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Strongly Stratified Limit for the 3D Inviscid Boussinesq Equations

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Abstract

We consider the initial value problem of the 3D inviscid Boussinesq equations for stably stratified fluids. We prove the long time existence of classical solutions for large initial data when the buoyancy frequency is sufficiently high. Furthermore, we consider the singular limit of the strong stratification, and show that the long time classical solution converges to that of 2D incompressible Euler equations in some space-time Strichartz norms.

1. Introduction

Let us consider the initial value problem for the 3D inviscid Boussinesq equations, describing the motion of perfect incompressible fluids in $\mathbb{R}^3$:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla q + \eta e_3 & t > 0,\ x \in \mathbb{R}^3, \\ \partial_t \eta + (v \cdot \nabla)\eta = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \nabla \cdot v = 0 & t \geqslant 0,\ x \in \mathbb{R}^3, \\ v(0, x) = v_0(x), \quad \eta(0, x) = \eta_0(x) & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

The unknown functions $v = (v_1(t, x), v_2(t, x), v_3(t, x))^T$, $\eta = \eta(t, x)$ and $q = q(t, x)$ represent the velocity field, the temperature and the scalar pressure of the fluids, respectively, while $v_0 = (v_{0,1}(x), v_{0,2}(x), v_{0,3}(x))^T$ is the given initial velocity field satisfying the compatibility condition $\nabla \cdot v_0 = 0$ and $\eta_0 = \eta_0(x)$ is the given initial temperature. The vertical unit vector is denoted by $e_3 = (0, 0, 1)^T$.

In this manuscript, we prove the long time existence of classical solutions to (1.1) around the explicit stratified solution $(v_s, \eta_s, q_s) = (0, N^2 x_3, N^2 x_3^2/2)$ when the constant temperature gradient $N = \sqrt{\mathrm{d}\eta_s/\mathrm{d}x_3} > 0$ is sufficiently large. More precisely, we shall show that for given initial disturbance $\phi = (v_0, (\eta_0 - N^2 x_3)/N)^T \in H^{s+4}(\mathbb{R}^3)$ with $s \geqslant 3$ and for given *finite* time T, there exists a positive parameter $N_{\phi,T}$ such that the 3D inviscid stratified Boussinesq system (1.3) admits a unique classical solution on the time interval $[0,T]$ provided $N \geqslant N_{\phi,T}$. Furthermore, we consider the singular limit of the strong stratification as $N \to \infty$, and show that the long time classical solution v^N to (1.3) strongly converges to that of the 2D incompressible Euler equations in the space-time norm $L^q(0,T;W^{1,\infty}(\mathbb{R}^3))$ with the convergence rate $O(N^{-\frac{1}{q}})$ for $4 \leqslant q < \infty$.

Before stating our result, we first review the local existence results on the inviscid Boussinesq equations. In the Sobolev spaces H^s-framework, it is known that for initial data $(v_0,\eta_0) \in H^s(\mathbb{R}^3)$ with $\nabla \cdot v_0 = 0$ and $s > 5/2$ there exists a $T_0 = T_0(s, \|(v_0,\eta_0)\|_{H^s}) > 0$ such that (1.1) possesses a unique classical solution (v,η) in the class $C([0,T_0];H^s(\mathbb{R}^3))$. See [7,8,16,34] for the local existence theory of (1.1) in function spaces embedded in C^1 class such as the Hölder spaces and the Besov spaces, and the blow-up criteria of local solutions. We also refer to [1,9,17] for the global existence results on the 2D Boussinesq systems.

Next, let us consider the solution of (1.1) around a stratified solution. It is known that the system (1.1) has an elementary explicit stationary solution (v_s,η_s,q_s) of the form

$$v_s \equiv 0, \quad \eta_s(x_3) = N^2 x_3, \quad q_s(x_3) = \frac{N^2}{2} x_3^2, \tag{1.2}$$

satisfying the hydrostatic balance $\dfrac{dq_s}{\mathrm{d}x_3} = \eta_s$, where $N > 0$ is called the buoyancy or the Brunt–Väisälä frequency and represents the strength of stable stratification. Let us set

$$\theta(t,x) = \eta(t,x) - \eta_s(x_3), \quad p(t,x) = q(t,x) - q_s(x_3),$$

where η_s and q_s are given by (1.2). We consider the time evolution of the perturbations around a stable state in hydrostatic balance, and then (v,θ,p) should satisfy

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla p + \theta e_3, \\ \partial_t \theta + (v\cdot\nabla)\theta = -N^2 v_3, \\ \nabla\cdot v = 0, \\ v(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x) = \eta_0(x) - N^2 x_3, \end{cases} \tag{1.3}$$

where θ_0 denotes the initial thermal disturbance. The system (1.3) exhibits a dispersive nature due to the presence of the skew-symmetric linear term $(\theta e_3, -N^2 v_3)^T$ by the stable stratification. This phenomenon is closely related to the dispersive estimates for the propagator $e^{\pm iNt|D_h|/|D|}$ defined by the Fourier integral

$$e^{\pm iNt\frac{|D_h|}{|D|}} f(x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm iNt\frac{|\xi_h|}{|\xi|}} \widehat{f}(\xi)\,\mathrm{d}\xi, \qquad (t,x) \in \mathbb{R}^{1+3}.$$

Here, $\xi_h = (\xi_1,\xi_2) \in \mathbb{R}^2$ so that $|\xi_h| = \sqrt{\xi_1^2+\xi_2^2}$ and $\widehat{f}$ denotes the Fourier transform of f. The sharp dispersive estimate for $e^{\pm iNt|D_h|/|D|}$ was established in [33]. WIDMAYER [39] proved the local well-posedness of (1.3) in $H^s(\mathbb{R}^3)$ with $s \geqslant 3$ for all $N \geqslant 0$. Furthermore, it is shown in [39] that for initial data $(v_0,\theta_0)^T \in$

$H^{s+3}(\mathbb{R}^3) \cap W^{5,1}(\mathbb{R}^3)$ with $s \geqslant 3$, the local solution (v^N, θ^N) to (1.3) on $[0, T_0]$ can be decomposed into two parts as

$$(v^N, \theta^N/N) = (w^N, 0, 0) + (u^N, \rho^N), \qquad w^N = (w_1^N, w_2^N), \quad u^N = (u_1^N, u_2^N, u_3^N),$$

and there holds for every $0 < t \leqslant T_0$ that

$$\|(u^N, \rho^N)(t)\|_{W^{1,\infty}(\mathbb{R}^3)} \to 0, \quad \|w^N(t) - \overline{w}(t)\|_{L^2(\mathbb{R}^3)} \to 0$$

as $N \to \infty$, where $\overline{w} = (\overline{w}_1(t,x), \overline{w}_2(t,x))$ solves the 2D incompressible Euler equations (see (1.7) below). For the related singular limit problems to the rotating Navier–Stokes equations and the viscous and inviscid rotating Boussinesq equations, we refer to [2–4,10–15,18,36] (see also [6,19–21,35] for compressible stratified flows).

To state our result more precisely, we firstly rewrite the sytem (1.3). Let us combine the velocity field with the rescaled thermal disturbance into the new unknown function

$$u := \left(v, \frac{\theta}{N}\right)^T = \left(v_1, v_2, v_3, \frac{\theta}{N}\right)^T.$$

Put

$$J := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \widetilde{\nabla} := (\nabla, 0)^T.$$

Then, the perturbed system (1.3) can be written as

$$\begin{cases} \partial_t u + NJu + (u \cdot \widetilde{\nabla})u + \widetilde{\nabla} p = 0, \quad \widetilde{\nabla} \cdot u = 0, \\ u(0,x) = \phi(x), \end{cases} \tag{1.4}$$

where $\phi := (v_0, \theta_0/N)^T$. Next, let $\mathbb{P}$ be the Helmholtz projection of the velocity v onto the divergence-free vector fields which is defined by

$$\mathbb{P} := \left(\begin{array}{c|c} \left(\delta_{jk} + R_j R_k\right)_{1 \leqslant j,k \leqslant 3} & 0 \\ \hline 0 & 1 \end{array} \right).$$

Here $\{R_j\}_{1 \leqslant j \leqslant 3}$ denote the Riesz transforms on $\mathbb{R}^3$. Applying the Helmholtz projection $\mathbb{P}$ to (1.4) gives the following evolution equation:

$$\begin{cases} \partial_t u + N\mathbb{P}J\mathbb{P}u + \mathbb{P}(u \cdot \widetilde{\nabla})u = 0, \quad \widetilde{\nabla} \cdot u = 0, \\ u(0,x) = \phi(x). \end{cases} \tag{1.5}$$

Here, we have used the facts that $\mathbb{P}\widetilde{\nabla} p = 0$ and $\mathbb{P}u = u$ since $\widetilde{\nabla} \cdot u = 0$.

In this paper, we address the *long time* existence of classical solutions to (1.5) when the buoyancy frequency N is sufficiently high, and then we show that the long time classical solution v^N to (1.5) converges to that of the 2D incompressible Euler equations in the space-time Strichartz norm $L^q(0, T; W^{1,\infty}(\mathbb{R}^3))$ for $4 \leqslant q < \infty$.

The main result of this paper reads as follows:

Theorem 1.1. *Let $s \in \mathbb{N}$ satisfy $s \geqslant 3$, and let $4 \leqslant q < \infty$. Then, for every $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in H^{s+4}(\mathbb{R}^3)$ satisfying $\widetilde{\nabla} \cdot \phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $N_{\phi,T}$ depending on s, q, T and $\|\phi\|_{H^{s+4}}$ such that if $N \geqslant N_{\phi,T}$ then (1.5) possesses a unique classical solution u^N in the class*

$$u^N \in C([0,T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0,T]; H^{s+3}(\mathbb{R}^3)).$$

Furthermore, there exists a positive constant $C = C(s, q, T, \|\phi\|_{H^{s+4}})$ such that

$$\|u^N - u^0\|_{L^q(0,T;W^{1,\infty})} \leqslant C N^{-\frac{1}{q}} \tag{1.6}$$

for all $N \geqslant N_{\phi,T}$, where $u^0 = (w, 0, 0)^T$ and $w = (w_1(t,x), w_2(t,x))^T$ is the classical solution of the two dimensional Euler equations

$$\begin{cases} \partial_t w + \mathbb{P}_h (w \cdot \nabla_h) w = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \nabla_h \cdot w = 0 & t \geqslant 0,\ x \in \mathbb{R}^3, \\ w(0,x) = \mathbb{P}_h \phi_h(x) & x \in \mathbb{R}^3, \\ w \in C([0,T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0,T]; H^{s+3}(\mathbb{R}^3)). \end{cases} \tag{1.7}$$

Here, $\phi_h = (\phi_1, \phi_2)^T$, $\nabla_h = (\partial_1, \partial_2)^T$ and $\mathbb{P}_h = \left(\delta_{jk} + \partial_j \partial_k (-\Delta_h)^{-1}\right)_{1 \leqslant j,k \leqslant 2}$ denotes the two dimensional Helmholtz projection.

This can be compared with the corresponding results for the 3D rotating Euler equations and the 2D inviscid stratified Boussinesq equations. In [31,38], the long time existence of classical solutions to those systems were proved for large Coriolis parameter and high buoyancy frequency; their proofs are by the contradiction arguments based on the Strichartz estimate for the linear propagator with the blow-up criteria of the Beale–Kato–Majda type. However, the situation is different for the 3D inviscid stratified Boussinesq equations. Indeed, the linear solution of (1.5) is given explicitly by

$$e^{-tN\mathbb{P}J\mathbb{P}}\phi = e^{iNt\frac{|D_h|}{|D|}} P_+\phi + e^{-iNt\frac{|D_h|}{|D|}} P_-\phi + P_0\phi \tag{1.8}$$

(see Proposition 2.1 in Section 2 for details), which has the stationary mode $P_0\phi$. Thus, the continuation arguments in [31,38] cannot be applied directly. To overcome this, we adapt the ideas in [10] for the viscous rotating stratified fluids and the arguments in [25,37] to extend the local solutions of the 3D Euler equations, and employ the stability method for the limit system. In the proof of Theorem 1.1, we first show the global regularity of the limit system (1.7) and give the global a priori $H^{s+3}(\mathbb{R}^3)$-estimate for the solution $u^0 = (w, 0, 0)^T$ to the limit system with $u^0(0) = P_0\phi$. Next, we introduce the modified linear dispersive equations

$$\begin{cases} \partial_t u^{\pm} \mp iN \dfrac{|D_h|}{|D|} u^{\pm} + P_{\pm}(u^0 \cdot \widetilde{\nabla}) u^0 = 0, \quad \widetilde{\nabla} \cdot u^{\pm} = 0, \\ u^{\pm}(0,x) = P_{\pm}\phi(x) \end{cases} \tag{1.9}$$

(see (4.2) in Section 4), and establish the space-time Strichartz estimates for the solutions $u^{\pm}$ in $L^q(0,T; W^{1,\infty}(\mathbb{R}^3))$ with the decay rate $N^{-\frac{1}{q}}$ for $4 \leqslant q < \infty$.

