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A Gradient Flow Approach to the Porous Medium Equation with Fractional Pressure

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Abstract

We consider a family of porous media equations with fractional pressure, recently studied by Caffarelli and Vázquez. We show the construction of a weak solution as the Wasserstein gradient flow of a square fractional Sobolev norm. The energy dissipation inequality, regularizing effect and decay estimates for the L^p norms are established. Moreover, we show that a classical porous medium equation can be obtained as a limit case.

1. Introduction

We consider the evolution problem

$$\begin{cases} \partial_t u - \operatorname{div}(u\nabla v) = 0 & \text{in } \mathbb{R}^d \times (0,+\infty), \\ (-\Delta)^s v = u & \text{in } \mathbb{R}^d \times (0,+\infty), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where the initial datum u_0 is a Borel probability measure on $\mathbb{R}^d$, $d \geq 1$, and $0 < s < \min\{1, \frac{d}{2}\}$. The linear operator $(-\Delta)^s$ is the s-fractional Laplacian on $\mathbb{R}^d$, defined by means of Fourier transform as

$$(\widehat{(-\Delta)^s v})(\xi) = |\xi|^{2s}\hat{v}(\xi).$$

We define the Riesz kernel K_s by the relation $\hat{K}_s(\xi) = |\xi|^{-2s}$, that is,

$$K_s(x) = C_{d,s}|x|^{-d+2s},$$

where $C_{d,s}$ is a normalization constant. With our convention for the Fourier transform, that is, $\hat{\varphi}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\,\varphi(x)\,\mathrm{d}x$, we have

$$C_{d,s} = \pi^{-d/2}2^{-2s}\Gamma(d/2-s)/\Gamma(s), \tag{1.2}$$

where Γ is the Euler Gamma function, see for instance [1, Section 1.2.2]. The relation between u and v, in the second equation of (1.1), is understood as $v = K_s * u$. Therefore, problem (1.1) corresponds to an evolution repulsive interaction equation, characterized by the Riesz kernel K_s.

Problem (1.1) has been studied by CAFFARELLI and Vázquez in [14], where the existence of solutions was proved for non-negative bounded initial data which decay exponentially fast at infinity. The existence result of [14] has been generalized to L^1 positive initial data in [11] and to positive finite measure data in [27,28]. Moreover, [11,16] contain comprehensive results about the Hölder regularity of solutions. Barenblatt profiles and asymptotic behavior are investigated in [15]. Exponential convergence towards stationary states in one space dimension, after changing to self similar variables, has been obtained in [18]. More general nonlocal porous media equations are considered in [6,28–30]. See also [32] and the references therein.

The system (1.1) is derived by starting from the continuity equation

$$\partial_t u + \operatorname{div}(u\mathbf{v}) = 0,$$

which governs the evolution of the density distribution u, driven by a velocity vector field $\mathbf{v}$. Now, as happens for the classical porous medium equation, we suppose that $\mathbf{v}$ is the gradient of a scalar function v, the pressure, which is assumed to be a function of the density u. The system (1.1) emerges by choosing the nonlocal closing relation $\mathbf{v} := -\nabla v = -\nabla(K_s * u)$.

Let us briefly discuss the extreme cases $s = 0$ and $s = 1$. When $s = 0$, the second equation formally reduces to the identity $v = u$ and thus the system in (1.1) becomes

$$\partial_t u - \frac{1}{2}\Delta u^2 = 0, \tag{1.3}$$

which is a classical (local) porous medium equation. Among the other results, in this paper we will make this transition rigorous (see Theorem 1.3). The other extreme situation corresponds to the case $s = 1, d \geq 2$, where the second equation becomes $-\Delta v = u$. The resulting system (1.1) is related to the Chapman–Rubinstein–Schatzman's mean field model in superconductivity (see [17]) and to the E's model in superfluidity, at least for positive solutions (see [20]). Existence for this system when $s = 1$ was first proved in two space dimensions in [23]. More recently, Serfaty and Vázquez [27] proved that the solutions of the system (1.1) converge in a proper way when $s \nearrow 1$ to the solutions of the corresponding system with $s = 1$.

The gradient flow structure. Our main contribution is the rigorous construction of non-negative solutions for the Cauchy problem (1.1) as trajectories of a gradient flow. More precisely, we consider the space $\mathcal{P}_2(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$ with finite second moment endowed with the 2-Wasserstein distance, here denoted by W (see Sect. 2). For $u \in \mathcal{P}_2(\mathbb{R}^d)$ we define the energy functional

$$\mathcal{F}_s(u) = \frac{1}{2}\|u\|^2_{\dot{H}^{-s}(\mathbb{R}^d)} := \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s}|\hat{u}(\xi)|^2 \,\mathrm{d}\xi,$$

that is, $\mathcal{F}_s$ is the square norm of the homogeneous Sobolev space $\dot{H}^{-s}(\mathbb{R}^d)$, see Sect. 2.2. We observe that this functional admits the alternative representation

$$\mathcal{F}_s(u) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_s(x-y)\,\mathrm{d}u(x)\,\mathrm{d}u(y),$$

enlightening the structure of an interaction energy, characterized by the Riesz convolution kernel K_s. Within the gradient flow interpretation, we prove that a solution to the Cauchy problem (1.1) can be obtained by means of the minimizing movement approximation scheme, applied to the functional $\mathcal{F}_s$ in the metric space $(\mathcal{P}_2(\mathbb{R}^d), W)$. A general theory of minimizing movements in metric spaces and its applications to the space $(\mathcal{P}_2(\mathbb{R}^d), W)$ is contained in the book of Ambrosio, Gigli and Savaré [2]. The gradient flow approach in $(\mathcal{P}_2(\mathbb{R}^d), W)$ was first exploited by Jordan–Kinderlehrer–Otto in the seminal paper [22]. Let us illustrate the strategy in our case: given $u_0 \in \dot{H}^{-s}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ and $\tau > 0$ we introduce the following time discretization scheme. We consider a uniform partition of size τ of the time interval $[0, +\infty)$ and we let u_τ^0 be a suitable approximation of the initial datum (see (3.2)). Then, we recursively define

$$u_\tau^k \in \mathrm{Argmin}_{u\in\mathcal{P}_2(\mathbb{R}^d)} \left\{\mathcal{F}_s(u) + \frac{1}{2\tau} W^2\left(u, u_\tau^{k-1}\right)\right\}, \qquad \text{for } k = 1, 2, \ldots \tag{1.4}$$

If $\{u_\tau^k\}_{k\in\mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ is a sequence defined by (1.4), we introduce the piecewise constant interpolation

$$u_\tau(t) := u_\tau^{\lceil t/\tau\rceil}, \qquad t \in [0, +\infty),$$

where $\lceil a\rceil := \min\{m \in \mathbb{N} : m > a\}$ is the upper integer part of the real number a. We refer to u_τ as discrete solution. We prove that this family of piecewise constant curves admits limit points as $\tau \to 0$, and that a limit curve is a weak solution to (1.1), satisfying some additional properties (see Theorem 1.1).

Nonlocal evolution equations with singular kernels appear in several mathematical models. However, up until now the corresponding gradient flow approach was limited to less singular interactions. Besides the works [3,4], dealing with the Chapman–Rubinstein–Schatzman superconductivity model, gradient flows of equations involving Newtonian interaction appear in the study of the Keller–Segel model for chemotaxis, see [9] and the reviews [7,8]. The approach we propose here is strictly related to the latter contributions, and problem (1.1), with the corresponding functional $\mathcal{F}_s$, turns out to be a remarkable example of Wasserstein gradient flow.

The main result. We shall now state the results. The main one is Theorem 1.1, which contains all the properties of the gradient flow solutions. Throughout the paper we denote by $\mathcal{H} : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ the entropy defined by $\mathcal{H}(u) := \int_{\mathbb{R}^d} u \log u \,\mathrm{d}x$ if u is absolutely continuous with respect to the Lebesgue measure and $\mathcal{H}(u) = +\infty$ otherwise. We use the notation $D(\mathcal{H}) = \{u \in \mathcal{P}_2(\mathbb{R}^d) : \mathcal{H}(u) < +\infty\}$ for the domain of $\mathcal{H}$. Moreover, in the statement of Theorem 1.1, the approximation u_τ^0 of the initial datum u_0 is not arbitrary, but given by the suitable

Gaussian regularization defined in Sect. 3 below, see (3.2). See also Sect. 2.1 for the definition of narrow convergence and Sect. 3.2 for the definition of the space $AC^2([0,+\infty);(\mathcal{P}_2(\mathbb{R}^d),W))$.

Theorem 1.1. *Let $d \geq 1$, $0 < s < \min\{1, \frac{d}{2}\}$ and $u_0 \in \dot{H}^{-s}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$. Then the following assertions hold:*

(i) **Existence and uniqueness of discrete solutions.** *For every $\tau > 0$, after having defined u_τ^0 by (3.2), there exists a unique sequence $\{u_\tau^k : k = 1,2,\ldots\}$ satisfying (1.4);*

(ii) **Convergence and regularity.** *For every vanishing sequence τ_n there exists a (not relabeled) subsequence τ_n and a curve $u \in AC^2([0,+\infty);(\mathcal{P}_2(\mathbb{R}^d),W))$ such that*

$$u_{\tau_n}(t) \to u(t) \quad \textit{narrowly as } n \to \infty, \textit{ for any } t \in [0,+\infty).$$

Moreover, $u \in L^2((T_0,T);H^{1-s}(\mathbb{R}^d))$ for every $0 < T_0 < T$, and

$$u_{\tau_n} \to u \quad \textit{strongly in } L^2((T_0,T);L^2_{loc}(\mathbb{R}^d)) \quad \textit{as } n \to \infty\,.$$

Defining $v_\tau(t) := K_s * u_\tau(t)$ and $v(t) := K_s * u(t)$ $\forall\, t > 0$, we have that $\nabla v \in L^2((T_0,T);L^2(\mathbb{R}^d))$ for every $0 < T_0 < T$, and

$$\nabla v_{\tau_n} \to \nabla v \quad \textit{weakly in } L^2((T_0,T);L^2(\mathbb{R}^d)) \quad \textit{as } n \to \infty;$$

(iii) **Solution of the equation.** *Given u, v from point* ii)*, the first equation in* (1.1) *is satisfied in the following weak form:*

$$\int_0^{+\infty}\int_{\mathbb{R}^d} (\partial_t\varphi - \nabla\varphi\cdot\nabla v)u \,\mathrm{d}x\,\mathrm{d}t = 0, \quad \textit{for all } \varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d);$$

(iv) **Energy dissipation inequality.** *Given u, v from point* ii)*, there holds*

$$\mathcal{F}_s(u(t)) + \int_0^t\int_{\mathbb{R}^d} |\nabla v(r)|^2 u(r)\,\mathrm{d}x\,\mathrm{d}r \leq \mathcal{F}_s(u_0), \qquad \forall\, t \in [0,+\infty); \quad (1.5)$$

(v) **Regularizing effect and decay estimates.** *For every $p \in [1,+\infty]$ there is a constant C_p depending only on p, d and s (independent of u_0) such that*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq C_p t^{-\gamma_p} \qquad \forall t > 0,$$

where $\gamma_p = \frac{p-1}{p}\frac{d}{d+2(1-s)}$ for $p < +\infty$ and $\gamma_\infty = \frac{d}{d+2(1-s)}$. In particular $u(t) \in D(\mathcal{H}) \cap L^p(\mathbb{R}^d)$ for every $t > 0$;

(vi) **Entropy estimates.** *If, in addition, $u_0 \in D(\mathcal{H})$, then*

$$\mathcal{H}(u(t)) \leq \mathcal{H}(u_0), \qquad \forall t > 0.$$

If $u_0 \in L^p(\mathbb{R}^d)$ for some $p \in [1,+\infty]$, then

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}, \qquad \forall t > 0.$$

Remark 1.2. The proof of Theorem 1.1 will be given as a collection of different results throughout the paper. Let us give some comments here.

- If $u_0 \in D(\mathcal{H})$, then the results of point (ii) also hold for $T_0 = 0$ and the results of points (i)-(ii)-(iii)-(iv) do not require the approximation of the initial datum (that is, we could define $u_\tau^0 = u_0$ in this case).
- The value of the constant C_p in point (v) is explicit, see Lemma 4.10 below for $p \in (1, +\infty)$ and Theorem 7.2 for $p = +\infty$. If $p = 1$ we have $C_1 = 1$ and equality holds in points (v) and (vi) because mass conservation is an automatic consequence of the Wasserstein gradient flow construction of solutions.
- For every $p \in [1, +\infty]$ the exponent γ_p in point (v) is sharp, since the Barenblatt-type solutions constructed in [15] have the same decay rate.
- The solutions that we construct are weak energy solutions in the terminology of Caffarelli and Vazquez. Consequently they are also Hölder continuous thanks to [11, Theorem 5.1]. The finite speed of propagation is obtained by Caffarelly and Vazquez in [14] and relies on their construction of weak solutions (see also [21] and [29]). It would be an interesting problem to obtain the finite speed of propagation directly from our discrete scheme.
- Theorem 1.1 holds if we consider positive measure data in $\dot{H}^{-s}(\mathbb{R}^d)$, with finite second moment and mass $M > 0$. In such case, the constant C_p from point (v) gets multiplied by M^{ℓ_p} where $\ell_p = \frac{2p(1-s)+d}{2p(1-s)+dp}$ if $p \in [1, +\infty)$ and $\ell_\infty = \frac{2(1-s)}{2(1-s)+d}$. This scaling is the same obtained in [11] for positive $L^1(\mathbb{R}^d)$ data. See also Remark 7.3 below.

Let us summarize the main techniques and the strategy that we shall use in the paper. We start with the analysis of the discrete variational problem (1.4) proving existence and uniqueness of the discrete solutions. Moreover we analyze the regularity of minimizers, which are indeed shown to belong to $\dot{H}^{1-s}(\mathbb{R}^d)$, and not only to $\dot{H}^{-s}(\mathbb{R}^d)$. In order to do this we make use of the flow interchange technique, described by McCann, Matthes and Savaré in [24]. The improved regularity of minimizers allows as to perform variations along transport maps and to derive a corresponding Euler–Lagrange equation, which yields a discrete formulation of problem (1.1). Moreover, the obtained regularity estimates entail sufficient compactness in order to pass to the limit in such discrete formulation, obtaining a weak solution to problem (1.1). Finally, in order to obtain the energy dissipation inequality of functional $\mathcal{F}_s$ along the solution we use the De Giorgi variational interpolation. In these steps we often work in Fourier variables; this approach reveals useful and appears quite natural, starting from the definition of the energy functional.

The other important features that we discuss are the regularizing effect and the decay rate at infinity of L^p norms stated in point v) of Theorem 1.1. We stress that the regularizing effect allows as to treat the case of general $\mathcal{P}_2 \cap \dot{H}^{-s}$ initial data. The decay rate of the L^p norms was already obtained in [11]. From our point of view, this relates to the interesting issue of finding general L^p estimates at the discrete level of the minimizing movements scheme, along with the corresponding decay rates for large times, which is new in this framework. At the discrete level, for $p < +\infty$, we obtain an estimate of the form

$$\|u_\tau^k\|_{L^p(\mathbb{R}^d)} \le \min\left\{\|u_\tau^0\|_{L^p(\mathbb{R}^d)}, C_p(k\tau)^{-\gamma_p}\right\} + R_\tau, \qquad k = 1, 2, 3, ...,$$

where $\gamma_p = \frac{p-1}{p}\frac{d}{d+2(1-s)}$ and R_τ is a suitable remainder term. Such an estimate is proved by combining the flow interchange technique with Sobolev inequalities. The term R_τ is then shown to vanish as $\tau \to 0$, thus yielding the desired decay estimates of the L^p norms for $p < +\infty$. However, it is not possible to directly pass to the limit as $p \to +\infty$, because the multiplicative constant C_p blows up. We note that an analogous difficulty for the case of the porous medium equation was observed for instance in [10], when trying to obtain the decay rate of the L^∞ norm by making use of Sobolev inequalities.

In order to obtain the L^∞ decay, a refined argument is indeed necessary. Here, we adapt the techniques of Caffarelli–Soria–Vázquez [11] to the discrete setting. Their approach for proving L^∞ decay estimates was previously introduced by Caffarelli and Vasseur [12,13] for the case of the semigeostrophic equation, and it is based on the De Giorgi technique for elliptic equations. In order to apply this technique within the discrete setting we introduce a sequence of minimizing movements approximations on a smaller scale. This construction represents one of the main novelties of the paper (see Section 7). The new approximation provides the required information on the solution, allowing for an L^2 to L^∞ argument to get L^∞ decay with the expected rate $\gamma_\infty = \lim_{p\to+\infty}\gamma_p$, corresponding to the one obtained in [6,11].

