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Global Classical and Weak Solutions to the Three-Dimensional Full Compressible Navier–Stokes System with Vacuum and Large Oscillations

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Abstract

For the three-dimensional full compressible Navier–Stokes system describing the motion of a viscous, compressible, heat-conductive, and Newtonian polytropic fluid, we establish the global existence and uniqueness of classical solutions with smooth initial data which are of small energy but possibly large oscillations where the initial density is allowed to vanish. Moreover, for the initial data, which may be discontinuous and contain vacuum states, we also obtain the global existence of weak solutions. These results generalize previous ones on classical and weak solutions for initial density being strictly away from a vacuum, and are the first for global classical and weak solutions which may have large oscillations and can contain vacuum states.

1. Introduction

The motion of a compressible viscous, heat-conductive, and Newtonian polytropic fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following full compressible Navier–Stokes system:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \nabla P = 0, \\ (\rho E)_t + \operatorname{div}(\rho E u + P u) = \Delta\left(\kappa\theta + \frac{1}{2}\mu|u|^2\right) + \operatorname{div}(\mu u \cdot \nabla u + \lambda u \operatorname{div} u). \end{cases} \tag{1.1}$$

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Here $t \geq 0$ is time, $x \in \Omega$ is the spatial coordinate, and $\rho, u = \left(u^1, u^2, u^3\right)^{\text{tr}}, e$, $P(\rho, e)$, and θ represent respectively the fluid density, velocity, specific internal energy, pressure, and absolute temperature, and $E = e + \frac{1}{2}|u|^2$ is the specific total energy. The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0; \tag{1.2}$$

and positive constant κ is the ratio of the heat conductivity coefficient over the heat capacity. The equations (1.1) then express respectively the conservation of mass, the balance of momentum, and the balance of energy under internal pressure, viscosity forces, and the conduction of thermal energy. We study the ideal polytropic fluids so that P and e are given by the state equations

$$P(\rho, e) = (\gamma - 1)\rho e = R\rho\theta, \quad e = \frac{R\theta}{\gamma - 1}, \tag{1.3}$$

where $\gamma > 1$ is the adiabatic constant, and R is a positive constant.

Let $\Omega = \mathbb{R}^3$ and $\tilde{\rho}, \tilde{\theta}$ both be fixed positive constants. We look for the solutions $(\rho(x,t), u(x,t), \theta(x,t))$, to the Cauchy problem for (1.1) with the far field behavior

$$(\rho, u, \theta)(x,t) \to (\tilde{\rho}, 0, \tilde{\theta}), \quad \text{as } |x| \to \infty, \ t > 0, \tag{1.4}$$

and initial data

$$(\rho, \rho u, \rho\theta)(x, t=0) = (\rho_0, \rho_0 u_0, \rho_0\theta_0)(x), \quad x \in \mathbb{R}^3, \tag{1.5}$$

with $\rho_0 \geq 0, \theta_0 \geq 0$. Note here that for classical solutions, (1.1) can be rewritten as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(u_t + u \cdot \nabla u) = \mu\Delta u + (\mu + \lambda)\nabla(\operatorname{div} u) - \nabla P, \\ \frac{R}{\gamma-1}\rho(\theta_t + u \cdot \nabla\theta) = \kappa\Delta\theta - P\operatorname{div} u + \lambda(\operatorname{div} u)^2 + 2\mu|\mathfrak{D}(u)|^2, \end{cases} \tag{1.6}$$

where $\mathfrak{D}(u) = (\nabla u + (\nabla u)^{\text{tr}})/2$ is the deformation tensor. Moreover, for classical solutions, we replace the initial condition (1.5) with

$$(\rho, u, \theta)(x, t=0) = (\rho_0, u_0, \theta_0), \quad x \in \mathbb{R}^3. \tag{1.7}$$

There is a lot of literature on the large time existence and behavior of solutions to (1.1). The one-dimensional problem with strictly positive initial density and temperature has been studied extensively by many people, see [1,11,12] and the references therein. For the multi-dimensional case, the local existence and uniqueness of classical solutions are known in [16,19] in the absence of vacuum. Recently, for the case that the initial density need not be positive and may vanish in open sets, Cho–Kim [4] obtained the local existence and uniqueness of strong solutions. The global classical solutions were first obtained by Matsumura–Nishida [15] for initial data close to a non-vacuum equilibrium in some Sobolev space H^s. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from vacuum and the gradient of the density remains bounded uniformly in time. Later, Hoff [8] studied the global weak solutions with strictly positive initial density and temperature for discontinuous initial data. On the other hand, in the presence of vacuum, this issue becomes much more complicated. Concerning viscous compressible fluids in a barotropic regime, where the state of these fluids at each instant $t > 0$ is completely determined by the density $\rho = \rho(x, t)$ and the velocity $u = u(x, t)$, the pressure P being an explicit function of the density, the major breakthrough is due to Lions [14] (see also Feireisl [5,7]), where he obtained global existence of weak solutions, defined as solutions with finite energy, when the pressure P satisfies $P(\rho) = a\rho^{\gamma}$ $(a > 0, \gamma > 1)$ with suitably large γ. The main restriction on initial data is that the initial energy is finite, so that the density vanishes at far fields, or even has compact support. Recently, Huang–Li–Xin [10] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier–Stokes equations in three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support. This result can be regarded as the uniqueness and regularity theory of Lions–Feireisl's weak solutions in [5,7,14] with small initial energy.

However, the global well-posedness of classical solutions, even the global existence of weak solutions to (1.1), remains completely open in the presence of vacuum. For specific pressure laws excluding the perfect gas equation of state, the question of the existence of so-called "variational" solutions in dimension $d \geq 2$ has been recently addressed in [5,6], where the temperature equation is satisfied only as an inequality which justifies the notion of variational solutions. Recently, for a very particular form of the viscosity coefficients depending on the density, Bresch–Desjardins [3] obtained global stability of weak solutions. It is worth noting here that Xin [20] first showed that in the case that the initial density has compact support, any smooth solution to the Cauchy problem of the full compressible Navier–Stokes system without heat conduction blows up in finite time. See also the recent generalizations to the case for non-compact but rapidly decreasing at far field initial densities [18].

Motivated by our previous work on the isentropic compressible Navier–Stokes equations [10], we try to look for the global existence of classical and weak solutions to the three-dimensional full compressible Navier–Stokes system (1.1); in particular, the initial density is allowed to vanish.

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f \mathrm{d}x = \int_{\mathbb{R}^3} f \mathrm{d}x.$$

For $1 \leq p \leq \infty$ and integer $k \geq 0$, we adopt the simplified notations for the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^p = L^p(\mathbb{R}^3), \quad W^{k,p} = W^{k,p}(\mathbb{R}^3), \quad H^k = W^{k,2}, \\ D^1 = \left\{ u \in L^6 \,\middle|\, \|\nabla u\|_{L^2} < \infty \right\}, \quad D^{1,p} = \left\{ u \in L^1_{loc}(\mathbb{R}^3) \,\middle|\, \|\nabla u\|_{L^p} < \infty \right\}. \end{cases}$$

Without loss of generality, we assume that $\tilde{\rho} = \tilde{\theta} = 1$. We define the initial energy C_0 as follows:

$$C_0 \triangleq \frac{1}{2}\int \rho_0|u_0|^2\mathrm{d}x + R\int (\rho_0\log\rho_0 - \rho_0 + 1)\,\mathrm{d}x + \frac{R}{\gamma-1}\int \rho_0\,(\theta_0 - \log\theta_0 - 1)\,\mathrm{d}x. \tag{1.8}$$

Then the first main result in this paper can be stated as follows:

Theorem 1.1. *For given numbers $M > 0$ (not necessarily small), $q \in (3,6)$, $\bar{\rho} > 2$, and $\bar{\theta} > 1$, suppose that the initial data (ρ_0, u_0, θ_0) satisfies*

$$\rho_0 - 1 \in H^2 \cap W^{2,q}, \quad u_0 \in H^2, \quad \theta_0 - 1 \in H^2, \tag{1.9}$$

$$0 \le \inf\rho_0 \le \sup\rho_0 < \bar{\rho}, \quad 0 \le \inf\theta_0 \le \sup\theta_0 \le \bar{\theta}, \quad \|\nabla u_0\|_{L^2} \le M, \tag{1.10}$$

and the compatibility conditions

$$-\mu\Delta u_0 - (\mu+\lambda)\nabla\mathrm{div}u_0 + R\nabla(\rho_0\theta_0) = \sqrt{\rho_0}g_1, \tag{1.11}$$

$$\kappa\Delta\theta_0 + \frac{\mu}{2}|\nabla u_0 + (\nabla u_0)^{\mathrm{tr}}|^2 + \lambda(\mathrm{div}u_0)^2 = \sqrt{\rho_0}g_2, \tag{1.12}$$

with $g_1, g_2 \in L^2$. Then there exists a positive constant ε depending only on μ, λ, κ, R, γ, $\bar{\rho}$, $\bar{\theta}$ and M such that if

$$C_0 \le \varepsilon, \tag{1.13}$$

the Cauchy problem (1.6) (1.4) (1.7) admits a unique global classical solution (ρ, u, θ) in $\mathbb{R}^3 \times (0,\infty)$ satisfying

$$0 \le \rho(x,t) \le 2\bar{\rho}, \quad \theta(x,t) \ge 0, \quad x \in \mathbb{R}^3,\ t \ge 0, \tag{1.14}$$

and

$$\begin{cases} \rho - 1 \in C([0,T]; H^2 \cap W^{2,q}), \quad (u, \theta - 1) \in C([0,T]; H^2), \\ u \in L^\infty(\tau, T; H^3 \cap W^{3,q}), \quad \theta - 1 \in L^\infty(\tau, T; H^4), \\ (u_t, \theta_t) \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1), \end{cases} \tag{1.15}$$

for any $0 < \tau < T < \infty$. Moreover, the following large-time behavior holds:

$$\lim_{t\to\infty} (\|\rho(\cdot,t) - 1\|_{L^p} + \|\nabla u(\cdot,t)\|_{L^r} + \|\nabla\theta(\cdot,t)\|_{L^r}) = 0, \tag{1.16}$$

with any

$$p \in (2,\infty), \quad r \in [2,6). \tag{1.17}$$

The next result of this paper will treat the weak solutions with better regularity due to the fact that discontinuous solutions are fundamental both in the physical theory of nonequilibrium thermodynamics as well as in the mathematical theory of inviscid models for compressible fluids. To begin with, we give the definition of weak solutions.

Definition 1.1. We say that $(\rho, u, E = \frac{1}{2}|u|^2 + \frac{R}{\gamma-1}\theta)$ is a weak solution to Cauchy problem (1.1) (1.4) (1.5) provided that

$$\rho - 1 \in L^\infty_{\rm loc}([0,\infty); L^2 \cap L^\infty(\mathbb{R}^3)), \quad u, \theta - 1 \in L^2(0,\infty; H^1(\mathbb{R}^3)),$$

and that for all test functions $\psi \in \mathcal{D}(\mathbb{R}^3 \times (-\infty,\infty))$,

$$\int_{\mathbb{R}^3} \rho_0 \psi(\cdot,0)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}^3} (\rho\psi_t + \rho u \cdot \nabla\psi)\, \mathrm{d}x\mathrm{d}t = 0, \tag{1.18}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} \rho_0 u_0^j \psi(\cdot,0)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}^3} \left(\rho u^j \psi_t + \rho u^j u \cdot \nabla\psi + P(\rho,\theta)\psi_{x_j}\right) \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} \left(\mu \nabla u^j \cdot \nabla\psi + (\mu+\lambda)(\mathrm{div} u)\psi_{x_j}\right) \mathrm{d}x\mathrm{d}t = 0, \quad j = 1,2,3, \end{aligned} \tag{1.19}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(\frac{1}{2}\rho_0|u_0|^2 + \frac{R}{\gamma-1}\rho_0\theta_0\right) \psi(\cdot,0)\mathrm{d}x \\ &= \int_0^\infty \int_{\mathbb{R}^3} (\rho E\psi_t + (\rho E + P)u \cdot \nabla\psi)\, \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} \left(\kappa\nabla\theta + \frac{1}{2}\mu\nabla(|u|^2) + \mu u \cdot \nabla u + \lambda u \mathrm{div} u\right) \cdot \nabla\psi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{1.20}$$

Then, denoting by

$$\begin{cases} \dot{f} \triangleq f_t + u \cdot \nabla f, \\ G \triangleq (2\mu+\lambda)\mathrm{div} u - R(\rho\theta - 1), \\ \omega \triangleq \nabla \times u, \end{cases} \tag{1.21}$$

which are the material derivative of f, the effective viscous flux, and the vorticity, respectively, we state our second main result as follows:

Theorem 1.2. *For given numbers $M > 0$ (not necessarily small), $\bar\rho > 2$, and $\bar\theta > 1$, there exists a positive constant ε depending only on μ, λ, κ, R, γ, $\bar\rho$, $\bar\theta$, and M such that if the initial data (ρ_0, u_0, θ_0) satisfies (1.10) and*

$$C_0 \le \varepsilon, \tag{1.22}$$

with C_0 as in (1.8), there is a global weak solution $(\rho, u, E = \frac{1}{2}|u|^2 + \frac{R}{\gamma-1}\theta)$ to the Cauchy problem (1.1) (1.4) (1.5) satisfying

$$\rho - 1 \in C([0,\infty); L^2 \cap L^p), \tag{1.23}$$

$$(\rho u,\ \rho|u|^2,\ \rho(\theta-1)) \in C([0,\infty); H^{-1}), \tag{1.24}$$

$$u \in C((0,\infty); L^2), \quad \theta - 1 \in C((0,\infty); W^{1,r}), \tag{1.25}$$

$$u(\cdot,t),\ \omega(\cdot,t),\ G(\cdot,t),\ \nabla\theta(\cdot,t) \in H^1, \quad t > 0, \tag{1.26}$$

$$\rho \in [0, 2\bar\rho] \quad \text{almost everywhere}, \quad \theta \ge 0 \quad \text{almost everywhere}, \tag{1.27}$$

and the following large-time behavior:

$$\lim_{t\to\infty} (\|\rho(\cdot,t) - 1\|_{L^p} + \|u(\cdot,t)\|_{L^p\cap L^\infty} + \|\nabla\theta(\cdot,t)\|_{L^r}) = 0, \tag{1.28}$$

with any p, r as in (1.17). In addition, there exists some positive constant C depending only on μ, λ, κ, R, γ, $\bar\rho, \bar\theta$, and M such that, for $\sigma(t) \triangleq \min\{1, t\}$, the following estimates hold:

$$\sup_{t\in(0,\infty)} \|u\|_{H^1}^2 + \int_0^\infty \int |(\rho u)_t + \operatorname{div}(\rho u \otimes u)|^2 \,\mathrm{d}x\mathrm{d}t \le C, \tag{1.29}$$

$$\begin{aligned} &\sup_{t\in(0,\infty)} \int \left((\rho-1)^2 + \rho|u|^2 + \rho(\theta-1)^2\right) \mathrm{d}x \\ &\quad + \int_0^\infty \left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \mathrm{d}t \le C C_0^{1/4}, \end{aligned} \tag{1.30}$$

$$\begin{aligned} &\sup_{t\in(0,\infty)} \left(\sigma^2 \|\nabla u\|_{L^6}^2 + \sigma^4 \|\theta - 1\|_{H^2}^2\right) \\ &\quad + \int_0^\infty \left(\sigma^2\|u_t\|_{L^2}^2 + \sigma^2\|\nabla \dot u\|_{L^2}^2 + \sigma^4\|\theta_t\|_{H^1}^2\right) \mathrm{d}t \le C C_0^{1/8}. \end{aligned} \tag{1.31}$$

Moreover, (ρ, u, θ) satisfies (1.6)$_3$ in the weak form, that is, for any test function $\psi \in \mathcal{D}(\mathbb{R}^3 \times (-\infty, \infty))$,

$$\begin{aligned} &\frac{R}{\gamma-1}\int \rho_0\theta_0\psi(\cdot,0)\mathrm{d}x + \frac{R}{\gamma-1}\int_0^\infty\int \rho\theta\,(\psi_t + u\cdot\nabla\psi)\,\mathrm{d}x\mathrm{d}t \\ &= \kappa\int_0^\infty\int \nabla\theta\cdot\nabla\psi \mathrm{d}x\mathrm{d}t + R\int_0^\infty\int \rho\theta \operatorname{div}u\psi \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\infty\int \left(\lambda(\operatorname{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)\psi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{1.32}$$

The following Corollary 1.3, whose proof is similar to that of [10, Theorem 1.2], shows that we can obtain from (1.16) the following large time behavior of the gradient of the density when vacuum states appear initially, which is completely in contrast to the classical theory [15].

Corollary 1.3. *In addition to the conditions of Theorem 1.1, assume further that there exists some point $x_0 \in \mathbb{R}^3$ such that $\rho_0(x_0) = 0$. Then the unique global classical solution (ρ, u, θ) to the Cauchy problem (1.6) (1.4) (1.7) obtained in Theorem 1.1 has to blow up as $t \to \infty$, in the sense that for any $r > 3$,*

$$\lim_{t\to\infty} \|\nabla\rho(\cdot,t)\|_{L^r} = \infty.$$

A few remarks are in order.

Remark 1.1. It follows from (1.15) that, for any $0 < \tau < T < \infty$,

$$(\rho - 1,\ \nabla\rho,\ u,\ \theta - 1) \in C(\overline{\mathbb{R}^3} \times [0, T]), \tag{1.33}$$

and

$$\nabla u,\ \nabla^2 u \in C([\tau, T]; L^2) \cap L^\infty(\tau, T; W^{1,q}) \hookrightarrow C(\overline{\mathbb{R}^3} \times [\tau, T]), \tag{1.34}$$

which together with $(1.6)_1$ and (1.33) gives

$$\rho_t \in C(\overline{\mathbb{R}^3} \times [\tau, T]). \tag{1.35}$$

Similarly, we deduce from (1.15) that

$$\nabla\theta,\ \nabla^2\theta \in C([\tau, T]; H^1) \cap L^\infty(\tau, T; H^2) \hookrightarrow C(\overline{\mathbb{R}^3} \times [\tau, T]),$$

which combined with (1.33)–(1.35) thus shows that the solution (ρ, u, θ) obtained in Theorem 1.1 is in fact a classical one to the Cauchy problem (1.6) (1.4) (1.7) in $\mathbb{R}^3 \times (0, \infty)$. Although it has small energy, yet its oscillations could be arbitrarily large. In particular, initial vacuum states are allowed.

Remark 1.2. Theorem 1.1 is the first result concerning the global existence of classical solutions with vacuum to the full compressible Navier–Stokes system. Moreover, the conclusions in Theorem 1.1 generalize the classical theory of Matsumura–Nishida [15] to the case of large oscillations since in this case, the requirement of small energy, (1.13), is equivalent to smallness of the mean-square norm of $(\rho_0 - 1, u_0, \theta_0 - 1)$. In addition, the initial density is allowed to vanish and the initial temperature may be zero. However, although the large-time asymptotic behavior (1.16) is similar to that in [15], yet our solution may contain vacuum states, whose appearance leads to the large time blowup behavior stated in Corollary 1.3, this is in sharp contrast to that in [15] where the gradients of the density are suitably small uniformly for all time.

Remark 1.3. It should be noted here that Theorem 1.2 is the first result concerning the global existence of weak solutions to (1.1) in the presence of vacuum and extends the global weak solutions of Hoff [8] to the case of large oscillations and non-negative initial density. Moreover, the initial temperature is allowed to be zero.

Remark 1.4. It follows from (1.29) and Sobolev's embedding theorem that u and θ obtained in Theorem 1.2 are in fact Hölder continuous away from $t = 0$, that is, for any $0 < \tau < \infty$,

$$\sup_{t\in[\tau,\infty)} \|u\|_{L^\infty} + \langle u\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\tau,\infty)} + \sup_{t\in[\tau,\infty)} \|\theta\|_{L^\infty} + \langle \theta\rangle^{1/2,1/8}_{\mathbb{R}^3\times[\tau,\infty)} < \infty,$$

where we employ the usual notation for Hölder norms:

$$\langle w\rangle^{1/2,1/8}_Q = \sup_{\substack{(x,t),(y,s)\in Q\\ (x,t)\neq(y,s)}} \frac{|w(x,t) - w(y,s)|}{|x-y|^{1/2} + |t-s|^{1/8}},$$

for functions $w : Q \subseteq \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^m$.

Remark 1.5. In fact, the weak solutions obtained by Theorem 1.2 have better regularity than just finite energy weak ones, and can be viewed as mild solutions to the full compressible Navier–Stokes system (1.1).

We now comment on the analysis of this paper. Note that though the local existence and uniqueness of strong solutions to (1.6) in the presence of vacuum was obtained by Cho–Kim [4], the local existence of classical solutions with vacuum to (1.6) still remains unknown. Some of the main new difficulties to obtain the classical solutions to (1.6) (1.4) (1.7) for initial data in the class satisfying (1.9)–(1.12) are due to the appearance of vacuum. Thus, we take the strategy that we first extend the standard local classical solutions with strictly positive initial density (see Lemma 2.1) globally in time just under the condition that the initial energy is suitably small (see Proposition 5.1), then let the lower bound of the initial density go to zero. To do so, one needs to establish global a priori estimates, which are independent of the lower bound of the density, on smooth solutions to (1.6) (1.4) (1.7) in suitable higher norms. It turns out that the key issue in this paper is to derive both the time-independent upper bound for the density and the time-dependent higher norm estimates of the smooth solution (ρ, u, θ). Compared to the isentropic case [10], the first main difficulty lies in the fact that the basic energy estimate cannot yield directly the bounds on the L^2-norm (in both time and space) of the spatial derivatives of both the velocity and the temperature since the super norm of the temperature is just assumed to satisfy the a priori bound $(\min\{1, t\})^{-3/2}$ (see (3.6)), which in fact could be arbitrarily large for small time. To overcome this difficulty, based on careful analysis on the basic energy estimate, we succeed in deriving a new estimate of the temperature which shows that the spatial L^2-norm of the deviation of the temperature from its far field value can be bounded by the combination of the initial energy with the spatial L^2-norm of the spatial derivatives of the temperature (see (3.10)). Combining this estimate, which will play a crucial role in the analysis of this paper, with elaborate analysis on the bounds of the energy, then yields the key energy-like estimate, provided that the initial energy is suitably small (see Lemma 3.3).

Next, the second main difficulty is to obtain the time-independent upper bound of the density. Based on careful initial layer analysis and making a full use of the structure of (1.6), we succeed in deriving the weighted spatial mean estimates of the material derivatives of both the velocity and the temperature, which are independent of the lower bound of density, provided that the initial energy is suitably small (see Lemmas 3.4 and 3.5). This approach is motivated by the basic estimates of the material derivatives of both the velocity and the temperature, which are developed by Hoff [8] in the theory of weak solutions with strictly positive initial density. Having all these estimates at hand, we get the desired estimates of $L^1(0, \min\{1, T\}; L^\infty(\mathbb{R}^3))$-norm and the time-independent ones of $L^2(\min\{1, T\}, T; L^\infty(\mathbb{R}^3))$-norm of both the effective viscous flux (see (1.21)) for the definition) and the deviation of the temperature from its far field value. Using these key estimates and a Grönwall-type inequality (see Lemma 2.5), we obtain a time-uniform upper bound of the density which is crucial for global estimates of classical solutions. This approach to estimate a uniform upper bound for the density is new compared to our previous analysis on the isentropic compressible Navier–Stokes equations in [10].

Then, the third main step is to bound the gradients of the density, the velocity, and the temperature. Motivated by our recent studies [9] on the blow-up criteria of strong (or classical) solutions to the barotropic compressible Navier–Stokes equations, such bounds can be obtained by solving a logarithm Grönwall inequality based on a Beale–Kato–Majda-type inequality (see Lemma 2.6) and the a priori estimates we have just derived. Moreover, such a derivation simultaneously yields the bound for $L^{3/2}(0, T; L^\infty(\mathbb{R}^3))$-norm of the gradient of the velocity(see Lemma 4.1 and its proof). It should be noted here that we do not require smallness of the gradient of the initial density which prevents the appearance of vacuum [15].

Finally, with these a priori estimates of the gradients of the solutions at hand, one can obtain the desired higher order estimates by careful initial layer analysis on the time derivatives and then the spatial ones of the density, the velocity and the temperature. It should be emphasized here that all these a priori estimates are independent of the lower bound of the density. Therefore, we can build proper approximate solutions with strictly positive initial density then take appropriate limits by letting the lower bound of the initial density go to zero. The limiting functions having exactly the desired properties are shown to be the global classical solutions to the Cauchy problem (1.6) (1.4) (1.7). In addition, the initial density is allowed to vanish. We can also establish the global weak solutions almost the same way as we established the classical one with a new modified approximating initial data.

The rest of the paper is organized as follows: in Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to deriving the lower-order a priori estimates on classical solutions which are needed to extend the local solution to all time. Based on the previous results, higher-order estimates are established in Section 4. Then finally, the main results, Theorems 1.1 and 1.2, are proved in Section 5.

2. Preliminaries

The following well-known local existence theory, where the initial density is strictly away from vacuum, can be shown by the standard contraction mapping argument (see for example [15,16], in particular, [15, Theorem 5.2]).

Lemma 2.1. *Assume that* (ρ_0, u_0, θ_0) *satisfies*

$$(\rho_0 - 1, u_0, \theta_0 - 1) \in H^3, \quad \inf_{x\in\mathbb{R}^3} \rho_0(x) > 0. \tag{2.1}$$

Then there exist a small time $T_0 > 0$ *and a unique classical solution* (ρ, u, θ) *to the Cauchy problem* (1.6) (1.4) (1.7) *on* $\mathbb{R}^3 \times (0, T_0]$ *such that*

$$\inf_{(x,t)\in\mathbb{R}^3\times(0,T_0]} \rho(x,t) \geq \frac{1}{2} \inf_{x\in\mathbb{R}^3} \rho_0(x), \tag{2.2}$$

$$\begin{cases} (\rho - 1, u, \theta - 1) \in C([0, T_0]; H^3), \quad \rho_t \in C([0, T_0]; H^2), \\ (u_t, \theta_t) \in C([0, T_0]; H^1), \quad (u, \theta - 1) \in L^2(0, T_0; H^4), \end{cases} \tag{2.3}$$

and

$$\begin{cases} (\sigma u_t, \sigma\theta_t) \in L^2(0, T_0; H^3), \quad (\sigma u_{tt}, \sigma\theta_{tt}) \in L^2(0, T_0; H^1), \\ (\sigma^2 u_{tt}, \sigma^2\theta_{tt}) \in L^2(0, T_0; H^2), \quad (\sigma^2 u_{ttt}, \sigma^2\theta_{ttt}) \in L^2(0, T_0; L^2), \end{cases} \tag{2.4}$$

where $\sigma(t) \triangleq \min\{1, t\}$. *Moreover, for any* $(x, t) \in \mathbb{R}^3 \times [0, T_0]$, *the following estimate holds*

$$\theta(x,t) \geq \inf_{x\in\mathbb{R}^3} \theta_0(x) \exp\left\{-(\gamma-1)\int_0^{T_0} \|\mathrm{div} u\|_{L^\infty} \mathrm{d}t\right\}, \tag{2.5}$$

provided $\inf_{x\in\mathbb{R}^3} \theta_0(x) \geq 0$.

Proof. We only have to show (2.4) and (2.5), which are not given in [15, Theorem 5.2].