Then, the difference $v^N = u^N - u^0 - u^+ - u^-$ of u^0, $u^\pm$ and the local solution u^N to (1.5) with $u^N(0) = \phi$ satisfies

$$\partial_t v^N + N\mathbb{P}J\mathbb{P}v^N + \mathbb{P}(u^N \cdot \widetilde{\nabla})v^N + \sum_{j=0,\pm} \mathbb{P}(v^N \cdot \widetilde{\nabla})u^j + \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \mathbb{P}(u^j \cdot \widetilde{\nabla})u^k = 0$$

with $v^N(0) = 0$ on some local time interval. We shall show that the H^s-norm of v^N can be taken arbitrarily small provided that the buoyancy frequency N is large enough depending only on the given data s, q, T and $\|\phi\|_{H^{s+4}}$. Then, the local solution u^N has a uniform H^s-bound, and can be continued to the given time interval $[0, T]$. Furthermore, the estimate (1.6) of the singular limit immediately follows from the H^s-bound for v^N and the space-time estimates for $u^\pm$.

This paper is organized as follows: in Section 2, we derive the explicit formula (1.8) of linear solutions $e^{-tN\mathbb{P}J\mathbb{P}}\phi$, and establish the space-time Strichartz estimates for the linear propagator $e^{\pm iNt|D_h|/|D|}$. In Section 3, we show the global regularity of the limit system (1.7). In Section 4, we introduce the modified linear dispersive systems (1.9) and show the space-time decay estimates for $u^\pm$. In Section 5, we present the proof of Theorem 1.1.

Throughout this paper, we denote by C the constants which may differ at each occurrence. In particular, $C = C(\cdot, \cdots, \cdot)$ will denote the constant which depends only on the quantities appearing in parentheses.

2. Linear Estimates

In this section, we derive the explicit representation for the time evolution semigroup generated by the linear operator $-N\mathbb{P}J\mathbb{P}$, and establish the homogeneous and inhomogeneous space-time Strichartz estimates for the linear propagator $e^{\pm iNt|D_h|/|D|}$.

Linear solutions

We follow the argument in [33, Section 2]. Let us consider the linear system associated to (1.5):

$$\begin{cases} \partial_t u + N\mathbb{P}J\mathbb{P}u = 0, \quad \widetilde{\nabla} \cdot u = 0, \\ u(0, x) = \phi(x). \end{cases} \tag{2.1}$$

Applying the Fourier transform to (2.1), we have

$$\begin{cases} \partial_t \widehat{u} + NP(\xi)JP(\xi)\widehat{u} = 0, \quad (\xi, 0)^T \cdot \widehat{u} = 0, \\ \widehat{u}(0, \xi) = \widehat{\phi}(\xi). \end{cases} \tag{2.2}$$

Here, $P(\xi)$ is the multiplier matrix of the projection $\mathbb{P}$ defined by $\widehat{\mathbb{P}u}(\xi) = P(\xi)\widehat{u}(\xi)$, which is given explicitly by

$$P(\xi) := \left(\begin{array}{c|c} \left(\delta_{jk} - \dfrac{\xi_j \xi_k}{|\xi|^2}\right)_{1\leqslant j,k\leqslant 3} & 0 \\ \hline 0 & 1 \end{array} \right).$$

Set $S(\xi) := -P(\xi)JP(\xi)$. Then, direct calculation yields

$$S(\xi) = \frac{1}{|\xi|^2}\begin{pmatrix} 0 & 0 & 0 & -\xi_1\xi_3 \\ 0 & 0 & 0 & -\xi_2\xi_3 \\ 0 & 0 & 0 & \xi_1^2+\xi_2^2 \\ \xi_1\xi_3 & \xi_2\xi_3 & -(\xi_1^2+\xi_2^2) & 0 \end{pmatrix},$$

and then

$$\det\{\lambda I - S(\xi)\} = \lambda^2\left(\lambda^2 + \frac{\xi_1^2+\xi_2^2}{|\xi|^2}\right).$$

Thus, the eigenvalues of $S(\xi)$ are $\left\{\pm i\frac{|\xi_h|}{|\xi|},\ 0,\ 0\right\}$, where $\xi_h = (\xi_1, \xi_2)$ and $|\xi_h| = \sqrt{\xi_1^2+\xi_2^2}$. Moreover, the corresponding eigenvectors are given by

$$a_\pm(\xi) = \frac{1}{\sqrt{2}|\xi_h||\xi|}\begin{pmatrix} \pm i\xi_1\xi_3 \\ \pm i\xi_2\xi_3 \\ \mp i|\xi_h|^2 \\ |\xi_h||\xi| \end{pmatrix},\quad a_0(\xi) = \frac{1}{|\xi_h|}\begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix},\quad b_0(\xi) = \frac{1}{|\xi|}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{pmatrix}. \tag{2.3}$$

We see that $\{a_+(\xi), a_-(\xi), a_0(\xi), b_0(\xi)\}$ is an orthonormal basis in $\mathbb{C}^4$ and satisfies

$$S(\xi)a_\pm(\xi) = \pm i\frac{|\xi_h|}{|\xi|}a_\pm(\xi), \quad S(\xi)a_0(\xi) = S(\xi)b_0(\xi) = 0.$$

Hence the solution to (2.2) can be written as

$$\widehat{u}(t,\xi) = e^{NtS(\xi)}\widehat{\phi}(\xi) = \sum_{\sigma\in\{\pm,0\}} e^{\sigma iNt\frac{|\xi_h|}{|\xi|}}\langle\widehat{\phi}(\xi), a_\sigma(\xi)\rangle_{\mathbb{C}^4}a_\sigma(\xi).$$

Here, we remark that $\langle\widehat{\phi}(\xi), b_0(\xi)\rangle_{\mathbb{C}^4} = 0$ by the divergence-free condition $\widetilde{\nabla}\cdot\phi = 0$. Therefore, the solution to (2.1) is explicitly given in terms of the evolution semigroup, and we obtain the following proposition.

Proposition 2.1. *For every $N \geqslant 0$ and for every $\phi \in L^2(\mathbb{R}^3)$ with $\widetilde{\nabla}\cdot\phi = 0$, there exists a unique solution u to (2.1) which is given explicitly by*

$$\begin{aligned} u(t,x) &= e^{-tN\mathbb{P}J\mathbb{P}}\phi(x) \\ &= e^{iNtp(D)}P_+\phi(x) + e^{-iNtp(D)}P_-\phi(x) + P_0\phi(x), \end{aligned}$$

where

$$P_j\phi := \mathcal{F}^{-1}[\langle\widehat{\phi}(\xi), a_j(\xi)\rangle_{\mathbb{C}^4}a_j(\xi)] \quad (j = \pm, 0), \tag{2.4}$$

$$e^{\pm iNtp(D)}f(x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi\pm iNtp(\xi)}\widehat{f}(\xi)\,d\xi \tag{2.5}$$

and

$$p(\xi) := \frac{|\xi_h|}{|\xi|} = \frac{\sqrt{\xi_1^2+\xi_2^2}}{|\xi|}, \qquad \xi \in \mathbb{R}^3\backslash\{0\}. \tag{2.6}$$

Strichartz estimates

In this subsection, we shall prove the homogeneous and inhomogeneous space-time Strichartz estimates for the linear propagator $e^{\pm iNt|D_h|/|D|}$ defined by (2.5)–(2.6). Since the phase $p(\xi) = |\xi_h|/|\xi|$ is homogeneous of degree 0, by the Littlewood-Paley decomposition and scaling, the matter is reduced to the frequency localized case. Also, the sign $\pm$ does not have any role. Hence we consider the operators

$$U_N(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi+iNtp(\xi)}\psi(\xi)^2\widehat{f}(\xi)\,\mathrm{d}\xi,$$
$$V_N(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi+iNtp(\xi)}\psi(\xi)\widehat{f}(\xi)\,\mathrm{d}\xi, \quad (t,x)\in\mathbb{R}^{1+3},$$

where ψ is a real-valued function in $\mathscr{S}(\mathbb{R}^3)$ satisfying $\operatorname{supp}\psi \subset \{2^{-2} \leqslant |\xi| \leqslant 2^2\}$ and $\psi(\xi) = 1$ on $\{2^{-1} \leqslant |\xi| \leqslant 2\}$. The sharp dispersive estimates for $U_N(t)$ and $V_N(t)$ are obtained in [33].

Lemma 2.2. [33, Theorem 1.1] *There exists a positive constant $C = C(\psi) > 0$ such that*

$$\|U_N(t)f\|_{L^\infty} \leqslant C(1+N|t|)^{-\frac{1}{2}}\|f\|_{L^1}$$

for all $t \in \mathbb{R}$ and $f \in L^1(\mathbb{R}^3)$. The same is true for $V_N(t)$. Also, the decay rate $1/2$ cannot be improved to a larger one.

Now we investigate the boundedness of $U_N(t)$. We use the notation for the space-time norm

$$\|f\|_{L^q_tL^r_x} := \|f\|_{L^q(\mathbb{R};L^r(\mathbb{R}^3))}.$$

The following results are the homogeneous and inhomogeneous space-time estimates for the linear operator $U_N(t)$:

Lemma 2.3. *Let the exponents $q, \tilde{q}, r, \tilde{r}$ satisfy*

$$\frac{2}{q}+\frac{1}{r} \leqslant \frac{1}{2}, \quad \frac{2}{\tilde{q}}+\frac{1}{\tilde{r}} \leqslant \frac{1}{2}, \quad 4 \leqslant q,\tilde{q} \leqslant \infty, \quad 2 \leqslant r,\tilde{r} \leqslant \infty. \tag{2.7}$$

Then, there exist positive constants $C_1 = C_1(\psi, q, r)$ and $C_2 = C_2(\psi, q, \tilde{q}, r, \tilde{r})$ such that

$$\|U_N(t)f\|_{L^q_tL^r_x} \leqslant C_1N^{-\frac{1}{q}}\|f\|_{L^2}, \tag{2.8}$$
$$\left\|\int_{-\infty}^t U_N(t-s)F(s)\,\mathrm{d}s\right\|_{L^q_tL^r_x} \leqslant C_2N^{-\frac{1}{q}-\frac{1}{\tilde{q}}}\|F\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x} \tag{2.9}$$

for $f \in L^2(\mathbb{R}^3)$ and $F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^3))$, where $1/\tilde{r}+1/\tilde{r}' = 1$ and $1/\tilde{q}+1/\tilde{q}' =$ 1.

Proof. We remark that the $L^1 - L^\infty$ decay rate of $U_N(t)$ is $-1/2$ and the admissible range (2.7) does not include the endpoint $q = 2$. Hence the proof is based on the standard TT^* argument and the interpolation (See for examples, [22,28,30]).

