Onsager's Conjecture for the Incompressible Euler Equations in Bounded Domains

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Abstract

The goal of this note is to show that, in a bounded domain $\Omega \subset \mathbb{R}^n$, with $\partial\Omega \in C^2$, any weak solution $(u(x,t), p(x,t))$, of the Euler equations of ideal incompressible fluid in $\Omega \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}_t$, with the impermeability boundary condition $u \cdot \vec{n} = 0$ on $\partial\Omega \times (0,T)$, is of constant energy on the interval $(0,T)$, provided the velocity field $u \in L^3((0,T); C^{0,\alpha}(\overline{\Omega}))$, with $\alpha > \frac{1}{3}$.

Contents

1. Introduction and Preliminary Remarks

The aim of this article is to prove the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary, $\partial\Omega$, and let $(u(x,t), p(x,t))$ be a weak solution of the incompressible Euler equations in $\Omega \times (0,T)$, that is,*

$$u \in L^\infty((0,T); L^2(\Omega)), \quad \nabla \cdot u = 0 \quad in \quad \Omega \times (0,T), \ and$$
$$u \cdot \vec{n} = 0 \quad on \quad \partial\Omega \times (0,T), \tag{1.1}$$

and for every test vector field $\Psi(x,t) \in \mathcal{D}(\Omega \times (0,T))$:

$$\langle u, \partial_t \Psi \rangle_x + \langle u \otimes u : \nabla\Psi \rangle_x + \langle p, \nabla \cdot \Psi \rangle_x = 0, \quad in \quad L^1(0,T). \tag{1.2}$$

Assume that

$$u \in L^3((0, T); C^{0,\alpha}(\overline{\Omega})), \tag{1.3}$$

with $\alpha > \frac{1}{3}$*, then the energy conservation holds true, that is:*

$$\|u(., t_2)\|_{L^2(\Omega)} = \|u(., t_1)\|_{L^2(\Omega)}, \quad \text{for every} \quad t_1, t_2 \in (0, T). \tag{1.4}$$

In the above statement $\langle \cdot, \cdot \rangle_x$ denotes the distributional duality with respect to the spatial variable x. For the justification of the weak formulation as it is stated in the above theorem, see, for example, LIONS AND MAGENES [13] page 8, and SCHWARTZ [15].

Notably, this theorem implies that to dissipate energy, a weak solution of Euler equations must not be in the space more regular than $L^3((0, T); C^{0,\alpha}(\overline{\Omega}))$, with $\alpha > \frac{1}{3}$. Such a fact was observed, with a formal proof, by ONSAGER in 1949 [14]. Hence it carries the name Onsager conjecture. In the absence of a physical boundary (that is, in the whole space $\Omega = \mathbb{R}^d$ or for the case of periodic boundary conditions in the torus $\Omega = \mathbb{T}^n$) this conjecture was proven in 1994 by CONSTANTIN, E AND TITI [4], after a first preliminary result of EYINK [7] (see also [3] and [6]). Moreover, the relevance of this issue has been underlined by a series of contributions (cf. Isett [11], Buckmaster, De Lellis , Székelyhidi and Vicol [2] and references therein) where weak solutions, $u \in C^{0,\alpha}((0, T); \mathbb{T}^n)$, with $\alpha < \frac{1}{3}$, that dissipate energy were constructed. These results concern the problem in domains without physical boundaries. However, due to the well recognized dominant role of the boundary in the generation of turbulence (cf. [1] and references therein) it seems very reasonable to investigate the analogue of the Onsager conjecture in bounded domains. Eventually, the need to localize in order to deal with the boundary effect, as will be shown below, stimulates the construction of a direct proof which may have further applications.

The proof of the theorem will consist of several fundamental steps presented in the following propositions.

Proposition 1.2. *Under the assumptions of Theorem 1.1 the pair* (u, p) *satisfies the following regularity properties:*

$$u \otimes u \in L^3((0, T); L^2(\Omega)), \quad p \in L^{\frac{3}{2}}((0, T); C^{0,\alpha}(\overline{\Omega})), \tag{1.5a}$$

$$\partial_t u = -\nabla \cdot (u \otimes u) - \nabla p \in L^{\frac{3}{2}}((0, T); H^{-1}(\Omega)). \tag{1.5b}$$