The limit as $s \to 0$. A final result that we prove is the convergence of the constructed solutions to a solution of the standard porous medium equation (1.3) as the fractional parameter s goes to zero. This complements the result of Serfaty and Vázquez [27], where the limiting case as $s \to 1$ (corresponding to the interaction with the Newtonian potential) is analyzed. More precisely, the result is stated in the following Theorem:

Theorem 1.3. *Let* $u_0 \in L^2(\mathbb{R}^d)$ *and* $\{u_0^s\}_{s\in(0,1)}$ *be a family of initial data such that* $u_0^s \in D(\mathcal{F}_s)$, u_0^s *converges narrowly to* u_0 *as* $s \to 0$, $\sup_{s\in(0,1)}\int_{\mathbb{R}^d}|x|^2\,\mathrm{d}u_0^s(x) < +\infty$ *and* $\lim_{s\to0}\mathcal{F}_s(u_0^s) = \mathcal{F}_0(u_0)$ *where* $\mathcal{F}_0(\cdot) := \frac{1}{2}\|\cdot\|_{L^2(\mathbb{R}^d)}$. *For each* $s \in (0,1)$, *let* u^s *be a solution to the corresponding equation* (1.1), *with initial datum* u_0^s, *given by* Theorem 1.1. *Moreover, let* u *be the unique solution of the Cauchy problem for the porous medium equation*

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u^2 = 0 & \text{in } \mathbb{R}^d \times (0,+\infty), \\ u(0) = u_0 \end{cases} \tag{1.6}$$

satisfying the energy identity

$$\mathcal{F}_0(u(T)) + \int_0^T\int_{\mathbb{R}^d}|\nabla u(t)|^2 u(t)\,\mathrm{d}x\,\mathrm{d}t = \mathcal{F}_0(u_0), \qquad \forall\, T > 0.$$

Then we have

$$u^s(t) \to u(t) \quad \text{narrowly as } s \to 0 \text{ for every } t \ge 0,$$

and, for every T_0 *and* T *such that* $T > T_0 > 0$,

$$u^s \to u \quad \text{strongly in } L^2((T_0, T); L^2_{loc}(\mathbb{R}^d)) \quad \text{as } s \to 0,$$
$$\nabla u^s \to \nabla u \quad \text{weakly in } L^2((T_0, T); L^2(\mathbb{R}^d)) \quad \text{as } s \to 0.$$

Plan of the paper. Section 2 introduces the basic framework for gradient flows in the Wasserstein space and for fractional Sobolev norms. Section 2 shows the convergence of the scheme to some absolutely continuous curve in $\mathcal{P}_2(\mathbb{R}^d)$, owing only to the general theory of minimizing movements, and not relating to the specific choice of functional $\mathcal{F}_s$. Section 4 introduces the flow interchange, which will be repeatedly used in order to obtain further regularity of minimizers, the regularizing effect of the dynamics, and the L^p decay estimates for $p \in (1, \infty)$. Section 5 is devoted to the Euler–Lagrange equation for discrete minimizers, thus building up the key element for the existence result. Section 6 proves existence, by showing that the limit curve found in Step 3 is in fact regular enough for giving sense to the term $u\nabla v$ and satisfies equation (1.1). This is moreover a gradient flow solution, so that (1.1) holds in the sense of distributions and an energy dissipation inequality for functional $\mathcal{F}_s$ holds. Section 7 introduces the double scale approximation and proves the L^∞ decay estimates, thus completing the proof of Theorem 1.1 Eventually, Section 8 contains the proof of Theorem 1.3.

2. Notation and Preliminary Results

2.1. Wasserstein Distance

We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$. The narrow convergence in $\mathcal{P}(\mathbb{R}^d)$ is defined in duality with continuous and bounded functions on $\mathbb{R}^d$, that is, a sequence $\{u_n\} \subset \mathcal{P}(\mathbb{R}^d)$ narrowly converges to $u \in \mathcal{P}(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} \phi \, \mathrm{d}u_n \to \int_{\mathbb{R}^d} \phi \, \mathrm{d}u$ for every $\phi \in C_b(\mathbb{R}^d)$, where $C_b(\mathbb{R}^d)$ is the set of continuous and bounded functions defined on $\mathbb{R}^d$.

We define $\mathcal{P}_2(\mathbb{R}^d) := \{u \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \, \mathrm{d}u(x) < +\infty\}$ the set of Borel probability measure with finite second moment. The Wasserstein distance W in $\mathcal{P}_2(\mathbb{R}^d)$ is defined as

$$W(u, v) := \min_{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)} \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \mathrm{d}\gamma(x, y) \right)^{1/2} : (\pi_1)_\# \gamma = u, \; (\pi_2)_\# \gamma = v \right\}, \tag{2.1}$$

where π_i, $i = 1, 2$, denote the canonical projections on the first and second factor respectively. Denoting by I the identity map in $\mathbb{R}^d$, when u is absolutely continuous with respect to the Lebesgue measure, the minimum problem (2.1) has a unique solution γ induced by a transport map T_u^v in the following way: $\gamma = (I, T_u^v)_\# u$. In particular, T_u^v is the unique solution of the Monge optimal transport problem

$$\min_{S : \mathbb{R}^d \to \mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} |S(x) - x|^2 \mathrm{d}u(x) : \; S_\# u = v \right\}.$$

Finally, we also recall that if v is absolutely continuous with respect to Lebesgue measure, then

$$T_v^u \circ T_u^v = I \quad u\text{-a.e.} \quad \text{and} \quad T_u^v \circ T_v^u = I \quad v\text{-a.e.}$$

The function $W : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a distance and the metric space $(\mathcal{P}_2(\mathbb{R}^d), W)$ is complete and separable. Moreover the distance W is sequentially lower semi continuous with respect to the narrow convergence, that is,

$$u_n \to u, \quad v_n \to v, \text{ narrowly} \implies \liminf_{n\to+\infty} W(u_n, v_n) \geq W(u, v),$$

and bounded sets in $(\mathcal{P}_2(\mathbb{R}^d), W)$ are narrowly sequentially relatively compact.

2.2. *Fourier Transform and Fractional Sobolev Spaces*

We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of smooth functions with rapid decay at infinity and by $\mathcal{S}'(\mathbb{R}^d)$ the dual space of tempered distributions. The Fourier transform of $u \in \mathcal{S}(\mathbb{R}^d)$ is defined by $\hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x$. The Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$ and by transposition it can be defined on $\mathcal{S}'(\mathbb{R}^d)$. Moreover, the Plancherel formula holds:

$$\int_{\mathbb{R}^d} \hat{u}(\xi)\overline{\hat{w}(\xi)}\,\mathrm{d}\xi = (2\pi)^d \int_{\mathbb{R}^d} u(x)w(x)\,\mathrm{d}x, \qquad \forall u, w \in L^2(\mathbb{R}^d).$$

Let $r \in \mathbb{R}$. For every tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$, we define

$$\|u\|^2_{H^r(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|^2)^r |\hat{u}(\xi)|^2\,\mathrm{d}\xi$$

and

$$\|u\|^2_{\dot{H}^r(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2r} |\hat{u}(\xi)|^2\,\mathrm{d}\xi.$$

The fractional Sobolev space $H^r(\mathbb{R}^d)$ is defined by

$$H^r(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \hat{u} \in L^1_{loc}(\mathbb{R}^d),\ \|u\|_{H^r(\mathbb{R}^d)} < +\infty \right\},$$

and the homogenous fractional Sobolev space $\dot{H}^r(\mathbb{R}^d)$ is defined by

$$\dot{H}^r(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \hat{u} \in L^1_{loc}(\mathbb{R}^d),\ \|u\|_{\dot{H}^r(\mathbb{R}^d)} < +\infty \right\}.$$

The next proposition summarizes some basic facts about fractional Sobolev spaces, which will be used many times in the sequel. We refer for instance to [5, Sections 1.3, 1.4].

Proposition 2.1. *The following assertions hold:*

- *Interpolation. If* $r_0 < r_1 < r_2$ *then*

$$\|u\|_{H^{r_1}(\mathbb{R}^d)} \le \|u\|^{1-\theta}_{H^{r_0}(\mathbb{R}^d)} \|u\|^{\theta}_{H^{r_2}(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{\dot H^{r_1}(\mathbb{R}^d)} \le \|u\|^{1-\theta}_{\dot H^{r_0}(\mathbb{R}^d)} \|u\|^{\theta}_{\dot H^{r_2}(\mathbb{R}^d)},$$

where θ *is defined by* $r_1 = (1-\theta)r_0 + \theta r_2$;
- *If* $r_1 < r_2$ *then* $\|u\|_{H^{r_1}(\mathbb{R}^d)} \le \|u\|_{H^{r_2}(\mathbb{R}^d)}$. *If* $r > 0$ *then* $\|u\|_{\dot H^{r}(\mathbb{R}^d)} \le \|u\|_{H^{r}(\mathbb{R}^d)}$. *If* $r < 0$ *then* $\|u\|_{H^{r}(\mathbb{R}^d)} \le \|u\|_{\dot H^{r}(\mathbb{R}^d)}$. *If* $r = 0$ *then* $\|u\|_{\dot H^{0}(\mathbb{R}^d)} = \|u\|_{H^{0}(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)}$;
- *If* $\phi \in \mathcal{S}(\mathbb{R}^d)$ *and* $u \in H^r(\mathbb{R}^d)$ *then there exists a constant c, depending only on* ϕ, r *and* d, *such that*

$$\|\phi\, u\|_{H^r(\mathbb{R}^d)} \le c\|u\|_{H^r(\mathbb{R}^d)};$$

- *If* $\phi \in \mathcal{S}(\mathbb{R}^d)$, $r_1 < r_2$ *and* $\sup_{n\in\mathbb{N}} \|u_n\|_{H^{r_2}(\mathbb{R}^d)} < +\infty$, *then* $\{\phi\, u_n : n \in \mathbb{N}\}$ *is relatively compact in* $H^{r_1}(\mathbb{R}^d)$.

Let $d \ge 1$ and $r \in (0, d/2)$. Then the fractional Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^d)} \le S_{d,r} \|u\|_{\dot H^r(\mathbb{R}^d)} \tag{2.2}$$

holds for any $u \in \dot H^r(\mathbb{R}^d)$, where $q := \frac{2d}{d-2r} > 2$ and (see for instance [19])

$$S_{d,r} = 2^{-2r}\pi^{-r}\frac{\Gamma(d/2-r)}{\Gamma(d/2+r)}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{2r/d}. \tag{2.3}$$

From (2.2) and interpolation of L^p norms we obtain that for q_1, q_2 such that $1 \le q_1 < q_2 < q = \frac{2d}{d-2r}$, the inequality

$$\|u\|_{L^{q_2}(\mathbb{R}^d)} \le S^{\theta}_{d,r} \|u\|^{1-\theta}_{L^{q_1}(\mathbb{R}^d)} \|u\|^{\theta}_{\dot H^r(\mathbb{R}^d)}$$

holds for any $u \in \dot H^r(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$, where $\theta = \frac{(q_1-q_2)q}{(q_1-q)q_2}$. In particular, for any $u \in \dot H^r(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $q_2 = 2 + \frac{2r}{d}$, there holds

$$\|u\|^{q_2}_{L^{q_2}(\mathbb{R}^d)} \le S^2_{d,r} \|u\|^{2r/d}_{L^1(\mathbb{R}^d)} \|u\|^2_{\dot H^r(\mathbb{R}^d)}. \tag{2.4}$$

Similarly, from (2.2) and the interpolation of L^p norms between the exponents $1 < p < \frac{d(p+1)}{d-2r}$, for $p \in (1, +\infty)$ and nonnegative $u \in L^1(\mathbb{R}^d)$ such that $u^{(p+1)/2} \in \dot H^r(\mathbb{R}^d)$, we have

$$\|u\|^{p+1}_{L^p(\mathbb{R}^d)} \le S^{2\theta}_{d,r} \|u\|^{(1-\theta)(p+1)}_{L^1(\mathbb{R}^d)} \|u^{(p+1)/2}\|^{2\theta}_{\dot H^r(\mathbb{R}^d)}, \tag{2.5}$$

where $\theta = \frac{d(p^2-1)}{p(2r+dp)}$.

In dimension $d = 1$, for $s \in (0, 1/2)$, we shall also need the following inequalities:

$$\|u\|^{4-2s}_{L^{4-2s}(\mathbb{R})} \le S^{2-2s}_{1,\frac{1-s}{4-2s}} \|u\|^{2-2s}_{L^1(\mathbb{R})} \|u\|^2_{\dot H^{1-s}(\mathbb{R})}, \tag{2.6}$$

and

$$\|u\|_{L^p(\mathbb{R})}^{p\beta_p} \le S_{1,\frac{1-s}{4-2s}}^{8-4s} \|u\|_{L^1(\mathbb{R})}^{(2p-2sp+1)/(p-1)} \|u^{(p+1)/2}\|_{\dot H^{1-s}(\mathbb{R})}^2, \tag{2.7}$$

where $\beta_p = \frac{2(1-s)+p}{p-1}$ and $p \in (1, +\infty)$. Indeed, by (2.2) and the interpolation property of Proposition 2.1, we have

$$\|u\|_{L^{\frac{2}{1-2r}}(\mathbb{R})} \le S_{1,r}\|u\|_{\dot H^r(\mathbb{R})} \le S_{1,r}\|u\|_{L^2(\mathbb{R})}^{(1-s-r)/(1-s)} \|u\|_{\dot H^{1-s}(\mathbb{R})}^{r/(1-s)} \tag{2.8}$$

for every $r, s \in (0, 1/2)$. Choosing $r = \frac{1-s}{4-2s}$ in (2.8) and interpolating the L^2 norm between L^1 and $L^{\frac{2}{1-2r}}$ we obtain (2.6), whereas similar interpolation arguments and (2.5) entail (2.7).

If $d \ge 1$ and $r \in (0, 1)$, the scalar product in the space $\dot H^r(\mathbb{R}^d)$, defined by

$$\langle v, w\rangle_r := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{2r}\hat v(\xi)\overline{\hat w(\xi)}\,\mathrm{d}\xi,$$

can also be expressed as

$$\langle v, w\rangle_r = \bar C_{d,r}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(v(x)-v(y))(w(x)-w(y))|x-y|^{-d-2r}\,\mathrm{d}x\,\mathrm{d}y. \tag{2.9}$$

This equivalence follows from [5, Proposition 1.37]. The value of the positive constant $\bar C_{d,r}$ can be obtained through the following formal computation. Since the Riesz kernel satisfies $\Delta K_r = -K_{r-1}$, using the Plancherel formula and integration by parts we have

$$\begin{aligned}\langle v, w\rangle_r &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{-2(1-r)}|\xi|^2\hat v(\xi)\overline{\hat w(\xi)}\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^d}(K_{1-r} * \nabla v)(x)\cdot\nabla w(x)\,\mathrm{d}x\\ &= \frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\Delta K_{1-r}(x-y))\,(v(x)-v(y))\,(w(x)-w(y))\,\mathrm{d}x\,\mathrm{d}y\\ &= -\frac12 C_{d,-r}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^{-2r-d}(v(x)-v(y))(w(x)-w(y))\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

thus (2.9) holds with $\bar C_{d,r} = -\frac12 C_{d,-r}$, where $C_{d,-r} < 0$ is given by extending formula (1.2) to values of the second index in $(-1, 0)$.

We also have the following:

Proposition 2.2. *Let $d \ge 1$ and $r \in (0, 1)$. Let $v \in \dot H^r(\mathbb{R}^d)$. If $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing, then $\langle v, F(v)\rangle_r \ge 0$. If, in addition, F is Lipschitz continuous on $\mathbb{R}^d$, then $F \circ v \in \dot H^r(\mathbb{R}^d)$ and there hold*

$$\langle v, F(v)\rangle_r \le L\langle v, v\rangle_r, \quad \langle F(v), F(v)\rangle_r \le L\langle F(v), v\rangle_r,$$

where L is the Lipschitz constant of F. Moreover, if v is nonnegative and $p \in (1, +\infty)$, the following Stroock–Varopoulos inequality holds:

$$\langle v, v^p\rangle_r \ge \frac{4p}{(p+1)^2}\|v^{(p+1)/2}\|_{\dot H^r(\mathbb{R}^d)}^2. \tag{2.10}$$

Proof. The first properties follow at once from the representation (2.9). (2.10) is also a consequence of (2.9), by means of the elementary inequality

$$(a-b)(a^p-b^p) \geq \frac{4p}{(p+1)^2}\left(a^{(p+1)/2}-b^{(p+1)/2}\right)^2,$$

which holds for any couple of nonnegative numbers a, b. □

3. Energy Functional and First Convergence Result

Henceforth it will be always assumed that $d \geq 1$ and $0 < s < \min\{1, \frac{d}{2}\}$.

3.1. Energy Functional

After noticing that a Borel probability measure u is a tempered distribution with $\hat{u}$ in $L^1_{loc}(\mathbb{R}^d)$, we define the energy functional $\mathcal{F}_s : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ by

$$\mathcal{F}_s(u) := \frac{1}{2}\|u\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}.$$

We state a basic property of functional $\mathcal{F}_s$:

Proposition 3.1. *The following assertions hold:*

- $D(\mathcal{F}_s) = \dot{H}^{-s}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$*;*
- $\mathcal{F}_s(u) \geq 0$ *for every* $u \in \mathcal{P}_2(\mathbb{R}^d)$*;*
- $\mathcal{F}_s$ *is sequentially lower semicontinuous with respect to the narrow convergence.*

Proof. The first two points are obvious. In order to prove the third one, let $\{u_n\} \subset \mathcal{P}_2(\mathbb{R}^d)$ be a sequence, narrowly converging to $u \in \mathcal{P}_2(\mathbb{R}^d)$, and such that $\sup_n \mathcal{F}_s(u_n) < +\infty$. Using the notation $U_n(\xi) := |\xi|^{-s}\hat{u}_n(\xi)$, the previous assumption reads as $\sup_n \|U_n\|_{L^2(\mathbb{R}^d)} < +\infty$. By L^2 weak compactness there exists a subsequence of $\{U_n\}$ that weakly converges in $L^2(\mathbb{R}^d)$ to some $U \in L^2(\mathbb{R}^d)$. By the narrow convergence of u_n we have that $\hat{u}_n(\xi) \to \hat{u}(\xi)$ for every $\xi \in \mathbb{R}^d$, and then $U_n(\xi) \to |\xi|^{-s}\hat{u}(\xi)$ for every $\xi \in \mathbb{R}^d$. By uniqueness of the weak limits and the lower semicontinuity of the L^2 norm we obtain that $\mathcal{F}_s(u) \leq \liminf_{n\to\infty} \mathcal{F}_s(u_n)$ and the statement holds. □

3.2. Wasserstein Gradient Flow, Minimizing Movements

Let $u_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $\tau > 0$. We let

$$\Gamma_t(x) := \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}, \quad x \in \mathbb{R}^d,\ t > 0, \tag{3.1}$$

and we define a regularized initial datum as

$$u^0_\tau := \Gamma_{\omega(\tau)} * u_0, \qquad \text{where} \quad \omega(\tau) := \begin{cases} -1/\log\tau & \text{if } \tau \in (0, 1/2) \\ -1/\log(1/2) & \text{if } \tau \in [1/2, +\infty). \end{cases} \tag{3.2}$$

We consider, for $k = 1, 2, \ldots$, the problem

$$\min_{u \in \mathcal{P}_2(\mathbb{R}^d)} \mathcal{F}_s(u) + \frac{1}{2\tau} W^2\left(u, u_\tau^{k-1}\right). \tag{3.3}$$

Proposition 3.2. *For every $\tau > 0$ and every $u_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a unique sequence $\{u_\tau^k : k = 0, 1, 2, \ldots\} \subset D(\mathcal{F}_s)$ satisfying $u_\tau^0 = \Gamma_{\omega(\tau)} * u_0$ and such that u_τ^k is a solution to problem (3.3) for $k = 1, 2, \ldots$.*

Proof. Let $\tau > 0$ and $k \in \mathbb{N}$. By Proposition 3.1 and the properties of the Wasserstein distance, the functional $u \mapsto \mathcal{F}_s(u) + \frac{1}{2\tau} W^2(u, u_\tau^{k-1})$ is nonnegative, lower semicontinuous with respect to the narrow convergence and with narrowly compact sublevels. The existence of minimizers follows by standard direct methods in calculus of variations. The uniqueness of minimizers follows from the strict convexity of the functional $u \mapsto \mathcal{F}_s(u) + \frac{1}{2\tau} W^2(u, u_\tau^{k-1})$ with respect to linear convex combinations in $\mathcal{P}_2(\mathbb{R}^d)$, since $\mathcal{F}_s$ is a square Hilbert norm. □

By Proposition 3.2, the piecewise constant curve

$$u_\tau(t) := u_\tau^{\lceil t/\tau \rceil} \tag{3.4}$$

is uniquely defined, where $\lceil a \rceil := \min\{m \in \mathbb{N} : m > a\}$ is the upper integer part.