For the homogeneous estimate (2.8), it suffices to show its adjoint estimate

$$\left\| \int_{\mathbb{R}} U_N(-s)F(s)\,\mathrm{d}s \right\|_{L^2} \leqslant CN^{-\frac{1}{q}} \|F\|_{L_t^{q'}L_x^{r'}}, \tag{2.10}$$

and also (2.10) follows from the estimate

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\langle U_N(-s)F(s),\ U_N(-t)G(t)\rangle_{L^2}\right| \mathrm{d}s\,\mathrm{d}t \leqslant CN^{-\frac{2}{q}} \|F\|_{L_t^{q'}L_x^{r'}} \|G\|_{L_t^{q'}L_x^{r'}}. \tag{2.11}$$

Now we shall show (2.11). By Lemma 2.2 and the L^2-boundedness of $U_N(t)$ with $\|U_N(t)f\|_{L^2} \leqslant C\|f\|_{L^2}$, we have for $2 \leqslant r \leqslant \infty$ that

$$\|U_N(t)f\|_{L^r} \leqslant C(1 + N|t|)^{-\frac{1}{2}(1-\frac{2}{r})} \|f\|_{L^{r'}} \tag{2.12}$$

for all $t \in \mathbb{R}$. Then, it follows from (2.12) and the Hausdorff-Young inequality that

$$\begin{aligned}
\left|\langle U_N(-s)F(s),\ U_N(-t)G(t)\rangle_{L^2}\right| &= \left| \int_{\mathbb{R}^3} U_N(t-s)F(s) \cdot \overline{\mathcal{F}^{-1}[\psi^2] * G(t)}\,\mathrm{d}x \right| \\
&\leqslant \|U_N(t-s)F(s)\|_{L^r} \|\mathcal{F}^{-1}[\psi^2] * G(t)\|_{L^{r'}} \\
&\leqslant \frac{C}{(1+N|t-s|)^{\frac{1}{2}-\frac{1}{r}}} \|F(s)\|_{L^{r'}} \|G(t)\|_{L^{r'}}.
\end{aligned} \tag{2.13}$$

For $(q, r) = (\infty, 2)$, we have by (2.13)

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\langle U_N(-s)F(s),\ U_N(-t)G(t)\rangle_{L^2}\right| \mathrm{d}s\,\mathrm{d}t \leqslant C\|F\|_{L_t^1L_x^2} \|G\|_{L_t^1L_x^2}. \tag{2.14}$$

In the case $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ with $(q, r) \neq (\infty, 2)$, it follows from (2.13) and the Hardy–Littlewood–Sobolev inequality that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\langle U_N(-s)F(s),\ U_N(-t)G(t)\rangle_{L^2}\right| \mathrm{d}s\,\mathrm{d}t \\
&\leqslant CN^{-(\frac{1}{2}-\frac{1}{r})} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|t-s|^{\frac{1}{2}-\frac{1}{r}}} \|F(s)\|_{L^{r'}} \|G(t)\|_{L^{r'}}\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant CN^{-\frac{2}{q}} \left\| \int_{\mathbb{R}} \frac{1}{|t-s|^{\frac{2}{q}}} \|F(s)\|_{L^{r'}}\,\mathrm{d}s \right\|_{L_t^q} \|G\|_{L_t^{q'}L_x^{r'}} \\
&\leqslant CN^{-\frac{2}{q}} \|F\|_{L_t^{q'}L_x^{r'}} \|G\|_{L_t^{q'}L_x^{r'}}.
\end{aligned} \tag{2.15}$$

In the case $\frac{2}{q}+\frac{1}{r}<\frac{1}{2}$, we have by (2.13) and the Hausdorff–Young inequality

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\langle U_N(-s)F(s),\ U_N(-t)G(t)\rangle_{L^2}\right| \mathrm{d}s\,\mathrm{d}t\\
&\quad\leqslant C\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(1+N|t-s|)^{\frac{1}{2}-\frac{1}{r}}}\|F(s)\|_{L^{r'}}\|G(t)\|_{L^{r'}}\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\leqslant C\left\|\int_{\mathbb{R}}\frac{1}{(1+N|t-s|)^{\frac{1}{2}-\frac{1}{r}}}\|F(s)\|_{L^{r'}}\,\mathrm{d}s\right\|_{L^q_t}\|G\|_{L^{q'}_tL^{r'}_x}\\
&\quad\leqslant C\left\|\frac{1}{(1+N|t|)^{\frac{1}{2}-\frac{1}{r}}}\right\|_{L_t^{\frac{q}{2}}}\|F\|_{L^{q'}_tL^{r'}_x}\|G\|_{L^{q'}_tL^{r'}_x}\\
&\quad= CN^{-\frac{2}{q}}\|F\|_{L^{q'}_tL^{r'}_x}\|G\|_{L^{q'}_tL^{r'}_x}.
\end{aligned}
\tag{2.16}
$$

Hence we obtain the homogeneous estimate (2.8) by (2.14)–(2.16). Note that it also holds that

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\right| \mathrm{d}s\,\mathrm{d}t\leqslant CN^{-\frac{2}{q}}\|F\|_{L^{q'}_tL^{r'}_x}\|G\|_{L^{q'}_tL^{r'}_x},
\tag{2.17}
$$

by the exactly same procedure as above.

Next, we shall prove the inhomogeneous estimate (2.9). Since we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t}U_N(t-s)F(s)\,\mathrm{d}s\right)\overline{G(t)}\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\qquad=\left|\int_{\mathbb{R}}\int_{-\infty}^{t}\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\,\mathrm{d}s\,\mathrm{d}t\right|,
\end{aligned}
$$

by duality, it suffices to show that

$$
\left|\int_{\mathbb{R}}\int_{-\infty}^{t}\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\,\mathrm{d}s\,\mathrm{d}t\right|\leqslant CN^{-\frac{1}{q}-\frac{1}{\tilde{q}}}\|F\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x}\|G\|_{L^{q'}_tL^{r'}_x}.
$$

Firstly, it easily follows from (2.17) that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{-\infty}^{t}\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\,\mathrm{d}s\,\mathrm{d}t\right|\\
&\quad\leqslant\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\right|\mathrm{d}s\,\mathrm{d}t\\
&\quad\leqslant CN^{-\frac{2}{q}}\|F\|_{L^{q'}_tL^{r'}_x}\|G\|_{L^{q'}_tL^{r'}_x}.
\end{aligned}
$$

Hence we have for the case $(q,r)=(\tilde{q},\tilde{r})$

$$
\left\|\int_{-\infty}^{t}U_N(t-s)F(s)\,\mathrm{d}s\right\|_{L^q_tL^r_x}\leqslant C_2N^{-\frac{2}{q}}\|F\|_{L^{q'}_tL^{r'}_x}.
\tag{2.18}
$$

Also, the estimate (2.10) for $V_N(t)$ gives that

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{-\infty}^{t}\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\,\mathrm{d}s\,\mathrm{d}t\right|\\ \leqslant \int_{\mathbb{R}}\left\|\int_{-\infty}^{t}V_N(-s)F(s)\,\mathrm{d}s\right\|_{L^2}\|V_N(-t)G(t)\|_{L^2}\,\mathrm{d}t\\ \leqslant CN^{-\frac{1}{\tilde q}}\|F\|_{L_t^{\tilde q'}L_x^{\tilde r'}}\|G\|_{L_t^1L_x^2},\end{aligned}$$

which yields for $\frac{2}{\tilde q}+\frac{1}{\tilde r}\leqslant\frac{1}{2}$ that

$$\left\|\int_{-\infty}^{t}U_N(t-s)F(s)\,\mathrm{d}s\right\|_{L_t^\infty L_x^2}\leqslant C_2N^{-\frac{1}{\tilde q}}\|F\|_{L_t^{\tilde q'}L_x^{\tilde r'}}. \tag{2.19}$$

Therefore, interpolating (2.18) and (2.19), and using the duality argument, we have for $\frac{2}{q}+\frac{1}{r}=\frac{1}{2},\ \frac{2}{\tilde q}+\frac{1}{\tilde r}=\frac{1}{2}$

$$\left\|\int_{-\infty}^{t}U_N(t-s)F(s)\,\mathrm{d}s\right\|_{L_t^qL_x^r}\leqslant C_2N^{-\frac{1}{q}-\frac{1}{\tilde q}}\|F\|_{L_t^{\tilde q'}L_x^{\tilde r'}}. \tag{2.20}$$

Next, we consider the case $2\leqslant r\leqslant\infty$ and $\frac{2}{\tilde q}+\frac{1}{\tilde r}\leqslant\frac{1}{2}$. Since there holds

$$\begin{aligned}\|V_N(-t)G(t)\|_{L^2}&=\|e^{-iNtp(\xi)}\psi(\xi)\widehat{G}(t)\|_{L_\xi^2}\\ &=\|\mathcal{F}^{-1}[\psi]*G(t)\|_{L^2}\\ &\leqslant\|\mathcal{F}^{-1}[\psi]\|_{L^{(\frac{1}{2}+\frac{1}{r})^{-1}}}\|G(t)\|_{L^{r'}},\end{aligned}$$

we have by (2.10) for $V_N(t)$ that

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{-\infty}^{t}\langle V_N(-s)F(s),\ V_N(-t)G(t)\rangle_{L^2}\,\mathrm{d}s\,\mathrm{d}t\right|\\ \leqslant \int_{\mathbb{R}}\left\|\int_{-\infty}^{t}V_N(-s)F(s)\,\mathrm{d}s\right\|_{L^2}\|V_N(-t)G(t)\|_{L^2}\,\mathrm{d}t\\ \leqslant CN^{-\frac{1}{\tilde q}}\|F\|_{L_t^{\tilde q'}L_x^{\tilde r'}}\|G\|_{L_t^1L_x^{r'}},\end{aligned}$$

which yields for $2\leqslant r\leqslant\infty$ and $\frac{2}{\tilde q}+\frac{1}{\tilde r}\leqslant\frac{1}{2}$ that

$$\left\|\int_{-\infty}^{t}U_N(t-s)F(s)\,\mathrm{d}s\right\|_{L_t^\infty L_x^r}\leqslant C_2N^{-\frac{1}{\tilde q}}\|F\|_{L_t^{\tilde q'}L_x^{\tilde r'}}. \tag{2.21}$$

Then, since every (q,r) satisfying $\frac{2}{q}+\frac{1}{r}\leqslant\frac{1}{2}$ is an interpolation between (∞,r) and (q_0,r) with $\frac{2}{q_0}+\frac{1}{r}=\frac{1}{2}$, it follows from (2.20), (2.21) and the interpolation argument that the inhomogeneous estimates (2.9) hold true for $\frac{2}{q}+\frac{1}{r}\leqslant\frac{1}{2}$ and $\frac{2}{\tilde{q}}+\frac{1}{\tilde{r}}=\frac{1}{2}$. By duality, we then have for $\frac{2}{q}+\frac{1}{r}=\frac{1}{2}$ and $\frac{2}{\tilde{q}}+\frac{1}{\tilde{r}}\leqslant\frac{1}{2}$ that

$$\left\|\int_{-\infty}^{t} U_N(t-s)F(s)\,\mathrm{d}s\right\|_{L_t^q L_x^r}\leqslant C_2 N^{-\frac{1}{q}-\frac{1}{\tilde{q}}}\|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}. \tag{2.22}$$

Again, since every (q,r) satisfying $\frac{2}{q}+\frac{1}{r}\leqslant\frac{1}{2}$ is an interpolation between (∞,r) and (q_0,r) with $\frac{2}{q_0}+\frac{1}{r}=\frac{1}{2}$, it follows from (2.21), (2.22) and the interpolation argument that the inhomogeneous estimates (2.9) hold for every (q,r) and $(\tilde{q},\tilde{r})$ satisfying (2.7). This completes the proof of Lemma 2.3. □

From (2.8), (2.9), the Littlewood–Paley theory and scaling, we can show the space-time Strichartz estimates for the original propagator $e^{\pm iNt|D_h|/|D|}$ as a corollary of Lemma 2.3. Let φ_0 be a function in $\mathscr{S}(\mathbb{R}^3)$ satisfying

$$0\leqslant\varphi_0(\xi)\leqslant 1\quad\text{for all }\xi\in\mathbb{R}^3,\quad \operatorname{supp}\varphi_0\subset\left\{\xi\in\mathbb{R}^3\;\middle|\;2^{-1}\leqslant|\xi|\leqslant 2\right\}$$

and

$$\sum_{j\in\mathbb{Z}}\varphi_j(\xi)=1\quad\text{for every }\xi\in\mathbb{R}^3\backslash\{0\},$$

where $\varphi_j(\xi):=\varphi_0(2^{-j}\xi)$. We set $\Delta_j f:=\mathcal{F}^{-1}[\varphi_j(\xi)\widehat{f}(\xi)]$ for $j\in\mathbb{Z}$. Then, for $s\in\mathbb{R}$ and $1\leqslant r,\sigma\leqslant\infty$, we define the semi-norm of the homogeneous Besov spaces $\dot{B}^s_{r,\sigma}(\mathbb{R}^3)$ as

$$\|f\|_{\dot{B}^s_{r,\sigma}}:=\left\|\left\{2^{sj}\,\|\Delta_j f\|_{L^r}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^\sigma(\mathbb{Z})}.$$