Without loss of generality, assume that $T_0 \leq 1$. We first prove $(2.4)_1$. Differentiating $(1.6)_2$ with respect to t leads to

$$\begin{aligned} &\rho u_{tt} + \rho_t u_t + \rho_t u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t + \nabla P_t \\ &\quad = \mu \Delta u_t + (\mu+\lambda)\nabla \mathrm{div} u_t. \end{aligned} \tag{2.6}$$

This shows that tu_t satisfies

$$\begin{cases} \rho(tu_t)_t - \mu\Delta(tu_t) - (\mu+\lambda)\nabla\mathrm{div}(tu_t) = F_1, \\ (tu_t)(x,0) = 0, \end{cases} \tag{2.7}$$

where

$$F_1 \triangleq \rho u_t - t\rho_t u_t - t\rho_t u\cdot\nabla u - t\rho u_t \cdot \nabla u - t\rho u \cdot \nabla u_t - Rt\nabla(\rho_t\theta + \rho\theta_t)$$

satisfies $F_1 \in L^2(0, T_0; L^2)$ due to (2.3). It thus follows from (2.3), (2.2), and standard L^2-theory for parabolic system (2.7) that

$$(tu_t)_t, \nabla^2(tu_t) \in L^2(0, T_0; L^2). \tag{2.8}$$

Similarly, we differentiate $(1.6)_3$ with respect to t to get

$$\begin{aligned} &-\frac{\kappa(\gamma-1)}{R}\Delta\theta_t + \rho\theta_{tt} \\ &\quad = -\rho_t\theta_t - \rho_t\left(u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div}u\right) - \rho\left(u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div}u\right)_t \\ &\qquad + \frac{\gamma-1}{R}\left(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)_t, \end{aligned} \tag{2.9}$$

which implies that $t\theta_t$ satisfies

$$\begin{cases} R\rho(t\theta_t)_t - \kappa(\gamma-1)\Delta(t\theta_t) = RF_2, \\ (t\theta_t)(x,0) = 0, \end{cases} \tag{2.10}$$

with

$$\begin{aligned} F_2 \triangleq &\rho\theta_t - t\rho_t\theta_t - t\rho_t\left(u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div}u\right) \\ &- t\rho\left(u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div}u\right)_t + \frac{\gamma-1}{R}t\left(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)_t. \end{aligned}$$

One derives from (2.3) that $F_2 \in L^2(0, T_0; L^2)$, which together with (2.3), (2.2), and standard L^2-theory for parabolic system (2.10) implies

$$(t\theta_t)_t, \nabla^2(t\theta_t) \in L^2(0, T_0; L^2). \tag{2.11}$$

It thus follows from (2.3), (2.8), and (2.11) that

$$F_1, F_2 \in L^2(0, T_0; H^1),$$

which together with (2.3), (2.2), (2.7), and (2.10) gives $(2.4)_1$.

Next, we prove $(2.4)_2$. Differentiating (2.6) with respect to t gives

$$\begin{aligned}&\rho u_{ttt} + \rho u \cdot \nabla u_{tt} - \mu \Delta u_{tt} - (\mu + \lambda)\nabla \mathrm{div} u_{tt}\\&\quad = 2\mathrm{div}(\rho u)u_{tt} + \mathrm{div}(\rho u)_t u_t - 2(\rho u)_t \cdot \nabla u_t - (\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u\\&\qquad -\rho u_{tt} \cdot \nabla u - \nabla P_{tt}.\end{aligned} \tag{2.12}$$

This together with $(2.4)_1$ and (2.3) implies that $t^2 u_{tt}$ satisfies

$$\begin{cases}\rho(t^2 u_{tt})_t - \mu\Delta(t^2 u_{tt}) - (\mu + \lambda)\nabla \mathrm{div}(t^2 u_{tt}) = F_3,\\(t^2 u_{tt})(x, 0) = 0,\end{cases} \tag{2.13}$$

where

$$\begin{aligned}F_3 \triangleq\, & 2t\rho u_{tt} - t^2 \rho u \cdot \nabla u_{tt} + 2t^2 \mathrm{div}(\rho u) u_{tt} + t^2 \mathrm{div}(\rho u)_t u_t\\& - 2t^2(\rho u)_t \cdot \nabla u_t - t^2(\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u - t^2 \rho u_{tt} \cdot \nabla u - t^2 \nabla P_{tt},\end{aligned} \tag{2.14}$$

satisfies $F_3 \in L^2(0, T_0; L^2)$ due to (2.3) and $(2.4)_1$. It follows from (2.2), (2.3), $(2.4)_1$, and standard L^2-estimate for (2.13) that

$$(t^2 u_{tt})_t, \nabla^2(t^2 u_{tt}) \in L^2(0, T_0; L^2). \tag{2.15}$$

Similarly, differentiating (2.9) with respect to t yields

$$\begin{aligned}&\rho\theta_{ttt} + \rho u \cdot \nabla\theta_{tt} - \frac{\kappa(\gamma - 1)}{R}\Delta\theta_{tt}\\&\quad = 2\mathrm{div}(\rho u)\theta_{tt} - \rho_{tt}\left(\theta_t + u \cdot \nabla\theta + (\gamma - 1)\theta \mathrm{div} u\right)\\&\qquad -2\rho_t\left(u \cdot \nabla\theta + (\gamma - 1)\theta \mathrm{div} u\right)_t\\&\qquad -\rho\left(u_{tt} \cdot \nabla\theta + 2u_t \cdot \nabla\theta_t + (\gamma - 1)(\theta \mathrm{div} u)_{tt}\right)\\&\qquad +\frac{\gamma - 1}{R}\left(\lambda(\mathrm{div} u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)_{tt}.\end{aligned} \tag{2.16}$$

We thus obtain from $(2.4)_1$, (2.3), and (2.16) that $t^2\theta_{tt}$ satisfies

$$\begin{cases}R\rho(t^2\theta_{tt})_t - \kappa(\gamma - 1)\Delta(t^2\theta_{tt}) = RF_4,\\(t^2\theta_{tt})(x, 0) = 0,\end{cases} \tag{2.17}$$

with

$$\begin{aligned} F_4 \triangleq & 2t\rho\theta_{tt} - t^2\rho u\cdot\nabla\theta_{tt} + 2t^2\mathrm{div}(\rho u)\theta_{tt} - t^2\rho_{tt}\left(\theta_t + u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div}u\right) \\ & - 2t^2\rho_t\left(u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div}u\right)_t - t^2\rho u_{tt}\cdot\nabla\theta - 2t^2\rho u_t\cdot\nabla\theta_t \\ & - (\gamma-1)t^2\rho(\theta\mathrm{div}u)_{tt} + \frac{\gamma-1}{R}t^2\left(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)_{tt}. \end{aligned}$$

It thus follows from (2.3) and $(2.4)_1$ that $F_4 \in L^2(0,T_0;L^2)$, which together with (2.2), (2.3), $(2.4)_1$, and standard L^2-estimate for (2.17) gives that

$$(t^2\theta_{tt})_t,\ \nabla^2(t^2\theta_{tt}) \in L^2(0,T_0;L^2). \tag{2.18}$$

One thus obtain $(2.4)_2$ directly from (2.3), $(2.4)_1$, (2.15), and (2.18).

Finally, we will show the lower bound of θ, (2.5), by maximum principle. In fact, it follows from $(1.6)_3$ and (1.4) that

$$\begin{aligned} & \rho\theta_t + \rho u\cdot\nabla\theta - \frac{\kappa(\gamma-1)}{R}\Delta\theta + (\gamma-1)\rho\theta\mathrm{div}u \geq 0, \\ & \theta \to 1 \ \text{ as } \ |x|\to\infty, \end{aligned}$$

where we have used

$$2\mu|\mathfrak{D}(u)|^2 + \lambda(\mathrm{div}u)^2 \geq 0. \tag{2.19}$$

By (2.3), we have

$$\int_0^{T_0} \|\mathrm{div}u\|_{L^\infty}\mathrm{d}t < \infty,$$

which together with the standard maximum principle thus gives (2.5). The proof of Lemma 2.1 is completed.

Next, the following well-known Gagliardo–Nirenberg–Sobolev-type inequality will be used later frequently (see [17]).

Lemma 2.2. *For $p\in(1,\infty)$ and $q\in(3,\infty)$, there exists some generic constant $C>0$ which may depend on p and q such that for $f\in D^1(\mathbb{R}^3)$, $g\in L^p(\mathbb{R}^3)\cap D^{1,q}(\mathbb{R}^3)$, and $\varphi,\psi\in H^2(\mathbb{R}^3)$, we have*

$$\|f\|_{L^6} \leq C\|\nabla f\|_{L^2}, \tag{2.20}$$

$$\|g\|_{C\left(\overline{\mathbb{R}^3}\right)} \leq C\|g\|_{L^p}^{p(q-3)/(3q+p(q-3))}\|\nabla g\|_{L^q}^{3q/(3q+p(q-3))}, \tag{2.21}$$

$$\|\varphi\psi\|_{H^2} \leq C\|\varphi\|_{H^2}\|\psi\|_{H^2}. \tag{2.22}$$

Then, the following inequality is an easy consequence of (2.20) and will play an important role in our analysis.

Lemma 2.3. *Let the function $g(x)$ defined in $\mathbb{R}^3$ be non-negative and satisfy $g(\cdot)-1\in L^2(\mathbb{R}^3)$. Then there exists a universal positive constant C such that for $r\in[1,2]$ and any open set $\Sigma\subset\mathbb{R}^3$, the following estimate holds*

$$\int_\Sigma |f|^r\mathrm{d}x \leq C\int_\Sigma g|f|^r\mathrm{d}x + C\|g-1\|_{L^2(\mathbb{R}^3)}^{(6-r)/3}\|\nabla f\|_{L^2(\mathbb{R}^3)}^r, \tag{2.23}$$

for all $f\in\left\{f\in D^1(\mathbb{R}^3)\,\middle|\, g|f|^r\in L^1(\Sigma)\right\}$.

Proof. In fact, Sobolev's inequality, (2.20), yields that

$$\begin{aligned}2\int_\Sigma |f|^r\mathrm{d}x &\le 2\int_\Sigma g|f|^r\mathrm{d}x + 2\int_\Sigma |g-1||f|^r\mathrm{d}x\\ &\le 2\int_\Sigma g|f|^r\mathrm{d}x + 2\|g-1\|_{L^2(\mathbb{R}^3)}\|f\|_{L^r(\Sigma)}^{r(3-r)/(6-r)}\|f\|_{L^6(\mathbb{R}^3)}^{3r/(6-r)}\\ &\le 2\int_\Sigma g|f|^r\mathrm{d}x + \int_\Sigma |f|^r\mathrm{d}x + C\|g-1\|_{L^2(\mathbb{R}^3)}^{(6-r)/3}\|\nabla f\|_{L^2(\mathbb{R}^3)}^r,\end{aligned}$$

which implies (2.23) directly. The proof of Lemma 2.3 is completed.

Next, it follows from $(1.6)_2$ that G and ω, defined in (1.21), satisfy

$$\triangle G = \operatorname{div}(\rho\dot{u}), \quad \mu\triangle\omega = \nabla\times(\rho\dot{u}). \tag{2.24}$$

Applying the standard L^p-estimate to the elliptic systems (2.24) together with (2.20) yields the following elementary estimates (see [10, Lemma 2.3]):

Lemma 2.4. *Let (ρ, u, θ) be a smooth solution of (1.6) (1.4). Then there exists a generic positive constant C depending only on μ, λ, and R such that, for any $p \in [2, 6]$,*

$$\|\nabla u\|_{L^p} \le C\left(\|G\|_{L^p} + \|\omega\|_{L^p}\right) + C\|\rho\theta - 1\|_{L^p}, \tag{2.25}$$

$$\|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{2.26}$$

$$\begin{aligned}\|G\|_{L^p} + \|\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)}\big(\|\nabla u\|_{L^2}\\ +\|\rho\theta-1\|_{L^2}\big)^{(6-p)/(2p)},\end{aligned} \tag{2.27}$$

$$\|\nabla u\|_{L^p} \le C\|\nabla u\|_{L^2}^{(6-p)/(2p)}\left(\|\rho\dot{u}\|_{L^2} + \|\rho\theta - 1\|_{L^6}\right)^{(3p-6)/(2p)}. \tag{2.28}$$

Next, the following Grönwall-type inequality will be used to get the uniform (in time) upper bound of the density ρ:

Lemma 2.5. *Let the function $y \in W^{1,1}(0, T)$ satisfy*

$$y'(t) + \alpha y(t) \le g(t) \text{ on } [0, T], \quad y(0) = y^0, \tag{2.29}$$

where α is a positive constant and $g \in L^p(0, T_1)\cap L^q(T_1, T)$ for some $p \ge 1$, $q \ge 1$, and $T_1 \in [0, T]$. Then

$$\sup_{0\le t\le T} y(t) \le |y^0| + (1+\alpha^{-1})\left(\|g\|_{L^p(0,T_1)} + \|g\|_{L^q(T_1,T)}\right). \tag{2.30}$$

Proof. Let p' and q' denote the conjugate numbers of p and q respectively. Multiplying (2.29) by $e^{\alpha t}$ and integrating the resulting inequality over $(0, t)$ yield that

$$\begin{aligned}e^{\alpha t}y(t) &\le y^0 + \int_0^{\min\{t,T_1\}} e^{\alpha s}|g(s)|ds + \int_{\min\{t,T_1\}}^t e^{\alpha s}|g(s)|ds\\ &\le |y^0| + \|g\|_{L^p(0,\min\{t,T_1\})}\|e^{\alpha s}\|_{L^{p'}(0,t)}\\ &\quad + \|g\|_{L^q(\min\{t,T_1\},t)}\|e^{\alpha s}\|_{L^{q'}(0,t)}\\ &\le |y^0| + \left(\|g\|_{L^p(0,T_1)} + \|g\|_{L^q(T_1,T)}\right)(1+\alpha^{-1})e^{\alpha t},\end{aligned}$$

due to $\|e^{\alpha s}\|_{L^r(0,t)} \le (1+\alpha^{-1})e^{\alpha t}$, for all $r \in [1,\infty]$. This yields (2.30) directly and finishes the proof of Lemma 2.5.

Finally, the following Beale–Kato–Majda-type inequality whose proof can be found in [2,9] will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\rho\|_{L^2\cap L^6}$.

Lemma 2.6. [2,9] *For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$:*

$$\begin{aligned}\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le{}& C\left(\|\mathrm{div} u\|_{L^\infty(\mathbb{R}^3)} + \|\nabla\times u\|_{L^\infty(\mathbb{R}^3)}\right)\log(e+\|\nabla^2 u\|_{L^q(\mathbb{R}^3)})\\ &+ C\|\nabla u\|_{L^2(\mathbb{R}^3)} + C.\end{aligned}$$

3. A Priori Estimates (I): Lower-Order Estimates

In this section, we will establish a priori bounds for the smooth, local-in-time solution to (1.6) (1.4) (1.7) obtained in Lemma 2.1. We thus fix a smooth solution (ρ, u, θ) of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ for some time $T > 0$, with initial data (ρ_0, u_0, θ_0) satisfying (2.1).

For $\sigma(t) \triangleq \min\{1, t\}$, we define $A_i(T)(i = 1, \ldots, 4)$ as follows:

$$A_1(T) = \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho|\dot{u}|^2 \mathrm{d}x\mathrm{d}t, \tag{3.1}$$

$$A_2(T) = \frac{R}{2(\gamma-1)} \sup_{t\in[0,T]} \int \rho(\theta-1)^2 \mathrm{d}x + \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \mathrm{d}t, \tag{3.2}$$

$$\begin{aligned}A_3(T) ={}& \sup_{t\in(0,T]} \left(\sigma\|\nabla u\|_{L^2}^2 + \sigma^2 \int \rho|\dot{u}|^2 \mathrm{d}x + \sigma^2\|\nabla\theta\|_{L^2}^2\right)\\ &+ \int_0^T \int \left(\sigma\rho|\dot{u}|^2 + \sigma^2|\nabla\dot{u}|^2 + \sigma^2\rho(\dot{\theta})^2\right) \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.3}$$

$$A_4(T) = \sup_{t\in(0,T]} \sigma^4 \int \rho|\dot{\theta}|^2 \mathrm{d}x + \int_0^T \int \sigma^4|\nabla\dot{\theta}|^2 \mathrm{d}x\mathrm{d}t. \tag{3.4}$$

We have the following key a priori estimates on (ρ, u, θ).

Proposition 3.1. *For given numbers $M > 0$ (not necessarily small), $\bar{\rho} > 2$, and $\bar{\theta} > 0$, assume that (ρ_0, u_0, θ_0) satisfies*

$$0 < \inf\rho_0 \le \sup\rho_0 < \bar{\rho}, \quad 0 < \inf\theta_0 \le \sup\theta_0 \le \bar{\theta}, \quad \|\nabla u_0\|_{L^2} \le M. \tag{3.5}$$

Then there exist positive constants K and ε_0 both depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}, \bar{\theta}$, and M such that if (ρ, u, θ) is a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying

$$0 < \rho \le 2\bar{\rho}, \quad A_1(\sigma(T)) \le 3K, \quad A_i(T) \le 2C_0^{1/(2i)} \ (i = 2, 3, 4), \tag{3.6}$$

the following estimates hold:

$$0 < \rho \le 3\bar{\rho}/2, \quad A_1(\sigma(T)) \le 2K, \quad A_i(T) \le C_0^{1/(2i)} \ (i = 2, 3, 4), \tag{3.7}$$

provided

$$C_0 \le \varepsilon_0. \tag{3.8}$$

Proof. Proposition 3.1 is an easy consequence of the following Lemmas 3.2, 3.3, and 3.6–3.8, with ε_0 as in (3.97).

In this section, we always assume that $C_0 \le 1$ and let C denote some generic positive constant depending only on $\mu, \lambda, \kappa, R, \gamma, \bar\rho, \bar\theta$, and M, and we write $C(\alpha)$ to emphasize that C may depend on α.

First, the following elementary L^2 bounds are crucial for deriving the desired estimate on $A_2(T)$ (see Lemma 3.3 below).

Lemma 3.1. *Under the conditions of Proposition 3.1, there exists a positive constant $C = C(\bar\rho)$ depending only on $\mu, \lambda, \kappa, R, \gamma$, and $\bar\rho$ such that if (ρ, u, θ) is a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying $0 < \rho \le 2\bar\rho$, the following estimates hold:*

$$\sup_{0\le t\le T} \int \left(\rho|u|^2 + (\rho-1)^2\right) \mathrm{d}x \le C(\bar\rho)C_0, \tag{3.9}$$

and

$$\|(\theta-1)(\cdot,t)\|_{L^2} \le C(\bar\rho)C_0^{1/2} + C(\bar\rho)C_0^{1/3}\|\nabla\theta(\cdot,t)\|_{L^2}, \tag{3.10}$$

for all $t \in (0, T]$.

Proof. First, it follows from (3.5) and (2.5) that, for all $(x,t) \in \mathbb{R}^3 \times (0, T)$,

$$\theta(x,t) > 0. \tag{3.11}$$

Adding $(1.6)_2$ multiplied by u to $(1.6)_3$ multiplied by $1-\theta^{-1}$, we obtain after integrating the resulting equality over $\mathbb{R}^3$ and using $(1.6)_1$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\frac{1}{2}\rho|u|^2 + R(1+\rho\log\rho-\rho) + \frac{R}{\gamma-1}\rho(\theta-\log\theta-1)\right)\mathrm{d}x \\
&= \int \Big[-\mu|\nabla u|^2 - (\lambda+\mu)(\mathrm{div}u)^2 - \kappa\theta^{-2}|\nabla\theta|^2 \\
&\quad + (1-\theta^{-1})(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2)\Big]\mathrm{d}x \\
&= -\int \left(\theta^{-1}(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2) + \kappa\theta^{-2}|\nabla\theta|^2\right)\mathrm{d}x,
\end{aligned} \tag{3.12}$$

where in the second equality we have used

$$2\int |\mathfrak{D}(u)|^2\mathrm{d}x = \int \left(|\nabla u|^2 + (\mathrm{div}u)^2\right)\mathrm{d}x. \tag{3.13}$$

Direct calculations yield that

$$\begin{aligned}
\rho\log\rho - \rho + 1 &= (\rho-1)^2\int_0^1 \frac{1-\alpha}{\alpha(\rho-1)+1}d\alpha \\
&\ge \frac{1}{2(2\bar\rho+1)}(\rho-1)^2,
\end{aligned} \tag{3.14}$$

and

$$\theta - \log\theta - 1 = (\theta-1)^2 \int_0^1 \frac{\alpha}{\alpha(\theta-1)+1} d\alpha \\ \geq \frac{1}{8}(\theta-1)1_{(\theta(\cdot,t)>2)} + \frac{1}{12}(\theta-1)^2 1_{(\theta(\cdot,t)<3)}, \tag{3.15}$$

where we denote

$$(\theta(\cdot,t)>2) \triangleq \left\{x\in\mathbb{R}^3 \middle| \theta(x,t)>2\right\},$$
$$(\theta(\cdot,t)<3) \triangleq \left\{x\in\mathbb{R}^3 \middle| \theta(x,t)<3\right\}.$$

Integrating (3.12) with respect to t over $(0,T)$ yields

$$\sup_{0\le t\le T}\int \left(\frac{1}{2}\rho|u|^2 + R(1+\rho\log\rho-\rho) + \frac{R}{\gamma-1}\rho(\theta-\log\theta-1)\right)\mathrm{d}x \\ + \int_0^T\int\left(\frac{1}{\theta}(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2) + \kappa\frac{|\nabla\theta|^2}{\theta^2}\right)\mathrm{d}x\mathrm{d}t \le 2C_0, \tag{3.16}$$

which together with (2.19), (3.11), (3.14), and (3.15) leads to

$$\sup_{0\le t\le T}\int\left(\rho|u|^2+(\rho-1)^2\right)\mathrm{d}x \\ + \sup_{0\le t\le T}\int\left(\rho(\theta-1)1_{(\theta(\cdot,t)>2)} + \rho(\theta-1)^2 1_{(\theta(\cdot,t)<3)}\right)\mathrm{d}x \\ \le C(\bar\rho)C_0. \tag{3.17}$$

This directly gives (3.9).

Next, we shall prove (3.10). Taking $g(x)=\rho(x,t)$, $f(x)=\theta(x,t)-1$, $r=2$ and $\Sigma=(\theta(\cdot,t)<3)$ in (2.23), we conclude after using (3.17) that

$$\|\theta(\cdot,t)-1\|_{L^2(\theta(\cdot,t)<3)} \le C(\bar\rho)C_0^{1/2} + C(\bar\rho)C_0^{1/3}\|\nabla\theta(\cdot,t)\|_{L^2(\mathbb{R}^3)}. \tag{3.18}$$

Similarly, taking $g(x)=\rho(x,t)$, $f(x)=\theta(x,t)-1$, $r=1$ and $\Sigma=(\theta(\cdot,t)>2)$ in (2.23), we obtain after using (3.17) that

$$\|\theta(\cdot,t)-1\|_{L^1(\theta(\cdot,t)>2)} \le C(\bar\rho)C_0 + C(\bar\rho)C_0^{5/6}\|\nabla\theta(\cdot,t)\|_{L^2(\mathbb{R}^3)},$$

which together with Hölder's inequality and (2.20) leads to

$$\|\theta(\cdot,t)-1\|_{L^2(\theta(\cdot,t)>2)} \\ \le \|\theta(\cdot,t)-1\|_{L^1(\theta(\cdot,t)>2)}^{2/5}\|\theta(\cdot,t)-1\|_{L^6(\mathbb{R}^3)}^{3/5} \\ \le C(\bar\rho)\left(C_0^{2/5}+C_0^{1/3}\|\nabla\theta(\cdot,t)\|_{L^2}^{2/5}\right)\|\nabla\theta(\cdot,t)\|_{L^2}^{3/5} \\ \le C(\bar\rho)C_0^{1/2} + C(\bar\rho)C_0^{1/3}\|\nabla\theta(\cdot,t)\|_{L^2}. \tag{3.19}$$

Combining (3.18) and (3.19) yields (3.10) directly. The proof of Lemma 3.1 is finished.

Next, the following lemma will give an estimate on the term $A_1(\sigma(T))$:

Lemma 3.2. *Under the conditions of Proposition 3.1, there exist positive constants $K \geq M+1$ and $\varepsilon_1 \leq 1$ both depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M such that if (ρ, u, θ) is a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying*

$$0 < \rho \leq 2\bar{\rho}, \quad A_2(\sigma(T)) \leq 2C_0^{1/4}, \tag{3.20}$$

the following estimate holds:

$$A_1(\sigma(T)) \leq 2K, \tag{3.21}$$

provided $A_1(\sigma(T)) \leq 3K$ and $C_0 \leq \varepsilon_1$.

Proof. First, multiplying $(1.6)_2$ by $2u_t$ and integrating the resulting equality over $\mathbb{R}^3$, we obtain after integration by parts that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)\mathrm{d}x+\int\rho|u_t|^2\mathrm{d}x\\
&\leq -2\int\nabla P\cdot u_t\mathrm{d}x+\int\rho|u\cdot\nabla u|^2\mathrm{d}x\\
&=2R\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho\theta-1)\mathrm{div}u\mathrm{d}x-2\int P_t\mathrm{div}u\mathrm{d}x+\int\rho|u\cdot\nabla u|^2\mathrm{d}x\\
&=2R\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho\theta-1)\mathrm{div}u\mathrm{d}x-\frac{R^2}{2\mu+\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho\theta-1)^2\mathrm{d}x\\
&\quad-\frac{2}{2\mu+\lambda}\int P_tG\mathrm{d}x+\int\rho|u\cdot\nabla u|^2\mathrm{d}x,
\end{aligned}\tag{3.22}$$

where in the last equality, we have used

$$\mathrm{div}u=\frac{1}{2\mu+\lambda}(G+R(\rho\theta-1)), \tag{3.23}$$

due to (1.21).

