of $\mathbb{R}^2$ and let the boundary $\partial\Omega$ be an image of smooth regular curve $\alpha \in C^3$ in (2.1). For $\partial\Omega \cap B(x, r) = \{\alpha(\tau) : \delta_1 < \tau < \delta_2\}$, if

$$\kappa(\tau) < 0, \quad \delta_1 < \tau < \delta_2,$$

then we say $\partial\Omega \cap B(x, r)$ is locally *convex*. Otherwise, if $\kappa(\tau) > 0$, we say it is locally *concave*.

We denote the phase boundary of the phase space $\Omega\times\mathbb{R}^3$ as $\gamma := \partial\Omega\times\mathbb{R}^3$, and split into the outgoing boundary γ_+, the incoming boundary γ_-, and the grazing boundary γ_0 :

$$\begin{aligned}\gamma_0 &:= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : \mathbf{n}(x)\cdot v = 0\},\\ \gamma_+ &:= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : \mathbf{n}(x)\cdot v > 0\},\\ \gamma_- &:= \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : \mathbf{n}(x)\cdot v < 0\}.\end{aligned} \tag{2.3}$$

Let us define trajectory. Given $(t,x,v)\in[0,\infty)\times cl(\Omega)\times\mathbb{R}^3$, we use $[X(s),V(s)] = [X(s;t,x,v),V(s;t,x,v)]$ to denote position and velocity of the particle at time s which was placed at x at time t. Along this trajectory, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}X(s;t,x,v) = V(s;t,x,v),\quad \frac{\mathrm{d}}{\mathrm{d}s}V(s;t,x,v)=0,$$

with the initial condition: $(X(t;t,x,v),V(t;t,x,v))=(x,v)$.

Definition 3. We recall the standard notations from [9]. We define

$$\begin{aligned}t_{\mathbf{b}}(t,x,v) &:= \sup\left\{s\geqq 0 : X(\tau;t,x,v)\in\Omega \text{ for all } \tau\in(t-s,t)\right\},\\ x_{\mathbf{b}}(t,x,v) &:= X(t-t_{\mathbf{b}}(t,x,v);t,x,v),\\ v_{\mathbf{b}}(t,x,v) &:= \lim_{s\to t_{\mathbf{b}}(t,x,v)} V(t-s;t,x,v),\end{aligned}$$

and similarly,

$$\begin{aligned}t_{\mathbf{f}}(t,x,v) &:= \sup\left\{s\geqq 0 : X(\tau;t,x,v)\in\Omega \text{ for all } \tau\in(t,t+s)\right\},\\ x_{\mathbf{f}}(t,x,v) &:= X(t+t_{\mathbf{f}}(t,x,v);t,x,v),\\ v_{\mathbf{f}}(t,x,v) &:= \lim_{s\to t_{\mathbf{f}}(t,x,v)} V(t+s;t,x,v).\end{aligned}$$

Here, $t_{\mathbf{b}}$ and $t_{\mathbf{f}}$ are called the backward exit time and the forward exit time, respectively. We also define the specular cycle as in [9].

We set $(t^0,x^0,v^0)=(t,x,v)$. When $t_{\mathbf{b}}>0$, we define, inductively,

$$\begin{aligned}t^k &= t^{k-1}-t_{\mathbf{b}}(t^{k-1},x^{k-1},v^{k-1}),\\ x^k &= X(t^k;t^{k-1},x^{k-1},v^{k-1}),\\ v^k &= R_{x^k}V(t^k;t^{k-1},x^{k-1},v^{k-1}),\end{aligned} \tag{2.4}$$

with reflection operator

$$\begin{aligned}R_{x^k}V(t^k;t^{k-1},x^{k-1},v^{k-1}) &= V(t^k;t^{k-1},x^{k-1},v^{k-1})\\ &\quad -2\big(\mathbf{n}(x^k)\cdot V(t^k;t^{k-1},x^{k-1},v^{k-1})\big)\mathbf{n}(x^k),\end{aligned}$$

where we used abbreviation $t^k=t^k(x,v)$, $x^k=x^k(x,v)$, and $v^k=v^k(x,v)$ for each $k\in\mathbb{N}$. We define the specular characteristics as

$$\begin{aligned}X_{\mathbf{cl}}(s;t,x,v) &= \sum_k \mathbf{1}_{s\in(t^{k+1},t^k]}X(s;t^k,x^k,v^k),\\ V_{\mathbf{cl}}(s;t,x,v) &= \sum_k \mathbf{1}_{s\in(t^{k+1},t^k]}V(s;t^k,x^k,v^k).\end{aligned} \tag{2.5}$$

For the sake of simplicity, we abuse the notation of (2.5) by dropping the subscription **cl** in this section.

2.2. Decomposition of the Grazing Set and the Boundary $\partial\Omega$

In order to study the effect of geometry on particle trajectory, we further decompose the grazing boundary γ_0 (which was defined in (2.3)) more carefully.

Definition 4. Using a disjoint union symbol $\sqcup$, we decompose the grazing set γ_0 as follows:

$$\gamma_0 = \gamma_0^C \sqcup \gamma_0^V \sqcup \gamma_0^I, \quad \gamma_0^I = \gamma_0^{I_+} \sqcup \gamma_0^{I_-}.$$

γ_0^C is a *concave(singular) grazing set*

$$\gamma_0^C := \{(x, v) \in \gamma_0 : t_{\mathbf{b}}(x, v) \neq 0 \text{ and } t_{\mathbf{b}}(x, -v) \neq 0\}.$$

γ_0^V is a *convex grazing set*

$$\gamma_0^V := \{(x, v) \in \gamma_0 : t_{\mathbf{b}}(x, v) = 0 \text{ and } t_{\mathbf{b}}(x, -v) = 0\}.$$

$\gamma_0^{I_+}$ is an *outward inflection grazing set*

$$\gamma_0^{I+} = \{(x, v) \in \gamma_0 : t_{\mathbf{b}}(x, v) \neq 0 \text{ and } t_{\mathbf{b}}(x, -v) = 0 \text{ and } \exists\, \delta > 0 \\ \text{such that } \ x + \tau v \in \mathbb{R}^2 \backslash cl(\Omega) \ \text{ for } \ \tau \in (0, \delta)\}.$$

$\gamma_0^{I_+}$ is an *inward inflection grazing set*

$$\gamma_0^{I-} = \{(x, v) \in \gamma_0 : t_{\mathbf{b}}(x, v) = 0 \text{ and } t_{\mathbf{b}}(x, -v) \neq 0 \text{ and } \exists\, \delta > 0 \\ \text{such that } \ x - \tau v \in \mathbb{R}^2 \backslash cl(\Omega) \ \text{ for } \ \tau \in (0, \delta)\}.$$

Recall that $\Omega := \Omega_0 \setminus \{\Omega_1 \cup \cdots \cup \Omega_M\}$, where each Ω_i is an image of a regularized curve $\alpha_i : [a_i, b_i] \to \mathbb{R}^2$. Recall that κ stands the signed curvature in Definition 2.2. Since the curvature κ is continuous, the set $\{\tau \in [a_i, b_i] : \kappa(\tau) > 0\}$ is an open subset of the interval $[a_i, b_i]$ and therefore it is a countable union of disjoint open intervals, that is,

$$\big\{\tau \in [a_i, b_i] : \kappa(\tau) > 0\big\} = \sqcup_{j=1}^{\infty} \{\tau \in (a_{i,j}, b_{i,j}) : a_{i,j} < \tau < b_{i,j}\}.$$

It is clear that $\kappa(a_{i,j}) = 0 = \kappa(b_{i,j})$ for all i, j : Suppose not, then there exists $\varepsilon > 0$ such that $(a_{i,j} - \varepsilon, b_{i,j}) \in \{\tau \in [a_i, b_i] : \kappa(\tau) > 0\}$ or $(a_{i,j}, b_{i,j} + \varepsilon) \in \{\tau \in [a_i, b_i] : \kappa(\tau) > 0\}$, which is a contradiction.

On the other hand, the signed curvature κ is analytic since the curve α_i is analytic. If κ is identically zero then α_i is a straight line so that $\partial\Omega_i$ cannot be a boundary of a bounded set Ω. By analyticity, κ have at most finite zeroes on a compact set $[a_i, b_i]$.

For fixed Ω_i, we can assume that there are $2N_i$ number of inflection points (where signed curvature changes sign near its neighborhood)

$$\bigsqcup_{j=1}^{N_i} \{\alpha_i(a_{i,j}), \alpha_i(b_{i,j})\},$$

where $a_{i,j}$ and $b_{i,j}$ are properly chosen so that concave intervals are written by

$$\kappa(\tau) \geqq 0 \quad \text{on } \tau \in \bigsqcup_{j=1}^{N_i} [a_{i,j}, b_{i,j}],$$

and convex intervals are written by

$$\kappa(\tau) \leqq 0 \quad \text{on } cl\left([a_i, b_i] \setminus \bigsqcup_{j=1}^{N_i} [a_{i,j}, b_{i,j}] \right),$$

where $\bigsqcup$ stands disjoint union.

Definition 5. Let $\Omega \subset \mathbb{R}^2$ be an analytic non-convex domain in Definition 1. We decompose the boundary $\partial\Omega$ into three parts:

$$\begin{aligned}
\partial\Omega^C &:= \bigsqcup_{i=0}^{M} \bigsqcup_{j=1}^{N_i} \left\{\alpha_i(\tau) : \tau \in (a_{i,j}, b_{i,j})\right\} := \bigsqcup_{l=1}^{M^C} \partial\Omega_l^C, && \text{(Concave boundary)}\\
\partial\Omega^I &:= \bigsqcup_{i=0}^{M} \bigsqcup_{j=1}^{N_i} \{\alpha_i(a_{i,j}), \alpha_i(b_{i,j})\}, && \text{(Inflection boundary)}\\
\partial\Omega^V &:= \partial\Omega \setminus (\partial\Omega^C \cup \partial\Omega^I). && \text{(Convex boundary)}
\end{aligned}$$

The number $M^C = \sum_{i=0}^{M} N_i$ and the lth concave part $\partial\Omega_l^C$ for $l = 1, 2, \ldots, M^C$ is renumbered sequence of $\left\{\alpha_i(\tau) : \tau \in (a_{i,j}, b_{i,j})\right\}$ for $i = 0, 1, \ldots, M$ and $j = 1, 2, \ldots, N_i$. Therefore, we can define M^C number of parametrization $\bar{\alpha}_l$ with $l = 1, \ldots, M^C$ such that

$$\partial\Omega_l^C = \{\bar{\alpha}_l(\tau) : \tau \in (\bar{a}_l, \bar{b}_l)\}. \tag{2.6}$$

We further split $\partial\Omega^I = \partial\Omega^{I+} \cup \partial\Omega^{I-}$ where $\partial\Omega^{I+} := \bigsqcup_{i=0}^{M} \partial\Omega_i^{I+}$ and $\partial\Omega^{I-} := \bigsqcup_{i=0}^{M} \partial\Omega_i^{I-}$ with

$$\begin{aligned}
\partial\Omega_i^{I+} &= \{\alpha_i(\tau) \in \partial\Omega_i^I : \exists\, \varepsilon > 0 \text{ such that } \kappa_i(\tau') < 0 \text{ for } \tau' \in (\tau - \varepsilon, \tau) \text{ and}\\
&\qquad \kappa_i(\tau') > 0 \text{ for } \tau' \in (\tau, \tau + \varepsilon)\},\\
\partial\Omega_i^{I-} &= \{\alpha_i(\tau) \in \partial\Omega_i^I : \exists\, \varepsilon > 0 \text{ such that } \kappa_i(\tau') > 0 \text{ for } \tau' \in (\tau - \varepsilon, \tau) \text{ and}\\
&\qquad \kappa_i(\tau') < 0 \text{ for } \tau' \in (\tau, \tau + \varepsilon)\}.
\end{aligned}$$

Note that the following decomposition is compatible with those of Definition 4:

$$\gamma_0^C = \{(x, v) \in \gamma_0 : x \in \partial\Omega^C\}, \qquad \text{(Concave grazing set)}$$
$$\gamma_0^I = \{(x, v) \in \gamma_0 : x \in \partial\Omega^I\}, \qquad \text{(Inflection grazing set)}$$
$$\gamma_0^V = \{(x, v) \in \gamma_0 : x \in \partial\Omega^V\}. \qquad \text{(Convex grazing set)}$$

Remark that from the definition, it is clear that

$$cl(\partial\Omega^C) = \bigcup_{i=0}^{M} \bigcup_{j=1}^{N_i} cl\big(\{\alpha_i(\tau) : \tau \in (a_{i,j}, b_{i,j})\}\big) = \bigsqcup_{i=0}^{M} \bigsqcup_{j=1}^{N_i} \{\alpha_i(\tau) : \tau \in [a_{i,j}, b_{i,j}]\}.$$

3. L^∞ Estimate

3.1. Inflection Grazing Set

In this section, we study some properties of inflection grazing phase and its neighborhood. We note that grazing trajectories are of measure zero in *phase space* $\Omega \times \mathbb{R}^2$.

First, let us split axial dynamics. The trajectory of a particle is very simple for axial direction:

$$V_2(s; t, x, v) = v_2, \quad X_2(s; t, x, v) = x_2 - (t - s)v_2.$$

Therefore, the characteristics of trajectories come from dynamics in the two-dimensional cross section Ω. In this subsection, we analyze trajectories in $\Omega \subset \mathbb{R}^2$. First, for fixed $N \gg 1$, we define the admissible set of velocity:

$$\mathbb{V}^N := \left\{ v \in \mathbb{R}^2 : \frac{2}{N} \leqq |v| \leqq \frac{N}{2} \right\},$$

and $\mathfrak{m}_2 : P(\Omega) \to \mathbb{R}$ is the standard Lebesgue measure in $\mathbb{R}^2$.

We control the collection of bad phase sets with those that are nearly grazing sets for each open cover containing a boundary $\partial\Omega$. Different studies have been done on the size of the sets of trajectories that reach the grazing set to derive a lower bound of the solution in [1,2].

Lemma 1. *Let $\Omega \subset \mathbb{R}^2$ be an analytic non-convex domain, defined in Definition 1. For $\varepsilon \ll 1$, $N \gg 1$, there exist finite points*

$$\left\{x_1^{nB}, \dots, x_{l_{nB}}^{nB}\right\} \subset cl(\Omega),$$

and their open neighborhoods

$$B\left(x_1^{nB}, r_1^{nB}\right), \dots, B\left(x_{l_{nB}}^{nB}, r_{l_{nB}}^{nB}\right) \subset \mathbb{R}^2,$$

as well as corresponding open sets

$$\mathcal{O}_1^{nB}, \dots, \mathcal{O}_{l_{nB}}^{nB} \subset \mathbb{V}^N,$$

with $\mathfrak{m}_2(\mathcal{O}_i^{nB}) \leqq \varepsilon$ *for all* $i = 1, \dots, l_{nB}$ *such that for every* $x \in cl(\Omega)$*, there exists* $i \in \{1, \dots, l_{nB}\}$ *with* $x \in B(x_i^{nB}, r_i^{nB})$ *and satisfies either*

$$B\left(x_i^{nB}, r_i^{nB}\right) \cap \partial\Omega = \emptyset \quad or \quad |v' \cdot \mathbf{n}(x')| > \varepsilon/N^4$$

for all $x' \in B(x_i^{nB}, r_i^{nB}) \cap \partial\Omega$ *and* $v' \in \mathbb{V}^N \backslash \mathcal{O}_i^{nB}$.

Proof. By Definition 1, $\partial\Omega \in \mathbb{R}^2$ is a compact set in $\mathbb{R}^2$ and a union of the images of finite curves. For $x \in \Omega$, we define $r_x > 0$ such that $B(x, r_x) \cap \partial\Omega = \emptyset$. For each $x \in \partial\Omega$, we can define the outward unit normal direction $\mathbf{n}(x)$ and the outward normal angle $\theta_n(x) \in [0, 2\pi)$ specified uniquely by $\mathbf{n}(x) = (\cos\theta_n(x),\ \sin\theta_n(x))$. Using the smoothness and uniform boundedness of curvature of the boundary $\partial\Omega$, there exists uniform $r_{\varepsilon,N} > 0$ such that

$$|\,\theta_n(x') - \theta_n(x)\,| < \varepsilon/2N^2 \quad \text{for all} \quad x' \in B(x, r_{\varepsilon,N}) \cap \partial\Omega, \tag{3.1}$$

and $B(x, r_{\varepsilon,N}) \cap \partial\Omega$ is a simply connected curve.

By compactness, we have finite integer $l_{nB} > 0$, points $\{x_i^{nB}\}_{i=1}^{l_{nB}}$, and positive numbers $\{r_i^{nB}\}_{i=1}^{l_{nB}}$ such that

$$cl(\Omega) \subset \bigcup_{i=1}^{l_{nB}} B\left(x_i^{nB}, r_i^{nB}\right), \quad r_i^{nB} \leqq r_{\varepsilon,N}.$$

By the above construction, for each $1 \leqq i \leqq l_{nB}$, we have either

$$B\left(x_i^{nB}, r_i^{nB}\right) \cap \partial\Omega = \emptyset \tag{3.2}$$

or

$$x_i^{nB} \in \partial\Omega \quad \text{and} \quad r_i^{nB} < r_{\varepsilon,N}, \quad \text{so that (3.1) holds.} \tag{3.3}$$

For i with case (3.2), we set $\mathcal{O}_i^{nB} = \emptyset$. For i with the case (3.3), we define

$$\mathcal{O}_i^{nB} := \Big\{ v \in \mathbb{V}^N : v = \big(|v|\cos\theta, |v|\sin\theta\big) \quad \text{where} \\ \theta \in \left(\left(\theta_i \pm \frac{\pi}{2}\right) - \frac{\varepsilon}{N^3}, \left(\theta_i \pm \frac{\pi}{2}\right) + \frac{\varepsilon}{N^3}\right) \Big\},$$

where we abbreviated $\theta_n(x_i^{nB}) = \theta_i$. Obviously, $\mathfrak{m}_2(\mathcal{O}_i^{nB}) \leqq \pi \frac{N^2}{4} \frac{\varepsilon/N^2}{\pi} \leqq \varepsilon$ and

$$\begin{aligned} |v' \cdot \mathbf{n}(x')| &\geqq |v'| \times \Big|(\cos\theta', \sin\theta') \cdot (\cos\theta_n(x'), \sin\theta_n(x'))\Big| \\ &\geqq \frac{2}{N} \times \left|\cos\left(\frac{\pi}{2} + \frac{\varepsilon}{N^3}\right)\right| = \frac{2}{N}\left|\sin\left(\frac{\varepsilon}{N^3}\right)\right|, \quad \varepsilon/N^3 \ll 1, \\ &\geqq \frac{\varepsilon}{N^4}, \end{aligned}$$

for $x' \in B(x_i^{nB}, r_i^{nB})$ and $v' = |v'|(\cos\theta', \sin\theta') \in \mathbb{V}^N \backslash \mathcal{O}_i^{nB}$. □

We state a critical property of the analytic boundary for non-convergence of consecutive specular bouncing points. We use the notation of the specular cycles (x^i, v^i) defined in (2.4).

Lemma 2. *Assume $\Omega \subset \mathbb{R}^2$ is the analytic non-convex domain of Definition 1. Choose $x \in cl(\Omega)$ and nonzero $v \in \mathbb{V}^N$. If $[x^i(x,v), v^{i-1}(x,v)] \notin \gamma_0^I$ for all $i = 0, 1, 2, \ldots$, then*

$$\sum_{i=0}^{\infty} |x^i(x,v) - x^{i+1}(x,v)| = \infty.$$

Proof. We prove this lemma by a contradiction argument: suppose $[x^i(x,v), v^i(x,v)] \notin \gamma_0^I$ for all $i = 0, 1, 2, \ldots$ and

$$\sum_{i=0}^{\infty} |x^i(x,v) - x^{i+1}(x,v)| < \infty, \tag{3.4}$$

then $x^i(x,v) \to x^\infty$ and $x^\infty = \lim_{i\to\infty} \alpha(\tau_i) = \alpha(\tau_\infty) \in \partial\Omega$ using that $\partial\Omega$ is closed set. For $i \gg 1$, we assume $x^i(x,v) \in \{\alpha_j(\tau) : \tau \in [a_j, b_j]\}$ for some fixed $j \in \mathbb{N}$ in Definition 1. Otherwise $x^i(x,v)$ cannot converge because $dist(\partial\Omega_{j_1}, \partial\Omega_{j_2}) > \delta > 0$ for $j_1 \neq j_2$. Therefore we drop index j and denote $\alpha(\tau_i) = \alpha_j(\tau_i) = x^i(x,v)$ in this proof.

Step 1. Let us drop the notation of fixed (x,v) and assume that

$$x^i = \alpha(\tau_i), \quad x^{i+1} = \alpha(\tau_{i+1}), \quad x^{i+2} = \alpha(\tau_{i+2}).$$

We claim that if $\tau_i < \tau_{i+1}$, then $\tau_{i+1} < \tau_{i+2}$ for sufficiently large $i \gg 1$. As explained in (2.1), we can find $r^* \ll 1$ such that if $r \leqq r^*$, then $B(x,r) \cap \partial\Omega$ is simply connected curve for $x \in \partial\Omega$. Also for $x \in \partial\Omega$, we can find $r^{**} \ll 1$ such that if $r \leqq r^{**}$, then $\{B(x,r) \cap \partial\Omega\} \cap N(x) = \{x\}$ where $N(x) = \{x + c\mathbf{n}(x) : c \in \mathbb{R}\}$, the normal line crossing $x \in \partial\Omega$. For $r = \min(r^*, r^{**})$ we can decompose

$$B(x^{i+1}, r) \cap \partial\Omega = \underbrace{\Big\{\{\alpha(\tau) : \tau < \tau_{i+1}\} \cap \partial\Omega\Big\}}_{:=B_-} \sqcup \{x^{i+1}\} \sqcup \underbrace{\Big\{\{\alpha(\tau) : \tau > \tau_{i+1}\} \cap \partial\Omega\Big\}}_{:=B_+}. \tag{3.5}$$

From (3.4), for any $\varepsilon < \frac{1}{2}\min(r^*, r^{**})$, we can choose $R \gg 1$ such that

$$|x^i - x^{i+1}| < \varepsilon, \quad \forall i > R. \tag{3.6}$$

If we consider $B(x^{i+1}, \min(r^*, r^{**}))$, both x^i and x^{i+2} are in $B(x^{i+1}, \min(r^*, r^{**})) \cap \partial\Omega$ by (3.6). If $\tau_i < \tau_{i+1}$, then $\tau_i \in B_-$. Combining this fact with disjoint decomposition (3.5), we know that $v^{i+1} \cdot \dot{\alpha}(\tau_{i+1}) > 0$. Therefore, $x^{i+2} \notin B_-$ and we already know that $x^{i+2} \neq x^{i+1}$. Finally we get

$$x^{i+2} \in \{B(x^{i+1}, \min(r^*, r^{**})) \cap \partial\Omega\} \backslash \{B_- \sqcup \{x^{i+1}\}\} := B_+.$$

By definition of B_+, $\tau_{i+1} < \tau_{i+2}$.

Step 2. We split τ_∞ into three cases and study possible cases for (3.4). Without loss of generality, we assume that ε and $i > R$ in the rest of this proof satisfy (3.6).

(i) If $\kappa(\tau_\infty) < 0$, $\exists\, \varepsilon > 0$ such that $\kappa(\tau) < 0$ for $\tau \in (\tau_\infty - \varepsilon, \tau_\infty + \varepsilon)$. While the boundary is convex, we can apply a velocity lemma; Lemma 1 in [11] or Lemma 2.6 in [17]. From the velocity lemma, the normal velocity at the bouncing points are equivalent, especially,

$$\begin{aligned} e^{C_\Omega(|v|+1)t^i}(v^i \cdot \mathbf{n}(x^i)) &\leqq e^{C_\Omega(|v|+1)t^{i+1}}(v^{i+1} \cdot \mathbf{n}(x^{i+1})), \\ e^{-C_\Omega(|v|+1)t^i}(v^i \cdot \mathbf{n}(x^i)) &\geqq e^{-C_\Omega(|v|+1)t^{i+1}}(v^{i+1} \cdot \mathbf{n}(x^{i+1})). \end{aligned} \tag{3.7}$$

Since nonzero speed $|v|$ is constant, (3.4) implies finite time stop of the trajectory. From (3.7), $v^i \cdot \mathbf{n}(x^i)$ cannot be zero at finite time. This is a contradiction.

(ii) If $\kappa(\tau_\infty) > 0$, $\exists\, \varepsilon > 0$ such that $\kappa(\tau) > 0$ for $\tau \in (\tau_\infty - \varepsilon, \tau_\infty + \varepsilon)$. Without loss of generality, we choose $\varepsilon \leqq \min(r^*, r^{**})$, as chosen in *Step 1*. By concavity,

$$\begin{aligned} &(\alpha(\tau) - x^{i+1}) \cdot \mathbf{n}(x^{i+1}) > 0 \quad \text{for} \quad \tau \in (\tau_\infty, \tau_\infty + \varepsilon) \\ &\quad \text{where } R_{x^{i+1}} v^{i+1} \cdot \mathbf{n}(x^{i+1}) < 0. \end{aligned}$$

This implies $\tau_i \in (\tau_{i+1} - \varepsilon, \tau_{i+1}]$, then $\tau_{i+2} \notin [\tau_{i+1}, \tau_{i+1} + \varepsilon)$. This is a contradiction.

(iii) If $\kappa(\tau_\infty) = 0$ and $\kappa(\tau) > 0$ for $\tau \in (\tau_\infty - \varepsilon, \tau_\infty]$, this case is exactly the same as case (ii).

(iv) If $\kappa(\tau_\infty) = 0$ and $\kappa(\tau) = 0$ for $\tau \in (\tau_\infty - \varepsilon, \tau_\infty]$, then $\kappa(\tau) = 0$ for $\tau \in [a_j, b_j]$ by analyticity. Thus, Ω must be a half plane and we get a contradiction.

(v) Assume $\kappa(\tau_\infty) = 0$ and $\kappa(\tau) < 0$ for $\tau \in (\tau_\infty - \varepsilon, \tau_\infty]$.

Step 3. We derive a contradiction for the last case (v) by claiming

$$l_{i+1} = |x^{i+1} - x^i| \leqq |x^{i+2} - x^{i+1}| = l_{i+2} < \varepsilon, \quad i \geqq R \tag{3.8}$$

for ε and R is what we have chosen in (3.6). As explained in (2.1), we can assume that $B(x^\infty, \varepsilon) \cap \partial\Omega$ is a graph of analytic function $\varphi(s)$. From the argument of *Step 1*, we assume $s_\infty - \varepsilon < s_i < s_{i+1} < s_{i+2} < s_\infty$. Moreover, up to translation and rotation, we can assume that $\varphi(s_{i+1}) = \varphi'(s_{i+1}) = 0$ and $\varphi''(s) > 0$ on $s \in (s_\infty - \varepsilon, s_\infty)$. There exist $n_0 \in \mathbb{N}$ such that

$$\frac{\mathrm{d}^{n_0}\varphi}{\mathrm{d}s^{n_0}}(s_\infty) \neq 0 \quad \text{and} \quad \frac{\mathrm{d}^i\varphi}{\mathrm{d}s^i}(s_\infty) = 0 \quad \text{for all } 0 \leqq i < n_0.$$

If $n_0 = \infty$, $\partial\Omega$ is a straight line so is a contradiction as explained in (iv) of *Step 2*. Also, by definition of the inflection point, $n_0 \geqq 3$. For finite $n_0 \in \mathbb{N}$, for $|s| < \varepsilon \ll 1$,

$$\varphi''(s) = c_{n_0-2}(s - s_\infty)^{n_0-2}\big(1 + O(|s - s_\infty|)\big) \to 0 \quad \text{as } s \to s_\infty^-. \tag{3.9}$$

To claim $|x^{i+1}-x^i| \leqq |x^{i+2}-x^{i+1}|$, it suffices to claim $s_{i+1}-s_i \leqq s_{i+2}-s_{i+1}$, because the absolute values of slopes of $\overline{x^i x^{i+1}}$ and $\overline{x^{i+1}x^{i+2}}$ are the same by the specular boundary condition. Since we assume $\varphi'(s_{i+1}) = 0$, from the specular boundary condition,

$$\frac{\varphi(s_{i+2})-\varphi(s_{i+1})}{s_{i+2}-s_{i+1}} = \frac{\varphi(s_i)-\varphi(s_{i+1})}{s_{i+1}-s_i},$$
$$\frac{1}{s_{i+2}-s_{i+1}}\int_{s_{i+1}}^{s_{i+2}}\int_{s_{i+1}}^{t}\varphi''(r)\mathrm{d}r\mathrm{d}t = \frac{1}{s_{i+1}-s_i}\int_{s_i}^{s_{i+1}}\int_{t}^{z_{i+1}}\varphi''(r)\,\mathrm{d}r\mathrm{d}t. \tag{3.10}$$

It is important that near the inflection point, from (3.9), $\varphi'' > 0$ is monotone decreasing to zero on $s \in (s_\infty - \varepsilon, s_\infty)$ for $\varepsilon \ll 1$. Therefore,

$$\begin{aligned}\frac{1}{s_{i+2}-s_{i+1}}\int_{s_{i+1}}^{s_{i+2}}\int_{s_{i+1}}^{t}\varphi''(r)\mathrm{d}r\mathrm{d}t &\leqq \frac{1}{2}(s_{i+2}-s_{i+1})\varphi''(s_{i+1}),\\ \frac{1}{s_{i+1}-s_i}\int_{s_i}^{s_{i+1}}\int_{t}^{z_{i+1}}\varphi''(r)\mathrm{d}r\mathrm{d}t &\geqq \frac{1}{2}(s_{i+1}-s_i)\varphi''(s_{i+1}).\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we get $s_{i+1}-s_i \leqq s_{i+2}-s_{i+1}$ and justify (3.8). We proved contradictions for all possible cases listed in *Step 2*, and finish the proof. □

Remark that this fact is non-trivial because we can observe the infinitely many bounces of the specular cycles in a finite time interval even in some convex domains [12]. Moreover in the case of non-convex domains we need to treat carefully the trajectories that hit the inflection part (Definition 5) tangentially. The analyticity assumption is essential in the proof.

Using Lemma 2, we define and control bad phase sets where their cycles may hit inflection grazing sets γ_0^I, defined in Definition 4 or 5.