We say that a curve $u : [0, +\infty) \to \mathcal{P}_2(\mathbb{R}^d)$ is absolutely continuous with finite energy, and we use the notation $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$, if there exists $m \in L^2([0, +\infty))$ such that $W(u(t_1), u(t_2)) \leq \int_{t_1}^{t_2} m(r)\, dr$ for every $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$. If $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$, then there exists its metric derivative defined by

$$|u'|(t) := \lim_{h \to 0} \frac{W(u(t+h), u(t))}{|h|} \qquad \text{for a.e. } t \in [0, +\infty),$$

and $|u'|(t) \leq m(t)$ for almost every $t \in [0, +\infty)$.

Theorem 3.3. (First convergence result). *Let $u_0 \in \dot{H}^{-s}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ and u_τ the piecewise constant curve defined in (3.4). For every vanishing sequence τ_n there exists a subsequence (not relabeled) τ_n and a curve $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ such that*

$$u_{\tau_n}(t) \to u(t) \quad \textit{narrowly as } n \to \infty, \textit{ for any } t \in [0, +\infty). \tag{3.5}$$

Proof. The proof is based on the compactness argument of minimizing movements, stated in [2].

Since $0 < \hat{\Gamma}_\tau(\xi) \leq 1$ we have $|\hat{u}_\tau^0(\xi)| = |\hat{\Gamma}_{\omega(\tau)}(\xi)\hat{u}_0(\xi)| \leq |\hat{u}_0(\xi)|$ and then

$$\mathcal{F}_s\left(u_\tau^0\right) \leq \mathcal{F}_s(u_0). \tag{3.6}$$

The first estimate given by the scheme (3.3) is the following:

$$\mathcal{F}_s\left(u_\tau^N\right) + \frac{1}{2} \sum_{k=1}^{N} \tau \frac{W^2\left(u_\tau^k, u_\tau^{k-1}\right)}{\tau^2} \leq \mathcal{F}_s\left(u_\tau^0\right) \leq \mathcal{F}_s\left(u_0\right), \qquad \forall N \in \mathbb{N}. \tag{3.7}$$

We show that for any $T > 0$ the set $\mathcal{A}_T := \{u_\tau^k : \tau > 0, N \in \mathbb{N}, N\tau \le T\}$ is bounded in $(\mathcal{P}_2(\mathbb{R}^d), W)$ and consequently sequentially narrowly compact. Indeed, recalling that $\int_{\mathbb{R}^d} |x|^2 \,\mathrm{d}u(x) = W^2(u, \delta_0)$ for any $u \in \mathcal{P}_2(\mathbb{R}^d)$, using the triangle inequality and Jensen's discrete inequality we have

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2 u_\tau^N\left(x\right) \mathrm{d}x = W^2\left(u_\tau^N, \delta_0\right) \quad &\le \left(\textstyle\sum_{k=1}^N W\left(u_\tau^k, u_\tau^{k-1}\right) + W\left(u_\tau^0, \delta_0\right)\right)^2 \\ &\le 2\left(\textstyle\sum_{k=1}^N \tau \frac{W\left(u_\tau^k, u_\tau^{k-1}\right)}{\tau}\right)^2 + 2W^2\left(u_\tau^0, \delta_0\right) \\ &\le 2N\tau \textstyle\sum_{k=1}^N \tau \frac{W^2\left(u_\tau^k, u_\tau^{k-1}\right)}{\tau^2} + 2W^2\left(u_\tau^0, \delta_0\right).\end{aligned} \tag{3.8}$$

Since, for suitable $c > 0$, we have

$$\begin{aligned}W^2\left(u_\tau^0, \delta_0\right) &\le 2W^2\left(u_\tau^0, \Gamma_{\omega(\tau)}\right) + 2W^2\left(\Gamma_{\omega(\tau)}, \delta_0\right) \\ &= 2W^2\left(\Gamma_{\omega(\tau)} * u_0, \Gamma_{\omega(\tau)} * \delta_0\right) + 2W^2\left(\Gamma_{\omega(\tau)}, \delta_0\right) \\ &\le 2W^2\left(u_0, \delta_0\right) + 2W^2\left(\Gamma_{\omega(\tau)}, \delta_0\right) = 2W^2\left(u_0, \delta_0\right) + c\omega\left(\tau\right),\end{aligned}$$

it follows from (3.7) and (3.8), since $\mathcal{F}_s \ge 0$, that

$$\int_{\mathbb{R}^d} |x|^2 u_\tau^N(x)\,\mathrm{d}x \le 4T\mathcal{F}_s(u_0) + 4\int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x + 2c, \tag{3.9}$$

and the boundedness of $\mathcal{A}_T$ follows.

We define the piecewise constant function $m_\tau : [0, +\infty) \to [0, +\infty)$ as

$$m_\tau(t) := \frac{W(u_\tau(t), u_\tau(t-\tau))}{\tau},$$

with the convention that $u_\tau(t-\tau) = u_\tau(0)$ if $t - \tau < 0$. Since $\mathcal{F}_s \ge 0$, from (3.7) it follows that

$$\frac{1}{2}\int_0^{+\infty} m_\tau^2(t)\,\mathrm{d}t \le \mathcal{F}_s(u_0).$$

It follows that there exists $m \in L^2(0, +\infty)$ such that m_τ weakly converges to m in $L^2(0, +\infty)$. Moreover for any $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$, setting $k_1(\tau) = [t_1/\tau]$ and $k_2(\tau) = [t_2/\tau]$, by triangle inequality it holds that

$$W\left(u_\tau\left(t_1\right), u_\tau\left(t_2\right)\right) \le \sum_{k=k_1(\tau)}^{k_2(\tau)-1} W\left(u_\tau^k, u_\tau^{k-1}\right) \le \int_{k_1(\tau)\tau}^{k_2(\tau)\tau} m_\tau\left(t\right)\,\mathrm{d}t.$$

By the L^2 weak convergence of m_τ the following equicontinuity estimate holds:

$$\limsup_{\tau\to 0} W(u_\tau(t_1), u_\tau(t_2)) \le \lim_{\tau\to 0}\int_{k_1(\tau)\tau}^{k_2(\tau)\tau} m_\tau(t)\,\mathrm{d}t = \int_{t_1}^{t_2} m(t)\,\mathrm{d}t. \tag{3.10}$$

Applying Proposition 3.3.1 of [2] we obtain the convergence (3.5). Passing to the limit in (3.10) we obtain

$$W(u(t_1), u(t_2)) \leq \int_{t_1}^{t_2} m(t)\,\mathrm{d}t, \qquad \forall\, t_1, t_2 \in [0, +\infty), \quad t_1 < t_2,$$

and then $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ and

$$\int_0^{+\infty} |u'|^2(t)\,\mathrm{d}t \leq 2\mathcal{F}_s(u_0) \tag{3.11}$$

holds. □

4. Flow Interchange and Entropy Decay Estimates

We briefly review the flow interchange technique introduced by Matthes, McCann and Savaré [24]. Then, with this technique, we obtain suitable regularity estimates for solutions to (3.3).

Definition 4.1. (Displacement convex entropy). Let $V : [0, +\infty) \to \mathbb{R}$ be a convex function with super linear growth at infinity, such that $V(0) = 0$, $V \in C^1(0, +\infty)$, V is continuous at 0, $\lim_{x\downarrow 0} \frac{V(x)}{x^\alpha} > -\infty$ for some $\alpha > \frac{d}{d+2}$ and the following McCann displacement convexity assumption (introduced in [25]) holds:

$$r \mapsto r^d V(r^{-d}) \qquad \text{is convex and decreasing in } (0, +\infty).$$

If V satisfies the above assumptions, we say that the functional $\mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$, defined by

$$\mathcal{V}(u) = \int_{\mathbb{R}^d} V(u(x))\,\mathrm{d}x$$

if u is absolutely continuous with respect to the Lebesgue measure and $\mathcal{V}(u) = +\infty$ otherwise, is a *displacement convex entropy*. We say that V is the density function of $\mathcal{V}$.

As usual we denote by $D(\mathcal{V})$ the set of all $u \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathcal{V}(u) < +\infty$.

Remark 4.2. The condition on the behavior of V at 0 is needed as usual to have the integrability of the negative part of $V \circ u$, as soon as u is a probability density with finite second moment. Moreover, if $u_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and u_τ^0 is the regularization defined by (3.2), it is clear that $u_\tau^0 \in D(\mathcal{V})$ for any displacement convex entropy $\mathcal{V}$, since u_τ^0 is bounded.

It is well known that a displacement convex entropy $\mathcal{V}$ generates a continuous semigroup $S_t : D(\mathcal{V}) \to D(\mathcal{V})$ satisfying the following family of *Evolution Variational Inequalities* (see [2, Theorem 11.2.5]):

$$\frac{1}{2}W^2(S_t(u), v) - \frac{1}{2}W^2(u, v) \leq t(\mathcal{V}(v) - \mathcal{V}(S_t(u))) \quad \forall u, v \in D(\mathcal{V}), \quad \forall t > 0, \tag{4.1}$$

and $S_t(\bar{u})$ is the unique distributional solution of the Cauchy problem

$$\partial_t u = \Delta(L_V(u)), \quad u(0) = \bar{u},$$

where $L_V(u) := uV'(u) - V(u)$, such that (4.1) holds. The semigroup is contractive with respect to W and extends to $\overline{D(\mathcal{V})} = \mathcal{P}_2(\mathbb{R}^d)$. Thanks to the regularizing effect $S_t(u) \in D(\mathcal{V})$ for any $u \in \mathcal{P}_2(\mathbb{R}^d)$ and any $t > 0$, we obtain that (4.1) holds for every $u, v \in \mathcal{P}_2(\mathbb{R}^d)$.

If $u \in D(\mathcal{F}_s)$ we define the dissipation of $\mathcal{F}_s$ along the flow S_t of $\mathcal{V}$ by

$$\mathfrak{D}_{\mathcal{V}}\mathcal{F}_s(u) := \limsup_{t \downarrow 0} \frac{\mathcal{F}_s(u) - \mathcal{F}_s(S_t(u))}{t}.$$

Proposition 4.3. (Flow interchange). *Let* $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ *be the sequence given by* Proposition 3.2 *and* $\mathcal{V}$ *a displacement convex entropy. If*

$$\mathfrak{D}_{\mathcal{V}}\mathcal{F}_s\left(u_\tau^k\right) > -\infty \quad \text{for } k \geq 1, \tag{4.2}$$

then $u_\tau^k \in D(\mathcal{V})$ *and*

$$\mathfrak{D}_{\mathcal{V}}\mathcal{F}_s\left(u_\tau^k\right) \leq \frac{\mathcal{V}\left(u_\tau^{k-1}\right) - \mathcal{V}\left(u_\tau^k\right)}{\tau}, \qquad k = 1, 2, \ldots .$$

Proof. We have $u_\tau^0 \in D(\mathcal{V})$, see Remark 4.2. For $t > 0$ and $k > 0$, by definition of minimizer there holds

$$\mathcal{F}_s\left(u_\tau^k\right) + \frac{1}{2\tau}W^2\left(u_\tau^k, u_\tau^{k-1}\right) \leq \mathcal{F}_s\left(S_t\left(u_\tau^k\right)\right) + \frac{1}{2\tau}W^2\left(S_t\left(u_\tau^k\right), u_\tau^{k-1}\right),$$

that is,

$$\tau\left(\mathcal{F}_s\left(u_\tau^k\right) - \mathcal{F}_s\left(S_t\left(u_\tau^k\right)\right)\right) \leq \frac{1}{2}W^2\left(S_t\left(u_\tau^k\right), u_\tau^{k-1}\right) - \frac{1}{2}W^2\left(u_\tau^k, u_\tau^{k-1}\right).$$

By using (4.1) we obtain

$$\tau\frac{\mathcal{F}_s\left(u_\tau^k\right) - \mathcal{F}_s\left(S_t\left(u_\tau^k\right)\right)}{t} \leq \mathcal{V}\left(u_\tau^{k-1}\right) - \mathcal{V}\left(S_t\left(u_\tau^k\right)\right).$$

As $u_\tau^0 \in D(\mathcal{V})$, we may now recursively apply the above inequality; thanks to (4.2), by passing to the limit as $t \downarrow 0$ and using the lower semicontinuity of $\mathcal{V}$ with respect to the narrow convergence we conclude. □

Remark 4.4. With the next lemmas we will characterize the dissipation and show that (4.2) holds true for any displacement convex entropy $\mathcal{V}$.

4.1. Improved Regularity

The following result makes use of flow interchange with the choice $\mathcal{V} = \mathcal{H}$, the entropy functional:

Lemma 4.5. *Let* $u_0 \in D(\mathcal{F}_s)$ *and* $\{u_\tau^k : k = 0, 1, 2, \dots\}$ *the sequence given by* Proposition 3.2. *Then* $u_\tau^k \in \dot H^{1-s}(\mathbb{R}^d) \cap D(\mathcal{H})$ *for any* $k \geq 0$ *and*

$$\|u_\tau^k\|^2_{\dot H^{1-s}(\mathbb{R}^d)} \leq \frac{\mathcal{H}\left(u_\tau^{k-1}\right) - \mathcal{H}\left(u_\tau^k\right)}{\tau}, \qquad k = 1, 2, \dots. \tag{4.3}$$

In particular,

$$\mathcal{H}\left(u_\tau^k\right) \leq \mathcal{H}\left(u_\tau^{k-1}\right), \qquad k = 1, 2, \dots.$$

Proof. By its definition in (3.2), it is clear that $u_\tau^0 \in D(\mathcal{H})$.

We denote by S_t the heat semigroup on $\mathbb{R}^d$, namely the flow generated by the entropy $\mathcal{H}$. For $k \geq 0$ we have $S_t(u_\tau^k) \in \dot H^{1-s}(\mathbb{R}^d)$ for any $t > 0$. Indeed, by the uniqueness of the solution of the heat equation the representation $S_t(u_\tau^k) = \Gamma_t * u_\tau^k$ holds, where Γ_t denotes the family of gaussian kernels (3.1). Then, using the notation $w_t := S_t(u_\tau^k)$, since $\hat\Gamma_t$ is a Gaussian, by (3.7) we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\xi|^{2(1-s)} |\hat w_t(\xi)|^2 \,\mathrm{d}\xi &= \int_{\mathbb{R}^d} |\xi|^{2(1-s)} |\hat\Gamma_t(\xi)|^2 |\hat u_\tau^k(\xi)|^2 \,\mathrm{d}\xi \\ &\leq C_t \|u_\tau^k\|^2_{\dot H^{-s}(\mathbb{R}^d)} \leq 2C_t \mathcal{F}_s\left(u_\tau^k\right) \\ &\leq 2C_t \mathcal{F}_s(u_0) < +\infty,\end{aligned}$$

where $C_t := \max_{\xi\in\mathbb{R}^d} |\xi|^2 |\hat\Gamma_t(\xi)|^2$. Since $u_\tau^0 := \Gamma_{\omega(\tau)} * u_0$ (see (3.2)), a similar argument shows that $u_\tau^0 \in \dot H^{1-s}(\mathbb{R}^d)$.

Next we let $k > 0$ and we consider the real function $t \mapsto \mathcal{F}_s(w_t)$ for $t \in [0, +\infty)$. We claim that this function is differentiable in $(0, +\infty)$ and continuous at $t = 0$, and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_s(w_t) = -\left\| S_t\left(u_\tau^k\right)\right\|^2_{\dot H^{1-s}(\mathbb{R}^d)} = -\|w_t\|^2_{\dot H^{1-s}(\mathbb{R}^d)} \qquad \forall\, t \in (0, +\infty). \tag{4.4}$$

To show this we recall that in Fourier variables the heat equation reads $\partial_t \hat w_t(\xi) + |\xi|^2 \hat w_t(\xi) = 0$ in $\mathbb{R}^d \times (0, +\infty)$. Taking into account the smoothness of w_t we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_s(w_t) &= \frac{1}{2(2\pi)^d}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\xi|^{-2s}\overline{\hat w_t(\xi)}\hat w_t(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s}\overline{\hat w_t(\xi)}\partial_t \hat w_t(\xi)\,\mathrm{d}\xi \\ &= -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s}\overline{\hat w_t(\xi)}|\xi|^2 \hat w_t(\xi)\,\mathrm{d}\xi = -\|w_t\|^2_{\dot H^{1-s}(\mathbb{R}^d)},\end{aligned}$$

and thus the desired differentiability and (4.4) follow. Now, we prove that the map $t \mapsto \mathcal{F}_s(w_t)$ is continuous at $t = 0$. Indeed, since $0 < \hat{\Gamma}_t(\xi) \leq 1$ we have $|\hat{w}_t(\xi)|^2 = |\hat{\Gamma}_t(\xi)\hat{u}_\tau^k(\xi)|^2 \leq |\hat{u}_\tau^k(\xi)|^2$ and it follows that $\mathcal{F}_s(w_t) \leq \mathcal{F}_s(u_\tau^k)$. Since $\mathcal{F}_s$ is lower semi continuous with respect to the narrow convergence, the continuity at 0 follows.