Proof. The first part of (1.5a) is an immediate consequence of the assumption that $u \in L^\infty((0, T); L^2(\Omega)) \cap L^3((0, T); C^{0,\alpha}(\overline{\Omega}))$. For the second part of (1.5a) we first observe that, from the definition of weak solutions of the Euler equations, the pressure, p, is a solution of the following elliptic boundary-value problem:

$$-\Delta p = \sum_{i,j=1}^{n} \partial_{x_i} \partial_{x_j}(u_i u_j) \quad \text{in} \quad \Omega, \quad \text{and} \quad \frac{\partial p}{\partial \vec{n}} = -\sum_{i,j=1}^{n} u_j u_j \partial_{x_i} \vec{n}_j \quad \text{on} \quad \partial\Omega. \tag{1.6}$$

Observe that the boundary condition in (1.6) follows from simple calculations for the case of classical solutions using the fact that $u \cdot \vec{n} = 0$ (see, for example, [16]),

which is considered here to be the suitable boundary condition in the definition of weak solution for the pressure. Applying the classical theory of elliptic equations in Hölder spaces applied to (1.6) (cf., for example, [12] chapters 5 and 6) implies the estimate

$$\|p(.,t)\|_{C^{0,\alpha}} \leqq C\|u(.,t)\|^2_{C^{0,\alpha}}, \tag{1.7}$$

from which one infers the second part of (1.5a). Eventually, (1.5b) follows from (1.2) and (1.5a). □

To investigate the boundary effect one introduces the distance to the boundary: for any $x \in \overline{\Omega}$, $d(x) = \inf\limits_{y \in \partial\Omega} |x-y|$, and the open set $\Omega_h = \{x \in \Omega|$ such that $d(x) < h\}$.

Since $\partial\Omega$ is assumed to be a C^2 compact manifold, there exists $h_0(\Omega) > 0$ with the following properties (for an explicit construction see, for example, [9] page 9):

1. For any $x \in \overline{\Omega_{h_0}}$, the function $x \mapsto d(x)$ belongs to $C^1(\overline{\Omega_{h_0}})$;
2. for any $x \in \overline{\Omega_{h_0}}$ there exists a unique point $\sigma(x) \in \partial\Omega$ such that

$$d(x) = |x - \sigma(x)| \quad \text{and one has} \quad \nabla d(x) = -\vec{n}(\sigma(x)). \tag{1.8}$$

Then one introduces a $C^\infty(\mathbb{R})$ nondecreasing function $\eta : \mathbb{R} \mapsto [0,1]$, with $\eta(s) = 0$, for $s \in (-\infty, \frac{1}{2}]$, and $\eta(s) = 1$, for $s \in [1,\infty)$. For $h \in (0, h_0)$, the function $\theta_h(x) = \eta(\frac{d(x)}{h})$ is a compactly supported $C^1(\Omega)$ function. We will also denote by θ_h its extension, by zero, outside Ω. Similarly, for any $w \in L^\infty(\Omega)$, the compactly supported function $\theta_h w$ is well defined in Ω, and its extension, by zero outside Ω, is also well defined over all $\mathbb{R}^n$, and will be also denoted by $\theta_h w$. Next, one has the following:

Lemma 1.3. *Let $h \in (0, h_0)$. For any vector field $w \in C^{0,\alpha}(\overline{\Omega})$, with $w \cdot \vec{n} = 0$ on $\partial\Omega$, one has the following estimates (with a constant C independant of h, but might depend on Ω):*

$$|w(x) \cdot \nabla\theta_h(x)| \leqq C\|w\|_{C^{0,\alpha}(\Omega)} h^{\alpha-1}, \quad \text{for every} \ x \in \mathbb{R}^n, \tag{1.9a}$$

$$\int_{\mathbb{R}^n} |w(x) \cdot \nabla\theta_h(x)| dx \leqq C\|w\|_{C^{0,\alpha}(\Omega)} h^\alpha. \tag{1.9b}$$

Proof. Observe that $w(x) \cdot \nabla\theta_h(x) = 0$, for every $x \in (\Omega_h)^c$. Moreover, for $x \in \Omega_h$, thanks to (1.8), one has:

$$\nabla\theta_h(x) = -\frac{1}{h}\eta'\left(\frac{d(x)}{h}\right)\vec{n}(\sigma(x)). \tag{1.10}$$