Lemma 3. *Let* $\Omega \subset \mathbb{R}^2$ *be an analytic non-convex domain in Definition* 1*. For* $T_0 > 0, \varepsilon \ll 1, N \gg 1$*, there exist finite points*

$$\left\{x_1^{nI},\ldots,x_{l_{nI}}^{nI}\right\} \subset cl(\Omega),$$

and open balls

$$B\left(x_1^{nI}, r_1^{nI}\right),\ldots,B\left(x_{l_{nI}}^{nI}, r_{l_{nI}}^{nI}\right) \subset \mathbb{R}^2,$$

as well as corresponding open sets

$$\mathcal{O}_1^{nI},\ldots,\mathcal{O}_{l_{nI}}^{nI} \subset \mathbb{V}^N,$$

with $\mathfrak{m}_2(\mathcal{O}_i^{nI}) \lesssim \varepsilon$ *for all* $i = 1,\ldots,l_{nI}$ *such that for every* $x \in cl(\Omega)$*, there exists* $i \in \{1,\ldots,l_{nI}\}$ *with* $x \in B(x_i^{nI}, r_i^{nI})$ *and, for* $v \in \mathbb{V}^N\backslash\mathcal{O}_i^{nI}$*, the following holds,*

$$[X(s;T_0,x,v), V(s;T_0,x,v)] \notin \gamma_0^I \quad \textit{for all } s \in [0,T_0]. \tag{3.12}$$

Proof. With the specular boundary condition, a particle trajectory is always reversible in time. Therefore, we track backward in time the trajectory which departs from the inflection grazing phase. Recall from Definition 5 that the inflection boundary $\partial\Omega^I$ is a set of finite points and denote $\partial\Omega^I = \{x_1^I, x_2^I, \dots, x_{M^I}^I\}$. Define

$$\bigcup_{j=1}^{M^I} \Big\{ \big(X\big(s; T_0, x_j^I, v\big), V\big(s; T_0, x_j^I, v\big)\big) \in cl(\Omega) \times \mathbb{R}^2 : s \in [0, T_0], \big(x_j^I, v\big) \in \gamma_0^I, v \in \mathbb{V}^N \Big\}.$$

Now we fix one point of the inflection boundary $x_j^I \in \partial\Omega^I$ and a velocity $v_j^I \in \mathbb{R}^2$ with $|v_j^I| = 1$ such that $(x_j^I, v_j^I) \in \gamma_0^I$. More precisely, for $x_j^I = \alpha_i(\tau) \in \partial\Omega_i^{I+}$ with some $i = 1, \dots, M$ in Definition 5, we choose $v_j^I = -\dot{\alpha}_i(\tau)$, and for $x_j^I = \alpha_i(\tau) \in \partial\Omega_i^{I-}$ we choose $v_j^I = \dot{\alpha}_i(\tau)$ so that $(x_j^I, v_j^I) \in \gamma_0^{I+}$ and backward in time trajectory is well-defined for short time $(T_0 - \varepsilon, T_0]$, $\varepsilon \ll 1$ at least.

Since $|V(s; T_0, x_j^I, v_j^I)| = |v_j^I| \leqq \frac{N}{2}$ for $v_j^I \in \mathbb{V}^N$, possible total length of the specular cycles is bounded by $\frac{NT_0}{2}$. By Lemma 2, the number of bounces cannot be infinite for finite travel length without hitting an inflection grazing phase. Moreover, if the trajectory hits an inward inflection grazing phase γ_0^{I-}, the particle cannot propagate anymore. Therefore, the number of bounces for finite travel length is always bounded. This implies

$$\mathtt{m}(x_j^I) := \inf \left\{ m \in \mathbb{N} : \sum_{i=1}^{m} \left| x^i\big(x_j^I, v_j^I\big) - x^{i+1}\big(x_j^I, v_j^I\big) \right| > \frac{NT_0}{2} \right\} < +\infty,$$

which actually depends on N for fixed Ω and $T_0 > 0$. Therefore the set (3.1) is a subset of

$$\mathcal{A} := \bigcup_{j=1}^{M^I} \bigcup_{i=0}^{\mathtt{m}(x_j^I)} \Big\{ (y, u) \in cl(\Omega) \times \mathbb{V}^N : y \in \overline{x^i\big(x_j^I, v_j^I\big) x^{i+1}\big(x_j^I, v_j^I\big)} \text{ and } \frac{u}{|u|} = \pm v^i\big(x_j^I, v_j^I\big) \Big\},$$

which is a set of all particle paths from all inflection grazing phase. Now, we define the projection of $\mathcal{A}$ on spatial a dimension

$$\mathcal{P}_x(\mathcal{A}) := \bigcup_{j=1}^{M^I} \bigcup_{i=0}^{\mathtt{m}(x_j^I)} \Big\{ y \in cl(\Omega) : y \in \overline{x^i\big(x_j^I, v_j^I\big) x^{i+1}\big(x_j^I, v_j^I\big)} \Big\}.$$

Now we construct open coverings. For $x \in cl(\Omega) \backslash \mathcal{P}_x(cl(\mathcal{A}))$, we pick $r_x > 0$ so that $B(x, r_x) \cap \mathcal{P}_x(cl(\mathcal{A})) = \emptyset$. For $x \in \mathcal{P}_x(cl(\mathcal{A}))$, we pick $r_x > 0$ to generate covering for $\mathcal{P}_x(cl(\mathcal{A}))$. By compactness, we have a finite open covering

$B(x_1^{nI}, r_1^{nI}), \ldots, B(x_{l_{nI}}^{nI}, r_{l_{nI}}^{nI})$. From the above construction, for each $1 \leqq i \leqq l_{nI}$, we have either

$$B\big(x_i^{nI}, r_i^{nI}\big) \cap \mathcal{P}_x(cl(\mathcal{A})) = \emptyset, \tag{3.13}$$

or

$$x_i^{nI} \in \mathcal{P}_x(cl(\mathcal{A})). \tag{3.14}$$

For i with the (3.13) case, we set $\mathcal{O}_i^{nI} = \emptyset$. For i with the (3.14) case, there are a finite number of straight segments (ones that may intersect each other) of $\mathcal{P}_x(\mathcal{A})$. This number of segments are bounded by $M^I \times \max_i \mathfrak{m}(x_i^I) < \infty$ for $i = 1, \ldots, M^I$. By saying that $\mathcal{O}_i^{nI}$ with i satisfies (3.14), we mean

$$\begin{aligned}\mathcal{O}_i^{nI} = \Big\{ u \in \mathbb{V}^N : \big|u/|u| \pm v^i\big(x_j^I, v_j^I\big)\big| < C_N\varepsilon,\\ \forall (i,j) \ \text{s.t} \ \overline{x^i\big(x_j^I, v_j^I\big)x^{i+1}\big(x_j^I, v_j^I\big)} \cap B\big(x_i^{nI}, r_i^{nI}\big) \neq \emptyset \Big\}.\end{aligned} \tag{3.15}$$

Obviously $\mathfrak{m}_2(\mathcal{O}_i^{nI}) \lesssim \pi \frac{N^2}{4} \frac{C_N}{2\pi} \varepsilon M^I \times \max_i \mathfrak{m}(x_i^I) \lesssim \varepsilon$ by choosing $C_N \lesssim \frac{1}{N^4}$ for sufficiently large $N \gg 1$.

Now we prove (3.12). Since the trajectory is reversible in time, $[X(s; T_0, x, v), V(s; T_0, x, v)] \notin \gamma_0^I$ if $(x, v) \notin \mathcal{A}$. By the definition of (3.15), if $x \in B(x_i^{nI}, r_i^{nI})$, $v \in \mathbb{V}^N \backslash \mathcal{O}_i^{nI}$, and $s \in [0, T_0]$, then $(x, v) \notin \mathcal{A}$. This finishes the proof. □

The following lemma comes from Lemmas 1 and 3:

Lemma 4. *Consider Ω as defined in Definition 1. For $\varepsilon \ll 1$, $N \gg 1$, and $T_0 > 0$, there exist finite points*

$$\Big\{x_1^{IB}, \ldots, x_{l_{IB}}^{IB}\Big\} \subset cl(\Omega),$$

and open balls

$$B\big(x_1^{IB}, r_1^{IB}\big), \ldots, B\big(x_{l_{IB}}^{IB}, r_{l_{IB}}^{IB}\big) \subset \mathbb{R}^2,$$

as well as corresponding open sets

$$\mathcal{O}_1^{IB}, \ldots, \mathcal{O}_{l_{IB}}^{IB} \subset \mathbb{V}^N,$$

with $\mathfrak{m}_2(\mathcal{O}_i^{IB}) < C\varepsilon$ (for uniform constant $C > 0$) for all $i = 1, \ldots, l_{IB}$ such that for every $x \in cl(\Omega)$, there exists $i \in \{1, \ldots, l_{IB}\}$ with $x \in B(x_i^{IB}, r_i^{IB})$ and, for $v \in \mathbb{V}^N \backslash \mathcal{O}_i^{IB}$,

$$|v \cdot \mathbf{n}(x)| > \frac{\varepsilon}{N^4},$$

for all $x \in \partial\Omega \cap B(x_i^{IB}, r_i^{IB})$ and

$$\big(X(t_k; T_0, x, v), V(t_k; T_0, x, v)\big) \notin \gamma_0^I \quad \textit{for all } t_k \in [0, T_0].$$

Using the above lemma, we define *the infinite-bounces set* $\mathfrak{IB}$ as

$$\mathfrak{IB} := \Big\{(x,v) \in cl(\Omega) \times \mathbb{V}^N : v \in \mathcal{O}_i^{IB} \text{ for some } i \in \{1,2,\dots,l_{IB}\} \\ \text{satisfying } x \in B\left(x_i^{IB}, r_i^{IB}\right)\Big\}. \tag{3.16}$$

The most important property of the infinite-bounces set (3.16) is that the bouncing number of the specular backward trajectories on $\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}$ is uniformly bounded.

Definition 6. When $L > 0$, $x \in cl(\Omega) \subset \mathbb{R}^2$, and nonzero $v \in \mathbb{R}^2$ are given, we consider a set

$$\left\{k \in \mathbb{N} : \sum_{j=1}^{k} \left|x^{j-1}(x,v) - x^j(x,v)\right| > L\right\} \subset \mathbb{N}.$$

If this set is not empty, then we define $\mathfrak{N}(x,v,L) \in \mathbb{N}$ as follows,

$$\mathfrak{N}(x,v,L) := \inf\left\{k \in \mathbb{N} : \sum_{j=1}^{k} \left|x^{j-1}(x,v) - x^j(x,v)\right| > L\right\}.$$

Otherwise, if the set is empty, it means the backward trajectory is trapped in γ_0^{I-}, so we define

$$\mathfrak{N}(x,v,L) := \inf\left\{i \in \mathbb{N} : (x^i(x,v), v^{i-1}(x,v)) \in \gamma_0^{I-}\right\}.$$

From Lemma 4, we have $\mathfrak{N}(x,v,\frac{NT_0}{2}) < \infty$ for $(x,v) \in \{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}$. To improve this finite result into a uniform bound, we use compactness and continuity arguments.

Lemma 5. *Let* $(x,v) \in cl(\Omega) \times \mathbb{V}^N$. *Then* $(x^k(x,v), v^k(x,v))$ *is a locally continuous function of* (x,v) *if*

$$(x^i(x,v), v^i(x,v)) \notin \gamma_0, \quad \forall i \in \{1,2,\dots,k\},$$

that is, for any $\varepsilon > 0$, *there exist* $\delta_{x,v,\varepsilon} > 0$ *such that if* $|(x,v)-(y,u)| < \delta_{x,v,\varepsilon}$, *then*

$$\left|(x^i(x,v), v^i(x,v)) - (x^i(y,u), v^i(y,u))\right| < \varepsilon, \quad \forall i \in \{1,2,\dots,k\}.$$

Moreover $(x^i(y,u), v^i(y,u)) \notin \gamma_0$ *for* $i \in \{1,\dots,k\}$.

Proof. First we claim continuity of $(x^1(x,v), v^1(x,v))$. Using trajectory notation and the lower bound of speed in $\mathbb{V}^N$, we know that

$$x^1(x,v) = X(t_{\mathbf{b}}(t,x,v); t,x,v), \quad t_{\mathbf{b}} \leqq CN$$

for uniform C which depend on the size of Ω. Let us assume that $|(x,v)-(y,u)| \leqq \delta$. Then

$$|x^1(x,v) - x^1(y,u)| \leqq |x^1(x,v) - x^1(x,u)| + |x^1(x,u) - x^1(y,u)|. \quad (3.17)$$

Let $x^1(x,v) = \alpha_j(\tau^*)$. Since $(x^1(x,v), v) \notin \gamma_0$, $\left|\frac{v}{|v|} \cdot \dot{\alpha}(\tau^*)\right| < 1$. Then we can choose sufficiently small $r_{x,v} \ll 1$ such that $\partial\Omega \cap B(x^1(x,v), r_{x,v})$ is simply connected and intersects with line $\{x + sv : s \in \mathbb{R}\}$ in only one point non-tangentially, because $\dot{\alpha}|_{x^1(x,v)}$ is not parallel to v. Since $x + sv$ is continuous on v, $x + su$ must intersect to $\partial\Omega \cap B(x^1(x,v), r_{x,v})$ at some $\alpha_j(\tau) \in \partial\Omega \cap B(x^1(x,v), r)$ whenever $|u - v| \ll \delta_{v,\varepsilon}$. This shows $|x^1(x,v) - x^1(x,u)| < O(\delta_{v,\varepsilon})$, and

$$\begin{aligned}\left|\frac{u}{|u|} \cdot \dot{\alpha}(\tau) - \frac{v}{|v|} \cdot \dot{\alpha}(\tau^*)\right| &\leqq \left|\frac{u}{|u|} \cdot \dot{\alpha}(\tau) - \frac{u}{|u|} \cdot \dot{\alpha}(\tau^*)\right| \\ &\quad + \left|\frac{u}{|u|} \cdot \dot{\alpha}(\tau^*) - \frac{v}{|v|} \cdot \dot{\alpha}(\tau^*)\right| \\ &\leqq C|\tau - \tau^*| + N\delta \\ &\leqq C_N\delta \leqq \frac{1}{2}\left(1 - \left|\frac{v}{|v|} \cdot \dot{\alpha}(\tau^*)\right|\right) \end{aligned} \quad (3.18)$$

for sufficiently small $\delta \ll 1$. This implies $\left|\frac{u}{|u|} \cdot \dot{\alpha}(\tau)\right| < 1$, that is, $(x^1(x,u), u) \notin \gamma_0$.

Now, there exists small $r_{x,u} \ll 1$ such that $\partial\Omega \cap B(x^1(x,u), r_{x,u})$ is simply connected and intersects with line $\{x+su : s \in \mathbb{R}\}$ in only one point non-tangentially by (3.18). Thus there exists $\delta_{x,u,\varepsilon} \ll 1$ such that line $y+su$ hits $\partial\Omega \cap B(x^1(x,u), r_{x,u})$ if $|x - y| < \delta_{x,u,\varepsilon}$. It is obvious that $|x^1(x,u) - x^1(y,u)| < r_{x,u}$. So far we have shown the continuity of $x^1(x,\cdot)$ and $x^1(\cdot,u)$, so the continuity of x^1 follows from (3.17).

We also note that $v^1(x,v)$ is continuous by the continuity of $x^1(x,v)$ and $\mathbf{n}(x,v)$, and the specular reflection BC. For the case of $i = 2, \ldots, k$ are easily gained by a chain rule, applying the above argument several times. □

Lemma 6. *Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 1. Then*

$$\mathfrak{N}^*_{\varepsilon,N,T_0} := \sup_{(x,v)\in\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}} \mathfrak{N}(x,v,NT_0) \leqq C_{\varepsilon,NT_0},$$

where $\mathfrak{N}(x,v,NT_0)$ is defined in Definition 6 and ε-dependence comes from $\{\mathcal{O}_i^{IB}\}_{i=0}^{l_{IB}}$, which was defined in Lemma 4.

Proof. From Lemmas 2 and 4, the trajectory does not belong to inflection grazing set during time $[0, T_0]$. $\mathfrak{N}(x,v,\cdot)$ is a nondecreasing function for fixed $(x,v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$ and we can assume $|v| = 1$, because NT_0 has a fixed maximal travel length during time interval $[0, T_0]$ with $v \in \mathbb{V}^N$.

Step 1. We study cases depending on concave grazing.

(*Case 1*) If $\mathbf{n}(x^i(x,v)) \cdot v^i(x,v) \neq 0$ for $i = 1, \ldots, \mathfrak{N}(x,v,NT_0)$, trajectory $(X(s;T_0,x,v), V(s;T_0,x,v))$ is continuous in (x,v) by Lemma 5. Therefore, we can choose $\delta_{x,v,\varepsilon,NT_0} \ll 1$ such that if $|(x,v)-(y,u)| < \delta_{x,v,\varepsilon,NT_0}$, then $|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,u)| < O(\delta_{x,v,\varepsilon,NT_0})$, where $O(\delta_{x,v,\varepsilon,NT_0}) \to 0$ as $\delta_{x,v,\varepsilon,NT_0} \to 0$. Therefore,

$$\mathfrak{N}(y,u,NT_0) \leqq 1 + \mathfrak{N}(x,v,NT_0),$$

for $|(x,v)-(y,u)| < \delta_{x,v,\varepsilon,NT_0} \ll 1$. Moreover, we have

$$|x^i(x,v) - x^i(y,u)| < O(\delta_{x,v,\varepsilon,NT_0})$$

for $i = 1, \ldots, \mathfrak{N}(x,v,NT_0)$.

(*Case 2*) Assume that $(x^i(x,v), v^i(x,v))$ belongs to grazing set γ_0 for some $i \in \{1, \ldots, \mathfrak{N}(x,v,NT_0)\}$. In particular, $(x^i(x,v), v^i(x,v)) \in \gamma_0^C$, because γ_0^I is not gained from $\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}$ as proved in Lemma 4, and γ_0^V is the stopping point for both forward/backward in time. Let us assume that $i \in \{0, \ldots, \mathfrak{N}(x,v,NT_0)\}$ is the smallest bouncing index satisfying $(x^i(x,v), v^i(x,v)) \in \gamma_0^C$. Even though there are consecutive convex grazings, it must stop at some $(x^k(x,v), v^k(x,v)$, because Ω is analytic and bounded domain, that is, there exist $i, k \in \mathbb{N}$ such that

$$\begin{cases} (x^j(x,v), v^j(x,v)) \notin \gamma_0^C, & \forall j < i, \\ (x^j(x,v), v^j(x,v)) \in \gamma_0^C, & i \leqq \forall j \leqq k-1, \\ (x^i(x,v), v^i(x,v)) \notin \gamma_0^C, & j = k. \end{cases} \tag{3.19}$$

When $j < i$, the bouncing number can be counted similarly to *Step 1*:

$$\mathfrak{N}(y,u,N(T_0 - t_{i-1}(x,v))) \leqq 1 + \mathfrak{N}(x,v,N(T_0 - t_{i-1}(x,v))),$$

for $|(x,v)-(y,u)| < \delta_{x,v,\varepsilon,NT_0}$ for some $\delta_{x,v,\varepsilon,NT_0} \ll 1$. Now we consider consecutive multiple grazing.

When $i \leqq j \leqq k-1$ (consecutive convex grazing), we split things into two cases: Case 2-1 and Case 2-2.

(*Case 2-1*) We assume $\mathbf{n}(x^i(x,v)) = \mathbf{n}(x^{i+1}(x,v)) = \cdots = \mathbf{n}(x^{k-1}(x,v))$. When $|(x,v)-(y,u)| < \delta_{x,v,\varepsilon,NT_0} \ll 1$, we have

$$\left|(x^{i-1}(x,v), v^{i-1}(x,v)) - (x^{i-1}(y,u), v^{i-1}(y,u))\right| < O(\delta_{x,v,\varepsilon,NT_0}) \ll 1,$$

from Lemma 5. When trajectory $\big(X(s;y,u,T_0), V(s;y,u,T_0)\big)$ passes near $x^i(x,v)$, we split things into several cases (Fig. 3).

We claim that

$$\mathfrak{N}(y,u,N(T_0 - t_k(x,v))) \leqq 1 + \mathfrak{N}(x,v,N(T_0 - t_k(x,v))) \tag{3.20}$$

holds for all of the following cases.

(i) If $\overline{x^{i-1}(y,u)x^k(y,u)}$ does not bounce near $x^j(x,v)$ for all $j \in \{i, \ldots, k-1\}$, then obviously we get (3.20).

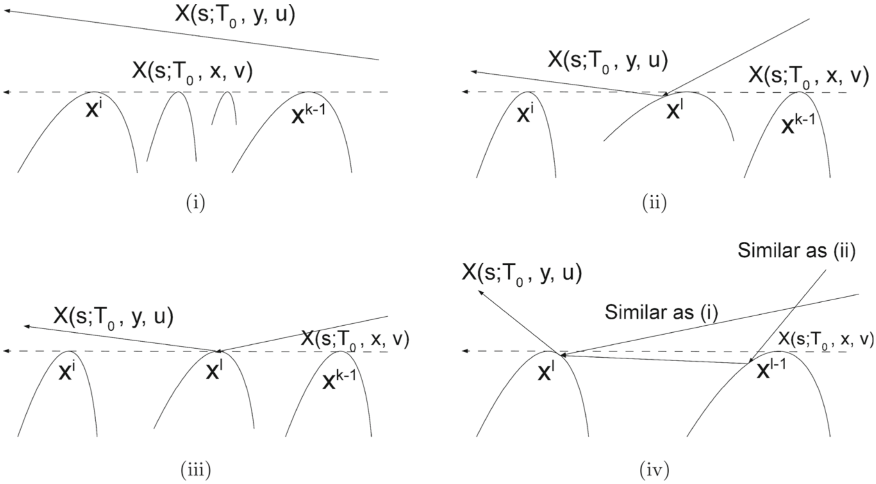


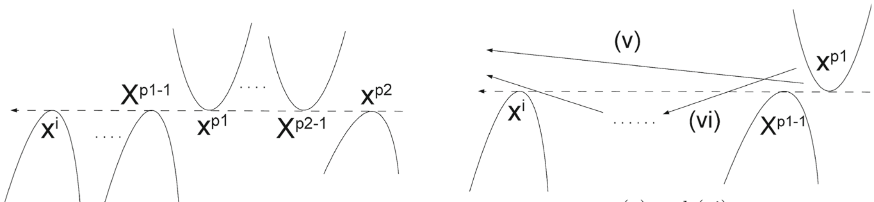
Fig. 3. Case 2-1

If case (i) does not hold, we can assume that the backward trajectory $\big(X(s; y, u, T_0), V(s; y, u, T_0)\big)$ hits near $x^\ell(x, v)$ without hitting near $x^j(x, v)$ for $i \leqq j \leqq \ell-1$. Without loss of generality, we parametrize $B(x^\ell(x, v), \varepsilon) \cap \partial\Omega$, $\varepsilon \ll 1$ by a regularized curve $\big\{\beta^\ell(\tau) : \tau^\ell - \delta_1 < \tau < \tau^\ell + \delta_2,\ \beta^\ell(\tau^\ell) = x^\ell(x, v)\big\}$, $0 \leqq \delta_1, \delta_2 \ll 1$.

(ii) Let $x^i(y, u) = \beta^\ell(\tau)$ with $\tau^\ell - \delta_1 < \tau < \tau^\ell$. Without loss of generality, we assume a multigrazing dashed line as x-axis. By the specular BC, the trajectory $\big(X(s; y, u, T_0), V(s; y, u, T_0)\big)$ must be above the tangential line $\{x^i(y, u) + s\dot{\beta}^\ell(\tau) : s \in \mathbb{R}\}$ near $x^i(y, u)$. Moreover, from the specular BC,

$$
\begin{aligned}
\left| \frac{v^i(y, u)}{|v^i(y, u)|} \cdot \dot{\beta}^\ell(\tau) \right| &= \left| \frac{v^{i-1}(y, u)}{|v^{i-1}(y, u)|} \cdot \dot{\beta}^\ell(\tau) \right| \\
&\geqq \left| \frac{v^\ell(x, v)}{|v^\ell(x, v)|} \cdot \dot{\beta}^\ell(\tau^\ell) \right| - \left| \left(\frac{v^{i-1}(y, u)}{|v^{i-1}(y, u)|} - \frac{v^\ell(x, v)}{|v^\ell(x, v)|} \right) \cdot \dot{\beta}^\ell(\tau) \right| \\
&\quad - \left| \frac{v^\ell(x, v)}{|v^\ell(x, v)|} \cdot \left(\dot{\beta}^\ell(\tau) - \dot{\beta}^\ell(\tau^\ell) \right) \right| \\
&\geqq 1 - O(\delta_{x,v,\varepsilon,NT_0}).
\end{aligned} \tag{3.21}
$$

This implies that the angle between $v^{i-1}(y, u)$ and tangential line $\{x^i(y, u) + s\dot{\beta}^\ell(\tau) \mid s \in \mathbb{R}\}$ is very small, so we can apply the argument of (i) again and we obtain (3.20).



(v) and (vi)

Fig. 4. Case 2-2

(iii) When $x^i(y,u) = \beta^\ell(\tau^\ell)$, we must have

$$\left|\frac{v^i(y,u)}{|v^i(y,u)|}\cdot\dot\beta^\ell(\tau^\ell)\right| = \left|\frac{v^{i-1}(y,u)}{|v^{i-1}(y,u)|}\cdot\dot\beta^\ell(\tau^\ell)\right| = \left|\frac{v^{i-1}(y,u)}{|v^{i-1}(y,u)|}\cdot\frac{v^\ell(x,v)}{|v^\ell(x,v)|}\right| \\ = 1 + O(\delta_{x,v,\varepsilon,NT_0}), \tag{3.22}$$

so the angle between $v^{i-1}(y,u)$ and $v^\ell(x,v)$ is very small. Moreover, trajectory $\big(X(s;y,u,T_0), V(s;y,u,T_0)\big)$ must be above the dash tagential line, So we can apply (i) to derive (3.20).

(iv) When $x^i(y,u) = \beta^\ell(\tau)$ with $\tau^\ell < \tau < \tau^\ell + \delta_2$, the angle between $\dot\beta^\ell(\tau)$ and $\dot\beta^\ell(\tau^\ell)$ is very small, since $\delta_2 \ll 1$. Moreover, the angle between $v^{i-1}(y,u)$ and $\dot\beta^\ell(\tau)$ is also small from (3.21). Therefore the angle between $v^{i-1}(y,u)$ and $\dot\beta^\ell(\tau^\ell)$ is also small, that is, $v^{i-1}(y,u)$ is nearly parallel with the dashed line in Fig. 1. Therefore only cases (i) and (ii) are possible for $x^{i+1}(y,u)$. For both cases, we gain (3.20).

(*Case 2-2*) Assume that there exist $\{p_1, p_2, \dots, p_q\} \in \{i+1, \dots, k-1\}$ with $p_1 < p_2 < \cdots < p_q$ such that

$$\begin{cases} \mathbf{N} := \mathbf{n}(x^i(x,v)) = \mathbf{n}(x^{i+1}(x,v)) = \cdots = \mathbf{n}(x^{p_1-1}(x,v)) \\ -\mathbf{N} = \mathbf{n}(x^{p_1}(x,v)) = \mathbf{n}(x^{p_1+1}(x,v)) = \cdots = \mathbf{n}(x^{p_2-1}(x,v)) \\ \mathbf{N} = \mathbf{n}(x^{p_2}(x,v)) = \mathbf{n}(x^{p_2+1}(x,v)) = \cdots = \mathbf{n}(x^{p_3-1}(x,v)) \\ -\mathbf{N} = \mathbf{n}(x^{p_3}(x,v)) = \mathbf{n}(x^{p_3+1}(x,v)) = \cdots = \mathbf{n}(x^{p_4-1}(x,v)) \\ \qquad\qquad\cdots \end{cases}$$

We split into cases and claim that

$$\mathfrak{N}(y,u,N(T_0 - t_k(x,v))) \leqq 1 + \mathfrak{N}(x,v,N(T_0 - t_k(x,v)))$$

holds for all cases (Fig. 4).

First we define $T_{p_\ell} := (t^{p_\ell - 1}(x,v) - t^{p_\ell}(x,v))/2,\ 1 \leqq \ell \leqq q$, and choose $\delta_{x,v,\varepsilon,NT_0}$ so that

$$\delta_{x,v,\varepsilon,NT_0} \ll \frac{T_{p_\ell}}{N}, \quad \text{for all } \ell \in \{1,\dots,q\}, \tag{3.23}$$

which implies that the traveling time (or distance) between $x^{p_1}(x,v)$ and $x^{p_1-1}(x,v)$ is sufficiently larger than the size of $\delta_{x,v,\varepsilon,NT_0}$. We split into two cases (v) and (vi) as follows:

(v) If $x^i(y,u)$ does not hit near any of $x^i(x,v),\dots,x^{p_1-1}(x,v)$, we have

$$\Big|\big(X(T_{p_1};y,u,T_0),V(T_{p_1};y,u,T_0)\big)-\big(X(T_{p_1};x,v,T_0),V(T_{p_1};x,v,T_0)\big)\Big| \\ \leqq O(\delta_{x,v,\varepsilon,NT_0}), \tag{3.24}$$

by Lemma 5.

(vi) If $x^i(y,u)$ hits near one of $x^i(x,v),\dots,x^{p_1-1}(x,v)$, then we can apply (ii), (iii), or (iv) of Case 2-1 to claim that there are *at most* 2 bouncings before trajectory $\big(X(s;y,u,T_0),V(s;y,u,T_0)\big)$ approaches $x^{p_1}(x,v)$. Moreover, in any case of (ii), (iii), and (iv), (assuming 2 bouncings WLOG),

$$|v^{i+2}(y,u)-v^i(x,v)|=|v^{i+2}(y,u)-v^{p_1}(x,v)|=O(\delta_{x,v,\varepsilon,NT_0}).$$

Since, trajectory $X(s;y,u,T_0)$ is very close to $X(s;x,v,T_0)$,

$$\Big|X(s;y,u,T_0)-X(s;x,v,T_0)\Big|\leqq O(\delta_{x,v,\varepsilon,NT_0}),\quad t^{i-1}(x,v)\leqq s\leqq T_{p_1}.$$

Using the above two estimates for both velocity and position, (3.24) also holds for case (vi).

Now let us derive a uniform number of bounces of the second case in (3.19). For (*Case 2-1*), we proved that (3.20) holds. For (*Case 2-2*) we change index $p_1-1\leftrightarrow k-1$, and then apply the same argument of (*Case 2-1*) to derive

$$\mathfrak{N}(y,u,N(T_0-T_{p_1}(x,v)))\leqq\mathfrak{N}(x,v,N(T_0-T_{p_1}(x,v))).$$

During $\big(t^{p_2}(x,v),t^{p_1}(x,v)\big)$, we can also apply the same argument as that of (*Case 2-1*) with the help of (3.23) and (3.24) to obtain

$$\mathfrak{N}(y,u,N(T_0-T_{p_1}(x,v)))\leqq\mathfrak{N}(x,v,N(T_0-T_{p_1}(x,v))).$$

We iterate this process until T_{p_q} to obtain

$$\mathfrak{N}(y,u,N(T_0-T_{p_q}(x,v)))\leqq\mathfrak{N}(x,v,N(T_0-T_{p_q}(x,v))).$$

Since $(x^k(x,v),v^k(x,v))$ is non-grazing, we have

$$\mathfrak{N}(y,u,N(T_0-t^k(x,v)))\leqq 1+\mathfrak{N}(x,v,N(T_0-t^k(x,v))) \tag{3.25}$$

by applying (*Case 2-1*) for traveling from near $x^{p_q}(x,v)$ to $x^k(x,v)$.

Step 2. When we encounter the second consecutive convex grazings after $t^k(x, v)$, we can follow *Step 1* to derive a similar estimate as to (3.25). Finally there exist $\delta_{x,v,\varepsilon,NT_0} \ll 1$ such that

$$\mathfrak{N}(y, u, NT_0) \leqq 1 + \mathfrak{N}(x, v, NT_0), \tag{3.26}$$

where $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$. Since $\mathfrak{IB}$ is an open set from (3.16), $\{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$ is a closed set. Then we use a compactness argument to derive uniform boundness from (3.26). For each $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$, we construct small balls $B((x, v), \delta_{x,v,\varepsilon,NT_0})$ near each point. For each $(y, u) \in B((x, v), \delta_{x,v,\varepsilon,NT_0})$, (3.26) holds. By compactness, there exists a finite covering $\bigcup_{i=1}^{\ell} B((x_i, v_i), \delta_{x_i,v_i,\varepsilon,NT_0})$ for some finite $\ell < \infty$. Therefore, for any $(y, u) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$,

$$\mathfrak{N}(y, u, NT_0) \leqq 1 + \max_{1\leqq i\leqq \ell} \mathfrak{N}(x_i, v_i, NT_0) \leqq C_{\varepsilon,NT_0}.$$

□

Lemma 7. *Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 1. For any $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$, trajectory $(X(s; T_0, x, v), V(s; T_0, x, v))$ for $s \in [0, T_0]$ is uniformly away from the inflection grazing set γ_0^I, that is, there exists $\rho_{\varepsilon,NT_0} > 0$ such that*

$$\begin{aligned} D_{\mathcal{I}}(s, x, v) &:= dist(\partial\Omega^{\mathcal{I}}, X(s; T_0, x, v)) + |\mathbf{n}(X(s; T_0, x, v)) \cdot V(s; T_0, x, v)| \\ &\geqq \rho_{\varepsilon,NT_0} > 0 \end{aligned} \tag{3.27}$$

for all $s \in [0, T_0]$ such that $X(s; T_0, x, v) \in \partial\Omega$.