By Lagrange's mean value Theorem, for every $t > 0$ there exists $\theta(t) \in (0, t)$ such that

$$\frac{\mathcal{F}_s\left(u_\tau^k\right) - \mathcal{F}_s\left(S_t\left(u_\tau^k\right)\right)}{t} = \left\| S_{\theta(t)}\left(u_\tau^k\right)\right\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)}.$$

By the lower semicontinuity of the $\dot{H}^{1-s}$ norm with respect to the narrow convergence it follows that

$$\left\| u_\tau^k \right\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)} \leq \mathfrak{D}_{\mathcal{H}}\mathcal{F}_s\left(u_\tau^k\right).$$

Then, by Proposition 4.3, we obtain that $u_\tau^k \in D(\mathcal{H}) \cap \dot{H}^{1-s}(\mathbb{R}^d)$ and (4.3) holds. □

Integrating the estimate (4.3) with respect to time, we obtain the following space-time bound on the discrete solution u_τ. For the integer part of the real number a we use the notation $[a] := \max\{m \in \mathbb{Z} : m \leq a\}$.

Corollary 4.6. *Let $u_0 \in D(\mathcal{F}_s)$, $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ the sequence given by* Proposition 3.2 *and u_τ the corresponding discrete piecewise constant approximate solution defined in* (3.4). *Then $u_\tau(t) \in \dot{H}^{1-s}(\mathbb{R}^d)$ for every $t > 0$ and*

$$\int_{T_0}^{T} \|u_\tau(t)\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)}\,\mathrm{d}t \leq \mathcal{H}\left(u_\tau^{N_0(\tau)}\right) + c\left(1 + T\mathcal{F}_s(u_0) + \int_{\mathbb{R}^d} |x|^2\,\mathrm{d}u_0(x)\right) \tag{4.5}$$

holds for any $T_0 \geq 0$ and $T > T_0$, where $N_0(\tau) := [T_0/\tau]$ and c is a constant depending only on the dimension d.

Proof. Let $T > 0$, $N = \lceil T/\tau \rceil$ and $N_0 = N_0(\tau)$. By (4.3) we obtain

$$\int_{T_0}^{T} \|u_\tau\,(t)\,\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)}\,\mathrm{d}t \leq \sum_{k=N_0+1}^{N} \tau\|u_\tau^k\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)} \leq \mathcal{H}\left(u_\tau^{N_0}\right) - \mathcal{H}\left(u_\tau^N\right).$$

By a Carleman type inequality there holds

$$-\mathcal{H}(u_\tau^N) \leq \tilde{c}\left(1 + \int_{\mathbb{R}^d} |x|^2 u_\tau^N(x)\,\mathrm{d}x\right)$$

for a suitable constant depending only on d. From (3.9) we obtain

$$-\mathcal{H}(u_\tau^N) \leq c\left(1 + T\mathcal{F}_s(u_0) + \int_{\mathbb{R}^d} |x|^2\,\mathrm{d}u_0(x)\right)$$

for c depending only on the dimension d and we conclude. □

4.2. Decay of the Entropies

In the next Lemma we apply the flow interchange to a general displacement convex entropy $\mathcal{G}$ and we compute a lower bound for the dissipation of the functional $\mathcal{F}_s$ along the flow of $\mathcal{G}$. This result is useful for the regularizing effect and the L^p estimates.

Lemma 4.7. *Let* $u_0 \in D(\mathcal{F}_s)$ *and* $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ *the sequence given by* Proposition 3.2. *Let* $\mathcal{G}$ *be a displacement convex entropy with density function* G*, according to* Definition 4.1. *Then* $u_\tau^k \in D(\mathcal{G})$ *for any* $k \geq 0$ *and there holds*

$$0 \leq \left\langle u_\tau^k, L_G\left(u_\tau^k\right)\right\rangle_{1-s} \leq \frac{\mathcal{G}\left(u_\tau^{k-1}\right) - \mathcal{G}\left(u_\tau^k\right)}{\tau}, \qquad k = 1, 2, \ldots. \tag{4.6}$$

In particular,

$$\mathcal{G}\left(u_\tau^k\right) \leq \mathcal{G}\left(u_\tau^{k-1}\right), \qquad k = 1, 2, \ldots.$$

Proof. The proof is based on the same argument of Lemma 4.5. First of all, we have $u_\tau^0 \in D(\mathcal{G})$ by Remark 4.2.

For $\varepsilon > 0$ we consider the displacement convex entropy

$$\mathcal{V}(u) := \mathcal{G}(u) + \varepsilon\mathcal{H}(u).$$

We denote by S_t the flow associated to $\mathcal{V}$ with respect to the Wasserstein distance. Let us fix $k > 0$ and define $w_t := S_t(u_\tau^k)$, thus w_t satisfies the equation

$$\partial_t w_t = \Delta L_G(w_t) + \varepsilon \Delta w_t = \Delta \Psi(w_t), \tag{4.7}$$

with initial datum u_τ^k, where $L_G(v) = vG'(v) - G(v)$ and $\Psi(v) = L_G(v) + \varepsilon v$. Equation (4.7) is a quasilinear non degenerate parabolic equation since Ψ satisfies $\Psi' > 0$. As a result, the solution w_t is bounded, smooth and strictly positive for $t > 0$ (see for example [31, Chapter 3]). Moreover since Lemma 4.5 gives $u_\tau^k \in \dot{H}^{1-s}(\mathbb{R}^d)$ for any $k > 0$ and $u_\tau^k \in L^1(\mathbb{R}^d)$ by construction, we have that $u_\tau^k \in L^2(\mathbb{R}^d)$ thanks to the Sobolev embedding (2.2). Now, if we test equation (4.7) with w_t, we immediately get (recall that L_G is monotone increasing)

$$\|w_t\|_{L^2(\mathbb{R}^d)} \leq \|u_\tau^k\|_{L^2(\mathbb{R}^d)}, \quad \forall t > 0. \tag{4.8}$$

Thus, the estimate above combined with the lower semi continuity of the norm, gives the strong continuity in $L^2(\mathbb{R}^d)$ of the semigroup.

By making use of the transformed version of (4.7), there holds, for any $t > 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_s(w_t) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s}\overline{\hat{w}_t(\xi)}\partial_t \hat{w}_t(\xi)\,\mathrm{d}\xi \\
&= -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s}\overline{\hat{w}_t(\xi)}|\xi|^2\left(\widehat{L_G(w_t)}(\xi) + \varepsilon\hat{w}_t(\xi)\right) \\
&= -\langle w_t, L_G(w_t)\rangle_{1-s} - \varepsilon\langle w_t, w_t\rangle_{1-s}.
\end{aligned}$$

Notice that L_G is non decreasing and locally Lipschitz, and since w_t is bounded and $w_t \in H^{1-s}(\mathbb{R}^d)$ for $t \in (0, +\infty)$, from Proposition 2.2 we obtain $L_G \circ w_t \in \dot{H}^{1-s}(\mathbb{R}^d)$ and $\langle w_t, L_G(w_t)\rangle_{1-s} \geq 0$ for t in $(0, +\infty)$. In particular, $t \mapsto \mathcal{F}_s(w_t)$ is differentiable in $(0, +\infty)$.

Next we shall prove that $t \mapsto \mathcal{F}_s(w_t)$ is continuous at $t = 0$. Since w_t is a probability density, we have that $|\hat{w}_t(\xi)| \leq 1$ for any $\xi \in \mathbb{R}^d$. Thus, for every $t \in [0, +\infty)$ and for some $\delta > 0$ we have

$$\begin{aligned}\|S_t(u_\tau^k)\|^2_{\dot{H}^{-s-\delta}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\xi|^{-2s-2\delta}|\hat{w}_t(\xi)|^2\,\mathrm{d}\xi \\ &\leq \int_{\{|\xi|\geq 1\}} |\hat{w}_t(\xi)|^2\,\mathrm{d}\xi + \int_{\{|\xi|<1\}} |\xi|^{-2s-2\delta}\,\mathrm{d}\xi.\end{aligned}$$

By (4.8) and Plancherel's Theorem, for $0 < \delta < d/2 - s$ the previous estimate shows that $\|S_t(u_\tau^k)\|_{\dot{H}^{-s-\delta}(\mathbb{R}^d)} \leq c$ for every $t \in [0, 1]$, where c is a constant not depending on t. Then, for a suitable $\theta \in (0, 1)$ and $0 < \delta < d/2 - s$, by interpolation we have

$$\begin{aligned}\left\|S_t\left(u_\tau^k\right) - u_\tau^k\right\|_{\dot{H}^{-s}(\mathbb{R}^d)} &\leq \left\|S_t\left(u_\tau^k\right) - u_\tau^k\right\|^{1-\theta}_{L^2(\mathbb{R}^d)} \left\|S_t\left(u_\tau^k\right) - u_\tau^k\right\|^{\theta}_{\dot{H}^{-s-\delta}(\mathbb{R}^d)} \\ &\leq (2c)^\theta \left\|S_t\left(u_\tau^k\right) - u_\tau^k\right\|^{1-\theta}_{L^2(\mathbb{R}^d)},\end{aligned}$$

and the obtained $L^2(\mathbb{R}^d)$ strong continuity of S_t implies that $t \mapsto \mathcal{F}_s(w_t)$ is continuous at $t = 0$.

By the same argument of Lemma 4.5, based on Lagrange mean value theorem, we obtain for suitable $\theta(t) \in (0, t)$

$$\begin{aligned}\frac{\mathcal{F}_s\left(u_\tau^k\right) - \mathcal{F}_s\left(S_t\left(u_\tau^k\right)\right)}{t} &= \varepsilon \left\|S_{\theta(t)}\left(u_\tau^k\right)\right\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)} \\ &\quad + \left\langle S_{\theta(t)}\left(u_\tau^k\right), L_G\left(S_{\theta(t)}\left(u_\tau^k\right)\right)\right\rangle_{1-s}.\end{aligned}$$

Notice that the map $u \mapsto \langle u, L_G(u)\rangle_{1-s}$ is lower semicontinuous with respect to the strong $L^2(\mathbb{R}^d)$ convergence. This follows by applying Fatou's lemma to the expression (2.9), where the integrand is nonnegative in this case, since L_G is nondecreasing (see Proposition 2.2). Therefore, by passing to the limit as $t \downarrow 0$ we obtain

$$0 \leq \left\langle u_\tau^k, L_G\left(u_\tau^k\right)\right\rangle_{1-s} \leq \liminf_{t\downarrow 0} \frac{\mathcal{F}_s\left(u_\tau^k\right) - \mathcal{F}_s\left(S_t\left(u_\tau^k\right)\right)}{t} \leq \mathfrak{D}_{\mathcal{V}}\mathcal{F}_s\left(u_\tau^k\right).$$

The latter estimate, together with Proposition 4.3, entails $u_\tau^k \in D(\mathcal{V})$ and

$$\begin{aligned}&\tau\left\langle u_\tau^k, L_G\left(u_\tau^k\right)\right\rangle_{1-s} + \mathcal{G}\left(u_\tau^k\right) + \varepsilon\mathcal{H}\left(u_\tau^k\right) \leq \mathcal{G}\left(u_\tau^{k-1}\right) + \varepsilon\mathcal{H}\left(u_\tau^{k-1}\right), \\ &\qquad k = 1, 2, \ldots.\end{aligned}$$

In particular, for $k = 1, 2, \ldots$ there is $u_\tau^k \in D(\mathcal{G})$ and $\langle u_\tau^k, L_G(u_\tau^k)\rangle_{1-s} < +\infty$. By letting $\varepsilon \to 0$ we find that (4.6) holds. □

4.3. Regularizing Effect

In order to obtain a quantitative decay of a positive logarithmic entropy and of the L^p norms of the discrete solution we need the two following propositions.

Proposition 4.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex C^1 function and $\tau > 0$. If a_k and b_k satisfies*

$$a_k - a_{k-1} \le -\tau\phi'(a_k), \qquad b_k - b_{k-1} = -\tau\phi'(b_k), \quad \forall k \in \mathbb{N}$$

and $a_0 \le b_0$, then $a_k \le b_k$ for every $k \in \mathbb{N}$.

Proof. By induction, assuming that $a_{k-1} \le b_{k-1}$ we have that

$$a_k + \tau\phi'(a_k) \le a_{k-1} \le b_{k-1} = b_k + \tau\phi'(b_k).$$

Since the function $r \mapsto r + \tau\phi'(r)$ is strictly increasing we conclude. □

Proposition 4.9. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex C^1 function and $\tau > 0$. Let $b_0 \in \mathbb{R}$ and b_k be satisfying*

$$b_k - b_{k-1} = -\tau\phi'(b_k), \quad \forall k \in \mathbb{N}$$

and $b : [0, +\infty) \to \mathbb{R}$ the solution of the Cauchy problem

$$b'(t) = -\phi'(b(t)), \qquad b(0) = b_0. \tag{4.9}$$

Then $|b_k - b(k\tau)| \le \frac{1}{\sqrt{2}}|\phi'(b_0)|\tau$.

Proof. The result is the error estimate for the Euler implicit discretization scheme. See for instance the general expression derived by Nochetto–Savaré–Verdi [26] and [2, Theorem 4.0.7]. □

In the following of the paper we denote by $\mathcal{K} : \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty]$ the positive entropy defined by $\mathcal{K}(u) := \int_{\mathbb{R}^d} u(x)\log(u(x) + 1)\,\mathrm{d}x$ if u is absolutely continuous with respect to the Lebesgue measure and $\mathcal{K}(u) = +\infty$ otherwise, which is a displacement convex entropy according to Definition 4.1.

Lemma 4.10. *Let $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ be the sequence given by* Proposition 3.2. *There holds*

$$\mathcal{K}\left(u_\tau^k\right) \le \min\left\{\mathcal{K}\left(u_\tau^0\right), C_0\,(k\tau)^{-\gamma_0}\right\} + \frac{\tilde{C}_0}{\sqrt{2}}\,\tau\left(\mathcal{K}\left(u_\tau^0\right)\right)^{\beta_0}, \qquad k = 1, 2, \ldots, \tag{4.10}$$

where $\gamma_0 := \frac{1}{2}\frac{d}{d+2(1-s)}$, $\beta_0 := \frac{3d+4(1-s)}{d}$, $\tilde{C}_0 := 2^{-\frac{3d+4(1-s)}{d}} A_{d,s}$, $C_0 = (\tilde{C}_0(\beta_0 - 1))^{-\gamma_0}$, $A_{d,s} := S_{d,1-s}^{-2}$ if $d \ge 2$, $A_{1,s} := S_{1,\frac{1-s}{4-2s}}^{2s-2}$ and $S_{d,r}$ is defined by (2.3). Moreover, for every $p \in (1, +\infty)$ there holds

$$\left\|u_\tau^k\right\|_{L^p(\mathbb{R}^d)}^p \le \min\left\{\left\|u_\tau^0\right\|_{L^p(\mathbb{R}^d)}^p, C_p^p\,(k\tau)^{-p\gamma_p}\right\} + \frac{\tilde{C}_p}{\sqrt{2}}\,\tau\,\left\|u_\tau^0\right\|_{L^p(\mathbb{R}^d)}^{p\beta_p}, \quad k = 1, 2, \ldots, \tag{4.11}$$

where $\gamma_p := \frac{p-1}{p}\frac{d}{d+2(1-s)}$, $\beta_p := \frac{pd+2(1-s)}{(p-1)d}$, $\tilde{C}_p := \frac{4p(p-1)}{(p+1)^2}B_{d,s}$, $C_p := (\tilde{C}_p(\beta_p - 1))^{-\gamma_p}$, $B_{d,s} := S_{d,1-s}^{-2}$ if $d \ge 2$ and $B_{1,s} := S_{1,\frac{1-s}{4-2s}}^{4s-8}$.

Proof. We shall apply Lemma 4.7 to the particular cases $\mathcal{G} = \mathcal{K}$ and $\mathcal{G} = \mathcal{G}_p$, where $\mathcal{G}_p$ is the displacement convex entropy with power density function $G_p(u) = \frac{1}{p-1}u^p$, for $p \in (1, +\infty)$.