Then for every $x \in \Omega_h$, we use the fact that $w(\sigma(x)) \cdot \vec{n}(\sigma(x) = 0$ to obtain

$$|w(x) \cdot \nabla\theta_h(x)| = \frac{1}{h}\eta'\left(\frac{d(x)}{h}\right)|(w(x) - w(\sigma(x)) \cdot \vec{n}(\sigma(x)|$$
$$\leqq \frac{C}{h}\|w\|_{C^{0,\alpha}} |x - \sigma(x)|^\alpha \leqq C\|w\|_{C^{0,\alpha}} h^{\alpha-1}. \tag{1.11}$$

Combining all of the above we conclude with (1.9a). Estimate (1.9b) follows by integrating (1.9a) over $\mathbb{R}^n$, taking into account the facts that the support of $\nabla\theta_h$ is a subset of $\overline{\Omega_h}$ and that $|\Omega_h| \leqq Ch$. □

As in [4], we introduce a nonnegative radially symmetric $C^\infty(\mathbb{R}^n)$ mollifier, $\phi(x)$, with support in $|x| \leqq 1$, and $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$. Furthermore, for any $\varepsilon > 0$, we denote by $\phi_\varepsilon = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$, and by $v^\varepsilon = v \star \phi_\varepsilon$, for any $v \in \mathcal{D}'(\mathbb{R}^n)$. Moreover, for $h \in (0, h_0)$, the distributions $(\theta_h v)^\varepsilon$ and $((\theta_h v)^\varepsilon)^\varepsilon$ belong to $\mathcal{D}(\mathbb{R}^n)$; in addition, they are compactly supported inside Ω, whenever $\varepsilon \in (0, \frac{h}{4})$.

2. Fundamental Steps Toward Proving Energy Conservation

In this section we work under the assumptions of Theorem 1.1, and we assume throughout that the regularization parameters h and ε satisfy $h \in (0, h_0)$ and $\varepsilon \in (0, \frac{h}{4})$. First observe that by virtue of Proposition 1.2, equation (1.2) remains valid for test vector field $\Psi \in W^{1,3}((0, T); H_0^1(\Omega))$. Therefore, we take in (1.2) $\Psi = \theta_h((\theta_h u)^\varepsilon)^\varepsilon \in W^{1,3}((0, T); H_0^1(\Omega))$ to obtain

$$\langle u, \partial_t(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x + \langle u \otimes u : \nabla(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x + \langle p, \nabla \cdot (\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x = 0 \tag{2.1}$$

in $L^1(0, T)$. The last equation involves three terms:

$$J_1 = \langle u, \partial_t(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x, \quad J_2 = \langle u \otimes u : \nabla(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x, \text{ and}$$
$$J_3 = \langle p, \nabla \cdot (\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x. \tag{2.2}$$

For the term J_1 we have the following:

Proposition 2.1. *Let u be as in Theorem 1.1. Then for any $(t_1, t_2) \in (0, T)$ one has*

$$\lim_{h \to 0} \int_{t_1}^{t_2} \langle u, \partial_t(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x \, dt = \frac{1}{2}\|u(t_2)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u(t_1)\|_{L^2(\Omega)}^2. \tag{2.3}$$

Proof. With the regularity estimates (1.5a) and (1.5b) the duality between $L^3((0, T); H_0^1(\Omega))$ and $L^{\frac{3}{2}}(0, T; H^{-1}(\Omega))$ gives:

$$\langle u, \partial_t(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x = \langle(\theta_h u)^\varepsilon, \partial_t(\theta_h u)^\varepsilon\rangle_x$$
$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} |((\theta_h u)^\varepsilon|^2\mathrm{d}x, \quad \text{in } L^1(0, T), \tag{2.4}$$

and the result follows, after integration in time, from the Lebesgue Dominant Convergence Theorem and the fact that $\varepsilon \in (0, \frac{h}{4})$. □

For the second term $J_2 = \langle u \otimes u : \nabla(\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x$ one has the following:

Proposition 2.2. *Let u be as in Theorem 1.1. Then*

$$|J_2| = |\langle u \otimes u : \nabla\left(\theta_h\left((\theta_h u)^\varepsilon\right)^\varepsilon\right)\rangle_x| \leqq Ch^\alpha \|u\|_{C^{0,\alpha}}\|u\|_{L^\infty}^2$$
$$+ C\|u\|_{C^{0,\alpha}}\varepsilon^{\alpha-1}\left(\|u\|_{C^{0,\alpha}}\varepsilon^\alpha + \|u\|_{L^\infty}\frac{\varepsilon}{h}\right)^2. \tag{2.5}$$