Proof. By definition of $\mathfrak{IB}$ and Lemma 4,

$$(X(s; T_0, x, v), V(s; T_0, x, v)) \notin \gamma_0^I.$$

Therefore,

$$\min_{t_j\in[0,T_0]} D_{\mathcal{I}}(t^j(x, v), x, v) > 0,$$

where $D_{\mathcal{I}}(t^j(x, v), x, v)$ is defined in (3.27). To derive uniform positivity, we use a compactness argument again. From Lemma 6, for $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$, we know that

$$\mathfrak{N}(x, v, NT_0) \leqq C_{\varepsilon,NT_0}.$$

Therefore,

$$\min_{t_j\in[0,T_0]} D_{\mathcal{I}}(t^j(x, v), x, v) = \min_{1\leqq j\leqq C_{x,v,\varepsilon,NT_0}} D_{\mathcal{I}}(t^j(x, v), x, v) := \rho_{x,v,\varepsilon,NT_0} > 0 \tag{3.28}$$

for some uniform positive constant $\rho_{x,v,\varepsilon,NT_0} > 0$. Now we split things into two cases.

Case 1. If $(X(s;T_0,x,v),V(s;T_0,x,v)) \notin \gamma_0$, we have local continuity from Lemma 5, so there exists $r_{x,v,\varepsilon,NT_0} \ll 1$ such that if $|(x,v)-(y,u)| < r_{x,v,\varepsilon,NT_0}$,

$$\min_{1\leqq j\leqq C_{x,v,\varepsilon,NT_0}} \left|D_{\mathcal{I}}(t^j(x,v),x,v) - D_{\mathcal{I}}(t^j(y,u),y,u)\right| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2}. \tag{3.29}$$

From (3.28) and (3.29),

$$\min_{1\leqq j\leqq C_{x,v,\varepsilon,NT_0}} D_{\mathcal{I}}(t^j(y,u),y,u) > \frac{\rho_{x,v,\varepsilon,NT_0}}{2},$$

which implies uniform nonzero on a ball $cl\big(B((x,v),r_{x,v,\varepsilon,NT_0})\big)$. By compactness, we have a finite open cover for $\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}$, which is written by $\bigcup_{i=1}^{\ell} B((x_i,v_i),r_{x_i,v_i,\varepsilon,NT_0})$ for some finite q. Finally, we pick a uniform positive number

$$\rho_{\varepsilon,NT_0} := \min_{1\leqq i\leqq \ell} \frac{\rho_{x_i,v_i,\varepsilon,NT_0}}{2} > 0$$

to finish the proof.

Case 2. If $(X(s;T_0,x,v),V(s;T_0,x,v)) \in \gamma_0$ for some $s\in[0,T_0]$, it must be concave grazing by definition of $\mathfrak{IB}$, and we consider consecutive concave grazing cases of (*Case 2-1*) in the proof of Lemma 6 again with Fig. 1. Let us assume (3.19).

When $j<i$, using Lemma 5, we have $r_{x,v,\varepsilon,NT_0} \ll 1$ such that if $|(x,v)-(y,u)| < r_{x,v,\varepsilon,NT_0}$,

$$\min_{1\leqq j\leqq i-1} \left|D_{\mathcal{I}}(t^j(x,v),x,v) - D_{\mathcal{I}}(t^j(y,u),y,u)\right| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2}.$$

When $i\leqq j\leqq k-1$, it is not reasonable to compare with the same bouncing index, because we have discontinuity by convex grazing. However, since $D_{\mathcal{I}}$ is uniformly bounded from below by (3.28), it suffices to compare $D_{\mathcal{I}}(t^j(y,u),y,u)$ with the nearest $D_{\mathcal{I}}(t^\ell(x,v),x,v)$ for some $j\leqq\ell$.

(i) If $\overline{x^{i-1}(x,v)x^k(x,v)}$ does not bounce near $x^j(x,v)$ for all $j\in\{i,\dots,k-1\}$, then from Lemma 5 again, we can redefine $r_{x,v,\varepsilon,NT_0}\ll 1$ so that if $|(x,v)-(y,u)|<r_{x,v,\varepsilon,N}$,

$$\left|D_{\mathcal{I}}(t^k(x,v),x,v) - D_{\mathcal{I}}(t^i(y,u),y,u)\right| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2}$$

holds. This implies

$$D_{\mathcal{I}}(t^i(y,u),y,u) \geqq D_{\mathcal{I}}(t^k(x,v),x,v) - \frac{\rho_{x,v,\varepsilon,NT_0}}{2} > \frac{\rho_{x,v,\varepsilon,NT_0}}{2}, \tag{3.30}$$

from (3.28).

(ii) From Lemma 5, there exist $r_{x,v,\varepsilon,NT_0} \ll 1$ so that if $|(x,v)-(y,u)| < r_{x,v,\varepsilon,N}$, $|x^i(y,u)-x^\ell(x,v)| = O(r_{x,v,\varepsilon,N})$. Moreover, from (3.21), $|v^i(y,u)-v^\ell(x,v)| = O(r_{x,v,\varepsilon,N})$ also holds, so

$$\left|D_{\mathcal{I}}(t^\ell(x,v),x,v) - D_{\mathcal{I}}(t^i(y,u),y,u)\right| < \frac{\rho_{x,v,\varepsilon,NT_0}}{2}$$

holds and therefore, (3.30) also holds by (3.28).

(iii) Obviously, $|x^i(y,u) - x^\ell(x,v)| = 0$ and $|v^i(y,u) - v^\ell(x,v)| = O(r_{x,v,\varepsilon,N})$ also holds by (3.22), and so yields (3.30), similarly.

(iv) Near $x^i(y,u)$ (near $x^\ell(x,v)$) and $x^{i+1}(y,u)$ (near $x^{\ell+1}(x,v)$), we use the argument of (ii) for both bouncings to claim that

$$D_{\mathcal{I}}(t^i(y,u),y,u),\ \ D_{\mathcal{I}}(t^{i+1}(y,u),y,u) \geqq \frac{\rho_{x,v,\varepsilon,NT_0}}{2},$$

if $|(x,v)-(y,u)| < r_{x,v,\varepsilon,NT_0}$ for some small $r_{x,v,\varepsilon,NT_0} \ll 1$.

From *Step 2* in proof of Lemma 6, the number of intervals of consecutive grazing is uniformly bounded because we assume Definition 1. Whenever we encounter consecutive grazing, we can split into cases (i)–(iv) to gain uniform positivity of $D_{\mathcal{I}}(t^j(y,u),y,u)$ for $0 \leqq t^j(y,u) \leqq T_0$. Then we apply the compactness argument of *Case 1* in the proof of this Lemma to finish the proof. □

3.2. Dichotomy of Sticky Grazing

Lemma 8. *Assume $\Omega \subset \mathbb{R}^2$ as defined in Definition 1. Assume that $(\alpha_j(\tau), \alpha_j'(\tau)) \in \gamma_0^C$ for some $j \in \{1,\dots,M\}$ and $\tau \in (\tau^*-\delta, \tau^*+\delta) \subset [a_j,b_j]$. Also we assume that*

$$(X(s;T_0,\alpha_j(\tau),\alpha_j'(\tau)),V(s;T_0,\alpha_j(\tau),\alpha_j'(\tau))) \notin \gamma_0$$

for $s \in [0,T_0]$. Let us simplify notation as follows

$$x^i(\tau) := x^i(\alpha_j(\tau),\alpha_j'(\tau)),\ \ v^i(\tau) := v^i(\alpha_j(\tau),\alpha_j'(\tau)),\ \ \ t^i(\tau) := t^i(\alpha_j(\tau),\alpha_j'(\tau)),$$

for $\tau \in (\tau^*-\delta,\tau^*+\delta) \subset [a_j,b_j]$. Then we have the following dichotomy: for each k, there exist unique $x^* \in cl(\Omega)$ such that $x^* \in \overline{x^k(\tau)x^{k+1}(\tau)}$ for all $\tau \in (\tau^*-\delta,\tau^*+\delta) \subset [a_j,b_j]$, and for each $x \in cl(\Omega)$, the following set is finite:

$$\left\{\frac{v^k(\tau)}{|v^k(\tau)|} \in \mathbb{S}^1 : x \in \overline{x^k(\tau)x^{k+1}(\tau)},\ \ \tau \in (\tau^*-\delta,\tau^*+\delta)\right\}.$$

Proof. Assume that we have some x^* satisfying (a). If there exists another $y^* \neq x^*$,

$$\begin{aligned} x^k(\tau) - x^* = |x^k(\tau)-x^*|\frac{v^k(\tau)}{|v^k(\tau)|},\quad x^k(\tau) - y^* \\ = |x^k(\tau)-y^*|\frac{v^k(\tau)}{|v^k(\tau)|},\quad \tau \in (\tau^*-\delta,\tau^*-\delta). \end{aligned}$$

This gives

$$x^* - y^* = \Big(|x^k(\tau) - y^*| - |x^k(\tau) - x^*|\Big)\frac{v^k(\tau)}{|v^k(\tau)|}.$$

Therefore, $\frac{v^k(\tau)}{|v^k(\tau)|}$ is a constant unit vector for $\tau \in (\tau^* - \delta, \tau^* - \delta)$. Since $(x^k(\tau), v^k(\tau))$ is not grazing, $x^k(\tau)$ is also constant for all $\tau \in (\tau^* - \delta, \tau^* - \delta)$. Since the trajectory is deterministic forward/backward in time, $\dot{\alpha}_j(\tau)$ should be constant for $\tau \in (\tau^* - \delta, \tau^* - \delta)$, which implies $\alpha_j(\tau)$ is part of a straight line locally. This is a contradiction, because Ω is an analytic bounded domain.

If there does not exist x^* which satisfies (a) for $\tau \in (\tau^* - \delta, \tau^* - \delta)$,

$$\Big\{\tau \in (\tau^* - \delta, \tau^* - \delta) : x^k(\tau) - x^* = |x^k(\tau) - x^*|\frac{v^k(\tau)}{|v^k(\tau)|}\Big\}$$

is a finite set for any $x^* \in cl(\Omega)$ by rigidity of the analytic function. This yields (b). □

3.3. Grazing Set

In this section, we characterize the points of $\{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$ whose specular backward cycle grazes the boundary (hits the boundary tangentially) at some moment. By definition of $\mathfrak{IB}$, this grazing cannot be an inflection grazing γ_0^I. Moreover, Lemma 4 guarantees that convex grazing does not happen either. Therefore, the only possible grazing is the concave grazing γ_0^C. We will classify these concave grazing sets depending on the first (backward in time) concave grazing time.

Definition 7. For $T_0 > 0$ and $(x, v) \in cl(\Omega) \times \mathbb{R}^2$, we define the grazing set

$$\begin{aligned}\mathfrak{G} := \Big\{&(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB} : \exists\, s \in [0, T_0)\\ &s.t \ \big(X(s; T_0, x, v), V(s; T_0, x, v)\big) \in \gamma_0\Big\},\end{aligned}$$

which is a set of phase (x, v) whose trajectory grazes at least once for time interval $[0, T_0]$. We also define $\mathfrak{G}^C$, $\mathfrak{G}^V$, and $\mathfrak{G}^I$ by their grazing type, that is,

$$\begin{aligned}\mathfrak{G}^C &:= \Big\{(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB} : \exists\, s \in [0, T_0)\\ &\qquad s.t \ \big(X(s; T_0, x, v), V(s; T_0, x, v)\big) \in \gamma_0^C\Big\},\\ \mathfrak{G}^V &:= \Big\{(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB} : \exists\, s \in [0, T_0)\\ &\qquad s.t \ \big(X(s; T_0, x, v), V(s; T_0, x, v)\big) \in \gamma_0^V\Big\},\\ \mathfrak{G}^I &:= \Big\{(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB} : \exists\, s \in [0, T_0)\\ &\qquad s.t \ \big(X(s; T_0, x, v), V(s; T_0, x, v)\big) \in \gamma_0^I\Big\}.\end{aligned}$$

By definition of $\mathfrak{IB}$, we know that $\mathfrak{G}^V = \mathfrak{G}^I = \emptyset$. Therefore, we rewrite and decompose $\mathfrak{G}$ as

$$\mathfrak{G} = \mathfrak{G}^C := \bigcup_j \mathfrak{G}^{C,j} := \bigcup_{l=1}^{M^C} \mathfrak{G}_l^C = \bigcup_j \bigcup_{l=1}^{M^C} \mathfrak{G}_l^{C,j},$$

where

$$\begin{aligned} \mathfrak{G}^{C,j} &:= \left\{ (x,v) \in \mathfrak{G}^C : (x^j(x,v), v^j(x,v)) \in \gamma_0^C \right\}, \\ \mathfrak{G}_l^C &:= \left\{ (x,v) \in \mathfrak{G}^C : \exists k \ s.t. \ (x^k(x,v), v^k(x,v)) \in \gamma_0^C \text{ and } x^k(x,v) \in \partial\Omega_l^C \right\}, \\ \mathfrak{G}_l^{C,j} &:= \left\{ (x,v) \in \mathfrak{G}^{C,j} : x^j(x,v) \ \in \ \partial\Omega_l^C \right\}, \end{aligned}$$

where $l \in \{1, \ldots, M^C\}$, which is defined in (2.6).

Remark 1. Let us use renumbered notation (2.6) and the sets defined in Definition 7. If $(x,v) \in \mathfrak{G}_l^C$ then there exists $\tau \in (\bar{a}_l, \bar{b}_l)$ and k such that $(x^k(x,v), v^k(x,v)) \in \gamma_0^C$ and $x^k(x,v) = \bar{\alpha}_l(\tau)$. Due to Lemma 7, such τ cannot be arbitrarily close to the end points $\bar{a}_l$, $\bar{b}_l$ which are inflection points $\kappa = 0$. Lemma 7 implies that there exists $\bar{a}_l^* > \bar{a}_l$ and $\bar{b}_l^* < \bar{b}_l$ for each $l \in \{1, \ldots, M^C\}$ such that

$$\begin{aligned} &\Big\{ \tau \in (\bar{a}_l, \bar{b}_l) : \big(X(s; T_0, x, v), V(s; T_0, x, v) \big) \in \gamma_0^C, X(s; T_0, x, v) \\ &\quad = \bar{\alpha}_l(\tau) \ \text{ for } \ (x,v) \in \mathfrak{G}_l^C \Big\} \subset [\, \bar{a}_l^*, \ \bar{b}_l^* \,]. \end{aligned} \tag{3.31}$$

Throughout this subsection, we use some temporary symbols. Inspired by (2.4), we can also define kth backward/forward exit time:

$$\begin{aligned} t_{\mathbf{b}}(x,v) &:= t_{\mathbf{b}}^1(t,x,v), \\ t_{\mathbf{b}}^k(x,v) &:= t - t^k(t,x,v), \\ x_{\mathbf{b}}^k(x,v) &:= x^k(t,x,v), \\ t_{\mathbf{f}}(x,v) &:= t_{\mathbf{f}}^1(t,x,v), \\ t_{\mathbf{f}}^k(x,v) &:= -t^k(0,x,-v), \\ x_{\mathbf{f}}^k(x,v) &:= x^k(t,x,-v). \end{aligned}$$

3.3.1. 1st-Grazing Set, $\mathfrak{G}^{C,1}$ Let us use renumbered notation for the concave part (2.6). From the definition of $\mathfrak{G}_l^{C,1}$ and (3.31),

$$\begin{aligned} \mathfrak{G}_l^{C,1} \subset \bigcup_{p=\pm 1} &\Big\{ \big(\bar{\alpha}_l(\tau) + sp|v|\dot{\bar{\alpha}}_l(\tau), p|v|\dot{\bar{\alpha}}_l(\tau) \big) \in \big\{ \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB} \big\} : \\ &\tau \in [\bar{a}_l^*, \bar{b}_l^*], \ v \in \mathbb{V}^N, \ s \in [0, t_{\mathbf{f}}(\bar{\alpha}_l(\tau), p|v|\dot{\bar{\alpha}}_l(\tau))] \Big\}. \end{aligned}$$

Since the signed curvature κ is positive and bounded with finite zero points, $\mathbb{S}^1 \cap \{v \in \mathbb{R}^2 : (x, v) \in \mathfrak{G}_l^{C,1}\}$ has at most two points for fixed x. Since M^C is uniformly bounded, $\mathbb{S}^1 \cap \{v \in \mathbb{R}^2 : (x, v) \in \mathfrak{G}_l^{C,1}\}$ contains at most $2 \times M^C$ points and therefore,

$$\mathfrak{m}_2 \left\{ v \in \mathbb{R}^2 : (x, v) \in \mathfrak{G}_l^{C,1} \right\} = 0. \tag{3.32}$$

Lemma 9. *For any $\varepsilon > 0$, there exist an open cover $\bigcup_{i=1}^{l_1} B(x_i^{C,1}, r_i^{C,1})$ for $\mathcal{P}_x\left(\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\right)$, where $\mathcal{P}_x$ is projection on spatial space, and corresponding velocity set $\mathcal{O}_i^{C,1} \subset \mathbb{V}^N$ with $\mathfrak{m}_2(\mathcal{O}_i^{C,1}) < \varepsilon$ such that:*

(1) For any $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}$, there exists $x_i^{C,1}$, $r_i^{C,1}$, and $\delta^{C,1} > 0$ such that $x \in B(x_i^{C,1}, r_i^{C,1})$ and

(2) $\phi^1(x, v) = |v \cdot \mathbf{n}(x_{\mathbf{b}}(x, v))\,| > \delta^1 > 0$ holds for $v \in \mathbb{V}^N \backslash \mathcal{O}_i^{C,1}$, for some uniformly positive $\delta^1 > 0$.

From the above, we define a ε-neighborhood of $\mathfrak{G}^{C,1}$:

$$(\mathfrak{G}^{C,1})_\varepsilon := \bigcup_{i=1}^{l_1} \left\{ B(x_i^{C,1}, r_i^{C,1}) \times \mathcal{O}_i^{C,1} \right\}.$$

Proof. Let $x \in \mathcal{P}_x\left(\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\right)$. Then, there exist at most $2M^C$ distinct unit velocities $\frac{v_i}{|v_i|}$, $i \in \{1, \dots, 2M^C\}$ such that $(x, v_i) \in \mathfrak{G}^{C,1}$. We define

$$\mathcal{O}_x^{C,1} := \left\{ v \in \mathbb{V}^N : \left| \frac{v_i}{|v_i|} - \frac{v}{|v|} \right| < C_1(N)\varepsilon, \ \forall i \in \{1, \dots, 2M^C\} \right\}. \tag{3.33}$$

When $v \in \mathbb{V}^N \backslash \mathcal{O}_x^{C,1}$, we can apply Lemma 5 to show that

$$\phi^1(x, v) := |v \cdot \mathbf{n}(x_{\mathbf{b}}(x, v))|$$

is well-defined and locally smooth since $(x, v) \in \left\{\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\right\} \backslash\ \mathfrak{G}^{C,1}$. Using the local continuity of Lemma 5 again, we can find $r_x^{C,1} \ll 1$ such that

$$\phi^1(x, v) > \delta_x^1 > 0, \quad \text{for} \quad (x, v) \in cl\big(B(x, r_x^{C,1})\big) \times \mathbb{V}^N \backslash \mathcal{O}_x^{C,1}.$$

By compactness, we can find a finite open cover $\bigcup_{i=1}^{l_1} B(x_i^{C,1}, r_i^{C,1})$ for $\mathcal{P}_x\left(\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\right)$ and a corresponding $\mathcal{O}_i^{C,1}$ with small measure $\mathfrak{m}_2(\mathcal{O}_i^{C,1}) < \varepsilon$ by choosing (3.33) with some proper small $C_1(N)$. Finally we choose

$$\delta^1 := \min_{1 \leqq i \leqq l_1} \delta^1_{x_i^{C,1}} > 0$$

to finish the proof. □

3.3.2. 2nd-Grazing Set $\mathfrak{G}^{C,2}$ From the definition of $\mathfrak{G}^{C,2}$ and (3.31), the set $\mathfrak{G}^{C,2} \setminus (\mathfrak{G}^{C,1})_\varepsilon$ is a subset of

$$\bigcup_{l=1}^{M^C} \bigcup_{p=\pm 1} \Big\{ \Big(x_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), p|v|\dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), p|v|\dot{\bar{\alpha}}_l(\tau)), v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), p|v|\dot{\bar{\alpha}}_l(\tau)) \Big) \\ \in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB} : \tau \in [\bar{a}_l^*, \bar{b}_l^*],\ v \in \mathbb{V}^N,\ s \in \big[0, \big(t_{\mathbf{f}}^2 - t_{\mathbf{f}}^1\big)\big(\bar{\alpha}_l(\tau), \\ p|v|\dot{\bar{\alpha}}_l(\tau)\big)\big] \Big\} \Big\backslash (\mathfrak{G}^{C,1})_\varepsilon. \tag{3.34}$$

Without loss of generality, it suffices to consider only the $p = 1$ case of (3.34), since $p = -1$ does not change any argument.

Step 1. Fix $p = 1$ and $l \in \{1, \ldots, M^C\}$. First, we remove 1st-grazing set by complementing $(\mathfrak{G}^{C,1})_\varepsilon$.

Let us consider $(\bar{x}, \bar{v}) \in \mathfrak{G}^{C,1} \cap \mathfrak{G}_l^{C,2}$ and we write $\bar{\alpha}_l(\bar{\tau}) = x^2(\bar{x}, \bar{v})$. Then, from Lemma 9 and Lemma 5, there exists $i \in \{1, \ldots, l_1\}$ such that $(\bar{x}, \bar{v}) \in B(x_i^{C,1}, r_i^{C,1}) \times \mathcal{O}_i^{C,1}$ and

$$\Big\{ x^2(x, v) \in \partial\Omega_l^C : \forall (x, v) \in cl\left(B(x_i^{C,1}, r_i^{C,1}) \times \mathcal{O}_i^{C,1} \right), \text{where } (\bar{x}, \bar{v}) \\ \in B(x_i^{C,1}, r_i^{C,1}) \times \mathcal{O}_i^{C,1} \Big\} \subset \left[\bar{\alpha}_l(\bar{\tau} - \delta_-), \bar{\alpha}_l(\bar{\tau} + \delta_+) \right], \\ \text{for } 0 < \delta_\pm = O(r_{1,i}, \varepsilon) \ll 1. \tag{3.35}$$

Excluding (3.35) from $[\bar{a}_l^*, \bar{b}_l^*]$ for all $(\bar{x}, \bar{v}) \in \mathfrak{G}^{C,1} \cap \mathfrak{G}_l^{C,2}$ yields a union of countable open connected intervals $\mathcal{I}$, that is,

$$\mathcal{I} := [\bar{a}_l^*, d_{l,1}) \cup (c_{l,2}, d_{l,2}) \cup \cdots \subset [\bar{a}_l^*, \bar{b}_l^*], \quad \bar{a}_l^* < d_1 < c_2 < d_2 < \cdots.$$

Now we claim that $\mathcal{I}$ contains only finite subintervals. If this union is not finite, there exist infinitely many distinct $\{\tau_i\}_{i=1}^\infty$, $\tau_1 < \tau_2 < \cdots$ such that

$$\dot{\bar{\alpha}}_l(\tau_i) \cdot \mathbf{n}(x^{\mathbf{f}}(\bar{\alpha}_l(\tau_i), \dot{\bar{\alpha}}_l(\tau_i))) = 0, \quad i \in \mathbb{N}.$$

We pick a monotone increasing sequence $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ by choosing a point τ_i for each disjoint closed interval. Since $\tau_n \leqq b_*^l$ for all $n \in \mathbb{N}$, there exist a τ_∞ such that $\tau_n \to \tau_\infty$ up to subsequence. Let us assume that

$$\big(x_{\mathbf{f}}(\bar{\alpha}_l(\tau_n), \dot{\bar{\alpha}}_l(\tau_n)), \dot{\bar{\alpha}}_l(\tau_n) \big) \in \gamma_0^C, \quad x_{\mathbf{f}}(\bar{\alpha}_l(\tau_n), \dot{\bar{\alpha}}_l(\tau_n)) \in \partial\Omega_p^C.$$

Since we have chosen τ_n's from each distinct interval, there exists τ', $\tau_n < \tau' < \tau_{n+1}$ such that

$$\big(x_{\mathbf{f}}(\bar{\alpha}_l(\tau'), \dot{\bar{\alpha}}_l(\tau')), \dot{\bar{\alpha}}_l(\tau') \big) \notin \gamma_0^C.$$

By the monotonicity of $\{\tau_1, \ldots, \tau_\infty\}$, the fact that τ_∞ is an accumulation implies that we have an accumulating concave grazing phase $\{\big(x_{\mathbf{f}}(\bar{\alpha}_l(\tau_n), \dot{\bar{\alpha}}_l(\tau_n)), \dot{\bar{\alpha}}_l(\tau_n)\big)\}_{i=1}^\infty$ near $\{\big(x_{\mathbf{f}}(\bar{\alpha}_l(\tau_\infty), \dot{\bar{\alpha}}_l(\tau_\infty)), \dot{\bar{\alpha}}_l(\tau_\infty)\big)\}$. This is a contradiction because

$\partial\Omega$ is an analytic domain. Finally we can write $\mathcal{I}$ as a disjoint union of finite m_2^l intervals, that is,

$$\mathcal{I} := [\bar{a}_l^*, d_{l,1}) \cup (c_{l,2}, d_{l,2}) \cup \cdots \cup (c_{l,m_2^l-1}, d_{l,m_2^l-1}) \cup (c_{l,m_2^l}, \bar{b}_l^*]. \tag{3.36}$$

Step 2. Since we have chosen $\delta_\pm$ as nonzero in (3.35), we can include boundary points of each subinterval of (3.36). Therefore, $\mathfrak{G}^{C,2} \setminus (\mathfrak{G}^{C,1})_\varepsilon$ is a subset of

$$\begin{aligned}\bigcup_{l=1}^{M^C} \Big\{ &\Big(x_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)), v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) \Big) \\ &\in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB} : \tau \in [\bar{a}_l^*, d_{l,1}] \cup [c_{l,2}, d_{l,2}] \cup \cdots \cup [c_{l,m_2^l-1}, d_{l,m_2^l-1}] \\ &\cup [c_{l,m_2^l}, \bar{b}_l^*], v \in \mathbb{V}^N, s \in \Big[0, \Big(t_{\mathbf{f}}^2 - t_{\mathbf{f}}^1\Big)(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau))\Big] \Big\},\end{aligned} \tag{3.37}$$

and for all $\tau \in [\bar{a}_l^*, d_{l,1}] \cup [c_{l,2}, d_{l,2}] \cup \cdots \cup [c_{l,m_2^l-1}, d_{l,m_2^l-1}] \cup [c_{l,m_2^l}, \bar{b}_l^*]$,

$$|\dot{\bar{\alpha}}_l(\tau) \cdot \mathbf{n}(x_{\mathbf{f}}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau)))| > \delta^{C,1} > 0,$$

where $\delta^{C,1}$ was found in Lemma 9. Moreover, we can choose these subintervals so that the measure of each puncture $\{(d_{l,i}, c_{l,i+1})\}_{i=1}^{m_2^l-1}$ is arbitrary small, because we can choose $\delta_\pm > 0$ arbitrarily small in (3.35).

Step 3. We construct a **2nd-Sticky Grazing Set** $\mathcal{SG}_l^{C,2}$ where all grazing rays from the non-measure zero subset of $[\bar{a}_l^*, d_{l,1}) \cup (c_{l,2}, d_{l,2}) \cup \cdots \cup (c_{l,m_2^l-1}, d_{l,m_2^l-1}) \cup (c_{l,m_2^l}, \bar{b}_l^*]$ intersect at a fixed point in $\mathcal{P}_x\Big(\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\Big)$ where $\mathcal{P}_x$ is projection on the spatial domain. Choose any $i \in \{1, \ldots, m_2^l\}$ and corresponding sub interval $[c_{l,i}, d_{l,i}]$. We define

$$\begin{aligned}\mathfrak{G}_{l,i}^{C,2} := \Big\{ &\Big(x_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)), v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) \Big) \\ &\in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB} : \tau \in [c_{l,i}, d_{l,i}], v \in \mathbb{V}^N, \\ &s \in \Big[0, \Big(t_{\mathbf{f}}^2 - t_{\mathbf{f}}^1\Big)(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau))\Big] \Big\}.\end{aligned}$$

Fix $x^* \in \overline{\Omega}$. If there does not exist $\tau \in [c_{l,i}, d_{l,i}]$ and $s \in [t_1^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), t_2^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))]$ satisfying $x^* = x_{\mathbf{f}}^1(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v_{\mathbf{f}}^1(\alpha^l(\tau), \dot{\alpha}^l(\tau))$ then $\{v \in \mathbb{R}^2 : (x^*, v) \in \mathfrak{G}_{l,i}^{C,2}\} = \emptyset$ with zero measure. Now suppose that there exist such τ and s. Due to Lemma 8, there are only two cases: (i) **sticky grazing:** for all $\tau \in [c_{l,i}, d_{l,i}]$, there exists $s = s(\tau) \in [t_{\mathbf{f}}^1(\alpha^l(\tau), \dot{\alpha}^l(\tau)),\ t_{\mathbf{f}}^2(\alpha^l(\tau), \dot{\alpha}^l(\tau))]$ and fixed $x^* \in cl(\Omega)$ such that

$$x^* = x_{\mathbf{f}}^1(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v_{\mathbf{f}}^1(\alpha^l(\tau)), \tag{3.38}$$

or (ii) **isolated grazing:** there exists $\delta_-, \delta_+ > 0$ so that for $\tau' \in (\tau - \delta_-, \tau + \delta_+) \backslash \{\tau\}$, there is no s satisfying (3.38). We define the 2nd-sticky grazing set $\mathcal{SG}^{C,2}$ as a collection of all such $x^* \in cl(\Omega)$ points.

Definition 8. Consider (3.37) and disjoint union of intervals $[\bar{a}_l^*, d_{l,1}]\cup[c_{l,2}, d_{l,2}]\cup \cdots \cup [c_{l,m_2^l-1}, d_{l,m_2^l-1}] \cup [c_{l,m_2^l}, \bar{b}_l^*]$. There are finite $i \in I^2_{\mathbf{sg},l} \subset \{1, 2, \ldots, m_2^l\}$ such that case (i) **sticky grazing** holds;

$$\bigcap_{\tau\in[c_{l,i},d_{l,i}]} \overline{x^1_{\mathbf{f}}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau))\, x^2_{\mathbf{f}}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau))} = x^2_{\mathbf{sg},l,i} \quad \text{which is a point in } cl(\Omega),$$

by writing $\bar{a}_l^* = c_{l,1},\ \bar{b}_l^* = d_{l,m_2^l}$. The 2nd-sticky grazing set is the collection of such points:

$$\mathcal{SG}^{C,2} := \bigcup_{l=1}^{M^C} \mathcal{SG}_l^{C,2} := \bigcup_{l=1}^{M^C} \big\{x^2_{\mathbf{sg},l,i} \in \overline{\Omega} : i \in I^2_{\mathbf{sg},l}\big\}. \tag{3.39}$$

Note that $\mathcal{SG}^{C,2}$ is a set of finite points, from the finiteness of M^C and Lemma 8.