Next, it follows from Hölder's inequality, (3.20), (2.20) and (3.9) that for $p \in [2, 6]$,

$$\begin{aligned}
\|\rho\theta-1\|_{L^p}&=\|\rho(\theta-1)+(\rho-1)\|_{L^p}\\
&\leq\|\rho(\theta-1)\|_{L^2}^{(6-p)/(2p)}\|\rho(\theta-1)\|_{L^6}^{3(p-2)/(2p)}+\|\rho-1\|_{L^p}\\
&\leq C(\bar{\rho})C_0^{(6-p)/(16p)}\|\nabla\theta\|_{L^2}^{3(p-2)/(2p)}+C(\bar{\rho})C_0^{1/p},
\end{aligned}\tag{3.24}$$

which together with (2.28) yields

$$\|\nabla u\|_{L^6}\leq C(\bar{\rho})\left(\|\rho^{1/2}\dot{u}\|_{L^2}+\|\nabla\theta\|_{L^2}+C_0^{1/6}\right). \tag{3.25}$$

Noticing that (1.6) implies

$$\begin{aligned}P_t =& -\operatorname{div}(Pu) - (\gamma-1)P\operatorname{div}u + (\gamma-1)\kappa\Delta\theta \\ &+ (\gamma-1)\left(\lambda(\operatorname{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right),\end{aligned} \tag{3.26}$$

we obtain after integration by parts and using (3.20), (2.20), (2.26), (3.25), (3.24), and (3.9) that

$$\begin{aligned}&\left|\int P_t G\mathrm{d}x\right| \\ &\le C\int P(|G||\nabla u| + |u||\nabla G|)\mathrm{d}x + \int\left(|\nabla\theta||\nabla G| + |\nabla u|^2|G|\right)\mathrm{d}x \\ &\le C\int \rho(|G||\nabla u| + |u||\nabla G|)\mathrm{d}x + C\int\rho|\theta-1|(|G||\nabla u| + |u||\nabla G|)\mathrm{d}x \\ &\quad + C\|\nabla G\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\nabla G\|_{L^2}\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{L^6}^{1/2} \\ &\le C(\bar\rho)(\|\nabla u\|_{L^2} + \|\rho\theta-1\|_{L^2})\|\nabla u\|_{L^2} + C\|\rho u\|_{L^2}\|\nabla G\|_{L^2} \\ &\quad + C(\bar\rho)\|\rho(\theta-1)\|_{L^2}^{1/2}\|\nabla\theta\|_{L^2}^{1/2}\|\nabla G\|_{L^2}\|\nabla u\|_{L^2} + C\|\nabla G\|_{L^2}\|\nabla\theta\|_{L^2} \\ &\quad + C(\bar\rho)\|\nabla G\|_{L^2}\|\nabla u\|_{L^2}^{3/2}\left(\|\rho\dot u\|_{L^2}^{1/2} + \|\nabla\theta\|_{L^2}^{1/2} + C_0^{1/12}\right) \\ &\le C(\delta,\bar\rho)C_0^{1/4} + C(\bar\rho,\delta)\|\nabla u\|_{L^2}^2 + \delta\|\nabla G\|_{L^2}^2 + C(\bar\rho,\delta)\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2}^2 \\ &\quad + C(\delta,\bar\rho)\|\nabla\theta\|_{L^2}^2 + \delta\|\rho^{1/2}\dot u\|_{L^2}^2 + C(\delta,\bar\rho)\|\nabla u\|_{L^2}^6 \\ &\le C(\bar\rho)\delta\|\rho^{1/2}\dot u\|_{L^2}^2 + C(\delta,\bar\rho)\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1\right) + C(\delta,\bar\rho)\|\nabla u\|_{L^2}^6.\end{aligned} \tag{3.27}$$

Finally, it follows from (2.20) and (3.25) that

$$\begin{aligned}\int\rho|u\cdot\nabla u|^2\mathrm{d}x &\le C(\bar\rho)\|u\|_{L^6}^2\|\nabla u\|_{L^2}\|\nabla u\|_{L^6} \\ &\le \delta\|\rho^{1/2}\dot u\|_{L^2}^2 + C(\bar\rho,\delta)\|\nabla u\|_{L^2}^6 \\ &\quad + C(\bar\rho,\delta)\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right).\end{aligned} \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.22) and choosing δ suitably small, we get after integrating (3.22) over $(0,\sigma(T))$ and using (3.20) that

$$\begin{aligned}&\sup_{0\le t\le\sigma(T)}\|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)}\int\rho|\dot u|^2\mathrm{d}x\mathrm{d}t \\ &\le CM + C(\bar\rho)C_0^{1/4} + C(\bar\rho)C_0^{1/4}\sup_{0\le t\le\sigma(T)}\|\nabla u\|_{L^2}^4 \\ &\le K + C(\bar\rho)C_0^{1/4}\sup_{0\le t\le\sigma(T)}\|\nabla u\|_{L^2}^4,\end{aligned}$$

where K is defined by

$$K \triangleq CM + C(\bar\rho) + 1, \tag{3.29}$$

depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}$, and M. We thus finish the proof of (3.21) by choosing $\varepsilon_1 \triangleq \min\left\{1, (9C(\bar{\rho})K)^{-4}\right\}$ and K as in (3.29). The proof of Lemma 3.2 is completed.

Next, the following energy-like bound of the local smooth solutions will be crucial for further estimates.

Lemma 3.3. *Under the conditions of Proposition 3.1, there exists a positive constant ε_2 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}, \bar{\theta}$, and M such that if (ρ, u, θ) is a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.6) with K as in Lemma 3.2, the following estimate holds:*

$$A_2(T) \le C_0^{1/4}, \tag{3.30}$$

provided $C_0 \le \varepsilon_2$.

Proof. First, multiplying $(1.6)_2$ by u and integrating the resulting equality over $\mathbb{R}^3$ give

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\frac{1}{2}\rho|u|^2 + R(1+\rho\log\rho-\rho)\right)\mathrm{d}x \\
&\quad +\mu\int|\nabla u|^2\mathrm{d}x + (\mu+\lambda)\int(\mathrm{div}u)^2\mathrm{d}x \\
&\le C(\bar{\rho})\left(\|\theta-1\|_{L^2} + \|\rho-1\|_{L^2}\right)\|\nabla u\|_{L^2} \\
&\le C(\bar{\rho})\left(C_0^{1/2} + C_0^{1/3}\|\nabla\theta\|_{L^2}\right)\|\nabla u\|_{L^2} \\
&\le C(\bar{\rho})C_0^{2/3} + C(\bar{\rho})C_0^{1/3}\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right),
\end{aligned} \tag{3.31}$$

where in the second inequality we have used (3.9) and (3.10).

Then, multiplying $(1.6)_3$ by $\theta-1$ and integrating the resulting equality over $\mathbb{R}^3$ lead to

$$\begin{aligned}
&\frac{R}{2(\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho(\theta-1)^2\mathrm{d}x + \kappa\|\nabla\theta\|_{L^2}^2 \\
&\le C(\bar{\rho})\int\theta|\theta-1||\mathrm{div}u|\mathrm{d}x + C\int|\nabla u|^2|\theta-1|\mathrm{d}x.
\end{aligned} \tag{3.32}$$

For the first term on the righthand side of (3.32), one has

$$\begin{aligned}
&\int\theta|\theta-1||\mathrm{div}u|\mathrm{d}x \\
&\le \int(\theta-1)^2|\mathrm{div}u|\mathrm{d}x + \int|\theta-1||\mathrm{div}u|\mathrm{d}x \\
&\le C\|\theta-1\|_{L^2}^{1/2}\|\theta-1\|_{L^6}^{3/2}\|\nabla u\|_{L^2} + C\|\theta-1\|_{L^2}\|\nabla u\|_{L^2} \\
&\le C(\bar{\rho}, M)\left(C_0^{1/4} + C_0^{1/6}\|\nabla\theta\|_{L^2}^{1/2}\right)\|\nabla\theta\|_{L^2}^{3/2} \\
&\quad +C(\bar{\rho})\left(C_0^{1/2} + C_0^{1/3}\|\nabla\theta\|_{L^2}\right)\|\nabla u\|_{L^2} \\
&\le C(\bar{\rho}, M)C_0^{1/2} + C(\bar{\rho}, M)C_0^{1/6}\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right),
\end{aligned} \tag{3.33}$$

where we have used (2.20), (3.10), (3.6), and the following simple fact:

$$\sup_{t\in[0,T]} \|\nabla u\|_{L^2} \le A_1(\sigma(T)) + A_3(T) \le C(\bar\rho, M), \tag{3.34}$$

due to (3.6). For the second one on the righthand side of (3.32), in light of (3.10), (2.20), (2.28), (3.25), and (3.6), we have

$$\begin{aligned}
&\int |\nabla u|^2|\theta-1|\mathrm{d}x\\
&\le C\|\theta-1\|_{L^2}^{1/2}\|\theta-1\|_{L^6}^{1/2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\\
&\le C(\bar\rho, M)\left(C_0^{1/4}\|\nabla\theta\|_{L^2}^{1/2}+C_0^{1/6}\|\nabla\theta\|_{L^2}\right)\left(\|\rho\dot u\|_{L^2}+\|\nabla\theta\|_{L^2}+C_0^{1/6}\right)\\
&\le C(\bar\rho, M,\delta)C_0^{1/3}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+1\right)+C(\bar\rho, M)\left(\delta+C_0^{1/6}\right)\|\nabla\theta\|_{L^2}^2. \qquad (3.35)
\end{aligned}$$

Substituting (3.33) and (3.35) into (3.32) leads to

$$\begin{aligned}
&\frac{R}{2(\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(\theta-1)^2\mathrm{d}x+\kappa\|\nabla\theta\|_{L^2}^2\\
&\le C(\bar\rho, M)\left(\delta+C_0^{1/6}\right)\left(\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\
&\quad+C(\bar\rho, M,\delta)C_0^{1/3}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+1\right).
\end{aligned}\tag{3.36}$$

Next, combining (3.31) and (3.36) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac12\rho|u|^2+R(1+\rho\log\rho-\rho)+\frac{R}{2(\gamma-1)}\rho(\theta-1)^2\right)\mathrm{d}x\\
&\quad+\mu\int|\nabla u|^2\mathrm{d}x+(\mu+\lambda)\int(\mathrm{div}u)^2\mathrm{d}x+\kappa\int|\nabla\theta|^2\mathrm{d}x\\
&\le C(\bar\rho, M)\left(\delta+C_0^{1/6}\right)\left(\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\
&\quad+C(\bar\rho, M,\delta)C_0^{1/3}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+1\right).
\end{aligned}\tag{3.37}$$

Letting

$$C_0\le\varepsilon_{2,1}\triangleq\min\left\{1,\left((4C(\bar\rho, M))^{-1}\min\{\mu,\kappa\}\right)^6\right\},$$

choosing $\delta\le(4C(\bar\rho, M))^{-1}\min\{\mu,\kappa\}$ and integrating (3.37) over $(0,\sigma(T))$, we obtain after using (3.6) that

$$\begin{aligned}
&\sup_{0\le t\le\sigma(T)}\int\left(\rho|u|^2+(\rho-1)^2+\frac{R}{2(\gamma-1)}\rho(\theta-1)^2\right)\mathrm{d}x\\
&\quad+\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\mathrm{d}t\le C(\bar\rho, M)C_0^{1/3},
\end{aligned}\tag{3.38}$$

due to

$$\int\rho_0(\theta_0-1)^2\mathrm{d}x\le2(\bar\theta+1)\int\rho_0\left(\theta_0-\log\theta_0-1\right)\mathrm{d}x.$$

□

Next, applying the standard L^2-estimate to the following elliptic problem

$$\begin{cases}\kappa\Delta\theta = \frac{R}{\gamma-1}\rho\dot{\theta} + R\rho\theta\mathrm{div}u - \lambda(\mathrm{div}u)^2 - 2\mu|\mathfrak{D}(u)|^2,\\ \theta\to 1 \quad \text{as } |x|\to\infty,\end{cases} \tag{3.39}$$

gives

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}^2 &\le C\left(\|\rho\dot{\theta}\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\theta\nabla u\|_{L^2}^2\right)\\ &\le C(\bar{\rho})\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)\left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + 1\right)\\ &\quad + C\left(\|\rho\dot{\theta}\|_{L^2}^2 + \|\nabla u\|_{L^4}^4\right),\end{aligned} \tag{3.40}$$

where we have used

$$\begin{aligned}\int\theta^2|\nabla u|^2\mathrm{d}x &\le C\|\theta-1\|_{L^6}^2\|\nabla u\|_{L^2}\|\nabla u\|_{L^6} + C\|\nabla u\|_{L^2}^2\\ &\le C(\bar{\rho})\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)\left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + 1\right),\end{aligned} \tag{3.41}$$

due to (2.20) and (3.25). Note that (3.25) and (3.6) give

$$\sup_{0\le t\le T}\sigma\|\nabla u\|_{L^6} \le C(\bar{\rho})C_0^{1/12}. \tag{3.42}$$

Combining this with (3.40) and (3.6) leads to

$$\begin{aligned}\sup_{0<t\le T}\sigma^4\|\nabla^2\theta\|_{L^2}^2 &\le C(\bar{\rho})\sup_{0<t\le T}\sigma^2\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)\times\\ &\quad\sup_{0<t\le T}\sigma^2\left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + 1\right)\\ &\quad + C\sup_{0<t\le T}\left(\sigma^4\|\rho\dot{\theta}\|_{L^2}^2 + \left(\sigma\|\nabla u\|_{L^2}\right)\left(\sigma\|\nabla u\|_{L^6}\right)^3\right)\\ &\le C(\bar{\rho})C_0^{1/8},\end{aligned} \tag{3.43}$$

which together with (2.20), (2.21), and (3.6) yields that

$$\begin{aligned}\sup_{0<t\le T}\sigma^2\|\theta-1\|_{L^\infty} &\le \sup_{0<t\le T}\sigma^2\left(\|\nabla\theta\|_{L^2} + \|\nabla^2\theta\|_{L^2}\right)\\ &\le C(\bar{\rho})C_0^{1/16} \le 1/2,\end{aligned} \tag{3.44}$$

provided $C_0 \le \varepsilon_{2,2} \triangleq \min\left\{1, (2C(\bar{\rho}))^{-16}\right\}$. Let $C_0 \le \min\{\varepsilon_{2,1}, \varepsilon_{2,2}\}$. It follows from (3.44) that, for all $(x,t)\in\mathbb{R}^3\times[\sigma(T), T]$,

$$1/2 \le \theta(x,t) \le 3/2,$$

which as well as (2.19) and (3.13)–(3.16) gives

$$\sup_{\sigma(T)\le t\le T}\int\left(\rho|u|^2+(\rho-1)^2+\frac{R}{2(\gamma-1)}\rho(\theta-1)^2\right)\mathrm{d}x \\ +\int_{\sigma(T)}^T\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\mathrm{d}t\le C(\bar\rho)C_0. \tag{3.45}$$

Finally, the combination of (3.38) with (3.45) yields

$$\begin{aligned}\sup_{0\le t\le T}\int&\left(\rho|u|^2+(\rho-1)^2+\frac{R}{2(\gamma-1)}\rho(\theta-1)^2\right)\mathrm{d}x\\ &+\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\mathrm{d}t\\ &\le\max\left\{C(\bar\rho)C_0,\,C(\bar\rho,M)C_0^{1/3}\right\},\end{aligned}$$

which in particular gives (3.30) provided

$$C_0\le\varepsilon_2\triangleq\min\left\{\varepsilon_{2,1},\varepsilon_{2,2},(C(\bar\rho))^{-4/3},(C(\bar\rho,M))^{-12}\right\}.$$

The proof of Lemma 3.3 is completed.

Next, to estimate $A_3(T)$, we establish the following Lemmas 3.4 and 3.5 concerning some elementary estimates on $\dot u$ and $\dot\theta$ for the case that the density may contain vacuum states. This approach is motivated by the basic estimates on $\dot u$ and $\dot\theta$ developed by Hoff [8] where the density is strictly away from vacuum. The estimate of $A_3(T)$ will be postponed to Lemma 3.6.

Lemma 3.4. *Under the conditions of Proposition 3.1, let (ρ,u,θ) be a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3\times(0,T]$ satisfying (3.6) with K as in Lemma 3.2. Then there exist positive constants C and C_1 both depending only on $\mu,\lambda,\kappa,R,\gamma,\bar\rho,\bar\theta$, and M such that, for any $\beta\in(0,1]$, the following estimates hold:*

$$\begin{aligned}(\sigma B_1)'(t)+\frac32\int\sigma\rho|\dot u|^2\mathrm{d}x\le{}& CC_0^{1/4}\sigma'+2\beta\sigma^2\|\rho^{1/2}\dot\theta\|_{L^2}^2\\ &+C\beta^{-1}\left(\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\ &+C\sigma^2\|\nabla u\|_{L^4}^4,\end{aligned}\tag{3.46}$$

and

$$\begin{aligned}&\left(\sigma^2\int\rho|\dot u|^2\mathrm{d}x\right)_t+\frac{3\mu}{2}\int\sigma^2|\nabla\dot u|^2\mathrm{d}x\\ &\le2\sigma\int\rho|\dot u|^2\mathrm{d}x+C_1\sigma^2\|\rho^{1/2}\dot\theta\|_{L^2}^2\\ &\quad+C\left(\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+C\sigma^2\|\nabla u\|_{L^4}^4,\end{aligned}\tag{3.47}$$

where

$$B_1(t)\triangleq\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+2R\int\mathrm{div}u(\rho\theta-1)\mathrm{d}x.\tag{3.48}$$

Proof. First, we prove (3.46). Multiplying $(1.6)_2$ by $\sigma\dot{u}$ and integrating the resulting equality over $\mathbb{R}^3$ lead to

$$\begin{aligned}\int\sigma\rho|\dot{u}|^2\mathrm{d}x &= \int(-\sigma\dot{u}\cdot\nabla P+\mu\sigma\triangle u\cdot\dot{u}+(\lambda+\mu)\sigma\nabla\mathrm{div}u\cdot\dot{u})\mathrm{d}x\\ &\triangleq\sum_{i=1}^{3}M_i.\end{aligned}\tag{3.49}$$

Noticing that $(1.6)_1$ leads to

$$P_t=R\rho\dot{\theta}-\mathrm{div}(Pu),\tag{3.50}$$

we get after integration by parts that, for any $\beta\in(0,1]$,

$$\begin{aligned}M_1 &= \int\sigma\left((P-R)\mathrm{div}u\right)_t\mathrm{d}x+\int\sigma\left(-P_t\mathrm{div}u+Pu\cdot\nabla\mathrm{div}u\right)\mathrm{d}x\\ &\quad+\int\sigma P\partial_iu^j\partial_ju^i\mathrm{d}x\\ &= R\left(\int\sigma(\rho\theta-1)\mathrm{div}u\mathrm{d}x\right)_t-R\sigma'\int(\rho\theta-1)\mathrm{div}u\mathrm{d}x\\ &\quad-R\int\sigma\mathrm{div}u\rho\dot{\theta}\mathrm{d}x+\int\sigma\mathrm{div}(Pu\mathrm{div}u)\mathrm{d}x\\ &\quad+\int\sigma P\partial_iu^j\partial_ju^i\mathrm{d}x\\ &\le R\left(\int\sigma(\rho\theta-1)\mathrm{div}u\mathrm{d}x\right)_t+C\sigma'\|\nabla u\|_{L^2}\|\rho\theta-1\|_{L^2}\\ &\quad+C(\bar{\rho})\sigma\|\nabla u\|_{L^2}\|\rho^{1/2}\dot{\theta}\|_{L^2}+C(\bar{\rho})\sigma\int\theta|\nabla u|^2\mathrm{d}x\\ &\le R\left(\int\sigma(\rho\theta-1)\mathrm{div}u\mathrm{d}x\right)_t+C(\bar{\rho})C_0^{1/4}\sigma'+\beta\sigma^2\|\rho^{1/2}\dot{\theta}\|_{L^2}^2\\ &\quad+C(\bar{\rho})\delta\|\rho^{1/2}\dot{u}\|_{L^2}^2+C(\bar{\rho},\delta,M)\beta^{-1}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right),\end{aligned}\tag{3.51}$$

where in the last inequality we have used (3.24) and the following simple fact:

$$\begin{aligned}\int\theta|\nabla u|^2\mathrm{d}x &\le \int|\theta-1||\nabla u|^2\mathrm{d}x+\int|\nabla u|^2\mathrm{d}x\\ &\le C\|\theta-1\|_{L^6}\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{L^6}^{1/2}+\|\nabla u\|_{L^2}^2\\ &\le C\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2}^{3/2}\left(\|\rho\dot{u}\|_{L^2}+\|\nabla\theta\|_{L^2}+1\right)^{1/2}+\|\nabla u\|_{L^2}^2\\ &\le \delta\left(\|\nabla\theta\|_{L^2}^2+\|\rho^{1/2}\dot{u}\|_{L^2}^2\right)+C(\bar{\rho},\delta,M)\|\nabla u\|_{L^2}^2,\end{aligned}\tag{3.52}$$

due to (2.20), (3.34), and (3.25). Integration by parts gives

$$
\begin{aligned}
M_2 &= \int \mu\sigma \triangle u \cdot \dot{u} \mathrm{d}x \\
&= -\frac{\mu}{2}\left(\sigma\|\nabla u\|_{L^2}^2\right)_t + \frac{\mu}{2}\sigma'\|\nabla u\|_{L^2}^2 - \mu\sigma\int \partial_i u^j \partial_i (u^k\partial_k u^j)\mathrm{d}x \\
&= -\frac{\mu}{2}\left(\sigma\|\nabla u\|_{L^2}^2\right)_t + \frac{\mu}{2}\sigma'\|\nabla u\|_{L^2}^2 - \mu\sigma\int \partial_i u^j \partial_i u^k\partial_k u^j \mathrm{d}x \\
&\quad + \frac{\mu}{2}\sigma \int |\nabla u|^2 \mathrm{div} u \mathrm{d}x \\
&\le -\frac{\mu}{2}\left(\sigma\|\nabla u\|_{L^2}^2\right)_t + C\|\nabla u\|_{L^2}^2 + C\sigma^2\|\nabla u\|_{L^4}^4.
\end{aligned} \tag{3.53}
$$

Similar to (3.53), we have

$$
M_3 \le -\frac{\lambda+\mu}{2}\left(\sigma\|\mathrm{div} u\|_{L^2}^2\right)_t + C\|\nabla u\|_{L^2}^2 + C\sigma^2\|\nabla u\|_{L^4}^4. \tag{3.54}
$$

Substituting (3.51), (3.53), and (3.54) into (3.49), we obtain (3.46) after choosing δ suitably small.

It remains to prove (3.47). For $m \ge 0$, operating $\sigma^m \dot{u}^j[\partial/\partial t + \mathrm{div}(u\cdot)]$ to $(1.6)_2^j$ and integrating the resulting equality over $\mathbb{R}^3$, we obtain after integration by parts that

$$
\begin{aligned}
&\left(\frac{\sigma^m}{2}\int \rho|\dot{u}|^2\mathrm{d}x\right)_t - \frac{m}{2}\sigma^{m-1}\sigma'\int \rho|\dot{u}|^2\mathrm{d}x \\
&= -\int \sigma^m \dot{u}^j[\partial_j P_t + \mathrm{div}(\partial_j P u)]\mathrm{d}x \\
&\quad + \mu\int \sigma^m \dot{u}^j[\triangle u_t^j + \mathrm{div}(u\triangle u^j)]\mathrm{d}x \\
&\quad + (\lambda+\mu)\int \sigma^m \dot{u}^j[\partial_t\partial_j \mathrm{div} u + \mathrm{div}(u\partial_j \mathrm{div} u)]\mathrm{d}x \\
&\triangleq \sum_{i=1}^3 N_i.
\end{aligned} \tag{3.55}
$$

We get after integration by parts and using (3.50) that

$$
\begin{aligned}
N_1 &= -\int \sigma^m \dot{u}^j[\partial_j P_t + \mathrm{div}(\partial_j P u)]\mathrm{d}x \\
&= -\int \sigma^m \dot{u}^j\left(R\partial_j(\rho\dot{\theta}) - \partial_j \mathrm{div}(Pu) + \mathrm{div}(\partial_j P u)\right)\mathrm{d}x \\
&= -\int \sigma^m \dot{u}^j\left(R\partial_j(\rho\dot{\theta}) - \mathrm{div}(P\partial_j u)\right)\mathrm{d}x \\
&\le \frac{\mu}{8}\int \sigma^m|\nabla\dot{u}|^2\mathrm{d}x + C(\bar{\rho})\sigma^m\left(\|\rho\dot{\theta}\|_{L^2}^2 + \int \theta^2|\nabla u|^2\mathrm{d}x\right).
\end{aligned} \tag{3.56}
$$

Integration by parts leads to

$$
\begin{aligned}
N_2 &= \mu \int \sigma^m \dot{u}^j [\triangle u_t^j + \mathrm{div}(u \triangle u^j)] \mathrm{d}x \\
&= -\mu \int \sigma^m \left(\partial_i \dot{u}^j \partial_i u_t^j + \triangle u^j u \cdot \nabla \dot{u}^j \right) \mathrm{d}x \\
&= -\mu \int \sigma^m \left(|\nabla \dot{u}|^2 - \partial_i \dot{u}^j u^k \partial_k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j + \triangle u^j u \cdot \nabla \dot{u}^j \right) \mathrm{d}x \\
&= -\mu \int \sigma^m \left(|\nabla \dot{u}|^2 + \partial_i \dot{u}^j \partial_i u^j \mathrm{div} u - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j - \partial_i u^j \partial_i u^k \partial_k \dot{u}^j \right) \mathrm{d}x \\
&\le -\frac{7\mu}{8} \int \sigma^m |\nabla \dot{u}|^2 \mathrm{d}x + C \int \sigma^m |\nabla u|^4 \mathrm{d}x. \qquad (3.57)
\end{aligned}
$$

Similarly, we have

$$
N_3 \le -\frac{7(\mu+\lambda)}{8} \int \sigma^m (\mathrm{div} \dot{u})^2 \mathrm{d}x + C \int \sigma^m |\nabla u|^4 \mathrm{d}x. \qquad (3.58)
$$

Substituting (3.56)–(3.58) into (3.55) yields that there exists some C_1 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}, \bar{\theta}$, and M such that

$$
\begin{aligned}
&\left(\sigma^m \int \rho |\dot{u}|^2 \mathrm{d}x \right)_t + \frac{3\mu}{2} \int \sigma^m |\nabla \dot{u}|^2 \mathrm{d}x \\
&\quad + \frac{7(\mu+\lambda)}{8} \int \sigma^m (\mathrm{div} \dot{u})^2 \mathrm{d}x \\
&\le m \sigma^{m-1} \sigma' \int \rho |\dot{u}|^2 \mathrm{d}x \\
&\quad + C_1 \sigma^m \|\rho^{1/2} \dot{\theta}\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^4}^4 \\
&\quad + C(\bar{\rho}) \sigma^m \|\theta \nabla u\|_{L^2}^2. \qquad (3.59)
\end{aligned}
$$

Taking $m = 2$ in (3.59) and using (3.41), (3.6), and (1.2), we obtain (3.47) and finish the proof of Lemma 3.4.

Lemma 3.5. *Under the conditions of Proposition 3.1, let (ρ, u, θ) be a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.6) with K as in Lemma 3.2. Then there exists a positive constant C depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}, \bar{\theta}$, and M such that the following estimate holds*

$$
\begin{aligned}
&\left(\sigma^2 \varphi \right)'(t) + \sigma^2 \int \left(\mu |\nabla \dot{u}|^2 + \rho (\dot{\theta})^2 \right) \mathrm{d}x \\
&\le C \left(\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \qquad (3.60) \\
&\quad + 2\sigma \int \rho |\dot{u}|^2 \mathrm{d}x + C \sigma^2 \|\nabla u\|_{L^4}^4,
\end{aligned}
$$

where $\varphi(t)$ is defined by

$$
\varphi(t) \triangleq \int \rho |\dot{u}|^2 (x, t) \mathrm{d}x + (C_1 + 1) B_2(t), \qquad (3.61)
$$

with C_1 *as in Lemma* 3.4 *and*

$$B_2(t) \triangleq \frac{\gamma-1}{R}\left(\kappa\|\nabla\theta\|_{L^2}^2 - 2\lambda\int(\mathrm{div}u)^2\theta\mathrm{d}x - 4\mu\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x\right). \tag{3.62}$$

Proof. For $m \ge 0$, multiplying (1.6)$_3$ by $\sigma^m\dot\theta$ and integrating the resulting equality over $\mathbb{R}^3$ yields that

$$\begin{aligned}&\frac{\kappa\sigma^m}{2}\left(\|\nabla\theta\|_{L^2}^2\right)_t + \frac{R\sigma^m}{\gamma-1}\int\rho|\dot\theta|^2\mathrm{d}x\\&= -\kappa\sigma^m\int\nabla\theta\cdot\nabla(u\cdot\nabla\theta)\mathrm{d}x + \lambda\sigma^m\int(\mathrm{div}u)^2\dot\theta\mathrm{d}x\\&\quad + 2\mu\sigma^m\int|\mathfrak{D}(u)|^2\dot\theta\mathrm{d}x - R\sigma^m\int\rho\theta\mathrm{div}u\dot\theta\mathrm{d}x \triangleq \sum_{i=1}^4 I_i.\end{aligned} \tag{3.63}$$

First, it follows from (2.20) and (3.6) that

$$\begin{aligned}|I_1| &\le C\sigma^m\int|\nabla u||\nabla\theta|^2\mathrm{d}x\\&\le C\sigma^m\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{1/2}\|\nabla\theta\|_{L^6}^{3/2}\\&\le \delta\sigma^m\|\rho\dot\theta\|_{L^2}^2 + C\sigma^m\|\nabla u\|_{L^4}^4 + C\sigma^m\|\theta\nabla u\|_{L^2}^2\\&\quad + C(\bar\rho,\delta,M)\sigma^m\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{3.64}$$

where in the last inequality we have used (2.20) and (3.40).