Proof. One writes $J_2 = J_{21} + J_{22}$ with

$$
\begin{aligned}
J_{21} &= \langle u \otimes u : (\nabla \theta_h) \otimes ((\theta_h u)^\varepsilon)^\varepsilon \rangle_x \\
J_{22} &= \langle u \otimes u : \theta_h \nabla (((\theta_h u)^\varepsilon)^\varepsilon) \rangle_x.
\end{aligned}
\tag{2.6}
$$

To estimate the term J_{21} one uses Lemma 1.3 to obtain

$$
|J_{21}| = \left| \langle u \otimes u : (\nabla \theta_h) \otimes ((\theta_h u)^\varepsilon))^\varepsilon \rangle_x \right| = \left| \int_{\Omega_h} (u \cdot \nabla \theta_h (x))(u(x) \cdot ((\theta_h u)^\varepsilon)^\varepsilon) dx \right|
$$

$$
\leqq C h^\alpha \|u\|_{C^{0,\alpha}} \|u\|_{L^\infty}^2.
\tag{2.7}
$$

Next, we turn into estimating the term J_{22}. First we observe that since $u^\varepsilon(x)$ is a divergence free smooth vector field for every $x \in \operatorname{supp}(\theta_h u)^\varepsilon \subset\subset \Omega$, therefore, one has:

$$
\langle (u^\varepsilon \otimes (\theta_h u)^\varepsilon) : \nabla (\theta_h u)^\varepsilon \rangle_x = \int_\Omega (u^\varepsilon \cdot \nabla (\theta_h u)^\varepsilon) \cdot (\theta_h u)^\varepsilon \, dx = 0.
\tag{2.8}
$$

Consequently, thanks to (2.8), one has the following estimate for J_{22}:

$$
\begin{aligned}
|J_{22}| &= \left| \langle u \otimes u : \theta_h \nabla ((\theta_h u)^\varepsilon)^\varepsilon \rangle_x \right| = \left| \langle (u \otimes \theta_h u) : \nabla ((\theta_h u)^\varepsilon)^\varepsilon \rangle_x \right| \\
&= \left| \langle (u \otimes \theta_h u)^\varepsilon : \nabla (\theta_h u)^\varepsilon \rangle_x \right| \\
&= \left| \left\langle \left((u \otimes \theta_h u)^\varepsilon - (u^\varepsilon \otimes (\theta_h u)^\varepsilon) \right) : \nabla (\theta_h u)^\varepsilon \right\rangle_x \right|.
\end{aligned}
\tag{2.9}
$$

To treat the term

$$
\left\langle \left((u \otimes \theta_h u)^\varepsilon - (u^\varepsilon \otimes (\theta_h u)^\varepsilon) \right) : \nabla (\theta_h u)^\varepsilon \right\rangle_x ,
$$

one uses similar computations to those in [4] (cf. Remark 3.1 below) which relate $(u \otimes \theta_h u)^\varepsilon$ to $(u^\varepsilon \otimes (\theta_h u)^\varepsilon)$. More precisely, for any two distributions, $v, w \in \mathcal{D}'(\mathbb{R}^n)$, one has the following identity:

$$
(v \otimes w)^\varepsilon(x) - (v^\varepsilon \otimes w^\varepsilon)(x) = \int_{\mathbb{R}_y^n} (\delta_y v \otimes \delta_y w)(x) \phi_\varepsilon (y) dy + (v - v^\varepsilon)(x) \otimes (w - w^\varepsilon)(x)
$$

with $(\delta_y v)(x) = v(x - y) - v(x)$, and $(\delta_y w)(x) = w(x - y) - w(x)$.

$$
\tag{2.10}
$$

Hence $J_{22} = J_{221} + J_{222}$ with

$$
\begin{aligned}
J_{221} &= \int_{\mathbb{R}_x^n} \left(\left(\int_{\mathbb{R}_y^n} (\delta_y u \otimes \delta_y (\theta_h u))(x) \phi_\varepsilon (y) dy \right) : \left(\int_{\mathbb{R}_z^n} \nabla \phi_\varepsilon (z) \otimes (\theta_h u)(x - z) dz \right) \right) dx \\
&= \int_\Omega \left(\left(\int_{\mathbb{R}_y^n} (\delta_y u \otimes \delta_y (\theta_h u))(x) \phi_\varepsilon (y) dy \right) : \left(\int_{\mathbb{R}_z^n} \nabla \phi_\varepsilon (z) \otimes (\theta_h u)(x - z) dz \right) \right) dx
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
J_{222} &= \int_{\mathbb{R}_x^n} \left(((u - u^\varepsilon) \otimes ((\theta_h u) - (\theta_h u)^\varepsilon)) : \nabla (\theta_h u)^\varepsilon \right) dx \\
&= \int_\Omega \left(((u - u^\varepsilon) \otimes ((\theta_h u) - (\theta_h u)^\varepsilon)) : \nabla (\theta_h u)^\varepsilon \right) dx
\end{aligned}
\tag{2.12}
$$