Also, we define the following space-time norm for $1\leqslant q\leqslant\infty$:

$$\|F\|_{\widetilde{L_t^q}\dot{B}^s_{r,\sigma}}:=\left\|\left\{2^{sj}\,\|\Delta_j F\|_{L_t^q L_x^r}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^\sigma(\mathbb{Z})}.$$

Lemma 2.4. *Let the exponents $q,\tilde{q},r,\tilde{r}$ satisfy*

$$\frac{2}{q}+\frac{1}{r}\leqslant\frac{1}{2},\quad\frac{2}{\tilde{q}}+\frac{1}{\tilde{r}}\leqslant\frac{1}{2},\quad 4\leqslant q,\tilde{q}\leqslant\infty,\quad 2\leqslant r,\tilde{r}\leqslant\infty.$$

Then, there exist positive constants $C_1=C_1(q,r)$ and $C_2=C_2(q,\tilde{q},r,\tilde{r})$ such that

$$\|e^{\pm iNtp(D)}f\|_{\widetilde{L_t^q}\dot{B}^0_{r,\sigma}}\leqslant C_1 N^{-\frac{1}{q}}\|f\|_{\dot{B}^{3(\frac{1}{2}-\frac{1}{r})}_{2,\sigma}}, \tag{2.23}$$

$$\left\|\int_{-\infty}^{t} e^{\pm iN(t-s)p(D)}F(s)\,\mathrm{d}s\right\|_{\widetilde{L_t^q}\dot{B}^0_{r,\sigma}}\leqslant C_2 N^{-\frac{1}{q}-\frac{1}{\tilde{q}}}\|F\|_{\widetilde{L_t^{\tilde{q}'}}\dot{B}^{3(1-\frac{1}{r}-\frac{1}{\tilde{r}})}_{\tilde{r}',\sigma}} \tag{2.24}$$

for all $1\leqslant\sigma\leqslant\infty$, $f\in\dot{B}^{3(\frac{1}{2}-\frac{1}{r})}_{2,\sigma}(\mathbb{R}^3)$ and $F\in\widetilde{L^{\tilde{q}'}}(\mathbb{R};\dot{B}^{3(1-\frac{1}{r}-\frac{1}{\tilde{r}})}_{\tilde{r}',\sigma}(\mathbb{R}^3))$.

Proof. Since $\psi(\xi) = 1$ on the support of φ_0, we see that

$$\begin{aligned} U_N(t)\Delta_0 f(x) &= \int_{\mathbb{R}^3} e^{ix\cdot\xi + iNtp(\xi)} \psi(\xi)^2 \varphi_0(\xi) \widehat{f}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^3 \Delta_0 e^{iNtp(D)} f(x). \end{aligned}$$

Hence we have by (2.8) and (2.9) in Lemma 2.3

$$\|\Delta_0 e^{iNtp(D)} f\|_{L^q_t L^r_x} \leqslant C N^{-\frac{1}{q}} \|\Delta_0 f\|_{L^2}, \tag{2.25}$$

$$\left\| \int_{-\infty}^t \Delta_0 e^{iN(t-s)p(D)} F(s)\, \mathrm{d}s \right\|_{L^q_t L^r_x} \leqslant C N^{-\frac{1}{q} - \frac{1}{\tilde{q}}} \|\Delta_0 F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}. \tag{2.26}$$

Note that $p(\xi) = |\xi_h|/|\xi|$ is homogeneous of degree 0. Hence scaling $\xi \mapsto 2^j \xi$ gives that for $j \in \mathbb{Z}$

$$\begin{aligned} \Delta_j e^{iNtp(D)} f(x) &= \Delta_0 e^{iNtp(D)} \left[f(2^{-j}\cdot) \right] (2^j x), \\ \Delta_j f(x) &= \Delta_0 \left[f(2^{-j}\cdot) \right] (2^j x). \end{aligned}$$

Therefore, we obtain by (2.25) and (2.26)

$$\begin{aligned} \|\Delta_j e^{iNtp(D)} f\|_{L^q_t L^r_x} &\leqslant C N^{-\frac{1}{q}} (2^j)^{3(\frac{1}{2} - \frac{1}{r})} \|\Delta_j f\|_{L^2}, \\ \left\| \int_{-\infty}^t \Delta_j e^{iN(t-s)p(D)} F(s)\, \mathrm{d}s \right\|_{L^q_t L^r_x} &\leqslant C_2 N^{-\frac{1}{q} - \frac{1}{\tilde{q}}} (2^j)^{3(1 - \frac{1}{r} - \frac{1}{\tilde{r}})} \|\Delta_j F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}. \end{aligned}$$

Taking the $\ell^\sigma(\mathbb{Z})$-norm, we complete the proof of Lemma 2.4. □

3. The Limit System

In this section, we shall show the global regularity of the limit system (1.7), and give the global a priori $H^{s+3}(\mathbb{R}^3)$-estimate for the solution to (1.7). We remark that the projection P_0 onto the stationary mode of the linear solution to (2.1) defined in (2.3) and (2.4) is also written as

$$\widehat{P_0 \phi}(\xi) = \left(\begin{array}{c|c} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi_h|^2} \right)_{1 \leqslant j,k \leqslant 2} & 0 \\ \hline 0 & 0 \end{array} \right) \widehat{\phi}(\xi).$$

Hence we see that P_0 corresponds to the two dimensional Helmholtz projection

$$\mathbb{P}_h = \left(\delta_{jk} + \partial_j \partial_k (-\Delta_h)^{-1} \right)_{1 \leqslant j,k \leqslant 2}, \quad P_0 = \left(\begin{array}{c|c} \mathbb{P}_h & 0 \\ \hline 0 & 0 \end{array} \right). \tag{3.1}$$

Then, considering a solution u to (1.5) of the form $u = P_0 u = (w, 0, 0)^T$ with $\mathbb{P}_h w = w$, we obtain the following limit system:

$$\begin{cases} \partial_t w + \mathbb{P}_h(w \cdot \nabla_h)w = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \nabla_h \cdot w = 0 & t \geqslant 0,\ x \in \mathbb{R}^3, \\ w(0, x) = \mathbb{P}_h \phi_h(x) & x \in \mathbb{R}^3, \end{cases} \tag{3.2}$$

where $w = (w_1(t, x), w_2(t, x))^T,\ \phi_h = (\phi_1(x), \phi_2(x))^T$ and $\nabla_h = (\partial_1, \partial_2)^T$.

The global regularity result for (3.2) reads as follows:

Theorem 3.1. *Let $s \in \mathbb{N}$ satisfy $s \geqslant 3$. Then, for every $\phi_h \in H^{s+3}(\mathbb{R}^3)$ and for every $0 < T < \infty$, there exists a unique classical solution w to (3.2) in the class*

$$w \in C([0, T]; H^{s+3}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+2}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C_L = C_L(s, T, \|\phi_h\|_{H^{s+3}})$ such that

$$\sup_{0 \leqslant t \leqslant T} \|w(t)\|_{H^{s+3}} \leqslant C_L(s, T, \|\phi_h\|_{H^{s+3}}). \tag{3.3}$$

Proof. We first remark that for fixed $x_3 \in \mathbb{R}$ the system (3.2) for $w = w(\cdot, x_3)$ is the two dimensional incompressible Euler equations. Hence it is well-known by [23,26] that for the initial data $\phi_h(\cdot, x_3) \in H^s(\mathbb{R}^2)$ there exists a unique classical global solution $w(\cdot, x_3)$ to (3.2) satisfying

$$w(\cdot, x_3) \in C([0, \infty); H^s(\mathbb{R}^2)) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}^2)).$$

Let us first derive the a priori estimate for the norm

$$\|w(t)\|_{L^\infty_{x_3} H^s_{x_h}} := \|w(t)\|_{L^\infty(\mathbb{R}_{x_3}; H^s(\mathbb{R}^2))}.$$

By the standard energy method and the Gronwall inequality (see [24,27]), we have

$$\|w(t, \cdot, x_3)\|_{H^s(\mathbb{R}^2)} \leqslant \|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)} \exp\left\{C \int_0^t \|\nabla_h w(\tau, \cdot, x_3)\|_{L^\infty(\mathbb{R}^2)}\, \mathrm{d}\tau\right\}. \tag{3.4}$$

By the Biot-Savart law, we have a representation of w in terms of the vorticity $\omega = \partial_1 w_2 - \partial_2 w_1$ as

$$w = -(-\Delta_h)^{-1} \nabla_h^\perp \omega, \quad \nabla_h^\perp = (-\partial_2, \partial_1)^T,$$

which yields

$$\nabla_h w = \begin{pmatrix} R_1^h R_2^h & (R_2^h)^2 \\ -(R_1^h)^2 & -R_1^h R_2^h \end{pmatrix} \omega.$$

Here, $R_j^h = -\partial_j(-\Delta_h)^{-\frac{1}{2}}$ $(j = 1, 2)$ denotes the two dimensional Riesz transform. Then, since the Riesz transform is bounded in $\mathrm{BMO}(\mathbb{R}^2)$ and there holds $L^\infty(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}(\mathbb{R}^2)$, it follows from the logarithmic Sobolev inequality by [32, Theorem 1] that

$$\begin{aligned} &\|\nabla_h w(\tau, \cdot, x_3)\|_{L^\infty(\mathbb{R}^2)} \\ &\quad \leqslant C\left\{1 + \|\omega(\tau, \cdot, x_3)\|_{\mathrm{BMO}(\mathbb{R}^2)}\left(1 + \log^+ \|\nabla_h w(\tau, \cdot, x_3)\|_{H^{s-1}(\mathbb{R}^2)}\right)\right\} \\ &\quad \leqslant C\left\{1 + \|\omega(\tau, \cdot, x_3)\|_{L^\infty(\mathbb{R}^2)} \log\left(\|w(\tau, \cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e\right)\right\}. \end{aligned} \tag{3.5}$$

Let $X_h(t)$ be the trajectory flow defined by the solution of the ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}X_h}{\mathrm{d}t}(t) = w(t, X_h(t), x_3), \\ X_h(0) = x_h \in \mathbb{R}^2. \end{cases}$$

Then, since the vorticity w satisfies $\partial_t \omega + w \cdot \nabla_h \omega = 0$, we have

$$\frac{\partial}{\partial t} \{\omega(t, X_h(t), x_3)\} = 0,$$

which yields $\omega(t, X_h(t), x_3) = \omega(0, x_h, x_3)$ for all $t \geqslant 0$. Hence we have

$$\begin{aligned} \|\omega(t, \cdot, x_3)\|_{L^\infty(\mathbb{R}^2)} &= \|\omega(0, \cdot, x_3)\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant C\|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)}. \end{aligned} \tag{3.6}$$

Then, it follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned} &\|w(t, \cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e \\ &\quad\leqslant \left(\|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e\right) \\ &\qquad \times \exp\left\{Ct + C\|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)} \int_0^t \log\left(\|w(\tau, \cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e\right) \mathrm{d}\tau\right\}. \end{aligned} \tag{3.7}$$

Defining $z(t) = \log(\|w(t, \cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e)$, we have from (3.7)

$$z(t) \leqslant z(0) + Ct + C\|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)} \int_0^t z(\tau)\, \mathrm{d}\tau.$$

The Gronwall inequality gives

$$z(t) \leqslant (z(0) + Ct) \exp\left\{Ct\|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)}\right\},$$

which implies that

$$\begin{aligned} &\|w(t, \cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e \\ &\quad\leqslant \left(\|\phi_h(\cdot, x_3)\|_{H^s(\mathbb{R}^2)} + e\right)^{\exp\left\{Ct\|\phi_h(\cdot,x_3)\|_{H^s(\mathbb{R}^2)}\right\}} \cdot e^{Ct\exp\left\{Ct\|\phi_h(\cdot,x_3)\|_{H^s(\mathbb{R}^2)}\right\}}. \end{aligned}$$

Therefore, we obtain for all $t \geqslant 0$

$$\|w(t)\|_{L^\infty_{x_3} H^s_{x_h}} \leqslant \left(\|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}} + e\right)^{\exp\left\{Ct\|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}}\right\}} \cdot e^{Ct\exp\left\{Ct\|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}}\right\}}. \tag{3.8}$$

Now, we shall show that w belongs to $C([0, T]; H^{s+3}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+2}(\mathbb{R}^3))$ and satisfies the estimate (3.3). We firstly observe that the limit system (3.2) can also be regarded as the following modified 3D Euler equations for the velocity v of the form $v = (w, 0)^T = (w_1(t, x), w_2(t, x), 0)^T$:

$$\begin{cases} \partial_t v + \overline{\mathbb{P}_h}(v \cdot \nabla)v = 0, \quad \nabla \cdot v = 0, \\ v(0, x) = (\mathbb{P}_h \phi_h(x), 0)^T, \end{cases}$$

where $\overline{\mathbb{P}_h}$ is the 3×3 matrix defined by $\overline{\mathbb{P}_h} = \left(\begin{array}{c|c} \mathbb{P}_h & 0 \\ \hline 0 & 0 \end{array} \right)$. Hence it follows from the local well-posedness theory of the 3D Euler equations by [5,24,27] that there exists a local time $T_0 \geqslant C/\|\phi_h\|_{H^{s+3}}$ such that the unique solution w to (3.2) belongs to the class $C([0, T_0]; H^{s+3}(\mathbb{R}^3)) \cap C^1([0, T_0]; H^{s+2}(\mathbb{R}^3))$. Hence it suffices to show the global a priori estimate (3.3) for the norm $\|w(t)\|_{H^{s+3}}$.