Next, integration by parts yields that, for any $\eta \in (0, 1]$,

$$\begin{aligned}I_2 &= \lambda\sigma^m\int(\mathrm{div}u)^2\theta_t\mathrm{d}x + \lambda\sigma^m\int(\mathrm{div}u)^2u\cdot\nabla\theta\mathrm{d}x\\&= \lambda\sigma^m\left(\int(\mathrm{div}u)^2\theta\mathrm{d}x\right)_t - 2\lambda\sigma^m\int\theta\mathrm{div}u\mathrm{div}(\dot u - u\cdot\nabla u)\mathrm{d}x\\&\quad + \lambda\sigma^m\int(\mathrm{div}u)^2u\cdot\nabla\theta\mathrm{d}x\\&= \lambda\sigma^m\left(\int(\mathrm{div}u)^2\theta\mathrm{d}x\right)_t - 2\lambda\sigma^m\int\theta\mathrm{div}u\mathrm{div}\dot u\mathrm{d}x\\&\quad + 2\lambda\sigma^m\int\theta\mathrm{div}u\partial_iu^j\partial_ju^i\mathrm{d}x + \lambda\sigma^m\int u\cdot\nabla\left(\theta(\mathrm{div}u)^2\right)\mathrm{d}x\\&\le \lambda\left(\sigma^m\int(\mathrm{div}u)^2\theta\mathrm{d}x\right)_t - \lambda m\sigma^{m-1}\sigma'\int(\mathrm{div}u)^2\theta\mathrm{d}x\\&\quad + \eta\sigma^m\|\nabla\dot u\|_{L^2}^2 + C\eta^{-1}\sigma^m\int\theta^2|\nabla u|^2\mathrm{d}x + \sigma^m\|\nabla u\|_{L^4}^4.\end{aligned} \tag{3.65}$$

Then, similar to (3.65), we obtain that, for any $\eta \in (0, 1]$,

$$\begin{aligned} I_3 \le\, & 2\mu \left(\sigma^m \int |\mathfrak{D}(u)|^2 \theta \mathrm{d}x \right)_t \\ & -2\mu m \sigma^{m-1} \sigma' \int |\mathfrak{D}(u)|^2 \theta \mathrm{d}x \\ & + C\eta \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C\eta^{-1} \sigma^m \int \theta^2 |\nabla u|^2 \mathrm{d}x \\ & + C\sigma^m \|\nabla u\|_{L^4}^4. \end{aligned} \tag{3.66}$$

Finally, Cauchy's inequality gives

$$|I_4| \le C(\bar{\rho}) \sigma^m \int \theta^2 |\nabla u|^2 \mathrm{d}x + \frac{R}{4(\gamma - 1)} \sigma^m \int \rho |\dot{\theta}|^2 \mathrm{d}x. \tag{3.67}$$

Substituting (3.64)–(3.67) into (3.63), we obtain after using (2.19), (3.11), (1.2) and choosing δ suitably small that, for any $\eta \in (0, 1]$,

$$\begin{aligned} (\sigma^m B_2)'(t) + \sigma^m \int \rho (\dot{\theta})^2 \mathrm{d}x \le\, & C\eta \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C(\bar{\rho}, M) \|\nabla \theta\|_{L^2}^2 \\ & + C\sigma^m \|\nabla u\|_{L^4}^4 + C(\bar{\rho}, \eta) \sigma^m \|\theta \nabla u\|_{L^2}^2, \end{aligned} \tag{3.68}$$

with B_2 as in (3.62). For C_1 as in Lemma 3.4 (see also (3.59)), adding (3.68) multiplied by $C_1 + 1$ to (3.59), we obtain after choosing η suitably small and using (3.41) that, for φ as in (3.61) and for $m \ge 0$,

$$\begin{aligned} & \left(\sigma^m \varphi\right)'(t) + \sigma^m \int \left(\mu |\nabla \dot{u}|^2 + \rho (\dot{\theta})^2 \right) \mathrm{d}x \\ & \le C(\bar{\rho}, M) \left(\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \left(\sigma^m \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \sigma^m \|\nabla \theta\|_{L^2}^2 + 1 \right) \\ & \quad + m \sigma' \sigma^{m-1} \int \rho |\dot{u}|^2 \mathrm{d}x + C(\bar{\rho}, M) \sigma^m \|\nabla u\|_{L^4}^4. \end{aligned} \tag{3.69}$$

Taking $m = 2$ in (3.69) together with (3.6) gives (3.60). The proof of Lemma 3.5 is completed.

Next, we will use Lemmas 3.4 and 3.5 to obtain the following estimate on $A_3(T)$:

Lemma 3.6. *Under the conditions of Proposition 3.1, there exists a positive constant ε_3 depending only on $\mu, \lambda, \kappa, R, \gamma, \bar{\rho}, \bar{\theta}$, and M such that if (ρ, u, θ) is a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.6) with K as in Lemma 3.2, the following estimate holds:*

$$A_3(T) \le C_0^{1/6}, \tag{3.70}$$

provided $C_0 \le \varepsilon_3$.

Proof. First, it follows from (2.25), (2.27), (2.20), (3.34), and (3.6) that

$$\begin{aligned}\|\nabla u\|_{L^4}^4 &\le C\|G\|_{L^4}^4 + C\|\omega\|_{L^4}^4 + C\|\rho\theta - 1\|_{L^4}^4 \\ &\le C(\bar\rho)\left(\|\nabla u\|_{L^2} + 1\right)\|\rho^{1/2}\dot u\|_{L^2}^3 + C\|\rho(\theta-1)\|_{L^4}^4 + C\|\rho - 1\|_{L^4}^4 \\ &\le C(\bar\rho, M)\|\rho^{1/2}\dot u\|_{L^2}^3 + C(\bar\rho)\|\rho(\theta-1)\|_{L^2}\|\nabla\theta\|_{L^2}^3 + C\|\rho - 1\|_{L^4}^4 \\ &\le C(\bar\rho, M)\|\rho^{1/2}\dot u\|_{L^2}^3 + C(\bar\rho)\|\nabla\theta\|_{L^2}^3 + C\|\rho - 1\|_{L^4}^4, \end{aligned} \tag{3.71}$$

which together with (3.6) yields

$$\sigma\|\nabla u\|_{L^4}^4 \le C(\bar\rho, M)C_0^{1/12}\|\rho^{1/2}\dot u\|_{L^2}^2 + C(\bar\rho)\|\nabla\theta\|_{L^2}^2 + C\sigma\|\rho - 1\|_{L^4}^4. \tag{3.72}$$

Combining this with (3.60) gives that, for $\varphi(t)$ as in (3.61),

$$\begin{aligned} &\left(\sigma^2\varphi\right)'(t) + \sigma^2\int\left(\mu|\nabla\dot u|^2 + \rho(\dot\theta)^2\right)\mathrm{d}x \\ &\le C(\bar\rho, M)\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \\ &\quad + \left(C(\bar\rho, M)C_0^{1/12} + 2\right)\sigma\int\rho|\dot u|^2\mathrm{d}x \\ &\quad + C(\bar\rho, M)\sigma^2\|\rho - 1\|_{L^4}^4 \\ &\le C(\bar\rho, M)\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \\ &\quad + 3\sigma\int\rho|\dot u|^2\mathrm{d}x + C(\bar\rho, M)\sigma^2\|\rho - 1\|_{L^4}^4, \end{aligned} \tag{3.73}$$

provided $C_0 \le \varepsilon_{3,1} \triangleq \min\left\{1, (C(\bar\rho, M))^{-12}\right\}$.

Next, to estimate the second term on the righthand side of (3.73), we substitute (3.72) into (3.46) to obtain that, for $B_1(t)$ as in (3.48),

$$\begin{aligned} &(\sigma B_1)'(t) + \int\sigma\rho|\dot u|^2\mathrm{d}x \\ &\quad \le C(\bar\rho, M)C_0^{1/4}\sigma' + 2\beta\sigma^2\|\rho^{1/2}\dot\theta\|_{L^2}^2 \\ &\qquad + C(\bar\rho, M)\beta^{-1}\left(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \\ &\qquad + C(\bar\rho, M)\sigma^2\|\rho - 1\|_{L^4}^4, \end{aligned} \tag{3.74}$$

provided $C_0 \le \varepsilon_{3,2} \triangleq \min\left\{1, (2C(\bar\rho, M))^{-12}\right\}$. From now on, we assume that $C_0 \le \min\left\{\varepsilon_{3,1}, \varepsilon_{3,2}\right\}$. It follows from (3.61), (3.62), and (3.52) that

$$\varphi(t) \ge \frac{1}{2}\int\rho|\dot u|^2\mathrm{d}x + \frac{\kappa(\gamma-1)}{2R}\|\nabla\theta\|_{L^2}^2 - C_2(\bar\rho, M)\|\nabla u\|_{L^2}^2, \tag{3.75}$$

which together with (3.34) directly gives

$$\begin{aligned}\int\rho|\dot u|^2(x,t)\mathrm{d}x + \|\nabla\theta(\cdot,t)\|_{L^2}^2 &\le 2\left(\frac{R}{\kappa(\gamma-1)} + 1\right)\varphi(t) \\ &\quad + C(\bar\rho, M). \end{aligned} \tag{3.76}$$

For C_2 as in (3.75), adding (3.74) multiplied by $2(C_2+2\mu+1)/\mu$ to (3.73), we obtain after choosing β suitably small that

$$\begin{aligned}
&B_3'(t)+\frac{1}{2}\int\left(\sigma\rho|\dot u|^2+\mu\sigma^2|\nabla\dot u|^2+\sigma^2\rho(\dot\theta)^2\right)\mathrm{d}x\\
&\quad\le C(\bar\rho,M)C_0^{1/4}\sigma'+C(\bar\rho,M)\left(\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\
&\qquad+C(\bar\rho,M)\sigma^2\|\rho-1\|_{L^4}^4,
\end{aligned}\tag{3.77}$$

where

$$B_3(t)\triangleq\sigma^2\varphi+2(C_2+2\mu+1)\mu^{-1}\sigma B_1$$

satisfies

$$\begin{aligned}
B_3(t)\ge\frac{\sigma^2}{2}\int\rho|\dot u|^2\mathrm{d}x+\frac{\kappa(\gamma-1)}{2R}\sigma^2\|\nabla\theta\|_{L^2}^2\\
+\sigma\|\nabla u\|_{L^2}^2-C(\bar\rho,M)C_0^{1/4},
\end{aligned}\tag{3.78}$$

due to (3.75) and the following simple fact:

$$\begin{aligned}
B_1(t)&\ge\frac{\mu}{2}\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2-C(\bar\rho)\|\rho\theta-1\|_{L^2}^2\\
&\ge\frac{\mu}{2}\|\nabla u\|_{L^2}^2-C(\bar\rho)C_0^{1/4},
\end{aligned}$$

which comes from (3.48) and (3.24).

Finally, we claim that

$$\int_0^T\sigma^2\|\rho-1\|_{L^4}^4\mathrm{d}t\le C(\bar\rho,M)C_0^{1/4}.\tag{3.79}$$

Combining this with (3.77), (3.78), and (3.6) yields

$$A_3(T)\le C(\bar\rho,M)C_0^{1/4},$$

which implies (3.70) provided $C_0\le\varepsilon_3\triangleq\min\left\{\varepsilon_{3,1},\varepsilon_{3,2},(C(\bar\rho,M))^{-12}\right\}$.

Then, it remains to prove (3.79). In fact, it follows from $(1.6)_1$ and (3.23) that $\rho-1$ satisfies

$$\begin{aligned}
&(\rho-1)_t+\frac{R}{2\mu+\lambda}(\rho-1)\\
&\quad=-u\cdot\nabla(\rho-1)-(\rho-1)\mathrm{div}u-\frac{G}{2\mu+\lambda}\\
&\qquad-\frac{R\rho(\theta-1)}{2\mu+\lambda}.
\end{aligned}\tag{3.80}$$

Multiplying (3.80) by $4(\rho-1)^3$ and integrating the resulting equality over $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\left(\|\rho-1\|_{L^4}^4\right)_t+\frac{4R}{2\mu+\lambda}\|\rho-1\|_{L^4}^4\\
&=-3\int(\rho-1)^4\mathrm{div}u\mathrm{d}x-\frac{4}{2\mu+\lambda}\int(\rho-1)^3G\mathrm{d}x\\
&\quad-\frac{4R}{2\mu+\lambda}\int(\rho-1)^3\rho(\theta-1)\mathrm{d}x\\
&\le\frac{2R}{2\mu+\lambda}\|\rho-1\|_{L^4}^4+C(\bar\rho)\|\nabla u\|_{L^2}^2\\
&\quad+C\|\rho-1\|_{L^4}^3\|G\|_{L^2}^{1/4}\|\nabla G\|_{L^2}^{3/4}\\
&\quad+C(\bar\rho)\|\rho-1\|_{L^4}^3\|\rho(\theta-1)\|_{L^2}^{1/4}\|\nabla\theta\|_{L^2}^{3/4}\\
&\le\frac{3R}{2\mu+\lambda}\|\rho-1\|_{L^4}^4+C(\bar\rho)\|\nabla u\|_{L^2}^2\\
&\quad+C(\bar\rho,M)\left(\|\rho^{1/2}\dot u\|_{L^2}^3+\|\nabla\theta\|_{L^2}^3\right),
\end{aligned}\tag{3.81}$$

where in the last inequality, we have used (3.6), (3.34), (3.24), (2.26), and (3.9). It thus follows from (3.81) that

$$\begin{aligned}
&\left(\|\rho-1\|_{L^4}^4\right)_t+\frac{R}{2\mu+\lambda}\|\rho-1\|_{L^4}^4\\
&\le C(\bar\rho,M)\left(\|\rho^{1/2}\dot u\|_{L^2}^3+\|\nabla\theta\|_{L^2}^3\right)+C(\bar\rho)\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.82}$$

Multiplying (3.82) by σ^n with $n\ge1$, integrating the resulting inequality over $(0,T)$, we obtain by using (3.9) and (3.6) that

$$\begin{aligned}
&\int_0^T\sigma^n\|\rho-1\|_{L^4}^4\mathrm{d}t\\
&\le C(\bar\rho,M)A_3^{1/2}(T)\int_0^T\sigma^{n-1}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\mathrm{d}t\\
&\quad+C(\bar\rho)C_0^{1/4}+C\int_0^{\sigma(T)}\|\rho-1\|_{L^4}^4\mathrm{d}t\\
&\le C(\bar\rho,M)C_0^{1/4}+C(\bar\rho,M)C_0^{1/12}\int_0^T\sigma^{n-1}\|\rho^{1/2}\dot u\|_{L^2}^2\mathrm{d}t,
\end{aligned}\tag{3.83}$$

which together with (3.6) directly gives (3.79). We thus complete the proof of Lemma 3.6.

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtain all the higher order estimates and thus to extend the classical solution globally.

Lemma 3.7. *Under the conditions of Proposition 3.1, there exists a positive constant ε_4 depending only on $\mu,\lambda,\kappa,R,\gamma,\bar\rho,\bar\theta$, and M such that if (ρ,u,θ) is a* *smooth solution of* (1.6) (1.4) (1.7) *on* $\mathbb{R}^3 \times (0, T]$ *satisfying* (3.6) *with* K *as in Lemma* 3.2*, the following estimate holds:*

$$\sup_{0\le t\le T} \|\rho(\cdot,t)\|_{L^\infty} \le \frac{3\bar\rho}{2}, \tag{3.84}$$

provided $C_0 \le \varepsilon_4$.

Proof. First, taking $n = 1$ in (3.83) as well as (3.6) yields

$$\int_0^T \sigma\|\rho - 1\|_{L^4}^4 \mathrm{d}t \le C(\bar\rho, M). \tag{3.85}$$

Choosing $m = 1$ in (3.69) together with (3.76) and (3.72) yields that, for $\varphi(t)$ as in (3.61),

$$\begin{aligned}
&(\sigma\varphi)'(t) + \sigma\int \left(\mu|\nabla\dot u|^2 + \rho(\dot\theta)^2\right)\mathrm{d}x\\
&\le C(\bar\rho, M)\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right)(\sigma\varphi) + C(\bar\rho, M)\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right)\\
&\quad + C(\bar\rho, M)\int \rho|\dot u|^2 \mathrm{d}x + C(\bar\rho, M)\sigma\|\rho - 1\|_{L^4}^4,
\end{aligned}$$

which combined with (3.6), (3.85), and Grönwall's inequality yields that

$$\sup_{0\le t\le\sigma(T)} \sigma\varphi(t) + \int_0^{\sigma(T)} \sigma\int\left(\mu|\nabla\dot u|^2 + \rho(\dot\theta)^2\right)\mathrm{d}x\mathrm{d}t \le C(\bar\rho, M). \tag{3.86}$$

The combination of (3.76) with (3.86) thus directly gives

$$\begin{aligned}
&\sup_{0\le t\le\sigma(T)} \sigma\left(\int \rho|\dot u|^2 \mathrm{d}x + \|\nabla\theta\|_{L^2}^2\right)\\
&\quad + \int_0^{\sigma(T)} \sigma\int (|\nabla\dot u|^2 + \rho(\dot\theta)^2)\mathrm{d}x\mathrm{d}t \le C(\bar\rho, M).
\end{aligned} \tag{3.87}$$

Next, it follows from (3.40), (3.87), (3.72) (3.6), and (3.85) that

$$\begin{aligned}
\int_0^T \sigma\|\nabla^2\theta\|_{L^2}^2 \mathrm{d}t &\le C(\bar\rho, M)\int_0^T \left(\sigma\|\rho\dot\theta\|_{L^2}^2 + \|\rho^{1/2}\dot u\|_{L^2}^2\right)\mathrm{d}t\\
&\quad + C(\bar\rho, M)\int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \sigma\|\rho - 1\|_{L^4}^4\right)\mathrm{d}t\\
&\le C(\bar\rho, M),
\end{aligned}$$

which together with (2.21), (2.20), and (3.6) gives

$$\begin{aligned}
&\int_0^{\sigma(T)} \|\theta-1\|_{L^\infty}\mathrm{d}t \\
&\le C\int_0^{\sigma(T)} \|\theta-1\|_{L^6}^{1/2}\|\nabla\theta\|_{L^6}^{1/2}\mathrm{d}t \\
&\le C\int_0^{\sigma(T)} \|\nabla\theta\|_{L^2}^{1/2}\left(\sigma\|\nabla^2\theta\|_{L^2}^2\right)^{1/4}\sigma^{-1/4}\mathrm{d}t \\
&\le C\left(\int_0^{\sigma(T)}\|\nabla\theta\|_{L^2}^2\mathrm{d}t\int_0^{\sigma(T)}\sigma\|\nabla^2\theta\|_{L^2}^2\mathrm{d}t\right)^{1/4}\left(\int_0^{\sigma(T)}\sigma^{-\frac{1}{2}}\mathrm{d}t\right)^{1/2} \\
&\le C(\bar\rho, M)C_0^{1/16},
\end{aligned} \tag{3.88}$$

and

$$\begin{aligned}
\int_{\sigma(T)}^T \|\theta-1\|_{L^\infty}^2\mathrm{d}t &\le C\int_{\sigma(T)}^T \|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}\mathrm{d}t \\
&\le C\left(\int_{\sigma(T)}^T\|\nabla\theta\|_{L^2}^2\mathrm{d}t\right)^{1/2}\left(\int_{\sigma(T)}^T\|\nabla^2\theta\|_{L^2}^2\mathrm{d}t\right)^{1/2} \\
&\le C(\bar\rho, M)C_0^{1/8}.
\end{aligned} \tag{3.89}$$

Next, (2.21), (2.26), (3.87) and (3.6) lead to

$$\begin{aligned}
&\int_0^{\sigma(T)} \|G\|_{L^\infty}\mathrm{d}t \\
&\le C\int_0^{\sigma(T)} \|\nabla G\|_{L^2}^{1/2}\|\nabla G\|_{L^6}^{1/2}\mathrm{d}t \\
&\le C(\bar\rho)\int_0^{\sigma(T)} \|\rho\dot u\|_{L^2}^{1/2}\|\nabla\dot u\|_{L^2}^{1/2}\mathrm{d}t \\
&\le C(\bar\rho)\int_0^{\sigma(T)} \left(\sigma\|\rho\dot u\|_{L^2}\right)^{1/4}\left(\sigma\|\rho\dot u\|_{L^2}^2\right)^{1/8}\left(\sigma\|\nabla\dot u\|_{L^2}^2\right)^{1/4}\sigma^{-5/8}\mathrm{d}t \\
&\le C(\bar\rho, M)C_0^{1/48}\left(\int_0^{\sigma(T)}\sigma\|\nabla\dot u\|_{L^2}^2\mathrm{d}t\right)^{1/4}\left(\int_0^{\sigma(T)}\sigma^{-5/6}\mathrm{d}t\right)^{3/4} \\
&\le C(\bar\rho, M)C_0^{1/48},
\end{aligned} \tag{3.90}$$

and

$$\begin{aligned}
\int_{\sigma(T)}^T \|G\|_{L^\infty}^2\mathrm{d}t &\le C\int_{\sigma(T)}^T \|\nabla G\|_{L^2}\|\nabla G\|_{L^6}\mathrm{d}t \\
&\le C(\bar\rho)\int_{\sigma(T)}^T\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla\dot u\|_{L^2}^2\right)\mathrm{d}t \\
&\le C(\bar\rho)C_0^{1/6}.
\end{aligned} \tag{3.91}$$

Finally, denoting $D_t\rho = \rho_t + u \cdot \nabla\rho$ and expressing $(1.6)_1$ in terms of the Lagrangian coordinates, we obtain by (3.23) that

$$
\begin{aligned}
(2\mu+\lambda)D_t\rho &= -R\rho(\rho-1) - R\rho^2(\theta-1) - \rho G \\
&\le -R(\rho-1) + C(\bar\rho)\|\theta-1\|_{L^\infty} + C(\bar\rho)\|G\|_{L^\infty},
\end{aligned}
$$

which gives

$$
D_t(\rho-1) + \frac{R}{2\mu+\lambda}(\rho-1) \le C(\bar\rho)\|\theta-1\|_{L^\infty} + C(\bar\rho)\|G\|_{L^\infty}. \tag{3.92}
$$

Taking

$$
\begin{aligned}
&y = \rho - 1, \quad \alpha = \frac{R}{2\mu+\lambda}, \\
&g(t) = C(\bar\rho)\|\theta-1\|_{L^\infty} + C(\bar\rho)\|G\|_{L^\infty}, \quad T_1 = \sigma(T),
\end{aligned}
$$

in Lemma 2.5, we thus deduce from (3.92), (3.88)–(3.91), and (2.30) that

$$
\begin{aligned}
\rho &\le \bar\rho + 1 + C\left(\|g\|_{L^1(0,\sigma(T))} + \|g\|_{L^2(\sigma(T),T)}\right) \\
&\le \bar\rho + 1 + C(\bar\rho, M)C_0^{1/48},
\end{aligned}
$$

which gives (3.84) provided

$$
C_0 \le \varepsilon_4 \triangleq \min\left\{1, \left(\frac{\bar\rho-2}{2C(\bar\rho, M)}\right)^{48}\right\}.
$$

We thus complete the proof of Lemma 3.7.

Next, the following Lemma will give an estimate on $A_4(T)$, which together with Lemmas 3.2, 3.3, 3.6 and 3.7 finishes the proof of Proposition 3.1:

Lemma 3.8. *Under the conditions of Proposition* 3.1*, there exists a positive constant* ε_0 *depending only on* $\mu, \lambda, \kappa, R, \gamma, \bar\rho, \bar\theta$*, and* M *such that if* (ρ, u, θ) *is a smooth solution of* (1.6) (1.4) (1.7) *on* $\mathbb{R}^3 \times (0, T]$ *satisfying* (3.6) *with* K *as in Lemma* 3.2*, the following estimate holds:*

$$
A_4(T) \le C_0^{1/8}, \tag{3.93}
$$

provided $C_0 \le \varepsilon_0$.

Proof. It follows from (3.40), (3.6), (3.72), and (3.79) that

$$
\begin{aligned}
\int_0^T \sigma^2\|\nabla^2\theta\|_{L^2}^2 dt &\le C(\bar\rho, M)\int_0^T \left(\sigma^2\|\rho\dot\theta\|_{L^2}^2 + \sigma\|\rho\dot u\|_{L^2}^2\right) dt \\
&\quad + C(\bar\rho, M)\int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \sigma^2\|\rho-1\|_{L^4}^4\right) dt \\
&\le C(\bar\rho, M)C_0^{1/6}.
\end{aligned} \tag{3.94}
$$

Applying the operator $\partial_t + \operatorname{div}(u\cdot)$ to (1.6)$_3$, we use (1.6)$_1$ to get

$$
\begin{aligned}
&\frac{R}{\gamma-1}\rho\left(\partial_t\dot\theta + u\cdot\nabla\dot\theta\right)\\
&\quad=\kappa\Delta\dot\theta + \kappa\left(\operatorname{div}u\Delta\theta - \partial_i\left(\partial_i u\cdot\nabla\theta\right) - \partial_i u\cdot\nabla\partial_i\theta\right)\\
&\qquad+\left(\lambda(\operatorname{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)\operatorname{div}u + R\rho\theta\partial_k u^l\partial_l u^k\\
&\qquad-R\rho\dot\theta\operatorname{div}u - R\rho\theta\operatorname{div}\dot u + 2\lambda\left(\operatorname{div}\dot u - \partial_k u^l\partial_l u^k\right)\operatorname{div}u\\
&\qquad+\mu(\partial_i u^j + \partial_j u^i)\left(\partial_i\dot u^j + \partial_j\dot u^i - \partial_i u^k\partial_k u^j - \partial_j u^k\partial_k u^i\right).
\end{aligned}
\tag{3.95}
$$

Multiplying (3.95) by $\dot\theta$, we obtain after integration by parts that

$$
\begin{aligned}
&\frac{R}{2(\gamma-1)}\left(\int\rho|\dot\theta|^2\mathrm{d}x\right)_t + \kappa\|\nabla\dot\theta\|_{L^2}^2\\
&\le C\int|\nabla u|\left(|\nabla^2\theta||\dot\theta| + |\nabla\theta||\nabla\dot\theta|\right)\mathrm{d}x\\
&\quad+C(\bar\rho)\int|\nabla u|^2|\dot\theta|\left(|\nabla u| + |\theta-1|\right)\mathrm{d}x\\
&\quad+C(\bar\rho)\int\left(|\nabla u|^2|\dot\theta| + \rho|\dot\theta|^2|\nabla u|\right)\mathrm{d}x + C\int|\nabla\dot u|\rho|\dot\theta|\mathrm{d}x\\
&\quad+C(\bar\rho)\int\rho|\theta-1||\nabla\dot u||\dot\theta|\mathrm{d}x + C(\bar\rho)\int|\nabla u||\nabla\dot u||\dot\theta|\mathrm{d}x\\
&\le C\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{L^6}^{1/2}\|\nabla^2\theta\|_{L^2}\|\nabla\dot\theta\|_{L^2}\\
&\quad+C(\bar\rho)\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\left(\|\nabla u\|_{L^6} + \|\nabla\theta\|_{L^2}\right)\|\nabla\dot\theta\|_{L^2}\\
&\quad+C(\bar\rho)\|\nabla u\|_{L^6}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla\dot\theta\|_{L^2}\left(\|\nabla u\|_{L^2} + \|\rho\dot\theta\|_{L^2}\right)\\
&\quad+C(\bar\rho)\|\nabla\dot u\|_{L^2}\|\rho\dot\theta\|_{L^2} + C(\bar\rho)\|\sqrt{\rho}(\theta-1)\|_{L^2}^{1/2}\|\nabla\theta\|_{L^2}^{1/2}\|\nabla\dot u\|_{L^2}\|\nabla\dot\theta\|_{L^2}\\
&\quad+C(\bar\rho)\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{L^6}^{1/2}\|\nabla\dot u\|_{L^2}\|\nabla\dot\theta\|_{L^2}.
\end{aligned}
\tag{3.96}
$$

Multiplying (3.96) by σ^4 and integrating the resulting inequality over $(0,t)$, we obtain after integration by parts and using (3.42), (3.6), (3.24), and (3.94) that

$$
\begin{aligned}
&\frac{R}{2(\gamma-1)}\sigma^4\int\rho|\dot\theta|^2\mathrm{d}x + \kappa\int_0^t\sigma^4\|\nabla\dot\theta\|_{L^2}^2 ds\\
&\le C\int_0^{\sigma(t)}\sigma^2\int\rho|\dot\theta|^2\mathrm{d}x ds + C(\bar\rho)\int_0^t\sigma^3\|\nabla^2\theta\|_{L^2}\|\nabla\dot\theta\|_{L^2} ds\\
&\quad+C(\bar\rho)\int_0^t\sigma^2\|\nabla u\|_{L^2}\|\nabla\dot\theta\|_{L^2} ds + C(\bar\rho)\int_0^t\sigma^3\|\nabla\dot\theta\|_{L^2}\|\rho\dot\theta\|_{L^2} ds\\
&\quad+C(\bar\rho)\int_0^t\sigma^3\|\nabla\dot u\|_{L^2}\|\rho^{1/2}\dot\theta\|_{L^2} ds + C(\bar\rho)\int_0^t\sigma^3\|\nabla\dot u\|_{L^2}\|\nabla\dot\theta\|_{L^2} ds\\
&\le C(\bar\rho)\int_0^t\left(\sigma^2\|\nabla^2\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \sigma^2\|\rho^{1/2}\dot\theta\|_{L^2}^2 + \sigma^2\|\nabla\dot u\|_{L^2}^2\right) ds
\end{aligned}
$$

$$+\frac{\kappa}{2}\int_0^t \sigma^4\|\nabla\dot\theta\|_{L^2}^2 ds$$
$$\le C(\bar\rho, M)C_0^{1/6}+\frac{\kappa}{2}\int_0^t \sigma^4\|\nabla\dot\theta\|_{L^2}^2 ds,$$

which yields that

$$\sup_{0\le t\le T}\sigma^4\int\rho|\dot\theta|^2\mathrm{d}x+\int_0^T\sigma^4\|\nabla\dot\theta\|_{L^2}^2 ds\le C(\bar\rho, M)C_0^{1/6}.$$

This gives (3.93) provided

$$C_0\le\varepsilon_0\triangleq\min\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_5\},\tag{3.97}$$

with $\varepsilon_5\triangleq(C(\bar\rho, M))^{-24}$. The proof of Lemma 3.8 is finished.