Let us start with $\mathcal{G} = \mathcal{K}$, so that the density function is $G(u) = u(\log u + 1)$. In this case $L_G(u) := uG'(u) - G(u) = \frac{u^2}{u+1}$. Since L_G is increasing on $[0, +\infty)$ and $L'_G(u) = \frac{u}{u+1} < 1$ by Proposition 2.2 we have, for any $k \in \mathbb{N}$,

$$+\infty > \left\langle u_\tau^k, L_G\left(u_\tau^k\right)\right\rangle_{1-s} \geq \left\langle L_G\left(u_\tau^k\right), L_G\left(u_\tau^k\right)\right\rangle_{1-s} = \left\| L_G\left(u_\tau^k\right)\right\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)}. \tag{4.12}$$

Since $0 \leq L_G(u) < u$ we have $\|L_G(u_\tau^k)\|_{L^1(\mathbb{R}^d)} \leq \|u_\tau^k\|_{L^1(\mathbb{R}^d)} = 1$. Therefore $L_G \circ u_\tau^k \in \dot{H}^{1-s}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Using (2.4) in the case $d \geq 2$ and (2.6) in the case $d = 1$ we obtain

$$\left\| L_G\left(u_\tau^k\right)\right\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)} \geq A_{d,s} \int_{\mathbb{R}^d} \left(L_G\left(u_\tau^k\right)\right)^q \,\mathrm{d}x \tag{4.13}$$

for $q := 2 + 2(1-s)/d$, where $A_{d,s} := S_{d,1-s}^{-2}$ if $d \geq 2$, $A_{1,s} := S^{2s-2}_{1,\frac{1-s}{4-2s}}$. By Jensen inequality we have

$$\begin{aligned}\int_{\mathbb{R}^d} \left(L_G\left(u_\tau^k\right)\right)^q \mathrm{d}x &= \int_{\mathbb{R}^d} \frac{\left(u_\tau^k\right)^{2q-1}}{\left(u_\tau^k + 1\right)^q}\, u_\tau^k \,\mathrm{d}x \\ &\geq \left(\int_{\mathbb{R}^d} \frac{u_\tau^k}{\left(u_\tau^k+1\right)^{q/(2q-1)}}\, u_\tau^k \,\mathrm{d}x\right)^{2q-1},\end{aligned}$$

and an elementary computation shows that, for any $u \in [0, +\infty)$, there holds

$$\frac{2u^2}{(u+1)^{q/(2q-1)}} \geq u \log(u+1),$$

then we have

$$\int_{\mathbb{R}^d} \left(L_G\left(u_\tau^k\right)\right)^q \,\mathrm{d}x \geq 2^{1-2q} \left(\mathcal{K}\left(u_\tau^k\right)\right)^{2q-1}. \tag{4.14}$$

Thanks to (4.12), (4.13) and (4.14) we find

$$\left\langle u_\tau^k, L_G\left(u_\tau^k\right)\right\rangle_{1-s} \geq \tilde{C}_0 \left(\mathcal{K}\left(u_\tau^k\right)\right)^{2q-1},$$

where $\tilde{C}_0 := 2^{1-2q} A_{d,s} = 2^{-\frac{3d+4(1-s)}{d}} A_{d,s}$. By applying Lemma 4.7 we obtain

$$\mathcal{K}\left(u_\tau^k\right) + \tilde{C}_0 \tau \left(\mathcal{K}\left(u_\tau^k\right)\right)^{\beta_0} \leq \mathcal{K}\left(u_\tau^{k-1}\right), \qquad k = 1, 2, \ldots, \tag{4.15}$$

where $\beta_0 := 2q - 1 = \frac{3d+4(1-s)}{d}$.

Let us now consider, for $p \in (1, +\infty)$ the case $\mathcal{G} = \mathcal{G}_p$, with density function $G = G_p$. Taking into account that $L_{G_p}(u) = u^p$, by Lemma 4.7 and the Stroock–Varopoulos inequality (Proposition 2.2), we have $(u_\tau^k)^{(p+1)/2} \in \dot{H}^{1-s}(\mathbb{R}^d)$ and

$$\tau \frac{4p(p-1)}{(p+1)^2} \left\| \left(u_\tau^k\right)^{(p+1)/2} \right\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)} + \left\| u_\tau^k \right\|^p_{L^p(\mathbb{R}^d)} \le \left\| u_\tau^{k-1} \right\|^p_{L^p(\mathbb{R}^d)} \qquad k = 1, 2, \dots .$$

By (2.5) with $r = 1 - s$ in the case $d \ge 2$ and (2.7) in the case $d = 1$, both with the choice $u = u_\tau^k$, we obtain

$$\tau \frac{4p(p-1)}{(p+1)^2} B_{d,s} \left(\left\| u_\tau^k \right\|^p_{L^p(\mathbb{R}^d)} \right)^{\beta_p} + \left\| u_\tau^k \right\|^p_{L^p(\mathbb{R}^d)} \le \left\| u_\tau^{k-1} \right\|^p_{L^p(\mathbb{R}^d)}, \qquad k = 1, 2, \dots, \tag{4.16}$$

where $\beta_p := \frac{pd+2(1-s)}{(p-1)d}$, $B_{d,s} := S_{d,1-s}^{-2}$ if $d \ge 2$ and $B_{1,s} := S^{4s-8}_{1,\frac{1-s}{4-2s}}$.

Now we are ready to conclude for both the cases $\mathcal{G} = \mathcal{K}$ and $\mathcal{G} = \mathcal{G}_p$. Setting $a_k := \mathcal{K}(u_\tau^k)$ in the first case and $a_k := \|u_\tau^k\|^p_{L^p(\mathbb{R}^d)}$ in the second case, the relations (4.15), (4.16) read

$$a_k - a_{k-1} \le -\tau C a_k^\beta,$$

where $C = \tilde{C}_0$, $\beta = \beta_0$ in the first case and $C = \tilde{C}_p := \frac{4p(p-1)}{(p+1)^2} B_{d,s}$, $\beta = \beta_p$ in the second case. In both cases, we apply Proposition 4.8 and Proposition 4.9 with the choice $\phi(a) = \frac{C}{\beta+1} a^{\beta+1}$. The solution of the Cauchy problem (4.9) is then $b(t) = (b_0 + C(\beta-1)t)^{1/(1-\beta)}$. Since $\beta > 1$, the function $y \mapsto y^{1/(1-\beta)}$ is decreasing in $(0, +\infty)$. Consequently we have $b(t) \le \min\{b_0, (C(\beta-1)t)^{1/(1-\beta)}\}$. Finally

$$a_k \le b_k \le b(k\tau) + |b_k - b(k\tau)| \le b(k\tau) + \frac{1}{\sqrt{2}} \phi'(b_0)\tau.$$

With the choice $b_0 = \mathcal{K}(u_\tau^0)$ in the first case, we obtain (4.10). With the choice $b_0 = \|u_\tau^0\|^p_{L^p(\mathbb{R}^d)}$ in the second case, we obtain (4.11). □

We may now pass to the limit as $\tau \to 0$ and prove the decay estimates for the solution.

Theorem 4.11. *Let* $\{u_\tau^k : k = 0, 1, 2, \dots\}$ *be the sequence given by* Proposition 3.2. *If* $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ *is a corresponding limit curve given by* Theorem 3.3, *then*

$$\mathcal{K}(u(t)) \le C_0 t^{-\gamma_0}, \qquad t > 0,$$

where C_0, γ_0 are positive constants, whose explicit value is found in Lemma 4.10*, and*

$$\mathcal{K}(u(t)) \leq \lim_{\tau\to 0} \mathcal{K}(u_\tau^0) \qquad t > 0.$$

Moreover, for every $p \in (1, +\infty)$ there holds

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq C_p t^{-\gamma_p}, \qquad t > 0,$$

where the positive constants C_p, γ_p are found in Lemma 4.10 *as well, and*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \lim_{\tau\to 0} \|u_\tau^0\|_{L^p(\mathbb{R}^d)}, \qquad t > 0.$$

Proof. With the choice of u_τ^0 from Sect. 3.2 we immediately have that

$$\lim_{\tau\to 0} \tau \left(\mathcal{K}\left(u_\tau^0\right)\right)^{\beta_0} = 0, \qquad \lim_{\tau\to 0} \tau^{1/p} \left\|u_\tau^0\right\|_{L^p(\mathbb{R}^d)}^{\beta_p} = 0,$$

since the L^p norms of u_τ^0 diverge at most logarithmically as $\tau \to 0$. The proof is now a consequence of Lemma 4.10, of the narrow convergence (3.5) and of the lower semi continuity of $\mathcal{K}$ and of the L^p norms with respect to the narrow convergence. □

5. Euler–Lagrange Equation for the Minimizers

Thanks to Lemma 4.5, we have enough regularity to obtain an Euler–Lagrange equation for discrete minimizers. This necessary condition (5.1) on the minimizers of the scheme is the first step towards a discrete version of a weak formulation of the equation (1.1), (see (6.5)).

Lemma 5.1. *Let $u_0 \in D(\mathcal{F}_s)$. Let $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ be the solution sequence to* (3.3) *given by* Proposition 3.2 *and $v_\tau^k := K_s * u_\tau^k$. Then, for any integer $k \geq 1$ there holds*

$$\int_{\mathbb{R}^d} \nabla v_\tau^k \cdot \eta\, u_\tau^k \,\mathrm{d}x = \frac{1}{\tau}\int_{\mathbb{R}^d} \left(T_{u_\tau^k}^{u_\tau^{k-1}} - I\right)\cdot \eta\, u_\tau^k \,\mathrm{d}x, \qquad \forall\, \eta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d), \tag{5.1}$$

where $T_{u_\tau^k}^{u_\tau^{k-1}}$ is the optimal transport map from u_τ^k to u_τ^{k-1} and I is the identity map on $\mathbb{R}^d$. Moreover, there holds

$$\int_{\mathbb{R}^d} \left|\nabla v_\tau^k\right|^2 u_\tau^k \,dx = \frac{1}{\tau^2} W^2\left(u_\tau^k, u_\tau^{k-1}\right), \quad k = 1, 2, 3, \ldots \tag{5.2}$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. For $\delta \geq 0$ we define $\Phi_\delta : \mathbb{R}^d \to \mathbb{R}^d$ by $\Phi_\delta(x) = x + \delta\eta(x)$. Clearly there exists $\delta_0 > 0$ such that

$$\frac{1}{2} \leq \det(\nabla\Phi_\delta(x)) \leq \frac{3}{2} \qquad \forall x \in \mathbb{R}^d, \quad \forall \delta \in [0,\delta_0],$$

and Φ_δ is a global diffeomorphism. In this proof, for simplicity, we use the following notation: $u := u_\tau^k$ and $u_\delta := (\Phi_\delta)_\# u$.

By the minimum problem (3.3) we have, for $\delta > 0$,

$$0 \leq \frac{1}{\delta}\left(\mathcal{F}_s(u_\delta) - \mathcal{F}_s(u)\right) + \frac{1}{\delta}\left(\frac{1}{2\tau}W^2\left(u_\delta, u_\tau^{k-1}\right) - \frac{1}{2\tau}W^2\left(u, u_\tau^{k-1}\right)\right). \tag{5.3}$$

A standard computation entails

$$\lim_{\delta\to 0}\frac{1}{\delta}\left(\frac{1}{2\tau}W^2\left(u_\delta, u_\tau^{k-1}\right) - \frac{1}{2\tau}W^2\left(u, u_\tau^{k-1}\right)\right) = -\frac{1}{\tau}\int_{\mathbb{R}^d}\left(T_\tau^k - I\right)\cdot\eta u\,\mathrm{d}x. \tag{5.4}$$

We have to compute

$$\lim_{\delta\to 0}\frac{1}{\delta}\left(\mathcal{F}_s(u_\delta) - \mathcal{F}_s(u)\right). \tag{5.5}$$

Since for $a, b \in \mathbb{C}$ it holds $|a|^2 - |b|^2 = (\bar{a}+\bar{b})(a-b) + \bar{a}b - \bar{b}a$ and $\bar{\hat{u}}(\xi) = \hat{u}(-\xi)$, we obtain

$$(2\pi)^d\left(\mathcal{F}_s(u_\delta) - \mathcal{F}_s(u)\right) = \frac{1}{2}\int_{\mathbb{R}^d}|\xi|^{-2s}\big(\hat{u}_\delta(-\xi) + \hat{u}(-\xi)\big)\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big)\,\mathrm{d}\xi,$$

because

$$\int_{\mathbb{R}^d}|\xi|^{-2s}\hat{u}_\delta(-\xi)\hat{u}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d}|\xi|^{-2s}\hat{u}_\delta(\xi)\hat{u}(-\xi)\,\mathrm{d}\xi.$$

After defining $\hat{v}_\delta(\xi) = |\xi|^{-2s}\hat{u}_\delta(\xi)$ and $\hat{v}(\xi) = |\xi|^{-2s}\hat{u}(\xi)$ we write

$$\begin{aligned}\frac{(2\pi)^d}{\delta}\left(\mathcal{F}_s(u_\delta) - \mathcal{F}_s(u)\right) &= \frac{1}{2}\int_{\mathbb{R}^d}\big(\hat{v}_\delta(-\xi) + \hat{v}(-\xi)\big)\frac{1}{\delta}\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big)\,\mathrm{d}\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^d}|\xi|\big(\hat{v}_\delta(-\xi) + \hat{v}(-\xi)\big)|\xi|^{-1}\frac{1}{\delta}\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big)\,\mathrm{d}\xi.\end{aligned} \tag{5.6}$$

We show that $|\xi|\hat{v}_\delta(-\xi)$ converges to $|\xi|\hat{v}(-\xi)$ strongly in $L^2(\mathbb{R}^d)$ as $\delta \to 0$. First of all we observe that there exists a constant c such that

$$\|u_\delta\|_{H^{1-s}(\mathbb{R}^d)} \leq c, \qquad \forall\delta \in [0,\delta_0].$$

In order to obtain this bound we write $u_\delta = \phi_\delta u \circ \Phi_\delta^{-1} + u \circ \Phi_\delta^{-1}$, where $\phi_\delta = \det\nabla\Phi_\delta^{-1} - 1$. Since Φ_δ^{-1} is a global diffeomorphism, close to the identity, and clearly there exists a constant $\tilde{c} > 0$ such that $|\Phi_\delta(x) - \Phi_\delta(y)| \geq \tilde{c}|x-y|$ for any

$x, y \in \mathbb{R}^d$ and any $\delta \in [0, \delta_0]$, we get $\|u \circ \Phi_\delta^{-1}\|_{H^{1-s}(\mathbb{R}^d)} \le \tilde{c}\|u\|_{H^{1-s}(\mathbb{R}^d)}$, see [5, Corollary 1.60]. A similar estimate holds true as well if we multiply by the smooth compactly supported function ϕ_δ, see also of [5, Theorem 1.62]. Then

$$\|u_\delta - u\|_{H^{1-s}(\mathbb{R}^d)} \le c + \|u\|_{H^{1-s}(\mathbb{R}^d)}, \qquad \forall \delta \in [0, \delta_0].$$

Since $\mathrm{supp}(u_\delta - u) = \mathrm{supp}\eta$ is compact we have that $\{u_\delta - u\}_{\delta \in [0,\delta_0]}$ is strongly compact in $H^r(\mathbb{R}^d)$ for any $r < 1-s$. Since $u_\delta \to u$ narrowly as $\delta \to 0$ we obtain that $\|u_\delta - u\|_{H^r(\mathbb{R}^d)} \to 0$ as $\delta \to 0$.

Since $-s < 1 - 2s < 1 - s$, choosing $r \in (1 - 2s, 1 - s) \cap (0, 1 - s)$, by interpolation we have

$$\|\nabla v_\delta - \nabla v\|_{L^2(\mathbb{R}^d)} = \|u_\delta - u\|_{\dot{H}^{1-2s}(\mathbb{R}^d)} \le \|u_\delta - u\|^{1-\theta}_{\dot{H}^{-s}(\mathbb{R}^d)} \|u_\delta - u\|^{\theta}_{\dot{H}^r(\mathbb{R}^d)},$$

where $1-2s = (1-\theta)(-s) + \theta r$. Since $\|u_\delta - u\|_{\dot{H}^{-s}(\mathbb{R}^d)}$ is uniformly bounded for $\delta \in (0, \delta_0)$ we obtain the strong convergence in $L^2(\mathbb{R}^d)$ of $|\xi|\hat{v}_\delta(-\xi)$ to $|\xi|\hat{v}(-\xi)$.

For every $\xi \in \mathbb{R}^d$ the function $g_\xi : [0, +\infty) \to \mathbb{R}$ defined by $g_\xi(\delta) = \hat{u}_\delta(\xi)$ is of class C^1 and

$$g'_\xi(\delta) = -i\xi \cdot \int e^{-i\xi\cdot(x+\delta\eta(x))}\eta(x)u(x)\,\mathrm{d}x.$$

The continuity of the derivative follows from its expression and dominated convergence Theorem. Indeed, by definition of image measure, that is,

$$\hat{u}_\delta(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot(x+\delta\eta(x))}u(x)\,\mathrm{d}x,$$

we have

$$\begin{aligned}\frac{1}{h}\big(\hat{u}_{\delta+h}(\xi) - \hat{u}_\delta(\xi)\big) &= \int_{\mathbb{R}^d} \frac{1}{h}\big(e^{-i\xi\cdot(h\eta(x))} - 1\big)e^{-i\xi\cdot(x+\delta\eta(x))}u(x)\,\mathrm{d}x \\ &\to -i\xi \cdot \int e^{-i\xi\cdot(x+\delta\eta(x))}\eta(x)u(x)\,\mathrm{d}x\end{aligned}$$

as $h \to 0$, by dominated convergence Theorem.

By Lagrange Theorem for every ξ and $\delta > 0$ there exist $\delta_\xi \in [0, \delta)$ such that $\frac{1}{\delta}\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big) = g'_\xi(\delta_\xi)$. Since $|g'_\xi(\delta_\xi)| \le |\xi|\|\eta\|_{L^\infty}$ we obtain that $|\xi|^{-1}\frac{1}{\delta}\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big)$ converges to $-i|\xi|^{-1}\xi \cdot (\widehat{\eta u})(\xi)$ in the sense of distributions, but

$$g'_\xi(\delta) = -i\xi \cdot \widehat{(\eta u)_\delta}(\xi),$$

where $(\eta u)_\delta = (\Phi_\delta)_\#(\eta u)$, and $\|(\eta u)_\delta\|_{L^2(\mathbb{R}^d)} \le 2\|\eta u\|_{L^2(\mathbb{R}^d)}$, so that $|\xi|^{-1}\frac{1}{\delta}\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big)$ is bounded in $L^2(\mathbb{R}^d)$. Consequently, $|\xi|^{-1}\frac{1}{\delta}\big(\hat{u}_\delta(\xi) - \hat{u}(\xi)\big)$ converges to $-i|\xi|^{-1}\xi \cdot (\widehat{\eta u})(\xi)$ weakly in $L^2(\mathbb{R}^d)$ as well.