Taking the $H^{s+3}(\mathbb{R}^3)$-inner product of (3.2) with w, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{H^{s+3}} + \langle (w(t) \cdot \nabla_h)w(t),\ w(t)\rangle_{H^{s+3}} = 0. \tag{3.9}$$

Since it holds that

$$\int_{\mathbb{R}^3} (w \cdot \nabla_h)(1-\Delta)^{\frac{s+3}{2}} w \cdot (1-\Delta)^{\frac{s+3}{2}} w \,\mathrm{d}x = 0$$

by the divergence-free condition $\nabla_h \cdot w = 0$, we see that

$$\begin{aligned} &\left|\langle (w \cdot \nabla_h)w,\ w\rangle_{H^{s+3}}\right| \\ &= \left| \int_{\mathbb{R}^3} \left\{ (1-\Delta)^{\frac{s+3}{2}} (w \cdot \nabla_h)w - (w \cdot \nabla_h)(1-\Delta)^{\frac{s+3}{2}} w \right\} \cdot (1-\Delta)^{\frac{s+3}{2}} w \,\mathrm{d}x \right| \\ &\leqslant \left\| (1-\Delta)^{\frac{s+3}{2}} (w \cdot \nabla_h)w - (w \cdot \nabla_h)(1-\Delta)^{\frac{s+3}{2}} w \right\|_{L^2} \left\| (1-\Delta)^{\frac{s+3}{2}} w \right\|_{L^2} \\ &\leqslant C \left(\|\nabla w\|_{L^\infty} \|\nabla_h w\|_{H^{s+2}} + \|\nabla_h w\|_{L^\infty} \|w\|_{H^{s+3}} \right) \|w\|_{H^{s+3}} \\ &\leqslant \|\nabla w\|_{L^\infty} \|w\|^2_{H^{s+3}}. \end{aligned} \tag{3.10}$$

Here, we have used the commutator estimates of the Klainerman–Majda [29] and the Kato–Ponce type [27]:

$$\|(1-\Delta)^{\frac{s}{2}}(fg) - f(1-\Delta)^{\frac{s}{2}} g\|_{L^2} \leqslant C \left(\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|g\|_{L^\infty} \|f\|_{H^s} \right).$$

Substituting (3.10) into (3.9) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{H^{s+3}} \leqslant C\|\nabla w(t)\|_{L^\infty} \|w(t)\|^2_{H^{s+3}}.$$

Therefore, we have

$$\begin{aligned} \|w(t)\|_{H^{s+3}} &\leqslant \|\phi_h\|_{H^{s+3}} + C\int_0^t \|\nabla w(\tau)\|_{L^\infty} \|w(\tau)\|_{H^{s+3}} \,\mathrm{d}\tau \\ &\leqslant \|\phi_h\|_{H^{s+3}} + C\int_0^t \|\nabla_h w(\tau)\|_{L^\infty} \|w(\tau)\|_{H^{s+3}} \,\mathrm{d}\tau \\ &\quad + C\int_0^t \|\partial_3 w(\tau)\|_{L^\infty} \|w(\tau)\|_{H^{s+3}} \,\mathrm{d}\tau. \end{aligned} \tag{3.11}$$

Let us set the right hand side of (3.8) by

$$A(t, \|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}}) := \left(\|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}} + e\right)^{\exp\left\{Ct\|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}}\right\}} e^{Ct\exp\left\{Ct\|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}}\right\}}.$$

Since $s \geqslant 3$, the Sobolev embedding $H^s(\mathbb{R}^2) \hookrightarrow C^1(\mathbb{R}^2)$ and (3.8) give that

$$\begin{aligned} \|\nabla_h w(t)\|_{L^\infty} &\leqslant \|w(t)\|_{L^\infty_{x_3} H^s_{x_h}} \\ &\leqslant A(t, \|\phi_h\|_{L^\infty_{x_3} H^s_{x_h}}). \end{aligned} \tag{3.12}$$

Next, we shall derive the estimate for $\|\partial_3 w(t)\|_{L^\infty}$. By the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we have

$$\|\partial_3 w(t)\|_{L^\infty} \leqslant C\|\partial_3 w(t)\|_{L^\infty_{x_3} H^2_{x_h}}. \tag{3.13}$$

Hence we consider the $H^2(\mathbb{R}^2)$-estimate for $\partial_3 w(t, \cdot, x_3)$. By (3.2), $\partial_3 w$ should satisfy

$$\partial_t(\partial_3 w) + \mathbb{P}_h(\partial_3 w \cdot \nabla_h)w + \mathbb{P}_h(w \cdot \nabla_h)\partial_3 w = 0. \tag{3.14}$$

Taking the $L^2(\mathbb{R}^2)$-inner product of (3.14) with $\partial_3 w$, we have by the divergence-free condition $\nabla_h \cdot w = 0$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 w(t, \cdot, x_3)\|^2_{L^2(\mathbb{R}^2)} + \langle(\partial_3 w \cdot \nabla_h)w,\ \partial_3 w\rangle_{L^2(\mathbb{R}^2)} = 0. \tag{3.15}$$

Since it holds that

$$\left|\langle(\partial_3 w \cdot \nabla_h)w,\ \partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| \leqslant \|\nabla_h w\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w\|^2_{L^2(\mathbb{R}^2)},$$

we have the $L^2(\mathbb{R}^2)$-estimate by (3.15)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 w(t, \cdot, x_3)\|^2_{L^2(\mathbb{R}^2)} \leqslant \|\nabla_h w(t, \cdot, x_3)\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w(t, \cdot, x_3)\|^2_{L^2(\mathbb{R}^2)}. \tag{3.16}$$

For the $\dot{H}^1(\mathbb{R}^2)$-estimate for $\partial_3 w$, it follows from (3.14) that

$$\begin{aligned} \partial_t(\partial_l\partial_3 w) + \mathbb{P}_h(\partial_l\partial_3 w \cdot \nabla_h)w + \mathbb{P}_h(\partial_3 w \cdot \nabla_h)\partial_l w \\ + \mathbb{P}_h(\partial_l w \cdot \nabla_h)\partial_3 w + \mathbb{P}_h(w \cdot \nabla_h)\partial_l\partial_3 w = 0 \end{aligned} \tag{3.17}$$

for $l = 1, 2$. Taking the $L^2(\mathbb{R}^2)$-inner product of (3.17) with $\partial_l\partial_3 w$ gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l\partial_3 w(t, \cdot, x_3)\|^2_{L^2} + \langle(\partial_l\partial_3 w \cdot \nabla_h)w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)} \\ &\quad + \langle(\partial_3 w \cdot \nabla_h)\partial_l w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)} + \langle(\partial_l w \cdot \nabla_h)\partial_3 w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)} = 0. \end{aligned} \tag{3.18}$$

Here, we have used the fact that $\langle(w \cdot \nabla_h)\partial_l\partial_3 w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)} = 0$ by $\nabla_h \cdot w = 0$. By the Hölder inequality, we have

$$\begin{aligned} \left|\langle(\partial_l\partial_3 w \cdot \nabla_h)w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w\|^2_{L^2(\mathbb{R}^2)}, \\ \left|\langle(\partial_3 w \cdot \nabla_h)\partial_l w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h^2 w\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w\|_{L^2(\mathbb{R}^2)}\|\nabla_h\partial_3 w\|_{L^2(\mathbb{R}^2)}, \\ \left|\langle(\partial_l w \cdot \nabla_h)\partial_3 w,\ \partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Substituting these estimates into (3.18) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad\leqslant C\|\nabla_h w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 \\
&\qquad + C\|\nabla_h^2 w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
\tag{3.19}
$$

For the $\dot{H}^2(\mathbb{R}^2)$-estimate for $\partial_3 w$, we have by (3.17)

$$
\begin{aligned}
\partial_t(\partial_k\partial_l\partial_3 w) &+ \mathbb{P}_h(\partial_k\partial_l\partial_3 w\cdot\nabla_h)w + \mathbb{P}_h(\partial_l\partial_3 w\cdot\nabla_h)\partial_k w \\
&+ \mathbb{P}_h(\partial_k\partial_3 w\cdot\nabla_h)\partial_l w + \mathbb{P}_h(\partial_3 w\cdot\nabla_h)\partial_k\partial_l w \\
&+ \mathbb{P}_h(\partial_k\partial_l w\cdot\nabla_h)\partial_3 w + \mathbb{P}_h(\partial_l w\cdot\nabla_h)\partial_k\partial_3 w \\
&+ \mathbb{P}_h(\partial_k w\cdot\nabla_h)\partial_l\partial_3 w + \mathbb{P}_h(w\cdot\nabla_h)\partial_k\partial_l\partial_3 w = 0
\end{aligned}
\tag{3.20}
$$

for $k, l = 1, 2$. Then, there holds by the Hölder inequality that

$$
\begin{aligned}
\left|\langle(\partial_k\partial_l\partial_3 w\cdot\nabla_h)w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}^2, \\
\left|\langle(\partial_l\partial_3 w\cdot\nabla_h)\partial_k w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h^2 w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}, \\
\left|\langle(\partial_k\partial_3 w\cdot\nabla_h)\partial_l w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h^2 w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}, \\
\left|\langle(\partial_3 w\cdot\nabla_h)\partial_k\partial_l w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h^3 w\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}, \\
\left|\langle(\partial_k\partial_l w\cdot\nabla_h)\partial_3 w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h^2 w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}, \\
\left|\langle(\partial_l w\cdot\nabla_h)\partial_k\partial_3 w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}^2, \\
\left|\langle(\partial_k w\cdot\nabla_h)\partial_l\partial_3 w,\ \partial_k\partial_l\partial_3 w\rangle_{L^2(\mathbb{R}^2)}\right| &\leqslant \|\nabla_h w\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}
$$

Taking the $L^2(\mathbb{R}^2)$-inner product of (3.20) with $\partial_k\partial_l\partial_3 w$ and substituting the above estimates, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad\leqslant C\|\nabla_h w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 \\
&\qquad + C\|\nabla_h^2 w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)} \\
&\qquad + C\|\nabla_h^3 w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
\tag{3.21}
$$