Step 4. We claim

$$\mathfrak{m}_2\{v \in \mathbb{R}^2 : (x, v) \in \mathfrak{G}^{C,2}\backslash(\mathfrak{G}^{C,1})_\varepsilon\} = 0, \tag{3.40}$$

for all $x \in \mathcal{P}_x(\{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB})\backslash\mathcal{SG}^{C,2}$. Consider again the set (3.37) and fix $l \in \{1, \ldots, M^C\}$. For any $i \in \{1, 2, \ldots, m_2^l\} \setminus I^2_{\mathbf{sg},l}$, we apply case (b) of Lemma 8 to say that

$$\{v \in \mathbb{R}^2 : (x, v) \in \mathfrak{G}^{C,2}\backslash(\mathfrak{G}^{C,1})_\varepsilon\} \cap \mathbb{S}^1 = \text{finite points},$$

which gives (3.40).

Lemma 10. *For any $\varepsilon > 0$, there exist an open cover*

$$\left\{\bigcup_{i=1}^{l_2} B(x_i^{C,2}, r_i^{C,2})\right\} \cup \left\{\bigcup_{y\in\mathcal{SG}^{C,2}} B(y, \varepsilon)\right\}$$

for $\mathcal{P}_x\big(\{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}\big)$ and corresponding velocity sets $\mathcal{O}_i^{C,2} \subset \mathbb{V}^N$ with $\mathfrak{m}_2(\mathcal{O}_i^{C,2}) < \varepsilon$ such that:

(1) for any $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\}\backslash\mathfrak{IB}$,

$$x \in B(x_i^{C,2}, r_i^{C,2}) \quad \textit{or} \quad x \in B(y, \varepsilon),$$

for some $x_i^{C,2}$, $r_i^{C,2}$, and $y \in \mathcal{SG}^{C,2}$;

(2) moreover, if $x \notin \bigcup_{y\in\mathcal{SG}^{C,2}} B(y, \varepsilon)$, $x \in B(x_i^{C,2}, r_i^{C,2})$, and $v \in \mathbb{V}^N\backslash\mathcal{O}_i^{C,2}$, then

$$\begin{aligned}\phi^2(x, v) = |v^1(x, v) \cdot \mathbf{n}(x^2(x, v))| > \delta^{C,2} > 0 \quad \textit{and} \quad \phi^1(x, v)\\ = |v \cdot \mathbf{n}(x_1(x, v))| > \delta^{C,1} > 0,\end{aligned}$$

for some uniformly positive $\delta^{C,1},\ \delta^{C,2} > 0$.

From the above, we define an ε-neighborhood of $\mathfrak{G}^{C,2}$:

$$(\mathfrak{G}^{C,2})_{\varepsilon} = \left\{ \bigcup_{i=1}^{l_2} B(x_i^{C,2}, r_i^{C,2}) \times \mathcal{O}_i^{C,2} \right\} \cup \left\{ \bigcup_{y \in \mathcal{SG}^{C,2}} B(y, \varepsilon) \times \mathbb{V}^N \right\}.$$

Proof. From (3.39), $\mathcal{SG}^{C,2}$ has only finite points so we make a cover with finite balls, $\bigcup_{y \in \mathcal{SG}^{C,2}} B(y, \varepsilon)$ for $\mathcal{SG}^{C,2}$.

For $x \in \mathcal{P}_x(\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}) \setminus \bigcup_{y \in \mathcal{SG}^{C,2}} B(y, \varepsilon)$, there are at most finite (at most $2M^C + 2\sum_{l=1}^{M^C} m_2^l$) unit vectors $\frac{v_i}{|v_i|}$ such that

$$(x, v_i) \in \mathfrak{G}_C^2 \cup \mathfrak{G}_C^1,$$

from (3.40) and (3.32). So we define

$$\mathcal{O}_x^{C,2} := \left\{ v \in \mathbb{V}^N : \left| \frac{v_i}{|v_i|} - \frac{v}{|v|} \right| < C_2(N)\varepsilon, \ \forall v_i \ \text{ s.t } \ (x, v_i) \in \mathfrak{G}^{C,1} \cup \mathfrak{G}^{C,2} \right\}.$$

When $v \in \mathbb{V}^N \backslash \mathcal{O}_x^{C,2}$, the trajectory does not graze within second bounces, so both

$$\phi^1(x, v) = |v \cdot \mathbf{n}(x_{\mathbf{b}}(x, v))| \quad \text{and} \quad \phi^2(x, v) := |v^1(x, v) \cdot \mathbf{n}(x^2(x, v))|$$

are well-defined and locally smooth, because $(x, v) \in \big\{\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\big\} \backslash \big(\mathfrak{G}^{C,1} \cup \mathfrak{G}^{C,2}\big)$ implies that the trajectory does not graze in the first two bounces. Using the local continuity of Lemma 5 again, we can find $r_x^{C,2} \ll 1$ such that

$$\phi^1(x, v) > \delta_x^1 > 0, \quad \phi^2(x, v) > \delta_x^2 > 0, \quad \text{for } (x, v) \in cl\big(B(x, r_x^{C,2})\big) \times \mathbb{V}^N \backslash \mathcal{O}_x^{C,2}.$$

By compactness, we can find finite open cover $\bigcup_{i=1}^{l_2} B(x_i^{C,2}, r_i^{C,2})$ for $\mathcal{P}_x\big(\{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}\big) \setminus \bigcup_{y \in \mathcal{SG}^{C,2}} B(y, \varepsilon)$ and the corresponding $\mathcal{O}_i^{C,2}$ with small measure $\mathfrak{m}_2(\mathcal{O}_i^{C,2}) < \varepsilon$ by choosing (3.33) with sufficiently small $C_2(N)$. Finally we choose

$$\delta^1 := \min_{1 \leqq i \leqq l_1} \delta^1_{x_i^{C,2}} > 0, \quad \delta^2 := \min_{1 \leqq i \leqq l_2} \delta^2_{x_i^{C,2}} > 0$$

to finish the proof. □

3.3.3. kth-Grazing Set, $\mathfrak{G}^{C,k}$ Now we are going to construct, for $k > 2$, the **kth-Grazing Set** and its ε-neighborhood. We construct such sets via mathematical induction. We assume that Lemma 10 holds for $\mathfrak{G}^{C,k-1}$, that is, we have

Assumption 1. For any $\varepsilon > 0$, there exist $\mathcal{SG}^{C,k-1}$, which contains finite points in $cl(\Omega)$, an open cover

$$\left\{ \bigcup_{i=1}^{l_{k-1}} B(x_i^{C,k-1}, r_i^{C,k-1}) \right\} \cup \left\{ \bigcup_{y \in \mathcal{SG}^{C,k-1}} B(y, \varepsilon) \right\}$$

for $\mathcal{P}_x\big(\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}\big)$, and corresponding velocity sets $\mathcal{O}_i^{C,k-1}\subset\mathbb{V}^N$ with $\mathfrak{m}_2(\mathcal{O}_i^{C,k-1})<\varepsilon$ such that

(1) for any $(x,v)\in\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}$,

$$x\in B(x_i^{C,k-1},r_i^{C,k-1})\quad\text{or}\quad x\in B(y,\varepsilon),$$

for some $x_i^{C,k-1}$, $r_i^{C,k-1}$, and $y\in\mathcal{SG}^{C,k-1}$;

(2) moreover, if $x\notin\bigcup_{y\in\mathcal{SG}^{C,k-1}}B(y,\varepsilon)$, $x\in B(x_i^{C,k-1},r_i^{C,k-1})$, and $v\in\mathbb{V}^N\backslash\mathcal{O}_i^{C,k-1}$, then

$$\phi^s(x,v)=|v^{s-1}(x,v)\cdot\mathbf{n}(x^s(x,v))|>\delta^{C,s}>0,$$

for all $s\in\{1,\dots,k-1\}$ some uniformly positive $\delta^{C,1},\ \delta^{C,2},\dots,\delta^{C,k-1}>0$.

We define the ε-neighborhood of $\mathfrak{G}^{C,k-1}$:

$$(\mathfrak{G}^{C,k-1})_\varepsilon=\left\{\bigcup_{i=1}^{l_{k-1}}B(x_i^{C,k-1},r_i^{C,k-1})\times\mathcal{O}_i^{C,k-1}\right\}\cup\left\{\bigcup_{y\in\mathcal{SG}^{C,k-1}}B(y,\varepsilon)\times\mathbb{V}^N\right\}.$$

Now, under the above assumption, we follow the steps in $\mathfrak{G}_C^2$. From the definition of $\mathfrak{G}^{C,k}$ and (3.31), the set $\mathfrak{G}^{C,k}\setminus(\mathfrak{G}^{C,k-1})_\varepsilon$ is a subset of

$$\begin{aligned}\bigcup_{l=1}^{M^C}\bigcup_{p=\pm1}\Big\{\Big(&x_{\mathbf{f}}^{k-1}(\bar\alpha_l(\tau),p|v|\dot{\bar\alpha}_l(\tau))+sv_{\mathbf{f}}^{k-1}(\bar\alpha_l(\tau),p|v|\dot{\bar\alpha}_l(\tau)),v_{\mathbf{f}}^{k-1}(\bar\alpha_l(\tau),\\&p|v|\dot{\bar\alpha}_l(\tau))\Big)\in\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}:\tau\in[\bar a_l^*,\bar b_l^*],v\in\mathbb{V}^N,\\&s\in\big[0,\big(t_{\mathbf{f}}^k-t_{\mathbf{f}}^{k-1}\big)(\bar\alpha_l(\tau),p|v|\dot{\bar\alpha}_l(\tau))\big]\Big\}\Big\backslash(\mathfrak{G}^{C,k-1})_\varepsilon.\end{aligned}\tag{3.41}$$

Without loss of generality, it suffices to consider only the $p=1$ case of (3.41).

Step 1. Fix $p=1$ and $l\in\{1,\dots,M^C\}$. First, we remove the $k-1$st-grazing set by complementing $(\mathfrak{G}^{C,k-1})_\varepsilon$.

Let us consider $(\bar x,\bar v)\in\mathfrak{G}^{C,k-1}\cap\mathfrak{G}_l^{C,k}$, and write $\bar\alpha_l(\bar\tau)=x^k(\bar x,\bar v)$. Then, from Assumption 1, there exists $i\in\{1,\dots,l_{k-1}\}$ such that $(\bar x,\bar v)\in B(x_i^{C,k-1},r_i^{C,k-1})\times\mathcal{O}_i^{C,k-1}$ and

$$\begin{aligned}&\Big\{x^k(x,v)\in\partial\Omega_l^C:\forall(x,v)\in cl\left(B(x_i^{C,k-1},r_i^{C,k-1})\times\mathcal{O}_i^{C,k-1}\right),\\&\quad\text{where }(\bar x,\bar v)\in B(x_i^{C,k-1},r_i^{C,k-1})\times\mathcal{O}_i^{C,k-1}\Big\}\\&\quad\subset\big[\bar\alpha_l(\bar\tau-\delta_-),\bar\alpha_l(\bar\tau+\delta_+)\big],\quad\text{for }0<\delta_\pm=O(r_{k-1,i},\varepsilon)\ll1.\end{aligned}\tag{3.42}$$

Excluding (3.42) from $[\bar a_l^*,\bar b_l^*]$ for all $(\bar x,\bar v)\in\mathfrak{G}^{C,k-1}\cap\mathfrak{G}_l^{C,k}$ yields a union of countable open connected intervals $\mathcal{I}^k$, that is,

$$\mathcal{I}:=[\bar a_l^*,d_{l,1}^k)\cup(c_{l,2}^k,d_{l,2}^k)\cup\cdots\subset[\bar a_l^*,\bar b_l^*],\quad\bar a_l^*<d_1<c_2<d_2<\cdots.$$

Using exactly the same argument of *Step 1* in the **2nd-Grazing Set** $\mathfrak{G}^{C,2}$, we know that this should be a finite union of sub intervals and write

$$\mathcal{I}^k := [\bar{a}_l^*, d_{l,1}^k) \cup (c_{l,2}^k, d_{l,2}^k) \cup \cdots \cup (c_{l,m_k^l-1}^k, d_{l,m_k^l-1}^k) \cup (c_{l,m_k^l}^k, \bar{b}_l^*]. \tag{3.43}$$

Step 2. Since we have chosen $\delta_\pm$ as nonzero in (3.42), we can include boundary points of each subinterval of (3.43). Therefore, $\mathfrak{G}^{C,k} \setminus (\mathfrak{G}^{C,k-1})_\varepsilon$ is a subset of

$$\begin{aligned}\bigcup_{l=1}^{M^C} \Bigg\{ & \Big(x_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)), v_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) \Big) \\ & \in \Big\{ cl(\Omega) \times \mathbb{V}^N \Big\} \setminus \mathfrak{IB} : \tau \in [\bar{a}_l^*, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c_{l,m_k^l-1}^k, d_{l,m_k^l-1}^k] \\ & \cup [c_{l,m_k^l}^k, \bar{b}_l^*], v \in \mathbb{V}^N, s \in [0, (t_{\mathbf{f}}^k - t_{\mathbf{f}}^{k-1})(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau))] \Bigg\},\end{aligned} \tag{3.44}$$

and for all $\tau \in [\bar{a}_l^*, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c_{l,m_k^l-1}^k, d_{l,m_k^l-1}^k] \cup [c_{l,m_k^l}^k, \bar{b}_l^*]$,

$$|\dot{\bar{\alpha}}_l(\tau) \cdot \mathbf{n}(x_{\mathbf{f}}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau)))| > \min_{i=1,\ldots,k-1} \delta^{C,i} > 0,$$

where $\delta^{C,i} > 0$ is found in Assumption 1. Moreover, we can choose these subintervals so that for the measure of each the punctures $\{(d_{l,i}^k, c_{l,i+1}^k)\}_{i=1}^{m_k^l-1}$ are arbitrary small, because we can choose $\delta_\pm > 0$ arbitrary small in (3.42).

Step 3. We construct the **kth-Sticky Grazing Set** $\mathcal{SG}_l^{C,k}$ where all grazing rays from the non-measure zero subset of $[\bar{a}_l^*, d_{l,1}^k) \cup (c_{l,2}^k, d_{l,2}^k) \cup \cdots \cup (c_{l,m_k^l-1}^k, d_{l,m_k^l-1}^k) \cup (c_{l,m_k^l}^k, \bar{b}_l^*]$ intersect at a fixed point in $\mathcal{P}_x\Big(\{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathfrak{IB}\Big)$, where $\mathcal{P}_x$ is projection onto a spatial domain. Choose any $i \in \{1, \ldots, m_k^l\}$ and corresponding sub interval $[c_{l,i}^k, d_{l,i}^k]$. We define

$$\begin{aligned}\mathfrak{G}_{l,i}^{C,k} := \Bigg\{ & \Big(x_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)), v_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau)) \Big) \\ & \in \{cl(\Omega) \times \mathbb{V}^N\} \setminus \mathfrak{IB} : \tau \in [c_{l,i}, d_{l,i}], v \in \mathbb{V}^N, \\ & s \in \Big[0, \big(t_{\mathbf{f}}^2 - t_{\mathbf{f}}^1\big)(\bar{\alpha}_l(\tau), |v|\dot{\bar{\alpha}}_l(\tau))\Big] \Bigg\}.\end{aligned}$$

Fix $x^* \in \overline{\Omega}$. If there does not exist $\tau \in [c_{l,i}^k, d_{l,i}^k]$ and $s \in [t_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau)), t_{\mathbf{f}}^k(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau))]$ satisfying $x^* = x_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^1(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau))$, then $\{v \in \mathbb{R}^2 : (x^*, v) \in \mathfrak{G}_{l,i}^{C,k}\} = \emptyset$ with zero measure. Now suppose there exist such τ and s. Due to Lemma 8, there are only two cases: (i) **sticky grazing:** for all $\tau \in [c_{l,i}^k, d_{l,i}^k]$, there exists $s = s(\tau) \in [t_{\mathbf{f}}^{k-1}(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau)),\ t_{\mathbf{f}}^k(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau))]$ and points $\{x^{*,r}\}_{r=1}^{k-1} \in cl(\Omega)$ such that

$$x^{*,r} = x_{\mathbf{f}}^r(\bar{\alpha}_l(\tau), \dot{\bar{\alpha}}_l(\tau)) + s v_{\mathbf{f}}^r(\bar{\alpha}_l(\tau)), \quad \text{for all} \quad \tau \in [c_{l,i}^k, d_{l,i}^k]; \tag{3.45}$$

(ii) **isolated grazing:** there exists $\delta_-, \delta_+ > 0$ so that for $\tau' \in (\tau-\delta_-, \tau+\delta_+)\backslash\{\tau\}$ there is no s satisfying (3.45). We define the kth-sticky grazing set $\mathcal{SG}^{C,k}$ as a collection of all such $x^{*,r} \in cl(\Omega)$ points.

Definition 9. Consider (3.44) and a disjoint union of intervals $[\bar{a}_l^*, d_{l,1}^k] \cup [c_{l,2}^k, d_{l,2}^k] \cup \cdots \cup [c^k_{l,m^l_k-1}, d^k_{l,m^l_k-1}] \cup [c^k_{l,m^l_k}, \bar{b}^*_l]$. There are finite $i \in I^k_{\mathbf{sg},l} \subset \{1,2,\dots,(k-1)m^l_k\}$ such that case (i) **sticky grazing** holds, that is,

$$\bigcap_{\tau\in[c^k_{l,i},d^k_{l,i}]} \overline{x^{r-1}_{\mathbf{f}}(\bar{\alpha}_l(\tau),\dot{\bar{\alpha}}_l(\tau))\, x^{r}_{\mathbf{f}}(\bar{\alpha}_l(\tau),\dot{\bar{\alpha}}_l(\tau))} = x^k_{\mathbf{sg},l,i,r} \in cl(\Omega),$$
$$\text{for some} \quad r = 1,\dots,k-1,$$

by writing $\bar{a}^*_l = c^k_{l,1},\ \bar{b}^*_l = d^k_{l,m^l_2}$. When the above intersection is nonempty we collect all of those points to obtain the kth-sticky grazing set:

$$\mathcal{SG}^{C,k} := \bigcup_{l=1}^{M^C} \mathcal{SG}^{C,k}_l := \bigcup_{l=1}^{M^C}\bigcup_{i=1}^{m^l_k}\bigcup_{r=1}^{k-1} \left\{x^k_{\mathbf{sg},l,i,r} \in cl(\Omega)\right\}. \tag{3.46}$$

Note that $\mathcal{SG}^{C,k}$ has at most $(k-1)M^C m^l_k$ points, from indexes i, l, and r.

Step 4. We claim

$$\mathfrak{m}_2\left\{v\in\mathbb{R}^2 : (x,v)\in\mathfrak{G}^{C,k}\backslash(\mathfrak{G}^{C,k-1})_\varepsilon\right\} = 0 \tag{3.47}$$

for all $x \in \mathcal{P}_x(\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB})\backslash\mathcal{SG}^{C,k}$. Consider again the set (3.44) and fix $l\in\{1,\dots,M^C\}$. For any point $x \in cl(\Omega)$ such that $i \in \{1,2,\dots,(k-1)m^l_k\}\setminus I^k_{\mathbf{sg},l}$, we apply case (b) of Lemma 8 to say that

$$\left\{v\in\mathbb{R}^2 : (x,v)\in\mathfrak{G}^{C,k}\backslash(\mathfrak{G}^{C,k-1})_\varepsilon\right\}\cap\mathbb{S}^1 = \text{finite points},$$

which gives (3.47).

Lemma 11. *We assume Assumption 1. Then, for any $\varepsilon > 0$, there exist an open cover*

$$\left\{\bigcup_{i=1}^{l_k} B(x^{C,k}_i, r^{C,k}_i)\right\} \cup \left\{\bigcup_{y\in\mathcal{SG}^{C,k}} B(y,\varepsilon)\right\}$$

for $\mathcal{P}_x\big(\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}\big)$ and corresponding velocity sets $\mathcal{O}^{C,k}_i \subset \mathbb{V}^N$ with $\mathfrak{m}_2(\mathcal{O}^{C,k}_i) < \varepsilon$ such that

(1) for any $(x,v)\in\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}$,

$$x \in B(x^{C,k}_i, r^{C,k}_i) \quad \text{or} \quad x\in B(y,\varepsilon),$$

for some $x^{C,k}_i$, $r^{C,k}_i$, and $y\in\mathcal{SG}^{C,k}$;

(2) moreover, if $x \notin \bigcup_{y\in\mathcal{SG}^{C,k}} B(y,\varepsilon)$, $x \in B(x_i^{C,k}, r_i^{C,k})$, *and* $v \in \mathbb{V}^N \backslash \mathcal{O}_i^{C,k}$, *then*

$$\phi^r(x,v) = |v^{r-1}(x,v)\cdot \mathbf{n}(x^r(x,v))| > \delta^{C,r} > 0, \quad r = 1,\dots,k,$$

for some uniformly positive $\delta^{C,r} > 0$, $r = 1,\dots,k$.

From the above, we define the ε*-neighborhood of* $\mathfrak{G}^{C,k}$ *as follows:*

$$(\mathfrak{G}^{C,k})_\varepsilon = \left\{\bigcup_{i=1}^{l_k} B(x_i^{C,k}, r_i^{C,k}) \times \mathcal{O}_i^{C,k}\right\} \cup \left\{\bigcup_{y\in\mathcal{SG}^{C,k}} B(y,\varepsilon)\times \mathbb{V}^N\right\}.$$

Proof. Is suffices to follow the scheme of the proof of Lemma 10. From (3.46), $\mathcal{SG}^{C,k}$ has finite points so we make a cover with finite balls, $\bigcup_{y\in\mathcal{SG}^{C,k}} B(y,\varepsilon)$ for $\mathcal{SG}^{C,k}$.

For $x \in \mathcal{P}_x(\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}) \setminus \bigcup_{y\in\mathcal{SG}^{C,k}} B(y,\varepsilon)$, there are at most finite (at most $2M^C + 2\sum_{r=1}^{k}\sum_{l=1}^{M^C} m_r^l$) unit vectors $\frac{v_i}{|v_i|}$ such that

$$(x, v_i) \in \mathfrak{G}_C^k \cup \mathfrak{G}_C^{k-1} \cup \cdots \cup \mathfrak{G}_C^1,$$

from (3.47). Thus we define

$$\mathcal{O}_x^{C,k} := \left\{ v\in\mathbb{V}^N : \Big|\frac{v_i}{|v_i|} - \frac{v}{|v|}\Big| < C_k(N)\varepsilon, \ \forall v_i \ \text{ s.t } (x,v_i) \in \bigcup_{r=1}^{k}\mathfrak{G}^{C,r}\right\}. \tag{3.48}$$

When $v \in \mathbb{V}^N\backslash\mathcal{O}_x^{C,k}$, the trajectory does not graze within second bounces, so

$$\phi^r(x,v) = |v\cdot\mathbf{n}(x_{\mathbf{b}}(x,v))|, \quad 1 \leqq r \leqq k$$

are well-defined and locally smooth, because $(x,v) \in \big\{\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}\big\}\backslash \big(\bigcup_{r=1}^{k}\mathfrak{G}^{C,r}\big)$ implies that the trajectory does not graze in the first k bounces. Using the local continuity of Lemma 5 again, we can find $r_x^{C,k} \ll 1$ such that

$$\phi^r(x,v) > \delta_x^r > 0, \quad \text{for } 1\leqq r\leqq k \text{ and } (x,v)\in cl\big(B(x, r_x^{C,k})\big)\times\mathbb{V}^N\backslash\mathcal{O}_x^{C,k}.$$

By compactness, we can find an open cover $\bigcup_{i=1}^{l_k} B(x_i^{C,k}, r_i^{C,k})$ for $\mathcal{P}_x\big(\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB}\big) \setminus \bigcup_{y\in\mathcal{SG}^{C,2}} B(y,\varepsilon)$ and corresponding $\mathcal{O}_i^{C,k}$ with small measure $\mathfrak{m}_2(\mathcal{O}_i^{C,k}) < \varepsilon$ by choosing (3.48) with sufficiently small $C_k(N)$. Finally we choose

$$\delta^r := \min_{1\leqq i\leqq l_k} \delta^r_{x_i^{C,k}} > 0, \quad 1\leqq r\leqq k$$

to finish the proof. □

Proposition 1. *For any $\varepsilon > 0$, we have the ε-neighborhood of $\mathfrak{G}$:*

$$(\mathfrak{G})_\varepsilon = \left\{ \bigcup_{i=1}^{l_G} B(x_i^C, r_i^C) \times \mathcal{O}_i^C \right\} \cup \left\{ \bigcup_{j=1}^{l_{sg}} B(y_j^C, \varepsilon) \times \mathbb{V}^N \right\},$$

with $\mathcal{O}_i^C \subset \mathbb{V}^N$, $\mathfrak{m}_2(\mathcal{O}_i^C) < \varepsilon$ for all $i = 1, 2, \dots, l_G < \infty$, and $j = 1, 2, \dots, l_{sg} < \infty$. For any $(x, v) \in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}$,

$$x \in B(x_i^C, r_i^C) \quad or \quad x \in B(y_j^C, \varepsilon),$$

for some x_i^C or y_j^C. Moreover, if $x \notin \bigcup_{j=1}^{l_{sg}} B(y_j^C, \varepsilon)$, $x \in B(x_i^C, r_i^C)$, and $v \in \mathbb{V}^N \backslash \mathcal{O}_i^C$, then

$$|v^{k-1}(T_0, x, v) \cdot \mathbf{n}(x^k(T_0, x, v))| > \delta > 0, \quad \forall t^k(T_0, x, v) \in [0, T_0].$$

Proof. We use mathematical induction. We already proved the $k = 1$ case in Lemma 9, for when there is no sticky grazing set. From $k = 2$, a sticky grazing set appears and we have proved Lemma 10. From Assumption 1 and Lemma 11, we know that Lemma 11 holds for any finite $k \in \mathbb{N}$. Moreover, number of bounces is uniformly bounded from Lemma 6, so we stop mathematical induction with the maximal possible number of bouncing on $[0, T_0]$. □

3.4. Transversality and Double Duhamel Trajectory

We introduce local parametrization for $U = \Omega \times (\mathbb{R}/H\mathbb{Z})$. Since we should treat three-dimensional trajectory from this subsection, we introduce the following notation to denote two-dimensional points in cross section:

$$\underline{x} = (x_1, x_3), \quad \underline{v} = (v_1, v_3),$$

where the missing x_2 and v_2 are components for the axis direction. Therefore we can write

$$x = (\underline{x}, x_2) \in \Omega, \quad v = (\underline{v}, v_2) \in \mathbb{R}^3.$$

Especially for the points near the boundary, we define local parametrization, that is, for $p \in \partial\Omega$,

$$\begin{aligned}
&\eta_p : \{\mathbf{x}_p \in \mathbb{R}^3 : \mathbf{x}_{p,3} < 0\} \cap B(0, \delta_1) \ \to \ \Omega \cap B(p, \delta_2), \\
&\mathbf{x}_p = (\underline{\mathbf{x}}_p, \mathbf{x}_{p,2}) \ \mapsto \ x := \eta_p(\mathbf{x}_p), \\
&\eta_p(0,0,0) = p, \quad x = \eta_p(\mathbf{x}_p) = (\underline{\eta}_p(\underline{\mathbf{x}}_p), \mathbf{x}_{p,2}), \\
&\underline{x} = \underline{\eta}_p(\underline{\mathbf{x}}_p) = \underline{\eta}_p(\mathbf{x}_{p,1}, 0) + \mathbf{x}_{p,3}\mathbf{n}(\underline{\eta}_p(\mathbf{x}_{p,1}, 0)),
\end{aligned} \tag{3.49}$$

and $\eta_p(\mathbf{x}_p) \in \partial\Omega$ if and only if $\mathbf{x}_{p,3} = 0$. $\mathbf{n}\big(\underline{\eta}_p(\mathbf{x}_{p,1}, 0)\big)$ is an outward unit normal vector at $\big(\underline{\eta}_p(\mathbf{x}_{p,1}, 0), \mathbf{x}_{p,2}\big) \in \partial\Omega$. Since Ω is cylindrical, the unit normal vector $\mathbf{n}$ is independent of $\mathbf{x}_{p,2}$. We use the following derivative symbols:

$$\partial_i \eta_p := \frac{\partial}{\partial \mathbf{x}_i}\eta_p, \quad \underline{\nabla} := (\partial_1, \partial_3) = \left(\frac{\partial}{\partial \mathbf{x}_{p,1}}, \frac{\partial}{\partial \mathbf{x}_{p,3}}\right), \quad \partial_{x_i} = \frac{\partial}{\partial x_i},$$
$$\underline{\nabla}_x := (\partial_{x_1}, \partial_{x_3}) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\right),$$
$$\nabla = (\underline{\nabla}, \partial_2), \quad \nabla_x = (\underline{\nabla}_x, \partial_2),$$

where $x \in cl(\Omega)$ and $\mathbf{x}_p \in \eta_p^{-1}(\Omega)$. Note that it is easy to check that η_p is a locally triple orthogonal system, that is,

$$\langle \partial_i \eta_p, \partial_j \eta_p \rangle = 0, \quad \text{for all} \quad i \neq j, \quad \mathbf{x} \in \left\{\mathbf{x}_p \in \mathbb{R}^3 : \mathbf{x}_{p,3} < 0\right\} \cap B(0, \delta_1). \tag{3.50}$$

We also use standard notations $g_{p,ij} := \langle \partial_i \eta_p, \partial_j \eta_p \rangle$, and transformed velocity $\mathbf{v}_p$ is defined by

$$\mathbf{v}_{p,i}(\mathbf{x}_p) := \frac{\partial_i \eta_p(\mathbf{x}_p)}{\sqrt{g_{p,ii}(\mathbf{x}_p)}} \cdot v,$$

or, equivalently,

$$\mathbf{v}_p = \begin{bmatrix} \mathbf{v}_{p,1} \\ \mathbf{v}_{p,2} \\ \mathbf{v}_{p,3} \end{bmatrix} = Q^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = Q^T v, \quad \text{where} \quad Q := \begin{bmatrix} \frac{\partial_1 \eta_{p,1}}{\sqrt{g_{p,11}}} & \frac{\partial_2 \eta_{p,1}}{\sqrt{g_{p,22}}} & \frac{\partial_3 \eta_{p,1}}{\sqrt{g_{p,33}}} \\ \frac{\partial_1 \eta_{p,2}}{\sqrt{g_{p,11}}} & \frac{\partial_2 \eta_{p,2}}{\sqrt{g_{p,22}}} & \frac{\partial_3 \eta_{p,2}}{\sqrt{g_{p,33}}} \\ \frac{\partial_1 \eta_{p,3}}{\sqrt{g_{p,11}}} & \frac{\partial_2 \eta_{p,3}}{\sqrt{g_{p,22}}} & \frac{\partial_3 \eta_{p,3}}{\sqrt{g_{p,33}}} \end{bmatrix}. \tag{3.51}$$

We compute transversality between two consecutive bouncings using local parametrization (3.49) and transformed velocity (3.51). To denote the bouncing index, we define

$$\begin{aligned} \mathbf{x}^k_{p^k} &:= \left(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\right) \quad \text{such that} \quad x^k = \eta_{p^k}\left(\mathbf{x}^k_{p^k}\right), \\ \mathbf{v}^k_{p^k,i} &:= \frac{\partial_i \eta_{p^k}\left(\mathbf{x}^k_{p^k}\right)}{\sqrt{g_{p^k,ii}\left(\mathbf{x}^k_{p^k}\right)}} \cdot v^k, \end{aligned} \tag{3.52}$$

where p^k is a point on $\partial\Omega$ near the bouncing point x^k.