Eventually, we may pass to the limit in (5.6) by strong vs weak convergence, and using Plancherel Theorem we obtain

$$
\begin{aligned}
(2\pi)^d \lim_{\delta\to 0} \frac{1}{\delta}\left(\mathcal{F}_s(u_\delta) - \mathcal{F}_s(u)\right) &= -i\int_{\mathbb{R}^d} \hat{v}(-\xi)\xi\cdot(\widehat{\eta u})(\xi)\,\mathrm{d}\xi \\
&= -i\sum_{j=1}^d \int_{\mathbb{R}^d} \hat{v}(-\xi)\xi_j\cdot(\widehat{\eta_j u})(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^d \sum_{j=1}^d \int_{\mathbb{R}^d} \partial_{x_j} v(x)\eta_j(x)u(x)\,\mathrm{d}x \\
&= (2\pi)^d \int_{\mathbb{R}^d} \nabla v(x)\cdot\eta(x)u(x)\,\mathrm{d}x. \qquad (5.7)
\end{aligned}
$$

In conclusion, by combining (5.3), (5.4), (5.5), (5.6) and (5.7), we get

$$
0 \le \int_{\mathbb{R}^d} \nabla v\cdot\eta u\,\mathrm{d}x - \frac{1}{\tau}\int_{\mathbb{R}^d}\left(T_\tau^k - I\right)\cdot\eta u\,\mathrm{d}x.
$$

The above inequality is valid also for $-\eta$ instead of η, so that it is indeed an equality and (5.1) holds. From (5.1), it follows that $\tau u_\tau^k \nabla v_\tau^k = (T_{u_\tau^k}^{u_\tau^{k-1}} - I)u_\tau^k$ holds almost everywhere in $\mathbb{R}^d$. Since $W^2(u_\tau^k, u_\tau^{k-1}) = \int_{\mathbb{R}^d} |T_{u_\tau^k}^{u_\tau^{k-1}} - I|^2\, u_\tau^k\,\mathrm{d}x$, (5.2) follows as well. □

6. Convergence and Energy Dissipation

In this Section we prove that the limit curve obtained by means of Theorem 3.3 is indeed a gradient flow solution to problem (1.1): it satisfies (1.1) in the sense of distributions and a corresponding energy dissipation inequality holds.

6.1. Convergence

Lemma 6.1. *Let $u_0 \in \dot{H}^{-s}(\mathbb{R}^d)\cap\mathcal{P}_2(\mathbb{R}^d)$, u_τ the piecewise constant curve defined in (3.4) and $v_\tau(t) := K_s * u_\tau(t)$ defined for $t \ge 0$. Given a vanishing sequence τ_n, let u_{τ_n} be a narrowly convergent subsequence (not relabeled) given by* Theorem 3.3*, u its limit curve and $v(t) := K_s * u(t)$ for $t \ge 0$.*

Then, for any $T_0 > 0$ and $T > T_0$ we have $u \in L^2((T_0,T); H^{1-s}(\mathbb{R}^d))$ and $\nabla v \in L^2((T_0,T); L^2(\mathbb{R}^d))$. Moreover the following convergences hold:

$$
\begin{aligned}
&\phi u_{\tau_n} \to \phi u \quad \text{strongly in } L^2((T_0,T); H^r(\mathbb{R}^d)) \quad \text{as } n\to\infty,\ \forall\phi\in\mathcal{S}(\mathbb{R}^d),\\
&\quad \forall r < 1-s,\\
&u_{\tau_n} \to u \quad \text{strongly in } L^2((T_0,T); L^2_{loc}(\mathbb{R}^d)) \quad \text{as } n\to\infty,\\
&\nabla v_{\tau_n} \to \nabla v \quad \text{weakly in } L^2((T_0,T); L^2(\mathbb{R}^d)) \quad \text{as } n\to\infty. \qquad (6.1)
\end{aligned}
$$

If, in addition, $u_0 \in D(\mathcal{H})$, then the above results also hold for $T_0 = 0$.

Proof. Let $T_0 > 0$. By the definition of u_τ^0 we have that the error in (4.10) vanishes as $\tau \to 0$, that is, $\lim_{\tau\to 0} \tau(\mathcal{K}(u_\tau^0))^{\beta_0} = 0$. As in Corollary 4.6 we let $N_0(\tau) = [T_0/\tau]$. By (4.10) and the inequality $\mathcal{H}(u) \le \mathcal{K}(u)$ we obtain that

$$\limsup_{\tau\to 0} \mathcal{H}\left(u_\tau^{N_0(\tau)}\right) \le C_0 T_0^{-\gamma_0}, \tag{6.2}$$

where the value of the constants C_0 and γ_0 is stated in Lemma 4.10. Since by interpolation, for $\theta = s$, it holds

$$\|u_\tau(t)\|_{L^2(\mathbb{R}^d)} \le \|u_\tau(t)\|_{\dot H^{-s}(\mathbb{R}^d)}^{1-s} \|u_\tau(t)\|_{\dot H^{1-s}(\mathbb{R}^d)}^{s},$$

then by Hölder's inequality, (3.7) and (4.5), we obtain

$$\begin{aligned}\int_{T_0}^{T} \|u_\tau(t)\|_{L^2(\mathbb{R}^d)}^2\, \mathrm{d}t &\le \left(\int_{T_0}^{T} \|u_\tau(t)\|_{\dot H^{-s}(\mathbb{R}^d)}^2\, \mathrm{d}t\right)^{1-s} \left(\int_{T_0}^{T} \|u_\tau(t)\|_{\dot H^{1-s}(\mathbb{R}^d)}^2\, \mathrm{d}t\right)^{s}\\
&\le \Big(2\mathcal{F}_s(u_0)(T-T_0)\Big)^{1-s} \left(\int_{T_0}^{T} \|u_\tau(t)\|_{\dot H^{1-s}(\mathbb{R}^d)}^2\, \mathrm{d}t\right)^{s}\\
&\le \Big(2\mathcal{F}_s(u_0)(T-T_0)\Big)^{1-s} \Big(\mathcal{H}\Big(u_\tau^{N_0(\tau)}\Big)\\
&\quad + c\Big(1 + T\mathcal{F}_s(u_0) + \int_{\mathbb{R}^d} |x|^2\, \mathrm{d}u_0(x)\Big)\Big)^{s}. \end{aligned}\tag{6.3}$$

From (4.5) and the last estimate, by lower semicontinuity we obtain that $u \in L^2((T_0, T); H^{1-s}(\mathbb{R}^d))$.

Taking into account that $-s < 1-2s < 1-s$, by interpolation we obtain, for $\theta = 1-s$,

$$\|u_\tau(t)\|_{\dot H^{1-2s}(\mathbb{R}^d)} \le \|u_\tau(t)\|_{\dot H^{-s}(\mathbb{R}^d)}^{s} \|u_\tau(t)\|_{\dot H^{1-s}(\mathbb{R}^d)}^{1-s},$$

then by Holder's inequality, (3.7) and (4.5) we obtain, as above,

$$\begin{aligned}&\int_{T_0}^{T} \|u_\tau(t)\|_{\dot H^{1-2s}(\mathbb{R}^d)}^2\, \mathrm{d}t\\
&\le \left(\int_{T_0}^{T} \|u_\tau(t)\|_{\dot H^{-s}(\mathbb{R}^d)}^2\, \mathrm{d}t\right)^{s} \left(\int_{T_0}^{T} \|u_\tau(t)\|_{\dot H^{1-s}(\mathbb{R}^d)}^2\, \mathrm{d}t\right)^{1-s}\\
&\le \Big(2\mathcal{F}_s(u_0)(T-T_0)\Big)^{s} \Big(\mathcal{H}\Big(u_\tau^{N_0(\tau)}\Big) + c\Big(1 + T\mathcal{F}_s(u_0) + \int_{\mathbb{R}^d} |x|^2\, \mathrm{d}u_0(x)\Big)\Big)^{1-s}. \end{aligned}\tag{6.4}$$

Since $\widehat{v_\tau(t)}(\xi) = |\xi|^{-2s}\widehat{u_\tau(t)}(\xi)$, by Plancherel Theorem we have $\|\nabla v_\tau(t)\|_{L^2(\mathbb{R}^d)} = \|u_\tau(t)\|_{\dot H^{1-2s}(\mathbb{R}^d)}$. From the previous estimate it follows that $\{\nabla v_\tau\}_{\tau>0}$ is weakly compact in $L^2((T_0, T); L^2(\mathbb{R}^d))$. Moreover ∇v_{τ_k} converges to ∇v in the sense of distributions in $\mathbb{R}^d \times (T_0, T)$. Indeed for $\varphi \in C_c^\infty(\mathbb{R}^d \times (T_0, T); \mathbb{R}^d)$, denoting by φ_t the function $x \mapsto \varphi(x, t)$, by Plancherel's Theorem we have

$$(2\pi)^d \int_{T_0}^{T} \int_{\mathbb{R}^d} \nabla v_{\tau_k} \cdot \varphi\, \mathrm{d}x\, \mathrm{d}t = -i \int_{T_0}^{T} \int_{\mathbb{R}^d} |\xi|^{-2s} \widehat{u_{\tau_k}(t)}(-\xi)\xi \cdot \widehat{\varphi_t}(\xi)\, \mathrm{d}\xi\, \mathrm{d}t.$$

Since $||\xi|^{-2s}\widehat{u_{\tau_k}(t)}(-\xi)\xi\cdot\widehat{\varphi}_t(\xi)| \le |\xi|^{1-2s}|\widehat{\varphi}_t(\xi)|$ and $\widehat{\varphi}_t \in \mathcal{S}(\mathbb{R}^d)$ for every $t\in(T_0,T)$, by (3.5) and Lebesgue dominated convergence the right hand side of the above formula converges to

$$-i\int_{T_0}^{T}\int_{\mathbb{R}^d}|\xi|^{-2s}\widehat{u(t)}(-\xi)\xi\cdot\widehat{\varphi}_t(\xi)\,\mathrm{d}\xi\,\mathrm{d}t = (2\pi)^d\int_{T_0}^{T}\int_{\mathbb{R}^d}\nabla v\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

For the stated compactness in $L^2((T_0,T);L^2(\mathbb{R}^d))$ we obtain (6.1).

Let $\phi\in\mathcal{S}(\mathbb{R}^d)$, $r\in[0,1-s)$ and $\varepsilon>0$. Since $\|u_\tau(t)\|^2_{H^{-s}(\mathbb{R}^d)}\le\|u_\tau(t)\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}$ $\le 2\mathcal{F}_s(u_0)$, then $\{\phi u_\tau(t)\}_{\tau>0}$ is compact in $H^{-s-\varepsilon}(\mathbb{R}^d)$ for any t. Thus, for any t we can select a subsequence $\tau_{n_{k(t)}}$ of τ_n such that $\phi u_{\tau_{n_{k(t)}}}(t)\to w_t$ strongly for some $w_t\in H^{-s-\varepsilon}(\mathbb{R}^d)$. Actually, the subsequence is shown not to depend on t thanks to (3.5) and the uniqueness of the limit. As a result, we have that $\phi u_{\tau_n}(t)\to\phi u(t)$ in $H^{-s-\varepsilon}(\mathbb{R}^d)$ for any $t>0$ and for any $\phi\in\mathcal{S}(\mathbb{R}^d)$. By Proposition 2.1 there exists a constant C such that

$$\begin{aligned}\|\phi u_\tau(t)-\phi u(t)\|^2_{H^{-s-\varepsilon}(\mathbb{R}^d)} &\le C\|u_\tau(t)-u(t)\|^2_{H^{-s-\varepsilon}(\mathbb{R}^d)}\\ &\le C\|u_\tau(t)-u(t)\|^2_{H^{-s}(\mathbb{R}^d)}\\ &\le 2C\|u_\tau(t)\|^2_{H^{-s}(\mathbb{R}^d)}+2C\|u(t)\|^2_{H^{-s}(\mathbb{R}^d)}\\ &\le 8C\mathcal{F}_s(u_0).\end{aligned}$$

Then by dominated convergence we have that $\int_{T_0}^T\|\phi u_{\tau_n}(t)-\phi u(t)\|^2_{H^{-s-\varepsilon}(\mathbb{R}^d)}\,\mathrm{d}t\to 0$ as $n\to+\infty$. For $\theta=(r+s+\varepsilon)/(1+\varepsilon)$, by interpolation we have

$$\begin{aligned}&\int_{T_0}^{T}\|\phi u_\tau(t)-\phi u(t)\|^2_{H^r(\mathbb{R}^d)}\,\mathrm{d}t\\ &\quad\le\left(\int_{T_0}^{T}\|\phi u_\tau(t)-\phi u(t)\|^2_{H^{-s-\varepsilon}(\mathbb{R}^d)}\,\mathrm{d}t\right)^{1-\theta}\\ &\quad\left(\int_{T_0}^{T}\|\phi u_\tau(t)-\phi u(t)\|^2_{H^{1-s}(\mathbb{R}^d)}\,\mathrm{d}t\right)^{\theta}.\end{aligned}$$

Since by Proposition 2.1 there holds

$$\begin{aligned}&\int_{T_0}^{T}\|\phi u_\tau(t)-\phi u(t)\|^2_{H^{1-s}(\mathbb{R}^d)}\,\mathrm{d}t\\ &\le C\int_{T_0}^{T}\|u_\tau(t)-u(t)\|^2_{H^{1-s}(\mathbb{R}^d)}\,\mathrm{d}t\\ &\le 4C\Big(\mathcal{H}\Big(u_\tau^{N_0(\tau)}\Big)+c\Big(1+T\mathcal{F}_s(u_0)+\int_{\mathbb{R}^d}|x|^2\,\mathrm{d}u_0(x)\Big),\end{aligned}$$

the first convergence result follows.

In order to show the second convergence result let $K\subset\mathbb{R}^d$ be a compact and we choose $\phi:\mathbb{R}^d\to\mathbb{R}$ such that $\phi\in C_c^\infty(\mathbb{R}^d)$, $0\le\phi\le1$, $\phi=1$ on K and $r=0$. Since $\|u_\tau(t)-u(t)\|^2_{L^2(K)}\le\|\phi u_\tau(t)-\phi u(t)\|^2_{L^2(\mathbb{R}^d)}$, we conclude.

If $T_0=0$ then $N_0(\tau)=0$, and the last assertion follows from the previous estimates taking into account that $\mathcal{H}(u_\tau^0)\le\mathcal{H}(u_0)$. □

Theorem 6.2. *If* $u \in AC^2([0,+\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ *is a limit curve given by* Theorem 3.3*, and* $v(t) := K_s * u(t)$ *for* $t \geq 0$*, then* u *satisfies the equation in* (1.1) *in the following weak form:*

$$\int_0^{+\infty}\int_{\mathbb{R}^d} (\partial_t\varphi - \nabla\varphi\cdot\nabla v)u\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \textit{for all } \varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d).$$

Proof. We fix $\varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d)$. By (5.1) with the choice of $\eta = \nabla_x\varphi$ (depending on time) and integrating we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^d} \nabla v_\tau\cdot\nabla\varphi\, u_\tau\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\tau}\int_0^{+\infty}\int_{\mathbb{R}^d}(T_\tau - I)\cdot\nabla\varphi\, u_\tau\,\mathrm{d}x\,\mathrm{d}t, \tag{6.5}$$

where T_τ is defined as $T_\tau(t) = T_{u_\tau^k}^{u_\tau^{k-1}}$ if $t \in ((k-1)\tau, k\tau]$. By Lemma 6.1 along a suitable sequence τ_n the left hand side of (6.5) converges to

$$\int_0^{+\infty}\int_{\mathbb{R}^d} \nabla\varphi\cdot\nabla v\, u\,\mathrm{d}x\,\mathrm{d}t$$

By a standard argument, the right hand side of (6.5) converges to

$$\int_0^{+\infty}\int_{\mathbb{R}^d} \partial_t\varphi\, u\,\mathrm{d}x\,\mathrm{d}t,$$

see for instance [2, Theorem 11.1.6]. □

6.2. De Giorgi Interpolant and Discrete Energy Dissipation

In order to obtain an energy dissipation estimate we introduce the so called De Giorgi variational interpolant (see for instance [2, Section 3.2]) as follows: $\tilde{u}_\tau(0) := u_\tau^0$ and

$$\tilde{u}_\tau(t) \in \mathrm{Argmin}_{u\in\mathcal{P}_2(\mathbb{R}^d)}\left\{\frac{1}{2(t-(k-1)\tau)}W^2\left(u, u_\tau^{k-1}\right) + \mathcal{F}_s(u)\right\}$$
$$\text{for } t \in ((k-1)\tau, k\tau], \quad k = 1, 2, \dots .$$

We observe that by the argument in the proof of Proposition 3.2 this interpolant is uniquely defined and $\tilde{u}_\tau(k\tau) = u_\tau^k$ for any $k \in \mathbb{N}$.