Therefore, we obtain from (3.8), (3.16), (3.19), (3.21) and $H^{s-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for $s \geqslant 3$ that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 w(t,\cdot,x_3)\|_{H^2(\mathbb{R}^2)}^2 \\
&\quad\leqslant \|\nabla_h w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 \\
&\qquad + C\|\nabla_h w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 \\
&\qquad + C\|\nabla_h^2 w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\|\nabla_h w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2\\
&+ C\|\nabla_h^2 w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\nabla_h\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\\
&+ C\|\nabla_h^3 w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\|\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\|\nabla_h^2\partial_3 w(t,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\\
\leqslant{}& C\|\partial_3 w(t,\cdot,x_3)\|_{H^2(\mathbb{R}^2)}^2\sum_{l=1}^{3}\|\nabla_h^l w(t,\cdot,x_3)\|_{L^\infty(\mathbb{R}^2)}\\
\leqslant{}& C\|w(t,\cdot,x_3)\|_{H^{s+2}(\mathbb{R}^2)}\|\partial_3 w(t,\cdot,x_3)\|_{H^2(\mathbb{R}^2)}^2\\
\leqslant{}& CA(t,\|\phi_h\|_{L^\infty_{x_3}H^{s+2}_{x_h}})\|\partial_3 w(t,\cdot,x_3)\|_{H^2(\mathbb{R}^2)}^2,
\end{aligned}
$$

which yields by the Gronwall inequality that

$$
\|\partial_3 w(t,\cdot,x_3)\|_{H^2(\mathbb{R}^2)} \leqslant \|\partial_3\phi_h(\cdot,x_3)\|_{H^2(\mathbb{R}^2)}\exp\left\{CtA(t,\|\phi_h\|_{L^\infty_{x_3}H^{s+2}_{x_h}})\right\}. \tag{3.22}
$$

Hence we obtain from (3.13) and (3.22)

$$
\|\partial_3 w(t)\|_{L^\infty} \leqslant C\|\partial_3\phi_h\|_{L^\infty_{x_3}H^2_{x_h}}\exp\left\{CtA(t,\|\phi_h\|_{L^\infty_{x_3}H^{s+2}_{x_h}})\right\}. \tag{3.23}
$$

Therefore, we have by (3.11), (3.12) and (3.23)

$$
\begin{aligned}
\|w(t)\|_{H^{s+3}} \leqslant{}& \|\phi_h\|_{H^{s+3}} + CA(T,\|\phi_h\|_{L^\infty_{x_3}H^s_{x_h}})\int_0^t\|w(\tau)\|_{H^{s+3}}\,d\tau\\
&+ C\|\partial_3\phi_h\|_{L^\infty_{x_3}H^2_{x_h}}\exp\left\{CTA(T,\|\phi_h\|_{L^\infty_{x_3}H^{s+2}_{x_h}})\right\}\int_0^t\|w(\tau)\|_{H^{s+3}}\,d\tau
\end{aligned} \tag{3.24}
$$

for all $0 \leqslant t \leqslant T$. Here, let us set

$$
B(t,\|\phi_h\|_{H^s}) := \left(\|\phi_h\|_{H^s} + e\right)^{\exp\{Ct\|\phi_h\|_{H^s}\}}e^{Ct\exp\{Ct\|\phi_h\|_{H^s}\}}.
$$

Then, it follows from the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ that

$$
A(t,\|\phi_h\|_{L^\infty_{x_3}H^s_{x_h}}) \leqslant CB(t,\|\phi_h\|_{H^{s+1}}). \tag{3.25}
$$

Hence we have by, (3.24) and (3.25),

$$
\begin{aligned}
\|w(t)\|_{H^{s+3}} \leqslant{}& \|\phi_h\|_{H^{s+3}} + CB(T,\|\phi_h\|_{H^{s+1}})\int_0^t\|w(\tau)\|_{H^{s+3}}\,d\tau\\
&+ C\|\phi_h\|_{H^4}\exp\left\{CTB(T,\|\phi_h\|_{H^{s+3}})\right\}\int_0^t\|w(\tau)\|_{H^{s+3}}\,d\tau.
\end{aligned}
$$

Therefore, we obtain by the Gronwall inequality that

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\|w(t)\|_{H^{s+3}}\\
&\leqslant \|\phi_h\|_{H^{s+3}}\exp\left[CB(T,\|\phi_h\|_{H^{s+1}}) + C\|\phi_h\|_{H^4}\exp\left\{CTB(T,\|\phi_h\|_{H^{s+3}})\right\}\right].
\end{aligned}
$$

This gives the global a priori estimate for $\|w(t)\|_{H^{s+3}}$, and we complete the proof of Theorem 3.1. □

4. Modified Linear Dispersive Solutions

In this section, we adapt the idea in [10] and introduce the modified linear dispersive equations (1.9) (and (4.2) below). Making use of Lemma 2.4, we shall establish the global space-time estimates for the solutions $u^{\pm}$ to those systems.

Let $s \in \mathbb{N}$ satisfy $s \geqslant 3$, and let $0 < T < \infty$. Then, for the initial data $\phi = (\phi_h, \phi_3, \phi_4)^T \in H^{s+4}(\mathbb{R}^3)$ with $\widetilde{\nabla} \cdot \phi = 0$, let $w = (w_1, w_2) \in C([0, T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+3}(\mathbb{R}^3))$ be the classical solution to (3.2) with $w(0, x) = \mathbb{P}_h \phi_h(x)$ constructed in Theorem 3.1 satisfying the H^{s+4}-estimate

$$\sup_{0 \leqslant t \leqslant T} \|w(t)\|_{H^{s+4}} \leqslant C_L(s, T, \|\phi_h\|_{H^{s+4}}). \tag{4.1}$$

Now, we put $u^0 = (w, 0, 0)^T$, and consider the solution to the following linear systems with the external forces $P_{\pm}(u^0 \cdot \widetilde{\nabla})u^0$:

$$\begin{cases} \partial_t u^{\pm} \mp i N p(D) u^{\pm} + P_{\pm}(u^0 \cdot \widetilde{\nabla})u^0 = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \widetilde{\nabla} \cdot u^{\pm} = 0 & t \geqslant 0,\ x \in \mathbb{R}^3, \\ u^{\pm}(0, x) = P_{\pm}\phi(x) & x \in \mathbb{R}^3, \end{cases} \tag{4.2}$$

where $p(D) = |D_h|/|D|$ is the Fourier multiplier, and the projections $P_{\pm}$ are defined in (2.3) and (2.4). By the Duhamel principle, the solutions to (4.2) are given by

$$u^{\pm}(t) = e^{\pm i N t p(D)} P_{\pm}\phi - \int_0^t e^{\pm i N (t-\tau) p(D)} P_{\pm}(u^0(\tau) \cdot \widetilde{\nabla})u^0(\tau)\, \mathrm{d}\tau. \tag{4.3}$$

Lemma 4.1. *Let $s \in \mathbb{N}$ satisfy $s \geqslant 3$, and let $0 < T < \infty$. Then, for every $\phi \in H^{s+4}(\mathbb{R}^3)$ satisfying $\widetilde{\nabla} \cdot \phi = 0$, there exists a unique classical solution $u^{\pm}$ to* (4.2) *in the class*

$$u^{\pm} \in C([0, T]; H^{s+3}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+2}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C = C(s, T, \|\phi\|_{H^{s+4}})$ such that

$$\sup_{0 \leqslant t \leqslant T} \|u^{\pm}(t)\|_{H^{s+3}} \leqslant \|\phi\|_{H^{s+3}} + C(s, T, \|\phi\|_{H^{s+4}}). \tag{4.4}$$

Also, for $4 \leqslant q < \infty$ there exist positive constants $C_q = C(q)$ and $C = C(s, q, T, \|\phi\|_{H^{s+4}})$ such that

$$\|\nabla^l u^{\pm}\|_{L^q(0,T;L^\infty)} \leqslant C_q N^{-\frac{1}{q}} \left(\|\phi\|_{H^{2+l}} + C(s, q, T, \|\phi\|_{H^{s+4}}) \right) \tag{4.5}$$

for $l = 0, 1, 2, \ldots, s+1$.

Proof. Let us first show the H^{s+3}-estimate (4.4). Taking the H^{s+3} inner product of (4.2) with $u^{\pm}$, and considering the real part, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u^{\pm}(t)\|_{H^{s+3}}^2 + \langle (u^0(t) \cdot \widetilde{\nabla})u^0(t),\ u^{\pm}(t) \rangle_{H^{s+3}} = 0. \tag{4.6}$$

It follows from the H^{s+4}-estimates (4.1) for $w(t)$ that

$$\begin{aligned}\left|\langle (u^0(t)\cdot\widetilde{\nabla})u^0(t),\ u^\pm(t)\rangle_{H^{s+3}}\right| &\leqslant \|(w(t)\cdot\nabla_h)w(t)\|_{H^{s+3}}\|u^\pm(t)\|_{H^{s+3}}\\ &\leqslant C\|w(t)\|_{H^{s+4}}^2\|u^\pm(t)\|_{H^{s+3}}\\ &\leqslant C(s,T,\|\phi_h\|_{H^{s+4}})\|u^\pm(t)\|_{H^{s+3}}. \qquad (4.7)\end{aligned}$$

Substituting (4.7) into (4.6), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\pm(t)\|_{H^{s+3}}^2 \leqslant C(s,T,\|\phi_h\|_{H^{s+4}})\|u^\pm(t)\|_{H^{s+3}},$$

which implies that

$$\|u^\pm(t)\|_{H^{s+3}} \leqslant \|P_\pm\phi\|_{H^{s+3}} + tC(s,T,\|\phi_h\|_{H^{s+4}})$$

for all $0\leqslant t\leqslant T$. This yields the desired estimate (4.4).

Next, we shall prove the space-time estimate (4.5). For the homogeneous term in (4.3), by the continuous embedding $\dot{B}^0_{\infty,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, the Minkowski inequality and (2.23) in Lemma 2.4, we have for $l=0,1,2,\dots,s+1$

$$\begin{aligned}\left\|\nabla^l e^{\pm iNtp(D)}P_\pm\phi\right\|_{L^q(0,T;L^\infty)} &\leqslant C\left\|\nabla^l e^{\pm iNtp(D)}P_\pm\phi\right\|_{L^q(0,T;\dot{B}^0_{\infty,1})}\\ &\leqslant C\sum_{j\in\mathbb{Z}}\left\|\Delta_j\nabla^l e^{\pm iNtp(D)}P_\pm\phi\right\|_{L^q(0,T;L^\infty)}\\ &= C\|\nabla^l e^{\pm iNtp(D)}P_\pm\phi\|_{\widetilde{L^q}(0,T;\dot{B}^0_{\infty,1})}\\ &\leqslant CN^{-\frac{1}{q}}\|\nabla^l P_\pm\phi\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\leqslant CN^{-\frac{1}{q}}\|\phi\|_{H^{2+l}}. \qquad (4.8)\end{aligned}$$

For the inhomogeneous term in (4.3), similarly to (4.8), it follows from (2.24) in Lemma 2.4 with $(\tilde{q},\tilde{r})=(\infty,2)$ that

$$\begin{aligned}&\left\|\nabla^l\int_0^t e^{\pm iN(t-\tau)p(D)}P_\pm(u^0(\tau)\cdot\widetilde{\nabla})u^0(\tau)\,\mathrm{d}\tau\right\|_{L^q(0,T;L^\infty)}\\ &\leqslant C\left\|\int_0^t \nabla^l e^{\pm iN(t-\tau)p(D)}P_\pm(u^0(\tau)\cdot\widetilde{\nabla})u^0(\tau)\,\mathrm{d}\tau\right\|_{\widetilde{L^q}(0,T;\dot{B}^0_{\infty,1})}\\ &\leqslant CN^{-\frac{1}{q}}\left\|\nabla^l P_\pm(u^0\cdot\widetilde{\nabla})u^0\right\|_{\widetilde{L^1}(0,T;\dot{B}^{\frac{3}{2}}_{2,1})}. \qquad (4.9)\end{aligned}$$

Here, we have by the H^{s+4}-estimates (4.1) for $w(t)$

$$\begin{aligned}\left\|\nabla^l P_\pm(u^0\cdot\widetilde{\nabla})u^0\right\|_{\widetilde{L^1}(0,T;\dot{B}^{\frac{3}{2}}_{2,1})} &= \sum_{j\in\mathbb{Z}}2^{\frac{3}{2}j}\int_0^T\left\|\Delta_j\nabla^l P_\pm(u^0(t)\cdot\widetilde{\nabla})u^0(t)\right\|_{L^2}\,\mathrm{d}t\\ &= \int_0^T\left\|\nabla^l P_\pm(u^0(t)\cdot\widetilde{\nabla})u^0(t)\right\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned} &\leqslant C \int_0^T \left\| (u^0(t) \cdot \widetilde{\nabla}) u^0(t) \right\|_{H^{2+l}} \, \mathrm{d}t \\ &\leqslant C \int_0^T \|w(t)\|_{H^{3+l}}^2 \, \mathrm{d}t \\ &\leqslant C \int_0^T \|w(t)\|_{H^{s+4}}^2 \, \mathrm{d}t \leqslant C(s, T, \|\phi_h\|_{H^{s+4}}). \end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) yields the desired estimate (4.5). □