To estimate J_{221}, first, one uses the facts that for every $|y| \leqq \varepsilon$ one has $|(\delta_y \theta_h)(x)| \leqq C\frac{\varepsilon}{h}$, and that the supp $\phi_\varepsilon \subset \{y| \; |y| \leqq \varepsilon\}$, together with the $C^{0,\alpha}$ regularity of u, to obtain that

$$\left| \int_{\mathbb{R}_y^n} (\delta_y u \otimes \delta_y(\theta_h u))(x)\phi_\varepsilon(y)\mathrm{d}y \right|$$

$$= \left| \int_{\mathbb{R}_y^n} (\delta_y u)(x) \otimes (\theta_h(x-y)(\delta_y u)(x) + (\delta_y \theta_h)(x)u(x-y))\phi_\varepsilon(y)\mathrm{d}y \right|$$

$$\leqq C\varepsilon^\alpha \|u\|_{C^{0,\alpha}} \int_{\mathbb{R}_y^n} (\varepsilon^\alpha \|u\|_{C^{0,\alpha}} + \frac{\varepsilon}{h}\|u\|_{L^\infty})\phi_\varepsilon(y)\mathrm{d}y$$

$$= C\varepsilon^\alpha \|u\|_{C^{0,\alpha}} \left(\varepsilon^\alpha \|u\|_{C^{0,\alpha}} + \frac{\varepsilon}{h}\|u\|_{L^\infty} \right). \tag{2.13}$$

Second,

$$\left| \int_{\mathbb{R}_z^n} \left(\nabla\phi_\varepsilon(z) \otimes (\theta_h u)(x-z) \right)\mathrm{d}z \right|$$

$$= \left| \int_{\mathbb{R}_z^n} \left(\nabla\phi_\varepsilon(z) \otimes ((\theta_h u)(x-z) - (\theta_h u)(x)) \right)\mathrm{d}z \right|$$

$$= \left| \int_{\mathbb{R}_z^n} \left(\nabla\phi_\varepsilon(z) \otimes (\delta_z \theta_h(x)u(x-z) + \theta_h(x)\delta_z u(x)) \right)\mathrm{d}z \right|$$

$$\leqq C\left(\frac{\varepsilon}{h}\|u\|_{L^\infty} + \varepsilon^\alpha \|u\|_{C^{0,\alpha}} \right) \int_{\mathbb{R}_z^n} |\nabla\phi_\varepsilon(z)|\mathrm{d}z \leqq C\varepsilon^{-1}\left(\frac{\varepsilon}{h}\|u\|_{L^\infty} + \varepsilon^\alpha \|u\|_{C^{0,\alpha}} \right), \tag{2.14}$$

where in the last inequality we used the fact that $\int_{\mathbb{R}_z^n} |\nabla\phi_\varepsilon(z)|\mathrm{d}z \leqq C\varepsilon^{-1}$. Hence from all the above one has

$$|J_{221}| \leqq C\varepsilon^{\alpha-1}\|u\|_{C^{0,\alpha}} \left(\frac{\varepsilon}{h}\|u\|_{L^\infty} + \varepsilon^\alpha \|u\|_{C^{0,\alpha}} \right)^2. \tag{2.15}$$

To complete the proof of Proposition 2.2, it remains to estimate the term

$$J_{222} = \int_{\mathbb{R}_x^n} \left(\left((u - u^\varepsilon) \otimes ((\theta_h u) - (\theta_h u)^\varepsilon) \right) : \nabla(\theta_h u)^\varepsilon \right)\mathrm{d}x$$

$$= \int_\Omega \left(\left((u - u^\varepsilon) \otimes ((\theta_h u) - (\theta_h u)^\varepsilon) \right) : \nabla(\theta_h u)^\varepsilon \right)\mathrm{d}x. \tag{2.16}$$