Finally, before closing this section, we summarize some estimates on (ρ, u, θ) which will be useful for higher order ones in the next section.

Corollary 3.9. *In addition to the conditions of Proposition 3.1, assume that (ρ_0, u_0, θ_0) satisfies (3.8) with ε_0 as in Proposition 3.1. Then there exists a positive constant C depending only on $\mu, \lambda, \kappa, R, \gamma, \bar\rho, \bar\theta$, and M such that if (ρ, u, θ) is a smooth solution of (1.6) (1.4) (1.7) on $\mathbb{R}^3\times(0,T]$ satisfying (3.6) with K as in Lemma 3.2, it holds that*

$$\begin{aligned}&\sup_{t\in(0,T]}\left(\sigma^2\|\nabla u\|_{L^6}^2+\sigma^4\|\theta-1\|_{H^2}^2\right)\\&+\int_0^T\sigma^2(\|\nabla u\|_{L^4}^4+\|\nabla\theta\|_{H^1}^2+\|u_t\|_{L^2}^2+\sigma^2\|\theta_t\|_{H^1}^2)\mathrm{d}t\\&+\int_0^T\sigma^2\|\rho-1\|_{L^4}^4\mathrm{d}t\le CC_0^{1/8}.\end{aligned}\tag{3.98}$$

Proof. It follows from (3.6), (2.23), (3.10), (3.42), (3.43), (3.72), (3.79) and (3.94) that

$$\begin{aligned}&\sup_{t\in(0,T]}\left(\sigma^2\|\nabla u\|_{L^6}^2+\sigma^4\|\theta-1\|_{H^2}^2\right)\\&+\int_0^T\sigma^2(\|\nabla u\|_{L^4}^4+\|\nabla\theta\|_{H^1}^2+\|\rho-1\|_{L^4}^4)\mathrm{d}t\le CC_0^{1/8},\end{aligned}\tag{3.99}$$

which together with (3.6) and (2.23) gives

$$\begin{aligned}\int_0^T\sigma^2\|u_t\|_{L^2}^2\mathrm{d}t&\le C\int_0^T\sigma^2\|\dot u\|_{L^2}^2\mathrm{d}t+C\int_0^T\sigma^2\|u\cdot\nabla u\|_{L^2}^2\mathrm{d}t\\&\le C\int_0^T\sigma^2\int\rho|\dot u|^2\mathrm{d}x\mathrm{d}t+C\int_0^T\sigma^2\|\nabla\dot u\|_{L^2}^2\mathrm{d}t\\&\quad+C\int_0^T\sigma^2\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2\mathrm{d}t\\&\le CC_0^{1/6},\end{aligned}\tag{3.100}$$

$$
\begin{aligned}
\int_0^T \sigma^4 \|\theta_t\|_{L^2}^2 \mathrm{d}t &\le C \int_0^T \sigma^4 \|\dot\theta\|_{L^2}^2 \mathrm{d}t + C \int_0^T \sigma^4 \|u \cdot \nabla\theta\|_{L^2}^2 \mathrm{d}t \\
&\le C \int_0^T \sigma^4 \int \rho|\dot\theta|^2 \mathrm{d}x\mathrm{d}t + C \int_0^T \sigma^4 \|\nabla\dot\theta\|_{L^2}^2 \mathrm{d}t \\
&\quad + C \int_0^T \sigma^4 \|u\|_{L^6}^2 \|\nabla\theta\|_{L^3}^2 \mathrm{d}t \\
&\le C C_0^{1/8} + C \int_0^T \|\nabla u\|_{L^2}^2 \mathrm{d}t \\
&\le C C_0^{1/8},
\end{aligned} \tag{3.101}
$$

and

$$
\begin{aligned}
\int_0^T \sigma^4 \|\nabla\theta_t\|_{L^2}^2 \mathrm{d}t &\le C \int_0^T \sigma^4 \|\nabla\dot\theta\|_{L^2}^2 \mathrm{d}t + C \int_0^T \sigma^4 \|\nabla(u \cdot \nabla\theta)\|_{L^2}^2 \mathrm{d}t \\
&\le C C_0^{1/8} + C \int_0^T \sigma^4 \left(\|\nabla u\|_{L^3}^2 + \|u\|_{L^\infty}^2\right) \|\nabla^2\theta\|_{L^2}^2 \mathrm{d}t \\
&\le C C_0^{1/8}.
\end{aligned} \tag{3.102}
$$

We thus obtain (3.98) directly from (3.99)–(3.102) and finish the proof of Corollary 3.9.

4. A Priori Estimates (II): Higher-Order Estimates

In this section, we will derive the higher order estimates of smooth solutions (ρ, u, θ) of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ with smooth (ρ_0, u_0, θ_0) satisfying (1.9) and (3.5). Moreover, we shall always assume that (3.6) and (3.8) both hold. To proceed, we define $\tilde{g_1}$ and $\tilde{g_2}$ as

$$
\tilde{g_1} \triangleq \rho_0^{-1/2} \left(-\mu\Delta u_0 - (\mu + \lambda)\nabla \mathrm{div} u_0 + R\nabla(\rho_0\theta_0)\right) \tag{4.1}
$$

and

$$
\tilde{g_2} \triangleq \rho_0^{-1/2} \left(\kappa\Delta\theta_0 + \frac{\mu}{2}|\nabla u_0 + (\nabla u_0)^{\mathrm{tr}}|^2 + \lambda(\mathrm{div} u_0)^2\right), \tag{4.2}
$$

respectively. It thus follows from (1.9) and (3.5) that

$$
\tilde{g_1} \in L^2, \quad \tilde{g_2} \in L^2. \tag{4.3}
$$

From now on, the generic constant C will depend only on

$$
T,\ \|\tilde{g_1}\|_{L^2},\ \|\tilde{g_2}\|_{L^2},\ \|u_0\|_{H^2},\ \|\rho_0 - 1\|_{H^2 \cap W^{2,q}},\ \|\theta_0 - 1\|_{H^2},
$$

besides $\mu, \lambda, \kappa, R, \gamma, \bar\rho, \bar\theta$, and M.

We begin with the important estimates on the spatial gradient of the smooth solution (ρ, u, θ).

Lemma 4.1. *The following estimates hold:*

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\theta-1\|_{H^1}^2\right)+\int_0^T\int\rho(\dot\theta)^2\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\left(\|\nabla\dot u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2+\|\mathrm{div}u\|_{L^\infty}^2+\|\omega\|_{L^\infty}^2\right)\mathrm{d}t\le C, \tag{4.4}$$

$$\sup_{0\le t\le T}\left(\|\rho-1\|_{H^1\cap W^{1,6}}+\|u\|_{H^2}\right)+\int_0^T\|\nabla u\|_{L^\infty}^{3/2}\mathrm{d}t\le C. \tag{4.5}$$

Proof. We first prove (4.4). Taking $m=0$ in (3.69) gives that, for $\varphi(t)$ as in (3.61),

$$\begin{aligned}
&\varphi'(t)+\int\left(\mu|\nabla\dot u|^2+\rho(\dot\theta)^2\right)\mathrm{d}x\\
&\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+1\right)\\
&\quad+C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3\\
&\le C\left(\|\rho^{1/2}\dot u\|_{L^2}^4+\|\nabla\theta\|_{L^2}^4\right)+C\\
&\le C\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\varphi(t)+C\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)+C,
\end{aligned} \tag{4.6}$$

due to (3.6), (3.25), and (3.76). It follows from $(1.6)_2$, (3.5), and (4.1) that

$$\lim_{t\to 0^+}\sqrt{\rho}\dot u(x,t)=\rho_0^{-1/2}\left(\mu\Delta u_0+(\mu+\lambda)\nabla\mathrm{div}u_0-R\nabla(\rho_0\theta_0)\right)=-\tilde g_1,$$

which together with (3.61), (3.62), (3.52), and (4.3) yields that

$$\lim_{t\to 0^+}|\varphi(t)|\le C\|\tilde g_1\|_{L^2}^2+C\le C. \tag{4.7}$$

Grönwall's inequality, together with (4.6), (4.7) and (3.6), leads to

$$\sup_{0\le t\le T}\varphi(t)+\int_0^T\int\left(|\nabla\dot u|^2+\rho(\dot\theta)^2\right)\mathrm{d}x\mathrm{d}t\le C,$$

which, as well as (3.76) and (3.10), implies

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\theta-1\|_{H^1}^2\right)+\int_0^T\int\left(|\nabla\dot u|^2+\rho(\dot\theta)^2\right)\mathrm{d}x\mathrm{d}t\le C. \tag{4.8}$$

One thus deduces from (3.40), (4.8), (3.71) and (3.6) that

$$\int_0^T\|\nabla^2\theta\|_{L^2}^2\mathrm{d}t\le C+C\int_0^T\|\rho^{1/2}\dot\theta\|_{L^2}^2\mathrm{d}t\le C, \tag{4.9}$$

which, together with (1.21), (2.26), (2.20), (4.8) and (3.6), gives

$$\begin{aligned}
&\int_0^T \left(\|\mathrm{div}u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2\right) \mathrm{d}t \\
&\le C\int_0^T \left(\|G\|_{L^\infty}^2 + \|\rho\theta - 1\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2\right) \mathrm{d}t + C \\
&\le C\int_0^T \left(\|G\|_{W^{1,6}}^2 + \|\theta - 1\|_{L^\infty}^2 + \|\omega\|_{L^6}^2 + \|\nabla\omega\|_{L^6}^2\right) \mathrm{d}t + C \\
&\le C\int_0^T \left(\|\nabla G\|_{L^2}^2 + \|\rho\dot{u}\|_{L^6}^2 + \|\nabla^2\theta\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right) \mathrm{d}t + C \\
&\le C\int_0^T \left(\|\rho\dot{u}\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 + \|\nabla\dot{u}\|_{L^2}^2\right) \mathrm{d}t + C \\
&\le C.
\end{aligned}$$

This fact, combined with (4.8) and (4.9), yields (4.4) directly.

Next, we will prove the key estimate (4.5). Standard calculations shows that for $2 \le p \le 6$,

$$\begin{aligned}
\partial_t \|\nabla\rho\|_{L^p} &\le C(1 + \|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p} \\
&\le C\left(1 + \|\nabla^2\theta\|_{L^2} + \|\nabla u\|_{L^\infty}\right)\|\nabla\rho\|_{L^p} \\
&\quad + C\left(1 + \|\nabla\dot{u}\|_{L^2} + \|\nabla^2\theta\|_{L^2}\right),
\end{aligned} \tag{4.10}$$

where we have used

$$\begin{aligned}
\|\nabla^2 u\|_{L^p} &\le C\left(\|\rho\dot{u}\|_{L^p} + \|\nabla P\|_{L^p}\right) \\
&\le C\left(\|\rho\dot{u}\|_{L^2} + \|\nabla\dot{u}\|_{L^2} + \|\nabla\theta\|_{L^p} + \|\theta\|_{L^\infty}\|\nabla\rho\|_{L^p}\right) \\
&\le C\left(1 + \|\nabla\dot{u}\|_{L^2} + \|\nabla^2\theta\|_{L^2} + (\|\nabla^2\theta\|_{L^2} + 1)\|\nabla\rho\|_{L^p}\right),
\end{aligned} \tag{4.11}$$

which comes from the standard L^p-estimate of the following elliptic system:

$$\mu\Delta u + (\mu + \lambda)\nabla\mathrm{div}u = \rho\dot{u} + \nabla P, \quad u \to 0 \ \text{ as } |x| \to \infty. \tag{4.12}$$

It follows from Lemma 2.6, (3.6) and (4.11) that

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\le C\left(\|\mathrm{div}u\|_{L^\infty} + \|\omega\|_{L^\infty}\right)\log(e + \|\nabla^2 u\|_{L^6}) + C \\
&\le C\left(\|\mathrm{div}u\|_{L^\infty} + \|\omega\|_{L^\infty}\right)\log(e + \|\nabla\dot{u}\|_{L^2} + \|\nabla^2\theta\|_{L^2}) \\
&\quad + C\left(\|\mathrm{div}u\|_{L^\infty} + \|\omega\|_{L^\infty}\right)\log\left(e + \|\nabla\rho\|_{L^6}\right) + C.
\end{aligned} \tag{4.13}$$

Set

$$\begin{cases}
f(t) \triangleq e + \|\nabla\rho\|_{L^6}, \\
g(t) \triangleq 1 + \|\mathrm{div}u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2 + \|\nabla\dot{u}\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2.
\end{cases}$$

Putting (4.13) into (4.10), where we set $p = 6$, gives

$$f'(t) \le Cg(t)f(t)\ln f(t),$$

which implies

$$(\ln f(t))' \le Cg(t)\ln f(t).$$

Combining this with Grönwall's inequality and (4.4) yields that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^6}\le C, \tag{4.14}$$

which together with (4.13) and (4.4) leads to

$$\int_0^T \|\nabla u\|_{L^\infty}^{3/2}\mathrm{d}t\le C. \tag{4.15}$$

Finally, taking $p=2$ in (4.10), we get by using (4.15), (4.4), and Grönwall's inequality that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^2}\le C, \tag{4.16}$$

which gives

$$\sup_{0\le t\le T}\|\nabla P\|_{L^2}\le C\sup_{0\le t\le T}\left(\|\nabla\theta\|_{L^2}+\|\nabla\rho\|_{L^2}+\|\theta-1\|_{L^6}\|\nabla\rho\|_{L^3}\right)\le C,$$

due to (4.4) and (4.14). Combining this with (4.11) and (4.4) leads to

$$\sup_{0\le t\le T}\|\nabla^2 u\|_{L^2}\le C\sup_{0\le t\le T}\left(\|\rho\dot u\|_{L^2}+\|\nabla P\|_{L^2}\right)\le C,$$

which together with (3.9), (3.6), (2.23) and (4.14)–(4.16) yields (4.5) and finishes the proof of Lemma 4.1.

Lemma 4.2. *The following estimates hold*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho_t\|_{H^1}+\|\theta-1\|_{H^2}+\|\rho-1\|_{H^2}+\|u\|_{H^2}\right)\\&\quad+\int_0^T\left(\|u_t\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\|\rho u_t\|_{H^1}^2+\|\rho\theta_t\|_{H^1}^2\right)\mathrm{d}t\le C, \end{aligned}\tag{4.17}$$

$$\int_0^T\left(\|(\rho u_t)_t\|_{H^{-1}}^2+\|(\rho\theta_t)_t\|_{H^{-1}}^2\right)\mathrm{d}t\le C. \tag{4.18}$$

Proof. First, it follows from (1.21), (4.4), (4.5), (2.24) and (2.22) that

$$\begin{aligned}\|\nabla u\|_{H^2}&\le C\left(\|\mathrm{div}u\|_{H^2}+\|\omega\|_{H^2}\right)\\&\le C\left(\|G\|_{H^2}+\|\omega\|_{H^2}+\|\rho\theta-1\|_{H^2}\right)\\&\le C+C\|\nabla(\rho\dot u)\|_{L^2}+C\|(\rho-1)(\theta-1)\|_{H^2}\\&\quad+C\|\rho-1\|_{H^2}+C\|\theta-1\|_{H^2}\\&\le C+C(\|\nabla\rho\|_{L^3}\|\dot u\|_{L^6}+\|\nabla\dot u\|_{L^2})+C\|\rho-1\|_{H^2}\|\theta-1\|_{H^2}\\&\quad+C\|\nabla^2\rho\|_{L^2}+C\|\nabla^2\theta\|_{L^2}\\&\le C+C(1+\|\nabla^2\theta\|_{L^2})\|\nabla^2\rho\|_{L^2}+C\|\nabla\dot u\|_{L^2}+C\|\nabla^2\theta\|_{L^2},\end{aligned}\tag{4.19}$$

which together with simple computations and $(1.6)_1$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\rho\|_{L^2}^2 &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla^2\rho\|_{L^2}^2 + C\|\nabla u\|_{H^2}^2 \\ &\le C(1+\|\nabla u\|_{L^\infty}+\|\nabla^2\theta\|_{L^2})\|\nabla^2\rho\|_{L^2}^2 \\ &\quad + C\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla^2\theta\|_{L^2}^2 + C.\end{aligned}$$

Combining this with (4.15), (4.4), and Grönwall's inequality, yields

$$\sup_{0\le t\le T}\|\nabla^2\rho\|_{L^2} \le C. \tag{4.20}$$

Next, it follows from $(1.6)_3$, (3.5) and (4.2) that

$$\frac{R}{\gamma-1}\lim_{t\to 0^+}\sqrt{\rho}\dot{\theta}(x,t) = -R\rho_0^{1/2}\theta_0\mathrm{div}u_0 + \tilde{g_2}. \tag{4.21}$$

Integrating (3.96) over $(0,T)$ together with (4.5), (4.4), (3.25), and (4.21) leads to

$$\begin{aligned}&\sup_{0\le t\le T}\int\rho(\dot{\theta})^2\mathrm{d}x + \int_0^T\|\nabla\dot{\theta}\|_{L^2}^2\mathrm{d}t \\ &\le C\int_0^T\left(\|\nabla^2\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\rho^{1/2}\dot{\theta}\|_{L^2}^2+\|\nabla\dot{u}\|_{L^2}^2\right)\mathrm{d}t \\ &\quad+\frac{1}{2}\int_0^T\|\nabla\dot{\theta}\|_{L^2}^2\mathrm{d}t + C\left(\|\theta_0-1\|_{L^6}^2\|\nabla u_0\|_{L^3}^2+\|\nabla u_0\|_{L^2}^2\right)+C\|\tilde{g_2}\|_{L^2}^2 \\ &\le C+\frac{1}{2}\int_0^T\|\nabla\dot{\theta}\|_{L^2}^2\mathrm{d}t,\end{aligned}$$

which shows

$$\sup_{0\le t\le T}\int\rho(\dot{\theta})^2\mathrm{d}x + \int_0^T\|\nabla\dot{\theta}\|_{L^2}^2\mathrm{d}t \le C. \tag{4.22}$$

One thus deduces from (3.40), (4.22), (4.5), and (4.4) that

$$\sup_{0\le t\le T}\|\nabla^2\theta\|_{L^2} \le C. \tag{4.23}$$

It follows from (4.4), (4.5) and (4.22) that

$$\begin{aligned}&\sup_{0\le t\le T}\int\rho\left(|u_t|^2+\theta_t^2\right)\mathrm{d}x + \int_0^T\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\mathrm{d}t \\ &\le C\sup_{0\le t\le T}\int\rho\left(|\dot{u}|^2+\dot{\theta}^2\right)\mathrm{d}x \\ &\quad + C\sup_{0\le t\le T}\int\rho\left(|u\cdot\nabla u|^2+|u\cdot\nabla\theta|^2\right)\mathrm{d}x \\ &\quad + C\int_0^T\left(\|\nabla u\|_{L^3}^2+\|u\|_{L^\infty}^2\right)\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2\right)\mathrm{d}t \\ &\quad + C\int_0^T\left(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla\dot{\theta}\|_{L^2}^2\right)\mathrm{d}t \le C,\end{aligned} \tag{4.24}$$

which together with (4.4) and (4.5) gives

$$\begin{aligned}
&\int_0^T \left(\|\nabla(\rho u_t)\|_{L^2}^2 + \|\nabla(\rho\theta_t)\|_{L^2}^2\right) \mathrm{d}t \\
&\le C \int_0^T \left(\|\nabla u_t\|_{L^2}^2 + \|\nabla\rho\|_{L^3}^2 \|u_t\|_{L^6}^2 + \|\nabla\theta_t\|_{L^2}^2 + \|\nabla\rho\|_{L^3}^2 \|\theta_t\|_{L^6}^2\right) \mathrm{d}t \\
&\le C.
\end{aligned} \tag{4.25}$$

Next, one deduces from $(1.6)_1$ and (4.5) that

$$\|\rho_t\|_{L^2} \le C\|u\|_{L^\infty}\|\nabla\rho\|_{L^2} + C\|\nabla u\|_{L^2} \le C. \tag{4.26}$$

Applying ∇ to $(1.6)_1$ yields

$$\nabla\rho_t + u^i\partial_i\nabla\rho + \nabla u^i\partial_i\rho + \nabla\rho\,\mathrm{div}u + \rho\nabla\mathrm{div}u = 0,$$

which leads to

$$\begin{aligned}
\|\nabla\rho_t\|_{L^2} &\le C\|u\|_{L^\infty}\|\nabla^2\rho\|_{L^2} \\
&\quad + C\|\nabla u\|_{L^3}\|\nabla\rho\|_{L^6} + C\|\nabla^2 u\|_{L^2} \le C,
\end{aligned} \tag{4.27}$$

due to (4.5). Combining (4.26) with (4.27) implies

$$\sup_{0\le t\le T} \|\rho_t\|_{H^1} \le C,$$

which together with (4.4), (4.20), (4.5), (4.23)–(4.25), and (2.23) gives (4.17).

Finally, differentiating $(1.6)_3$ with respect to t yields that

$$\begin{aligned}
\frac{R}{\gamma-1}(\rho\theta_t)_t = &-\frac{R}{\gamma-1}(\rho u\cdot\nabla\theta)_t - R(\rho\theta\,\mathrm{div}u)_t + \kappa\Delta\theta_t \\
&+ \lambda((\mathrm{div}u)^2)_t + 2\mu(|\mathfrak{D}(u)|^2)_t.
\end{aligned} \tag{4.28}$$

It follows from (4.17) that

$$\begin{aligned}
&\|(\rho u\cdot\nabla\theta)_t\|_{L^2} \\
&\quad = \|\rho_t u\cdot\nabla\theta + \rho u_t\cdot\nabla\theta + \rho u\cdot\nabla\theta_t\|_{L^2} \\
&\quad \le C\|\rho_t\|_{L^6}\|\nabla\theta\|_{L^3} + C\|u_t\|_{L^6}\|\nabla\theta\|_{L^3} + C\|u\|_{L^\infty}\|\nabla\theta_t\|_{L^2} \\
&\quad \le C + C\|u_t\|_{H^1} + C\|\theta_t\|_{H^1},
\end{aligned} \tag{4.29}$$

$$\|(\rho\theta\,\mathrm{div}u)_t\|_{L^2} \le C + C\|u_t\|_{H^1} + C\|\theta_t\|_{H^1}, \tag{4.30}$$

and

$$\begin{aligned}
\|((\mathrm{div}u)^2)_t\|_{L^{6/5}} + \|(|\mathfrak{D}(u)|^2)_t\|_{L^{6/5}} &\le C\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2} \\
&\le C + C\|u_t\|_{H^1}.
\end{aligned} \tag{4.31}$$

Combining (4.28)–(4.31) with (4.17) shows

$$\int_0^T \|(\rho\theta_t)_t\|_{H^{-1}}^2 \mathrm{d}t \le C. \tag{4.32}$$

Similarly, we have

$$\int_0^T \|(\rho u_t)_t\|_{H^{-1}}^2 \mathrm{d}t \le C,$$

which combined with (4.32) implies (4.18). The proof of Lemma 4.2 is completed.