Proposition 6.3. *For every* $t > 0$, $\tilde{u}_\tau(t) \in H^{1-s}(\mathbb{R}^d)$ *and, denoting by* $\tilde{v}_\tau(t) := K_s * \tilde{u}_\tau(t)$*, the following discrete energy identity holds for all* $N \in \mathbb{N}$ *and* $\tau > 0$*:*

$$\begin{aligned}&\frac{1}{2}\int_0^{N\tau}\int_{\mathbb{R}^d}|\nabla v_\tau|^2 u_\tau\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^{N\tau}\int_{\mathbb{R}^d}|\nabla\tilde{v}_\tau|^2\tilde{u}_\tau\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \mathcal{F}_s(u_\tau(N\tau)) = \mathcal{F}_s\left(u_\tau^0\right).\end{aligned} \tag{6.6}$$

Moreover,

$$W^2(\tilde{u}_\tau(t), u_\tau(t)) \leq 8\tau\mathcal{F}_s(u_0), \qquad \forall t \in [0,+\infty). \tag{6.7}$$

Proof. Fixing $t > 0$, by the definition of $\tilde{u}_\tau(t)$, the same proof of Lemma 4.5 shows that $\tilde{u}_\tau(t) \in H^{1-s}(\mathbb{R}^d)$. For k such that $t \in ((k-1)\tau, k\tau]$, the same argument of Lemma 5.1 shows that

$$\int_{\mathbb{R}^d} |\nabla \tilde{v}_\tau(t)|^2 \tilde{u}_\tau(t)\, \mathrm{d}x = \frac{1}{(t-(k-1)\tau)^2} W^2\left(\tilde{u}_\tau(t), u_\tau^{k-1}\right). \tag{6.8}$$

From [2, Lemma 3.2.2] we have the one step energy identity

$$\frac{1}{2}\frac{W^2\left(u_\tau^k, u_\tau^{k-1}\right)}{\tau} + \frac{1}{2}\int_{(k-1)\tau}^{k\tau} \frac{W^2\left(\tilde{u}_\tau(t), u_\tau^{k-1}\right)}{(t-(k-1)\tau)^2}\, \mathrm{d}t + \mathcal{F}_s\left(u_\tau^k\right) = \mathcal{F}_s\left(u_\tau^{k-1}\right).$$

Defining the function $G_\tau : (0, +\infty) \to \mathbb{R}$ as

$$G_\tau(t) = \frac{W\left(\tilde{u}(t), u_\tau^{k-1}\right)}{t-(k-1)\tau}, \qquad t \in ((k-1)\tau, k\tau], \quad k = 1, 2, \dots$$

and summing from $k = 1$ to N, we obtain

$$\frac{1}{2}\sum_{k=1}^{N} \tau \frac{W^2\left(u_\tau^k, u_\tau^{k-1}\right)}{\tau^2} + \frac{1}{2}\int_0^{N\tau} G_\tau^2(t)\, \mathrm{d}t + \mathcal{F}_s\left(u_\tau^N\right) = \mathcal{F}_s\left(u_\tau^0\right).$$

Finally 6.6 follows by (5.2) and (6.8).

The estimate (6.7) follows by the definition of $\tilde{u}(t)$, (3.7), the non-negativity of $\mathcal{F}_s$ and the triangle inequality (see also [2, Remark 3.2.3]). □

In order to pass to the limit by lower semicontinuity in (6.6) we recall the following result, see [2, Theorem 5.4.4].

Lemma 6.4. *If $\{\mu_n\}$ is a sequence in $\mathcal{P}(\mathbb{R}^d \times [0, T])$ that narrowly converges to μ and $\{w_n\}$ is a sequence of vector fields in $\mathrm{L}^2(\mathbb{R}^d \times [0, T], \mu_n; \mathbb{R}^d)$ satisfying*

$$\sup_n \int_{\mathbb{R}^d \times [0,T]} |w_n|^2\, \mathrm{d}\mu_n < +\infty, \tag{6.9}$$

then there exists a vector field $w \in \mathrm{L}^2(\mathbb{R}^d \times [0, T], \mu; \mathbb{R}^d)$ and a subsequence (not relabeled here) such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d \times [0,T]} \varphi \cdot w_n\, \mathrm{d}\mu_n = \int_{\mathbb{R}^d \times [0,T]} \varphi \cdot w\, \mathrm{d}\mu, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d \times [0, T]; \mathbb{R}^d),$$

and moreover

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d \times [0,T]} |w_n|^2\, \mathrm{d}\mu_n \geq \int_{\mathbb{R}^d \times [0,T]} |w|^2\, \mathrm{d}\mu. \tag{6.10}$$

Theorem 6.5. *If $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ is a limit curve given by* Theorem 3.3, *and $v(t) := K_s * u(t)$ for $t \geq 0$, then u satisfies the following energy dissipation inequality:*

$$\mathcal{F}_s(u(T)) + \int_0^T \int_{\mathbb{R}^d} |\nabla v(t)|^2 u(t)\, \mathrm{d}x\, \mathrm{d}t \leq \mathcal{F}_s(u_0), \qquad \forall\, T > 0.$$

Proof. Let u_{τ_n} be the sequence of Lemma 6.1. We fix $T > 0$ and we apply Lemma 6.4 to the sequences $\mu_n := \frac{1}{T} u_{\tau_n}$, $w_n := \nabla v_{\tau_n}$ and $\tilde{\mu}_n := \frac{1}{T}\tilde{u}_{\tau_n}$, $\tilde{w}_n := \nabla \tilde{v}_{\tau_n}$. By (6.6) with $N = N_{\tau_n} := \lceil T/\tau_n \rceil$, and by (3.6), the assumption (6.9) is satisfied for both the couples (μ_n, w_n) and $(\tilde{\mu}_n, \tilde{w}_n)$. By (3.5) and (6.7) we have that μ_n and $\tilde{\mu}_n$ converge narrowly to $\mu := \frac{1}{T} u$. By (6.1) we have that the limit point of w_n and $\tilde{w}_n$ is the same $w = \tilde{w} = \nabla v$. Since $\lim_{n\to+\infty} \tau_n N_{\tau_n} = T$, by (3.6), the lower semi continuity of $\mathcal{F}_s$, (6.10) and (3.6) we conclude. □

7. Boundedness of Solutions and L^∞ Decay

In this section we show how to get an L^∞ decay rate starting from the discrete variational approach. We have indeed to extend the estimate of Theorem 4.11 to $p = \infty$. Notice that γ_p therein converges as $p \to \infty$, but the constant C_p blows up. Therefore, we have to go through a more refined argument.

We start by introducing a simple recursive estimate.

Proposition 7.1. *Let $Q > 0$, $R > 0$ and $q > 1$. If a sequence of positive numbers $\{A_j\}_{\{j\geq 0\}}$ satisfies $A_j \leq Q R^j A_{j-1}^q$ for every $j \geq 1$, then*

$$A_j \leq Q^{\beta(j-j_0,q)} R^{\gamma(j-j_0,q)} A_{j_0}^{q^{j-j_0}}, \qquad \forall\, j > j_0 \geq 0, \tag{7.1}$$

where

$$\beta(j,q) = \frac{q^j - 1}{q - 1}, \qquad \gamma(j,q) = \frac{q(q^j - 1)}{(q-1)^2} - \frac{j}{q-1}.$$

Proof. Let $j_0 = 0$. By recursively using the assumption we obtain that

$$A_j \leq \prod_{i=0}^{j-1} (Q R^{j-i})^{q^i} A_0^{q^j} = Q^{\beta(j,q)} R^{\gamma(j,q)} a_0^{q^j}, \qquad j > 0,$$

where indeed

$$\beta(j,q) = \sum_{i=0}^{j-1} q^i = \frac{q^j - 1}{q - 1},$$
$$\gamma(j,q) = \sum_{i=0}^{j-1} (j-i) q^i = \frac{1}{q-1} \sum_{i=1}^{j} (q^i - 1) = \frac{q(q^j - 1)}{(q-1)^2} - \frac{j}{q-1}.$$

If $j_0 > 0$ we apply the previous formula by shifting the indexes. □

Theorem 7.2. *If $u \in AC^2([0,+\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ is a limit curve given by* Theorem 3.3*, then there exists a constant C_∞ depending only on d and s such that*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C_\infty t^{-\gamma_\infty}, \qquad t > 0,$$

where $\gamma_\infty := \frac{d}{d+2(1-s)}$.

Proof. Fix $t > 0$ throughout. We let $\tau > 0$ and we define

$$T_j := t(1 - 2^{-j}), \quad j = 0, 1, 2, \ldots$$

and $j(\tau)$ as the smallest integer j such that $T_j > \tau\lfloor t/\tau \rfloor$, where $\lfloor a \rfloor := \max\{m \in \mathbb{Z} : m < a\}$ denotes the left continuous lower integer part of the real number a. The sequence $\{T_j\}$ satisfies

$$\tau\lfloor t/\tau \rfloor \leq T_{j(\tau)} < T_{j(\tau)+1} < T_{j(\tau)+2} < \cdots < \lim_{j\to+\infty} T_j = t,$$

and $T_j - T_{j-1} = t2^{-j}$. We recursively define $\tilde{u}_{\tau,j}$ by $\tilde{u}_{\tau,j(\tau)} := u_\tau(t)$ and

$$\tilde{u}_{\tau,j} = \operatorname{argmin}_{u\in\mathcal{P}_2(\mathbb{R}^d)} \left\{ \mathcal{F}_s(u) + \frac{1}{2(T_j - T_{j-1})} W_2^2(u, \tilde{u}_{\tau,j-1}) \right\}, \quad j > j(\tau) \tag{7.2}$$

For given $M > 0$ we define $G(u) := (u - M)_+^2$ and $\mathcal{V}$ as the displacement convex entropy with density function G, according to Definition 4.1. By the definition of $\tilde{u}_{\tau,j}$ in (7.2), Lemma 4.10 can be applied and yields

$$(T_j - T_{j-1})\langle \tilde{u}_{\tau,j}, L_G(\tilde{u}_{\tau_j})\rangle_{1-s} \leq \mathcal{V}(\tilde{u}_{\tau,j-1}) - \mathcal{V}(\tilde{u}_{\tau,j}), \qquad j > j(\tau). \tag{7.3}$$

Since $L_G(u) = (u - M)_+^2 + 2M(u - M)_+$, $u \mapsto (u - M)_+^2$ is nondecreasing and $u \mapsto (u - M)_+$ is 1-Lipschitz continuous, by Proposition 2.2 we have

$$\begin{aligned}
\langle u, L_G(u)\rangle_{1-s} &= \left\langle u, (u - M)_+^2 \right\rangle_{1-s} + 2M\langle u, (u - M)_+\rangle_{1-s} \\
&\geq 2M\langle u, (u - M)_+\rangle_{1-s} \\
&\geq 2M\langle (u - M)_+, (u - M)_+\rangle_{1-s} = 2M\|(u - M)_+\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)}.
\end{aligned}$$

Then, since $\mathcal{V} \geq 0$, from (7.3) we find

$$\int_{\mathbb{R}^d} (\tilde{u}_{\tau,j-1}(x) - M)_+^2 \,\mathrm{d}x \geq 2M(T_j - T_{j-1})\|(\tilde{u}_{\tau,j} - M)_+\|^2_{\dot{H}^{1-s}(\mathbb{R}^d)}, \quad j > j(\tau). \tag{7.4}$$

Next, we define

$$A_{j(\tau)} := \|\tilde{u}_{\tau,j(\tau)}\|^2_{L^2(\mathbb{R}^d)} = \|u_\tau(t)\|^2_{L^2(\mathbb{R}^d)}$$

and we separately treat the cases $d \geq 2$ and $d = 1$ in the rest of the proof.

The case $d \geq 2$. We let $q := d/(d - 2 + 2s)$, so that $2q$ is the critical exponent corresponding to the Sobolev inequality (2.2) with $r = 1 - s$ and constant denoted by $S_{d,1-s}$. We define the constant

$$\begin{aligned}
M_\tau(t) &:= \left(\frac{S_{d,1-s}^2}{t}\right)^{\frac{q}{3q-2}} \left(A_{j(\tau)}\, 2^{\frac{q(3q-2)}{(q-1)^2}}\right)^{\frac{q-1}{3q-2}} \\
&= 2^{q/(q-1)} S_{d,1-s}^{2q/(3q-2)} A_{j(\tau)}^{(q-1)/(3q-2)}\, t^{-q/(3q-2)},
\end{aligned}$$

and $M_{\tau,j} := (2-2^{-j})M_\tau(t)$ for $j > j(\tau)$. Finally we define

$$A_j := \int (\tilde{u}_{\tau,j} - M_{\tau,j})_+^2 \,\mathrm{d}x, \quad j > j(\tau).$$

Since $f - M_{\tau,j} > 0$ implies $f - M_{\tau,j-1} = f - M_{\tau,j} + 2^{-j}M_\tau(t) > 2^{-j}M_\tau(t) > 0$, a direct computation and the Sobolev inequality (2.2) entail, for any $j > j(\tau)$

$$\begin{aligned} A_j &\le \left(\frac{2^j}{M_\tau(t)}\right)^{2q-2} \int_{\mathbb{R}^d} (\tilde{u}_{\tau,j}(x) - M_{\tau,j-1})_+^{2q} \,\mathrm{d}x \\ &\le \left(\frac{2^j}{M_\tau(t)}\right)^{2q-2} S_{d,1-s}^{2q} \,\|(\tilde{u}_{\tau,j} - M_{\tau,j-1})_+\|_{\dot{H}^{1-s}(\mathbb{R}^d)}^{2q}. \end{aligned} \tag{7.5}$$

Now we make use of (7.4), with $M_{\tau,j}$ in place of M, and we get for any $j > j(\tau)$, since $M_\tau \le M_{\tau,j}$,

$$\begin{aligned} A_j &\le \left(\frac{2^j}{M_\tau(t)}\right)^{2q-2} S_{d,1-s}^{2q} \left(\frac{2^j}{tM_\tau(t)}\right)^{q} \left(\int_{\mathbb{R}^d} (\tilde{u}_{\tau,j-1}(x) - M_{\tau,j-1})_+^2 \,\mathrm{d}x\right)^q \\ &\le \frac{S_{d,1-s}^{2q}}{t^q M_\tau(t)^{3q-2}} (2^{3q-2})^j \; A_{j-1}^q. \end{aligned} \tag{7.6}$$

We may apply the recursion formula (7.1), with $Q = S_{d,1-s}^{2q} t^{-q} M_\tau(t)^{2-3q}$ and $R = 2^{3q-2}$, starting from $j_0 = j(\tau)$, and we get

$$\begin{aligned} A_j &\le \left(\frac{S_{d,1-s}^{2q}}{t^q M_\tau(t)^{3q-2}}\right)^{\frac{q^{j-j(\tau)}-1}{q-1}} \left(2^{3q-2}\right)^{\frac{q(q^{j-j(\tau)}-1)}{(q-1)^2} - \frac{j-j(\tau)}{q-1}} A_{j(\tau)}^{q^{j-j(\tau)}} \\ &= \left(\frac{S_{d,1-s}^{2q}\, 2^{q(3q-2)/(q-1)}\, A_{j(\tau)}^{q-1}}{t^q M_\tau(t)^{3q-2}}\right)^{\frac{q^{j-j(\tau)}-1}{q-1}} 2^{-(j-j(\tau))(3q-2)/(q-1)}\, A_{j(\tau)} \\ &= 2^{-(j-j(\tau))(3q-2)/(q-1)}\, A_{j(\tau)}, \end{aligned}$$

where we have used the definition of M_τ. As $q > 1$, we have $\lim_{j\to+\infty} A_j = 0$.

Notice that, for $j > j(\tau)$, there holds as in Theorem 3.3 the basic estimate

$$\mathcal{F}_s(\tilde{u}_{\tau,j}) + \frac{W^2(\tilde{u}_{\tau,j}, \tilde{u}_{\tau,j-1})}{2(T_j - T_{j-1})} \le \mathcal{F}_s(\tilde{u}_{\tau,j-1}) \le \mathcal{F}_s(u_0),$$

so that

$$W^2(\tilde{u}_{\tau,j}, \tilde{u}_{\tau,j-1}) \le 2\mathcal{F}_s(u_0)(T_j - T_{j-1}) = 2t\mathcal{F}_s(u_0)\, 2^{-j},$$

then

$$W(\tilde{u}_{\tau,n}, \tilde{u}_{\tau,m}) \le \sqrt{2t\mathcal{F}_s(u_0)} \sum_{j=m+1}^{n} 2^{-j/2}.$$

Therefore, $\{\tilde{u}_{\tau,j}\}_{j\geq j(\tau)}$ is a Cauchy sequence, converging in $\mathcal{P}_2(\mathbb{R}^d)$ as $j \to +\infty$ to a limit point that we denote by $\tilde{u}_\tau(t)$, such that

$$W(\tilde{u}_{\tau,m}, \tilde{u}_\tau(t)) \leq \sqrt{2t\mathcal{F}_s(u_0)} \sum_{j=m+1}^{+\infty} 2^{-j/2}. \tag{7.7}$$

Since $\tilde{u}_{\tau,j}$ narrowly converges to $\tilde{u}_\tau(t)$ as $j \to +\infty$, the lower semicontinuity of $\mathcal{V}$ with respect to the narrow convergence entails (together with $2M_\tau(t) > M_{\tau,j}$)

$$\begin{aligned}\int_{\mathbb{R}^d} (\tilde{u}_\tau(t) - 2M_\tau(t))_+^2 \,\mathrm{d}x &\leq \liminf_{j\to+\infty} \int_{\mathbb{R}^d} (\tilde{u}_{\tau,j} - 2M_\tau(t))_+^2 \,\mathrm{d}x \\ &\leq \liminf_{j\to+\infty} \int_{\mathbb{R}^d} (\tilde{u}_{\tau,j} - M_{\tau,j})_+^2 \,\mathrm{d}x = \lim_{j\to+\infty} A_j = 0,\end{aligned}$$

that is

$$\|\tilde{u}_\tau(t)\|_{L^\infty(\mathbb{R}^d)} \leq 2M_\tau(t) = 2^{(2q-1)/(q-1)} S_{d,1-s}^{2q/(3q-2)} A_{j(\tau)}^{(q-1)/(3q-2)} t^{-q/(3q-2)}. \tag{7.8}$$

However, we apply the estimate (4.11) for $p = 2$ to see that

$$A_{j(\tau)} = \|u_\tau(t)\|_{L^2(\mathbb{R}^d)}^2 \leq C_2^2(\tau\lceil t/\tau\rceil)^{-2\gamma_2} + \frac{\tilde{C}_2}{\sqrt{2}}\,\tau \left\|u_\tau^0\right\|_2^{2\beta_2},$$

where $C_2, \tilde{C}_2, \gamma_2, \beta_2$ are defined in Lemma 4.10 and where the right hand side converges, as $\tau \to 0$, to $C_2^2 t^{-2\gamma_2}$, see Theorem 4.11. Hence, from (7.8) we obtain

$$\limsup_{\tau\to 0} \|\tilde{u}_\tau(t)\|_{L^\infty(\mathbb{R}^d)} \leq K_{s,d}\, t^{-2\gamma_2 \frac{(q-1)}{3q-2} - \frac{q}{3q-2}} = K_{s,d}\, t^{-\frac{d}{d+2-2s}},$$

where

$$K_{s,d} := 2^{(2q-1)/(q-1)} S_{d,1-s}^{2q/(3q-2)} C_2^{2(q-1)/(3q-2)},$$

and where we used $2\gamma_2 = d/(d+2-2s)$ and $q = d/(d-2+2s)$ to compute the exponent of t.