First, as in (2.14), one has

$$|\nabla(\theta_h u)^\varepsilon(x)| \leqq C\varepsilon^{-1}\left(\frac{\varepsilon}{h}\|u\|_{L^\infty} + \varepsilon^\alpha \|u\|_{C^{0,\alpha}} \right). \tag{2.17}$$

Moreover, following similar arguments as to the above estimates for J_{221} one can show that for every $x \in \operatorname{supp}\theta_h$ one has

$$|(u - u^\varepsilon)(x)| \leqq \varepsilon^\alpha \|u\|_{C^{0,\alpha}} \quad \text{and} \quad |(\theta_h u)(x) - (\theta_h u)^\varepsilon(x)|$$

$$\leqq C\left(\varepsilon^\alpha \|u\|_{C^{0,\alpha}} + \frac{\varepsilon}{h}\|u\|_{L^\infty} \right). \tag{2.18}$$

Summing up, one has the following estimate:

$$|J_{222}| \leqq C\varepsilon^{\alpha-1}\|u\|_{C^{0,\alpha}} \left(\frac{\varepsilon}{h}\|u\|_{L^\infty} + \varepsilon^\alpha\|u\|_{C^{0,\alpha}}\right)^2. \qquad (2.19)$$

Collecting the estimates on J_2 from J_{21} and J_{22}, one obtains (2.5). □

Eventually, the introduction of the localized cutoff-function θ_h affects the divergence free property of the velocity field u of the solution (u, p). Therefore, to estimate the term J_3 in (2.2), which involves the pressure, p, one needs the following:

Proposition 2.3. *Let $h \in (0, h_0)$ and $\varepsilon \in (0, \frac{h}{4})$. Suppose (u, p) is a weak solution of the Euler equations with $u \in L^3((0, T); C^{0,\alpha}(\overline{\Omega}))$. Then one has the following estimate:*

$$|\langle p, \nabla \cdot (\theta_h((\theta_h u)^\varepsilon)^\varepsilon)\rangle_x| \leqq C\|u(t)\|_{C^{0,\alpha}}^3 (h^\alpha + \varepsilon^\alpha). \qquad (2.20)$$

Proof. Thanks to Proposition 1.2 one can write

$$\left\langle p, \nabla \cdot \left(\theta_h((\theta_h u)^\varepsilon)^\varepsilon\right)\right\rangle_x = \int_\Omega p \nabla \cdot \left(\theta_h((\theta_h u)^\varepsilon)^\varepsilon\right) dx = J_{31} + J_{32}$$

with $\quad J_{31} = \int_\Omega (p\,\theta_h)\nabla \cdot ((\theta_h u)^\varepsilon)^\varepsilon \, dx \quad$ and $\quad J_{32} = \int_\Omega p\,(\nabla\theta_h) \cdot ((\theta_h u)^\varepsilon)^\varepsilon \, dx.$
$$(2.21)$$

For the term J_{31} one obtains the following sequence of equalities by integration by parts and successive use of the fact that $\nabla_x\phi_\varepsilon(x - y) = -\nabla_y\phi_\varepsilon(x - y)$:

$$J_{31} = \int_\Omega \left((p(x)\theta_h(x))\nabla_x \cdot \left(\int_{\mathbb{R}_y^n}\int_{\mathbb{R}_z^n} u(z)\theta_h(z)\phi_\varepsilon(z - y)\phi_\varepsilon(x - y)dzdy\right)\right) dx$$

$$= \int_\Omega \left(p(x)\theta_h(x)\int_{\mathbb{R}_z^n}\int_{\mathbb{R}_y^n} u(z)\theta_h(z)\phi_\varepsilon(z - y) \cdot \nabla_x\phi_\varepsilon(x - y)dydz\right) dx$$

$$= -\int_\Omega \left(p(x)\theta_h(x)\int_{\mathbb{R}_z^n}\int_{\mathbb{R}_y^n} u(z)\theta_h(z)\phi_\varepsilon(z - y) \cdot \nabla_y\phi_\varepsilon(x - y)dydz\right) dx$$

$$= \int_\Omega \left(p(x)\theta_h(x)\int_{\mathbb{R}_z^n}\int_{\mathbb{R}_y^n} u(z)\theta_h(z)\phi_\varepsilon(x - y) \cdot \nabla_y\phi_\varepsilon(z - y)dydz\right) dx$$