5. Proof of Theorem 1.1

We are now ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $s \in \mathbb{N}$ with $s \geqslant 3$, and let $\phi = (\phi_h, \phi_3, \phi_4)^T \in H^{s+4}(\mathbb{R}^3)$ satisfying $\widetilde{\nabla} \cdot \phi = 0$. Since $\mathbb{P}J\mathbb{P}$ is skew-symmetric and then $\langle \mathbb{P}J\mathbb{P}u, u \rangle_{H^s} = 0$, it follows from the standard local well-posedness theory for the 3D Euler equations in $H^s(\mathbb{R}^3)$ by [24,27,31] that there exists a local time $T_0 = T_0(s, \|\phi\|_{H^s}) > 0$ such that (1.5) possesses a unique classical solution u^N for all $N \geqslant 0$ in the class

$$u^N \in C([0, T_0]; H^s(\mathbb{R}^3)) \cap C^1([0, T_0]; H^{s-1}(\mathbb{R}^3)). \tag{5.1}$$

In particular, there exist positive constants $C_0 = C_0(s)$ and $C_1 = C_1(s)$ such that

$$T_0 \geqslant \frac{C_0}{\|\phi\|_{H^s}}, \qquad \sup_{0 \leqslant t \leqslant T_0} \|u^N(t)\|_{H^s} \leqslant C_1 \|\phi\|_{H^s}. \tag{5.2}$$

Let $0 < T < \infty$. We shall first show that the local solution u^N in the class (5.1) can be extended to the arbitrary finite time interval $[0, T]$ provided that the buoyancy frequency N is sufficiently high.

Let $w = (w_1, w_2) \in C([0, T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+3}(\mathbb{R}^3))$ be the classical solution to the limit system (3.2) with $w(0, x) = \mathbb{P}_h \phi_h(x)$ constructed in Theorem 3.1. We put $u^0 = (w, 0, 0)^T$. Then, by (3.1), we see that u^0 is the classical solution to the system

$$\begin{cases} \partial_t u^0 + P_0(u^0 \cdot \widetilde{\nabla}) u^0 = 0, \quad \widetilde{\nabla} \cdot u^0 = 0, \\ u^0(0, x) = P_0 \phi. \end{cases}$$

Also, let $u^\pm \in C([0, T]; H^{s+3}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+2}(\mathbb{R}^3))$ be the classical solutions to the linear systems (4.2) constructed in Lemma 4.1 satisfying (4.4) and (4.5).

Now we set

$$v^N := u^N - u^+ - u^- - u^0.$$

Then, since there hold $\phi = \mathbb{P}\phi = P_+\phi + P_-\phi + P_0\phi$ and $P_j u^j = u^j$ for $j \in \{0, \pm\}$, the perturbation v^N should solve

$$\begin{cases} \partial_t v^N + N\mathbb{P}J\mathbb{P}v^N + \mathbb{P}(u^N \cdot \widetilde{\nabla})v^N + \sum_{j=0,\pm} \mathbb{P}(v^N \cdot \widetilde{\nabla})u^j + \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \mathbb{P}(u^j \cdot \widetilde{\nabla})u^k = 0, \\ \widetilde{\nabla} \cdot v^N = 0, \\ v^N(0,x) = 0 \end{cases} \tag{5.3}$$

on the local time interval $[0, T_0]$. Let us derive the H^s-estimate for $v^N(t)$. Taking the H^s inner product of (5.3) with v^N gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^N(t)\|_{H^s}^2 &+ \langle (u^N(t) \cdot \widetilde{\nabla})v^N(t),\ v^N(t)\rangle_{H^s} \\ &+ \sum_{j=0,\pm} \langle (v^N(t) \cdot \widetilde{\nabla})u^j(t),\ v^N(t)\rangle_{H^s} \\ &+ \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \langle (u^j(t) \cdot \widetilde{\nabla})u^k(t),\ v^N(t)\rangle_{H^s} = 0. \end{aligned} \tag{5.4}$$

Since it holds that

$$\int_{\mathbb{R}^3} (u^N \cdot \widetilde{\nabla})\partial^\alpha v^N \cdot \partial^\alpha v^N \,\mathrm{d}x = 0$$

for $\alpha \in (\mathbb{N} \cup \{0\})^3$ with $|\alpha| \leqslant s$ by the divergence-free condition, we have

$$\begin{aligned} \left|\langle (u^N \cdot \widetilde{\nabla})v^N,\ v^N\rangle_{H^s}\right| &= \left| \sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^3} \partial^\alpha (u^N \cdot \widetilde{\nabla})v^N \cdot \partial^\alpha v^N \,\mathrm{d}x \right| \\ &= \left| \sum_{|\alpha|\leqslant s} \sum_{0<\beta\leqslant\alpha} C_{\alpha,\beta} \int_{\mathbb{R}^3} (\partial^\beta u^N \cdot \widetilde{\nabla})\partial^{\alpha-\beta} v^N \cdot \partial^\alpha v^N \,\mathrm{d}x \right| \\ &\leqslant \sum_{|\alpha|\leqslant s} \sum_{0<\beta\leqslant\alpha} C_{\alpha,\beta} \left\| (\partial^\beta u^N \cdot \widetilde{\nabla})\partial^{\alpha-\beta} v^N \right\|_{L^2} \left\| \partial^\alpha v^N \right\|_{L^2} \\ &\leqslant C\|u^N\|_{H^s}\|v^N\|_{H^s}^2. \end{aligned} \tag{5.5}$$

Here, we have used the estimates (see [24, Lemma in page 302])

$$\left\| (\partial^\beta u^N \cdot \widetilde{\nabla})\partial^{\alpha-\beta} v^N \right\|_{L^2} \leqslant \begin{cases} C\|u^N\|_{H^3}\|v^N\|_{H^{|\alpha|}} & 0 < \beta \leqslant \alpha,\ |\beta| = 1, 2, \\ C\|u^N\|_{H^{|\beta|}}\|v^N\|_{H^{|\alpha|-|\beta|+3}} & 0 < \beta \leqslant \alpha,\ |\beta| \geqslant 3. \end{cases}$$

For the third term in the left hand side of (5.4), since $s \geqslant 3$ and $H^s(\mathbb{R}^3)$ is a Banach algebra, we see that

$$\begin{aligned} \left| \sum_{j=0,\pm} \langle (v^N \cdot \widetilde{\nabla})u^j,\ v^N\rangle_{H^s} \right| &\leqslant \sum_{j=0,\pm} \left\| (v^N \cdot \widetilde{\nabla})u^j \right\|_{H^s} \left\| v^N \right\|_{H^s} \\ &\leqslant C \sum_{j=0,\pm} \|u^j\|_{H^{s+1}}\|v^N\|_{H^s}^2. \end{aligned} \tag{5.6}$$

For the fourth term in the left hand side of (5.4), the Schwartz inequality gives

$$\left| \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \langle (u^j \cdot \widetilde{\nabla}) u^k,\ v^N \rangle_{H^s} \right| \leqslant \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \|(u^j \cdot \widetilde{\nabla}) u^k\|_{H^s} \|v^N\|_{H^s}. \tag{5.7}$$

Let us derive the estimates for $\|(u^j \cdot \widetilde{\nabla}) u^k\|_{H^s}$. It follows from the the Leibniz rule that

$$\begin{aligned} &\|(u^j \cdot \widetilde{\nabla}) u^k\|_{H^s}^2 \\ &\quad = \sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^3} \partial^\alpha (u^j \cdot \widetilde{\nabla}) u^k \cdot \partial^\alpha (u^j \cdot \widetilde{\nabla}) u^k \, \mathrm{d}x \\ &\quad = \sum_{|\alpha|\leqslant s} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma} \int_{\mathbb{R}^3} (\partial^\beta u^j \cdot \widetilde{\nabla}) \partial^{\alpha-\beta} u^k \cdot (\partial^\gamma u^j \cdot \widetilde{\nabla}) \partial^{\alpha-\gamma} u^k \, \mathrm{d}x. \end{aligned} \tag{5.8}$$

For $(j,k) = (\pm,\pm)$, we have by the Hölder inequality

$$\begin{aligned} &\sum_{|\alpha|\leqslant s} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma} \int_{\mathbb{R}^3} (\partial^\beta u^\pm \cdot \widetilde{\nabla}) \partial^{\alpha-\beta} u^\pm \cdot (\partial^\gamma u^\pm \cdot \widetilde{\nabla}) \partial^{\alpha-\gamma} u^\pm \, \mathrm{d}x \\ &\quad \leqslant \sum_{|\alpha|\leqslant s} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma} \|\partial^\beta u^\pm\|_{L^\infty} \|\partial^\gamma u^\pm\|_{L^\infty} \|\nabla \partial^{\alpha-\beta} u^\pm\|_{L^2} \|\nabla \partial^{\alpha-\gamma} u^\pm\|_{L^2} \\ &\quad \leqslant C \|u^\pm\|_{H^{s+1}}^2 \left(\sum_{l=0}^{s} \|\nabla^l u^\pm\|_{L^\infty} \right)^2. \end{aligned} \tag{5.9}$$

Similarly to (5.9), we see that for $(j,k) = (\pm,\mp)$

$$\begin{aligned} &\sum_{|\alpha|\leqslant s} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma} \int_{\mathbb{R}^3} (\partial^\beta u^\pm \cdot \widetilde{\nabla}) \partial^{\alpha-\beta} u^\mp \cdot (\partial^\gamma u^\pm \cdot \widetilde{\nabla}) \partial^{\alpha-\gamma} u^\mp \, \mathrm{d}x \\ &\quad \leqslant C \|u^\mp\|_{H^{s+1}}^2 \left(\sum_{l=0}^{s} \|\nabla^l u^\pm\|_{L^\infty} \right)^2. \end{aligned} \tag{5.10}$$

For $(j,k) = (\pm,0)$, it follows from the Hölder inequality that

$$\begin{aligned} &\sum_{|\alpha|\leqslant s} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma} \int_{\mathbb{R}^3} (\partial^\beta u^\pm \cdot \widetilde{\nabla}) \partial^{\alpha-\beta} u^0 \cdot (\partial^\gamma u^\pm \cdot \widetilde{\nabla}) \partial^{\alpha-\gamma} u^0 \, \mathrm{d}x \\ &\quad \leqslant \sum_{|\alpha|\leqslant s} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma} \|\partial^\beta u^\pm\|_{L^\infty} \|\partial^\gamma u^\pm\|_{L^\infty} \|\nabla \partial^{\alpha-\beta} u^0\|_{L^2} \|\nabla \partial^{\alpha-\gamma} u^0\|_{L^2} \\ &\quad \leqslant C \|u^0\|_{H^{s+1}}^2 \left(\sum_{l=0}^{s} \|\nabla^l u^\pm\|_{L^\infty} \right)^2. \end{aligned} \tag{5.11}$$

Similarly to (5.11), we have for $(j,k)=(0,\pm)$ that

$$\sum_{|\alpha|\leqslant s}\sum_{\beta\leqslant\alpha}\sum_{\gamma\leqslant\alpha} C_{\alpha,\beta,\gamma}\int_{\mathbb{R}^3}(\partial^\beta u^0\cdot\widetilde{\nabla})\partial^{\alpha-\beta}u^\pm\cdot(\partial^\gamma u^0\cdot\widetilde{\nabla})\partial^{\alpha-\gamma}u^\pm\,\mathrm{d}x \leqslant C\|u^0\|_{H^s}^2\left(\sum_{l=0}^{s+1}\|\nabla^l u^\pm\|_{L^\infty}\right)^2. \tag{5.12}$$