By (7.7) with $m = j(\tau)$ we have

$$W(u_\tau(t), \tilde{u}_\tau(t)) \leq \sqrt{2t\mathcal{F}_s(u_0)} \sum_{j=j(\tau)+1}^{+\infty} 2^{-j/2}. \tag{7.9}$$

Since $j(\tau) \to +\infty$ as $\tau \to 0$, by (7.9) it follows that along a sequence τ_n given by Lemma 6.1 we have that $\{\tilde{u}_{\tau_n}(t)\}_{n\in\mathbb{N}}$ is tight and converges to the same limit point $u(t)$ of $\{u_{\tau_n}(t)\}_{n\in\mathbb{N}}$.

By lower semicontinuity we conclude that

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq K_{s,d}\, t^{-d/(d+2-2s)}.$$

The result is achieved with $C_\infty = K_{s,d}$.

The case $d = 1$ and $0 < s < 1/2$. The argument is analogous to the previous one for $d \geq 2$, we shall only mention the main differences. Instead of defining $q = d/(d-2+2s)$, we fix $r \in (0, 1/2)$ and we let $q := 1/(1-2r)$. We define $\theta := r/(1-s)$, and we change the definition of $M_\tau(t)$ by letting

$$M_\tau(t) := 2^{q/(q-1)} S_{1,r}^{2q/(2q-2+q\theta)} A_{j(\tau)}^{(q-1)/(2q-2+q\theta)} t^{-q\theta/(2q-2+q\theta)}.$$

Using (2.8) instead of (2.2), the analogue of (7.5) is

$$\begin{aligned} A_j &\leq \left(\frac{2^j}{M_\tau(t)}\right)^{2q-2} \int_{\mathbb{R}^d} (\tilde{u}_{\tau,j}(x) - M_{\tau,j-1})_+^{2q} \, dx \\ &\leq \left(\frac{2^j}{M_\tau(t)}\right)^{2q-2} S_{1,r}^{2q} \left\| (\tilde{u}_{\tau,j} - M_{\tau,j-1})_+ \right\|_{L^2(\mathbb{R}^d)}^{2q(1-\theta)} \\ &\left\| (\tilde{u}_{\tau,j} - M_{\tau,j-1})_+ \right\|_{\dot{H}^{1-s}(\mathbb{R}^d)}^{2q\theta} . \end{aligned} \tag{7.10}$$

Moreover by (7.3) we have

$$\|(\tilde{u}_{\tau,j} - M_{\tau,j-1})_+\|_{L^2(\mathbb{R}^d)}^2 \leq \|(\tilde{u}_{\tau,j-1} - M_{\tau,j-1})_+\|_{L^2(\mathbb{R}^d)}^2 = A_{j-1}. \tag{7.11}$$

Using (7.4) and (7.11) in (7.10) we obtain the analogue of (7.6):

$$A_j \leq \left(\frac{2^j}{M_\tau(t)}\right)^{2q-2} S_{1,r}^{2q} \left(\frac{2^j}{t M_\tau(t)}\right)^{q\theta} A_{j-1}^q. \tag{7.12}$$

Then we can apply the recursion formula with the choice of $Q = S_{1,r}^{2q} t^{-q\theta} M_\tau(t)^{2-2q-q\theta}$ and $R = 2^{2q-2+q\theta}$ and we obtain, recalling the choice of $M_\tau(t)$,

$$A_j \leq 2^{-(j-j(\tau))(2q-2+q\theta)/(q-1)} A_{j(\tau)}.$$

The rest of the proof carries over along the line of the case $d \geq 2$. □

Proof of **Theorem 1.1.** We collect all the results that give the proof of the main Theorem. Point (i) follows from Proposition 3.2. Points (ii) and (iii) follow from Theorem 3.3, Lemma 6.1 and Theorem 6.2. Theorem 6.5 yields point (iv). Point (v) is a consequence of Theorem 4.11 and Theorem 7.2 for the case $p < +\infty$ and the case $p = +\infty$, respectively. Finally, point (vi) follows from Lemma 4.5 and Lemma 4.7 by letting $\tau \to 0$ and taking into account the lower semicontinuity of $\mathcal{H}$ and of the L^p norms with respect to the narrow convergence. This gives the result for $p < +\infty$. The case $p = +\infty$ follows by passing to the limit as $p \to +\infty$ in the inequality $\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}$. □

Remark 7.3. If we consider positive measure data with mass $M > 0$, according to Remark 1.2, the constant C_p in point v) has to be multiplied by M^{ℓ_p}, where ℓ_p is given therein. This scaling is deduced from Lemma 4.10 if $p < +\infty$, when making use of (2.5) and (2.7) for obtaining (4.16). We similarly obtain the value of ℓ_∞, since the constant C_∞ in Theorem 7.2 depends on the mass only through C_2.

8. The Limit for $s \to 0$

In this last section we are interested in the asymptotic analysis when $s \to 0$. We start by proving the following lemma which identifies the limit of the sequence of solutions u_s of the equation in (1.1) as $s \to 0$ with the solutions of the porous medium equation (1.3):

Lemma 8.1. *Let $u_0 \in L^2(\mathbb{R}^d)$ and $\{u_0^s\}_{s\in(0,1)}$ be a family of initial data such that $u_0^s \in D(\mathcal{F}_s)$, u_0^s converges narrowly to u_0 as $s \to 0$, $\sup_{s\in(0,1)} \int_{\mathbb{R}^d} |x|^2 \, \mathrm{d}u_0^s(x) < +\infty$ and $\lim_{s\to 0} \mathcal{F}_s(u_0^s) = \mathcal{F}_0(u_0)$. We denote by u^s a solution of problem (1.1) with initial datum u_0^s given by* Theorem 1.1.

If $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ is a vanishing sequence, then there exist a curve $u \in AC^2([0,+\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$ and a subsequence (not relabeled) $\{s_n\}$ such that

$$u^{s_n}(t) \to u(t) \quad \text{narrowly as } n \to \infty \text{ for every } t \geq 0. \tag{8.1}$$

Furthermore, for every T_0, T such that $T > T_0 > 0$ we have

$$u^{s_n} \to u \quad \text{strongly in } L^2((T_0,T); L^2_{loc}(\mathbb{R}^d)) \quad \text{as } n \to \infty, \tag{8.2}$$

and, setting $v^{s_n} = K_{s_n} * u^{s_n}$, we have

$$\nabla v^{s_n} \to \nabla u \quad \text{weakly in } L^2((T_0,T); L^2(\mathbb{R}^d)) \quad \text{as } n \to \infty. \tag{8.3}$$

Moreover, the curve u is a solution of the porous medium equation (1.3) in the following sense:

$$\int_0^{+\infty} \int_{\mathbb{R}^d} (\partial_t \varphi - \nabla\varphi \cdot \nabla u) u \, \mathrm{d}x \, \mathrm{d}t = 0, \quad \text{for all } \varphi \in C_c^\infty((0,+\infty) \times \mathbb{R}^d),$$

and the following energy dissipation inequality holds:

$$\mathcal{F}_0(u(T)) + \int_0^T \int_{\mathbb{R}^d} |\nabla u(t)|^2 u(t) \, \mathrm{d}x \, \mathrm{d}t \leq \mathcal{F}_0(u_0), \qquad \forall\, T > 0. \tag{8.4}$$

Proof. Since $\lim_{s\to 0} \mathcal{F}_s(u_0^s) = \mathcal{F}_0(u_0)$ we fix $s_0 \in (0,1)$ such that $\mathcal{F}_s(u_0^s) \leq \mathcal{F}_0(u_0) + 1/2$ for any $s \in (0, s_0)$. Denoting by $|(u^s)'|(t)$ the Wasserstein metric derivative of the curve $t \mapsto u^s(t)$, by (3.11) it holds that

$$\int_0^{+\infty} |(u^s)'|^2(r) \, \mathrm{d}r \leq 2\mathcal{F}_s(u_0^s) \leq 2\mathcal{F}_0(u_0) + 1. \tag{8.5}$$

We have tightness and equicontinuity of the family $\{u^s\}_{s\in(0,s_0)}$. Indeed, fixing $T > 0$, the estimate

$$\begin{aligned} W^2\left(u^s(t), \delta_0\right) &\leq 2W^2\left(u^s(t), u_0^s\right) + 2W^2\left(u_0^s, \delta_0\right) \\ &\leq 2t \int_0^t |\left(u^s\right)'|^2(r) \, \mathrm{d}r + 2\int |x|^2 u_0^s(x) \, \mathrm{d}x \\ &\leq 2T\left(2\mathcal{F}_0(u_0) + 1\right) + 2 \sup_{s\in(0,s_0)} \int |x|^2 u_0^s(x) \, \mathrm{d}x \end{aligned}$$

implies that the set $\{u^s(t) : s \in (0, s_0), t \in [0, T]\}$ is tight, and consequently, by the Prokhorov Theorem, narrowly compact.

By (8.5) there exists $m \in L^2(0, +\infty)$ such that the sequence $\{|(u^{s_n})'|\}$ converges to m (up to subsequences) weakly in $L^2(0, +\infty)$. Then, for every $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$, it holds that

$$\limsup_{n\to\infty} W(u^{s_n}(t_2), u^{s_n}(t_1)) \le \lim_{n\to\infty} \int_{t_1}^{t_2} |(u^{s_n})'|(r)\, \mathrm{d}r = \int_{t_1}^{t_2} m(r)\, \mathrm{d}r, \quad (8.6)$$

and the equicontinuity is proved. By the compactness argument of [2, Proposition 3.3.1], we obtain the existence of a continuous limit curve u such that (8.1) holds. In particular, since for $t > 0$, $u^s(t)$ is absolutely continuous with respect to the Lebesgue measure, (8.1) translates (for $t > 0$) into

$$\int_{\mathbb{R}^d} u^{s_n}(t, x)\phi(x)\, \mathrm{d}x \to \int_{\mathbb{R}^d} u(t, x)\phi(x)\, \mathrm{d}x, \quad \forall t > 0 \;\; \forall \phi \in C_b(\mathbb{R}^d). \quad (8.7)$$

Passing to the limit in (8.6) we obtain

$$W(u(t_2), u(t_1)) \le \int_{t_1}^{t_2} m(r)\, \mathrm{d}r, \qquad \forall t_1, t_2 \in [0, +\infty), \quad t_1 < t_2,$$

and $u \in AC^2([0, +\infty); (\mathcal{P}_2(\mathbb{R}^d), W))$.

We fix $\sigma > 0$ such that $\sigma < \min\{s_0, 1/2\}$. For $s \in (0, \sigma]$, the energy inequality (1.5) yields

$$\|u^s(t)\|^2_{H^{-\sigma}(\mathbb{R}^d)} \le 2\mathcal{F}_0(u_0) + 1, \qquad \forall s \in (0, \sigma], \quad \forall t \in [0, +\infty). \quad (8.8)$$

We fix a compact $K \subset \mathbb{R}^d$ and a compactly supported smooth cutoff function $\phi : \mathbb{R}^d \to [0, 1]$ such that $\phi = 1$ on K. By interpolation we have

$$\begin{aligned}
\|u^s(t) - u(t)\|_{L^2(K)} &\le \|\phi u^s(t) - \phi u(t)\|_{L^2(\mathbb{R}^d)} \\
&\le \|\phi u^s(t) - \phi u(t)\|^{1/2}_{H^{-1/2}(\mathbb{R}^d)} \|\phi u^s(t) - \phi u(t)\|^{1/2}_{H^{1/2}(\mathbb{R}^d)} \\
&\le C\|\phi u^s(t) - \phi u(t)\|^{1/2}_{H^{-1/2}(\mathbb{R}^d)} \|u^s(t) - u(t)\|^{1/2}_{H^{1/2}(\mathbb{R}^d)}.
\end{aligned}$$

By (8.8), (8.7) and the compact embedding of Sobolev spaces it follows that (up to subsequences) $\lim_{n\to+\infty} \|\phi u^{s_n}(t) - \phi u(t)\|^{1/2}_{H^{-1/2}(\mathbb{R}^d)} = 0$.

We fix $T_0 > 0$ and $T > T_0$. By (6.3), (4.5) and (6.2), for $s \le 1/2$ we have

$$\begin{aligned}
&\int_{T_0}^{T} \|u^s(t)\|^2_{H^{1/2}(\mathbb{R}^d)}\, \mathrm{d}t \le \int_{T_0}^{T} \|u^s(t)\|^2_{H^{1-s}(\mathbb{R}^d)}\, \mathrm{d}t \\
&\le \left(1 + C_0 T_0^{-\gamma_0} + T\mathcal{F}_s\left(u_0^s\right) + \int_{\mathbb{R}^d} |x|^2 u_0^s(x)\, \mathrm{d}x\right) \\
&+ \left(2\mathcal{F}_s\left(u_0^s\right)(T - T_0)\right)^{1-s} \left(C_0 T_0^{-\gamma_0}\right. \\
&\left.+ c\left(1 + T\mathcal{F}_s\left(u_0^s\right) + \int_{\mathbb{R}^d} |x|^2\, \mathrm{d}u_0^s(x)\right)\right)^s,
\end{aligned}$$

where the dependence of the constants C_0 and γ_0 on s is stated in Lemma 4.10. Since C_0 is bounded with respect to s, it follows that

$$\sup_{n\in\mathbb{N}} \int_{T_0}^{T} \|u^{s_n}(t)\|^2_{H^{1/2}(\mathbb{R}^d)}\,\mathrm{d}t < +\infty, \qquad \int_{T_0}^{T} \|u(t)\|^2_{H^{1/2}(\mathbb{R}^d)}\,\mathrm{d}t < +\infty.$$

By the previous estimates and dominated convergence theorem we obtain (8.2).

Analogously, from (6.4) we obtain

$$\int_{T_0}^{T} \|u^s(t)\|^2_{\dot H^{1-2s}(\mathbb{R}^d)}\,\mathrm{d}t \\ \leq \big(2\mathcal{F}_s\big(u_0^s\big)(T-T_0)\big)^s \Big(C_0 T_0^{-\gamma_0} + c\big(T\mathcal{F}_s\big(u_0^s\big) + \int_{\mathbb{R}^d} |x|^2 u_0^s(x)\,\mathrm{d}x\big)\Big)^{1-s}.$$

Since $\|\nabla v^s(t)\|_{L^2(\mathbb{R}^d)} = \|u^s(t)\|_{\dot H^{1-2s}(\mathbb{R}^d)}$, taking into account that C_0 is bounded as $s\to 0$, from the previous estimate it follows that $\{\nabla v^s\}_{s\in(0,\sigma)}$ is weakly compact in $L^2((T_0,T);L^2(\mathbb{R}^d))$. Moreover ∇v_{s_n} converges to ∇u in the sense of distributions in $\mathbb{R}^d\times(T_0,T)$. Indeed for $\varphi\in C_c^\infty(\mathbb{R}^d\times(T_0,T);\mathbb{R}^d)$, denoting by φ_t the function $x\mapsto\varphi(x,t)$, by Plancherel's Theorem we have

$$(2\pi)^d \int_{T_0}^{T}\int_{\mathbb{R}^d} \nabla v^{s_n}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t = -i\int_{T_0}^{T}\int_{\mathbb{R}^d} |\xi|^{-2s_n}\widehat{u^{s_n}(t)}(-\xi)\xi\cdot\widehat{\varphi}_t(\xi)\,\mathrm{d}\xi\,\mathrm{d}t.$$

Since $||\xi|^{-2s_n}\widehat{u^{s_n}(t)}(-\xi)\xi\cdot\widehat{\varphi}_t(\xi)| \leq \max\{1,|\xi|\}|\widehat{\varphi}_t(\xi)|$ and $\widehat{\varphi}_t\in\mathcal{S}(\mathbb{R}^d)$ for every $t\in(T_0,T)$, by (8.1) and Lebesgue dominated convergence the right hand side of the above formula converges to

$$-i\int_{T_0}^{T}\int_{\mathbb{R}^d} \widehat{u(t)}(-\xi)\xi\cdot\widehat{\varphi}_t(\xi)\,\mathrm{d}\xi\,\mathrm{d}t = (2\pi)^d\int_{T_0}^{T}\int_{\mathbb{R}^d}\nabla u\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

For the stated compactness in $L^2((T_0,T);L^2(\mathbb{R}^d))$ we obtain (8.3).

As a result, we can easily pass to the limit in the weak formulation of the equation. Concerning the limit procedure in the energy inequality, we observe that by (8.1) and Fatou's lemma we obtain

$$\liminf_{s\to 0}\mathcal{F}_s(u^s(t)) \geq \mathcal{F}_0(u(t)).$$

Moreover by Lemma 6.4 and the stated weak convergence we obtain

$$\liminf_{s\to 0}\int_0^T\int_{\mathbb{R}^d} |\nabla v^s(t)|^2 u^s(t)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_{\mathbb{R}^d}|\nabla u(t)|^2 u(t)\,\mathrm{d}x\,\mathrm{d}t,$$

and we conclude. □

*Proof of Theorem 1.3***.** The proof follows by the previous Lemma and the uniqueness of the solution of equation (1.6) with initial datum in $L^2(\mathbb{R}^d)$ satisfying the energy inequality (see [2, Theorem 11.2.5], which also shows that this unique solution satisfies all the properties of [2, Theorem 11.2.1], in particular the energy identity). □

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