Lemma 4.3. *The following estimate holds:*

$$\sup_{0\le t\le T} \sigma \left(\|\nabla u_t\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2\right) + \int_0^T \sigma \int \rho |u_{tt}|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{4.33}$$

Proof. Multiplying (2.6) by u_{tt}, one gets after integrating the resulting equality by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\mu|\nabla u_t|^2 + (\mu+\lambda)(\mathrm{div} u_t)^2\right)\mathrm{d}x + \int \rho|u_{tt}|^2 \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}\int \rho_t |u_t|^2 \mathrm{d}x - \int \rho_t u\cdot\nabla u\cdot u_t \mathrm{d}x + \int P_t \mathrm{div} u_t \mathrm{d}x\right) \\
&\quad + \frac{1}{2}\int \rho_{tt}|u_t|^2 \mathrm{d}x + \int (\rho_t u\cdot\nabla u)_t\cdot u_t \mathrm{d}x - \int \rho u_t\cdot\nabla u\cdot u_{tt}\mathrm{d}x \\
&\quad - \int \rho u\cdot\nabla u_t\cdot u_{tt}\mathrm{d}x - \int (P_{tt} - \kappa(\gamma-1)\Delta\theta_t)\,\mathrm{div} u_t \mathrm{d}x \\
&\quad + \kappa(\gamma-1)\int \nabla\theta_t\cdot\nabla \mathrm{div} u_t \mathrm{d}x \triangleq \frac{\mathrm{d}}{\mathrm{d}t} I_0 + \sum_{i=1}^6 I_i.
\end{aligned} \tag{4.34}$$

We estimate each term on the righthand side of (4.34) as follows: first, it follows from $(1.6)_1$, (4.17), (4.24), and (2.23) that

$$\begin{aligned}
|I_0| &= \left| -\frac{1}{2}\int \rho_t |u_t|^2 \mathrm{d}x - \int \rho_t u\cdot\nabla u\cdot u_t\mathrm{d}x \right.\\
&\quad \left. + \int P_t \mathrm{div} u_t \mathrm{d}x \right| \\
&\le \left| \int \mathrm{div}(\rho u)|u_t|^2 \mathrm{d}x \right| + C\|\rho_t\|_{L^3}\|u\cdot\nabla u\|_{L^2}\|u_t\|_{L^6} \\
&\quad + C\|(\rho\theta)_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le C\int \rho|u||u_t||\nabla u_t|\mathrm{d}x + C(1 + \|\sqrt{\rho}\theta_t\|_{L^2} \\
&\quad + \|\rho_t\|_{L^2}\|\theta\|_{L^\infty})\|\nabla u_t\|_{L^2} \\
&\le C\|u\|_{L^6}\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2} + C\|\nabla u_t\|_{L^2} \\
&\le \frac{\mu}{4}\|\nabla u_t\|_{L^2}^2 + C,
\end{aligned} \tag{4.35}$$

$$
\begin{aligned}
2|I_1| &= \left|\int \rho_{tt}|u_t|^2 \mathrm{d}x\right| \\
&\le C\|\rho_{tt}\|_{L^2}\|u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{3/2} \\
&\le C\|\rho_{tt}\|_{L^2}\left(1+\|\nabla u_t\|_{L^2}\right)^{1/2}\|\nabla u_t\|_{L^2}^{3/2} \\
&\le C\|\rho_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^4 + C,
\end{aligned}
\tag{4.36}
$$

and

$$
\begin{aligned}
|I_2| &= \left|\int (\rho_t u\cdot\nabla u)_t\cdot u_t \mathrm{d}x\right| \\
&= \left|\int (\rho_{tt}u\cdot\nabla u\cdot u_t + \rho_t u_t\cdot\nabla u\cdot u_t + \rho_t u\cdot\nabla u_t\cdot u_t)\,\mathrm{d}x\right| \\
&\le C\|\rho_{tt}\|_{L^2}\|u\cdot\nabla u\|_{L^3}\|u_t\|_{L^6} + C\|\rho_t\|_{L^2}\||u_t|^2\|_{L^3}\|\nabla u\|_{L^6} \\
&\quad + C\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6} \\
&\le C\|\rho_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2.
\end{aligned}
\tag{4.37}
$$

Next, Cauchy's inequality gives

$$
\begin{aligned}
|I_3|+|I_4| &= \left|\int \rho u_t\cdot\nabla u\cdot u_{tt}\mathrm{d}x\right| + \left|\int \rho u\cdot\nabla u_t\cdot u_{tt}\mathrm{d}x\right| \\
&\le C\|\rho^{1/2}u_{tt}\|_{L^2}\left(\|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\right) \\
&\le \frac{1}{4}\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2.
\end{aligned}
\tag{4.38}
$$

Next, it follows from (4.17) that

$$
\begin{aligned}
\|\nabla P_t\|_{L^2} &\le C\|\nabla(\rho\theta_t + \theta\rho_t)\|_{L^2} \\
&\le C\|\nabla\rho\|_{L^3}\|\theta_t\|_{L^6} + C\|\nabla\theta_t\|_{L^2} + C\|\nabla\theta\|_{L^6}\|\rho_t\|_{L^3} \\
&\quad + C\|\theta\|_{L^\infty}\|\nabla\rho_t\|_{L^2} \\
&\le C + C\|\nabla\theta_t\|_{L^2}
\end{aligned}
\tag{4.39}
$$

which together with (3.26) and (4.17) gives

$$
\begin{aligned}
&\|P_{tt} - \kappa(\gamma-1)\Delta\theta_t\|_{L^2} \\
&\quad\le C\|(u\cdot\nabla P)_t\|_{L^2} + C\|(P\mathrm{div}u)_t\|_{L^2} + C\||\nabla u||\nabla u_t|\|_{L^2} \\
&\quad\le C\|u_t\|_{L^6}\|\nabla P\|_{L^3} + C\|u\|_{L^\infty}\|\nabla P_t\|_{L^2} + C\|P_t\|_{L^6}\|\nabla u\|_{L^3} \\
&\qquad + C\|P\|_{L^\infty}\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2} \\
&\quad\le C + C\left(1+\|\nabla u\|_{L^\infty}\right)\|\nabla u_t\|_{L^2} + C\|\nabla\theta_t\|_{L^2}.
\end{aligned}
$$

This directly yields

$$
\begin{aligned}
|I_5| &= \left|\int (P_{tt} - \kappa(\gamma-1)\Delta\theta_t)\,\mathrm{div}u_t\mathrm{d}x\right| \\
&\le \|P_{tt} - \kappa(\gamma-1)\Delta\theta_t\|_{L^2}\|\mathrm{div}u_t\|_{L^2} \\
&\le C + C\left(1+\|\nabla u\|_{L^\infty}\right)\|\nabla u_t\|_{L^2}^2 + C\|\nabla\theta_t\|_{L^2}^2.
\end{aligned}
\tag{4.40}
$$

Finally, it follows from (4.39), (4.17), and the standard L^2-estimate for elliptic system (2.6) that

$$\begin{aligned}\|\nabla^2 u_t\|_{L^2} &\le C\|\rho u_{tt} + \rho_t u_t + \rho_t u\cdot\nabla u + \rho u_t\cdot\nabla u + \rho u\cdot\nabla u_t + \nabla P_t\|_{L^2}\\ &\le C\|\rho^{1/2}u_{tt}\|_{L^2} + C\|\rho_t\|_{L^3}\|u_t\|_{L^6} + C\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}\\ &\quad + C\|u_t\|_{L^6}\|\nabla u\|_{L^3} + C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2} + C\|\nabla P_t\|_{L^2}\\ &\le C + C\|\rho^{1/2}u_{tt}\|_{L^2} + C\|\nabla\theta_t\|_{L^2} + C\|\nabla u_t\|_{L^2},\end{aligned} \tag{4.41}$$

which combined with Cauchy inequality thus leads to

$$\begin{aligned}|I_6| &= \left|\kappa(\gamma-1)\int\nabla\theta_t\cdot\nabla\mathrm{div}u_t\mathrm{d}x\right|\\ &\le C\|\nabla^2 u_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\ &\le C\left(1 + \|\rho^{1/2}u_{tt}\|_{L^2} + \|\nabla\theta_t\|_{L^2} + \|\nabla u_t\|_{L^2}\right)\|\nabla\theta_t\|_{L^2}\\ &\le C + \frac{1}{4}\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2.\end{aligned} \tag{4.42}$$

Substituting (4.36)–(4.38), (4.40), and (4.42) into (4.34) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\mu|\nabla u_t|^2 + (\mu+\lambda)(\mathrm{div}u_t)^2 - 2I_0\right)\mathrm{d}x + \int\rho|u_{tt}|^2\mathrm{d}x\\ &\le C\|\rho_{tt}\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^\infty} + \|\nabla u_t\|_{L^2}^2)\|\nabla u_t\|_{L^2}^2\\ &\quad + C\|\nabla\theta_t\|_{L^2}^2 + C.\end{aligned} \tag{4.43}$$

Then, differentiating $(1.6)_1$ with respect to t shows

$$\rho_{tt} + \rho_t\mathrm{div}u + \rho\mathrm{div}u_t + u_t\cdot\nabla\rho + u\cdot\nabla\rho_t = 0,$$

which combined with (4.17) implies

$$\begin{aligned}\|\rho_{tt}\|_{L^2} &\le C\left(\|\rho_t\|_{L^6}\|\nabla u\|_{L^3} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}\|\nabla\rho\|_{L^3} + \|\nabla\rho_t\|_{L^2}\right)\\ &\le C + C\|\nabla u_t\|_{L^2}.\end{aligned} \tag{4.44}$$

This yields

$$\int_0^T \|\rho_{tt}\|_{L^2}^2\mathrm{d}t \le C, \tag{4.45}$$

due to (4.24). One thus deduces from (4.43), (4.35), (4.17), (4.5), (4.45), and Grönwall's inequality that

$$\sup_{0\le t\le T}\sigma\|\nabla u_t\|_{L^2}^2 + \int_0^T\sigma\int\rho|u_{tt}|^2\mathrm{d}x\mathrm{d}t \le C,$$

which together with (4.44) gives (4.33). We complete the proof of Lemma 4.3.

Lemma 4.4. *For $q \in (3, 6)$ as in Theorem 1.1, it holds that*

$$\sup_{0 \le t \le T} \left(\|\rho - 1\|_{W^{2,q}} + \sigma \|u\|_{H^3}^2 \right) + \int_0^T \left(\|u\|_{H^3}^2 + \|\nabla^2 u\|_{W^{1,q}}^{p_0} + \sigma \|\nabla u_t\|_{H^1}^2 \right) dt \le C, \tag{4.46}$$

where

$$p_0 \triangleq \frac{1}{2} \min \left\{ \frac{5q-6}{3(q-2)}, \frac{9q-6}{5q-6} \right\} \in (1, 7/6). \tag{4.47}$$

Proof. First, it follows from (4.19), (4.17), and (4.33) that

$$\sup_{0 \le t \le T} \sigma \|u\|_{H^3}^2 + \int_0^T \|u\|_{H^3}^2 dt \le C. \tag{4.48}$$

The standard H^1-estimate for elliptic problem (3.39) together with (4.17) leads to

$$\begin{aligned} \|\nabla^2 \theta\|_{H^1} \le & C\|\nabla(\rho\theta_t)\|_{L^2} + C\||\nabla u||\nabla^2 u|\|_{L^2} \\ & + C\|\nabla(\rho u \cdot \nabla\theta)\|_{L^2} + C\|\nabla(\rho\theta \mathrm{div} u)\|_{L^2} + C \\ \le & C \left(\|\nabla\rho\|_{L^3} + 1 \right) \|\nabla\theta_t\|_{L^2} + C\|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^6}^{1/2} \\ & + C(1 + \|\rho - 1\|_{H^2})(1 + \|\theta - 1\|_{H^2})\|u\|_{H^2} + C \\ \le & C\|\nabla\theta_t\|_{L^2} + C\|\nabla^2 u\|_{L^6}^{1/2} + C, \end{aligned} \tag{4.49}$$

which combined with (4.17), (4.41), (4.33), and (4.48) yields that

$$\int_0^T \left(\|\theta - 1\|_{H^3}^2 + \|u\|_{H^3}^2 + \sigma \|\nabla u_t\|_{H^1}^2 \right) dt \le C. \tag{4.50}$$

Next, it follows from standard $W^{1,p}$-estimate for elliptic systems (2.24) that

$$\begin{aligned} \|\nabla^2 u\|_{W^{1,q}} \le & C\|u\|_{H^3} + C\|\nabla^2 \mathrm{div} u\|_{L^q} + C\|\nabla^2 \omega\|_{L^q} + C \\ \le & C\|u\|_{H^3} + C\|\nabla(\rho\dot{u})\|_{L^q} + C\|\nabla^2(\rho\theta)\|_{L^q} + C \\ \le & C\|u\|_{H^3} + C\|\nabla(\rho\dot{u})\|_{L^q} + C\|\theta\nabla^2\rho\|_{L^q} + C\|\nabla\rho\nabla\theta\|_{L^q} \\ & + C\|\rho\nabla^2\theta\|_{L^q} + C \\ \le & C\|u\|_{H^3} + C\|\nabla(\rho\dot{u})\|_{L^q} + C\|\nabla^2\rho\|_{L^q} + C\|\nabla^2\theta\|_{H^1} + C. \end{aligned} \tag{4.51}$$

Applying operator Δ to $(1.6)_1$ gives

$$(\Delta\rho)_t + \mathrm{div}(u\Delta\rho) + \mathrm{div}(\rho\Delta u) + 2\mathrm{div}(\partial_i\rho \cdot \partial_i u) = 0. \tag{4.52}$$

Multiplying (4.52) by $q|\Delta\rho|^{q-2}\Delta\rho$ and integrating the resulting equality over $\mathbb{R}^3$, we obtain after using (4.17) and (4.51) that

$$\begin{aligned} (\|\Delta\rho\|_{L^q}^q)_t \le & C(1 + \|\nabla u\|_{L^\infty})\|\Delta\rho\|_{L^q}^q \\ & + C\left(\|\nabla\rho\|_{L^q} + 1\right)\|\nabla^2 u\|_{W^{1,q}}\|\Delta\rho\|_{L^q}^{q-1} \\ \le & C(1 + \|u\|_{H^3} + \|\nabla(\rho\dot{u})\|_{L^q} + \|\nabla^2\theta\|_{H^1})\left(\|\Delta\rho\|_{L^q}^q + 1\right). \end{aligned} \tag{4.53}$$

Note that (4.17) and (4.33) give

$$\begin{aligned}\|\nabla(\rho\dot{u})\|_{L^q} &\le C\|\nabla\rho\|_{L^6}\|\nabla\dot{u}\|_{L^2}^{\frac{q}{3(q-2)}}\|\nabla\dot{u}\|_{L^q}^{\frac{2(q-3)}{3(q-2)}}+C\|\nabla\dot{u}\|_{L^q}\\ &\le C\|\nabla\dot{u}\|_{L^2}+C\|\nabla u_t\|_{L^q}+C\|\nabla(u\cdot\nabla u)\|_{L^q}\\ &\le C\|\nabla u_t\|_{L^2}+C+C\|\nabla u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^6}^{\frac{3(q-2)}{2q}}\\ &\quad+C\|\nabla u\|_{L^6}^{\frac{6-q}{q}}\|u\|_{H^3}^{\frac{3(q-2)}{q}}+C\|u\|_{L^\infty}\|\nabla^2 u\|_{L^q}\\ &\le C\sigma^{-\frac12}+C\|u\|_{H^3}^{\frac{3(q-2)}{q}}+C\sigma^{-\frac12}\left(\sigma\|\nabla u_t\|_{H^1}^2\right)^{\frac{3(q-2)}{4q}},\end{aligned}\tag{4.54}$$

which combined with (4.50) shows that, for p_0 as in (4.47),

$$\int_0^T \|\nabla(\rho\dot{u})\|_{L^q}^{p_0}\mathrm{d}t \le C.\tag{4.55}$$

Applying Grönwall's inequality to (4.53), we obtain after using (4.50) and (4.55) that

$$\sup_{0\le t\le T}\|\Delta\rho\|_{L^q}\le C,$$

which combined with (4.17), (4.48), (4.51), (4.55), and (4.50) gives (4.46). We finish the proof of Lemma 4.4.

Lemma 4.5. *For $q\in(3,6)$ as in Theorem 1.1, the following estimate holds*

$$\begin{aligned}&\sup_{0\le t\le T}\sigma\left(\|\theta_t\|_{H^1}+\|\nabla^3\theta\|_{L^2}+\|u_t\|_{H^2}+\|u\|_{W^{3,q}}\right)\\ &\quad+\int_0^T\sigma^2\|\nabla u_{tt}\|_{L^2}^2\mathrm{d}t\le C.\end{aligned}\tag{4.56}$$

Proof. First, multiplying (2.12) by u_{tt} and integrating the resulting equality over $\mathbb{R}^3$, one gets after integration by parts that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u_{tt}|^2\mathrm{d}x+\int\left(\mu|\nabla u_{tt}|^2+(\mu+\lambda)(\mathrm{div}u_{tt})^2\right)\mathrm{d}x\\ &=-4\int u_{tt}^i\rho u\cdot\nabla u_{tt}^i\mathrm{d}x-\int(\rho u)_t\cdot[\nabla(u_t\cdot u_{tt})+2\nabla u_t\cdot u_{tt}]\,\mathrm{d}x\\ &\quad-\int(\rho_{tt}u+2\rho_t u_t)\cdot\nabla u\cdot u_{tt}\mathrm{d}x-\int\rho u_{tt}\cdot\nabla u\cdot u_{tt}\mathrm{d}x\\ &\quad+\int P_{tt}\mathrm{div}u_{tt}\mathrm{d}x\triangleq\sum_{i=1}^5 J_i.\end{aligned}\tag{4.57}$$

Hölder's inequality and (4.17) give

$$\begin{aligned}|J_1|&\le C\|\rho^{1/2}u_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}\|u\|_{L^\infty}\\ &\le\frac{\mu}{8}\|\nabla u_{tt}\|_{L^2}^2+C\|\rho^{1/2}u_{tt}\|_{L^2}^2.\end{aligned}\tag{4.58}$$

It follows from (4.24), (4.17), (4.33) and (4.46) that

$$
\begin{aligned}
|J_2| &\le C\left(\|\rho u_t\|_{L^3}+\|\rho_t u\|_{L^3}\right)\left(\|\nabla u_{tt}\|_{L^2}\|u_t\|_{L^6}+\|u_{tt}\|_{L^6}\|\nabla u_t\|_{L^2}\right)\\
&\le C\left(\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}+\|\rho_t\|_{L^6}\|u\|_{L^6}\right)\|\nabla u_{tt}\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le \frac{\mu}{8}\|\nabla u_{tt}\|_{L^2}^2+C\|\nabla u_t\|_{L^2}^3+C\\
&\le \frac{\mu}{8}\|\nabla u_{tt}\|_{L^2}^2+C\sigma^{-3/2},
\end{aligned}
\tag{4.59}
$$

$$
\begin{aligned}
|J_3| &\le C\left(\|\rho_{tt}\|_{L^2}\|u\|_{L^6}+\|\rho_t\|_{L^2}\|u_t\|_{L^6}\right)\|\nabla u\|_{L^6}\|u_{tt}\|_{L^6}\\
&\le \frac{\mu}{8}\|\nabla u_{tt}\|_{L^2}^2+C\|\rho_{tt}\|_{L^2}^2+C\|\nabla u_t\|_{L^2}^2,
\end{aligned}
\tag{4.60}
$$

and

$$
\begin{aligned}
|J_4|+|J_5| &\le C\|\rho u_{tt}\|_{L^2}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6}\\
&\quad +C\|(\rho_t\theta+\rho\theta_t)_t\|_{L^2}\|\nabla u_{tt}\|_{L^2}\\
&\le \frac{\mu}{8}\|\nabla u_{tt}\|_{L^2}^2+C\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\|\rho_{tt}\theta\|_{L^2}^2\\
&\quad +C\|\rho_t\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_{tt}\|_{L^2}^2\\
&\le \frac{\mu}{8}\|\nabla u_{tt}\|_{L^2}^2+C\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\|\rho_{tt}\|_{L^2}^2\\
&\quad +C\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_{tt}\|_{L^2}^2.
\end{aligned}
\tag{4.61}
$$

Substituting (4.58)–(4.61) into (4.57) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u_{tt}|^2\mathrm{d}x+\mu\int|\nabla u_{tt}|^2\mathrm{d}x\\
&\quad\le C\sigma^{-3/2}+C\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\|\rho_{tt}\|_{L^2}^2+C\|\nabla\theta_t\|_{L^2}^2+C_3\|\sqrt{\rho}\theta_{tt}\|_{L^2}^2.
\end{aligned}
\tag{4.62}
$$

Then, to estimate the last term on the righthand side of (4.62), we multiply (2.9) by θ_{tt} and integrate the resulting equality over $\mathbb{R}^3$ to get

$$
\begin{aligned}
&\left(\frac{\kappa(\gamma-1)}{2R}\|\nabla\theta_t\|_{L^2}^2+H_0\right)_t+\int\rho\theta_{tt}^2\mathrm{d}x\\
&=\frac{1}{2}\int\rho_{tt}\left(\theta_t^2+2\left(u\cdot\nabla\theta+(\gamma-1)\theta\mathrm{div}u\right)\theta_t\right)\mathrm{d}x\\
&\quad+\int\rho_t\left(u\cdot\nabla\theta+(\gamma-1)\theta\mathrm{div}u\right)_t\theta_t\mathrm{d}x\\
&\quad-\int\rho\left(u\cdot\nabla\theta+(\gamma-1)\theta\mathrm{div}u\right)_t\theta_{tt}\mathrm{d}x\\
&\quad-\frac{\gamma-1}{R}\int\left(\lambda(\mathrm{div}u)^2+2\mu|\mathfrak{D}(u)|^2\right)_{tt}\theta_t\mathrm{d}x\triangleq\sum_{i=1}^4 H_i,
\end{aligned}
\tag{4.63}
$$

where

$$H_0 \triangleq \frac{1}{2}\int \rho_t\theta_t^2 dx + \int \rho_t \left(u\cdot\nabla\theta + (\gamma-1)\theta \mathrm{div} u\right)\theta_t dx - \frac{\gamma-1}{R}\int \left(\lambda(\mathrm{div} u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)_t \theta_t dx$$

satisfies

$$\begin{aligned} |H_0| \le& C\int \rho|u||\theta_t||\nabla\theta_t|dx + C\|\rho_t\|_{L^3}\|\theta_t\|_{L^6}\left(\|\nabla\theta\|_{L^2} + \|\nabla u\|_{L^2}\right) \\ &+ C\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2}\|\theta_t\|_{L^6} \\ \le& C\|\rho\theta_t\|_{L^2}\|u\|_{L^\infty}\|\nabla\theta_t\|_{L^2} + C\|\nabla\theta_t\|_{L^2} + C\|\nabla\theta_t\|_{L^2}\|\nabla u_t\|_{L^2} \\ \le& \frac{\kappa(\gamma-1)}{4R}\|\nabla\theta_t\|_{L^2}^2 + C\sigma^{-1}, \end{aligned} \tag{4.64}$$

due to $(1.6)_1$, (4.17), (4.24) and (4.33). Note that (4.24) and (2.23) yield

$$\|\theta_t\|_{L^2} \le C + C\|\nabla\theta_t\|_{L^2}, \tag{4.65}$$

which, as well as (4.17), gives

$$\begin{aligned} |H_1| &\le C\|\rho_{tt}\|_{L^2}\left(\|\theta_t\|_{L^2}^{1/2}\|\theta_t\|_{L^6}^{3/2} + \|\theta_t\|_{L^6}\left(\|u\cdot\nabla\theta\|_{L^3} + \|\theta\mathrm{div} u\|_{L^3}\right)\right) \\ &\le C\|\nabla\theta_t\|_{L^2}^4 + C\|\rho_{tt}\|_{L^2}^2 + C. \end{aligned} \tag{4.66}$$

It follows from (4.17) that

$$\begin{aligned} &\|\left(u\cdot\nabla\theta + (\gamma-1)\theta\mathrm{div} u\right)_t\|_{L^2} \\ &\le C\left(\|u_t\|_{L^6}\|\nabla\theta\|_{L^3} + \|u\|_{L^\infty}\|\nabla\theta_t\|_{L^2}\right) \\ &\quad + C\left(\|\theta_t\|_{L^6}\|\nabla u\|_{L^3} + \|\theta\|_{L^\infty}\|\nabla u_t\|_{L^2}\right) \\ &\le C\|\nabla\theta_t\|_{L^2} + C\|\nabla u_t\|_{L^2}, \end{aligned} \tag{4.67}$$

which together with (4.17) shows

$$\begin{aligned} |H_2| + |H_3| &\le C(\|\nabla\theta_t\|_{L^2} + \|\nabla u_t\|_{L^2})\left(\|\rho_t\|_{L^3}\|\theta_t\|_{L^6} + \|\rho\theta_{tt}\|_{L^2}\right) \\ &\le \frac{1}{2}\int \rho\theta_{tt}^2 dx + C\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2. \end{aligned} \tag{4.68}$$

One deduces from (4.17) and (4.33) that

$$\begin{aligned} |H_4| &\le C\int \left(|\nabla u_t|^2 + |\nabla u||\nabla u_{tt}|\right)|\theta_t| dx \\ &\le C\left(\|\nabla u_t\|_{L^2}^{3/2}\|\nabla u_t\|_{L^6}^{1/2} + \|\nabla u\|_{L^3}\|\nabla u_{tt}\|_{L^2}\right)\|\theta_t\|_{L^6} \\ &\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C\|\nabla^2 u_t\|_{L^2}^2 + C(\delta)\|\nabla\theta_t\|_{L^2}^2 + C\sigma^{-2}\|\nabla u_t\|_{L^2}^2. \end{aligned} \tag{4.69}$$

Substituting (4.66), (4.68), and (4.69) into (4.63) gives

$$\begin{aligned}&\left(\frac{\kappa(\gamma-1)}{2R}\|\nabla\theta_t\|_{L^2}^2+H_0\right)_t+\frac{1}{2}\int\rho\theta_{tt}^2\mathrm{d}x\\&\le\delta\|\nabla u_{tt}\|_{L^2}^2+C(\delta)\|\nabla\theta_t\|_{L^2}^4+C\|\nabla^2u_t\|_{L^2}^2+C\sigma^{-2}\|\nabla u_t\|_{L^2}^2\\&\quad+C\|\rho_{tt}\|_{L^2}^2+C.\end{aligned}\tag{4.70}$$

Finally, for C_3 as in (4.62), adding (4.70) multiplied by $2(C_3+1)$ to (4.62), we obtain after choosing δ suitably small that

$$\begin{aligned}&\left(2(C_3+1)\left(\frac{\kappa(\gamma-1)}{2R}\|\nabla\theta_t\|_{L^2}^2+H_0\right)+\int\rho|u_{tt}|^2\mathrm{d}x\right)_t\\&\quad+\int\rho\theta_{tt}^2\mathrm{d}x+\frac{\mu}{2}\int|\nabla u_{tt}|^2\mathrm{d}x\\&\le C\sigma^{-3/2}+C\|\nabla\theta_t\|_{L^2}^4+C\|\nabla^2u_t\|_{L^2}^2+C\sigma^{-2}\|\nabla u_t\|_{L^2}^2+C\|\rho_{tt}\|_{L^2}^2\\&\quad+C\|\rho^{1/2}u_{tt}\|_{L^2}^2.\end{aligned}\tag{4.71}$$

Multiplying (4.71) by σ^2 and integrating the resulting inequality over $(0,T)$, we obtain by using (4.64), (4.46), (4.33), (4.24), and Grönwall's inequality that

$$\sup_{0\le t\le T}\sigma^2\int\left(|\nabla\theta_t|^2+\rho|u_{tt}|^2\right)\mathrm{d}x+\int_0^T\sigma^2\int\left(\rho\theta_{tt}^2+|\nabla u_{tt}|^2\right)\mathrm{d}x\mathrm{d}t\le C,\tag{4.72}$$

which together with (4.41), (4.33), (4.49), (4.48), (4.51), and (4.54) gives

$$\sup_{0\le t\le T}\sigma\left(\|\nabla u_t\|_{H^1}+\|\nabla^3\theta\|_{L^2}+\|\nabla^2u\|_{W^{1,q}}\right)\le C.\tag{4.73}$$

We thus derive (4.56) from (4.72), (4.65), (4.73), and (4.46). The proof of Lemma 4.5 is completed.