Combining (5.7)–(5.12), we obtain

$$\left|\sum_{\substack{j,k=0,\pm\\(j,k)\neq(0,0)}}\langle(u^j\cdot\widetilde{\nabla})u^k,\ v^N\rangle_{H^s}\right| \leqslant C\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\sum_{l=0}^{s+1}\Bigl(\|\nabla^l u^+\|_{L^\infty}+\|\nabla^l u^-\|_{L^\infty}\Bigr)\|v^N\|_{H^s}. \tag{5.13}$$

Substituting (5.5), (5.6) and (5.13) into (5.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^N(t)\|_{H^s}^2 &\leqslant C\|u^N\|_{H^s}\|v^N\|_{H^s}^2+C\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\|v^N\|_{H^s}^2\\ &\quad+C\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\sum_{l=0}^{s+1}\Bigl(\|\nabla^l u^+\|_{L^\infty}+\|\nabla^l u^-\|_{L^\infty}\Bigr)\|v^N\|_{H^s},\end{aligned}$$

which yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v^N(t)\|_{H^s} &\leqslant C\left(\|u^N\|_{H^s}+\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\right)\|v^N\|_{H^s}\\ &\quad+C\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\sum_{l=0}^{s+1}\Bigl(\|\nabla^l u^+\|_{L^\infty}+\|\nabla^l u^-\|_{L^\infty}\Bigr).\end{aligned} \tag{5.14}$$

Here, it follows from the uniform H^{s+3} estimates (3.3), (4.4) and (5.2) that there exists a positive constant $C=C(s,T,\|\phi\|_{H^{s+4}})$ such that

$$\begin{aligned}\|u^N(t)\|_{H^s}+\sum_{j=0,\pm}\|u^j(t)\|_{H^{s+1}} &\leqslant \sup_{0\leqslant t\leqslant T_0}\|u^N(t)\|_{H^s}+\sum_{j=0,\pm}\sup_{0\leqslant t\leqslant T}\|u^j(t)\|_{H^{s+3}}\\ &\leqslant C(s,T,\|\phi\|_{H^{s+4}}).\end{aligned} \tag{5.15}$$

for $0\leqslant t\leqslant T_0$. Then, by (5.14), (5.15) and $v^N(0)=0$, we have

$$\begin{aligned}\|v^N(t)\|_{H^s} &\leqslant C(s,T,\|\phi\|_{H^{s+4}})\sum_{l=0}^{s+1}\int_0^t\Bigl(\|\nabla^l u^+(\tau)\|_{L^\infty}+\|\nabla^l u^-(\tau)\|_{L^\infty}\Bigr)\mathrm{d}\tau\\ &\quad+C(s,T,\|\phi\|_{H^{s+4}})\int_0^t\|v^N(\tau)\|_{H^s}\,\mathrm{d}\tau.\end{aligned} \tag{5.16}$$

Here, it follows from the Hölder inequality and the space-time estimates (4.5) in Lemma 4.1 that for $4 \leqslant q < \infty$,

$$\sum_{l=0}^{s+1} \int_0^t \|\nabla^l u^{\pm}(\tau)\|_{L^\infty} \, d\tau \leqslant T^{1-\frac{1}{q}} \sum_{l=0}^{s+1} \|\nabla^l u^{\pm}\|_{L^q(0,T;L^\infty)}$$
$$\leqslant C(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} \tag{5.17}$$

for $0 \leqslant t \leqslant T_0 < T$. Hence we have by (5.16) and (5.17) that

$$\|v^N(t)\|_{H^s} \leqslant C(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} + C(s, T, \|\phi\|_{H^{s+4}}) \int_0^t \|v^N(\tau)\|_{H^s} \, d\tau. \tag{5.18}$$

The Gronwall inequality yields

$$\sup_{0 \leqslant t \leqslant T_0} \|v^N(t)\|_{H^s} \leqslant C(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} e^{C(s,T,\|\phi\|_{H^{s+4}})T}. \tag{5.19}$$

Therefore, there exists a positive constant $N_0 = N_0(s, q, T, \|\phi\|_{H^{s+4}}) > 0$ such that there holds

$$\sup_{0 \leqslant t \leqslant T_0} \|v^N(t)\|_{H^s} \leqslant 1 \tag{5.20}$$

for all $N \geqslant N_0$. Then, since $v^N = u^N - u^0 - u^+ - u^-$, it follows from (3.3), (4.4) and (5.20) that there exists a positive constant $C_* = C_*(s, T, \|\phi\|_{H^{s+4}})$ such that

$$\|u^N(T_0)\|_{H^s} \leqslant \|v^N(T_0)\|_{H^s} + \sum_{j=0,\pm} \|u^j(T_0)\|_{H^s}$$
$$\leqslant \sup_{0 \leqslant t \leqslant T_0} \|v^N(t)\|_{H^s} + \sum_{j=0,\pm} \sup_{0 \leqslant t \leqslant T} \|u^j(t)\|_{H^{s+3}}$$
$$\leqslant 1 + C_*(s, T, \|\phi\|_{H^{s+4}}). \tag{5.21}$$

Note that the constant $C_*(s, T, \|\phi\|_{H^{s+4}})$ is independent of the local time T_0. Therefore, the local solution u^N can be extended to $[T_0, T_1]$, where

$$T_1 - T_0 \geqslant \frac{C_0}{1 + C_*(s, T, \|\phi\|_{H^{s+4}})}, \tag{5.22}$$

and there holds

$$\sup_{T_0 \leqslant t \leqslant T_1} \|u^N(t)\|_{H^s} \leqslant C_1 \left(1 + C_*(s, T, \|\phi\|_{H^{s+4}})\right). \tag{5.23}$$

We repeat the same procedure as (5.4)–(5.19) on the time interval $[T_0, T_1]$. Since we have the global estimates for u^j $(j = 0, \pm)$ on $[0, T]$, it suffices to modify the above argument for the initial data $\|v(T_0)\|_{H^s}$ and the H^s estimates for u^N as in (5.2) and (5.23). Then, similarly to (5.18), we have

$$\|v^N(t)\|_{H^s} \leqslant \|v^N(T_0)\|_{H^s} + \tilde{C}(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}}$$
$$+ \tilde{C}(s, T, \|\phi\|_{H^{s+4}}) \int_{T_0}^t \|v^N(\tau)\|_{H^s} \, d\tau \tag{5.24}$$

for $T_0 \leqslant t \leqslant T_1$. Therefore, it follows from (5.24), (5.19) and the Gronwall inequality that

$$\sup_{T_0 \leqslant t \leqslant T_1} \|v^N(t)\|_{H^s} \leqslant \tilde{C}(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} e^{\tilde{C}(s,T,\|\phi\|_{H^{s+4}})T}$$

for $N \geqslant N_0$. Hence one can take $N_1 = N_1(s, q, T, \|\phi\|_{H^{s+4}}) \geqslant N_0$ so that there holds

$$\sup_{T_0 \leqslant t \leqslant T_1} \|v^N(t)\|_{H^s} \leqslant 1 \tag{5.25}$$

for all $N \geqslant N_1$. Then, we have by (3.3), (4.4) and (5.25)

$$\begin{aligned} \|u^N(T_1)\|_{H^s} &\leqslant \|v^N(T_1)\|_{H^s} + \sum_{j=0,\pm} \|u^j(T_1)\|_{H^s} \\ &\leqslant \sup_{T_0 \leqslant t \leqslant T_1} \|v^N(t)\|_{H^s} + \sum_{j=0,\pm} \sup_{0 \leqslant t \leqslant T} \|u^j(t)\|_{H^{s+3}} \\ &\leqslant 1 + C_*(s, T, \|\phi\|_{H^{s+4}}) \end{aligned} \tag{5.26}$$

for all $N \geqslant N_1$. Note that the above bound (5.26) is exactly same as (5.21). Hence the local solution u^N can be uniquely extended to the solution of (1.5) on the time interval $[T_1, T_1 + (T_1 - T_0)]$ (defined in (5.22)) for $N \geqslant N_1$ and satisfies

$$\sup_{T_1 \leqslant t \leqslant 2T_1 - T_0} \|u^N(t)\|_{H^s} \leqslant C_1 \left(1 + C_*(s, T, \|\phi\|_{H^{s+4}})\right). \tag{5.27}$$

Also note that the bound (5.27) is exactly same as (5.23). Since T is arbitrary *finite* time, we repeat a finite number of the same procedures in the above, and continue the local solution u^N to the given time interval $[0, T]$ in the class $C([0, T]; H^s(\mathbb{R}^3)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^3))$ for $N \geqslant N_{\phi,T}$, where $N_{\phi,T} = N(s, q, T, \|\phi\|_{H^{s+4}})$ is some large positive constant.

Next, we shall show that the solution u^N belongs to the class $C([0, T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+3}(\mathbb{R}^3))$. Since the initial data ϕ is in $H^{s+4}(\mathbb{R}^3)$ and $\mathbb{P}J\mathbb{P}$ is skew-symmetric, it follows from the standard local existence theory for the 3D Euler equations in $H^s(\mathbb{R}^3)$ by [24,27,31] that u^N belongs to

$$u^N \in C([0, T_L]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0, T_L]; H^{s+3}(\mathbb{R}^3))$$

with some local time $T_L \geqslant C_s/\|\phi\|_{H^{s+4}}$ for all $N \geqslant 0$. Hence it suffices to show the global a priori estimate for $\|u^N(t)\|_{H^{s+4}}$ on $[0, T]$ when $N \geqslant N_{\phi,T}$.

By the above procedure on the extension of solutions, we see that the long time solution u^N on $[0, T]$ satisfies the uniform H^s estimate as

$$\sup_{0 \leqslant t \leqslant T} \|u^N(t)\|_{H^s} \leqslant C(s, q, T, \|\phi\|_{H^{s+4}}), \tag{5.28}$$

with some positive constant $C(s, q, T, \|\phi\|_{H^{s+4}})$ for $N \geqslant N_{\phi,T}$. Therefore, the standard energy method, the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$ and (5.28) give that

$$\begin{aligned}\|u^N(t)\|_{H^{s+4}} &\leqslant \|\phi\|_{H^{s+4}} + C\int_0^t \|\nabla u^N(\tau)\|_{L^\infty}\|u^N(\tau)\|_{H^{s+4}}\,\mathrm{d}\tau \\ &\leqslant \|\phi\|_{H^{s+4}} + C(s,q,T,\|\phi\|_{H^{s+4}})\int_0^t \|u^N(\tau)\|_{H^{s+4}}\,\mathrm{d}\tau,\end{aligned}$$

which yields with the Gronwall inequality that

$$\|u^N(t)\|_{H^{s+4}} \leqslant \|\phi\|_{H^{s+4}} e^{C(s,q,T,\|\phi\|_{H^{s+4}})T}$$

for $0 \leqslant t \leqslant T$ and $N \geqslant N_{\phi,T}$. This completes the proof of the long time existence of classical solution to (1.5).

It remains to prove the convergence result (1.6). Let $N \geqslant N_{\phi,T}$. Since there holds the uniform H^s estimate (5.28) for $u^N(t)$, we have similarly to (5.19)

$$\sup_{0\leqslant t\leqslant T}\|v^N(t)\|_{H^s} \leqslant C(s,q,T,\|\phi\|_{H^{s+4}})N^{-\frac{1}{q}}e^{C(s,T,\|\phi\|_{H^{s+4}})T}. \tag{5.29}$$

Recall that $v^N = u^N - u^0 - u^+ - u^-$. Therefore, by (4.5), (5.29) and the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, we obtain for $4 \leqslant q < \infty$

$$\begin{aligned}\|u^N - u^0\|_{L^q(0,T;W^{1,\infty})} &\leqslant \|v^N\|_{L^q(0,T;W^{1,\infty})} + \sum_{j=\pm}\|u^j\|_{L^q(0,T;W^{1,\infty})} \\ &\leqslant T^{\frac{1}{q}}\sup_{0\leqslant t\leqslant T}\|v^N(t)\|_{H^s} + \sum_{j=\pm}\|u^j\|_{L^q(0,T;W^{1,\infty})} \\ &\leqslant C(s,q,T,\|\phi\|_{H^{s+4}})N^{-\frac{1}{q}}\end{aligned}$$

for all $N \geqslant N_{\phi,T}$. This completes the proof of Theorem 1.1. □

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