Since the dynamics in the x_2 direction is independent of the dynamics in a cross section, we focus on the dynamics of a two-dimensional cross section Ω, for fixed x_2. We hope to compute the Jacobian between two adjacent bounces. Since we use local orthogonal parametrization, the following lemma is a basic tool to compute the Jacobian:

Lemma 12. *Assume that* Ω *are* C^2 *(not necessarily convex) and* $|\mathbf{v}^k_{p^k,3}|,\ |\mathbf{v}^{k+1}_{p^{k+1},3}| > 0$. *Consider* $(t^{k+1}, \mathbf{x}^{k+1}_{p^{k+1},1}, \underline{\mathbf{v}}^{k+1}_{p^{k+1}})$ *as a function of* $(t^{k+1}, \mathbf{x}^k_{p^k,1}, \underline{\mathbf{v}}^k_{p^k})$.

$$\begin{aligned}\frac{\partial(t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,1}} &= \frac{-1}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} \\ &\quad \cdot \left[\partial_1 \underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1}, 0\right) - (t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}} \right], \end{aligned} \tag{3.53}$$

$$\begin{aligned}\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} &= \frac{1}{\sqrt{g_{p^k,11}(\underline{x}^{k+1})}} \left[\frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^k,11}(\underline{x}^{k+1})}} + \frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} \right] \\ &\quad \cdot \left[\partial_1 \underline{\eta}_{p^k}(\underline{x}^k) - (t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}} \right], \end{aligned} \tag{3.54}$$

$$\begin{aligned}\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} &= \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}} \cdot \frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} + \underline{v}^k \\ &\quad \cdot \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}} \left(\frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \right), \end{aligned} \tag{3.55}$$

$$\begin{aligned}\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{x}^k_{p^k,1}} &= -\frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}} \cdot \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} - \underline{v}^k \\ &\quad \cdot \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}} \left(\frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} \right), \end{aligned} \tag{3.56}$$

where

$$\frac{\partial v^k_i}{\partial \mathbf{x}^k_{p^k,1}} = \sum_{\ell=1}^{3} \mathbf{v}^k_{p^k,\ell} \sum_{r(\neq \ell)} \frac{\sqrt{g_{p^k,rr}(\underline{x}^k)}}{\sqrt{g_{p^k,\ell\ell}(\underline{x}^k)}} \Gamma^r_{p^k,\ell 1}(\underline{x}^k) \frac{\partial_r \underline{\eta}_{p^k,i}(\underline{x}^k)}{\sqrt{g_{p^k,rr}(\underline{x}^k)}}, \quad i = 1, 3. \tag{3.57}$$

For $i = 1$ *and* $j = 1, 3$,

$$\frac{\partial(t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} = \frac{(t^k - t^{k+1})}{\mathbf{v}^{k+1}_{p^{k+1},3}} \left[\frac{\partial_j \underline{\eta}_{p^k}(\underline{x}^k)}{\sqrt{g_{p^k,jj}(\underline{x}^k)}} \right] \cdot \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}}, \tag{3.58}$$

$$\begin{aligned}\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} &= -(t^k - t^{k+1}) \frac{\partial_j \underline{\eta}_{p^k}(\underline{x}^k)}{\sqrt{g_{p^k,jj}(\underline{x}^k)}} \cdot \frac{1}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \\ &\quad \left[\frac{\partial_1 \underline{\eta}^{k+1}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} + \frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} \right], \end{aligned} \tag{3.59}$$

$$\begin{aligned}\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^{k}_{p^k,j}} &= \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^{k}_{p^k,j}} \partial_\ell \left(\frac{\partial_1 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}} \right)\Bigg|_{\underline{x}^{k+1}} \cdot \underline{v}^k \\ &\quad + \frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \cdot \frac{\partial_j \underline{\eta}_{p^k}(\underline{x}^k)}{\sqrt{g_{p^k,jj}(\underline{x}^k)}},\end{aligned} \tag{3.60}$$

$$\begin{aligned}\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{v}^{k}_{p^k,j}} &= -\sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^{k}_{p^k,j}} \partial_\ell \left(\frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}} \right)\Bigg|_{\underline{x}^{k+1}} \cdot \underline{v}^k \\ &\quad - \frac{\partial_3 \eta_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} \cdot \frac{\partial_j \underline{\eta}_{p^k}(\underline{x}^k)}{\sqrt{g_{p^k,jj}(\underline{x}^k)}}.\end{aligned} \tag{3.61}$$

Proof of (3.53). By the definitions (3.49), (2.4), and our setting (3.52) and (2.1),

$$\underline{x}^{k+1}\left(\mathbf{x}^{k+1}_{p^{k+1},1}, 0\right) = \underline{\eta}_{p^k}\left(\mathbf{x}^{k}_{p^k,1}, 0\right) + \int_{t^k}^{t^{k+1}} \underline{v}^k. \tag{3.62}$$

We take $\frac{\partial}{\partial \mathbf{x}^k_{p^k,1}}$ to the above equality to get

$$\begin{aligned}\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^{k}_{p^k,1}} \frac{\partial \underline{\eta}^{k+1}_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}\Bigg|_{\underline{x}^{k+1}} &= -(t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}} \\ - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,1}} \underline{v}^k &+ \partial_1 \underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1}, 0\right),\end{aligned} \tag{3.63}$$

and then take an inner product with $\frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{\underline{x}^{k+1}}$ to have

$$\begin{aligned}&\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^{k}_{p^k,1}} \frac{\partial \underline{\eta}^{k+1}_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}\Bigg|_{\underline{x}^{k+1}} \cdot \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}} \\ &= -(t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}} \cdot \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}} - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,1}} \underline{v}^k \cdot \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}} \\ &\quad + \partial_1 \underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1}, 0\right) \cdot \frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}},\end{aligned}$$

where we abbreviated $\underline{X}(s) = \underline{X}(s; t^k, \underline{x}^k, \underline{v}^k)$ and $\underline{V}(s) = \underline{V}(s; t^k, \underline{x}^k, \underline{v}^k)$. Due to (3.50), the LHS equals zero. Now we consider the RHS. From (3.51), we prove (3.57).

We also note that

$$\lim_{s\downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k)=\underline{v}^k. \tag{3.64}$$

Therefore, from (2.4) and (3.52),

$$v^k\cdot\frac{\partial_3\underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}}=-\mathbf{v}^{k+1}_{p^{k+1},3}.$$

Dividing both sides by $\underline{v}^k\cdot\partial_3\underline{\eta}^{k+1}_{p^{k+1}}\big|_{\underline{x}^{k+1}}=\mathbf{v}^{k+1}_{p^{k+1},3}$, we get (3.53).

Proof of (3.54). We take the inner product with $\frac{\partial_1\underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Big|_{\underline{x}^{k+1}}$ to (3.63) to have

$$\begin{aligned}
&\frac{\partial\mathbf{x}^{k+1}_{p^{k+1},1}}{\partial\mathbf{x}^k_{p^k,1}}\frac{\partial\underline{\eta}^{k+1}_{p^{k+1}}}{\partial\mathbf{x}^{k+1}_{p^{k+1},1}}\Bigg|_{\underline{x}^{k+1}}\cdot\frac{\partial_1\underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}=\frac{\partial\mathbf{x}^{k+1}_{p^{k+1},1}}{\partial\mathbf{x}^k_{p^k,1}}\\
&\quad=-(t^k-t^{k+1})\frac{\partial\underline{v}^k}{\partial\mathbf{x}^k_{p^k,1}}\cdot\frac{\partial_1\underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}-\frac{\partial(t^k-t^{k+1})}{\partial\mathbf{x}^k_{p^k,1}}\underline{v}^k\cdot\frac{\partial_1\underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}\\
&\qquad+\partial_1\underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1},0\right)\cdot\frac{\partial_1\underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}.
\end{aligned}$$

Since

$$\underline{v}^k\cdot\frac{\partial_1\underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}=-\frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\sqrt{g_{p^{k+1},11}}},$$

from (3.50) and (3.53),

$$\begin{aligned}
\frac{\partial\mathbf{x}^{k+1}_{p^{k+1},1}}{\partial\mathbf{x}^k_{p^k,1}}&=\frac{1}{\mathbf{v}^{k+1}_{p^{k+1},3}}\frac{\partial_3\underline{\eta}^{k+1}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}}\cdot\left[\partial_1\underline{\eta}_{p^k}(\underline{x}^k)-(t^k-t^{k+1})\frac{\partial\underline{v}^k}{\partial\mathbf{x}^k_{p^k,1}}\right]\\
&\quad\frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\sqrt{g_{p^{k+1},11}}}\Bigg|_{\underline{x}^{k+1}}+\frac{\partial_1\underline{\eta}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}\cdot\left[\partial_1\underline{\eta}_{p^k}(\underline{x}^k)-(t^k-t^{k+1})\frac{\partial\underline{v}^k}{\partial\mathbf{x}^k_{p^k,1}}\right].
\end{aligned}$$

This ends the proof of (3.54).

Proof of (3.55) and (3.56). From (2.4) and (3.52),

$$\mathbf{v}^{k+1}_{p^{k+1},1} = \frac{\partial_1 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}}\Bigg|_{\underline{x}^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k),$$
$$\mathbf{v}^{k+1}_{p^{k+1},3} = -\frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k). \tag{3.65}$$

From (3.65),

$$\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} = \frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \cdot \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}}$$
$$+ \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}} \left(\frac{\partial_i \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}} \right)\Bigg|_{\underline{x}^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k),$$
$$\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{x}^k_{p^k,1}} = -\frac{\partial_3 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},33}(\underline{x}^{k+1})}} \cdot \frac{\partial \underline{v}^k}{\partial \mathbf{x}^k_{p^k,1}}$$
$$- \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{x}^k_{p^k,1}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}} \left(\frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}} \right)\Bigg|_{\underline{x}^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k).$$

From (3.64), we prove (3.55) and (3.56).

Now we consider (3.58)–(3.61) for v-derivatives.

Proof of (3.58). We take $\frac{\partial}{\partial \mathbf{v}^k_{p^k,j}}$ to (3.62) for $j = 1, 3$ to get

$$\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} \frac{\partial \underline{\eta}^{k+1}_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}\Bigg|_{\underline{x}^{k+1}} = -(t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{v}^k_{p^k,j}} - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \underline{v}^k, \tag{3.66}$$

and then take an inner product with $\frac{\partial_3 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{\underline{x}^{k+1}}$ to have

$$\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} \frac{\partial \underline{\eta}^{k+1}_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}\Bigg|_{\underline{x}^{k+1}} \cdot \frac{\partial_3 \underline{\eta}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}}$$
$$= \left\{ -(t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{v}^k_{p^k,j}} - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,1}} \lim_{s\downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k) \right\} \cdot \frac{\partial_3 \underline{\eta}_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{\underline{x}^{k+1}}. \tag{3.67}$$

Due to (3.50), the LHS equals zero. Now we consider the RHS. From (3.51),

$$\frac{\partial \underline{v}^k}{\partial \mathbf{v}^k_{p^k,j}} = \frac{\partial_j \underline{\eta}_{p^k}(\mathbf{x}^k_{p^k,1},0)}{\sqrt{g_{p^k,jj}(\mathbf{x}^k_{p^k,1},0)}}. \tag{3.68}$$

Using (3.65), (3.67) and (3.68), we prove (3.58).

Proof of (3.59). For $j = 1,3$, we take the inner product with $\frac{\partial_i \underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},ii}}\Big|_{\underline{x}^{k+1}}$ to (3.66) to have

$$\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} = \left\{ -\frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \lim_{s \downarrow t^{k+1}} \underline{V}(s;t^k,\underline{x}^k,\underline{v}^k) - (t^k - t^{k+1}) \frac{\partial \underline{v}^k}{\partial \mathbf{v}^k_{p^k,j}} \right\} \cdot \frac{\partial_1 \underline{\eta}^{k+1}_{p^{k+1}}}{g_{p^{k+1},11}}\Bigg|_{\underline{x}^{k+1}}.$$

From (3.68) and (3.58), we prove (3.59).

Proof of (3.60) and (3.61). For $j = 1,3$, from (3.65),

$$\begin{aligned}
\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} &= \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} \partial_1 \left(\frac{\partial_1 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}} \right)\Bigg|_{\underline{x}^{k+1}} \cdot \lim_{s \downarrow t^{k+1}} V(s;t^k,\underline{x}^k,\underline{v}^k) \\
&\quad + \frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \cdot \frac{\partial \underline{v}^k}{\partial \mathbf{v}^k_{p^k,j}} \\
&= \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},1}}{\partial \mathbf{v}^k_{p^k,j}} \partial_1 \left(\frac{\partial_1 \underline{\eta}^{k+1}_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}} \right)\Bigg|_{\underline{x}^{k+1}} \cdot \underline{v}^k + \frac{\partial_1 \underline{\eta}_{p^{k+1}}(\underline{x}^{k+1})}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \cdot \frac{\partial_j \underline{\eta}^k_{p^k}(\underline{x}^k)}{\sqrt{g_{p^k,jj}(\underline{x}^k)}}.
\end{aligned}$$

From (3.58) and (3.59), we prove (3.60). The proof of (3.61) is also very similar to the above, from (3.65). □

For the first bounce backward in time, we need similar results as to the previous lemma which connect the first backward bounce and the interior phase.

Lemma 13. *Assume that $x \in \Omega$ (not necessarily convex) and that $x_{\mathbf{b}}(t,x,v)$ is in the neighborhood of $p^1 \in \partial\Omega$. When $|\mathbf{v}^1_{p^1,3}| > 0$, locally, for $i, j = 1,3$,*

$$\frac{\partial t_{\mathbf{b}}}{\partial x_j} = -\frac{1}{\mathbf{v}^1_{p^1,3}} e_j \cdot \frac{\partial_3 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,33}(\underline{x}^1)}}, \tag{3.69}$$

$$\frac{\partial t_{\mathbf{b}}}{\partial v_j} = \frac{t_{\mathbf{b}}}{\mathbf{v}^1_{p^1,3}} e_j \cdot \frac{\partial_3 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,33}(\underline{x}^1)}}, \tag{3.70}$$

$$\frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_j} = e_j \cdot \frac{1}{\sqrt{g_{p^1,11}(\underline{x}^1)}} \left[\frac{\partial_1 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,11}(\underline{x}^1)}} + \frac{\mathbf{v}^1_{p^1,1}}{\mathbf{v}_{p^1,3}} \frac{\partial_3 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,33}(\underline{x}^1)}} \right], \tag{3.71}$$

$$\frac{\partial \mathbf{x}^1_{p^1,1}}{\partial v_j} = -t_{\mathbf{b}} e_j \cdot \frac{1}{\sqrt{g_{p^1,11}(\underline{x}^1)}} \left[\frac{\partial_1 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,11}(\underline{x}^1)}} + \frac{\mathbf{v}^1_{p^1,1}}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,33}(\underline{x}^1)}} \right], \tag{3.72}$$

$$\frac{\partial \mathbf{v}^1_{p^1,i}}{\partial x_j} = \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_j} \partial_1 \left(\frac{\partial_i \underline{\eta}_{p^1}}{\sqrt{g_{p^1,ii}}} \right)\Bigg|_{\underline{x}^1} \cdot V(t - t_{\mathbf{b}}) = \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_j} \partial_1 \left(\frac{\partial_i \underline{\eta}_{p^1}}{\sqrt{g_{p^1,ii}}} \right)\Bigg|_{\underline{x}^1} \cdot \underline{v}, \tag{3.73}$$

$$\frac{\partial \mathbf{v}^1_{p^1,i}}{\partial v_j} = \frac{\partial_i \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,ii}(\underline{x}^1)}} \cdot e_j + \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial v_j} \partial_1 \left(\frac{\partial_i \underline{\eta}_{p^1}}{\sqrt{g_{p^1,ii}}} \right)\Bigg|_{\underline{x}^1} \cdot \underline{v}. \tag{3.74}$$

Here, e_j is the jth directional standard unit vector in $\mathbb{R}^3$.

Moreover,

$$\frac{\partial |\underline{\mathbf{v}}^1_{p^1}|}{\partial x_j} = 0, \tag{3.75}$$

$$\frac{\partial |\underline{\mathbf{v}}^1_{p^1}|}{\partial v_j} = \lim_{s \downarrow t^1} \frac{V_j(s; t, \underline{x}, \underline{v})}{|\underline{V}(s; t, \underline{x}, \underline{v})|}. \tag{3.76}$$

Proof. We have

$$\lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}) = \underline{v}, \quad \underline{X}(\tau; t, \underline{x}, \underline{v}) = \underline{x} + \underline{v}(\tau - t). \tag{3.77}$$

In particular, when $\tau = t^1$, we get

$$\underline{X}(t^1; t, \underline{x}, \underline{v}) = \underline{x} + \underline{v}(t^1 - t). \tag{3.78}$$

From (3.77), we have

$$\lim_{s \downarrow t^1} \frac{\partial \underline{V}(s; t, \underline{x}, \underline{v})}{\partial x_j} = 0.$$

To prove (3.69)–(3.74), these estimates are very similar to those of Lemma 12. It suffices to choose global euclidean coordinates instead of $\underline{\eta}_{p^k}$. Therefore we should replace

$$\underline{\eta}_{p^{k+1}} \to \underline{\eta}_{p^1}, \quad \underline{\eta}_{p^k} \to \underline{x}, \quad t^k \to t, \quad t^{k+1} \to t - t_{\mathbf{b}} = t^1, \quad \partial_{x_j} \underline{x} = e_j.$$

Proof of (3.69). For $j = 1, 3$, we apply ∂x_j to (3.78) and take $\cdot \frac{\partial_3 \underline{\eta}_p^1}{\sqrt{g_{p^1,33}}}\Big|_{\underline{x}^1}$. In this case, we have $\frac{\partial \underline{v}}{\partial x_j} = 0$. Then we get

$$\frac{\partial t_{\mathbf{b}}}{\partial x_j} = -\frac{1}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,33}(\underline{x}^1)}} \cdot e_j.$$

Proof of (3.70). For $j = 1, 2$, we apply ∂v_j to (3.78) and take $\cdot \frac{\partial_3 \underline{\eta}_p^1}{\sqrt{g_{p^1,33}}}\Big|_{\underline{x}^1}$. Then we get

$$\begin{aligned} 0 &= \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial v_j} \frac{\partial \underline{\eta}_{p^1}}{\partial \mathbf{x}^k_{p^1,1}}\Bigg|_{\underline{x}^1} \cdot \frac{\partial_3 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,33}}}\Bigg|_{\underline{x}^1} \\ &= \left\{ -(t - t^1) e_j - \frac{\partial (t - t^1)}{\partial v_j} \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}) \right\} \cdot \frac{\partial_3 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,33}}}\Bigg|_{\underline{x}^1}. \end{aligned}$$

We use (3.77) to get (3.70).

Proof of (3.71). For $j = 1, 3$, we apply ∂x_j to (3.78) and take $\cdot \frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}}\Big|_{\underline{x}^1}$, and then

$$\frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_j} = \frac{1}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \underline{\eta}_{p^1}(\underline{x}^1)}{\sqrt{g_{p^1,33}(\underline{x}^1)}} \cdot e_j \; \frac{\mathbf{v}^1_{p^1,1}}{\sqrt{g_{p^1,11}}}\Bigg|_{\underline{x}^1} + \frac{\partial_1 \underline{\eta}_{p^1}}{g_{p^1,11}}\Bigg|_{\underline{x}^1} \cdot e_j.$$

This yields (3.71).

Proof of (3.72). For $j = 1, 3$, we apply ∂v_j to (3.78) and take $\cdot \frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}}\Big|_{\underline{x}^1}$, so we have

$$\frac{\partial \mathbf{x}^1_{p^1,1}}{\partial v_j} = \left\{ -\frac{\partial (t - t^1)}{\partial v_j} \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}) - (t - t^1) \frac{\partial \underline{v}}{\partial v_j} \right\} \cdot \frac{\partial_1 \underline{\eta}_{p^1}}{g_{p^1,11}}\Bigg|_{\underline{x}^1}.$$

Then we get (3.72).

Proof of (3.73). For $j = 1, 3$, we apply ∂x_j to

$$\begin{aligned} \mathbf{v}^1_{p^1,1} &= \frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}}\Bigg|_{\underline{x}^1} \cdot \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}), \\ \mathbf{v}^1_{p^1,3} &= -\frac{\partial_3 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,33}}}\Bigg|_{\underline{x}^1} \cdot \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}). \end{aligned} \tag{3.79}$$

From (3.65),

$$\frac{\partial \mathbf{v}^1_{p^1,1}}{\partial x_j} = \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_j} \frac{\partial}{\partial \mathbf{x}^1_{p^1,1}} \left(\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} \right)\Bigg|_{\underline{x}^1} \cdot \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}),$$

$$\frac{\partial \mathbf{v}^1_{p^1,3}}{\partial x_j} = - \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_j} \frac{\partial}{\partial \mathbf{x}^1_{p^1,1}} \left(\frac{\partial_3 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,33}}} \right)\Bigg|_{\underline{x}^1} \cdot \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}).$$

From (3.77), (3.71) and (3.69), we prove (3.58).

Proof of (3.74). In a fashion similar to the above, we apply ∂v_j to (3.79) and then use (3.77), (3.72) and (3.70). We skip the details.

Proof of (3.75). Since there is no external force, speed is constant, so the result is obvious.

Proof of (3.76). Note that $|\underline{\mathbf{v}}^1_{p^1}| = \lim_{s \downarrow t^1} |\underline{V}(s; t, \underline{x}, \underline{v})|$ and

$$2|\underline{\mathbf{v}}^1_{p^1}| \frac{\partial |\underline{\mathbf{v}}^1_{p^1}|}{\partial v_j} = 2 \lim_{s \downarrow t^1} \underline{V}(s; t, \underline{x}, \underline{v}) \cdot \lim_{s \downarrow t^1} \partial_{v_j} \underline{V}(s; t, \underline{x}, \underline{v}),$$

so we have

$$\frac{\partial |\underline{\mathbf{v}}^1_{p^1}|}{\partial v_j} = \lim_{s \downarrow t^1} \frac{\underline{V}(s; t, \underline{x}, \underline{v})}{|\underline{V}(s; t, \underline{x}, \underline{v})|} \cdot \lim_{s \downarrow t^1} \partial_{v_j} \underline{V}(s; t, \underline{x}, \underline{v}). \tag{3.80}$$

Since

$$\lim_{s \downarrow t^1} \frac{\partial \underline{V}(s; t, \underline{x}, \underline{v})}{\partial v_j} = e_j, \tag{3.81}$$

we combine (3.80), (3.81) and (3.70) to derive (3.76). □

Now we obtain the Jacobian between two bouncing phases when the trajectory is not grazing.

Lemma 14. *Assume Ω satisfies Definition 1 and $\frac{1}{N} \leqq |v| \leqq N$, for $1 \ll N$. We also assume $|t^k - t^{k+1}| \leqq 1$ and $|\mathbf{v}^k_{p^k,3}|,\ |\mathbf{v}^{k+1}_{p^{k+1},3}| > 0$. Then*

$$\left| \det \begin{bmatrix} \partial_{\mathbf{x}^k_{p^k,1}} \mathbf{x}^{k+1}_{p^{k+1},1} & \nabla_{\underline{\mathbf{v}}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1},1} \\ \partial_{\mathbf{x}^k_{p^k,1}} \underline{\mathbf{v}}^{k+1}_{p^{k+1}} & \nabla_{\underline{\mathbf{v}}^k_{p^k}} \underline{\mathbf{v}}^{k+1}_{p^{k+1}} \end{bmatrix}_{3\times 3} \right| = \frac{\sqrt{g_{p^k,11}(\underline{x}^k)}}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \frac{\left|\mathbf{v}^k_{p^k,3}\right|}{\left|\mathbf{v}^{k+1}_{p^{k+1},3}\right|}$$

for the mapping $(\mathbf{x}^k_{p^k,1}, \underline{\mathbf{v}}^k_{p^k}) \mapsto (\mathbf{x}^{k+1}_{p^{k+1},1}, \underline{\mathbf{v}}^k_{p^{k+1}})$.

Proof. We note that Lemma 12 holds for a nonconvex domain and the result is exactly the same as Lemma 26 in [17], without the external potential. Then a simplified two-dimensional version directly yields the above result. □

Using Lemma 14, we prove the lower bound of the Jacobian between the first bouncing phase and the general kth bouncing phase.

Lemma 15. *We define,*

$$\hat{\mathbf{v}}^k_{p^k,1} := \frac{\mathbf{v}^k_{p^k,1}}{\left|\underline{\mathbf{v}}^k_{p^k}\right|}, \quad \left|\underline{\mathbf{v}}^k_{p^k}\right| = \sqrt{\left(\mathbf{v}^k_{p^k,1}\right)^2 + \left(\mathbf{v}^k_{p^k,3}\right)^2},$$

where $\underline{\mathbf{v}}^k_{p^k} = \underline{\mathbf{v}}^k_{p^k}(t, \underline{x}, \underline{v})$ *are defined in* (3.52)*. Assume* $\frac{1}{N} \leqq |v| \leqq N$ *and* $|\mathbf{v}^k_{p^k,3}|$, $|\mathbf{v}^{k+1}_{p^{k+1},3}| > \delta_2 > 0$ *for* $1 \ll N$ *and* $k \lesssim_{\Omega,N,\delta_2} 1$*. If* $|t - t^k| \leqq 1$*, then*

$$\left|\det \begin{bmatrix} \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} \\ \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} \end{bmatrix}\right| > \varepsilon_{\Omega,N,\delta_2} > 0, \tag{3.82}$$

where $t^1 = t^1(t, x, v)$, $\mathbf{x}^1_{p^1,i} = \mathbf{x}^1_{p^1,i}(t, x, v)$, $\hat{\mathbf{v}}^1_{p^1,i} = \hat{\mathbf{v}}^1_{p^1,i}(t, x, v)$*, and*

$$\begin{aligned} \mathbf{x}^k_{p^k,i} &= \mathbf{x}^k_{p^k,i}\left(t^1, \mathbf{x}^1_{p^1,1}, \hat{\mathbf{v}}^1_{p^1,1}, |\underline{\mathbf{v}}^1_{p^1}|\right), \\ \hat{\mathbf{v}}^k_{p^k,i} &= \hat{\mathbf{v}}^k_{p^k,i}\left(t^1, \mathbf{x}^1_{p^1,1}, \hat{\mathbf{v}}^1_{p^1,1}, |\underline{\mathbf{v}}^1_{p^1}|\right). \end{aligned}$$

Here, the constant $\varepsilon_{\Omega,N,\delta_2} > 0$ *does not depend on* t *and* x*.*

Proof. *Step 1.* We compute

$$\begin{aligned} J^{i+1}_i &:= \frac{\partial\left(\mathbf{x}^{i+1}_{p^{i+1},1}, \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}, \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|\right)}{\partial\left(\mathbf{x}^i_{p^i,1}, \hat{\mathbf{v}}^i_{p^i,1}, \left|\underline{\mathbf{v}}^i_{p^i}\right|\right)} \\ &= \underbrace{\frac{\partial\left(\mathbf{x}^i_{p^i,1}, \underline{\mathbf{v}}^i_{p^i}\right)}{\partial\left(\mathbf{x}^i_{p^i,1}, \hat{\mathbf{v}}^i_{p^i,1}, |\underline{\mathbf{v}}^i_{p^i}|\right)}}_{=Q_i} \underbrace{\frac{\partial\left(\mathbf{x}^{i+1}_{p^{i+1},1}, \underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right)}{\partial\left(\mathbf{x}^i_{p^i,1}, \underline{\mathbf{v}}^i_{p^i}\right)}}_{=P_i} \underbrace{\frac{\partial\left(\mathbf{x}^{i+1}_{p^{i+1},1}, \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}, |\underline{\mathbf{v}}^{i+1}_{p^{i+1}}|\right)}{\partial\left(\mathbf{x}^{i+1}_{p^{i+1},1}, \underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right)}}_{=Q_{i+1}}. \end{aligned} \tag{3.83}$$

For Q_i,

$$Q_i = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial \hat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial \left|\underline{\mathbf{v}}^i_{p^i}\right|} \\ 0 & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \hat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \left|\underline{\mathbf{v}}^i_{p^i}\right|} \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & |\underline{\mathbf{v}}^i_{p^i}| & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial \left|\underline{\mathbf{v}}^i_{p^i}\right|} \\ 0 & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \hat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \left|\underline{\mathbf{v}}^i_{p^i}\right|} \end{array}\right].$$

For Q_{i+1},

$$Q_{i+1} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \\ 0 & \frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & |\underline{\mathbf{v}}^{i+1}_{p^{i+1}}|^{-1} & \frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \\ 0 & \frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \end{array}\right]. \tag{3.84}$$

Note that

$$\frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} = \mathbf{v}^{i+1}_{p^{i+1},1} \frac{\partial}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \left(\frac{1}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|} \right) = \frac{\mathbf{v}^{i+1}_{p^{i+1},1} \mathbf{v}^{i+1}_{p^{i+1},3}}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|^3}, \tag{3.85}$$

and for $k = 1, 3$,

$$\frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},k}} = -\frac{\mathbf{v}^{i+1}_{p^{i+1},k}}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}. \tag{3.86}$$

From (3.84), (3.85) and (3.86),

$$\det Q_{i+1} = \frac{1}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|} \left(-\frac{\mathbf{v}^{i+1}_{p^{i+1},3}}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|} \right) + \frac{\mathbf{v}^{i+1}_{p^{i+1},1} \mathbf{v}^{i+1}_{p^{i+1},3}}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|^3} \frac{\mathbf{v}^{i+1}_{p^{i+1},1}}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|} = -\frac{\left(\mathbf{v}^{i+1}_{p^{i+1},3}\right)^3}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|^4}. \tag{3.87}$$

By taking the inverse, we get

$$\det Q_i = -\frac{\left|\underline{\mathbf{v}}^{i}_{p^{i}}\right|^4}{\left(\mathbf{v}^{i}_{p^{i},3}\right)^3}. \tag{3.88}$$

From (3.83), (3.88), (3.87) and Lemma 14, we get

$$\left| \det \begin{bmatrix} \partial_{\mathbf{x}^k_{p^k,1}} \mathbf{x}^{k+1}_{p^{k+1},1} & \nabla_{\underline{\mathbf{v}}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1},1} \\ \partial_{\mathbf{x}^k_{p^k,1}} \underline{\mathbf{v}}^{k+1}_{p^{k+1}} & \nabla_{\underline{\mathbf{v}}^k_{p^k}} \underline{\mathbf{v}}^{k+1}_{p^{k+1}} \end{bmatrix}_{3\times 3} \right| = \frac{\sqrt{g_{p^k,11}(\underline{x}^k)}}{\sqrt{g_{p^{k+1},11}(\underline{x}^{k+1})}} \frac{\left|\mathbf{v}^k_{p^k,3}\right|}{\left|\mathbf{v}^{k+1}_{p^{k+1},3}\right|}.$$

Therefore,

$$\begin{aligned} |\det J^{i+1}_i| &= |\det Q_i \det P_i \det Q_{i+1}| \\ &= \frac{\left|\underline{\mathbf{v}}^{i}_{p^{i}}\right|^4}{\left(\mathbf{v}^{i}_{p^{i},3}\right)^3} \frac{\sqrt{g_{p^i,11}}\big|_{\underline{x}^i}}{\sqrt{g_{p^{i+1},11}}\big|_{\underline{x}^{i+1}}} \frac{\left|\mathbf{v}^{i}_{p^{i},3}\right|}{\left|\mathbf{v}^{i+1}_{p^{i+1},3}\right|} \frac{\left(\mathbf{v}^{i+1}_{p^{i+1},3}\right)^3}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|^4} \\ &= \frac{\sqrt{g_{p^i,11}}\big|_{\underline{x}^i}}{\sqrt{g_{p^{i+1},11}}\big|_{\underline{x}^{i+1}}} \frac{\left|\mathbf{v}^{i+1}_{p^{i+1},3}\right|^2}{\left|\mathbf{v}^{i}_{p^{i},3}\right|^2}, \end{aligned}$$

and we get

$$| \det J_1^k | = \frac{\sqrt{g_{p^1,11}}\big|_{\underline{x}^1}}{\sqrt{g_{p^k,11}}\big|_{\underline{x}^k}} \frac{\left|\mathbf{v}^k_{p^k,3}\right|^2}{\left|\mathbf{v}^1_{p^1,3}\right|^2}. \tag{3.89}$$