Lemma 4.6. *The following estimate holds*

$$\sup_{0\le t\le T}\sigma^2\left(\|\nabla^2\theta\|_{H^2}+\|\theta_t\|_{H^2}\right)+\int_0^T\sigma^4\|\nabla\theta_{tt}\|_{L^2}^2\mathrm{d}t\le C.\tag{4.74}$$

Proof. First, multiplying (2.16) by θ_{tt} and integrating the resulting equality over $\mathbb{R}^3$ yield that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho(\theta_{tt})^2\mathrm{d}x+\frac{\kappa(\gamma-1)}{R}\int|\nabla\theta_{tt}|^2\mathrm{d}x\\&=-4\int\theta_{tt}\rho u\cdot\nabla\theta_{tt}\mathrm{d}x-\int\rho_{tt}\left(\theta_t+u\cdot\nabla\theta+(\gamma-1)\theta\mathrm{div}u\right)\theta_{tt}\mathrm{d}x\\&\quad-2\int\rho_t\left(u\cdot\nabla\theta+(\gamma-1)\theta\mathrm{div}u\right)_t\theta_{tt}\mathrm{d}x\\&\quad-\int\rho\left(u_{tt}\cdot\nabla\theta+2u_t\cdot\nabla\theta_t+(\gamma-1)(\theta\mathrm{div}u)_{tt}\right)\theta_{tt}\mathrm{d}x\\&\quad+\frac{\gamma-1}{R}\int\left(\lambda(\mathrm{div}u)^2+2\mu|\mathfrak{D}(u)|^2\right)_{tt}\theta_{tt}\mathrm{d}x\triangleq\sum_{i=1}^5K_i.\end{aligned}\tag{4.75}$$

Then, Hölder's inequality and (4.17) give

$$\begin{aligned}\sigma^4|K_1| &\le C\sigma^4\|\rho^{1/2}\theta_{tt}\|_{L^2}\|\nabla\theta_{tt}\|_{L^2}\|u\|_{L^\infty}\\ &\le \delta\sigma^4\|\nabla\theta_{tt}\|_{L^2}^2 + C(\delta)\sigma^4\|\rho^{1/2}\theta_{tt}\|_{L^2}^2.\end{aligned} \tag{4.76}$$

It follows from (4.33), (4.56), and (4.17) that

$$\begin{aligned}\sigma^4|K_2| &\le C\sigma^4\|\rho_{tt}\|_{L^2}\|\theta_{tt}\|_{L^6}\left(\|\theta_t\|_{H^1} + \|\nabla\theta\|_{L^6}\|u\|_{L^6} + \|\nabla u\|_{L^3}\|\theta\|_{L^\infty}\right)\\ &\le C\sigma^2\|\nabla\theta_{tt}\|_{L^2}\\ &\le C\delta\sigma^4\|\nabla\theta_{tt}\|_{L^2}^2 + C(\delta),\end{aligned} \tag{4.77}$$

$$\begin{aligned}\sigma^4|K_4| &\le C\sigma^4\|\theta_{tt}\|_{L^6}\left(\|\nabla\theta\|_{L^3}\|\rho u_{tt}\|_{L^2} + \|\nabla\theta_t\|_{L^2}\|\rho u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}\right)\\ &\quad + C\sigma^4\|\theta_{tt}\|_{L^6}\left(\|\nabla u\|_{L^3}\|\rho\theta_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2}\|\rho\theta_t\|_{L^2}^{1/2}\|\theta_t\|_{L^6}^{1/2}\right)\\ &\quad + C\sigma^4\|\theta\|_{L^\infty}\|\rho\theta_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}\\ &\le \delta\sigma^4\|\nabla\theta_{tt}\|_{L^2}^2 + C(\delta)\sigma^4\left(\|\rho\theta_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2\right) + C(\delta),\end{aligned} \tag{4.78}$$

and

$$\begin{aligned}\sigma^4|K_5| &\le C\sigma^4\|\theta_{tt}\|_{L^6}\left(\|\nabla u_t\|_{L^2}^{3/2}\|\nabla u_t\|_{L^6}^{1/2} + \|\nabla u\|_{L^3}\|\nabla u_{tt}\|_{L^2}\right)\\ &\le \delta\sigma^4\|\nabla\theta_{tt}\|_{L^2}^2 + C(\delta)\sigma^4\|\nabla u_{tt}\|_{L^2}^2 + C(\delta).\end{aligned} \tag{4.79}$$

For K_3, one deduces from (4.67), (4.56), and (4.17) that

$$\begin{aligned}\sigma^4|K_3| &\le C\sigma^4\|\rho_t\|_{L^3}\|\theta_{tt}\|_{L^6}\left(\|\nabla u_t\|_{L^2} + \|\nabla\theta_t\|_{L^2}\right)\\ &\le C\delta\sigma^4\|\nabla\theta_{tt}\|_{L^2}^2 + C(\delta).\end{aligned} \tag{4.80}$$

Then, multiplying (4.75) by σ^4, substituting (4.76)–(4.80) into the resulting equality and choosing δ suitably small lead to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\sigma^4\rho(\theta_{tt})^2\mathrm{d}x + \frac{\kappa(\gamma-1)}{R}\sigma^4\int|\nabla\theta_{tt}|^2\mathrm{d}x\\ &\quad\le C\sigma^2\left(\|\rho^{1/2}\theta_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2\right) + C,\end{aligned}$$

which together with (4.72) gives

$$\sup_{0\le t\le T}\sigma^4\int\rho|\theta_{tt}|^2\mathrm{d}x + \int_0^T\sigma^4\int|\nabla\theta_{tt}|^2\mathrm{d}x\mathrm{d}t \le C. \tag{4.81}$$

Finally, applying the standard L^2-estimate to (2.9), by (4.67), (4.17), (4.81), and (4.56), we get

$$\begin{aligned}&\sup_{0\le t\le T}\sigma^2\|\nabla^2\theta_t\|_{L^2}\\ &\le C\sup_{0\le t\le T}\sigma^2\left(\|\rho\theta_{tt}\|_{L^2} + \|\rho_t\|_{L^3}\|\theta_t\|_{L^6} + \|\rho_t\|_{L^6}\left(\|\nabla\theta\|_{L^3} + \|\nabla u\|_{L^3}\right)\right)\\ &\quad + C\sup_{0\le t\le T}\sigma^2\left(\|\nabla\theta_t\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla u_t\|_{L^6}\right) + C\\ &\le C.\end{aligned} \tag{4.82}$$

Moreover, it follows from the standard H^2-estimate of (3.39) that

$$\begin{aligned}\|\nabla^2\theta\|_{H^2} &\le C\left(\|\rho\theta_t\|_{H^2}+\|\rho u\cdot\nabla\theta\|_{H^2}+\|\rho\theta\mathrm{div}u\|_{H^2}+\||\nabla u|^2\|_{H^2}\right)\\ &\le C\left(\left(\|\rho-1\|_{H^2}+1\right)\|\theta_t\|_{H^2}+\left(\|\rho-1\|_{H^2}+1\right)\|u\|_{H^2}\|\nabla\theta\|_{H^2}\right)\\ &\quad +C\left(\|\rho\theta-1\|_{H^2}+1\right)\|\mathrm{div}u\|_{H^2}+C\|\nabla u\|_{H^2}^2\\ &\le C+C\|\nabla^3u\|_{L^2}+C\|\nabla^3\theta\|_{L^2}+C\|\theta_t\|_{H^2},\end{aligned}$$

due to (2.22) and (4.17). Combining this with (4.46), (4.56), (4.82), and (4.81) shows (4.74). The proof of Lemma 4.6 is completed. □

5. Proofs of Theorems 1.1 and 1.2

With all the a priori estimates in Sects. 3 and 4 at hand, we are ready to prove the main results of this paper in this section.

Proposition 5.1. *For given numbers $M>0$ (not necessarily small), $\bar\rho>2$, and $\bar\theta>1$, assume that (ρ_0,u_0,θ_0) satisfies (2.1), (3.5), and (3.8). Then there exists a unique classical solution (ρ,u,θ) of (1.6) (1.4) (1.7) in $\mathbb{R}^3\times(0,\infty)$ satisfying (2.3)–(2.5) with T_0 replaced by any $T\in(0,\infty)$. Moreover, (3.9), (3.6), and (3.98) hold for any $T\in(0,\infty)$.*

Proof. First, the standard local existence result (Lemma 2.1) shows that the Cauchy problem (1.6) (1.4) (1.7) with initial data (ρ_0,u_0,θ_0) has a unique local solution (ρ,u,θ), defined up to a positive T_0 which may depend on $\inf\limits_{x\in\mathbb{R}^3}\rho_0(x)$, and satisfying (2.3)–(2.5), and $\inf\limits_{x\in\mathbb{R}^3}\rho_0(x)/4\le\rho\le 2\overline{\rho}$. One deduces from (3.1)–(3.5) that

$$\begin{aligned}&A_1(0)\le M,\quad A_2(0)\le C_0\le C_0^{1/4},\\ &A_3(0)=A_4(0)=0,\quad \rho_0<\bar\rho,\quad \theta_0\le\bar\theta.\end{aligned}$$

Then there exists a $T_1\in(0,T_0]$ such that (3.6) holds for $T=T_1$. We set

$$T^*=\sup\left\{T\;\middle|\;\sup_{t\in[0,T]}\|(\rho-1,u,\theta-1)\|_{H^3}<\infty\right\},$$

and

$$T_*=\sup\{T\le T^*\,|\,(3.6)\text{ holds}\}. \tag{5.1}$$

Then $T^*\ge T_*\ge T_1>0$. We claim that

$$T_*=\infty. \tag{5.2}$$

Otherwise, $T_*<\infty$. Proposition 3.1 implies that (3.7) holds for all $0<T<T_*$, which together with (3.8) yields Lemmas 4.1–4.6 still hold for all $0<T<T_*$. Note here that all constants C in Lemmas 4.1–4.6 depend on T_* and $\inf\limits_{x\in\mathbb{R}^3}\rho_0(x)$, and

are in fact independent of T. Then, we claim that there exists a positive constant $\tilde{C}$ which may depend on T_* and $\inf_{x\in\mathbb{R}^3} \rho_0(x)$ such that, for all $0 < T < T_*$,

$$\sup_{0\le t\le T} \|\rho - 1\|_{H^3} \le \tilde{C}, \tag{5.3}$$

which together with Lemmas 4.4–4.6 and (3.5) gives

$$\|(\rho(x,T_*) - 1, u(x,T_*), \theta(x,T_*) - 1)\|_{H^3} \le \tilde{C}, \quad \inf_{x\in\mathbb{R}^3} \rho(x,T_*) > 0.$$

Lemma 2.1 thus implies that there exists some $T^{**} > T_*$, such that (3.6) holds for $T = T^{**}$, which contradicts (5.1). Hence, we obtain (5.2) which together with Lemma 2.1 finishes the proof of Proposition 5.1.

Finally, it remains to prove (5.3). It follows from (3.5), $(1.6)_2$, and (2.2) that we can define

$$u_t(\cdot,0) \triangleq -u_0\cdot\nabla u_0 + \rho_0^{-1}\left(\mu\Delta u_0 + (\mu+\lambda)\nabla\mathrm{div}u_0 - R\nabla(\rho_0\theta_0)\right),$$

which together with (2.1) gives

$$\|\nabla u_t(\cdot,0)\|_{L^2} \le \tilde{C}. \tag{5.4}$$

It thus follows from (4.43), (5.4), (4.35), (4.45), (4.17), (4.5), and Grönwall's inequality that

$$\sup_{0\le t\le T} \|\nabla u_t\|_{L^2} + \int_0^T\int \rho|u_{tt}|^2\mathrm{d}x\mathrm{d}t \le \tilde{C}, \tag{5.5}$$

which as well as (4.19) and (4.17) yields

$$\sup_{0\le t\le T} \|u\|_{H^3} \le \tilde{C}. \tag{5.6}$$

Combining this with (4.49), (4.41), (5.5) and (4.17) gives

$$\int_0^T \left(\|\nabla^3\theta\|_{L^2}^2 + \|\nabla u_t\|_{H^1}^2\right)\mathrm{d}t \le \tilde{C}. \tag{5.7}$$

Applying the H^2-estimate to elliptic systems (2.24) leads to

$$\begin{aligned} \|\nabla^2 u\|_{H^2} &\le \tilde{C}\|\nabla\mathrm{div}u\|_{H^2} + \tilde{C}\|\nabla\omega\|_{H^2} \\ &\le \tilde{C}\|\rho\dot{u}\|_{H^2} + \tilde{C}\|\nabla P\|_{H^2} \\ &\le \tilde{C} + \tilde{C}\|\nabla^2 u_t\|_{L^2} + \tilde{C}\|\nabla^3\rho\|_{L^2} + \tilde{C}\|\nabla^3\theta\|_{L^2}, \end{aligned} \tag{5.8}$$

where one has used (4.17) and the following simple facts:

$$\begin{aligned} \|\rho u_t\|_{H^2} &\le \tilde{C}\|(\rho-1)u_t\|_{H^2} + \tilde{C}\|u_t\|_{H^2} \\ &\le \tilde{C}\|\rho-1\|_{H^2}\|u_t\|_{H^2} + \tilde{C}\|\nabla^2 u_t\|_{L^2} + \tilde{C} \\ &\le \tilde{C} + \tilde{C}\|\nabla^2 u_t\|_{L^2}, \end{aligned}$$

$$\begin{aligned}\|\rho u\cdot\nabla u\|_{H^2}&\le\tilde C\left(\|(\rho-1)u\|_{H^2}+\|u\|_{H^2}\right)\|\nabla u\|_{H^2}\\&\le\tilde C\|\rho-1\|_{H^2}\|u\|_{H^2}+\tilde C\\&\le\tilde C,\end{aligned}$$

and

$$\begin{aligned}\|\nabla^3(\rho\theta)\|_{L^2}\le&\tilde C\|\nabla^3\rho\|_{L^2}\|\theta\|_{L^\infty}+\tilde C\|\nabla^2\rho\|_{L^6}\|\nabla\theta\|_{L^3}\\&+\tilde C\|\nabla\rho\|_{L^3}\|\nabla^2\theta\|_{L^6}+\tilde C\|\nabla^3\theta\|_{L^2}\\\le&\tilde C\|\nabla^3\rho\|_{L^2}+\tilde C\|\nabla^3\theta\|_{L^2},\end{aligned}$$

due to (2.22), (4.17), (5.5), (4.33), and (5.6). Then, standard calculations lead to

$$\begin{aligned}&\left(\|\nabla^3\rho\|_{L^2}\right)_t\\&\le\tilde C\left(\||\nabla^3u||\nabla\rho|\|_{L^2}+\||\nabla^2u||\nabla^2\rho|\|_{L^2}+\||\nabla u||\nabla^3\rho|\|_{L^2}+\|\nabla^4u\|_{L^2}\right)\\&\le\tilde C\left(\|\nabla u\|_{H^2}\|\nabla\rho\|_{H^2}+\|\nabla^2u\|_{L^3}\|\nabla^2\rho\|_{L^6}\right)\\&\quad+\tilde C\left(1+\|\nabla^2u_t\|_{L^2}+\|\nabla^3\rho\|_{L^2}+\|\nabla^3\theta\|_{L^2}\right)\\&\le\tilde C+\tilde C\|\nabla^3\rho\|_{L^2}+\tilde C\|\nabla^2u_t\|_{L^2}^2+\tilde C\|\nabla^3\theta\|_{L^2}^2,\end{aligned}$$

where we have used (4.17), (5.6), and (5.8). Combining this with (5.7) and Grönwall's inequality yields

$$\sup_{0\le t\le T}\|\nabla^3\rho\|_{L^2}\le\tilde C,$$

which together with (4.17) gives (5.3). The proof of Proposition 5.1 is completed. □

With Proposition 5.1 at hand, we are now in a position to prove our main results, Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let (ρ_0,u_0,θ_0) satisfying (1.9)–(1.12) be initial data as described in Theorem 1.1. Assume that C_0 satisfies (1.13), where

$$\varepsilon\triangleq\varepsilon_0/2,\tag{5.9}$$

with ε_0 as in Proposition 3.1. For constants

$$\delta,\eta\in(0,\min\{1,\bar\rho-\sup_{x\in\mathbb{R}^3}\rho_0(x)\}),\tag{5.10}$$

we define

$$\rho_0^{\delta,\eta}\triangleq\frac{j_\delta*\rho_0+\eta}{1+\eta},\quad u_0^{\delta,\eta}\triangleq j_\delta*u_0,\quad\theta_0^{\delta,\eta}\triangleq\frac{j_\delta*\theta_0+\eta}{1+\eta},$$

where j_δ is the standard mollifying kernel of width δ. Then, $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$ satisfies

$$\begin{cases}(\rho_0^{\delta,\eta}-1, u_0^{\delta,\eta}, \theta_0^{\delta,\eta}-1)\in H^\infty,\\ \dfrac{\eta}{1+\eta}\le \rho_0^{\delta,\eta}\le \dfrac{\bar\rho+\eta}{1+\eta}<\bar\rho,\ \dfrac{\eta}{1+\eta}\le\theta_0^{\delta,\eta}\le\bar\theta,\ \|\nabla u_0^{\delta,\eta}\|_{L^2}\le M,\end{cases}\tag{5.11}$$

and

$$\begin{cases}\lim\limits_{\delta+\eta\to 0}\left(\|\rho_0^{\delta,\eta}-\rho_0\|_{H^2\cap W^{2,q}}+\|u_0^{\delta,\eta}-u_0\|_{H^2}+\|\theta_0^{\delta,\eta}-\theta_0\|_{H^2}\right)=0,\\ \|\nabla(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})\|_{H^1}\le\|\nabla(\rho_0,u_0,\theta_0)\|_{H^1},\\ \|\nabla\rho_0^{\delta,\eta}\|_{W^{1,q}}\le\|\nabla\rho_0\|_{W^{1,q}},\end{cases}\tag{5.12}$$

due to (1.9) and (1.10). Moreover, the initial norm $C_0^{\delta,\eta}$ for $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$, i.e., the right hand side of (1.8) with (ρ_0,u_0,θ_0) replaced by $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$, satisfies

$$\lim_{\eta\to0}\lim_{\delta\to0}C_0^{\delta,\eta}=C_0.$$

Therefore, there exists an $\eta_0\in(0,\min\{1,\bar\rho-\sup\limits_{x\in\mathbb{R}^3}\rho_0(x)\})$ such that, for any $\eta\in(0,\eta_0)$, we can find some $\delta_0(\eta)>0$ such that

$$C_0^{\delta,\eta}\le C_0+\varepsilon_0/2\le\varepsilon_0,\tag{5.13}$$

provided that

$$0<\eta\le\eta_0,\quad 0<\delta\le\delta_0(\eta).\tag{5.14}$$

We assume that δ,η satisfy (5.14). Proposition 5.1 together with (5.13) and (5.11) thus yields that there exists a smooth solution $(\rho^{\delta,\eta}, u^{\delta,\eta}, \theta^{\delta,\eta})$ of (1.6) (1.4) (1.7) with initial data $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$ on $\mathbb{R}^3\times[0,T]$ for all $T>0$. Moreover, (3.9) and (3.6) both hold with (ρ,u,θ) being replaced by $(\rho^{\delta,\eta}, u^{\delta,\eta}, \theta^{\delta,\eta})$.

Next, for the initial data $(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}, \theta_0^{\delta,\eta})$, $\tilde g_1$ in (4.1) in fact is

$$\begin{aligned}\tilde g_1&\triangleq(\rho_0^{\delta,\eta})^{-1/2}\left(-\mu\Delta u_0^{\delta,\eta}-(\mu+\lambda)\nabla\mathrm{div}u_0^{\delta,\eta}+R\nabla(\rho_0^{\delta,\eta}\theta_0^{\delta,\eta})\right)\\&=(\rho_0^{\delta,\eta})^{-1/2}(j_\delta*\rho_0)^{1/2}g_1+(\rho_0^{\delta,\eta})^{-1/2}\left(j_\delta*(\sqrt{\rho_0}g_1)-\sqrt{j_\delta*\rho_0}g_1\right)\\&\quad+R(\rho_0^{\delta,\eta})^{-1/2}\nabla\left(j_\delta*(\rho_0\theta_0)-(1+\eta)^{-2}(j_\delta*\rho_0)(j_\delta*\theta_0)\right)\\&\quad+R\eta(1+\eta)^{-2}(\rho_0^{\delta,\eta})^{-1/2}\nabla(\rho_0^{\delta,\eta}+\theta_0^{\delta,\eta}),\end{aligned}\tag{5.15}$$

where in the second equality we have used (1.11). Similarly, $\tilde g_2$ in (4.2) is

$$\begin{aligned}\tilde g_2&\triangleq(\rho_0^{\delta,\eta})^{-\frac12}\left(\kappa\Delta\theta_0^{\delta,\eta}+\frac{\mu}{2}|\nabla u_0^{\delta,\eta}+(\nabla u_0^{\delta,\eta})^{\mathrm{tr}}|^2+\lambda(\mathrm{div}u_0^{\delta,\eta})^2\right)\\&=(\rho_0^{\delta,\eta})^{-\frac12}(j_\delta*\rho_0)^{\frac12}g_2+(\rho_0^{\delta,\eta})^{-\frac12}\left(j_\delta*(\sqrt{\rho_0}g_2)-\sqrt{j_\delta*\rho_0}g_2\right)\\&\quad-\frac{\mu}{2}(\rho_0^{\delta,\eta})^{-\frac12}\left(j_\delta*|\nabla u_0+(\nabla u_0)^{\mathrm{tr}}|^2-|\nabla(j_\delta*u_0)+(\nabla(j_\delta*u_0))^{\mathrm{tr}}|^2\right)\\&\quad-\lambda(\rho_0^{\delta,\eta})^{-\frac12}\left(j_\delta*\left((\mathrm{div}u_0)^2\right)-(\mathrm{div}(j_\delta*u_0))^2\right),\end{aligned}\tag{5.16}$$

due to (1.12). Since $g_1, g_2 \in L^2$, one deduces from (5.15), (5.16), (5.11), (5.12), and (1.9) that there exists some positive constant C independent of δ and η such that

$$\begin{cases} \|\tilde{g}_1\|_{L^2} \le (1+\eta)^{1/2}\|g_1\|_{L^2} + C\eta^{-1/2}m_1(\delta) + C\sqrt{\eta}, \\ \|\tilde{g}_2\|_{L^2} \le (1+\eta)^{1/2}\|g_2\|_{L^2} + C\eta^{-1/2}m_2(\delta), \end{cases} \tag{5.17}$$

with $0 \le m_i(\delta) \to 0$ $(i = 1, 2)$ as $\delta \to 0$. Hence, for any $0 < \eta < \eta_0$, there exists some $0 < \delta_1(\eta) \le \delta_0(\eta)$ such that

$$m_1(\delta) + m_2(\delta) < \eta, \tag{5.18}$$

for any $0 < \delta < \delta_1(\eta)$. We thus obtain from (5.17) and (5.18) that there exists some positive constant C independent of δ and η such that

$$\|\tilde{g}_1\|_{L^2} + \|\tilde{g}_2\|_{L^2} \le 2\|g_1\|_{L^2} + 2\|g_2\|_{L^2} + C, \tag{5.19}$$

provided that

$$0 < \eta < \eta_0, \quad 0 < \delta < \delta_1(\eta). \tag{5.20}$$

Now, we assume that η, δ satisfy (5.20). It thus follows from (5.13), Proposition 3.1, Corollary 3.9, (5.12), (5.19), and Lemmas 4.1–4.6 that for any $T > 0$, there exists some positive constant C independent of δ and η such that (3.9), (3.6), (3.98), (4.17), (4.18), (4.46), (4.56), and (4.74) hold for $(\rho^{\delta,\eta}, u^{\delta,\eta}, \theta^{\delta,\eta})$. Then passing to the limit first $\delta \to 0$, then $\eta \to 0$, together with standard arguments yields that there exists a solution (ρ, u, θ) of (1.6) (1.4) (1.7) on $\mathbb{R}^3 \times (0, T]$ for all $T > 0$, such that (ρ, u, θ) satisfies (3.9), (3.6), (3.98), (4.17), (4.18), (4.46), (4.56) and (4.74). Hence, (ρ, u, θ) satisfies (1.14), $(1.15)_2$, $(1.15)_3$, and

$$\rho - 1 \in L^\infty(0, T; H^2 \cap W^{2,q}), \quad (u, \theta - 1) \in L^\infty(0, T; H^2). \tag{5.21}$$

Moreover, (4.52) holds in $\mathcal{D}'(\mathbb{R}^3 \times (0, T))$.

Next, to finish the existence part of Theorem 1.1, it remains to prove

$$\rho - 1 \in C([0, T]; H^2 \cap W^{2,q}), \quad u, \ \theta - 1 \in C([0, T]; H^2). \tag{5.22}$$

Indeed, it follows from (4.17) and (5.21) that

$$\rho - 1 \in C([0, T]; H^1 \cap W^{1,\infty}) \cap C([0, T]; H^2 \cap W^{2,q} \text{ -weak}), \tag{5.23}$$

and for all $r \in [2, 6)$,

$$u, \ \theta - 1 \in C([0, T]; H^1 \cap W^{1,r}). \tag{5.24}$$

Since (4.52) holds in $\mathcal{D}'(\mathbb{R}^3 \times (0, T))$ for all $T \in (0, \infty)$, one derives from [13, Lemma 2.3] that, for $j_\nu(x)$ being the standard mollifying kernel of width ν, $\rho^\nu \triangleq \rho * j_\nu$ satisfies

$$(\Delta\rho^\nu)_t + \operatorname{div}(u\Delta\rho^\nu) = -\operatorname{div}(\rho\Delta u) * j_\nu - 2\operatorname{div}(\partial_i\rho \cdot \partial_i u) * j_\nu + R_\nu, \tag{5.25}$$

where R_ν satisfies

$$\int_0^T \|R_\nu\|_{L^2\cap L^q}^{3/2} \mathrm{d}t \le C \int_0^T \|u\|_{W^{1,\infty}}^{3/2} \|\Delta\rho\|_{L^2\cap L^q}^{3/2} \mathrm{d}t \le C, \tag{5.26}$$

due to (4.5), (4.17), and (4.46). Multiplying (5.25) by $q|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu$, we obtain after integration by parts that

$$\begin{aligned}
&(\|\Delta\rho^\nu\|_{L^q}^q)'(t) \\
&= (1-q)\int |\Delta\rho^\nu|^q \mathrm{div} u \mathrm{d}x - q\int (\mathrm{div}(\rho\Delta u) * j_\nu)|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu \mathrm{d}x \\
&\quad - 2q\int (\mathrm{div}(\partial_i\rho\cdot\partial_i u) * j_\nu)|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu \mathrm{d}x + q\int R_\nu|\Delta\rho^\nu|^{q-2}\Delta\rho^\nu \mathrm{d}x,
\end{aligned}$$

which together with (4.17), (4.46), and (5.26) yields that, for p_0 as in (4.47),

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\Delta\rho^\nu\|_{L^q} + \int_0^T |(\|\Delta\rho^\nu\|_{L^q}^q)'(t)|^{p_0} \mathrm{d}t \\
&\le C + C\int_0^T \left(\|\nabla u\|_{W^{2,q}}^{p_0} + \|R_\nu\|_{L^2\cap L^q}^{p_0}\right) \mathrm{d}t \le C.
\end{aligned}$$

This combined with the Ascoli-Arzela theorem thus leads to

$$\|\Delta\rho^\nu(\cdot,t)\|_{L^q} \to \|\Delta\rho(\cdot,t)\|_{L^q} \text{ in } C([0,T]), \text{ as } \nu\to 0^+.$$

In particular, we have

$$\|\nabla^2\rho(\cdot,t)\|_{L^q} \in C([0,T]). \tag{5.27}$$

Similarly, one can obtain

$$\|\nabla^2\rho(\cdot,t)\|_{L^2} \in C([0,T]),$$

which together with (5.23) and (5.27) shows

$$\nabla^2\rho \in C([0,T]; L^2\cap L^q). \tag{5.28}$$

To prove the second part of (5.22), it follows from (4.17) and (4.18) that

$$\rho u_t, \rho\theta_t \in C([0,T]; L^2), \tag{5.29}$$

which together with (4.12), (5.23), (5.24), and (5.28) gives

$$u \in C([0,T]; H^2). \tag{5.30}$$

Combining this with (3.39), (5.29), (5.28), (5.24), and (4.17) leads to

$$\theta - 1 \in C([0,T]; H^2),$$

which as well as (5.23), (5.28), and (5.30) gives (5.22).