$$= -\int_\Omega p(x)\theta_h(x)(\int_{\mathbb{R}_z^n}\int_{\mathbb{R}_y^n} u(z)\theta_h(z)\phi_\varepsilon(x - y) \cdot \nabla_z\phi_\varepsilon(z - y)dydz) dx$$

$$= -\int_\Omega \left(p(x)\theta_h(x)\int_{\mathbb{R}_y^n}\int_{\mathbb{R}_z^n} u(z)\phi_\varepsilon(x - y) \cdot \left(\nabla_z\left(\theta_h(z)\phi_\varepsilon(z - y)\right)\right.\right.$$

$$\left.\left. -\phi_\varepsilon(z - y)\nabla\theta_h(z)\right)dzdy\right) dx. \qquad (2.22)$$

Observe that for every fixed $y \in \mathbb{R}^n$, the function $\theta_h(z)\phi_\varepsilon(z - y)$, as a function of z, is compactly supported in Ω, and that there exists a sequence $\chi_k(\cdot, y) \in \mathcal{D}(\Omega)$, $k = 1, 2, \cdots$, such that

$$\lim_{k\to\infty} \|\chi_k(\cdot, y) - \theta_h(\cdot)\phi_\varepsilon(\cdot - y)\|_{C^1(\Omega)} = 0. \qquad (2.23)$$

Therefore, since $\nabla \cdot u = 0$ in $\mathcal{D}'(\Omega)$, one has:

$$\int_{\mathbb{R}_z^n} u(z) \cdot \nabla_z \chi_k(z, y) \mathrm{d}z = 0. \tag{2.24}$$

Thus, for every fixed $y \in \mathbb{R}^n$, by virtue of (2.23) and the fact that $u \in C^{0,\alpha}(\Omega)$, one infers from (2.24), by letting $k \to \infty$, that

$$\int_{\mathbb{R}_z^n} u(z) \cdot \nabla_z (\theta_h(z) \phi_\varepsilon(z - y)) \mathrm{d}z = 0. \tag{2.25}$$

Hence, as a result of (2.22) and (2.25) one has

$$J_{31} = \int_\Omega \left(p(x)\theta_h(x) \int_{\mathbb{R}_y^n} \int_{\mathbb{R}_z^n} \phi_\varepsilon(x - y)\phi_\varepsilon(z - y)u(z) \cdot \nabla\theta_h(z) \mathrm{d}z \mathrm{d}y \right) \mathrm{d}x.$$

Consequently, by virtue of Lemma 1.3, one has

$$|J_{31}| \leqq C\|p\|_{L^\infty} \|u\|_{C^{0,\alpha}} h^\alpha. \tag{2.26}$$

Concerning the term J_{32}, observe again that the support of $\nabla\theta_h$ is contained in $\overline{\Omega_h}$, therefore, one has

$$J_{32} = \int_{\Omega_h} \left(p(x)\nabla\theta_h(x) \cdot \int_{\mathbb{R}_z^n} \int_{\mathbb{R}_y^n} \theta_h(x - y + z)u(x - y + z)\phi_\varepsilon(y)\phi_\varepsilon(z) \mathrm{d}y \mathrm{d}z \right) \mathrm{d}x$$

$$= \int_{\Omega_h} p(x) \left(\int_{\mathbb{R}_y^n} \int_{\mathbb{R}_z^n} \phi_\varepsilon(y)\phi_\varepsilon(z)\theta_h(x - y + z)\left(u(x - y + z) - u(x)\right) \right.$$

$$\left. \cdot \nabla\theta_h(x) \mathrm{d}y \mathrm{d}z \right) \mathrm{d}x$$

$$+ \int_{\Omega_h} p(x) \left(\int_{\mathbb{R}_y^n} \int_{\mathbb{R}_z^n} \phi_\varepsilon(y)\phi_\varepsilon(z)u(x) \cdot \nabla\theta_h(x) \mathrm{d}y \mathrm{d}z \right) \mathrm{d}x =: J_{321} + J_{322}. \tag{2.27}$$

In order to estimate the term J_{321}, one observes that for the relevant x, y, z for which the integrand in the definition of J_{321} is not zero, one has $|(u(x - y + z) - u(x))| \leqq C\|u\|_{C^{0,\alpha}}\varepsilon^\alpha$, and that $\int_{\Omega_h} |\nabla\theta_h(x)| \, \mathrm{d}x < C$. As a result, one obtains

$$|J_{321}| \leqq C\|p\|_{L^\infty} \|u\|_{C^{0,\alpha}} \varepsilon^\alpha. \tag{2.28}$$