Step 2. From (3.64),

$$\begin{aligned} 2\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right| \frac{\partial |\underline{\mathbf{v}}^{i+1}_{p^{i+1}}|}{\partial \mathbf{v}^i_{p^i,n}} &= \frac{\partial |\underline{V}(t^{i+1}; t^i, \underline{x}^i, \underline{v}^i)|^2}{\partial \mathbf{v}^i_{p^i,n}} = 2\frac{\partial \underline{V}(t^{i+1}; t^i, \underline{x}^i, \underline{v}^i)}{\partial \mathbf{v}^i_{p^i,n}} \\ &\quad \cdot \underline{V}(t^{i+1}; t^i, \underline{x}^i, \underline{v}^i) \\ &= 2 \left.\frac{\partial_n \underline{\eta}_{p^i}}{\sqrt{g_{p^i,nn}}}\right|_{\underline{x}^i} \cdot \underline{V}(t^{i+1}; t^i, \underline{x}^i, \underline{v}^i) = 2\mathbf{v}^i_{p^i,n}. \end{aligned}$$

Therefore, we get

$$\frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^i_{p^i,n}} = \frac{\mathbf{v}^i_{p^i,n}}{\left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}, \quad \text{for } n = 1, 3. \tag{3.90}$$

Since speed is conserved, for $n = 1, 3$,

$$\frac{\partial \left|\underline{\mathbf{v}}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{x}^i_{p^i,n}} = 0, \quad \text{for } n = 1, 3. \tag{3.91}$$

Also, by conservation,

$$\frac{\partial |\underline{\mathbf{v}}^{i+1}_{p^{i+1}}|}{\partial \left|\underline{\mathbf{v}}^i_{p^i}\right|} = 1. \tag{3.92}$$

Step 3. From (3.89), (3.90), (3.91) and (3.92),

$$\begin{aligned} | \det J_1^k | &= \frac{\sqrt{g_{p^1,11}}\big|_{\underline{x}^1}}{\sqrt{g_{p^k,11}}\big|_{\underline{x}^k}} \frac{\left|\mathbf{v}^k_{p^k,3}\right|^2}{\left|\mathbf{v}^1_{p^1,3}\right|^2} \\ &= \left| \det \begin{bmatrix} \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} & 0 \\ \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \right| = \left| \det \left[\frac{\partial \left(\mathbf{x}^k_{p^k,1}, \hat{\mathbf{v}}^k_{p^k,1}\right)}{\partial \left(\mathbf{x}^1_{p^1,1}, \hat{\mathbf{v}}^1_{p^1,1}\right)} \right]_{2\times 2} \right|. \end{aligned}$$

Therefore, we conclude (3.82), by (3.89). □

Now we study the lower bound of $\det\left(\frac{\mathrm{d}\underline{X}}{\mathrm{d}\underline{v}}\right)$. Instead of Euclidean variable $\underline{v} = (v_1, v_3)$, we introduce new variables via geometric decomposition. In a two-dimensional cross section, we split velocity $\underline{v}$ into speed and direction:

$$|\underline{v}| \quad \text{and} \quad \hat{v}_1 := \frac{v_1}{|\underline{v}|}.$$

Note that $\{\partial_{|v|}, \partial_{\hat{v}_1}\}$ are independent if $v_3 \geqq \frac{1}{N} > 0$. Thus, under the assumption of $v_3 \geqq \frac{1}{N} > 0$, we perform $\partial_{|v|}, \partial_{\hat{v}_1}$, instead of $\partial_{x_1}, \partial_{x_2}$. We assume $\frac{1}{N} \leqq |\underline{v}| \leqq N$, $t^{k+1}(t, \underline{x}, \underline{v}) < s < t^k(t, \underline{x}, \underline{v})$, and $|\mathbf{v}^i_{p^i,3}| > \delta_2 > 0$ (nongrazing) for $1 \leqq \forall i \leqq k$. When we differentiate $\underline{X}$ by speed $|\underline{v}|$, we have

$$\begin{aligned}
\partial_{|\underline{v}|}\underline{X}(s; t, \underline{x}, \underline{v}) &= \partial_{|\underline{v}|}\Big(\underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1}, 0\right) - (t^k - s)||\underline{v}||\hat{v}^k\Big), \quad \hat{v}^k := \frac{\underline{v}^k}{|\underline{v}|} \\
&= \partial_{|\underline{v}|}\mathbf{x}^k_{p^k,1}\partial_1\underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1}, 0\right) + \partial_{|\underline{v}|}\big[(t - t^k)|\underline{v}^k|\big]\hat{v}^k \\
&\quad - (t - s)\big[\partial_{|\underline{v}|}|\underline{v}^k|\big]\hat{v}^k - (t^k - s)|\underline{v}^k|\partial_{|\underline{v}|}\hat{v}^k = -(t - s)\hat{v}^k,
\end{aligned} \tag{3.93}$$

where we used $\partial_{|\underline{v}|}\mathbf{x}^k_{p^k,1} = 0$, $\partial_{|\underline{v}|}\big[(t - t^k)|\underline{v}^k|\big] = 0$, $\partial_{|\underline{v}|}\hat{v}^k = 0$, and $\partial_{|\underline{v}|}|\underline{v}^k| = 1$. Note that this is because the bouncing position x^k, the travel length until x^k, and the direction of v^k are independent of $|\underline{v}|$.

On the other hand, differentiating $\underline{X}$ by $\hat{v}_1$,

$$\partial_{\hat{v}_1}[\underline{X}(s; t, \underline{x}, \underline{v})] = \partial_{\hat{v}_1}\mathbf{x}^k_{p^k,1}\partial_1\underline{\eta}_{p^k}\left(\mathbf{x}^k_{p^k,1}, 0\right) - \partial_{\hat{v}_1}t^k\left|\underline{\mathbf{v}}^k_{p^k}\right|\underline{\hat{v}}^k - (t^k - s)\left|\underline{\mathbf{v}^k_{p^k}}\right|\partial_{\hat{v}_1}\underline{\hat{v}}^k. \tag{3.94}$$

To compute the last term $\partial_{\hat{v}_1}\underline{\hat{v}}^k$, we use $\hat{\mathbf{v}}^k_{p^k,3} = \sqrt{1 - |\hat{\mathbf{v}}^k_{p^k,1}|^2}$ and $|\hat{\mathbf{v}}^k_{p^k,3}| > 0$ to get

$$\begin{aligned}
\partial_{\hat{v}_1}\underline{\hat{v}}^k &= \partial_{\hat{v}_1}\left[\frac{\partial_1\underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}}\left(\mathbf{x}^k_{p^k,1}, 0\right)\hat{\mathbf{v}}^k_{p^k,1} + \frac{\partial_3\underline{\eta}_{p^k}}{\sqrt{g_{p^k,33}}}\left(\mathbf{x}^k_{p^k,1}, 0\right)\sqrt{1 - |\hat{\mathbf{v}}^k_{p^k,1}|^2}\right] \\
&= \partial_{\hat{v}_1}\mathbf{x}^k_{p^k,1}\sum_{\ell=1,3}\partial_1\left[\frac{\partial_\ell\underline{\eta}_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\left(\mathbf{x}^k_{p^k,1}, 0\right)\hat{\mathbf{v}}^k_{p^k,\ell} + \frac{\partial_1\underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}}\left(\mathbf{x}^k_{p,1}, 0\right)\partial_{\hat{v}_1}\hat{\mathbf{v}}^k_{p^k,1} \\
&\quad - \frac{\partial_3\underline{\eta}_{p^k}}{\sqrt{g_{p^k,33}}}\left(\mathbf{x}^k_{p^k,1}, 0\right)\frac{1}{\sqrt{1 - |\hat{\mathbf{v}}^k_{p^k,1}|^2}}\left[\hat{\mathbf{v}}^k_{p^k,1}\partial_{\hat{v}_1}[\hat{\mathbf{v}}^k_{p^k,1}]\right] \\
&= \left(\sum_{\ell=1,3}\frac{\partial}{\partial\mathbf{x}^k_{p^k,1}}\left[\frac{\partial_\ell\underline{\eta}_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\left(\mathbf{x}^k_{p^k,1}, 0\right)\hat{\mathbf{v}}^k_{p^k,\ell}\right)\partial_{\hat{v}_1}\mathbf{x}^k_{p^k,1} \\
&\quad + \left[\frac{\partial_1\underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}}\left(\mathbf{x}^k_{p^k,1}, 0\right) - \frac{\partial_3\underline{\eta}_{p^k}}{\sqrt{g_{p^k,33}}}\left(\mathbf{x}^k_{p^k,1}, 0\right)\frac{\hat{\mathbf{v}}^k_{p^k,1}}{\hat{\mathbf{v}}^k_{p^k,3}}\right]\partial_{\hat{v}_1}\left[\hat{\mathbf{v}}^k_{p^k,1}\right].
\end{aligned} \tag{3.95}$$

Combining (3.94) and (3.95), we get

$$\begin{aligned}\partial_{\hat{v}_1}\left[\underline{X}(s;t,\underline{x},\underline{v})\right] = &-\left(\partial_{\hat{v}_1}t^k\right)v^k + \partial_{\hat{v}_1}\mathbf{x}^k_{p^k,1}\partial_1\underline{\eta}_{p^k}(\mathbf{x}^k_{p^k,1},0)\\ &-(t^k-s)\left|\underline{\mathbf{v}}^k_{p^k}\right|\left(\sum_{\ell=1,3}\frac{\partial}{\partial\mathbf{x}^k_{p^k,1}}\left[\frac{\partial_\ell\underline{\eta}_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\left(\mathbf{x}^k_{p^k,1},0\right)\hat{\mathbf{v}}^k_{p^k,\ell}\right)\partial_{\hat{v}_1}\mathbf{x}^k_{p^k,1}\\ &-(t^k-s)\left|\underline{\mathbf{v}}^k_{p^k}\right|\left[\frac{\partial_1\underline{\eta}_{p^k,i}}{\sqrt{g_{p^k,11}}}\left(\mathbf{x}^k_{p^k,1},0\right)-\frac{\partial_3\underline{\eta}_{p^k,i}}{\sqrt{g_{p^k,33}}}\left(\mathbf{x}^k_{p^k,1},0\right)\frac{\hat{\mathbf{v}}^k_{p^k,1}}{\hat{\mathbf{v}}^k_{p^k,3}}\right]\partial_{\hat{v}_1}\left[\hat{\mathbf{v}}^k_{p^k,1}\right].\end{aligned} \tag{3.96}$$

Definition 10. (*Specular Basis and Matrix*) Recall the specular cycles $(t^k, \underline{x}^k, \underline{v}^k)$ in (2.4). Assume

$$\mathbf{n}(\underline{x}^k)\cdot\underline{v}^k \neq 0. \tag{3.97}$$

Recall $\underline{\eta}_{p^k}$ in (3.49). We define the *specular basis*, which is an orthonormal basis of $\mathbb{R}^2$, as

$$\mathbf{e}^k_0 := \frac{\underline{v}^k}{|\underline{v}^k|} = \frac{1}{|\underline{v}^k|}\left(v^k_1, v^k_3\right), \quad \mathbf{e}^k_{\perp,1} := \frac{1}{|\underline{v}^k|}\left(v^k_3, -v^k_1, 0\right). \tag{3.98}$$

Also, for fixed $k \in \mathbb{N}$, assume (3.97) with $\underline{x}^k = \underline{x}^k(t,\underline{x},|\underline{v}|,\hat{v}_1)$ and $\underline{v}^k = \underline{v}^k(t,\underline{x},|\underline{v}|,\hat{v}_1)$. We define the 2×2 specular transition matrix $\mathcal{S}^{k,p^k} = \mathcal{S}^{k,p^k}(t,\underline{x},|\underline{v}|,\hat{v}_1)$ as

$$\mathcal{S}^{k,p^k} := \begin{bmatrix}\mathcal{S}^{k,p^k}_1 & 0\\ \mathcal{S}^{k,p^k}_2 & \mathcal{S}^{k,p^k}_3\end{bmatrix}_{2\times 2}, \tag{3.99}$$

where

$$\begin{aligned}\mathcal{S}^{k,p^k}_1 &:= \partial_1\underline{\eta}_{p^k}\cdot\mathbf{e}^k_{\perp,1},\\ \mathcal{S}^{k,p^k}_2 &:= \left(\sum_{\ell=1}^{3}\partial_1\left[\frac{\partial_\ell\underline{\eta}_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\hat{\mathbf{v}}^k_{p^k,\ell}\right)\cdot\mathbf{e}^k_{\perp,1},\\ \mathcal{S}^{k,p^k}_3 &:= \left[\frac{\partial_1\underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}} - \frac{\partial_3\underline{\eta}_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}^k_{p^k,1}}{\hat{\mathbf{v}}^k_{p^k,3}}\right]\cdot\mathbf{e}^k_{\perp,1},\end{aligned}$$

and where η_{p^k} and g_{p^k} are evaluated at $\underline{x}^k(t,\underline{x},|\underline{v}|,\hat{v}_1)$. We also define

$$\begin{bmatrix}\mathcal{R}^{k,p^k}_1\\ \mathcal{R}^{k,p^k}_2\end{bmatrix} := \mathcal{S}^{k,p^k}\begin{bmatrix}\frac{\partial\mathbf{x}^k_{p^k,1}}{\partial\hat{v}_1}\\ \frac{\partial\hat{\mathbf{v}}^k_{p^k,1}}{\partial\hat{v}_1}\end{bmatrix}, \tag{3.100}$$

where $\underline{\mathbf{x}}^k_{p^k} = \underline{\mathbf{x}}^k_{p^k}(t,\underline{x},|\underline{v}|,\hat{v}_1)$ and $\underline{\mathbf{v}}^k_{p^k} = \underline{\mathbf{v}}^k_{p^k}(t,\underline{x},|\underline{v}|,\hat{v}_1)$.

The following Lemmas (16 and 17) are necessary to prove uniform non-degeneracy in the two dimensional cross section Ω (Lemma 18).

Lemma 16. *Fix $k \in \mathbb{N}$ with $|t - t^k| \leqq 1$. Assume $\frac{1}{N} \leqq |\underline{v}| \leqq N$ and $\frac{1}{N} \leqq |v_3|$, for $N \gg 1$. We also assume the non-grazing condition*

$$\left|\mathbf{v}^i_{p^i,3}\right| = |\underline{v}^i(t, \underline{x}, \underline{v}) \cdot \mathbf{n}(\underline{x}^i(t, \underline{x}, \underline{v}))| > \delta_2 > 0, \quad \forall 1 \leqq i \leqq k, \tag{3.101}$$

and

$$\left| \frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} \cdot e_1 \right| > \frac{1}{N} > 0 \tag{3.102}$$

for some uniform $\delta_2 > 0$. Then there exists at least one $i \in \{1, 2\}$ such that

$$\left|\mathcal{R}^{k,p^k}_i(t, x, v)\right| > \varrho_{\Omega,N,\delta_2} \tag{3.103}$$

for some constant $\varrho_{\Omega,N,\delta_2} > 0$.

Proof. First we claim that

$$|\det(\mathcal{S}^{k,p^k})| > \varrho_{\Omega,N,\delta_2}.$$

It suffices to compute diagonal entries. From (3.98),

$$\begin{aligned}
|\mathcal{S}^{k,p^k}_1| &= \left| \sqrt{g_{p^k,11}} \frac{\partial_1 \underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}} \cdot \frac{\underline{v}^k_\perp}{|\underline{v}^k|} \right| \\
&= \left| \sqrt{g_{p^k,11}} \frac{1}{|\underline{v}^k|} \left(\frac{\partial_1 \underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}} \cdot \underline{v}^k_\perp \right) \right| = \sqrt{g_{p^k,11}} \frac{\left|\mathbf{v}^k_{p^k,3}\right|}{|\underline{v}^k|}
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{S}^{k,p^k}_3| &= \left| \left[\frac{\partial_1 \underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}} - \frac{\partial_3 \underline{\eta}_{p^k}}{\sqrt{g_{p^k,33}}} \frac{\hat{\mathbf{v}}^k_{p^k,1}}{\hat{\mathbf{v}}^k_{p^k,3}} \right] \cdot \mathbf{e}^k_{\perp,1} \right| \\
&= \left| \frac{1}{\hat{\mathbf{v}}^k_{p^k,3}} \left[\hat{\mathbf{v}}^k_{p^k,3} \frac{\partial_1 \underline{\eta}_{p^k}}{\sqrt{g_{p^k,11}}} - \hat{\mathbf{v}}^k_{p^k,1} \frac{\partial_3 \underline{\eta}_{p^k}}{\sqrt{g_{p^k,33}}} \right] \cdot \mathbf{e}^k_{\perp,1} \right| = \frac{\left|\underline{\mathbf{v}}^k_{p^k}\right|}{\left|\mathbf{v}^k_{p^k,3}\right|},
\end{aligned}$$

which implies uniform invertibility of 2×2 matrix $\mathcal{S}^{k,p^k}$. To consider a 2×1 vector on the RHS of (3.100), we compute

$$\begin{aligned}
\begin{bmatrix} \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial x_1} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \hat{v}_1} \\ \frac{\partial \mathbf{v}^k_{p^k,1}}{\partial x_1} & \frac{\partial \mathbf{v}^k_{p^k,1}}{\partial \hat{v}_1} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{v}^1_{p^1,1}} \\ \frac{\partial \mathbf{v}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{v}^k_{p^k,1}}{\partial \mathbf{v}^1_{p^1,1}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_1} & \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial \hat{v}_1} \\ \frac{\partial \mathbf{v}^1_{p^1,1}}{\partial x_1} & \frac{\partial \mathbf{v}^1_{p^1,1}}{\partial \hat{v}_1} \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{v}^1_{p^1,1}} \\ \frac{\partial \mathbf{v}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{v}^k_{p^k,1}}{\partial \mathbf{v}^1_{p^1,1}} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_1} & \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial \hat{v}_1} \\ \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_1} \partial_1 \left(\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} \right) \cdot \underline{v} & |v| \left(\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} \right) \cdot e_1 + \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial \hat{v}_1} \partial_1 \left(\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} \right) \cdot \underline{v} \end{bmatrix}}_{B},
\end{aligned} \tag{3.104}$$

where we used (3.73) and (3.74). The determinant of A is uniformly nonzero from (3.82) in Lemma 15. From the elementary row operation for B,

$$\det B = \det \begin{bmatrix} \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_1} & \frac{\partial \mathbf{x}^1_{p^1,1}}{\partial \hat{v}_1} \\ 0 & |\underline{v}|\Big(\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}}\Big) \cdot e_1 \end{bmatrix}.$$

From (3.71), the (1, 1) entry of matrix B is computed by

$$\begin{aligned} \left|\frac{\partial \mathbf{x}^1_{p^1,1}}{\partial x_1}\right| &= \left| \frac{e_1}{\sqrt{g_{p^1,11}}} \cdot \left[\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} + \frac{\mathbf{v}^1_{p^1,1}}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,33}}} \right] \right| \\ &= \left| \frac{1}{\sqrt{g_{p^1,11}}} \frac{1}{|\mathbf{v}^1_{p^1,3}|} \frac{e_1}{\sqrt{g_{p^1,11}}} \cdot \left(\mathbf{v}^1_{p^1,3} \frac{\partial_1 \eta_{p^1}}{\sqrt{g_{p^1,11}}} + \mathbf{v}^1_{p^1,1} \frac{\partial_3 \eta_{p^1}}{\sqrt{g_{p^1,33}}} \right) \right| \\ &= \left| \frac{1}{\sqrt{g_{p^1,11}}} \frac{1}{|\mathbf{v}^1_{p^1,3}|} \frac{e_1}{\sqrt{g_{p^1,11}}} \cdot (v_1 e_3 + v_3 e_1) \right| = \frac{1}{g_{p^1,11}(\underline{x}^1)} \frac{|v_3|}{\left|\mathbf{v}^1_{p^1,3}\right|}. \end{aligned}$$

Therefore, from (3.101) and (3.102), the determinant of B is uniformly nonzero and thus the LHS of (3.104) also has a uniformly nonzero determinant. This yields uniform nonzeroness of the second column, that is, $\begin{bmatrix} \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \hat{v}_1} \\ \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \hat{v}_1} \end{bmatrix}$. From the uniform invertibility of matrix $\mathcal{S}^{k,p^k}$ and (3.100), we finish the proof. $\square$

Lemma 17. *Assume that $b(z)$, $c(z)$ are continuous-functions of $z \in \mathbb{R}^n$ locally. We consider $G(z,s) := b(z)s + c(z)$.*

(i) Assume $\min|b| > 0$. *Define*

$$\varphi_1(z) := \frac{-c(z)}{b(z)}. \tag{3.105}$$

Then $\varphi_1(z) \in C^1_{t,x,v}$ with $\|\varphi_1\|_{C^1_{t,x,v}} \leqq C(\min|b|, \|b\|_{C^1_{t,x,v}}, \|c\|_{C^1_{t,x,v}})$. Moreover, if $|s| \leqq 1$ and $|s - \varphi_1(z)| > \delta$, then $|G(z,s)| \gtrsim \min|b| \times \delta$.

(ii) Assume $\min|c| > 0$. *Define*

$$\varphi_2(z) := \mathbf{1}_{|b(z)| > \frac{\min|c|}{4}} \frac{-c(z)}{b(z)}. \tag{3.106}$$

Then $\varphi_2(z) \in C^1_{t,x,v}$ with $\|\varphi_2\|_{C^1_{t,x,v}} \leqq C(\min|b|, \|b\|_{C^1_{t,x,v}}, \|c\|_{C^1_{t,x,v}})$. Moreover, if $|s| \leqq 1$ and $|s - \varphi_5(z)| > \delta$, then $|G(z,s)| \geqq \min\left\{\frac{\min|c|}{2}, \frac{\min|c|}{4} \times \delta\right\}$.

Proof. Now we consider (i). Clearly φ_1 is C for this case, and

$$\begin{aligned}|G(z,s)| &\geqq \min\left\{\left|b(z)\left(\frac{-c(z)}{b(z)}+\delta\right)+c(z)\right|, \left|b(z)\left(\frac{-c(z)}{b(z)}\delta\right)+c(z)\right|\right\}\\ &\geqq \min|b|\times\delta.\end{aligned}$$

Now we consider (ii). First, if $|b| < \frac{\min|c|}{2}$, then $|\varphi_2(z)| \geqq \frac{|c(z)|}{\min|c|/2} \geqq 2$. Therefore,

$$|G(z,s)| \geqq \min\{|G(z,1)|, |G(z,-1)|\} \geqq |c(z)| - |b(z)| \geqq \frac{\min|c|}{2}.$$

Consider the case of $|b| > \frac{\min|c|}{4}$. If $|s-\varphi_2(s)| > \delta$, then

$$\begin{aligned}|G(z,s)| &\geqq \min\left\{\left|b(z)\left(\frac{-c(z)}{b(z)}+\delta\right)+c(z)\right|, \left|b(z)\left(\frac{-c(z)}{b(z)}-\delta\right)+c(z)\right|\right\}\\ &= \min|b|\times\delta \geqq \frac{\min|c|}{2}\times\delta.\end{aligned}$$

□

Now we obtain uniform non-degeneracy in a two dimensional cross section Ω away from small sets.

Lemma 18. *Fix $k\in\mathbb{N}$ with $t^k \geqq t-1$. Assume that Ω is C^2 and (3.49). Let $t^0\geqq 0$, $\underline{x}^0\in\bar{\Omega}$, $\underline{v}^0\in\mathbb{R}^2$, and assume*

$$\frac{1}{N}\leqq|\underline{v}^0|\leqq N,\ \ \frac{1}{N}\leqq|v_3^0|,\ \ \left|\mathbf{v}^i_{p^i,3}\right|>\delta_2>0,\quad \forall 1\leqq i\leqq k, \tag{3.107}$$

and we have (3.102) in Lemma 16, where $(\underline{x}^1,\underline{v}^1) = \big(\underline{x}^1(t^0,\underline{x}^0,\underline{v}^0), \underline{v}^1(t^0,\underline{x}^0,\underline{v}^0)\big)$. Then there exists $\varepsilon>0$ and $C^1_{t,\underline{x},\underline{v}}$-functions $\psi^k_1,\ \psi^k_2 : B_\varepsilon(t,\underline{x},\underline{v})\to\mathbb{R}$ with $\max_{i=1,2}\|\psi^k_i\|_{C^1_{t,\underline{x},\underline{v}}}\lesssim_{\delta_2,\Omega,N} 1$ *and there exists a constant $\varepsilon_{\delta_2,\Omega,N}>0$, such that*

$$\begin{aligned}&\text{if}\ \min_{i=1,2}|s-\psi^k_i(t,\underline{x},\underline{v})|>\delta_*\\ &\quad\text{and}\ (s;t,\underline{x},\underline{v})\in\left[\max\{t-1,t^{k+1}\},\min\left\{t-\frac{1}{N},t^k\right\}\right]\times B_\varepsilon(t^0,\underline{x}^0,\underline{v}^0),\\ &\text{then}\ \left|\partial_{|\underline{v}|}\underline{X}(s;t,\underline{x},\underline{v})\times\partial_{\hat{v}_1}\underline{X}(s;t,\underline{x},\underline{v})\right|>\varepsilon_{\delta_2,\Omega,N,\delta_*}.\end{aligned}$$

It is important that this lower bound $\varepsilon_{\delta_2,\Omega,N}$ does not depend on time t.

Proof. *Step 1.* Fix k with $|t^k(t,\underline{x},\underline{v})-t|\leqq 1$. Then we fix the orthonormal basis $\{\mathbf{e}^k_0,\mathbf{e}^k_{\perp,1}\}$ of (3.98) with $\underline{x}^k=\underline{x}^k(t,\underline{x},\underline{v})$, $\underline{v}^k=\underline{v}^k(t,\underline{x},\underline{v})$. Note that this orthonormal basis $\{\mathbf{e}^k_0,\mathbf{e}^k_{\perp,1}\}$ depends on $(t,\underline{x},\underline{v})$.

For $t^{k+1}<s<t^k$, recall the forms of $\frac{\partial\underline{X}(s)}{\partial|\underline{v}|}$ and $\frac{\partial\underline{X}(s)}{\partial\hat{v}_j}$ in (3.93) and (3.96), where $\underline{X}(s)=\underline{X}(s;t^k,\underline{x}^k,\underline{v}^k)$. Using the *specular basis* (3.98), we rewrite (3.93) and (3.96) as

$$\begin{bmatrix}\frac{\partial\underline{X}(s)}{\partial|\underline{v}|}\cdot\mathbf{e}^k_0 & \frac{\partial\underline{X}(s)}{\partial\hat{v}_1}\cdot\mathbf{e}^k_0\\ \frac{\partial\underline{X}(s)}{\partial|\underline{v}|}\cdot\mathbf{e}^k_{\perp,1} & \frac{\partial\underline{X}(s)}{\partial\hat{v}_1}\cdot\mathbf{e}^k_{\perp,1}\end{bmatrix} = \begin{bmatrix}-(t-s) & \frac{\partial\underline{X}(s)}{\partial\hat{v}_1}\cdot\mathbf{e}^k_0\\ 0 & \frac{\partial\underline{X}(s)}{\partial\hat{v}_1}\cdot\mathbf{e}^k_{\perp,1}\end{bmatrix}.$$

Note that the (2, 2) component is written by

$$\frac{\partial \underline{X}(s)}{\partial \hat{v}_1} \cdot \mathbf{e}^k_{\perp,1} = \mathcal{R}^{k,p^k}_1 - (t^k - s)\mathcal{R}^{k,p^k}_2,$$

by (3.96) and (3.99), where $\mathcal{R}^{k,p^k}_i$ are defined in (3.100). By direct computation, the determinant becomes,

$$\partial_{|\underline{v}|}\underline{X}(s) \times \partial_{\hat{v}_1} X(s) = -(t-s)\left\{\mathcal{R}^{k,p^k}_1 - (t^k - s)|\underline{\mathbf{v}}^k_{p^k}|\mathcal{R}^{k,p^k}_2\right\}. \tag{3.108}$$

Here $\mathcal{R}^{k,p^k}_i$, t^k, $\underline{\mathbf{v}}^k_{p^k}$ and $\mathbf{e}^k_{\perp,i}$ depend on $(t, \underline{x}, \underline{v})$, but not s.

Step 2. Recall Lemma 16. From (3.107), we can choose non-zero constants δ_2 for a large $N \gg 1$. Applying Lemma 16 and (3.103), we conclude that, for some $i \in \{1, 2\}$,

$$\left|\mathcal{R}^{k,p^k}_i(t, \underline{x}, \underline{v})\right| > \varrho_{\Omega,N,\delta_2} > 0. \tag{3.109}$$

Also, we can claim that $\mathcal{R}^{k,p^k}_i(t, \underline{x}, \underline{v}) \in C^1_{t,\underline{x},\underline{v}}$. From (3.107), all bouncings are non-grazing. We use Lemma 5, (3.72) and (3.74) in Lemma 13, and (3.100) with the regularity of Ω to derive $\mathcal{R}^{k,p^k}_i(t, \underline{x}, \underline{v}) \in C^1_{t,\underline{x},\underline{v}}$. Finally we choose a small constant $\varepsilon > 0$ such that, for some $i \in \{1, 2\}$ satisfying (3.109),

$$\left|\mathcal{R}^{k,p^k}_i(t, \underline{x}, \underline{v})\right| > \frac{\varrho_{\Omega,N,\delta_2}}{2} \quad \text{for } |(t, \underline{x}, \underline{v}) - (t^0, \underline{x}^0, \underline{v}^0)| < \varepsilon. \tag{3.110}$$

Step 3. With $N \gg 1$, from (3.110), we divide the cases into the following:

$$\left|\mathcal{R}^{k,p^k}_1\right| > \frac{\varrho_{\Omega,N,\delta_2}}{2} \quad \text{and} \quad \left|\mathcal{R}^{k,p^k}_2\right| \geqq \frac{\varrho_{\Omega,N,\delta_2}}{2}. \tag{3.111}$$

We split the first case (3.111) further into two cases:

$$\left|\mathcal{R}^{k,p^k}_1\right| > \frac{\varrho_{\Omega,N,\delta_2}}{2} \quad \text{and} \quad \left|\mathcal{R}^{k,p^k}_2\right| < \frac{\varrho_{\Omega,N,\delta_2}}{4N} \tag{3.112}$$

and

$$\left|\mathcal{R}^{k,p^k}_1\right| > \frac{\varrho_{\Omega,N,\delta_2}}{2} \quad \text{and} \quad \left|\mathcal{R}^{k,p^k}_2\right| \geqq \frac{\varrho_{\Omega,N,\delta_2}}{4N}.$$

Set the other case

$$\left|\mathcal{R}^{k,p^k}_2\right| \geqq \frac{\varrho_{\Omega,N,\delta_2}}{2}. \tag{3.113}$$

Then clearly (3.112) and (3.113) cover all the cases.