Finally, since the proof of the uniqueness of (ρ, u, θ) is similar to that of [4, Theorem 1], to finish the proof of Theorem 1.1, it remains to prove (1.16). We will only show

$$\lim_{t\to\infty} \|\nabla u\|_{L^2} = 0, \tag{5.31}$$

since the other terms in (1.16) follow directly from (1.28). It follows from (3.98) and (3.6) that

$$\begin{aligned}
&\int_1^\infty |(\|\nabla u\|_{L^2}^2)'(t)|\mathrm{d}t \\
&= 2\int_1^\infty \left|\int \partial_j u^i \partial_j u_t^i \mathrm{d}x\right| \mathrm{d}t \\
&= 2\int_1^\infty \left|\int \partial_j u^i \partial_j (\dot{u}^i - u^k\partial_k u^i)\mathrm{d}x\right| \mathrm{d}t \\
&= \int_1^\infty \left|\int (2\partial_j u^i \partial_j \dot{u}^i - 2\partial_j u^i \partial_j u^k \partial_k u^i + |\nabla u|^2 \mathrm{div} u)\mathrm{d}x\right| \mathrm{d}t \\
&\le C\int_1^\infty \left(\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^3}^3\right) \mathrm{d}t \\
&\le C\int_1^\infty \left(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4\right) \mathrm{d}t \le C,
\end{aligned}$$

which together with (3.6) implies (5.31). We finish the proof of Theorem 1.1. □

Proof of Theorem 1.2. We will prove Theorem 1.2 in three steps.

Step 1: Construction of approximate solutions Let (ρ_0, u_0, θ_0) satisfying (1.10) be initial data as described in Theorem 1.2. Assume that C_0 satisfies (1.22) with ε as in (5.9). Let δ and η be as in (5.10) and j_δ be the standard mollifier. We define

$$\hat{\rho}_0^{\delta,\eta} \triangleq \frac{j_\delta * \rho_0 + \eta}{1+\eta}, \quad \hat{u}_0^{\delta,\eta} \triangleq j_\delta * u_0, \quad \hat{\theta}_0^{\delta,\eta} \triangleq \frac{j_\delta * (\rho_0\theta_0) + \eta}{j_\delta * \rho_0 + \eta}.$$

Then, $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$ satisfies

$$\begin{cases} (\hat{\rho}_0^{\delta,\eta} - 1, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta} - 1) \in H^\infty, \\ \dfrac{\eta}{1+\eta} \le \hat{\rho}_0^{\delta,\eta} \le \dfrac{\bar{\rho}+\eta}{1+\eta} < \bar{\rho}, \ \dfrac{\eta}{\bar{\rho}+\eta} \le \hat{\theta}_0^{\delta,\eta} \le \bar{\theta}, \ \|\nabla \hat{u}_0^{\delta,\eta}\|_{L^2} \le M, \end{cases} \tag{5.32}$$

due to (1.10). Moreover, it follows from (1.10) and (1.22) that

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\left(\|\hat{\rho}_0^{\delta,\eta} - \rho_0\|_{L^2} + \|\hat{u}_0^{\delta,\eta} - u_0\|_{H^1} + \|\hat{\rho}_0^{\delta,\eta}\hat{\theta}_0^{\delta,\eta} - \rho_0\theta_0\|_{L^2}\right) = 0. \tag{5.33}$$

We claim that the initial norm $\hat{C}_0^{\delta,\eta}$ for $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$, i.e., the right hand side of (1.8) with (ρ_0, u_0, θ_0) replaced by $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$, satisfies

$$\lim_{\eta\to 0}\lim_{\delta\to 0} \hat{C}_0^{\delta,\eta} \le C_0, \tag{5.34}$$

which yields that there exists an $\hat{\eta} > 0$ such that, for any $\eta \in (0, \hat{\eta})$, there exists some $\hat{\delta}(\eta) > 0$ such that

$$\hat{C}_0^{\delta,\eta} \le C_0 + \varepsilon_0/2 \le \varepsilon_0, \tag{5.35}$$

provided

$$0 < \eta \le \hat{\eta}, \quad 0 < \delta \le \hat{\delta}(\eta). \tag{5.36}$$

We assume that δ, η always satisfy (5.36). Proposition 5.1 as well as (5.32) and (5.35) thus yields that there exists a smooth solution $(\hat{\rho}^{\delta,\eta}, \hat{u}^{\delta,\eta}, \hat{\theta}^{\delta,\eta})$ of (1.6) (1.4) (1.7) with initial data $(\hat{\rho}_0^{\delta,\eta}, \hat{u}_0^{\delta,\eta}, \hat{\theta}_0^{\delta,\eta})$ on $\mathbb{R}^3 \times [0, T]$ for all $T > 0$. Moreover, for any $T > 0$, $(\hat{\rho}^{\delta,\eta}, \hat{u}^{\delta,\eta}, \hat{\theta}^{\delta,\eta})$ satisfies (3.9), (3.6), and (3.98) with (ρ, u, θ) replaced by $(\hat{\rho}^{\delta,\eta}, \hat{u}^{\delta,\eta}, \hat{\theta}^{\delta,\eta})$.

It remains to prove (5.34). In fact, we only have to show

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\int \hat{\rho}_0^{\delta,\eta}\left(\hat{\theta}_0^{\delta,\eta} - \log\hat{\theta}_0^{\delta,\eta} - 1\right)\mathrm{d}x \le \int \rho_0\left(\theta_0 - \log\theta_0 - 1\right)\mathrm{d}x, \tag{5.37}$$

since the other terms in (5.34) can be proved in a similar and even simpler way. Noticing that

$$\begin{aligned}
&\hat{\rho}_0^{\delta,\eta}\left(\hat{\theta}_0^{\delta,\eta} - \log\hat{\theta}_0^{\delta,\eta} - 1\right)\\
&\quad = \hat{\rho}_0^{\delta,\eta}(\hat{\theta}_0^{\delta,\eta} - 1)^2\int_0^1 \frac{\alpha}{\alpha(\hat{\theta}_0^{\delta,\eta} - 1) + 1}d\alpha\\
&\quad = \frac{(j_\delta * (\rho_0\theta_0 - \rho_0))^2}{1+\eta}\int_0^1 \frac{\alpha}{\alpha(j_\delta * (\rho_0\theta_0) - j_\delta * \rho_0) + j_\delta * \rho_0 + \eta}d\alpha\\
&\quad \in \left[0,\ \eta^{-1}(j_\delta * (\rho_0\theta_0 - \rho_0))^2\right],
\end{aligned}$$

we deduce from (5.33) and Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&\lim_{\delta\to 0}\int \hat{\rho}_0^{\delta,\eta}\left(\hat{\theta}_0^{\delta,\eta} - \log\hat{\theta}_0^{\delta,\eta} - 1\right)\mathrm{d}x\\
&\quad = \int \frac{\rho_0+\eta}{1+\eta}\left(\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - \log\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - 1\right)\mathrm{d}x\\
&\quad = \frac{1}{1+\eta}\int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)}\left(\rho_0\theta_0 - \rho_0 - (\rho_0+\eta)\log\frac{\rho_0\theta_0+\eta}{\rho_0+\eta}\right)\mathrm{d}x\\
&\qquad + \frac{1}{1+\eta}\int_{(1/2\le\rho_0\theta_0\le 2)}(\rho_0+\eta)\left(\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - \log\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - 1\right)\mathrm{d}x\\
&\quad \triangleq \frac{1}{1+\eta}(I_1 + I_2),
\end{aligned} \tag{5.38}$$

where we have used the following simple fact that, for $f \in L^p (1 \le p < \infty)$,

$$\lim_{\delta\to 0}\|j_\delta * f - f\|_{L^p} = 0, \quad \lim_{\delta\to 0} j_\delta * f(x) = f(x), \text{ almost everywhere} x \in \mathbb{R}^3.$$

It follows from (1.22) that

$$
\begin{aligned}
|(\rho_0\theta_0 < 1/2) \cup (\rho_0\theta_0 > 2)| &\le 4\int (\rho_0\theta_0 - 1)^2 \mathrm{d}x \\
&\le 8\int (\rho_0\theta_0 - \rho_0)^2 \mathrm{d}x + 8\int (\rho_0 - 1)^2 \mathrm{d}x \\
&\le C,
\end{aligned}
$$

which combined with Lebesgue's dominated convergence theorem yields

$$
\begin{aligned}
I_1 &= \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} (\rho_0\theta_0 - \rho_0 \log(\rho_0\theta_0+\eta) - \eta\log(\rho_0\theta_0+\eta))\,\mathrm{d}x \\
&\quad + \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} ((\rho_0+\eta)\log(\rho_0+\eta) - \rho_0)\,\mathrm{d}x \\
&\le \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} (\rho_0\theta_0 - \rho_0\log(\rho_0\theta_0) - \eta\log\eta)\,\mathrm{d}x \\
&\quad + \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} (\rho_0\log(\rho_0+\eta) + \eta\log(\rho_0+\eta) - \rho_0)\,\mathrm{d}x \\
&\to \int_{(\rho_0\theta_0<1/2)\cup(\rho_0\theta_0>2)} \rho_0\,(\theta_0 - \log\theta_0 - 1)\,\mathrm{d}x, \quad \text{as } \eta \to 0.
\end{aligned}
\tag{5.39}
$$

Noticing that

$$
\begin{aligned}
&(\rho_0+\eta)\left(\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - \log\frac{\rho_0\theta_0+\eta}{\rho_0+\eta} - 1\right) \\
&= (\rho_0\theta_0-\rho_0)^2 \int_0^1 \frac{\alpha}{\alpha(\rho_0\theta_0-\rho_0)+\rho_0+\eta} d\alpha \\
&\in \left[0, 2\,(\rho_0\theta_0-\rho_0)^2\right],
\end{aligned}
$$

provided $\rho_0\theta_0 \ge 1/2$, we deduce from Lebesgue's dominated convergence theorem that

$$
\lim_{\eta\to 0} I_2 = \int_{(1/2\le\rho_0\theta_0\le 2)} \rho_0\,(\theta_0 - \log\theta_0 - 1)\,\mathrm{d}x,
$$

which together with (5.38) and (5.39) gives (5.37). □

Step 2: Compactness results For the approximate solutions $(\hat\rho^{\delta,\eta}, \hat u^{\delta,\eta}, \hat\theta^{\delta,\eta})$ obtained in the previous step, we will pass to the limit first $\delta \to 0$, then $\eta \to 0$ and apply (3.6) and (3.98) to obtain the global existence of weak solutions. Since the two steps are similar, we will only sketch the arguments for $\delta \to 0$. Thus, we fix $\eta \in (0, \hat\eta)$ and simply denote $(\hat\rho^{\delta,\eta}, \hat u^{\delta,\eta}, \hat\theta^{\delta,\eta})$ by $(\rho^\delta, u^\delta, \theta^\delta)$. For $R \in (0,\infty)$, let $B_R(x_0) \triangleq \{x \in \mathbb{R}^3 \,|\, |x - x_0| < R\}$ denote a ball centered at $x_0 \in \mathbb{R}^3$ with radius R.

We claim that there exists some appropriate subsequence $\delta_j \to 0$ of $\delta \to 0$ such that, for any $0 < \tau < T < \infty$ and $0 < R < \infty$, we have

$$\begin{cases} \theta^{\delta_j} - 1 \rightharpoonup \theta - 1 \ \text{ weakly in } \ L^2(0,T;H^1(\mathbb{R}^3)), \\ u^{\delta_j} \rightharpoonup u \ \text{ weakly star in } \ L^\infty(0,T;H^1(\mathbb{R}^3)), \end{cases} \tag{5.40}$$

$$\begin{cases} \rho^{\delta_j} - 1 \to \rho - 1 \quad \text{in } \ C([0,T];L^2(\mathbb{R}^3)\text{-weak}), \\ \rho^{\delta_j} - 1 \to \rho - 1 \quad \text{in } \ C([0,T];H^{-1}(B_R(0))), \end{cases} \tag{5.41}$$

$$\begin{cases} \rho^{\delta_j} u^{\delta_j} \to \rho u, \ \rho^{\delta_j}(\theta^{\delta_j} - 1) \to \rho(\theta - 1) \ \text{ in } C([0,T];L^2(\mathbb{R}^3)\text{-weak}), \\ \rho^{\delta_j} u^{\delta_j} \to \rho u \ \text{ in } \ C([0,T];H^{-1}(B_R(0))), \end{cases} \tag{5.42}$$

$$\rho^{\delta_j}|u^{\delta_j}|^2 \to \rho|u|^2 \ \text{ in } \ C([0,T];L^3\text{-weak}), \tag{5.43}$$

and

$$\begin{cases} u^{\delta_j} \to u, \ G^{\delta_j} \to G, \ \omega^{\delta_j} \to \omega, \ \nabla\theta^{\delta_j} \to \nabla\theta \ \text{ in } C([\tau,T];H^1(\mathbb{R}^3)\text{-weak}), \\ u^{\delta_j} \to u, \ G^{\delta_j} \to G, \ \omega^{\delta_j} \to \omega, \ \nabla\theta^{\delta_j} \to \nabla\theta \ \text{ in } \ C([\tau,T];L^2(B_R(0))). \end{cases} \tag{5.44}$$

We thus write (1.1) in the weak forms for the approximate solutions $(\rho^\delta, u^\delta, \theta^\delta)$, then let $\delta = \delta_j$ and take appropriate limits. Standard arguments as well as (5.40)–(5.44) thus yield that the limit (ρ, u, θ) is a weak solution of (1.1) (1.4) (1.5) in the sense of Definition 1.1 and satisfies (1.23)–(1.27) except $\rho - 1 \in C([0,\infty), L^2)$ which in fact can be obtained by similar arguments leading to (5.28). In addition, the estimates (1.29)–(1.31) follows directly from (3.9), (3.98), (3.6), and (5.40)–(5.44).

It remains to prove (5.41)–(5.44) since (5.40) is a direct consequence of (3.6). It follows from (3.9), (3.6), and $(1.6)_1$ that

$$\sup_{t\in[0,\infty)} \|\rho_t^\delta\|_{H^{-1}(\mathbb{R}^3)} \le C,$$

which as well as (3.6), [13, Lemma C.1], and the Aubin-Lions lemma yields that there exists a subsequence of $\delta_j \to 0$, still denoted by δ_j, such that (5.41) holds. Moreover, one deduces from (3.98) that (extract a subsequence)

$$\rho^{\delta_j} - 1 \rightharpoonup \rho - 1, \ \nabla u^{\delta_j} \rightharpoonup \nabla u \ \text{ weakly in } \ L^4(\mathbb{R}^3 \times (1,\infty)),$$

with $\rho - 1$ and ∇u satisfying

$$\int_1^\infty \left(\|\rho - 1\|_{L^4}^4 + \|\nabla u\|_{L^4}^4 \right) \mathrm{d}t \le C. \tag{5.45}$$

Then, simple calculations together with (3.6) yield that, for any $0 < T < \infty$, there exists some $C(T)$ independent of δ and η such that

$$\|(\rho^\delta u^\delta)_t\|_{L^2(0,T;H^{-1}(\mathbb{R}^3))} + \|(\rho^\delta \theta^\delta)_t\|_{L^2(0,T;H^{-1}(\mathbb{R}^3))} \le C(T), \tag{5.46}$$

which together with (3.6), (5.41), and (5.40) gives (5.42).

Next, to prove (5.43), one deduces from (3.6) and $(1.6)_1$ that, for any $\zeta \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}
&\left|\int(\rho^\delta|u^\delta|^2)_t\zeta\,\mathrm{d}x\right|\\
&=\left|-\int \mathrm{div}(\rho^\delta u^\delta)|u^\delta|^2\zeta\,\mathrm{d}x+2\int\rho^\delta u^\delta\cdot u_t^\delta\zeta\,\mathrm{d}x\right|\\
&=\left|\int\rho^\delta u^\delta\cdot\nabla(|u^\delta|^2\zeta)\mathrm{d}x+2\int\rho^\delta u^\delta\cdot(\dot u^\delta-u^\delta\cdot\nabla u^\delta)\zeta\,\mathrm{d}x\right|\\
&\le C\int\rho^\delta|u^\delta|^3|\nabla\zeta|\mathrm{d}x+C\int\rho^\delta|u^\delta|^2|\nabla u^\delta||\zeta|\mathrm{d}x+C\int\rho^\delta|u^\delta||\dot u^\delta||\zeta|\mathrm{d}x\\
&\le C\|u^\delta\|_{L^6}^3\|\nabla\zeta\|_{L^2}+C\|u^\delta\|_{L^6}^2\|\nabla u^\delta\|_{L^2}\|\zeta\|_{L^6}\\
&\quad+C\|u^\delta\|_{L^6}\|(\rho^\delta)^{1/2}\dot u^\delta\|_{L^2}\|\zeta\|_{L^3}\\
&\le C\left(\|\nabla u^\delta\|_{L^2}+\|(\rho^\delta)^{1/2}\dot u^\delta\|_{L^2}\right)\|\zeta\|_{H^1},
\end{aligned}$$

which together with (3.6) gives

$$\int_0^\infty\|(\rho^\delta|u^\delta|^2)_t\|_{H^{-1}}^2\mathrm{d}t\le C. \tag{5.47}$$

It follows from (3.6) that

$$\sup_{t\in[0,\infty)}\|\rho^\delta|u^\delta|^2\|_{L^1\cap L^3}\le C,$$

which combined with (5.47), (5.40), and (5.42) yields (5.43).

Finally, we prove (5.44) which implies the strong limits of u^δ and θ^δ. We deduce from (3.6), (2.26), (5.46), and (3.98) that

$$\sup_{t\in[0,\infty)}(\|u^\delta\|_{H^1}+\sigma^2\|G^\delta\|_{H^1}+\sigma^2\|\omega^\delta\|_{H^1}+\sigma^2\|\nabla\theta^\delta\|_{H^1})\le C, \tag{5.48}$$

and

$$\int_0^T\sigma^4\left(\|u_t^\delta\|_{L^2(\mathbb{R}^3)}^2+\|G_t^\delta\|_{H^{-1}(\mathbb{R}^3)}^2+\|\omega_t^\delta\|_{H^{-1}(\mathbb{R}^3)}^2+\|\theta_t^\delta\|_{H^1(\mathbb{R}^3)}^2\right)\mathrm{d}t\le C. \tag{5.49}$$

The Aubin-Lions lemma together with (5.48) and (5.49) thus gives (5.44).

Step 3: Proofs of (1.32) *and* (1.28) We first prove that (ρ, u, θ) satisfies (1.32). We rewrite the energy equation $(1.6)_3$ in the form

$$\begin{aligned}
&\frac{R}{\gamma-1}\left((\rho\theta)_t+\mathrm{div}(\rho u\theta)\right)-\kappa\Delta\theta\\
&=G\mathrm{div}u-R\mathrm{div}u+2\mu\mathrm{div}(u\cdot\nabla u-u\mathrm{div}u)+\frac{\mu}{2}|\omega|^2.
\end{aligned} \tag{5.50}$$

Thus, for any $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (0,\infty))$, we have

$$\begin{aligned}
&\frac{R}{\gamma-1}\int_0^\infty\int \rho^\delta\theta^\delta\left(\varphi_t + u^\delta\cdot\nabla\varphi\right)\mathrm{d}x\mathrm{d}t - \kappa\int_0^\infty\int \nabla\theta^\delta\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_0^\infty\int G^\delta \mathrm{div}u^\delta\varphi\mathrm{d}x\mathrm{d}t + R\int_0^\infty\int \mathrm{div}u^\delta\varphi\mathrm{d}x\mathrm{d}t \qquad (5.51)\\
&\qquad + 2\mu\int_0^\infty\int\left(u^\delta\cdot\nabla u^\delta - u^\delta\mathrm{div}u^\delta\right)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t - \frac{\mu}{2}\int_0^\infty\int |\omega^\delta|^2\varphi\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Letting $\delta = \delta_j$ in (5.51) and taking appropriate limits, we thus deduce from (5.41), (5.40), (5.42), and (5.44) that

$$\begin{aligned}
&\frac{R}{\gamma-1}\int_0^\infty\int \rho\theta\left(\varphi_t + u\cdot\nabla\varphi\right)\mathrm{d}x\mathrm{d}t - \kappa\int_0^\infty\int \nabla\theta\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_0^\infty\int G\mathrm{div}u\varphi\mathrm{d}x\mathrm{d}t + R\int_0^\infty\int \mathrm{div}u\varphi\mathrm{d}x\mathrm{d}t\\
&\qquad + 2\mu\int_0^\infty\int (u\cdot\nabla u - u\mathrm{div}u)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t - \frac{\mu}{2}\int_0^\infty\int|\omega|^2\varphi\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_0^\infty\int(\lambda\mathrm{div}u - P)\mathrm{div}u\varphi\mathrm{d}x\mathrm{d}t - 2\mu\int_0^\infty\int|\mathfrak{D}(u)|^2\varphi\mathrm{d}x\mathrm{d}t \qquad (5.52)\\
&\qquad + 2\mu\int_0^\infty\int\left(\partial_k u^i\partial_i(u^k\varphi) - \mathrm{div}(u\varphi)\mathrm{div}u\right)\mathrm{d}x\mathrm{d}t\\
&\quad= \int_0^\infty\int P\mathrm{div}u\varphi\mathrm{d}x\mathrm{d}t - \int_0^\infty\int\left(\lambda(\mathrm{div}u)^2 + 2\mu|\mathfrak{D}(u)|^2\right)\varphi\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where in the last equality, we have used the following simple fact that, for standard mollifier $j_\nu(x)$,

$$\begin{aligned}
&\left|\int_0^\infty\int\left(\partial_k u^i\partial_i(u^k\varphi) - \mathrm{div}(u\varphi)\mathrm{div}u\right)\mathrm{d}x\mathrm{d}t\right|\\
&\quad= \left|\int_0^\infty\int \partial_k(u^i - u^i * j_\nu)\partial_i\left(u^k\varphi\right)\mathrm{d}x\mathrm{d}t\right.\\
&\qquad \left. + \int_0^\infty\int\left(\partial_k(u^i * j_\nu)\partial_i(u^k\varphi) - \mathrm{div}(u\varphi)\mathrm{div}u\right)\mathrm{d}x\mathrm{d}t\right|\\
&\quad= \left|\int_0^\infty\int\left(\partial_k(u^i - u^i * j_\nu)\partial_i(u^k\varphi) + \mathrm{div}(u\varphi)\mathrm{div}(u * j_\nu - u)\right)\mathrm{d}x\mathrm{d}t\right|\\
&\quad\le C\int_0^\infty\int|\nabla(u\varphi)||\nabla(u - u * j_\nu)|\mathrm{d}x\mathrm{d}t \to 0, \text{ as } \nu\to 0,
\end{aligned}$$

due to (1.30). We thus derive (1.32) directly from (5.52), (5.42), and (5.41).

Finally, to finish the proof of Theorem 1.2, it remains to prove (1.28). Since (ρ, u) satisfies (1.18), for the standard mollifier $j_\nu(x)(\nu > 0)$, $\rho^\nu \triangleq \rho * j_\nu$ satisfies

$$\begin{cases}\rho_t^\nu + \mathrm{div}(u\rho^\nu) = r_\nu,\\ \rho^\nu(x, t=0) = \rho_0 * j_\nu,\end{cases} \qquad (5.53)$$

where r_ν satisfies, for any $T > 0$,

$$\lim_{\nu\to 0^+} \int_0^T \|r_\nu\|_{L^2}^2 \mathrm{d}t = 0, \tag{5.54}$$

due to (3.9), (3.6), and [13, Lemma 2.3]. Multiplying (5.53) by $4(\rho^\nu - 1)^3$, we obtain after integration by parts that, for $t \geq 1$,

$$\begin{aligned}
&(\|\rho^\nu - 1\|_{L^4}^4)' \\
&= -4\int (\rho^\nu - 1)^3 \mathrm{div} u \mathrm{d}x - 3\int (\rho^\nu - 1)^4 \mathrm{div} u \mathrm{d}x + 4\int r_\nu (\rho^\nu - 1)^3 \mathrm{d}x \\
&\leq C\|\rho^\nu - 1\|_{L^4}^4 + C\|\nabla u\|_{L^4}^4 + C\|r_\nu\|_{L^2},
\end{aligned}$$

which implies that, for all $1 \leq N \leq s \leq N+1 \leq t \leq N+2$,

$$\begin{aligned}
\|\rho^\nu(\cdot, t) - 1\|_{L^4}^4 \leq \|\rho^\nu(\cdot, s) - 1\|_{L^4}^4 & \\
+ C\int_N^{N+2} \left(\|\rho^\nu - 1\|_{L^4}^4 + \|\nabla u\|_{L^4}^4\right) \mathrm{d}t & \\
+ C\int_N^{N+2} \|r_\nu\|_{L^2} \mathrm{d}t. &
\end{aligned} \tag{5.55}$$

Letting $\nu \to 0^+$ in (5.55) together with (5.54) and (1.23) yields that

$$\|\rho(\cdot, t) - 1\|_{L^4}^4 \leq \|\rho(\cdot, s) - 1\|_{L^4}^4 + C\int_N^{N+2} \left(\|\rho - 1\|_{L^4}^4 + \|\nabla u\|_{L^4}^4\right) \mathrm{d}t. \tag{5.56}$$

Integrating (5.56) with respect to s over $[N, N+1]$ leads to

$$\begin{aligned}
\sup_{t\in[N+1,N+2]} \|\rho(\cdot, t) - 1\|_{L^4}^4 &\leq C\int_N^{N+2} \left(\|\rho - 1\|_{L^4}^4 + \|\nabla u\|_{L^4}^4\right) \mathrm{d}t \\
&\to 0, \text{ as } N \to \infty,
\end{aligned}$$

due to (5.45). Combining this with (1.27) and (1.30) implies that, for all $p \in (2, \infty)$,

$$\lim_{t\to\infty} \int |\rho - 1|^p \mathrm{d}x = 0. \tag{5.57}$$

Finally, we will prove

$$\lim_{t\to\infty} \left(\|u\|_{L^4} + \|\nabla\theta\|_{L^2}\right) = 0, \tag{5.58}$$

which, combined with (5.57), (1.27), (1.29)–(1.31) and the Gagliardo-Nirenberg inequality, thus gives (1.28). In fact, one deduces from (1.29)–(1.31) that

$$\begin{aligned}\int_1^\infty \left(\|u\|_{L^4}^4 + \|\nabla\theta\|_{L^2}^2\right) \mathrm{d}t &\le C \int_1^\infty \|u\|_{L^2} \|\nabla u\|_{L^2}^3 \mathrm{d}t \\ &\quad + \int_1^\infty \|\nabla\theta\|_{L^2}^2 \mathrm{d}t \le C,\end{aligned} \tag{5.59}$$

$$\begin{aligned}\int_1^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(\cdot,t)\|_{L^4}^4\right)\right| \mathrm{d}t &= 4\int_1^\infty \left|\int |u|^2 u\cdot u_t \mathrm{d}x\right| \mathrm{d}t \\ &\le C\int_1^\infty \|u\|_{L^\infty}\|u\|_{L^4}^2\|u_t\|_{L^2}\mathrm{d}t \\ &\le C,\end{aligned} \tag{5.60}$$

and

$$\begin{aligned}\int_1^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\theta(\cdot,t)\|_{L^2}^2\right)\right| \mathrm{d}t &= 2\int_1^\infty \left|\int \nabla\theta\cdot\nabla\theta_t \mathrm{d}x\right| \mathrm{d}t \\ &\le C\int_1^\infty \|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}\mathrm{d}t \\ &\le C.\end{aligned} \tag{5.61}$$

Thus, we derive (5.58) easily from (5.59)–(5.61). The proof of Theorem 1.2 is finished.

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