As for estimating J_{322}, Lemma 1.3 is used to obtain

$$|J_{322}| \leqq \int_{\Omega_h} |p(x)| \int_{\mathbb{R}_y^n} \int_{\mathbb{R}_z^n} |u(x) \cdot \nabla\theta_h(x)|\phi_\varepsilon(y)\phi_\varepsilon(z) \mathrm{d}y \mathrm{d}z \, \mathrm{d}x$$

$$\leqq C\|p(x)\|_{L^\infty} \|u(t)\|_{C^{0,\alpha}} h^\alpha. \tag{2.29}$$

$\square$

Now we are ready to complete the proof of Theorem 1.1. Let us integrate equation (2.1) over the interval $(t_1, t_2) \subset (0, T)$ to obtain

$$\int_{t_1}^{t_2} \langle u, \partial_t (\theta_h((\theta_h u)^\varepsilon)^\varepsilon) \rangle_x \, \mathrm{d}t = - \int_{t_1}^{t_2} \langle u \otimes u, \nabla(\theta_h((\theta_h u)^\varepsilon)^\varepsilon) \rangle_x \, \mathrm{d}t$$

$$- \int_{t_1}^{t_2} \langle p, \nabla(\theta_h((\theta_h u)^\varepsilon)^\varepsilon) \rangle_x \, \mathrm{d}t. \qquad (2.30)$$

At this stage we choose $\varepsilon = o(h^{\frac{2}{1+\alpha}})$, and since $\alpha > \frac{1}{3}$, Theorem 1.1 follows from Propositions 1.2, 2.1, 2.2 and 2.3 by letting $h \to 0$.

3. Remarks

Remark 3.1. The proof of Proposition 2.2 is an adaptation, to a domain with a boundary, of the main argument of [4]. The proof involves the expression

$$((u \otimes \theta_h u)^\varepsilon - (u^\varepsilon \otimes (\theta_h u)^\varepsilon),$$

which is reminiscent of the Reynolds stress tensor as it appears in statistical theory of turbulence, or in the vanishing viscosity weak limit of solutions of the Navier-Stokes equations according to the formula

$$\overline{(u_\varepsilon \otimes v_\varepsilon)} - \overline{u}_\varepsilon \otimes \overline{v}_\varepsilon = \overline{((\overline{u}_\varepsilon - u_\varepsilon) \otimes (\overline{v}_\varepsilon - v_\varepsilon))}. \qquad (3.1)$$

However, in the present work the localization and regularization do not exactly behave as an average and this is the reason for the presence (both in [4] and in this work) of the term

$$J_{221} = \int_{\mathbb{R}_x^n} \left(\left(\int_{\mathbb{R}_y^n} (\delta_y u \otimes \delta_y (\theta_h u))(x) \phi_\varepsilon(y) \mathrm{d}y \right) : \left(\int_{\mathbb{R}_z^n} \nabla \phi_\varepsilon(z) \otimes (\theta_h u)(x-z) \mathrm{d}z \right) \right) \mathrm{d}x,$$
$$(3.2)$$

which has to be estimated.

Remark 3.2. As expected, the impermeability boundary condition ($u \cdot \vec{n} = 0$ on $\partial\Omega$) plays an essential role in the arguments presented in this work. It is the main hypothesis in Lemma 1.3, which is then used for the estimation of J_{21}, in formula (2.7), and in the estimation of the pressure contribution term in formula (2.29).

Remark 3.3. Besides corresponding to physical situations that appear in nature, the introduction of boundary and boundary conditions is a stimulus for the construction of a direct proof avoiding the use of, for instance, the Besov space. However, the arguments presented in this work may well be adapted to provide similar results while replacing the Hölder spaces $C^{0,\alpha}$ by some "exotic" function spaces. Moreover, the ideas introduced in this article may also be well adapted to considering Onsager's conjecture for compressible fluids in bounded domains, and hence extending some preliminary results of [5,8,10,17] and references therein.

Remark 3.4. It is worth mentioning that, with a comparable amount of effort and similar ideas to the ones presented in this work, one should be able to extend the Duchon and Robert [6] local energy approach to the case of Euler equations with physical boundaries, and consequently establish our result. The authors are thankful to the anonymous referee for suggesting this alternative approach.

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