Step 4. We consider the case of (3.112). From (3.108),

$$\begin{aligned} |\partial_{|\underline{v}|}\underline{X}(s) \times \partial_{\hat{v}_1}\underline{X}(s)| &\geqq \left| |\underline{v}^k|\mathcal{R}^{k,p^k}_2(t^k - s) - \mathcal{R}^{k,p^k}_1 \right| (t-s) \\ &= \left| \underbrace{|\underline{v}^k|\mathcal{R}^{k,p^k}_{2,2}(t-s) + \left[-\mathcal{R}^{k,p^k}_1 + (t^k - t)|\underline{v}^k|\mathcal{R}^{k,p^k}_2\right]} \right| (t-s). \end{aligned} \tag{3.114}$$

We define

$$\tilde{s} = t - s, \tag{3.115}$$

and set

$$b := |\underline{v}^k|\mathcal{R}_2^{k,p^k} \quad \text{and} \quad c := -\mathcal{R}_1^{k,p^k} + (t^k - t)|\underline{v}^k|\mathcal{R}_2^{k,p^k}.$$

Note that $\mathcal{R}_1^{k,p^k}$, $\mathcal{R}_2^{k,p^k}$, $|\underline{v}^k|$ and t^k only depend on $(t, \underline{x}, \underline{v})$:

Hence we regard the underbraced term of (3.114) as an affine function of $\tilde{s}$:

$$b(t, \underline{x}, \underline{v})\tilde{s} + c(t, \underline{x}, \underline{v}).$$

Note that from (3.112),

$$|c(t, \underline{x}, \underline{v})| \geqq \frac{\varrho_{\Omega,N,\delta_2}}{2} - N\frac{\varrho_{\Omega,N,\delta_2}}{4N} \geqq \frac{\varrho_{\Omega,N,\delta_2}}{4}.$$

Now we apply (ii) of Lemma 17. With $\varphi_2(t, \underline{x}, \underline{v})$ in (3.106), if $|\tilde{s} - \varphi_2(t, \underline{x}, \underline{v})| > \delta_*$, then $|b(t, \underline{x}, \underline{v})\tilde{s} + c(t, \underline{x}, \underline{v})| \geqq \frac{\varrho_{\Omega,N,\delta}}{4} \times \delta_*$. We set

$$\psi_2(t, \underline{x}, \underline{v}) = t - \varphi_2(t, \underline{x}, \underline{v}).$$

From (3.115),

$$\text{if } |s - \psi_2(t, \underline{x}, \underline{v})| > \delta_*, \text{ then } |b(t, \underline{x}, \underline{v})(t - s) + c(t, \underline{x}, \underline{v})| \geqq \frac{\varrho_{\Omega,N,\delta_2}}{4} \times \delta_*. \tag{3.116}$$

Now we consider the case of (3.113). From (3.108),

$$|\partial_{|\underline{v}|}\underline{X}(s) \times \partial_{\hat{v}_1} X(s)| \geqq \left| |\underline{v}^k|\mathcal{R}_2^{k,p^k}(t-s) + \left[-\mathcal{R}_1^{k,p^k} + (t^k - t)|\underline{v}^k|\mathcal{R}_2^{k,p^k}\right]\right| (t - s). \tag{3.117}$$

We set $\tilde{s}$ as (3.115) and

$$b := |\underline{v}^k|\mathcal{R}_2^{k,p^k} \quad \text{and} \quad c := -\mathcal{R}_1^{k,p^k} + (t^k - t)|\underline{v}^k|\mathcal{R}_2^{k,p^k}. \tag{3.118}$$

From (3.113) and (3.118),

$$|b(t, \underline{x}, \underline{v})| \geqq \frac{\varrho_{\Omega,N,\delta_2}}{8N^2}.$$

We apply (i) of Lemma 17 to the following case: with $\varphi_1(t, \underline{x}, \underline{v})$ in (3.105), we set

$$\psi_4(t, \underline{x}, \underline{v}) = t - \varphi_1(t, \underline{x}, \underline{v})$$

and

$$\text{if } |s - \psi_1(t, \underline{x}, \underline{v})| > \delta_*, \text{ then } |b(t, \underline{x}, \underline{v})(t - s) + c(t, \underline{x}, \underline{v})| \gtrsim \frac{\varrho_{\Omega,N,\delta_2}}{8N^2} \times \delta_*. \tag{3.119}$$

Finally, from (3.116), (3.114), (3.119) and (3.117), we conclude the proof of Lemma 18. □

Now we return to three-dimensional cylindrical domain $U := \Omega \times (0, H) \subset \mathbb{R}^3$. We state a theorem about the uniform positivity of the determinant of $\frac{dX}{dv}$.

Proposition 2. *Let $t \in [T, T+1]$, then*

$$(x, v) = (x, \underline{v}, v_2) \in U \times \mathbb{V}^N \times \left\{ v_2 \in \mathbb{R} : \frac{1}{N} \leqq v_2 \leqq N \right\}.$$

Recall ε, δ in Lemma 1. For each $i = 1, 2, \dots, l_G$, there exists $\delta_2 > 0$ and a $C^1_{t,\underline{x},\underline{v}}$-function $\psi^{\ell_0, \vec{\ell}, i, k}$ for uniform bound $k \leqq C_{\varepsilon, N}$, where $\psi^{\ell_0, \vec{\ell}, i, k}$ is defined locally around $(T + \delta_2 \ell_0, X(T + \delta_2 \ell_0; t, x, v), (\delta_2 \vec{\ell}, u_2))$ with $(\ell_0, \vec{\ell}) = (\ell_0, \ell_1, \ell_3) \in \{0, 1, \dots, \lfloor \frac{1}{\delta_2} \rfloor + 1\} \times \{-\lfloor \frac{N}{\delta_2} \rfloor - 1, \dots, 0, \dots, \lfloor \frac{N}{\delta_2} \rfloor + 1\}^2$ and $\|\psi^{\ell_0, \vec{\ell}, i, k}\|_{C^1_{t,\underline{x},\underline{v}}} \leqq C_{N, \Omega, \delta, \delta_2} < \infty$.

For $(\underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) \in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}$, if

$$u_3 \geqq \frac{1}{N}, \tag{3.120}$$

$$\left| \left. \frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}} \right|_{x^1(\underline{X}(s;t,\underline{x},\underline{v}),\underline{u})} \cdot e_1 \right| > \frac{1}{N} > 0, \tag{3.121}$$

$$\underline{X}(s; t, \underline{x}, \underline{v}) \notin \bigcup_{j=1}^{l_{sg}} B\left(y_j^C, \varepsilon\right), \quad \textit{sticky grazing set defined in Lemma 1}, \tag{3.122}$$

$$(\underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) \in B\left(x_i^C, r_i^C\right) \times \mathbb{V}^N \backslash \mathcal{O}_i^C \quad \textit{for some} \quad i = 1, 2, \dots, l_G, \tag{3.123}$$

$$(s, \underline{u}) \in [T + (\ell_0 - 1)\delta_2, T + (\ell_0 + 1)\delta_2] \times B(\delta_2 \vec{\ell}, 2\delta_2), \tag{3.124}$$

$$|s - s'| \geqq \delta_2, \tag{3.125}$$

$$\begin{aligned} s' \in \Big[& t^{k+1}(T + \delta_2 \ell_0; \underline{X}(T + \delta_2 \ell_0; t, \underline{x}, \underline{v}), \delta_2 \vec{\ell}) + \frac{1}{N}, \\ & t^k(T + \delta_2 \ell_0; \underline{X}(T + \delta_2 \ell_0; t, \underline{x}, \underline{v}), \delta_2 \vec{\ell}) - \frac{1}{N} \Big], \end{aligned} \tag{3.126}$$

and

$$\begin{aligned} & \left| s' - \psi^{\ell_0, \vec{\ell}, i, k}(T + \delta_2 \ell_0, \underline{X}(T + \delta_2 \ell_0; t, \underline{x}, \underline{v}), \delta_2 \vec{\ell}) \right| \\ & \quad > N^2 \left(1 + \|\psi^{\ell_0, \vec{\ell}, i, k}\|_{C^1_{t,\underline{x},\underline{v}}} \right) \delta_2, \end{aligned} \tag{3.127}$$

then

$$\det \left(\frac{\partial X(s'; s, X(s; t, x, v), u)}{\partial u} \right) > \varepsilon'_{\Omega, N, \delta, \delta_2} > 0, \tag{3.128}$$

where $B(x_i^C, r_i^C) \times \mathbb{V}^N \backslash \mathcal{O}_i^C$ was constructed in Lemma 1. Also note that $\varepsilon'_{\Omega, N, \delta, \delta_2}$ does not depend on T, t, x, v.

Proof. *Step 1.* First we extend two-dimensional analysis into the three dimension case. For the v_2 direction, the dynamics is very simple, that is,

$$X_2(s; t, x, v) = x_2 - (t - s)v_2,$$

so we have

$$\frac{\mathrm{d}X_2}{\mathrm{d}v_2} = -(t-s).$$

Note that it is obvious that v_2 directional dynamics is independent of the two-dimensional trajectory which is projected on cross section Ω, because of the cylindrical domain with the specular boundary condition.

Step 2. Fix $t \in [T, T+1]$, $(x, v) \in \Omega \times \mathbb{V}^N$ and assume $(\underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) \in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}$. Assume that $s \in [T, t]$, and

$$\underline{X}(s; t, \underline{x}, \underline{v}) \notin \bigcup_{j=1}^{l_{sg}} B\left(y_j^C, \varepsilon\right) \quad \text{and} \quad (\underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) \in B\left(x_i^C, r_i^C\right) \times \mathbb{V}^N \backslash \mathcal{O}_i^C$$

for some $i = 1, \ldots, l_G$. Due to Lemma 1, $(\underline{X}(s'; s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}), \underline{V}(s'; s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}))$ is well-defined for all $s' \in [T, s]$ and

$$|\mathbf{n}(x^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})) \cdot v^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})| > \delta$$

for all k with $|t - t^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})| \leqq 1$.

From $\underline{X}(s; t, \underline{x}, \underline{v}) = \underline{X}(\bar{s}; t, \underline{x}, \underline{v}) + \int_{\bar{s}}^{s} \underline{V}(\tau; t, \underline{x}, \underline{v})\mathrm{d}\tau$, we have

$$\begin{aligned} &|\psi^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) - \psi^k(\bar{s}, \underline{X}(\bar{s}; t, \underline{x}, \underline{v}), \bar{\underline{u}})| \\ &\quad \leqq \|\psi^k\|_{C^1_{t,\underline{x},\underline{v}}} \{|s-\bar{s}| + |\underline{X}(s; t, \underline{x}, \underline{v}) - \underline{X}(\bar{s}; t, \underline{x}, \underline{v})| + |\underline{u} - \bar{\underline{u}}|\} \\ &\quad \leqq \|\psi^k\|_{C^1_{t,\underline{x},\underline{v}}} \{|s-\bar{s}| + (1+N)|\underline{u} - \bar{\underline{u}}|\}. \end{aligned} \tag{3.129}$$

For $0 < \delta_2 \ll 1$ we split

$$\begin{aligned} [T, T+1] &= \bigcup_{\ell_0=0}^{[\delta_2^{-1}]+1} \left[T + (\ell_0 - 1)\delta_2, T + (\ell_0+1)\delta_2\right], \\ \mathbb{V}^N \backslash \mathcal{O}_i^C &= \bigcup_{|\ell_i|=0}^{[N/\delta_2^{-2}]+1} B\big((\ell_1\delta_2, \ell_3\delta_2), 2\delta_2\big) \cap \mathbb{V}^N \backslash \mathcal{O}_i^C. \end{aligned}$$

From (3.129), if

$$(s, \underline{u}) \in \left[T + (\ell_0 - 1)\delta_2, T + (\ell_0+1)\delta_2\right] \times \{B\big((\ell_1\delta_2, \ell_3\delta_2), 2\delta_2\big) \cap \mathbb{V}^N \backslash \mathcal{O}_i^C\},$$

then

$$\begin{aligned} &|\psi^k(T + \ell_0\delta, \underline{X}(T + \ell_0\delta; t, \underline{x}, \underline{v}), (\ell_1\delta, \ell_3\delta)) - \psi^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})| \\ &\quad \leqq \|\psi^k\|_{C^1_{t,\underline{x},\underline{v}}} (2+N)\delta_2. \end{aligned}$$

Therefore, if (3.127) holds,

$$
\begin{aligned}
&|s' - \psi^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})| \\
&\quad \geqq |s' - \psi^k(T + \ell_0\delta, \underline{X}(T + \ell_0\delta; t, \underline{x}, \underline{v}), (\ell_1\delta, \ell_3\delta))| \\
&\qquad - |\psi^k(T + \ell_0\delta, \underline{X}(T + \ell_0\delta; t, \underline{x}, \underline{v}), (\ell_1\delta, \ell_3\delta)) - \psi^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})| \\
&\quad \gtrsim (N^2 - N)\|\psi^k\|_{C^1_{t,\underline{x},\underline{v}}}\delta_2 \gtrsim_N \|\psi^k\|_{C^1_{t,\underline{x},\underline{v}}}\delta_2.
\end{aligned}
$$

Step 3. Consider the three-dimensional mapping $u \mapsto X(s'; s, X(s; t, x, v), u)$. Note that from Lemma 1 we verify the condition of Lemma 18. From Lemma 18 and 6, we construct $C^1_{t,\underline{x},\underline{v}}$-function $\psi^k : B_\varepsilon(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) \to \mathbb{R}$ for a uniform bound $k \leqq C_{\varepsilon,N}$ such that if $|s' - \psi^k(s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})| \gtrsim_{N,\Omega,\delta} \delta_2$, then

$$
\begin{aligned}
&\left|\det\left(\frac{\partial X(s'; s, X(s; t, x, v), u)}{\partial u}\right)\right| \\
&\quad = \left|\frac{\mathrm{d}X_2}{\mathrm{d}v_2}\right| \left||\partial_{|u|}\underline{X}(s'; s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}) \times \partial_{\hat{u}_1}\underline{X}(s'; s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u})\right| \\
&\quad > |s - s'|\, \varepsilon_{\Omega,N,\delta,\delta_2} > \varepsilon'_{\Omega,N,\delta,\delta_2} > 0.
\end{aligned}
$$

□

3.5. Duhamel's Principle and L^∞ Estimate

Now we study the L^∞ estimate via the trajectory and Duhamel's principle.

Lemma 19. *Let f solve the linearized Boltzmann equation (1.15). For $h := wf$ with $w = (1 + |v|)^\beta$, $\beta > 5/2$, we have the following estimate:*

$$
\|h(t)\|_\infty \lesssim e^{-\frac{\nu_0}{2}t}\|h(0)\|_\infty + C_t \int_0^t \|f(s)\|_2\, \mathrm{d}s.
$$

Proof. Since $L = \nu(v) - K$,

$$
\partial_t f + v \cdot \nabla f + \nu f = Kf.
$$

For $h := wf$,

$$
\partial_t h + v \cdot \nabla_x h + \nu h = K_w h, \quad K_w h := wK\left(\frac{h}{w}\right).
$$

We define

$$
E(v, t, s) := \exp\left\{-\int_s^t \nu(V(\tau))\right\}.
$$

Along the trajectory,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\Big(E(v, t, s)h(s, X(s; t, x, v), V(s; t, x, v))\Big) \\
&\quad = E(v, t, s)\big[K_w h\big](s, X(s; t, x, v), V(s; t, x, v)).
\end{aligned}
$$

By integrating from 0 to t, we obtain

$$h(t,x,v) = E(v,t,0)h(0,X(0),V(0)) + \int_0^t E(v,t,s)\int_{\mathbb{R}^3} k_w(u,V(s))h(s,X(s;t,x,v),u)\,\mathrm{d}u\mathrm{d}s. \tag{3.130}$$

Recalling the standard estimates (see Lemmas 4 and 5 in [9]),

$$\int_{\mathbb{R}^3} |k_w(v,u)|\mathrm{d}u \leqq C_K\langle v\rangle^{-1}. \tag{3.131}$$

We apply Duhamel's formula (3.130) *two times*, for sufficiently small $0<\bar{\delta}\ll 1$, and cut a part of domain where a change of variable does not work. In particular, we use Lemma 1 and a split sticky grazing set to get

$$\begin{aligned} h(t,x,v) &= E(v,t,0)h(0) + \int_0^t E(v,t,s)\int_u k_w(u,v)h(s,X(s),u)\,\mathrm{d}u\mathrm{d}s \\ &\leqq E(v,t,0)h(0) + \int_0^t E(v,t,s)\int_u k_w(u,v)E(u,s,0)h(0) \\ &\quad +|(\mathcal{E}_1)|+|(\mathcal{E}_2)|+|(\mathcal{E}_3)|+|(\mathcal{E}_4)|+|(\mathcal{E}_5)|, \end{aligned} \tag{3.132}$$

where

$$(\mathcal{E}_k) := \int_0^t E(v,t,s)\int_u k_w(u,v)\int_0^s E(u,s,s')\int_{u'} k_w(u',u)h(s',X(s'),u') \mathbf{1}_{E_k}(X(s),u), \quad k=1,2,3,4,5. \tag{3.133}$$

Note that we abbreviated the notations

$$X(s) := X(s;t,x,v), \quad X(s') := X'(s';s,X(s;t,x,v),u),$$

and E_k in characteristic functions in (3.133) are defined as

$$E_1 := \left\{(X(s),u)\in\mathbb{R}^3\times\mathbb{R}^3 : \underline{u}\in\mathbb{R}^2\backslash\mathbb{V}^N \text{ or } |u_2|\in\mathbb{R}\backslash\left[\frac{1}{N},N\right]\right\},$$

$$E_2 := \left\{(X(s),u)\in\mathbb{R}^3\times\mathbb{R}^3 : (\underline{u},u_2)\in\mathbb{V}^N\times\left[\frac{1}{N},N\right],\ (\underline{X}(s),\underline{u})\in\mathfrak{IB}\right\},$$

$$E_3 := \left\{(X(s),u)\in\mathbb{R}^3\times\mathbb{R}^3 : (\underline{u},u_2)\in\mathbb{V}^N\times\left[\frac{1}{N},N\right],\right. \left.(\underline{X}(s),\underline{u})\in\{cl(\Omega)\times\mathbb{V}^N\}\backslash\mathfrak{IB},\ \underline{X}(s)\in\bigcup_{j=1}^{l_{sg}} B\left(y_j^C,\varepsilon\right)\right\},$$

$$E_4 := \Bigg\{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : (\underline{u}, u_2) \in \mathbb{V}^N \times \left[\frac{1}{N}, N\right],$$
$$(\underline{X}(s), \underline{u}) \in \{cl(\Omega) \times \mathbb{V}^N\} \backslash \mathfrak{IB}, (\underline{X}(s), \underline{u})$$
$$\in \left\{ \bigcup_{i=1}^{l_G} B\left(x_i^C, r_i^C\right) \times \mathcal{O}_i^C \right\} \Big\backslash \left\{ \bigcup_{j=1}^{l_{sg}} B\left(y_j^C, \varepsilon\right) \times \mathbb{V}^N \right\} \Bigg\},$$

and

$$E_5 := \Big\{ (X(s), u) \in \mathbb{R}^3 \times \mathbb{R}^3 : (\underline{u}, u_2) \in \mathbb{V}^N$$
$$\times \left[\frac{1}{N}, N\right],\ (\underline{X}(s), \underline{u}) \in \left\{ cl(\Omega) \times \mathbb{V}^N \right\} \Big\backslash \mathfrak{IB},$$
$$(\underline{X}(s), \underline{u}) \in \left\{ B\left(x_i^C, r_i^C\right) \times \left\{ \mathbb{V}^N \backslash \mathcal{O}_i^C \right\} \right\} \Big\backslash \left\{ \bigcup_{j=1}^{l_{sg}} B\left(y_j^C, \varepsilon\right) \times \mathbb{V}^N \right\}$$
$$\text{for some } i = 1, \ldots, l_{sg} \Big\}. \tag{3.134}$$

Also note that

$$(\mathfrak{G})_\varepsilon := \left\{ \bigcup_{i=1}^{l_G} B\left(x_i^C, r_i^C\right) \times \mathcal{O}_i^C \right\} \cup \left\{ \bigcup_{j=1}^{l_{sg}} B\left(y_j^C, \varepsilon\right) \times \mathbb{V}^N \right\}$$

was defined in Lemma 1. From $|V(\tau; t, x, v)| = |v|$ with the rotational symmetry of $\nu(v)$, we have

$$E(v, t, s) \leqq e^{-\nu(v)(t-s)}.$$

On the RHS of (3.132), every term except $(\mathcal{E}_1)$, $(\mathcal{E}_2)$, $(\mathcal{E}_3)$, $(\mathcal{E}_4)$ and $(\mathcal{E}_5)$ is controlled by

$$E(v, t, 0)h(0) \leqq e^{-\nu_0 t} \|h_0\|_\infty,$$
$$\int_0^t E(v, t, s) \int_u k_w(u, v) E(u, s, 0) h(0) \leqq \|h_0\|_\infty \int_0^t e^{-\nu_0 t} \int_u k_w(u, v)\, \mathrm{d}u \mathrm{d}s$$
$$\lesssim t e^{-\nu_0 t} \|h_0\|_\infty \lesssim e^{-\frac{\nu_0}{2} t} \|h_0\|_\infty, \tag{3.135}$$

where we used (3.131).

We claim the smallness of $(\mathcal{E}_1) \sim (\mathcal{E}_4)$.
From

$$\int_u \mathbf{1}_{\{\underline{u} \in \mathbb{R}^2 \backslash \mathbb{V}^N \text{ or } |u_2| \in \mathbb{R} \backslash [\frac{1}{N}, N]\}}(u) \sqrt{\mu} \mathrm{d}u = O(\frac{1}{N}),$$
$$(\mathcal{E}_1) \leqq O\left(\frac{1}{N}\right) \sup_{0 \leqq s \leqq t} \|h(s)\|_\infty. \tag{3.136}$$

From Lemma 4, $\mathfrak{m}_2(\mathcal{O}_i^{IB}) \lesssim \varepsilon$ for $1 \leqq i \leqq l_{IB}$. Therefore,

$$(\mathcal{E}_2) \leqq O(\varepsilon) \sup_{0\leqq s\leqq t} \|h(s)\|_\infty. \tag{3.137}$$

For $(\mathcal{E}_3)$, we also have a similar estimate, because

$$\begin{aligned}(\mathcal{E}_3) &\leqq \int_0^t \mathrm{d}s \mathbf{1}_{\underline{X}(s)\in\bigcup_{j=1}^{l_{sg}} B\left(y_j^C,\varepsilon\right)}(s)\|h(s)\|_\infty \\ &\leqq C\frac{\varepsilon}{1/N} \sup_{0\leqq s\leqq t} \|h(s)\|_\infty, \quad \text{since} \quad |\underline{v}| \geqq \frac{2}{N}, \\ &\leqq C\varepsilon N \sup_{0\leqq s\leqq t} \|h(s)\|_\infty \leqq O\left(\frac{1}{N}\right) \sup_{0\leqq s\leqq t} \|h(s)\|_\infty. \end{aligned} \tag{3.138}$$

For the estimate for $(\mathcal{E}_4)$, since $\mathfrak{m}_2(\mathcal{O}_i^C) < \varepsilon$ from Lemma 1,

$$(\mathcal{E}_4) \leqq O(\varepsilon) \sup_{0\leqq s\leqq t} \|h(s)\|_\infty. \tag{3.139}$$

For $(\mathcal{E}_5)$, we choose $m(N)$ so that

$$k_{w,m}(u,v) := \mathbf{1}_{\{|u-v|\geqq \frac{1}{m},\ |u|\leqq m\}} k_w(u,v)$$

satisfies $\int_{\mathbb{R}^3} |k_{w,m}(u,v) - k_w(u,v)|\,\mathrm{d}u \leqq \frac{1}{N}$ for sufficiently large $N \geqq 1$. Then, by splitting k_w,

$$\begin{aligned}(\mathcal{E}_5) \leqq & \underbrace{\int_0^t \int_0^s e^{-\nu(v)(t-s')} \int_u k_{w,m}(u,v) \int_{u'} k_{w,m}(u',u) h(s', X'(s'), u')\, \mathbf{1}_{E_5}(X(s),u)\, \mathrm{d}u'\mathrm{d}u\mathrm{d}s'\mathrm{d}s}_{(**)} \\ &+ O_\Omega\left(\frac{1}{N}\right) \sup_{0\leqq s\leqq t} \|h(s)\|_\infty. \end{aligned} \tag{3.140}$$

We define following sets for fixed $n, \vec{n}, i, k$, where Proposition 2 does not work:

$$\begin{aligned} R_1 &:= \{u \mid \underline{u} \notin B(\vec{n}\delta, 2\delta) \cap \{\mathbb{R}^2 \backslash \mathcal{O}_{i_s}^C\}\}, \\ R_2 &:= \{s' \mid |s-s'| \leqq \delta\}, \\ R_3 &:= \left\{s' \mid \max_{i=1,2} \left|s' - \psi_1^{n,\vec{n},i,k}(n\delta, \underline{X}(n\delta; t, \underline{x}, \underline{v}), (\vec{n}\delta, u_2))\right| \lesssim_N \delta \|\psi_1\|_{C^1_{t,\underline{x},\underline{v}}}\right\}, \\ R_4 &:= \left\{s' \mid |s' - t^k(n\delta, \underline{X}(n\delta; t, \underline{x}, \underline{v}), (\vec{n}\delta, u_2))| \lesssim_N \delta \|\psi_1\|_{C^1_{t,\underline{x},\underline{v}}}\right\}, \\ R_5 &:= \left\{u \mid |u_3| \leqq \frac{1}{N}\right\}, \\ R_6 &:= \left\{\underline{u} \in \mathbb{R}^2 \mid \left|\left.\frac{\partial_1 \underline{\eta}_{p^1}}{\sqrt{g_{p^1,11}}}\right|_{x^1(\underline{X}(s;t,\underline{x},\underline{v}),\underline{u})} \cdot e_1\right| \leqq \frac{1}{N}\right\}. \end{aligned} \tag{3.141}$$

Using (3.141), we write $(**)$ as

$$(**) = \underline{\sum_{n=0}^{[t/\delta]+1} \sum_{|\vec{n}|\leqq N} \sum_{k}^{C_{\varepsilon,N}} \int_{(n-1)\delta}^{(n+1)\delta} \int_{t^{k+1}}^{t^k} e^{-\nu(v)(t-s')}}$$
$$\underline{\times \int_{|u|\leqq N, |u'|\leqq N} k_{w,m}(u,v) k_{w,m}(u',u)\, |h(s', X(s'), u')|}$$
$$\underline{\times \mathbf{1}_{R_1^c \cap R_2^c \cap R_3^c \cap R_4^c \cap R_5^c \cap R_6^c} \mathbf{1}_{E_5}(X(s), u)}_{(\mathrm{MAIN})} + R, \tag{3.142}$$

where R corresponds to where (u, s') is in one of $R_1 \sim R_6$. We replace $\mathbf{1}_{\bigcap_{i=1}^6 R_i^c}$ in $\mathbf{1}_{\bigcup_{i=1}^6 R_i}$ in (MAIN). For R, we have the following smallness estimate:

$$\begin{aligned} R &\leqq \int_0^t \int_0^s e^{-\frac{1}{2}\nu(v)(t-s')} \int_{|u|\leqq N} k_{w,m}(u,v) \int_{|u'|\leqq N} k_{w,m}(u',u) h(s', X'(s'), u')\, \mathbf{1}_{\bigcup_{i=1}^6 R_i} \\ &\leqq C_N \big(\delta + \varepsilon + O\big(\frac{1}{N}\big)\big) \sup_{0\leqq s\leqq t} \|h(s)\|_\infty, \end{aligned} \tag{3.143}$$

by choosing sufficiently small $\delta \ll \frac{1}{N}$. Note that smallness from R_1 to R_5 is trivial. For R_6, we note that by the analyticity and boundness of Ω, there are only finite points $\underline{x}$ such that $\frac{\partial_1 \eta_{\underline{p}^1}}{\sqrt{g_{p^1,11}}}\Big|_{\underline{x}\in\partial\Omega} \cdot e_1 = 0$, so R_6 gives smallness $O(\frac{1}{N})$.

Let us focus on (MAIN) in (3.142). From (3.134) and (3.141), all conditions (3.120)–(3.127) in Proposition 2 are satisfied and

$$\exists i_s \in \{1, 2, \ldots, l_G\} \quad \text{such that } \underline{X}(s) \in B\left(x_{i_s}^C, r_{i_s}^C\right).$$

Under the condition of $(u, s') \in \bigcap_{i=1}^6 R_i^c$, indices $n, \vec{n}, i_s, k$ are determined so that

$$\begin{aligned} t &\in [(n-1)\delta, (n+1)\delta], \\ \underline{X}(s; t, \underline{x}, \underline{v}) &\in B\left(x_{i_s}^C, r_{i_s}^C\right), \\ \underline{u} &\in B(\vec{n}\delta, 2\delta) \cap \left\{\mathbb{V}^N \backslash \mathcal{O}_{i_s}^C\right\}, \end{aligned}$$

and (3.128) in Proposition 2 gives local time-independent lower bound

$$\left|\det\left(\frac{\partial X(s')}{\partial u}\right)\right| > \varepsilon'_\delta > 0.$$

If we choose sufficiently small δ, there exist small $r_{\delta,n,\vec{n},i,k}$ such that there exists a one-to-one map $\mathcal{M}$:

$$\mathcal{M} : B(\vec{n}\delta, 2\delta) \cap \{\mathbb{V}^N \backslash \mathcal{O}_{i_s}^C\} \mapsto B(\underline{X}(s'; s, \underline{X}(s; t, \underline{x}, \underline{v}), \underline{u}), r_{\delta,n,\vec{n},i,k}).$$

We perform a change of variable for (MAIN) in (3.142) to obtain

$$
\begin{aligned}
&(\text{MAIN})\\
&\leqq \sum_{n=0}^{[t/\delta]+1} \sum_{|\vec{n}|\leqq N} \sum_{k}^{C_{\varepsilon,N}} \int_{\max\{(n-1)\delta,0\}}^{\min\{(n+1)\delta,t\}} \int_{t^{k+1}}^{t^k} e^{-\nu(v)(t-s')}\\
&\quad\times \int_u k_{w,m}(u,v) \int_{u'} k_{w,m}(u',u)\, \mathbf{1}_{|u|\leqq N,|u'|\leqq N}\, |h(s',X(s'),u')|\\
&\quad \mathbf{1}_{E_5}(X(s),u)\mathrm{d}u'\,\mathrm{d}u\mathrm{d}s'\mathrm{d}s\\
&\leqq \sum_{n=0}^{[t/\delta]+1} \sum_{|\vec{n}|\leqq N} \sum_{k}^{C_{\varepsilon,N}} \int_{\max\{(n-1)\delta,0\}}^{\min\{(n+1)\delta,t\}} \int_{t^{k+1}}^{t^k} e^{-\nu(v)(t-s')}\\
&\quad\times \int_u k_{w,m}(u,v)\mathbf{1}_{|u|\leqq N}\,\mathrm{d}u\, \|f(s',X(s'),u')\|_{L^2_{|u'|\leqq N}} \mathbf{1}_{E_5}(X(s),u)\mathrm{d}u\mathrm{d}s'\,\mathrm{d}s\\
&\leqq \sum_{n=0}^{[t/\delta]+1} \sum_{|\vec{n}|\leqq N} \sum_{k}^{C_{\varepsilon,N}} \int_{\max\{(n-1)\delta,0\}}^{\min\{(n+1)\delta,t\}} \int_{t^{k+1}}^{t^k} e^{-\nu(v)(t-s')}\\
&\quad\left\{\int_{|u|\leqq N} \|f(s',X(s'),u')\|^2_{L^2_{|u'|\leqq N}}\mathrm{d}u\right\}^{1/2}\\
&\quad\times \mathbf{1}_{E_5}(X(s),u)\mathrm{d}s'\,\mathrm{d}s\\
&\leqq \sum_{n=0}^{[t/\delta]+1} \int_{\max\{(n-1)\delta,0\}}^{\min\{(n+1)\delta,t\}} \sum_{|\vec{n}|\leqq N} \sum_{k}^{C_{\varepsilon,N}} \int_{t^{k+1}}^{t^k} e^{-\nu(v)(t-s')}\\
&\quad\times \left\{\int_{B(X(s'),r_{\delta,n,\vec{n},i,k})} \|f(s',X(s'),u')\|^2_{L^2_{|u'|\leqq N}} \frac{1}{\varepsilon'_\delta}\mathrm{d}x\right\}^{1/2}\mathrm{d}s'\mathrm{d}s\\
&\lesssim \sum_{n=0}^{[t/\delta]+1} \int_{\max\{(n-1)\delta,0\}}^{\min\{(n+1)\delta,t\}} \sum_{k}^{C_{\varepsilon,N}} \int_{t^{k+1}}^{t^k} e^{-\nu(v)(t-s')}\\
&\quad\left\{\int_{\Omega} \|f(s',X(s'),u')\|^2_{L^2_{|u'|\leqq N}}\mathrm{d}x\right\}^{1/2}\mathrm{d}s'\,\mathrm{d}s\\
&\lesssim C_t \sum_{k}^{C_{\varepsilon,N}} \int_{t^{k+1}}^{t^k} \|f(s',y,u')\|_{L^2_{y,u'}}\,\mathrm{d}s' \lesssim C_t \int_0^t \|f\|_2 \mathrm{d}s,
\end{aligned}
\tag{3.144}
$$

since $w(u')$ is bounded for $|u'| \leqq N$ and $\sum_k^{C_{\varepsilon,N}} \int_{t^{k+1}}^{t^k} \leqq \int_0^t$, where $t^k = t^k(X(s;t,x,v),u)$ and $X(s') = X(s';s,X(s;t,x,v),u)$. We collect (3.132), (3.135), (3.136)–(3.139), (3.140), (3.142), (3.143) and (3.144) with sufficiently large $N \gg 1$ and small $\varepsilon, \delta \ll \frac{1}{N^2}$ to conclude

$$\|h(t)\|_\infty \lesssim e^{-\frac{\nu_0}{2}t}\|h(0)\|_\infty + C_t \int_0^t \|f(s)\|_2 \, \mathrm{d}s. \tag{3.145}$$

□

4. L^2-Coercivity via Contradiction Method

We start with a lemma which was proved in Lemma 5.1 in [17].

Lemma 20. *Let g be a (distributional) solution to*

$$\partial_t g + v \cdot \nabla_x g = G.$$

Then, for a sufficiently small $\varepsilon > 0$,

$$\int_\varepsilon^{1-\varepsilon} \|\mathbf{1}_{\mathrm{dist}(x,\partial U)<\varepsilon^4}\mathbf{1}_{|\mathbf{n}(x)\cdot v|>\varepsilon} g(t)\|_2^2 \mathrm{d}t \lesssim \int_0^1 \|\mathbf{1}_{\mathrm{dist}(x,\partial U)>\varepsilon^3/2} g(t)\|_2^2 \, \mathrm{d}t + \int_0^1 \iint_{U\times\mathbb{R}^3} |gG|.$$

Proposition 3. *Assume that f solves the linearized Boltzmann equation*

$$\partial_t f + v \cdot \nabla f + Lf = 0, \tag{4.1}$$

and satisfies the specular reflection BC and (1.3) for $F = \mu + \sqrt{\mu} f$. Furthermore, for an axis-symmetric domain, we assume (1.6). Then there exists $C > 0$ such that, for all $N \in \mathbb{N}$,

$$\int_N^{N+1} \|\mathbf{P}f(t)\|_2^2 \mathrm{d}t \leqq C \int_N^{N+1} \|(\mathbf{I}-\mathbf{P})f(t)\|_\nu^2 \, \mathrm{d}t, \tag{4.2}$$

where $\mathbf{P}f$ is hydrodynamic part (projection on the null space of L, $N(L)$), $\mathbf{P}f := (a + b\cdot v + c\frac{|v|^2-3}{2})\sqrt{\mu}$ and $\|\cdot\|_\nu = \|\cdot\sqrt{\nu}\|_2$.

Proof. We will use the contradiction method which is used in [11] and also in [17] with some modification. Instead of giving full details, we describe the scheme of proof following [17].

Step 1. First, (4.1) is translation invariant in time, so it suffices to prove coercivity for a finite time interval $t \in [0, 1]$ and so we claim (4.2) for $N = 0$. Now assume that Proposition 3 is wrong. Then, for any $m \gg 1$, there exists a solution f^m to (4.1) with specular reflection BC, which solves

$$\partial_t f^m + v \cdot \nabla_x f^m + Lf^m = 0, \quad \text{for } t \in [0, 1] \tag{4.3}$$

and satisfies

$$\int_0^1 \|\mathbf{P}f^m(t)\|_2^2 \mathrm{d}t \geqq m \int_0^1 \|(\mathbf{I}-\mathbf{P})f^m(t)\|_\nu^2 \, \mathrm{d}t.$$

Define the normalized form of f^m by

$$Z^m(t,x,v) := \frac{f^m(t,x,v)}{\sqrt{\int_0^1 \|\mathbf{P}f^m(t)\|_2^2 \mathrm{d}t}}, \quad \int_0^1 \|\mathbf{P}Z^m\|_2^2 = 1. \tag{4.4}$$

Then Z^m also solves (4.3) with the specular BC and

$$\frac{1}{m} \geqq \int_0^1 \|(\mathbf{I}-\mathbf{P})Z^m(t)\|_\nu^2 \,\mathrm{d}t. \tag{4.5}$$

Step 2. We claim that

$$\sup_m \sup_{0\leqq t\leqq 1} \|Z^m(t)\|_2^2 < \infty. \tag{4.6}$$

Since Z_m solves (4.3) with the specular BC, for $0 \leqq t \leqq 1$,

$$\sup_{0\leqq t\leqq 1} \|Z^m(t)\|_2^2 \leqq \|Z^m(0)\|_2^2,$$

from the non-negativity of L. Moreover, by integration $\int_0^1$ and using (4.5) and (4.4),

$$\|Z^m(0)\|_2^2 \lesssim \int_0^1 \|\mathbf{P}Z^m\|_2^2 + \int_0^1 \|(\mathbf{I}-\mathbf{P})Z^m\|_\nu^2 \lesssim 1 + \frac{1}{m}.$$

Therefore, we have proved the claim (4.6).

Step 3. Therefore, the sequence $\{Z^m\}_{m\gg 1}$ is uniformly bounded in $\sup_{0\leqq t\leqq 1} \|g(t)\|_\nu^2$. By the weak compactness of L^2-space, there exists a weak limit Z such that

$$Z^m \rightharpoonup Z \quad \text{in} \quad L^\infty([0,1]; L^2_\nu(U\times\mathbb{R}^3)) \cap L^2([0,1]; L^2_\nu(U\times\mathbb{R}^3)).$$

Therefore, in the sense of distributions, Z solves (4.1) with the specular BC. See the proof of Proposition of 1.4 in [17] to see that Z also satisfies the specular BC. Moreover, it is easy to check that the weak limit Z satisfies conservation laws as follows:

$$\iint_{U\times\mathbb{R}^3} Z(t)\sqrt{\mu} = 0, \quad \iint_{U\times\mathbb{R}^3} Z(t)\frac{|v|^2}{2}\sqrt{\mu} = 0, \quad 0 \leqq t \leqq 1. \tag{4.7}$$

In the case of axis-symmetry (1.5),

$$\iint_{U\times\mathbb{R}^3} \{(x-x^0)\times\varpi\}\cdot vZ(t)\sqrt{\mu} = 0. \tag{4.8}$$

On the other hand, since

$$\mathbf{P}Z^m \rightharpoonup \mathbf{P}Z \quad \text{and} \quad (\mathbf{I}-\mathbf{P})Z^m \to 0 \quad \text{in} \int_0^1 \|\cdot\|_\nu^2 \mathrm{d}t,$$

we know that the weak limit Z has only a hydrodynamic part, that is,

$$Z(t,x,v) = \left\{a(t,x) + v\cdot b(x,v) + \frac{|v|^2-3}{2}c(t,x)\right\}\sqrt{\mu}, \tag{4.9}$$

and

$$\int_0^1 \|Z\|_\nu^2 \mathrm{d}t \leqq \liminf_{m\to\infty} \int_0^1 \|Z^m\|_\nu^2 \mathrm{d}t \leqq 1 + \frac{1}{m} \to 1.$$

Step 4. Compactness. For interior compactness, let $\chi_\varepsilon : cl(U) \to [0,1]$ be a smooth function such that $\chi_\varepsilon(x) = 1$ if $\mathrm{dist}(x, \partial U) > 2\varepsilon^4$ and $\chi_\varepsilon(x) = 0$ if $\mathrm{dist}(x, \partial U) < \varepsilon^4$. From (4.1) with Z^m ,

$$[\partial_t + v \cdot \nabla_x](\chi_\varepsilon Z^m) = v \cdot \nabla_x \chi_\varepsilon Z^m - L(\chi_\varepsilon Z^m).$$

From the standard Average lemma, $\chi_\varepsilon Z^m$ is compact, that is,

$$\chi_\varepsilon Z^m \to \chi_\varepsilon Z \quad \text{strongly in } L^2([0,1]; L^2_\nu(U \times \mathbb{R}^3)). \tag{4.10}$$

For the near boundary compactness for the non-grazing part, we claim that

$$\begin{aligned} &\int_\varepsilon^{1-\varepsilon} \left\| \big(Z^m(t,x,v) - Z(t,x,v)\big) \mathbf{1}_{\mathrm{dist}(x,\partial U) < \varepsilon^4} \mathbf{1}_{|\mathbf{n}(x)\cdot v| > \varepsilon} \right\|_2^2 \\ &\lesssim \int_0^1 \left\| \big(Z^m(t,x,v) - Z(t,x,v)\big)\, \mathbf{1}_{\mathrm{dist}(x,\partial U) > \frac{\varepsilon^3}{2}} \right\|_2^2 + O\left(\frac{1}{\sqrt{m}}\right). \end{aligned} \tag{4.11}$$

Looking at the equation of $Z^m - Z$, from (4.9),

$$[\partial_t + v \cdot \nabla_x](Z^m - Z) + LZ^m = 0. \tag{4.12}$$

We apply Lemma 20 to (4.12) by equating g and G with $Z^m - Z$ and the last term of the LHS in (4.12), respectively. Then

$$\begin{aligned} &\int_\varepsilon^{1-\varepsilon} \left\| \mathbf{1}_{\mathrm{dist}(x,\partial U) < \varepsilon^4} \mathbf{1}_{|\mathbf{n}(x)\cdot v| > \varepsilon} (Z^m - Z)(t) \right\|_2^2 \mathrm{d}t \\ &\lesssim \int_0^1 \left\| \mathbf{1}_{\mathrm{dist}(x,\partial U) > \varepsilon^3/2} (Z^m - Z)(t) \right\|_2^2 \mathrm{d}t + \int_0^1 \iint_{U\times\mathbb{R}^3} |(Z^m - Z) L Z^m|. \end{aligned}$$

Since $\int_0^1 \iint_{U\times\mathbb{R}^3} |(Z^m - Z)LZ^m|$ is bounded by $C\Big(\sqrt{m}\int_0^1 \|(\mathbf{I} - \mathbf{P})Z^m\|_\nu^2 + \frac{1}{\sqrt{m}} \int_0^1 \|Z^m\|_\nu^2 + \|Z\|_\nu^2\Big)$, we conclude (4.11) using (4.6) and (4.5).

On the other hand,

$$\begin{aligned} \int_\varepsilon^{1-\varepsilon} \left\| (Z^m - Z) \mathbf{1}_{\mathrm{dist}(x,\partial U) < \varepsilon^4} \mathbf{1}_{|\mathbf{n}(x)\cdot v| \leqq \varepsilon} \right\|_2^2 &\leqq \int_\varepsilon^{1-\varepsilon} \left\| Z^m \mathbf{1}_{\mathrm{dist}(x,\partial U) < \varepsilon^4} \mathbf{1}_{|\mathbf{n}(x)\cdot v| \leqq \varepsilon} \right\|_2^2 \\ &\quad + \int_\varepsilon^{1-\varepsilon} \left\| Z \mathbf{1}_{\mathrm{dist}(x,\partial U) < \varepsilon^4} \mathbf{1}_{|\mathbf{n}(x)\cdot v| \leqq \varepsilon} \right\|_2^2 \\ &\leqq O(\varepsilon), \end{aligned} \tag{4.13}$$

where the smallness of the first term on the RHS comes from Lemma 9 of GUO [11]. For the second term, we use Lemma 6 of GUO [11] to obtain $\int Z^2 dv < \infty$ with $|U \backslash U_\varepsilon| \lesssim \varepsilon$, that is, a small measure in spatial phase.

Step 6. Strong convergence. To simplify notation, we write $U_\varepsilon := \{x \in U : \chi_\varepsilon > 0\}$. Then, for given $\varepsilon > 0$, we can choose $m \gg_\varepsilon 1$ such that

$$\begin{aligned}
&\int_0^1 \iint_{U\times\mathbb{R}^3} |Z^m - Z|^2 \\
&\quad \leqq \int_{1-\varepsilon}^1 \iint_{U\times\mathbb{R}^3} + \int_0^\varepsilon \iint_{U\times\mathbb{R}^3} + \int_\varepsilon^{1-\varepsilon} \iint_{U_\varepsilon\times\mathbb{R}^3} \\
&\quad\quad + \int_\varepsilon^{1-\varepsilon} \iint_{\{U\setminus U_\varepsilon\times\mathbb{R}^3\}\cap\,\{|\mathbf{n}(x)\cdot v|<\varepsilon\}} + \int_\varepsilon^{1-\varepsilon} \iint_{\{U\setminus U_\varepsilon\times\mathbb{R}^3\}\cap\,\{|\mathbf{n}(x)\cdot v|\geqq\varepsilon\}}, \\
&\quad < C\varepsilon,
\end{aligned}$$

where we have used (4.6), (4.10), (4.11) and (4.13). Therefore, we conclude that $Z^m \to Z$ strongly in $L^2([0, 1] \times U \times \mathbb{R}^3)$ and hence

$$\int_0^1 \|Z\|_2^2 = 1. \tag{4.14}$$

Step 7. We claim $Z = 0$. Plugging (4.9) into the linearized Boltzmann equation, we get

$$\begin{aligned}
\partial_i c &= 0, \\
\partial_t c + \partial_i b_i &= 0, \\
\partial_i b_j + \partial_j b_i &= 0, \quad i \neq j, \\
\partial_t b_i + \partial_i a &= 0, \\
\partial_t a &= 0.
\end{aligned} \tag{4.15}$$

Using the first equation and a direct computation of Lemma 12 in [11],

$$b(t, x) = -\partial_t c(t)x + \varpi(t) \times x + m(t).$$

From the second equation in (4.15) and the specular BC,

$$c(t, x) = c_0, \quad b = \varpi(t) \times x + m(t).$$

We split things into two cases: $\varpi = 0$ and $\varpi \neq 0$.

Case of $\varpi = 0$. From $b(t) = m(t)$ and from the specular BC, we deduce that

$$b(t) \equiv m(t) \equiv 0.$$

From, the fourth and final equations of (4.15), we can derive

$$a(t, x) = a_0.$$

Since $a(t, x)$ and $c(t, x)$ are constant, from (4.7), we derive $a_0 = c_0 = 0$, and hence $Z = 0$.

Case of $\varpi \neq 0$. From the specular BC,

$$b(t,x)\cdot n(x) = \big(\varpi(t)\times x + m(t)\big)\cdot n(x) = 0.$$

Since $m(t)$ is a fixed vector for a given t, we decompose $m(t)$ into the parallel and orthogonal components to $\varpi(t)$ as

$$m(t) = \alpha(t)\varpi(t) - \varpi(t)\times x_0(t).$$

Then

$$\begin{aligned} b(t,x)\cdot n(x) &= \big(\varpi(t)\times x + m(t)\big)\cdot n(x) \\ &= \big(\varpi(t)\times (x - x_0(t))\big)\cdot n(x) + \alpha(t)\varpi(t)\cdot n(x) = 0, \\ &\quad \forall x\in\partial U. \end{aligned} \tag{4.16}$$

Choose t with $\varpi(t)\neq 0$. We can pick $x'\in\partial U$ such that $\varpi(t)\parallel n(x')$. Then the first term of the RHS in (4.16) is zero. Hence we deduce

$$\alpha(t) = 0 \quad\text{and}\quad b(t,x) = \varpi(t)\times\big(x - x^0(t)\big). \tag{4.17}$$

This yields

$$\big(\varpi(t)\times(x - x_0(t))\big)\cdot n(x) = 0,\quad \forall x\in\partial U. \tag{4.18}$$

The equality (4.18) implies that U is axis-symmetric with the origin $x_0(t)$ and the axis $\varpi(t)$. From (4.8) and (4.17),

$$0 = \iint_U |\varpi\times(x - x_0(t))\cdot v|^2\mu\,\mathrm{d}x\mathrm{d}v.$$

Therefore, we conclude that $b(t,x)\equiv 0$. Then using conservation laws (mass and energy) again, we deduce $Z = 0$.

Step 8. Finally we deduce a contradiction from (4.14) and $Z = 0$. This finishes the proof. □

5. Linear and Nonlinear Decay

5.1. Linear L^2 Decay

We use the coercivity estimate Proposition 3 to derive the exponential linear L^2 decay of the linearized Boltzmann equation (1.15) with the specular boundary condition.

Corollary 1. *Assume that f solves the linearized Boltzmann equation with the specular BC so that f satisfies Proposition 3. Then there exists $\lambda > 0$ such that a solution of (1.15) satisfies*

$$\sup_{0\leqq t} e^{\lambda t}\|f(t)\|_2 \lesssim \|f_0\|_2. \tag{5.1}$$

Proof. Assume that $N \leqq t < N+1$. From the energy estimate of (4.1) in a time interval $[N, t]$,

$$\|f(t)\|_2^2 + \int_N^t \iint_{U\times\mathbb{R}^3} fLf \leqq \|f(N)\|_2^2. \tag{5.2}$$

From (4.1), for any $\lambda > 0$,

$$\left[\partial_t + v\cdot\nabla_x\right](e^{\lambda t} f) + L(e^{\lambda t} f) = \lambda e^{\lambda t} f.$$

By the energy estimate,

$$\|e^{\lambda N} f(N)\|_2^2 + 2\int_0^N \iint_{U\times\mathbb{R}^3} e^{2\lambda s} fLf - 2\lambda \int_0^N \iint_{U\times\mathbb{R}^3} |e^{\lambda s} f(s)|^2 \leqq \|f(0)\|_2^2. \tag{5.3}$$

Splitting for each time interval we have,

$$\begin{aligned} &\|e^{\lambda N} f(N)\|_2^2 + \sum_{k=0}^{N-1} 2e^{2\lambda k}\int_k^{k+1} \nu_0 \|(\mathbf{I}-\mathbf{P})f\|_\nu^2 \\ &\quad - \sum_{k=0}^{N-1} 2\lambda e^{2\lambda(k+1)} \int_k^{k+1} \|f(s)\|_2^2 \leqq \|f(0)\|_2^2. \end{aligned}$$

Using (4.2), there exist $C_{\nu_0} > 0$ such that

$$\|e^{\lambda N} f(N)\|_2^2 + \left(C_{\nu_0} - 2\lambda e^{2\lambda}\right)\sum_{k=0}^{N-1} e^{2\lambda k}\int_k^{k+1} \|f\|_2^2 \leqq \|f(0)\|_2^2.$$

Choosing ($\lambda \ll 1$) sufficiently small, we get

$$e^{2\lambda N}\|f(N)\|_2^2 \leqq \|f_0\|_2^2. \tag{5.4}$$

From the non-negativeness of L, we have $\|f(t)\|_2 \leqq \|f(N)\|_2$ from 5.2. Using 5.4, we conclude that

$$e^{2\lambda t}\|f(t)\|_2^2 \leqq e^{2\lambda t}\|f(N)\|_2^2 \leqq e^{2\lambda t} e^{-2\lambda N}\|f_0\|_2^2 \leqq e^{2\lambda(t-N)}\|f_0\|_2^2,$$

and obtain (5.1). □

5.2. Nonlinear L^∞ Decay

We use a L^2–L^∞ bootstrap from (3.145), Duhamel's principle, and Corollary 1 to derive nonlinear L^∞ decay.

Proof of Theorem 1. Let $h = wf$, where f solves the linearized Boltzmann equation. Then from (3.145),

$$\sup_{s\in[T,t]} \|h(s)\|_\infty \lesssim e^{-\frac{\nu_0}{2}(t-T)}\|h(T)\|_\infty + \int_T^t \|f(s)\|_2\, \mathrm{d}s.$$

We assume that $m \leqq t < m+1$ and define $\lambda^* := \min\{\frac{\nu_0}{2}, \lambda\}$, where λ is some constant from Corollary 1. We use (3.145) repeatedly for each time step, $[k, k+1),\ k \in \mathbb{N}$ and Corollary 1 to perform a L^2–L^∞ bootstrap:

$$\begin{aligned}
\|h(t)\|_\infty &\lesssim e^{-m\frac{\nu_0}{2}}\|h(0)\|_\infty + \sum_{k=0}^{m-1} e^{-k\nu_0}\int_{m-1-k}^{m-k}\|f(s)\|\,\mathrm{d}s \\
&\lesssim e^{-m\frac{\nu_0}{2}}\|h(0)\|_\infty + \sum_{k=0}^{m-1} e^{-k\nu_0}\int_{m-1-k}^{m-k} e^{-\lambda(m-1-k)}\|f(0)\|\,\mathrm{d}s \\
&\lesssim e^{-\lambda^* t}\|h(0)\|_\infty.
\end{aligned}$$

Now we solve the nonlinear problem. From the Duhamel principle, when f solves the nonlinear Boltzmann, $h = wf$ solves

$$\begin{aligned}
h &:= U(t)h_0 + \int_0^t U(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\,\mathrm{d}s, \\
\|h(t)\|_\infty &\lesssim e^{-\lambda^* t}\|h(0)\|_\infty + \left\|\int_0^t U(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s\right\|_\infty,
\end{aligned} \tag{5.5}$$

where $U(t)$ is a linear solution for the linearized Boltzmann equation. Inspired by [11], we use Duhamel's principle again to get

$$U(t-s) = G(t-s) + \int_s^t G(t-s_1)K_w U(s_1-s)\,\mathrm{d}s_1,$$

where $G(t)$ is linear solution for the system

$$\partial_t h + v\cdot\nabla_x h + \nu h = 0, \quad \text{and} \quad |G(t)h_0| \leqq e^{-\nu_0 t}|h_0|.$$

For the last term in (5.5),

$$\begin{aligned}
&\left\|\int_0^t U(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s\right\|_\infty \\
&\quad\leqq \left\|\int_0^t G(t-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s\right\|_\infty \\
&\qquad + \left\|\int_0^t\int_s^t G(t-s_1)K_w U(s_1-s)w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s_1\mathrm{d}s\right\|_\infty \\
&\quad\leqq Ce^{-\lambda^* t}\left(\sup_{0\leqq s\leqq\infty} e^{\lambda^* s}\|h(s)\|_\infty\right)^2,
\end{aligned}$$

where we used the nonlinear estimate $|w\Gamma(\frac{h}{w}, \frac{h}{w})| \leqq C\langle v\rangle^\zeta\|h\|_\infty^2$ (see Lemma 5 in [11]). Therefore, for sufficiently small $\|h_0\|_\infty \ll 1$, we have the uniform bound

$$\sup_{0\leqq t\leqq\infty} e^{\lambda^* t}\|h(t)\|_\infty \ll 1,$$

hence we get global decay and uniqueness. Also note that the positivity of F is standard by linear solvability and the solution sequence F^{ℓ}:

$$\partial_t F^{\ell+1} + v \cdot \nabla F^{\ell+1} = Q_+(F^{\ell}, F^{\ell}) - \nu(F^{\ell})F^{\ell+1}, \quad F|_{t=0} = F_0,$$
$$F^{\ell+1}(t, x, v) = F^{\ell+1}(t, x, R_x v) \quad \text{on } \partial U.$$

From $F_0 \geqq 0$ and $F^{\ell} \geqq 0$, we have $F^{\ell+1} \geqq 0$. □

Acknowledgements. The authors thank Yan Guo for stimulating discussions. Their research was supported in part by NSF-DMS 1501031, 151030, WARF, and the Herchel Smith fund. They thank the KAIST Center for Mathematical Challenges and ICERM for their kind hospitality. The authors also thank the anonymous reviewers for their valuable suggestions.

6. Appendix: Example of Sticky Grazing Point

Let us consider backward in time trajectories which start from $(1, 1)$ with velocity $v = (1, 1 + \delta)$, with $0 \leqq \delta \leqq \varepsilon \ll 1$. All the trajectories are part of the set of rays

$$\{(x, y) : y = (1 + \delta)(x - 1) + 1, \ 0 \leqq \delta \leqq \varepsilon \ll 1\}.$$

We consider that the trajectories bounce on the curve $f(x) = \frac{1}{2}x^2$. When $\delta = 0$, the trajectory bounces on $(0, 0)$ with collision angle $\frac{\pi}{4}$. When $0 < \delta \ll 1$, the bouncing point on $f(x) = \frac{1}{2}x^2$ is

$$\left(\delta_*, \frac{1}{2}\delta_*^2\right), \quad \text{where } \delta_* = (1 + \delta) - \sqrt{(1 + \delta)^2 - 2\delta}.$$

Using the specular BC, the bounced trajectory with v^1 direction is part of the set of rays

$$\left\{(x, y) : y = L(\delta)(x - \delta_*) + \frac{1}{2}\delta_*^2\right\}, \quad L(\delta) = \frac{(1 + \delta)(1 + \delta_*^2) - 2\sqrt{1 + \delta^2}}{1 + \delta_*^2 + 2\delta_*\sqrt{1 + \delta^2}}.$$

We parametrize the convex grazing boundary with parameter δ as follows:

$$\big(X(\delta), Y(\delta)\big), \quad X(0) = -Y(0) < 0.$$

Considering the tangential line on $(X(\delta), Y(\delta))$, it is easy to derive two conditions from concave grazing:

$$\begin{aligned} \frac{Y'(\delta)}{X'(\delta)} &= L(\delta), \\ -L(\delta)\delta_* + \frac{1}{2}\delta_*^2 &= -\frac{Y'(\delta)}{X'(\delta)}X(\delta) + Y(\delta). \end{aligned} \tag{6.1}$$

We differentiate the second equation and combine with first equation to get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\delta}\Big(- L(\delta)\delta_* + \frac{1}{2}\delta_*^2\Big) &= -L'(\delta)X(\delta) - L(\delta)X'(\delta) + Y'(\delta) \\ &= -L'(\delta)X(\delta). \end{aligned} \tag{6.2}$$

It is easy to check $L' > 0$ locally to see $0 < \delta \ll 1$. (6.2) gives $X(\delta)$, and this is analytic from the analyticity of $L(\delta)$ and δ_*^2; $X(\delta)$ is an analytic function of δ for local $0 < \delta \ll 1$. Using the first equation of (6.1), we obtain ODE for $Y(\delta)$ with $Y(0) = -X(0)$. Since $X(\delta)$ is analytic, $Y(\delta)$ is also analytic. Moreover, we can check the concavity of $\big(X(\delta), Y(\delta)\big)$ by

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{Y'(\delta)}{X'(\delta)}\right) = L'(\delta) > 0.$$

Finally, we see that the set of all lines grazing on $(X(\delta), Y(\delta))$ pass the sticky grazing point $(1, 1)$ after bouncing on the convex region $y = \frac{1}{2}x^2$.

References

1. BRIANT, M.: Instantaneous filling of the vacuum for the full Boltzmann equation in convex domains. *Arch. Ration. Mech. Anal.* **218**(2), 985–1041 (2015). https://doi.org/10.1007/s00205-015-0874-x
2. BRIANT, M.: Perturbative theory for the Boltzmann equation in bounded domains with different boundary conditions. *Kinet. Relat. Models* **10**(2), 329–371 (2017). https://doi.org/10.3934/krm.2017014
3. CERCIGNANI, C., ILLNER, R., PULVIRENTI, M.: *The Mathematical Theory of Dilute Gases. Applied Mathematical Sciences*, Vol. 106. Springer, New York, 1994. https://doi.org/10.1007/978-1-4419-8524-8
4. CHERNOV, N., MARKARIAN, R.: *Chaotic Billiards. Mathematical Surveys and Monographs*, Vol. 127. American Mathematical Society, Providence, 2006. https://doi.org/10.1090/surv/127
5. DEVILLETTES, L., VILLANI, C.: On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.* **159**(2), 245–316 (2005). https://doi.org/10.1007/s00222-004-0389-9
6. DO CARMO, M.: *Differential Geometry of Curves and Surfaces*. Prentice Hall, Englewood Cliffs, 1976
7. ESPOSITO, R., GUO, Y., KIM, C., MARRA, R.: Non-isothermal boundary in the Boltzmann theory and Fourier law. *Commun. Math. Phys.* **323**(1), 177–239 (2003). https://doi.org/10.1007/s00220-013-1766-2
8. ESPOSITO, R., GUO, Y., KIM, C., MARRA, R.: Stationary solutions to the Boltzmann equation in the Hydrodynamic limit. *Ann. PDE* **4**(1), 1–119 (2018). https://doi.org/10.1007/s40818-017-0037-5
9. GUO, Y., KIM, C., TONON, D., TRESCASES, A.: Regularity of the Boltzmann equation in convex domains. *Invent. Math.* **207**(1), 115–290 (2017). https://doi.org/10.1007/s00222-016-0670-8
10. GUO, Y., KIM, C., TONON, D., TRESCASES, A.: BV-regularity of the Boltzmann equation in non-convex domains. *Arch. Ration. Mech. Anal.* **220**(3), 1045–1093 (2016). https://doi.org/10.1007/s00205-015-0948-9
11. GUO, Y.: Decay and continuity of Boltzmann equation in bounded domains. *Arch. Ration. Mech. Anal.* **197**(3), 713–809 (2010). https://doi.org/10.1007/s00205-009-0285-y
12. HALPERN, B.: Strange billiard tables. *Tran. Am. Math. Soc.* **232**, 297–305 (1977). https://doi.org/10.1090/S0002-9947-1977-0451308-7
13. KIM, C.: Formation and propagation of discontinuity for Boltzmann equation in non-convex domains. *Commun. Math. Phys.* **308**(3), 641–701 (2011). https://doi.org/10.1007/s00220-011-1355-1
14. KIM, C.: Boltzmann equation with a large potential in a periodic box. *Commun. PDE* **39**(8), 1393–1423 (2014). https://doi.org/10.1080/03605302.2014.903278

15. Kim, C., Yun, S.: The Boltzmann equation near a rotational local maxwellian. *SIAM J. Math. Anal.* **44**(4), 2560–2598 (2012). https://doi.org/10.1137/11084981X
16. Kim, C., Lee, D.: Hölder regularity and decay of the Boltzmann equation in some non-convex domain (in preparation)
17. Kim, C., Lee, D.: The Boltzmann equation with specular boundary condition in convex domains. *Comm. Pure Appl. Math.* **71**, 411–504 (2018). https://doi.org/10.1002/cpa.21705
18. Shizuta, Y., Asano, K.: Global solutions of the Boltzmann equation in a bounded convex domain. *Proc. Jpn. Acad.* **53A**, 3–5 (1977). https://doi.org/10.3792/pjaa.53.3

Chanwoo Kim and Donghyun Lee
University of Wisconsin Madison,
Madison, WI, USA.
e-mail: ckim@math.wisc.edu
e-mail: dlee374@wisc.edu

(*Received March 24, 2017 / Accepted March 15, 2018*)
Published online April 26, 2018